Exact solution of the Riemann problem for the shallow water equations

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Abstract

This report presents the exact solution of the Riemann problem for the non-linear one-dimensional shallow water equations (SWEs) with zero bottom topography. The considered system is hyperbolic and admits a fully conservative form which would not be the case with non-zero bottom topography. This shallow water system is then extended to a one-dimensional symmetric system in which spatial variation in the y-direction is ignored at leading order. As a final extension, rotation will be added to the momentum equations via Coriolis terms. There is no exact closed-form solution for this final case. Solving the Riemann problem is an essential element of the implementation of the (Godunov) finite volume numerical scheme and other modern numerical upwind schemes.

1 One-dimensional shallow water flows

1.1 Governing equations

Shallow water (SW) flows are ubiquitous in nature and their governing equations have wide applications in fluid dynamics. The shallow water equations (SWEs) approximately describe invisicid, incompressible free-surface fluid flows under the assumption that the depth of the fluid is much smaller than the wavelength of any disturbances to the free surface. The one-dimensional (1D) SWEs are:

$$\partial_t h + \partial_x(hu) = 0, \tag{1}$$

$$\partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) = 0, \tag{2}$$

where $h = h(x,t)$ is the fluid depth from the bottom to the free surface, $u = u(x,t)$ is the velocity of the fluid, and g is the (constant) acceleration due to gravity. These equations can be written in a coupled, fully conservative system:

$$\partial_t \mathsf{u} + \partial_x(\mathsf{f}(\mathsf{u})) = 0 \tag{3}$$

with $\mathsf{u} = (h, hu)^T$ and the flux function $\mathsf{f}(\mathsf{u}) = (hu, hu^2 + \frac{1}{2}gh^2)$. In such a conservative formulation, the SW system admits correct discontinuous solutions while ensuring that energy is lost across the discontinuities. It is useful here to note the equivalence with the isentropic equations of gas dynamics, or $P(\sigma)$-flow, defined as:

$$\partial_t \sigma + \partial_x m = 0, \tag{4}$$

$$\partial_t m + \partial_x \left(\frac{m^2}{\sigma} + P(\sigma) \right) = 0, \tag{5}$$

where P is an (effective) pressure as a function of a (pseudo-) denstity $\sigma = \sigma(x,t)$ and $m = m(x,t)$ is the momentum density defined as $m = \sigma u$. Setting $\sigma = h$ and $P(\sigma) = \frac{1}{2}g\sigma^2$ in (4) recovers the shallow water system (1). It is possible to construct the solution of the Riemann problem for this generalised $P(\sigma)$-flow.

The Riemann problem itself is a special initial value problem, defined as the given equations (3) together with special initial data consisting of a piecewise constant function with a single jump discontinuity:

$$\mathsf{u}(x, t_0) = \begin{cases} \mathsf{u}_l, & \text{for } x < x_0; \\ \mathsf{u}_r, & \text{for } x \geq x_0. \end{cases} \tag{6}$$

For a strictly hyperbolic system (real and distinct eigenvalues, say $\lambda_1 < ... < \lambda_m$), the structure of the solution of the Riemann problem in the x-t-plane consists of m waves emanating from the origin corresponding to each eigenvalue λ_i. To the left of the λ_1-wave the solution is simply the left initial state u_l, while to the right of the λ_2-wave the solution is the left initial state u_r. Each wave carries a jump discontinuity in u travlleing at its characteristic speed λ_i. The aim is then to find the solution between each wave pair, λ_i and λ_{i+1}.

Thus, the Riemann solution depends naturally on the eigen-structure of the system, in this case with $m = 2$. In regions of the domain where the solution to the SWEs is continuous, the SWEs can be written in the quasi-linear form, $\partial_t \mathsf{u} + \mathsf{f}'(\mathsf{u})\partial_x \mathsf{u} = 0$. The Jacobian matrix is:

$$\mathsf{f}'(\mathsf{u}) = \begin{bmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{bmatrix}.$$

The eigenvalues of $\mathsf{f}'(\mathsf{u})$ are $\lambda_1 = u - \sqrt{gh}$ and $\lambda_2 = u + \sqrt{gh}$ with the corresponding eigenvectors $r_1 = (1, u - \sqrt{gh})^T$ and $r_2 = (1, u + \sqrt{gh})^T$. The 1-characteristics (or simply left characteristics) λ_1 are associated with a 1-wave (or left wave), and similarly for λ_2. The type of wave (rarefaction or shock) is determined by the Riemann initial data (6); identifying these waves and the structure between them is central to the Riemann problem.

1.2 Solving the Riemann problem

A solution to the general Riemann problem is sought with arbitrary left and right states. To aid understanding of the problem, a specific example of a Riemann solution is shown in figure 1. This is known as the dam-break Riemann problem; it models what happens when a dam separating two levels of water bursts at $x = 0$ and time $t = 0$. It consists of the shallow water system with the piecewise constant initial data:

$$h(x,0) = \begin{cases} h_l, & \text{for } x < 0; \\ h_r, & \text{for } x \geq 0, \end{cases} \qquad u(x,0) = 0, \tag{7}$$

where $h_l > h_r \geq 0$. Between the left and right initial states, u_l and u_r, emerges a constant star state u_*. This star state is connected to the left and right states via a rarefaction or shock wave. The value of the star state and the mathematical structure of this connection are essential components of the problem and are derived in the following section for arbitrary left and right initial states u_l and u_r.

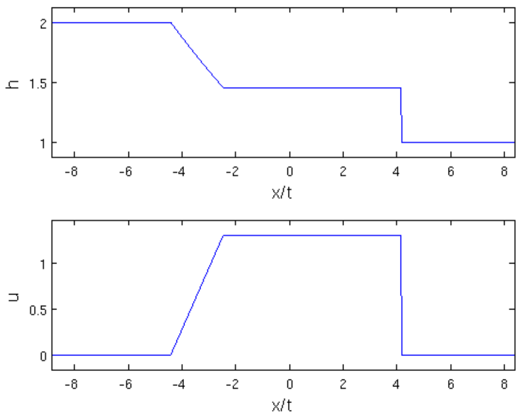
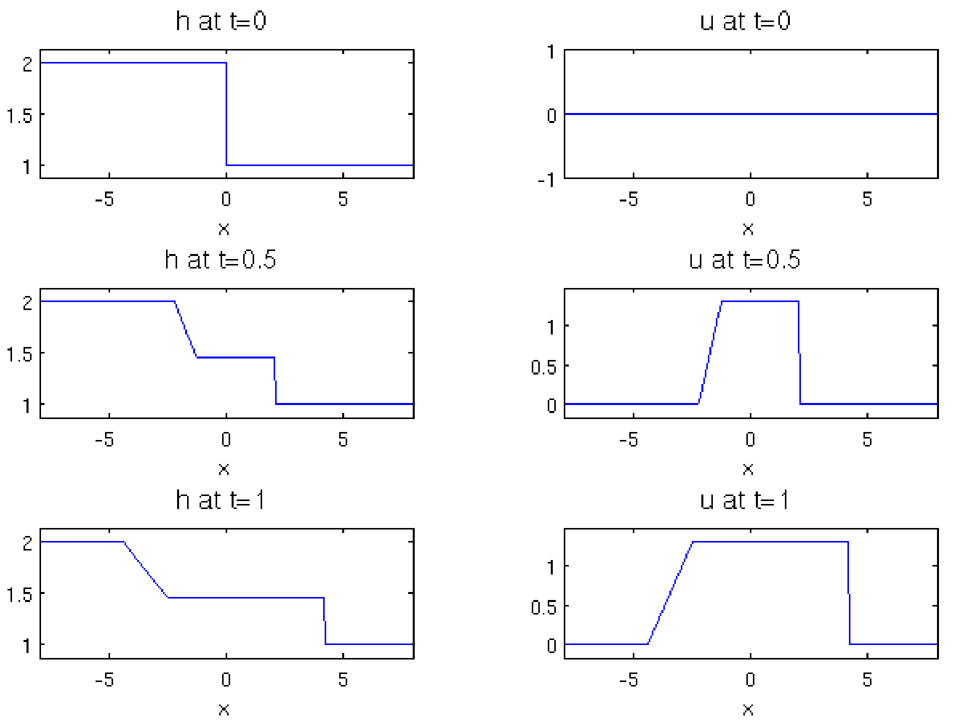


Figure 1: Similarity solution x/t of the dam-break Riemann problem for the shallow water equations: $h_l = 2, h_r = 1, u_l = u_r = 0$. Between the left and right initial states emerges a constant star state; in this case $h_l > h_* > h_r$ corresponding to a 'left rarefaction - right shock' Riemann solution.

Figure 2 illustrates the evolution of the depth and fluid velocity for the dam-break problem with $h_l = 2, h_r = 1$, and $u_l = u_r = 0$. The fluid flows from left to right (i.e. from the region of higher depth to the region of lower depth) in a body that expands from the dam location at $x = 0$. On the left hand side of this body, the fluid is moved away from the deeper stationary fluid through a rarefaction wave. On the right, fluid with intermediate depth h_* and velocity u_* collides with the stationary fluid at a lower depth, instantaneously accelerating it through a shock wave.

The dam-break Riemann solution is just one type of solution, resulting in a left rarefaction and right shock wave. In general, the Riemann solution consists of four states (made up of shock and/or rarefaction waves) which in principle allow 16 combinations. However, 12 cases can be excluded using physical arguments relating to entropy. This leaves four admissible solutions (see figure 3): (a) left shock and right shock, (b) left rarefaction and right shock, (c) left shock and right rarefaction, and finally (d) left rarefaction and right rarefaction. These four solutions are denoted LS-RS, LS-RW, LW-RS, and LW-RW respectively. The Riemann initial data determines



(a) Fluid depth $h(x, t)$ for given times t.

(b) Fluid velocity $u(x, t)$ for given times t.

Figure 2: Solution of the dam break Riemann problem for the SWEs.

uniquely which one of the four is correct.

2 The Riemann solution for 1D shallow water flows

A general strategy for solving the Riemann problem is as follows (after LeVeque):

- determine whether the 1-wave and 2-wave are shock or rarefaction waves;
- determine the intermediate star state u_* between the two waves;
- determine the location of shocks via shock speed and the head and tail of the rarefaction waves via the characteristic speeds;
- determine the structure of the solution through any rarefaction wave.

It is first necessary to consider an isolated wave separating two constant states and how these two states are connected. For shock waves, this connection is simply a step discontinuity, the location of which must be determined. For rarefaction waves, this connection is more complicated: the location of start and end of wave must be determined along with the curve connecting them.

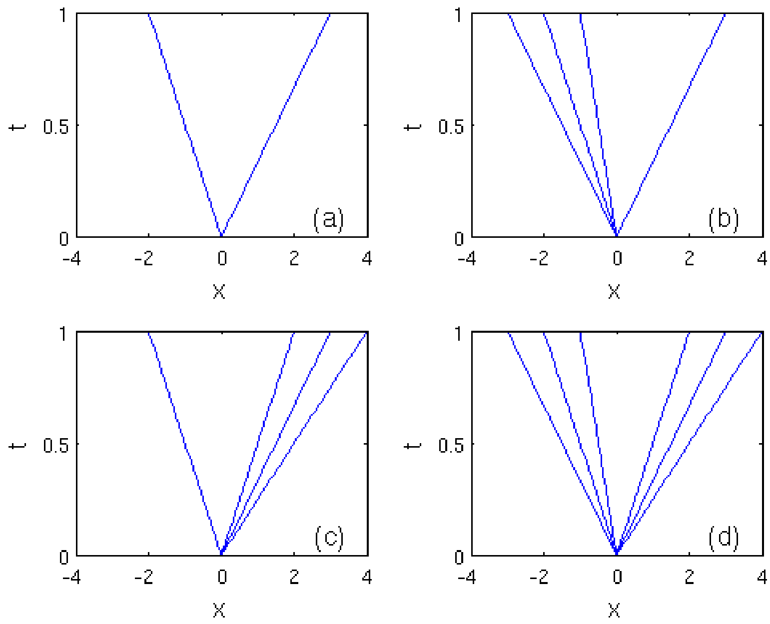


Figure 3: The four admissible solutions of the Riemann problem: (a) left shock and right shock, (b) left rarefaction and right shock, (c) left shock and right rarefaction, and (d) left rarefaction and right rarefaction. Note that the characteristics need not be positioned to the left and right of the t-axis, as illustrated here. As long as the left wave lies to the left of the right wave, the characteristics may lie anywhere on the x-t-plane with $t > 0$.

2.1 Shock wave solution

Consider first an isolated shock wave, in particular two constant states u_l and u_* which lead to a left shock wave. Across any shock the Rankine-Hugoniot relations must be satisfied. For a left shock with shock speed S_l, these relations read:

$$\mathsf{f}(\mathsf{u}_*) - \mathsf{f}(\mathsf{u}_l) = S_l(\mathsf{u}_* - \mathsf{u}_l). \tag{8}$$

For the SWEs, i.e. $\mathsf{u} = (h, hu)^T$ and $\mathsf{f}(\mathsf{u}) = (hu, hu^2 + \frac{1}{2}gh^2)$, these relations yield two equations to be solved simultaneously:

$$h_*u_* - h_lu_l = S_l(h_* - h_l), \tag{9}$$

$$h_*u_*^2 - h_lu_l^2 + \frac{1}{2}g(h_*^2 - h_l^2) = S_l(h_*u_* - h_lu_l). \tag{10}$$

We seek an expression for u^* and S_l in terms of u_l, h_l, and h_*. After introducing a cooridinate system moving with shock speed S_l, i.e. $\hat{u}_* = u_* - S_l$ and $\hat{u}_l = u_l - S_l$, the relations (9) transform into:

$$h_*\hat{u}_* = h_l\hat{u}_l, \tag{11}$$

$$h_*\hat{u}_*^2 + \frac{1}{2}gh_*^2 = h_l\hat{u}_l^2 + \frac{1}{2}gh_l^2. \tag{12}$$

Many manipulations later, we obtain the following expression for u_*:

$$u_* = u_l - (h_* - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_l} + \frac{1}{h_*}\right)}, \tag{13}$$

and for the left shock speed:

$$S_l = u_l - \frac{1}{h_l}\sqrt{\frac{g}{2}\left(h_l h_*(h_l + h_*)\right)}. \tag{14}$$

A similar expressions for u_* and S_r are obtained by considering a right shock wave separating two constant states u_* and u_r:

$$u_* = u_r + (h_* - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_*} + \frac{1}{h_r}\right)}, \tag{15}$$

$$S_r = u_r + \frac{1}{h_r}\sqrt{\frac{g}{2}\left(h_r h_*(h_r + h_*)\right)}. \tag{16}$$

Thus, for shock wave solutions we have:

$$u_* = \begin{cases} u_l - f_l(h_*, h_l), & \text{for left moving shock;} \\ u_r + f_r(h_*, h_r), & \text{for right moving shock,} \end{cases} \tag{17}$$

where:

$$f_k(h_*, h_k) = (h_* - h_k)\sqrt{\frac{g}{2}\left(\frac{1}{h_*} + \frac{1}{h_k}\right)}, \text{ with } k = l \text{ or } r. \tag{18}$$

This is a system of two equations (17) for two unknowns h_* and u_*; solving gives the star state in the Riemann solution. The location of the shock can be calculated directly from the shock speed $S_{l,r}$.

2.2 Rarefaction wave solution

Consider now an isolated rarefaction wave, i.e. a constant left state u_l and right state u_* and assume these values are such that a (left) rarefaction wave occurs. Unlike shock waves, rarefaction waves are continuous solutions of the Riemann problem and are associated with one of the characteristics of the system (i.e. one of the eigenvalues $\lambda_{1,2}$). To solve the Riemann problem for rarefaction waves, the concept of a Riemann invariant must be understood. The quasi-linear formulation of the system may be written:

$$\partial_t h + h\partial_x u + u\partial_x h = 0, \tag{19}$$
$$\partial_t u + u\partial_x u + g\partial_x h = 0, \tag{20}$$

and is identical to the conservative system (1) when dealing with continuous solutions. Eigenanalysis carried out in section 1.1 revealed that the system has two eigenvalues: $\lambda_1 = u - \sqrt{gh}$ and $\lambda_2 = u + \sqrt{gh}$. Multiplying (19) by $\sqrt{g/h}$ and noting that $\partial_\chi(2\sqrt{gh}) = \sqrt{g/h}\partial_\chi h$ for $\chi = x$ and t, the equations become:

$$\partial_t(2\sqrt{gh}) + \sqrt{gh}\partial_x u + u\partial_x(2\sqrt{gh}) = 0, \tag{21}$$
$$\partial_t u + u\partial_x u + \sqrt{gh}\partial_x(2\sqrt{gh}) = 0. \tag{22}$$

Then, adding or subtracting these equations yields:

$$\partial_t(u \pm 2\sqrt{gh}) + (u \pm \sqrt{gh})\partial_x(u \pm (2\sqrt{gh})) = 0. \tag{23}$$

Define $r_\pm = u \pm 2\sqrt{gh}$. Then $r_\pm$ are constant along the characterisics, i.e.

$$\frac{dr_\pm}{dt} = \frac{d}{dt}(u \pm 2\sqrt{gh}) = 0 \text{ along characteristics } \frac{dx}{dt} = \lambda_\pm = (u \pm \sqrt{gh}). \tag{24}$$

The function $r_\pm(u,h)$ is called a Riemann invariant: its value is invariant along the corresponding characterisic curve. This property allows one to find a relationship between the left and right state in a rarefaction wave.

Returning to the situation of an isolated rarefaction wave, a left wave is associated with the left eigenvalue $\lambda_1 = u - \sqrt{gh}$: the head of the wave is the characteristic $dx/dt = u_l - \sqrt{gh_l}$ while the tail of the wave is given by the characteristic $dx/dt = u_* - \sqrt{gh_*}$. Now we can exploit the property of the Riemann invariant, i.e. that one of the Riemann invariants is constant in the rarefaction wave, namely the one associated with the 2-characteristics. Thus, $u_* + 2\sqrt{gh_*} = u_l + 2\sqrt{gh_l}$ and an expression for u_* is obtained:

$$u_* = u_l - 2\sqrt{g}\left(\sqrt{h_*} - \sqrt{h_l}\right). \tag{25}$$

Again, a similar expression for u_* is obtained for a right rarefaction wave separated by two constant states u_* and u_r and related to the 2-eigenvalue $\lambda_2 = u + \sqrt{gh}$:

$$u_* = u_r + 2\sqrt{g}\left(\sqrt{h_*} - \sqrt{h_r}\right). \tag{26}$$

Thus, for rarefaction wave solutions we have:

$$u_* = \begin{cases} u_l - f_l(h_*, h_l), & \text{for left moving rarefaction;} \\ u_r + f_r(h_*, h_r), & \text{for right moving rarefaction,} \end{cases} \tag{27}$$

where:

$$f_k(h_*, h_k) = 2\sqrt{g}\left(\sqrt{h_*} - \sqrt{h_k}\right), \text{ with } k = l \text{ or } r. \tag{28}$$

Again, this is a system of two equations (27) for two unknowns h_* and u_*; solving gives the star state in the Riemann solution. The location of the rarefaction is determined by the speed of the head and tail of the wave and can be readily calculated given $\mathsf{u}_l, \mathsf{u}_r$ and u_*. For a left moving wave: $S_l^{\text{head}} = u_l - \sqrt{gh_l}$ and $S_l^{\text{tail}} = u_* - \sqrt{gh_*}$; and for a right moving wave: $S_r^{\text{head}} = u_r + \sqrt{gh_r}$ and $S_r^{\text{tail}} = u_* + \sqrt{gh_*}$.

2.3 General solution

By combining the solution for isolated rarefaction and shock waves, a general expression for the four admissible combinations (LS-RS, LS-RW, LW-RS, and LW-RW) can be stated:

$$u_* = \begin{cases} u_l - f_l(h_*, h_l), \\ u_r + f_r(h_*, h_r), \end{cases} \tag{29}$$

with:

$$f_k(h_*, h_k) = \begin{cases} 2\sqrt{g}\left(\sqrt{h_*} - \sqrt{h_k}\right), & \text{for a rarefaction wave;} \\ (h_* - h_k)\sqrt{\frac{g}{2}\left(\frac{1}{h_*} + \frac{1}{h_k}\right)}, & \text{for a shock wave,} \end{cases} \quad \text{with } k = l \text{ or } r. \tag{30}$$

2.4 Determining the star state u_*

In the previous two sections, the structure of the Riemann solution has been presented for isolated shock and rarefaction waves. What has not yet been shown however is how to determine whether a wave is a rarefaction or shock in the first place. Consider first a rarefaction wave. For a left moving wave, the characteristic speeds must satisfy $S_l^{\text{head}} < S_l^{\text{tail}}$, i.e. $u_l - \sqrt{gh_l} < u_* - \sqrt{gh_*}$ (as is apparent in figure 3 (b) and (d)). Substituting u_* from equation (25) yields the inequality:

$$u_l - \sqrt{gh_l} < u_l - 2\sqrt{g}\left(\sqrt{h_*} - \sqrt{h_l}\right) - \sqrt{gh_*} \tag{31}$$

$$3\sqrt{gh_*} < 3\sqrt{gh_l} \tag{32}$$

$$h_* < h_l. \tag{33}$$

Similarly, for a right moving rarefaction wave we must have $h_* < h_r$. For shocks, it can be shown that $h_* \geq h_l$ and $h_* \geq h_r$ using the Lax entropy condition in conjunction with the Rankine-Hugoniot relations (details not shown here). Thus, to summarise:

- a left moving wave is a shock if $h_* \geq h_l$ and a rarefaction if $h_* < h_l$;

- a right moving wave is a shock if $h_* \geq h_r$ and a rarefaction if $h_* < h_r$.

The value of h_* follows from (29) and (30) by subtracting u_* for a left wave from that of a right wave:

$$u_* - u_* = f_l(h_*, h_l) - f_r(h_*, h_r) + u_r - u_l = 0. \tag{34}$$

By considering each of the four possible combinations of waves separately, (34) produces four candidate h_* values, only one of which is viable when checked against h_l and h_r. For three cases (LS-RS, LS-RW, and LW-RS), h_* must be computed numerically via an iterative root finding routine (e.g. Newton). For the LW-RW case, an explicit expression for h_* exists:

$$h_* = \frac{1}{16g}\left(u_l - u_r + 2\sqrt{g}(\sqrt{h_l} + \sqrt{h_r})\right)^2. \tag{35}$$

Given h_* for the correct combination of waves, u_* is calculated using (29) and (30):

$$u_* = \frac{1}{2}(u_* + u_*) = \frac{1}{2}\left(f_r(h_*, h_r) - f_l(h_*, h_l)\right) + \frac{1}{2}(u_r + u_l). \tag{36}$$

2.5 Structure inside a rarefaction wave

The final step to complete the Riemann problem is to find the solution inside the so-called expansion fan of any rarefaction waves. In an expansion fan centred at the origin (as in figure 3), the characteristics are $dx/dt = x/t = u - \sqrt{gh}$ for a left wave and $dx/dt = x/t = u + \sqrt{gh}$ for a right wave where u and h are unknown values in the fan. However, since the rarefaction is a continuous solution we may use the Riemann invariants to determine the values of u and h. Thus, for a left wave we have:

$$\text{(i) } \frac{x}{t} = u - \sqrt{gh}, \text{ and (ii) } u + 2\sqrt{gh} = u_l + 2\sqrt{gh_l}, \tag{37}$$

and for a right fan:

$$\text{(i) } \frac{x}{t} = u + \sqrt{gh}, \text{ and (ii) } u - 2\sqrt{gh} = u_r - 2\sqrt{gh_r}. \tag{38}$$

Solving for h yields an expression for h as a function of the similarity variable x/t:

$$h\left(\frac{x}{t}\right) = \begin{cases} \frac{1}{9g}\left(u_l + 2\sqrt{gh_l} - \frac{x}{t}\right)^2, & \text{for a left fan;} \\ \frac{1}{9g}\left(\frac{x}{t} - u_r + 2\sqrt{gh_r}\right)^2, & \text{for a right fan.} \end{cases} \tag{39}$$

Substituting this function into the Riemann invariant equalities leads similarly to an expression for u as a function of x/t:

$$u\left(\frac{x}{t}\right) = \begin{cases} u_l + \frac{2}{3}\left(\frac{x}{t} - u_l + \sqrt{gh_l}\right), & \text{for a left fan;} \\ u_r + \frac{2}{3}\left(\frac{x}{t} - u_r - \sqrt{gh_r}\right), & \text{for a right fan.} \end{cases} \tag{40}$$

The complete solution to the Riemann problem can now be specified for all four cases given arbitrary left and right sates, u_l and u_r.

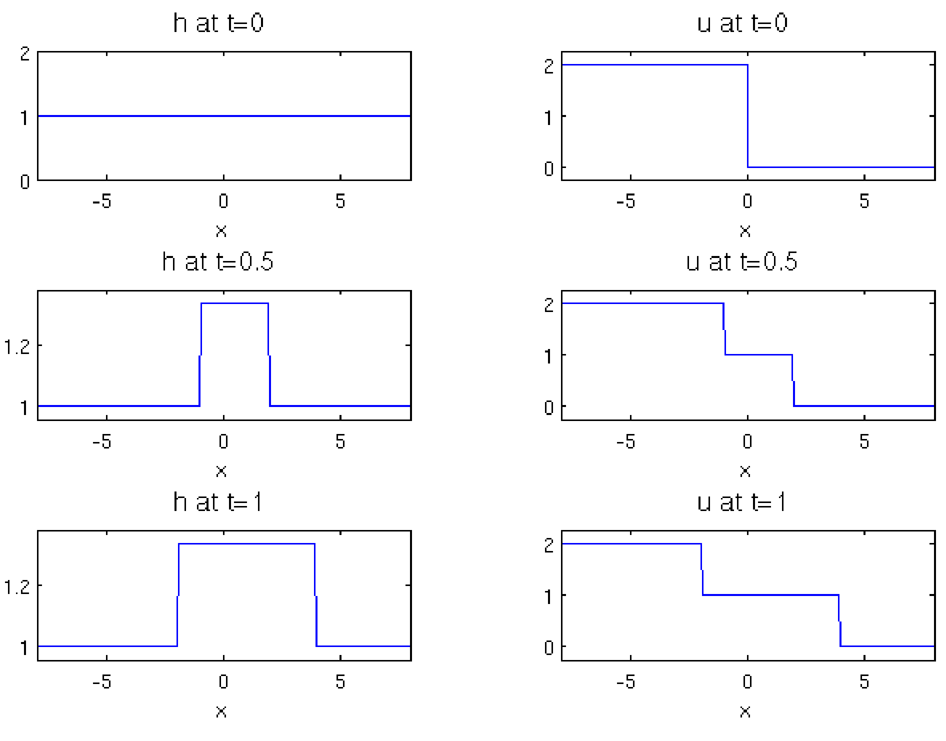
3 The 1D shallow water Riemann problem: examples

3.1 All-shock Riemann solution

Consider the 1D SWEs with Riemann initial data:

$$h(x,0) = 1, \quad u(x,0) = \begin{cases} 2, & \text{for } x < 0; \\ 0, & \text{for } x \geq 0. \end{cases} \tag{41}$$

This set-up corresponds to a moving fluid crashing into a static fluid of equal depth. The solution is shown in figure 4. A shock wave moves in both directions at different speeds (slower to the left, faster to the right), resulting in a wedge of fluid of increased depth expanding asymmetrically about $x = 0$ with constant intermediate velocity $u_* = 1$. Note that $h_* > h_l = h_r$ corresponding to a left and right shock, as determined in section 2.4. The location of the discontinuities follows from $x = S_l t$ and $x = S_r t$ at time $t > 0$.



(a) Fluid depth $h(x,t)$ for given times t.   (b) Fluid velocity $u(x,t)$ for given times t.

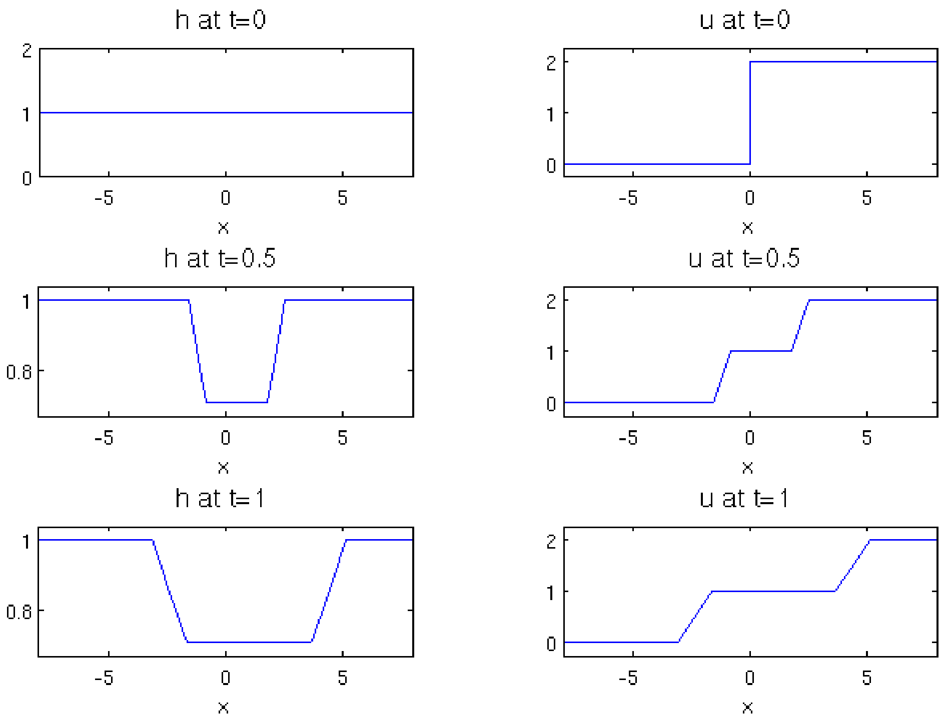
Figure 4: Solution of the shallow water Riemann problem with $h_l = h_r = 1$ and $u_l = 2, u_r = 0$.

3.2 All-rarefaction Riemann solution

Consider the 1D SWEs with Riemann initial data:

$$h(x,0) = 1, \quad u(x,0) = \begin{cases} 0, & \text{for } x < 0; \\ 2, & \text{for } x \geq 0. \end{cases} \tag{42}$$

This corresponds to a fluid moving away from a static fluid of equal depth. The solution is shown in figure 5 and consists of two rarefaction waves moving away from each other at different speeds. As the fluid in $x > 0$ moves to the right, a depression in the fluid depth forms about $x = 0$ and expands in width. This depression is fronted by two rarefaction waves and travels at constant intermediate speed $u_* = 1$ speed. The structure of the expansion fans is given by (39) and (40) for the given times $t > 0$.



(a) Fluid depth $h(x,t)$ for given times t.

(b) Fluid velocity $u(x,t)$ for given times t.

Figure 5: Solution of the shallow water Riemann problem with $h_l = h_r = 1$ and $u_l = 0, u_r = 2$.

3.3 Mixed Riemann solution

The dam-break problem illustrated in figure 2 is a mixed Riemann solution, i.e. the solution consists of one rarefaction and one shock wave, in that case left rarefaction and right shock wave.

It is constructed by considering the left and right waves in isolation and then combining the left and right solution.

4 One-dimensional symmetric shallow water flow

4.1 Governing equations

The one-dimensional symmetric SW system is an augmented system of SWEs in which spatial variation in the y-direction is ignored at leading order. Such anisotropic flow appears in many natural situations in which the flow is in a narrow channel, e.g. rivers or narrow jets. When modelling these anisotropic flows, it is assumed that the aspect ratio of length scale in the y-direction to that in the x-direction is very small, and likewise for the velocity scale. The starting point for such a system is the two-dimensional SWEs:

$$\partial_t h + \partial_x(hu) = 0, \tag{43}$$

$$\partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) + \partial_y(huv) = 0, \tag{44}$$

$$\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2 + \frac{1}{2}gh^2) = 0. \tag{45}$$

The assumption of no y-dependence implies that all y-derivatives are zero, resulting in the 1D symmetric SWEs:

$$\partial_t h + \partial_x(hu) = 0, \tag{46}$$

$$\partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) = 0, \tag{47}$$

$$\partial_t(hv) + \partial_x(huv) = 0. \tag{48}$$

The first two equations are simply the 1D SWEs. The additional equation gives information about the evolution of y-velocity v via the shallow water variable hv. The conservative formulation reads:

$$\partial_t \mathsf{u} + \partial_x(\mathsf{f}(\mathsf{u})) = 0, \text{ with } \mathsf{u} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \text{ and } \mathsf{f}(\mathsf{u}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}. \tag{49}$$

To illustrate the behaviour of v in this system, the augmented v-equation is expanded:

$$h\partial_t v + v\partial_t h + (hu)\partial_x v + v\partial_x(hu) = 0. \tag{50}$$

Since $\partial_t h + \partial_x(hu) = 0$ it follows that $h\partial_t v + (hu)\partial_x v = 0$. Further, dividing by h yields: $\partial_t v + u\partial_x v = 0$. This means that v is advected by the variable $u(x,t)$, i.e. v acts as a passive scalar in the system (46). In quasi-linear form, $\partial_t \mathsf{u} + \mathsf{f}'(\mathsf{u})\partial_x \mathsf{u} = 0$, with Jacobian matrix:

$$\mathsf{f}'(\mathsf{u}) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{bmatrix}.$$

Thus, the system has three distinct eigenvalues: $\lambda_1 = u - \sqrt{gh}$, $\lambda_2 = u$ and $\lambda_3 = u + \sqrt{gh}$ with corresponding eigenvectors $r_1 = (1, u - \sqrt{gh}, v)^T$, $r_2 = (0, 0, 1)^T$ and $r_3 = (1, u + \sqrt{gh}, v)^T$. The 1- and 3-field correspond to the non-linear waves of the 1D SWEs while the 2-field consists of a jump discontinuity in v (since the first two components of r_2 are zero). The speed of the 2-wave, $\lambda_2 = u$, depends on the shallow water flow. This shows that the 'scalar' v is clearly decoupled from the 1D SW system, in the sense that it is linearly degenerate.

4.2 Solving the Riemann problem

Consider the 1D symmetric SWEs (46) together with arbitrary Riemann initial data:

$$\mathsf{u}(x, t_0) = \begin{cases} \mathsf{u}_l, & \text{for } x < x_0; \\ \mathsf{u}_r, & \text{for } x \geq x_0. \end{cases} \tag{51}$$

Since v does not affect h or u, the 1D shallow water Riemann problem is valid for the 1- and 3-wave. Thus, for h and u the Riemann solution is the same as that described previously for the 1D SWEs. The left wave moves into the fluid on the left ($x < 0$), in which $v = v_l$, while the right wave moves into the fluid on the right ($x > 0$), in which $v = v_r$. Between these two waves is the 2-wave with constant velocity $\lambda_2 = u_*$, the value of which is calculated in the 1D Riemann problem. Across this 2-wave, $h = h_*$ and $u = u_*$ while v jumps from $v = v_l$ to $v = v_r$. Such a wave is known as a 'contact discontinuity' - it marks the point of contact between two 'different' fluids, i.e. one with $v = v_l$ and the other with $v = v_r$. This contact discontinuity must be added to the Riemann solution for the 1D SWEs to give the complete Riemann solution for the 1D symmetric SWEs.

An example dam-break solution for the symmetric system is illustrated in figure 6. As before, the solution consists of a left rarefaction and right shock wave, but has the additional structure of the contact discontinuity due to the introduction of v. The variables h and u are continuous over the contact discontinuity while v is continuous of the left and right waves. The similarity solution is shown in figure 7.

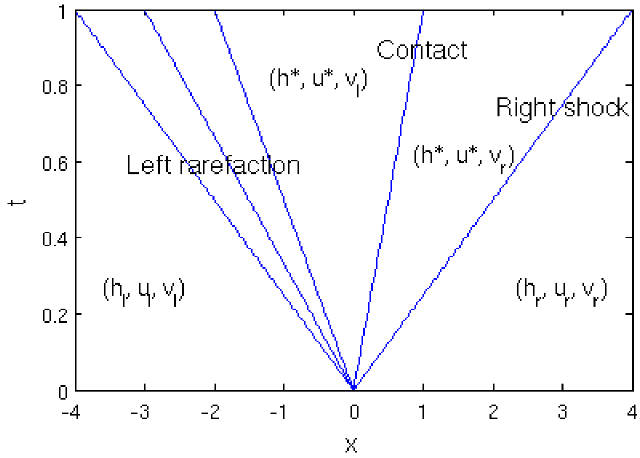


Figure 6: Schematic solution in the x-t-plane of the dam-break problem. The solution consists of a left rarefaction wave, contact discontinuity, and right shock wave. Note that: h and u are continuous over the contact discontinuity while v is continuous of the left and right waves.

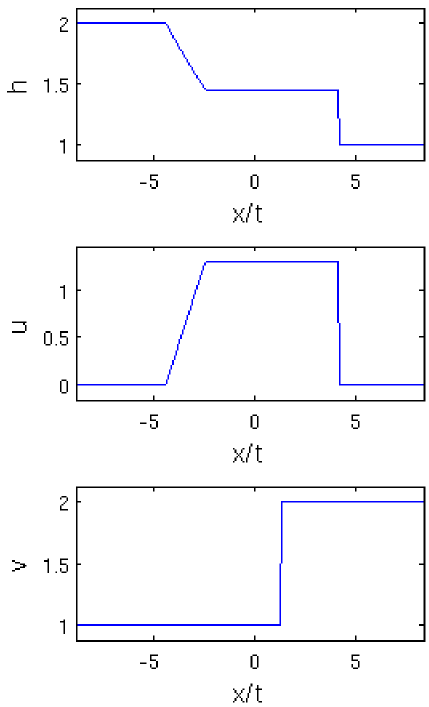


Figure 7: Similarity solution to the 1D symmetric shallow water Riemann problem with initial data: $h_l = 2, h_r = 1, u_l = u_r = 0$, and $v_l = 1, v_r = 2$. The contact discontinuity in v is located at u_*.