Three high performance simplex solvers

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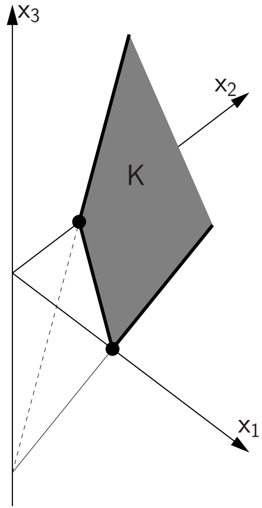
Overview

- Background
  - Dual vs primal simplex method
- Three high performance simplex solvers
  - `EMSOL` (1992–2004)
  - `PIPS-S` (2011–date)
  - `hsol` (2011–date)
- The future

Solving LP problems: Characterizing a basis



$$\begin{array}{ll} \text{minimize} & f = \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

- A **vertex** of the **feasible region** $K \subset \mathbb{R}^n$ has
  - m **basic** components, $i \in \mathcal{B}$ given by $A\boldsymbol{x} = \boldsymbol{b}$
  - $n - m$ zero **nonbasic** components, $j \in \mathcal{N}$

  where $\mathcal{B} \cup \mathcal{N}$ partitions $\{1, \ldots, n\}$
- Equations partitioned according to $\mathcal{B} \cup \mathcal{N}$ as

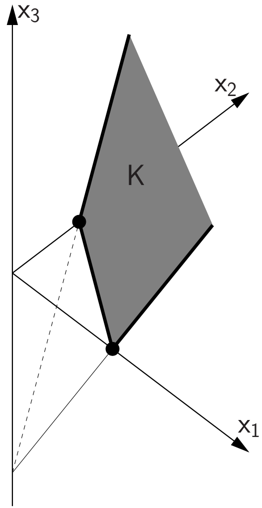
  $$B\boldsymbol{x}_B + N\boldsymbol{x}_N = \boldsymbol{b}$$

  with nonsingular **basis matrix** B
- Solution set of $A\boldsymbol{x} = \boldsymbol{b}$ characterized by

  $$\boldsymbol{x}_B = \widehat{\boldsymbol{b}} - B^{-1}N\boldsymbol{x}_N$$

  where $\widehat{\boldsymbol{b}} = B^{-1}\boldsymbol{b}$

Solving LP problems: Optimality conditions



$$\begin{array}{ll}\text{minimize} & f = \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0}\end{array}$$

- Objective partitioned according to $\mathcal{B} \cup \mathcal{N}$ as

$$\begin{aligned} f &= \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N \\ &= \widehat{f} + \widehat{\boldsymbol{c}}_N^T\boldsymbol{x}_N \end{aligned}$$

  where $\widehat{f} = \boldsymbol{c}_B^T\widehat{\boldsymbol{b}}$ and $\widehat{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T B^{-1}N$ is the vector of **reduced costs**

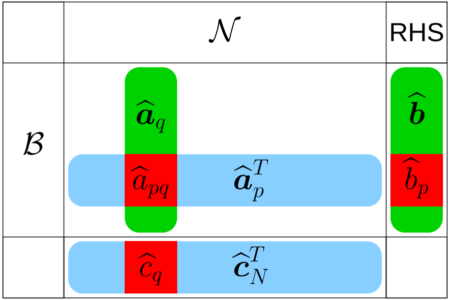
- Partition yields an optimal solution if there is
  - Primal feasibility $\widehat{\boldsymbol{b}} \geq \boldsymbol{0}$
  - Dual feasibility $\widehat{\boldsymbol{c}}_N \geq \boldsymbol{0}$

Primal simplex algorithm

Assume $\widehat{\boldsymbol{b}} \geq \mathbf{0}$ Seek $\widehat{\boldsymbol{c}}_N \geq \mathbf{0}$

Scan $\widehat{c}_j$ for q to leave $\mathcal{N}$

Scan $\widehat{b}_i/\widehat{a}_{iq} < 0$ for p to leave $\mathcal{B}$



Update: Exchange p and q between $\mathcal{B}$ and $\mathcal{N}$

Update $\widehat{\boldsymbol{b}} := \widehat{\boldsymbol{b}} - \alpha_p \widehat{\boldsymbol{a}}_q$ $\quad \alpha_p = \widehat{b}_p/\widehat{a}_{pq}$

Update $\widehat{\boldsymbol{c}}_N^T := \widehat{\boldsymbol{c}}_N^T - \alpha_d \widehat{\boldsymbol{a}}_p^T$ $\quad \alpha_d = \widehat{c}_q/\widehat{a}_{pq}$

Data required

- Pivotal row $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{e}_p^T B^{-1} N$
- Pivotal column $\widehat{\boldsymbol{a}}_q = B^{-1}\boldsymbol{a}_q$

Why does it work?

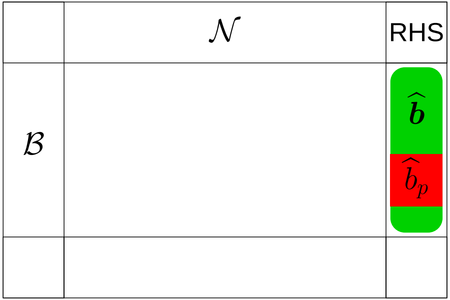
Objective improves by $-\dfrac{\widehat{b}_p \times \widehat{c}_q}{\widehat{a}_{pq}}$ each iteration

Dual simplex algorithm

Assume $\widehat{\boldsymbol{c}}_N \geq \mathbf{0}$ Seek $\widehat{\boldsymbol{b}} \geq \mathbf{0}$

Scan $\widehat{b}_i < 0$ for p to leave $\mathcal{B}$

Scan $\widehat{c}_j / \widehat{a}_{pj}$ for q to leave $\mathcal{N}$



Dual simplex algorithm

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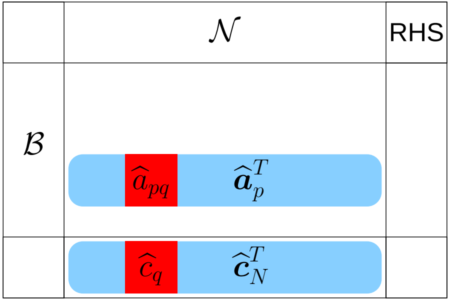
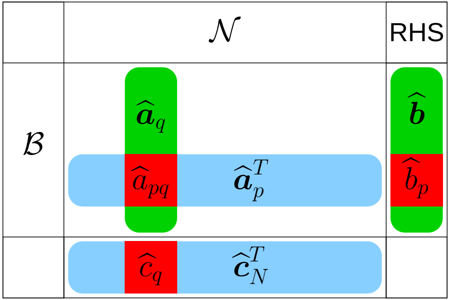


Dual simplex algorithm



Assume $\widehat{\boldsymbol{c}}_N \geq \mathbf{0}$ Seek $\widehat{\boldsymbol{b}} \geq \mathbf{0}$

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Scan $\widehat{c}_j / \widehat{a}_{pj}$ for q to leave $\mathcal{N}$

Update: Exchange p and q between $\mathcal{B}$ and $\mathcal{N}$

Update $\widehat{\boldsymbol{b}} := \widehat{\boldsymbol{b}} - \alpha_p \widehat{\boldsymbol{a}}_q$ $\quad \alpha_p = \widehat{b}_p / \widehat{a}_{pq}$

Update $\widehat{\boldsymbol{c}}_N^T := \widehat{\boldsymbol{c}}_N^T - \alpha_d \widehat{\boldsymbol{a}}_p^T$ $\quad \alpha_d = \widehat{c}_q / \widehat{a}_{pq}$

Data required

- Pivotal row $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{e}_p^T B^{-1} N$
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Why does it work?

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Simplex method: Computation

Standard simplex method (SSM): Major computational component

	$\mathcal{N}$	RHS
$\mathcal{B}$	$\widehat{N}$	$\widehat{\boldsymbol{b}}$
	$\widehat{\boldsymbol{c}}_N^T$	

Update of tableau: $\widehat{N} := \widehat{N} - \dfrac{1}{\widehat{a}_{pq}}\widehat{\boldsymbol{a}}_q\widehat{\boldsymbol{a}}_p^T$

- Hopelessly inefficient for sparse LP problems
- Prohibitively expensive for large LP problems

Revised simplex method (RSM): Major computational components

Pivotal row via $B^T\boldsymbol{\pi}_p = \boldsymbol{e}_p$ `BTRAN` and $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$ `PRICE`

Pivotal column via $B\,\widehat{\boldsymbol{a}}_q = \boldsymbol{a}_q$ `FTRAN` Invert B

Simplex algorithm: Primal or dual?

Primal simplex algorithm

- Traditional variant
- Solution is generally not primal feasible when LP is tightened

Dual simplex algorithm

- Preferred variant
- Easier to get dual feasibility
- More progress in many iterations
- Solution is dual feasible when LP is tightened

EMSOL (1992–2004)

Overview

- Written in FORTRAN to study parallel simplex
- Primal simplex
- Product form update for $B := B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T$
  - simple and easy to parallelize
  - inefficient and numerically unstable

Output: Parallel codes

- ASYNPLEX and PARSMI developed for Cray T3D
- Slave processors send attractive tableau columns to a master processor which keeps them up-to-date and determines basis changes
- Modest speed-up on general sparse LP problems
- Non-deterministic

H and McKinnon (1995–1998)

EMSOL: Theoretical and serial outputs

Output: Simplest LPs which cycle

- Each simplex iteration improves the objective by $-\frac{\widehat{b}_p \times \widehat{c}_q}{\widehat{a}_{pq}}$
  - Termination guaranteed if all $\widehat{b}_p > 0$
  - **Cycling** can occur if some $\widehat{b}_p = 0$ (degeneracy)
- Found the family of (provably) simplest LPs which cycle, *even* with the leading practical anti-degeneracy technique H and McKinnon (1996–2004)

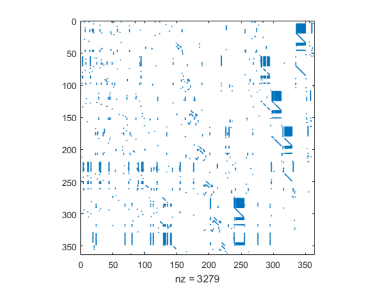
Output: Hyper-sparsity

- Solution of $B\boldsymbol{x} = \boldsymbol{r}$ is $\boldsymbol{x} = B^{-1}\boldsymbol{r}$
- In the simplex method, B and $\boldsymbol{r}$ are sparse
- For some LPs $\boldsymbol{x}$ is typically sparse: **hyper-sparsity**

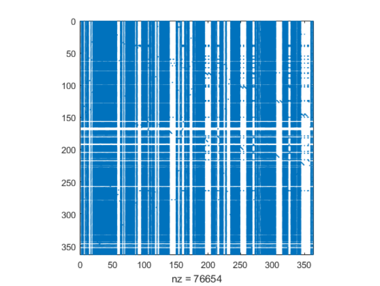
Remarkable?

Inverse of a sparse matrix and solution of $B\boldsymbol{x} = \boldsymbol{r}$

Optimal B for LP problem `stair`

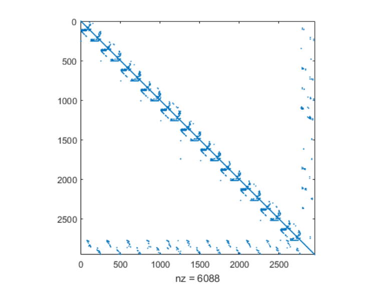


B^{-1} has density of 58%, so $B^{-1}\boldsymbol{r}$ is typically dense

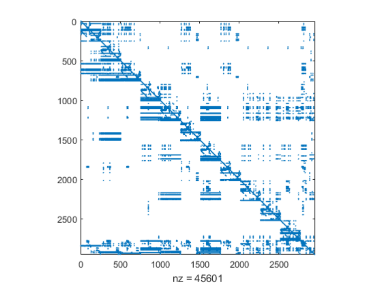


Inverse of a sparse matrix and solution of $Bx = r$

Optimal B for LP problem pds-02



B^{-1} has density of 0.52%, so $B^{-1}\boldsymbol{r}$ is typically sparse



EMSOL: Exploiting hyper-sparsity

To solve $B\boldsymbol{x} = \boldsymbol{r}$

- Using decomposition of B given by $\{p_k, \mu_k, \boldsymbol{\eta}_k\}_{k=1}^{K}$
- Traditional technique transforms RHS into solution $\boldsymbol{x}$

do $k = 1,\ K$

$$r_{p_k} := r_{p_k}/\mu_k$$
$$\boldsymbol{r} := \boldsymbol{r} - r_{p_k}\boldsymbol{\eta}_k$$

end do

- When initial RHS is sparse: inefficient until $\boldsymbol{r}$ fills in

EMSOL: Exploiting hyper-sparsity

To solve $B\boldsymbol{x} = \boldsymbol{r}$

- Using decomposition of B given by $\{p_k, \mu_k, \boldsymbol{\eta}_k\}_{k=1}^{K}$
- When $\boldsymbol{r}$ is sparse skip $\boldsymbol{\eta}_k$ if r_{p_k} is zero

```
do k = 1, K
   if (r_{p_k} .ne. 0) then
      r_{p_k} := r_{p_k}/μ_k
      r := r − r_{p_k} η_k
   end if
end do
```

- When $\boldsymbol{x}$ is sparse, the dominant cost is the test for zero
- Requires efficient identification of vectors $\boldsymbol{\eta}_k$ to be applied

Gilbert and Peierls (1988)
H and McKinnon (1998–2005)
COAP best paper prize (2005)

PIPS-S (2011–date)

Overview

- Written in C++ to solve stochastic MIP relaxations in parallel
- Dual simplex
- Based on NLA routines in `clp`
- Product form update

Concept

- Exploit data parallelism due to block structure of LPs
- Distribute problem over processes

Paper: Lubin, H, Petra and Anitescu (2013)

- COIN-OR INFORMS 2013 Cup
- COAP best paper prize (2013)

PIPS-S: Stochastic MIP problems

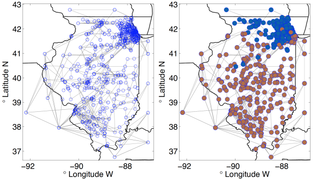
Two-stage stochastic LPs have column-linked block angular (BALP) structure

$$
\begin{array}{rccccccccccc}
\text{minimize} & \mathbf{c}_0^T\mathbf{x}_0 & + & \mathbf{c}_1^T\mathbf{x}_1 & + & \mathbf{c}_2^T\mathbf{x}_2 & + & \dots & + & \mathbf{c}_N^T\mathbf{x}_N & & \\
\text{subject to} & A\mathbf{x}_0 & & & & & & & & & = & \mathbf{b}_0 \\
& T_1\mathbf{x}_0 & + & W_1\mathbf{x}_1 & & & & & & & = & \mathbf{b}_1 \\
& T_2\mathbf{x}_0 & & & + & W_2\mathbf{x}_2 & & & & & = & \mathbf{b}_2 \\
& \vdots & & & & & & \ddots & & & & \vdots \\
& T_N\mathbf{x}_0 & & & & & & & + & W_N\mathbf{x}_N & = & \mathbf{b}_N \\
& \mathbf{x}_0 \geq \mathbf{0} & & \mathbf{x}_1 \geq \mathbf{0} & & \mathbf{x}_2 \geq \mathbf{0} & & \dots & & \mathbf{x}_N \geq \mathbf{0} & &
\end{array}
$$

- Variables $\mathbf{x}_0 \in \mathbb{R}^{n_0}$ are **first stage** decisions
- Variables $\mathbf{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \dots, N$ are **second stage** decisions
  Each corresponds to a **scenario** which occurs with modelled probability
- The objective is the expected cost of the decisions
- In stochastic MIP problems, some/all decisions are discrete

PIPS-S: Stochastic MIP problems

- Power systems optimization project at Argonne
- Integer second-stage decisions
- Stochasticity from wind generation
- Solution via branch-and-bound
  - Solve root using parallel IPM solver `PIPS`
    Lubin, Petra *et al.* (2011)
  - Solve nodes using parallel dual simplex solver `PIPS-S`



PIPS-S: Stochastic MIP problems

Convenient to permute the LP thus:

$$
\begin{array}{rccccccccccc}
\text{minimize} & \boldsymbol{c}_1^T\boldsymbol{x}_1 & + & \boldsymbol{c}_2^T\boldsymbol{x}_2 & + & \dots & + & \boldsymbol{c}_N^T\boldsymbol{x}_N & + & \boldsymbol{c}_0^T\boldsymbol{x}_0 & & \\
\text{subject to} & W_1\boldsymbol{x}_1 & & & & & & & + & T_1\boldsymbol{x}_0 & = & \boldsymbol{b}_1 \\
 & & & W_2\boldsymbol{x}_2 & & & & & + & T_2\boldsymbol{x}_0 & = & \boldsymbol{b}_2 \\
 & & & & & \ddots & & & \vdots & & & \vdots \\
 & & & & & & & W_N\boldsymbol{x}_N & + & T_N\boldsymbol{x}_0 & = & \boldsymbol{b}_N \\
 & & & & & & & & & A\boldsymbol{x}_0 & = & \boldsymbol{b}_0 \\
 & \boldsymbol{x}_1 \geq \boldsymbol{0} & & \boldsymbol{x}_2 \geq \boldsymbol{0} & & \dots & & \boldsymbol{x}_N \geq \boldsymbol{0} & & \boldsymbol{x}_0 \geq \boldsymbol{0} & &
\end{array}
$$

PIPS-S: Exploiting problem structure

- Inversion of the basis matrix B is key to revised simplex efficiency
- For column-linked BALP problems

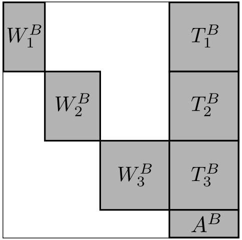
$$B = \begin{bmatrix} W_1^B & & & T_1^B \\ & \ddots & & \vdots \\ & & W_N^B & T_N^B \\ & & & A^B \end{bmatrix}$$

- W_i^B are columns corresponding to n_i^B basic variables in scenario i
- $\begin{bmatrix} T_1^B \\ \vdots \\ T_N^B \\ A^B \end{bmatrix}$ are columns corresponding to n_0^B basic first stage decisions

PIPS-S: Exploiting problem structure

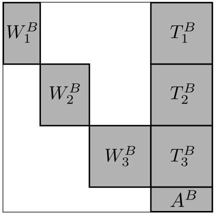
- Inversion of the basis matrix B is key to revised simplex efficiency
- For column-linked BALP problems

$$B = \begin{bmatrix} W_1^B & & & T_1^B \\ & \ddots & & \vdots \\ & & W_N^B & T_N^B \\ & & & A^B \end{bmatrix}$$

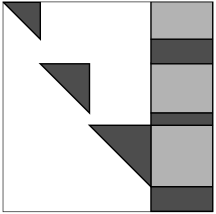


- B is nonsingular so
  - W_i^B are "tall": full column rank
  - $\begin{bmatrix} W_i^B & T_i^B \end{bmatrix}$ are "wide": full row rank
  - A^B is "wide": full row rank
- Scope for parallel inversion is immediate and well known

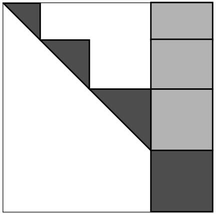
PIPS-S: Exploiting problem structure



- Eliminate sub-diagonal entries in each W_i^B (independently)



- Apply elimination operations to each T_i^B (independently)



- Accumulate non-pivoted rows from the W_i^B with A^B and complete elimination

PIPS-S: Overview

Scope for parallelism

- Parallel Gaussian elimination yields **block LU** decomposition of B
- Scope for parallelism in block forward and block backward substitution
- Scope for parallelism in PRICE

Implementation

- Distribute problem data over processes
- Perform data-parallel BTRAN, FTRAN and PRICE over processes
- Used MPI

PIPS-S: Results

On Fusion cluster: Performance relative to clp

Dimension	Cores	Storm	SSN	UC12	UC24
$m+n=O(10^6)$	1	0.34	0.22	0.17	0.08
	32	8.5	6.5	2.4	0.7
$m+n=O(10^7)$	256	299	45	67	68

On Blue Gene

- Instance of UC12
- $m+n=O(10^8)$
- Requires 1 TB of RAM
- Runs from an advanced basis

Cores	Iterations	Time (h)	Iter/sec
1024	Exceeded execution time limit		
2048	82,638	6.14	3.74
4096	75,732	5.03	4.18
8192	86,439	4.67	5.14

`hsol` (2011–date)

Overview

- Written in C++ to study parallel simplex
- Dual simplex with steepest edge and BFRT
- Forrest-Tomlin update
  - complex and inherently serial
  - efficient and numerically stable

Concept

- Exploit limited task and data parallelism in standard dual RSM iterations (`sip`)
- Exploit greater task and data parallelism via minor iterations of dual SSM (`pami`)
- Test-bed for research
- Work-horse for consultancy

Huangfu, H and Galabova (2011–date)

`hsol`: A high performance simplex solver

Features

- Model management: Add/delete/modify problem data
- Efficiency: High performance serial and parallel computational components
- Open-source C++

Presolve

- Presolve (and corresponding postsolve) has been implemented efficiently

  Remove redundancies in the LP to reduce problem dimension

  Galabova (2016)

Crash

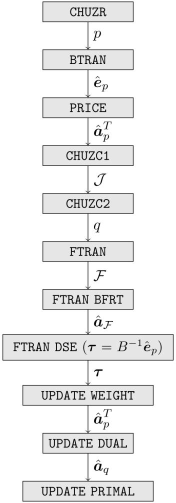
- Dual simplex "triangular basis" crash

  Can be valuable, even with dual steepest edge weights

- Alternative crash techniques being studied

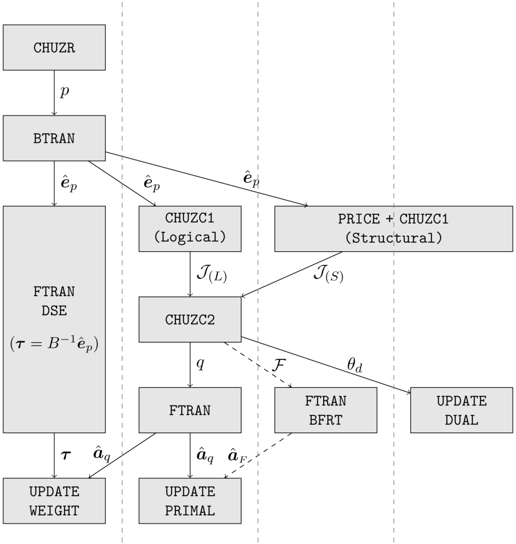
  H and Galabova (2016–date)

hsol: Single iteration parallelism with sip option



- Computational components appear sequential
- Each has highly-tuned sparsity-exploiting serial implementation
- Exploit "slack" in data dependencies

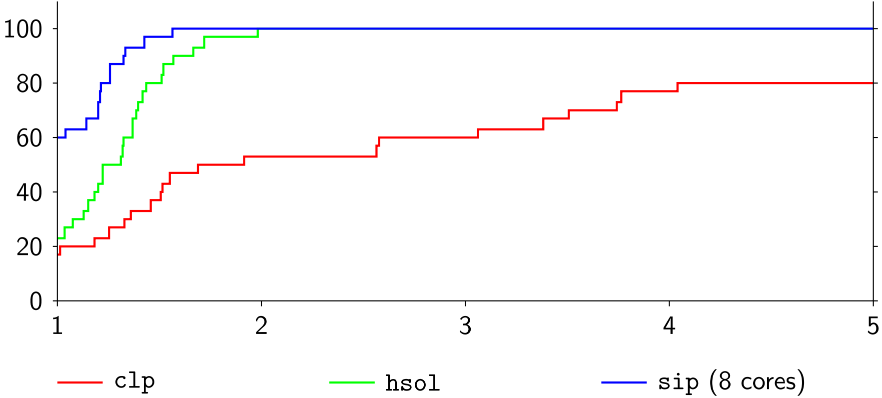
hsol: Single iteration parallelism with sip option



- Parallel PRICE to form $\hat{\mathbf{a}}_p^T = \boldsymbol{\pi}_p^T N$
- Other computational components serial
- Overlap any independent calculations
- Only four worthwhile threads unless $n \gg m$ so PRICE dominates
- More than Bixby and Martin (2000)
- Better than Forrest (2012)

Huangfu and H (2014)

hsol: clp vs hsol vs sip

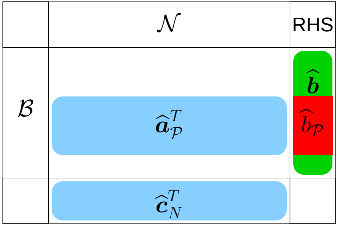


Performance on spectrum of 30 significant LP test problems

- hsol is 2.3 times faster than clp
- sip on 8 cores is 1.2 times faster than hsol; 2.6 times faster than clp

`hsol`: Multiple iteration parallelism with `pami` option

- Perform standard dual simplex minor iterations for rows in set $\mathcal{P}$ ($|\mathcal{P}| \ll m$)
- Suggested by Rosander (1975) but never implemented efficiently *in serial*



- Task-parallel multiple `BTRAN` to form $\boldsymbol{\pi}_{\mathcal{P}} = B^{-1}\boldsymbol{e}_{\mathcal{P}}$
- Data-parallel `PRICE` to form $\widehat{\boldsymbol{a}}_p^T$ (as required)
- Task-parallel multiple `FTRAN` for primal, dual and weight updates

Huangfu and H (2011–2014)
COAP best paper prize (2015)

`hsol`: Performance and reliability

Extended testing using 159 test problems

- 98 Netlib
- 16 Kennington
- 4 Industrial
- 41 Mittelmann

Exclude 7 which are "hard"

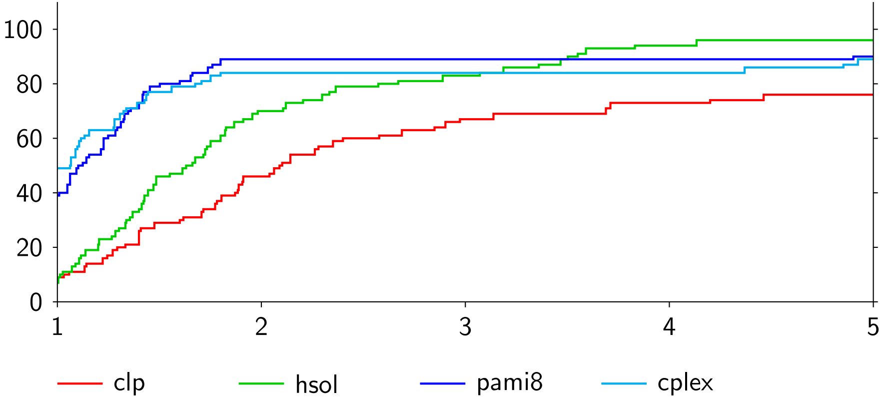
Performance

Benchmark against `clp` (v1.16) and `cplex` (v12.5)

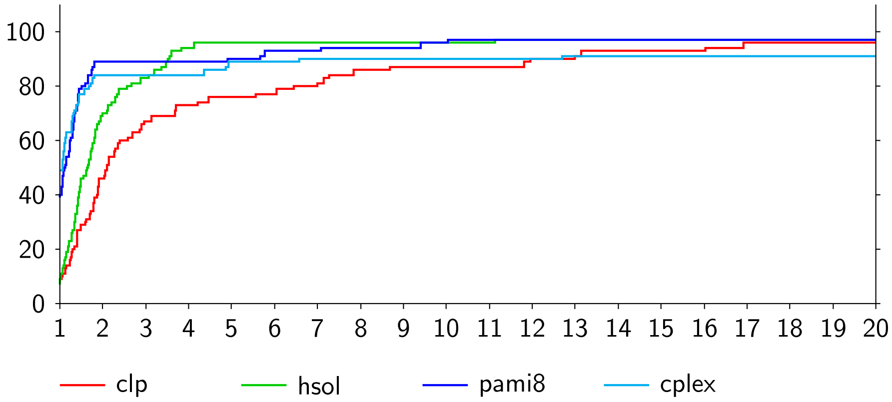
- Dual simplex
- No presolve
- No crash

Ignore results for 82 LPs with minimum solution time below 0.1s

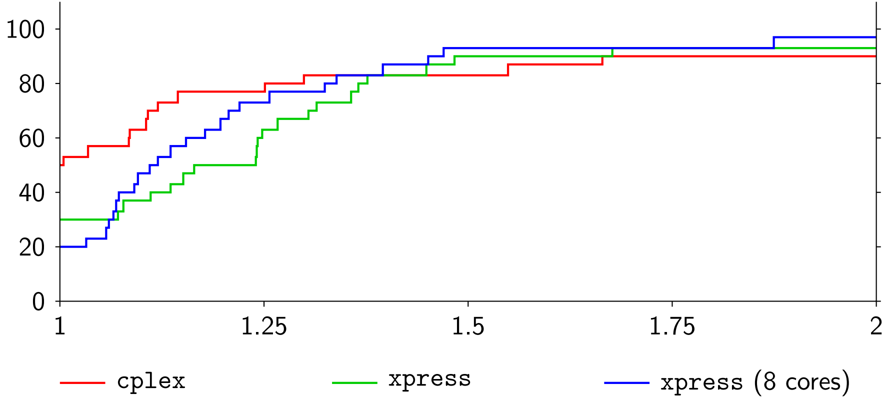
hsol: Performance



hsol: Reliability



hsol: Impact



- pami ideas incorporated in FICO Xpress (Huangfu 2014)
- Xpress simplex solver now fastest commercial simplex solver

`hsol`: Future

For 2017

- SCIP interface
- QP solver
- MIP solver

For 2018

- MIP presolve
- MIQP solver

Long term

Replacement for `clp`?

Commercial involvement

- Cargill (feed formulation)
- Google (techniques in `glop`)
- Financial services

Conclusions

- `EMSOL`: Parallel variants; Advance in serial simplex
- `PIPS-S`: Large scale data-parallelism for special problems
- `hsol`: Task and data parallelism; test-bed for further research

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Hyper-sparsity in the revised simplex method and how to exploit it.
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M. Lubin, J. A. J. Hall, C. G. Petra, and M. Anitescu.
Parallel distributed-memory simplex for large-scale stochastic LP problems.
Computational Optimization and Applications, 55(3):571–596, 2013.

Slides: http://www.maths.ed.ac.uk/hall/Argonne17