



ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.1 UNDIRECTED GRAPHS

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- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *connected components*
- ▶ *challenges*

# Algorithms

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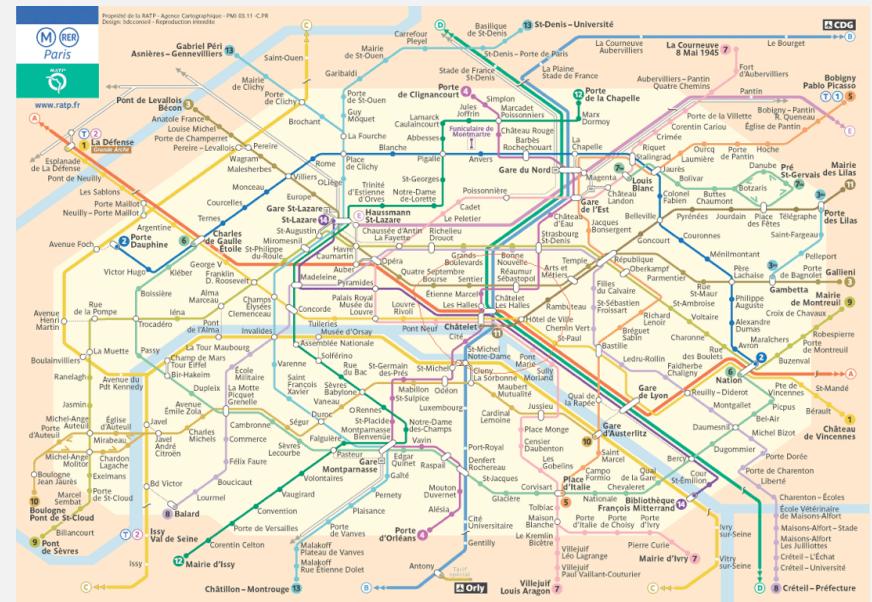
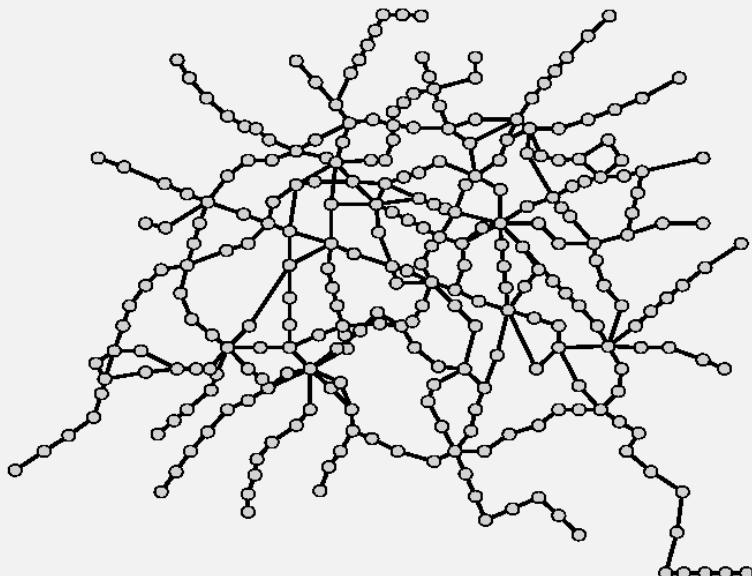
# Undirected graphs



Graph. Set of **vertices** connected pairwise by **edges**.  
→ edges가 서로 상호로 연결된 vertices

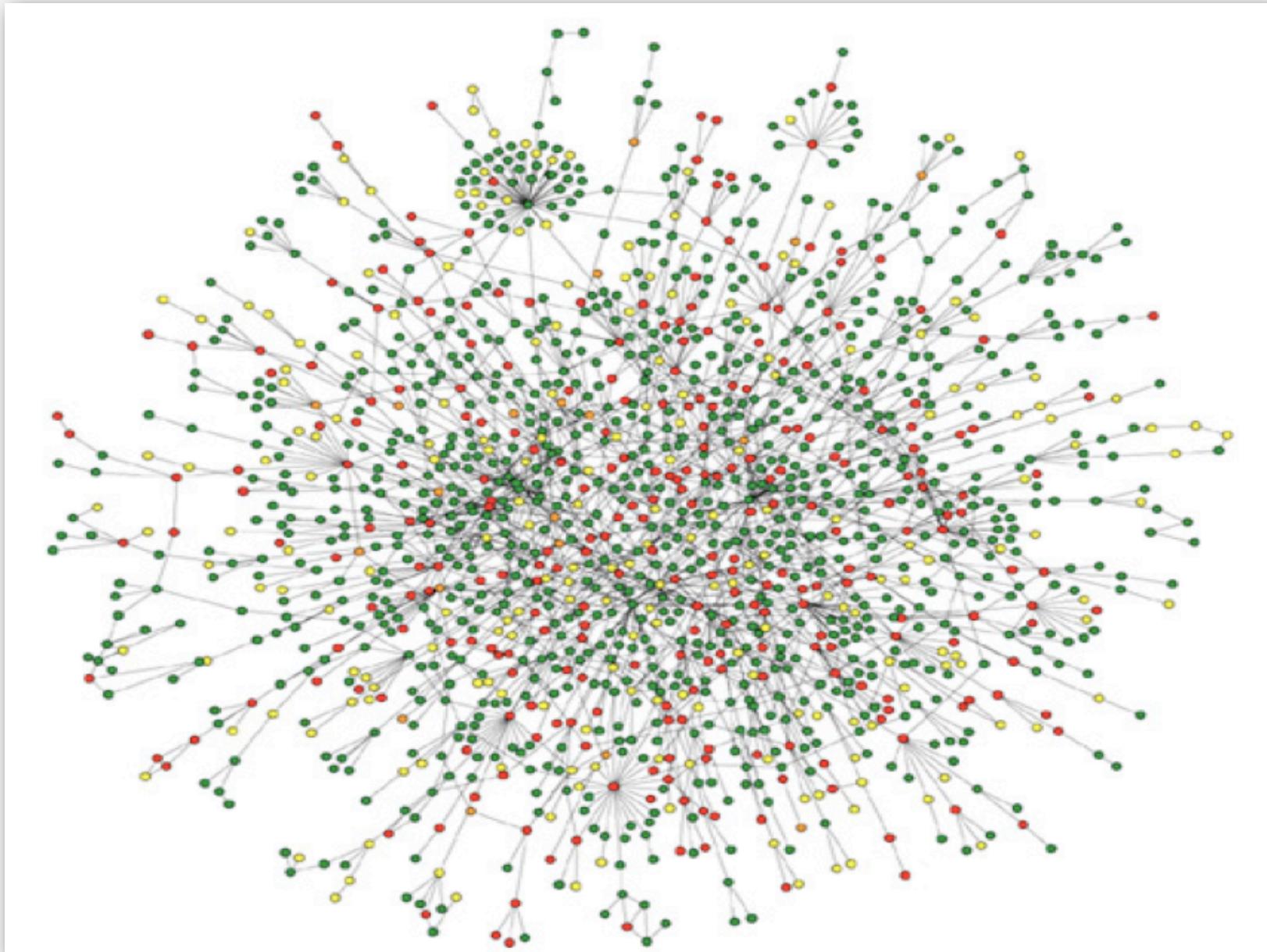
## Why study graph algorithms?

- Thousands of practical applications.
- Hundreds of graph algorithms known.
- Interesting and broadly useful abstraction.
- Challenging branch of computer science and discrete math.



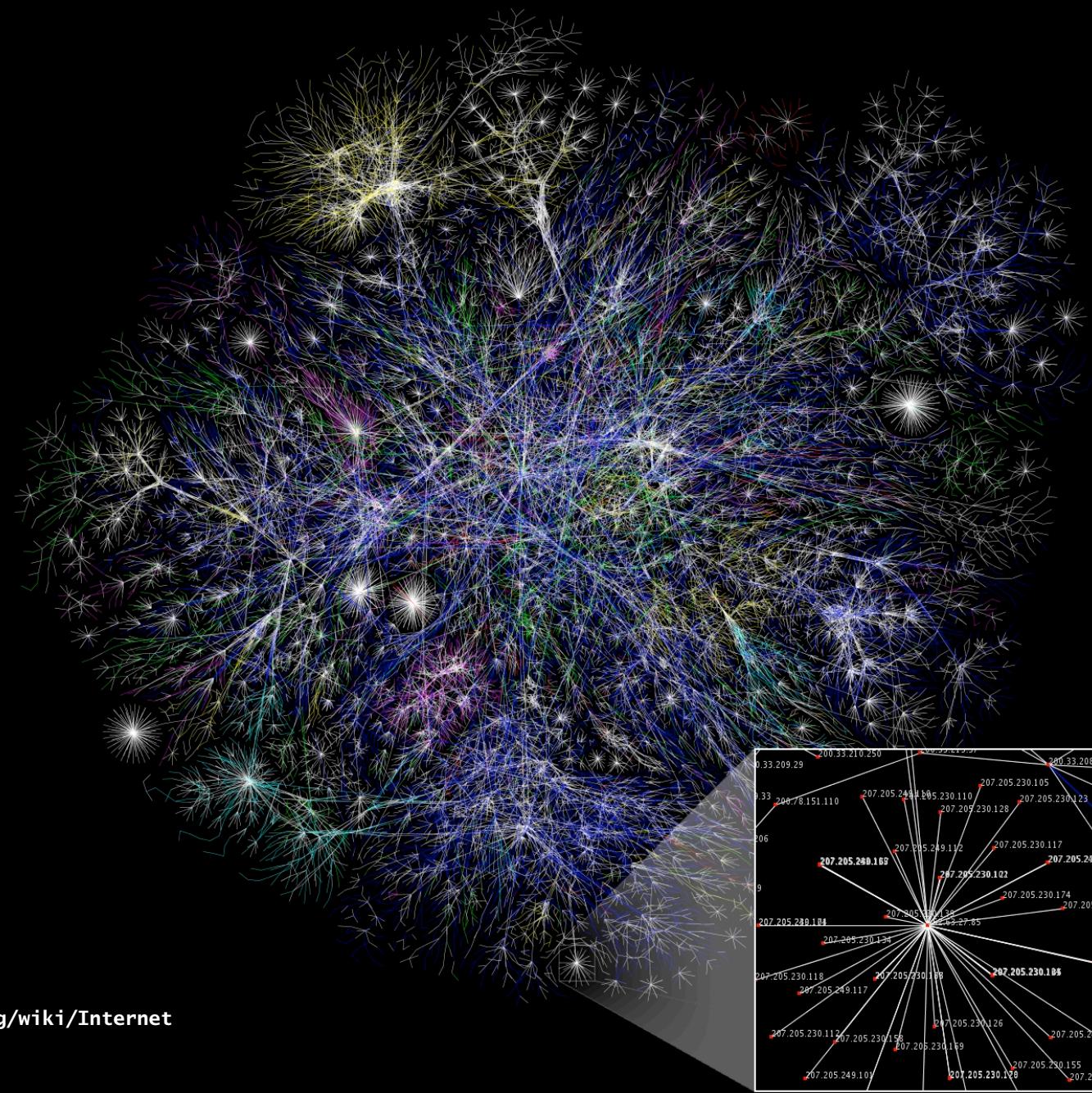
# Protein-protein interaction network

vertices : 단백질  
edges : 단백질 간의 상호작용



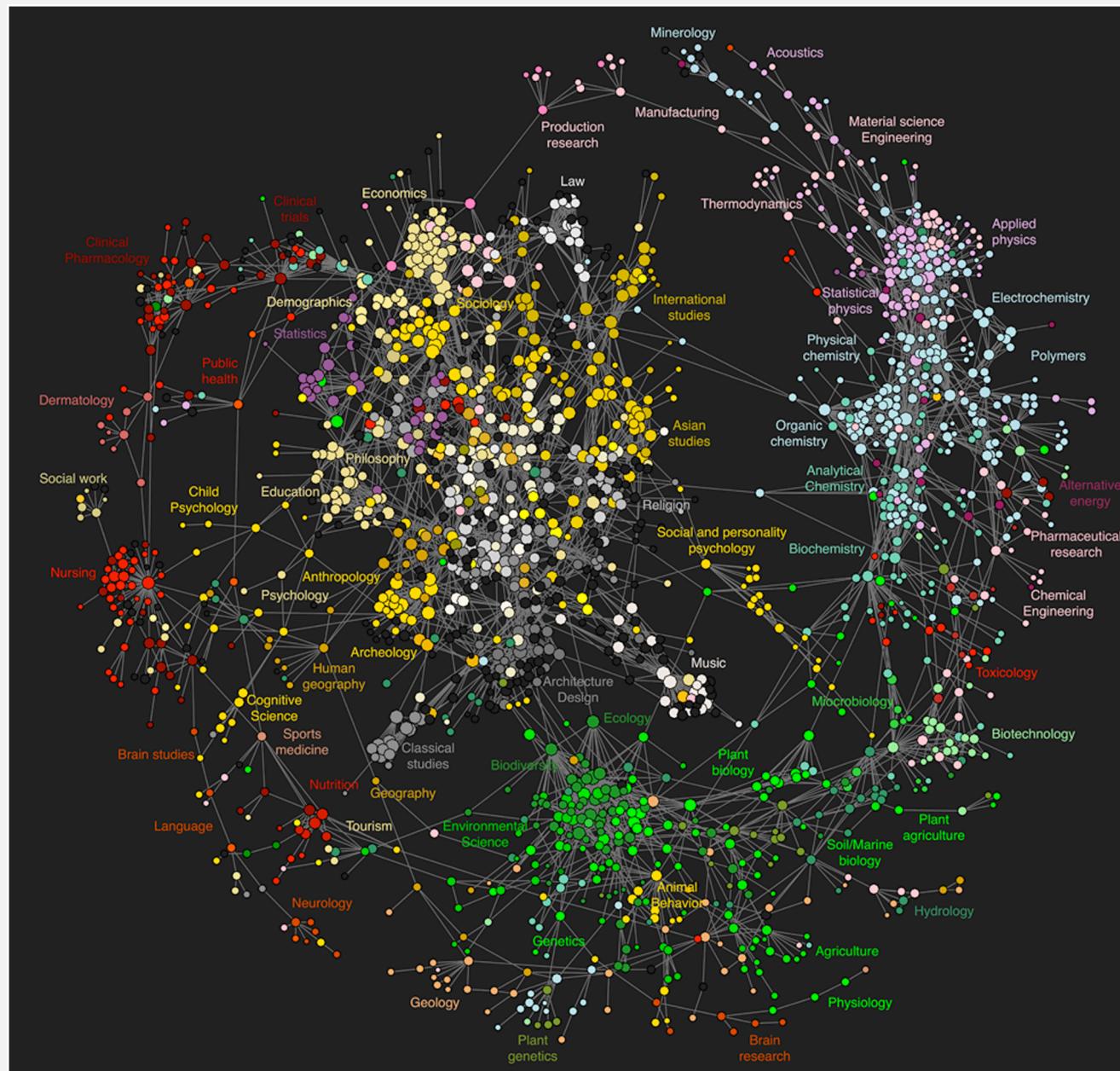
Reference: Jeong et al, Nature Review | Genetics

# The Internet as mapped by the Opte Project



<http://en.wikipedia.org/wiki/Internet>

# Map of science clickstreams



<http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803>

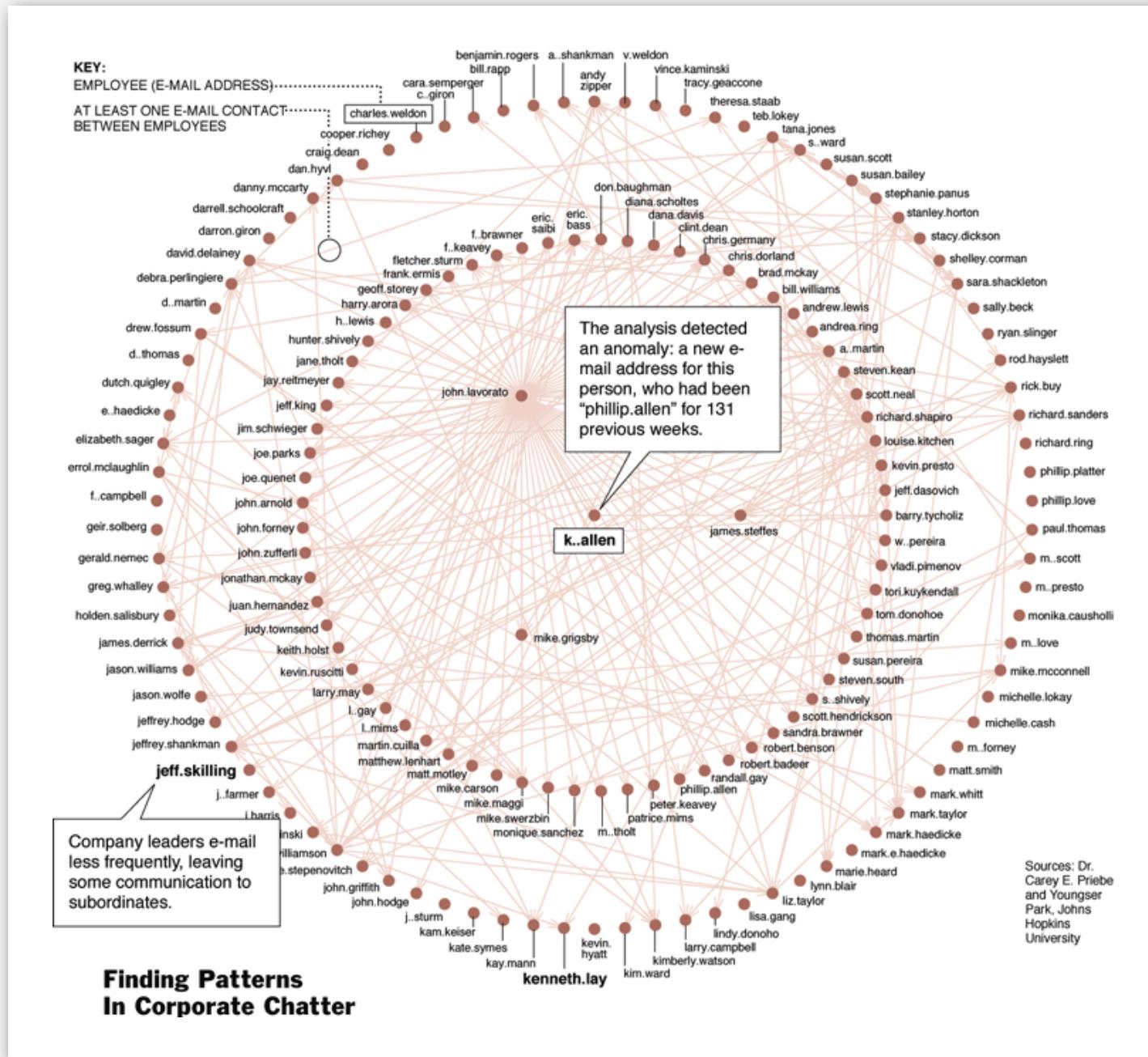
# 10 million Facebook friends

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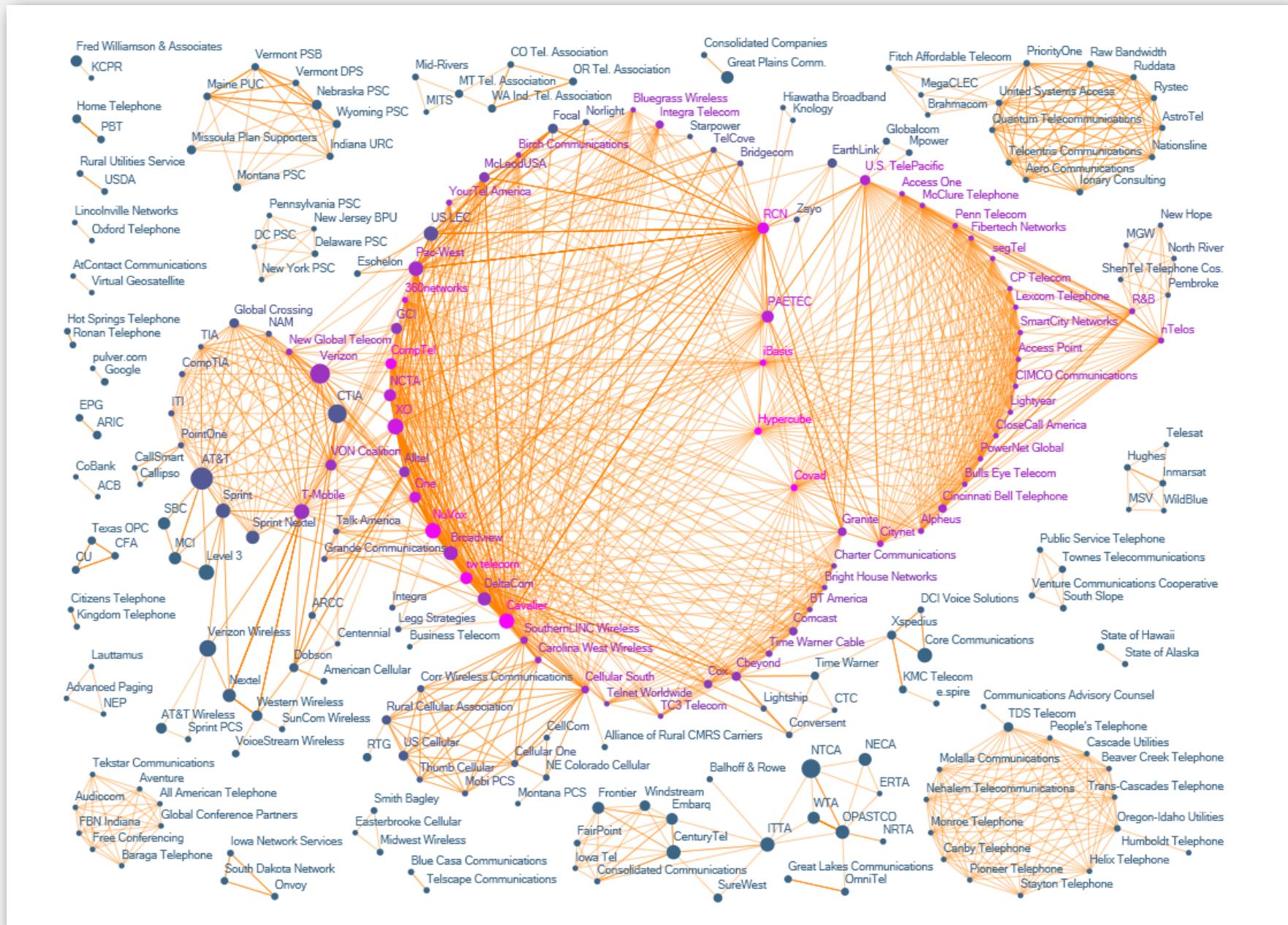


"Visualizing Friendships" by Paul Butler

# One week of Enron emails

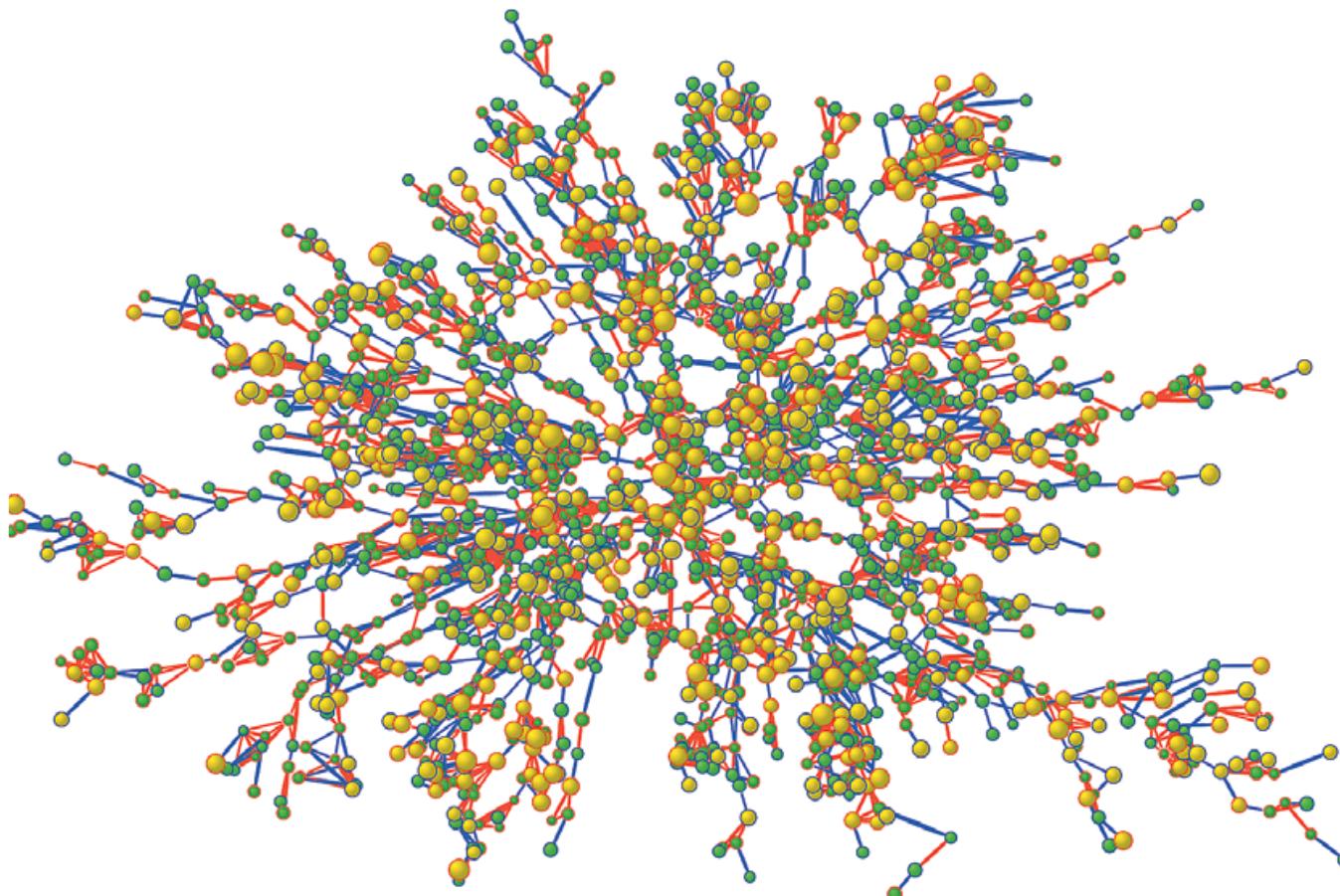


# The evolution of FCC lobbying coalitions



**"The Evolution of FCC Lobbying Coalitions" by Pierre de Vries in Ioss Visualization Symposium 2010**

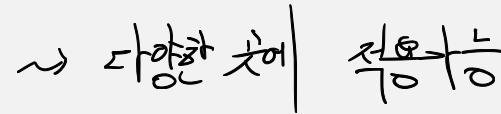
# Framingham heart study



**Figure 1.** Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000.

Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index,  $\geq 30$ ) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

# Graph applications

→  

| graph               | vertex                       | edge                        |
|---------------------|------------------------------|-----------------------------|
| communication       | telephone, computer          | fiber optic cable           |
| circuit             | gate, register, processor    | wire                        |
| mechanical          | joint                        | rod, beam, spring           |
| financial           | stock, currency              | transactions                |
| transportation      | street intersection, airport | highway, airway route       |
| internet            | class C network              | connection                  |
| game                | board position               | legal move                  |
| social relationship | person, actor                | friendship, movie cast      |
| neural network      | neuron                       | synapse                     |
| protein network     | protein                      | protein-protein interaction |
| molecule            | atom                         | bond                        |

# Graph terminology

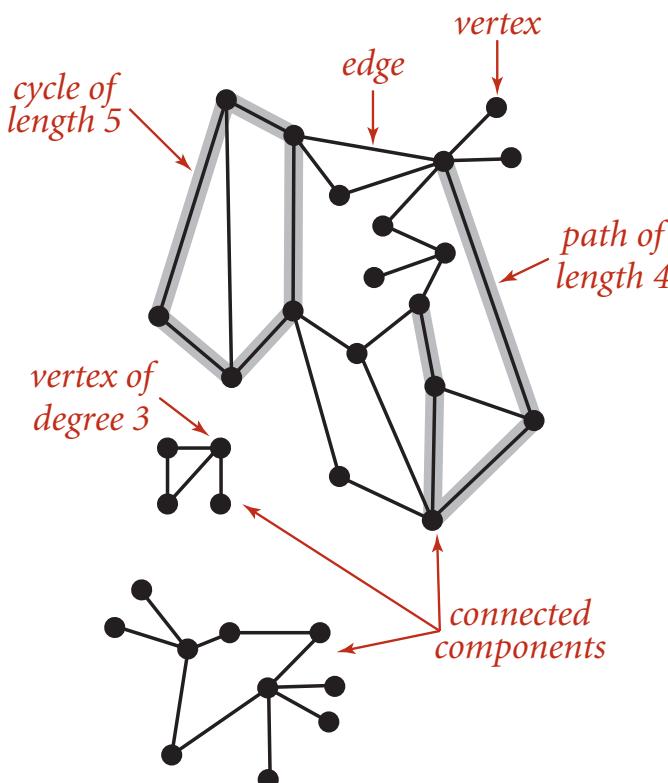
**Path.** Sequence of vertices connected by edges.

**Cycle.** Path whose first and last vertices are the same.

Two vertices are **connected** if there is a path between them.

path : edge  $\geq$  2 vertices  
sequence .

cycle :  $v_1 = v_n$  vertices path



## Some graph-processing problems

---

**Path.** Is there a path between  $s$  and  $t$ ?

**Shortest path.** What is the shortest path between  $s$  and  $t$ ?

**Cycle.** Is there a cycle in the graph?

**Euler tour.** Is there a cycle that uses each edge exactly once?

**Hamilton tour.** Is there a cycle that uses each vertex exactly once.

**Connectivity.** Is there a way to connect all of the vertices?

**MST.** What is the best way to connect all of the vertices?

**Biconnectivity.** Is there a vertex whose removal disconnects the graph?

**Planarity.** Can you draw the graph in the plane with no crossing edges?

**Graph isomorphism.** Do two adjacency lists represent the same graph?

**Challenge.** Which of these problems are easy? difficult? intractable?

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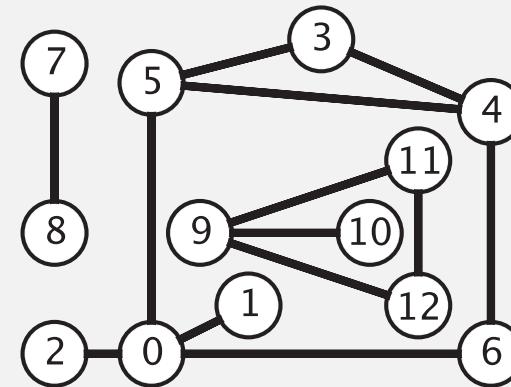
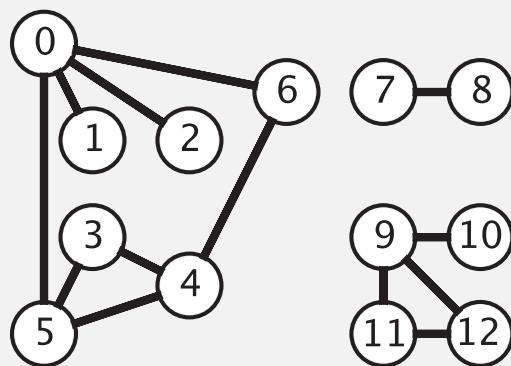
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# Graph representation

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Graph drawing. Provides intuition about the structure of the graph.



two drawings of the same graph

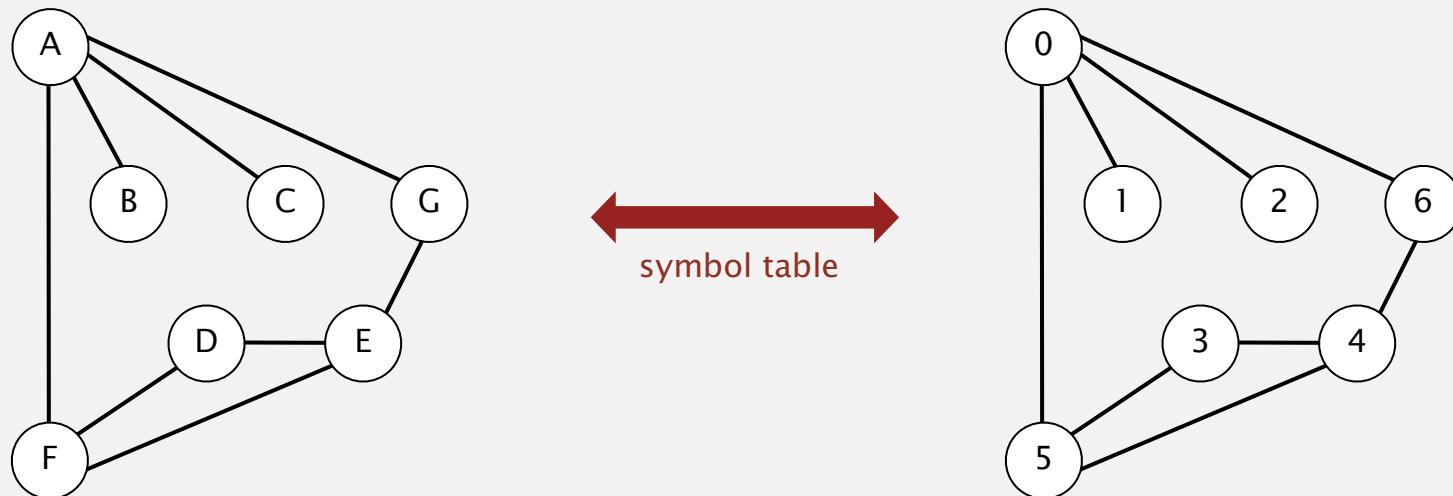
Caveat. Intuition can be misleading.

# Graph representation

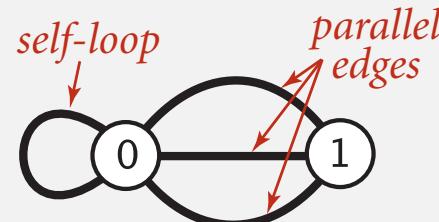
---

## Vertex representation.

- This lecture: use integers between 0 and  $V - 1$ .
- Applications: convert between names and integers with symbol table.



## Anomalies.



# Graph API

```
public class Graph  
    Constructor 2nd {  
        Graph(int V)  
        Graph(In in)  
  
        void addEdge(int v, int w)  
  
        Iterable<Integer> adj(int v)  
        int V()  
        int E()  
        String toString()  
    }
```

*create an empty graph with V vertices*

*create a graph from input stream*

*add an edge v-w*

*vertices adjacent to v*

*number of vertices*

*number of edges*

*string representation*

```
In in = new In(args[0]);  
Graph G = new Graph(in);
```

*read graph from  
input stream*

```
for (int v = 0; v < G.V(); v++)  
    for (int w : G.adj(v))  
        StdOut.println(v + "-" + w);
```

*print out each  
edge (twice)*

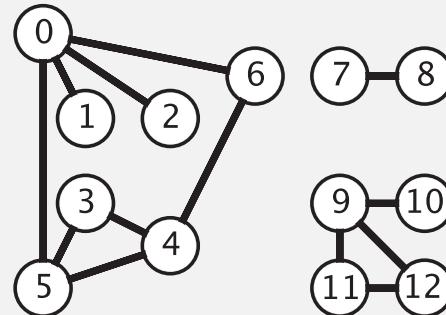
# Graph API: sample client

Graph input format.

**tinyG.txt**

$V \rightarrow 13$   
 $13 \leftarrow E$

```
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```



% java Test tinyG.txt

```
0-6
0-2
0-1
0-5
1-0
2-0
3-5
3-4
...
12-11
12-9
```

```
In in = new In(args[0]);
Graph G = new Graph(in);

for (int v = 0; v < G.V(); v++)
    for (int w : G.adj(v))
        StdOut.println(v + "-" + w);
```

read graph from  
input stream

print out each  
edge (twice)

# Typical graph-processing code

```
public static int degree(Graph G, int v)
{
    int degree = 0;
    for (int w : G.adj(v)) degree++;
    return degree;
}

public static int maxDegree(Graph G)
{
    int max = 0;
    for (int v = 0; v < G.V(); v++)
        if (degree(G, v) > max)
            max = degree(G, v);
    return max;
}

public static double averageDegree(Graph G)
{   return 2.0 * G.E() / G.V(); }

public static int numberOfSelfLoops(Graph G)
{
    int count = 0;
    for (int v = 0; v < G.V(); v++)
        for (int w : G.adj(v))
            if (v == w) count++;
    return count/2; // each edge counted twice
}
```

*compute the degree of v*

*compute maximum degree*

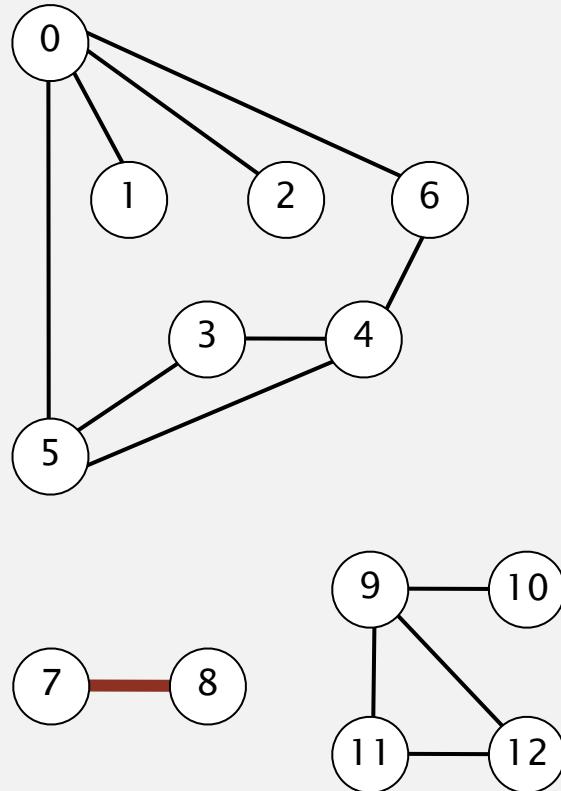
*compute average degree*

*count self-loops*

## Set-of-edges graph representation

---

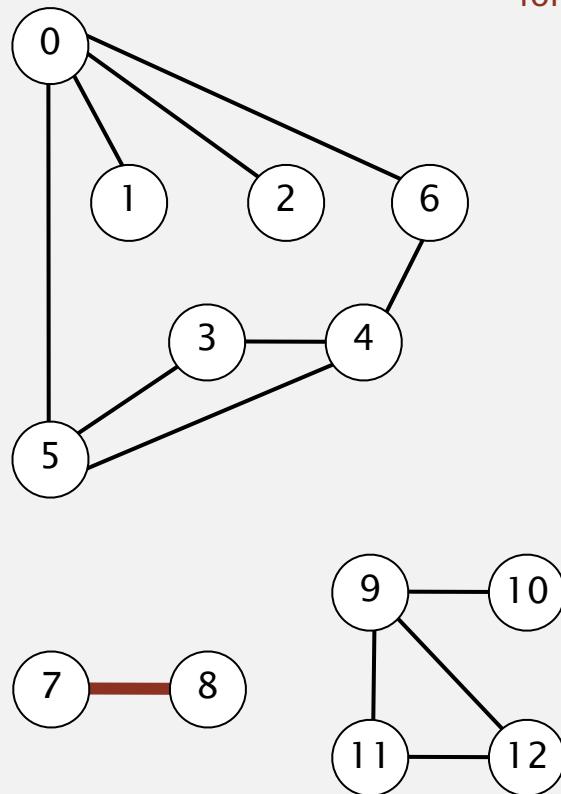
Maintain a list of the edges (linked list or array).



|    |    |
|----|----|
| 0  | 1  |
| 0  | 2  |
| 0  | 5  |
| 0  | 6  |
| 3  | 4  |
| 3  | 5  |
| 4  | 5  |
| 4  | 6  |
| 7  | 8  |
| 9  | 10 |
| 9  | 11 |
| 9  | 12 |
| 11 | 12 |

# Adjacency-matrix graph representation

Maintain a two-dimensional  $V$ -by- $V$  boolean array;  
for each edge  $v-w$  in graph:  $\text{adj}[v][w] = \text{adj}[w][v] = \text{true}$ .



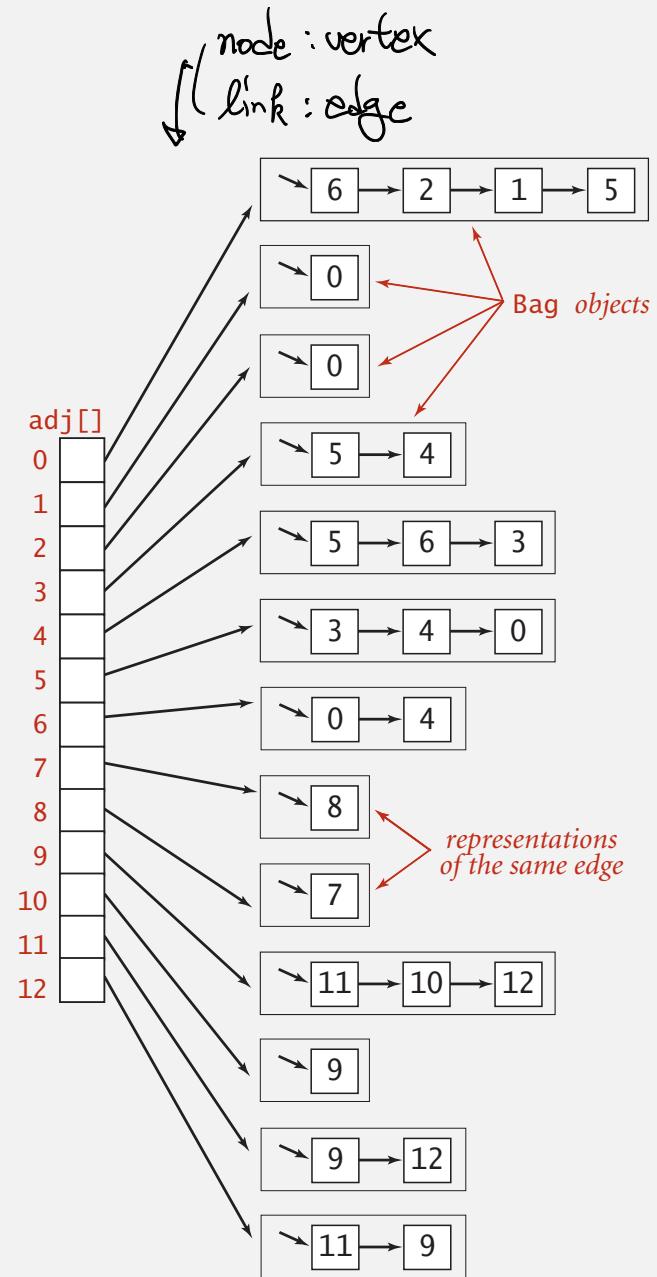
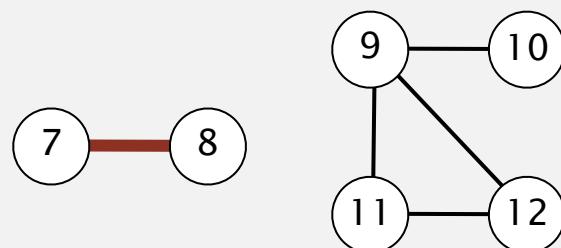
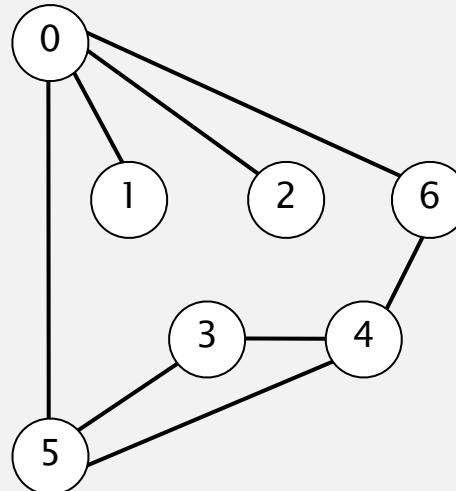
graph의는 행렬로  
→ 행렬로 표기 X

|    | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|----|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 0  | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0  | 0  | 0  |
| 1  | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  |
| 2  | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  |
| 3  | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0  | 0  | 0  |
| 4  | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0  | 0  | 0  |
| 5  | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  |
| 6  | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0  | 0  | 0  |
| 7  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0  | 0  | 0  |
| 8  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0  | 0  | 0  |
| 9  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1  | 1  | 1  |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0  | 0  | 0  |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0  | 0  | 1  |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0  | 1  | 0  |

# Adjacency-list graph representation



Maintain vertex-indexed array of lists.





# Adjacency-list graph representation: Java implementation

```
public class Graph
```

```
{
```

```
    private final int V;
```

```
    private Bag<Integer>[] adj;
```

adjacency lists  
( using Bag data type )

```
    public Graph(int V)
```

```
{
```

```
    this.V = V;
```

```
    adj = (Bag<Integer>[]) new Bag[V];
```

```
    for (int v = 0; v < V; v++)
```

```
        adj[v] = new Bag<Integer>();
```

```
}
```

create empty graph  
with V vertices

```
    public void addEdge(int v, int w)
```

```
{
```

```
    adj[v].add(w);
```

```
    adj[w].add(v);
```

```
}
```

add edge v-w  
(parallel edges and  
self-loops allowed)

```
    public Iterable<Integer> adj(int v)
```

```
    { return adj[v]; }
```

```
}
```

iterator for vertices adjacent to v

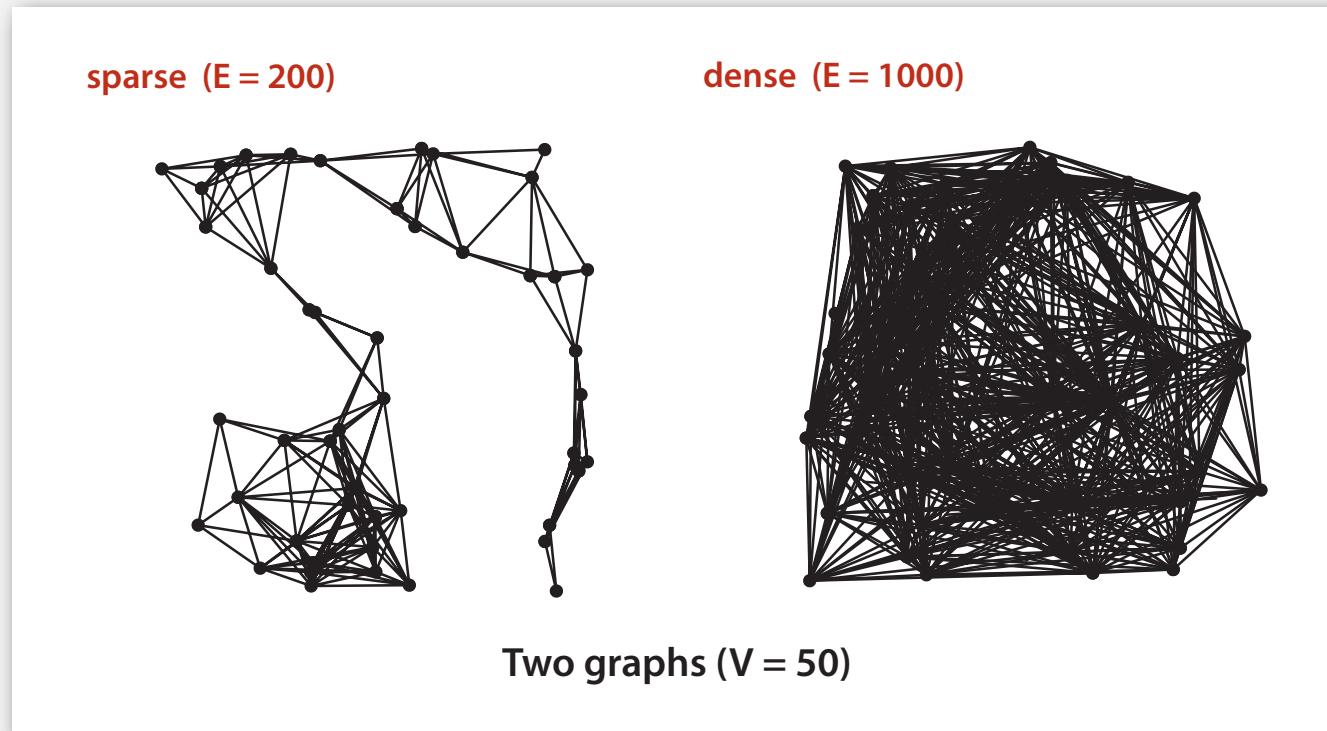
# Graph representations

---

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to  $v$ .
- Real-world graphs tend to be **sparse**.

huge number of vertices,  
small average vertex degree



# Graph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to  $v$ .
- Real-world graphs tend to be **sparse**.

huge number of vertices,  
small average vertex degree

| representation   | space   | add edge | edge between<br>$v$ and $w$ ? | iterate over vertices<br>adjacent to $v$ ? |
|------------------|---------|----------|-------------------------------|--|
| list of edges    | $E$     | 1        | $E$                           | $E$  |
| adjacency matrix | $V^2$   | $1^*$    | 1                             | $V$  |
| adjacency lists  | $E + V$ | 1        | $\text{degree}(v)$            | $\text{degree}(v)$                         |

\* disallows parallel edges

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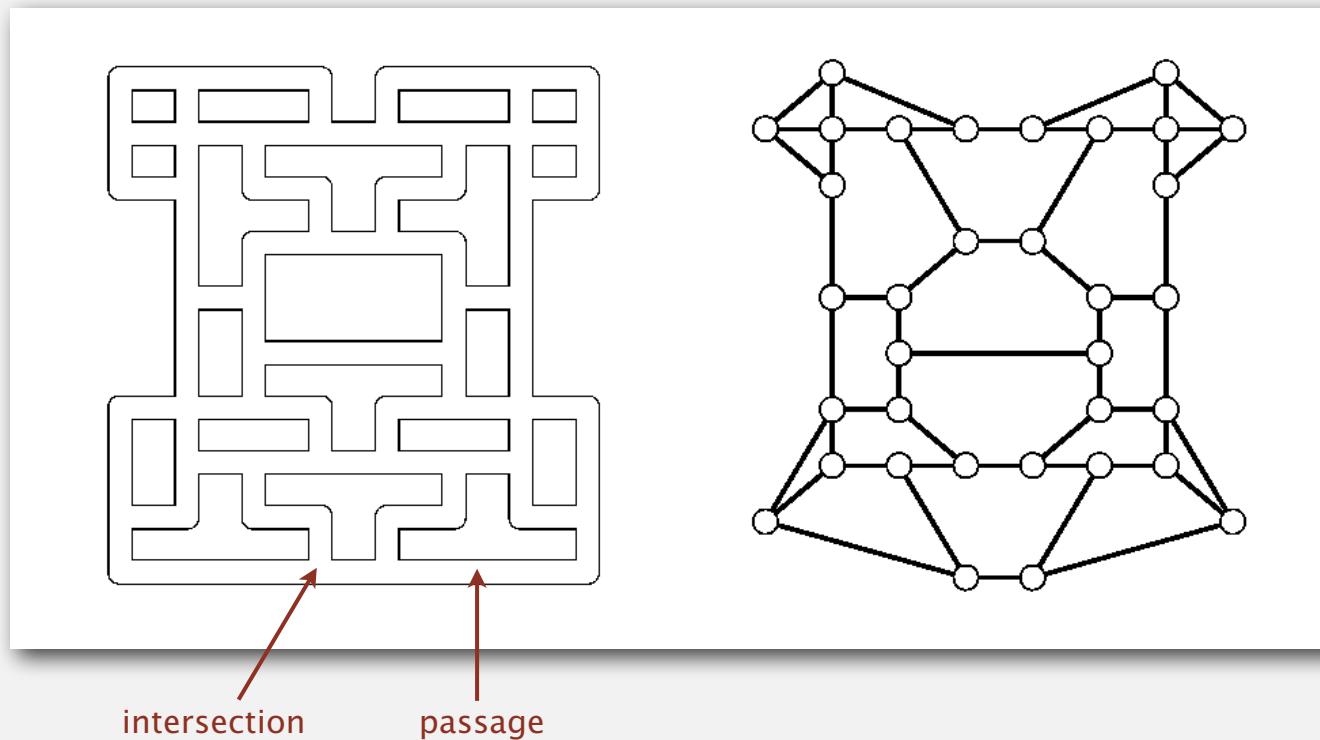
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# Maze exploration

---

## Maze graph.

- Vertex = intersection.
- Edge = passage.



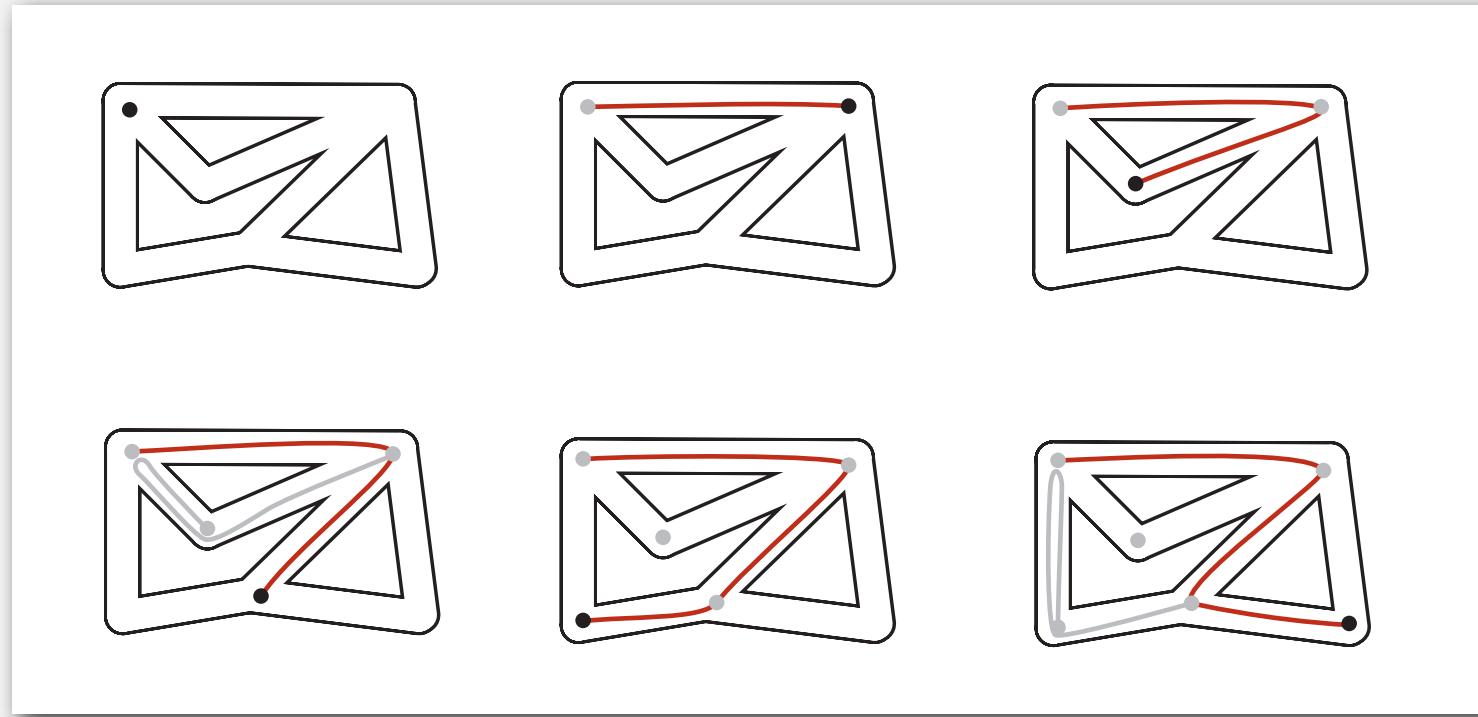
**Goal.** Explore every intersection in the maze.

## Trémaux maze exploration

Algorithm.

→ 한 번을 두 번 방울 X, 방울을 풀면 거리에 이르면 다른 경로들이 있는  
지전까지 backtrace 시 다른 경로 선택

- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.



# Trémaux maze exploration

---

## Algorithm.

- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.

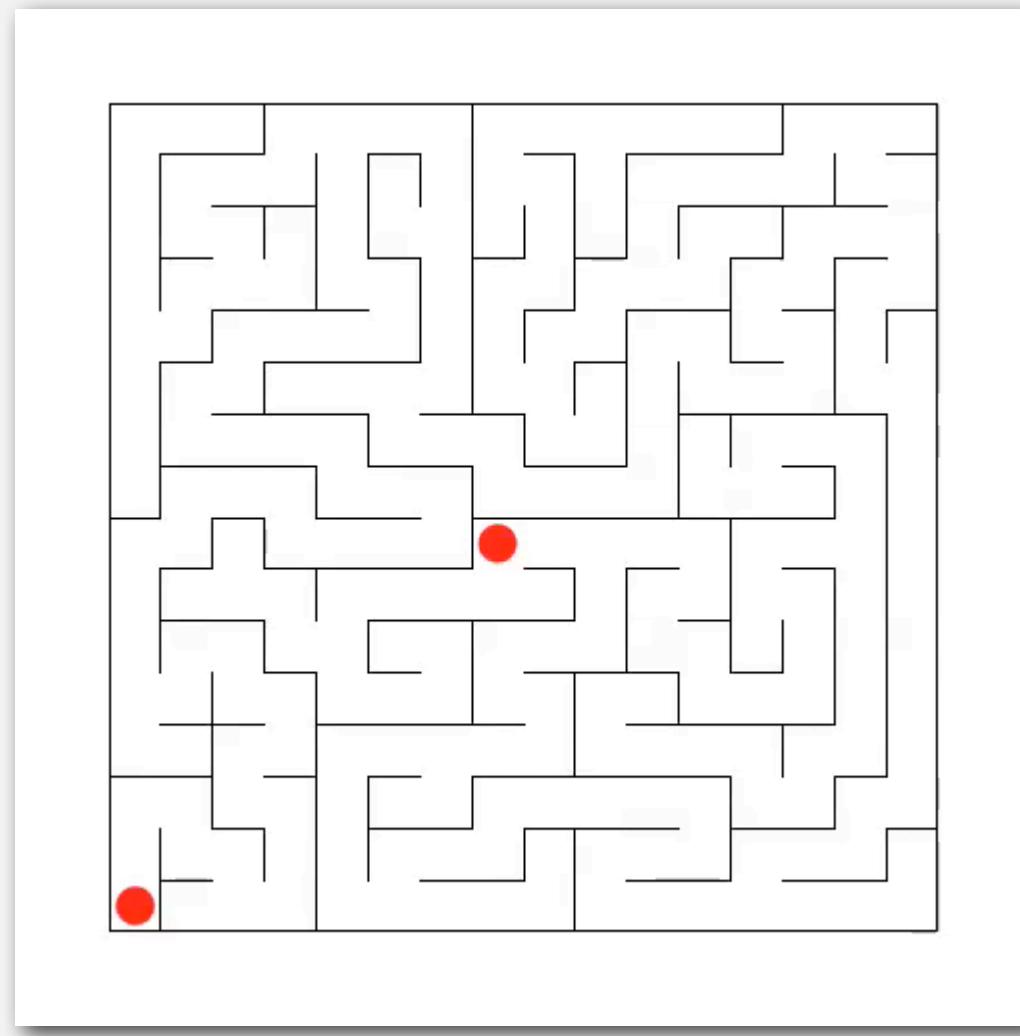
**First use?** Theseus entered Labyrinth to kill the monstrous Minotaur; Ariadne instructed Theseus to use a ball of string to find his way back out.



Claude Shannon (with Theseus mouse)

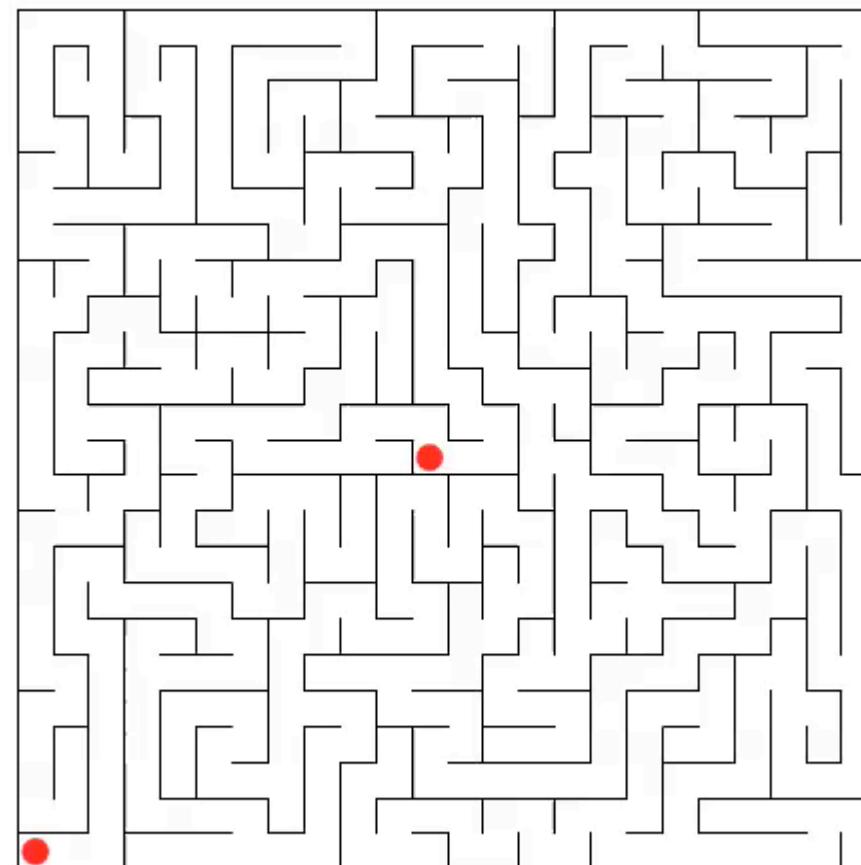
## Maze exploration

---



## Maze exploration

---



# Depth-first search

---

**Goal.** Systematically search through a graph.

**Idea.** Mimic maze exploration.

## **DFS (to visit a vertex v)**

---

1. **Mark v as visited.**
  2. **Recursively visit all unmarked vertices w adjacent to v.**
- 

**Typical applications.**

- Find all vertices connected to a given source vertex.
- Find a path between two vertices.

**Design challenge.** How to implement?

# Design pattern for graph processing

Design pattern. Decouple graph data type from graph processing.

- Create a Graph object.
- Pass the Graph to a graph-processing routine.
- Query the graph-processing routine for information.

```
public class Paths
```

```
    Paths(Graph G, int s)
```

*find paths in G from source s*

```
    boolean hasPathTo(int v)
```

*is there a path from s to v?*

```
    Iterable<Integer> pathTo(int v)
```

*path from s to v; null if no such path*

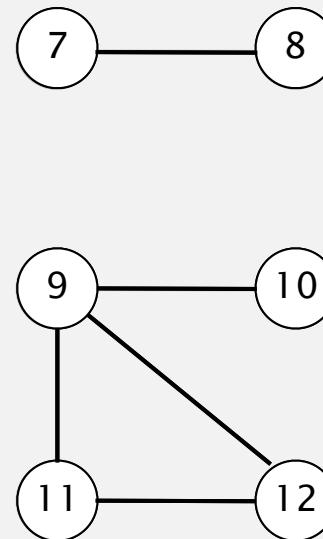
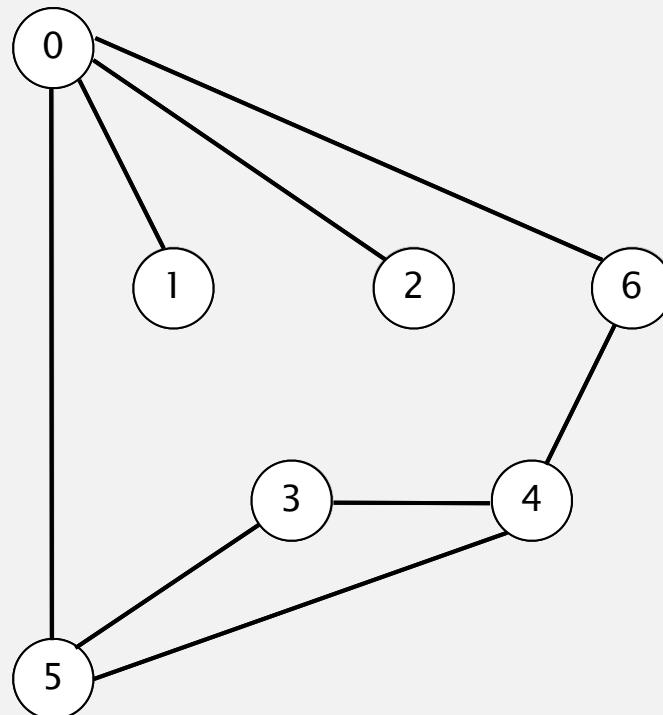
```
Paths paths = new Paths(G, s);
for (int v = 0; v < G.V(); v++)
    if (paths.hasPathTo(v))
        StdOut.println(v);
```

print all vertices  
connected to s

# Depth-first search demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



graph G

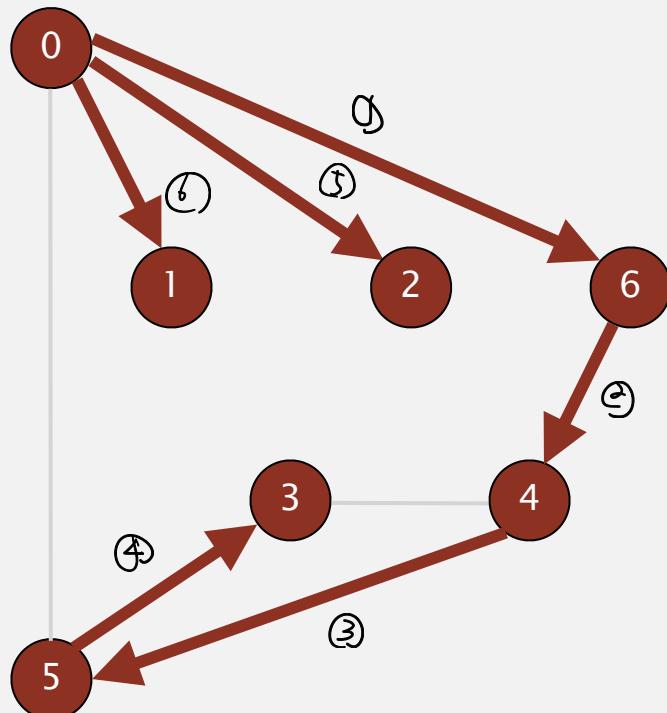
tinyG.txt  
V → 13  
E → 13  
0 5  
4 3  
0 1  
9 12  
6 4  
5 4  
0 2  
11 12  
9 10  
0 6  
7 8  
9 11  
5 3

# Depth-first search demo

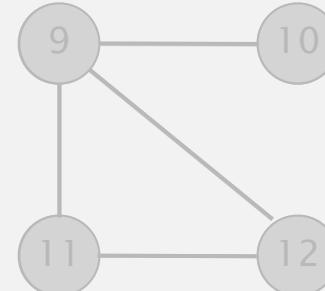
*DFS*

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



vertices reachable from 0



| $v$ | marked[] | edgeTo[ $v$ ] |
|-----|----------|---------------|
| 0   | T        | -             |
| 1   | T        | 0             |
| 2   | T        | 0             |
| 3   | T        | 5             |
| 4   | T        | 5             |
| 5   | T        | 6             |
| 6   | T        | 4             |
| 7   | F        | -             |
| 8   | F        | -             |
| 9   | F        | -             |
| 10  | F        | -             |
| 11  | F        | -             |
| 12  | F        | -             |

5가 0에 연결?

$5 \rightarrow 4 / 4 \rightarrow 6 / 6 \rightarrow 0$ .

# Depth-first search

---

서치 트리의 모든 점 찾기!

Goal. Find all vertices connected to  $s$  (and a corresponding path).

Idea. Mimic maze exploration.

Cost 최적화

Algorithm.

- Use recursion (ball of string).
- Mark each visited vertex (and keep track of edge taken to visit it).
- Return (retrace steps) when no unvisited options.

Data structures.

- boolean[] marked to mark visited vertices.
- int[] edgeTo to keep tree of paths.  
( $\text{edgeTo}[w] == v$ ) means that edge  $v-w$  taken to visit  $w$  for first time

# Depth-first search

```
public class DepthFirstPaths
{
    private boolean[] marked;
    private int[] edgeTo;
    private int s;
```

marked[v] = true  
if v connected to s  
edgeTo[v] = previous  
vertex on path from s to v

```
public DepthFirstPaths(Graph G, int s)
{
    ...
    dfs(G, s);
}
```

initialize data structures  
find vertices connected to s

```
private void dfs(Graph G, int v)
{
    marked[v] = true;
    for (int w : G.adj(v))
        if (!marked[w])
        {
            dfs(G, w);
            edgeTo[w] = v;
        }
}
```

recursive DFS does the work

# Depth-first search properties

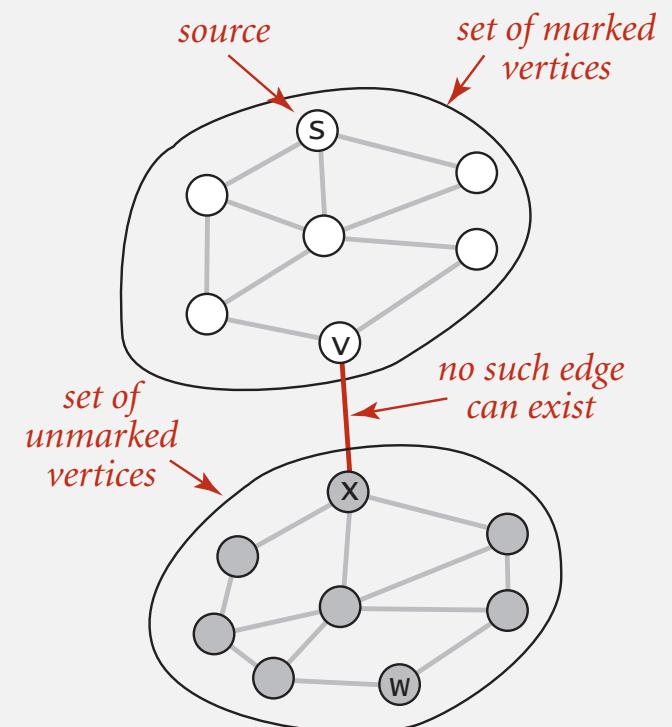
**Proposition.** DFS marks all vertices connected to  $s$  in time proportional to the sum of their degrees.

Pf. [correctness]

- If  $w$  marked, then  $w$  connected to  $s$  (why?)
- If  $w$  connected to  $s$ , then  $w$  marked.  
(if  $w$  unmarked, then consider last edge on a path from  $s$  to  $w$  that goes from a marked vertex to an unmarked one).

Pf. [running time]

Each vertex connected to  $s$  is visited once.



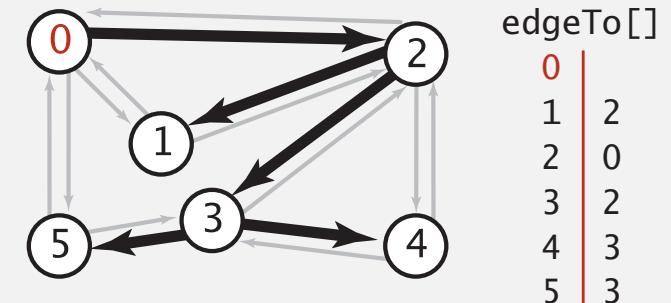
## Depth-first search properties

**Proposition.** After DFS, can find vertices connected to  $s$  in constant time and can find a path to  $s$  (if one exists) in time proportional to its length.

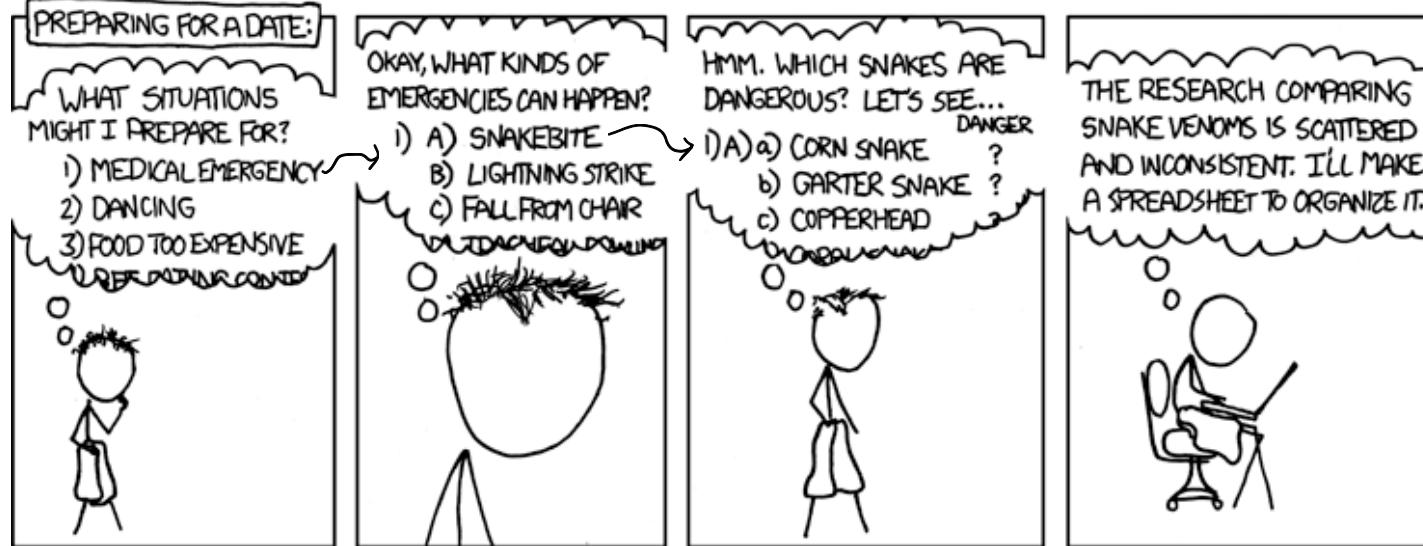
Pf. `edgeTo[]` is parent-link representation of a tree rooted at  $s$ .

```
public boolean hasPathTo(int v)
{   return marked[v]; }

public Iterable<Integer> pathTo(int v)
{
    if (!hasPathTo(v)) return null;
    Stack<Integer> path = new Stack<Integer>();
    Stack !  
DFSc    for (int x = v; x != s; x = edgeTo[x])
        path.push(x);
    path.push(s);
    return path;
}
```



# Depth-first search application: preparing for a date



xkcd

<http://xkcd.com/761/>

# Depth-first search application: flood fill

---

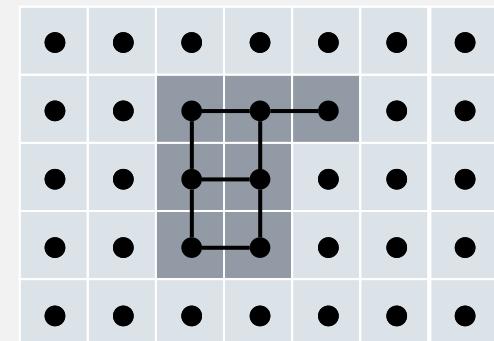
**Challenge.** Flood fill (Photoshop magic wand).

**Assumptions.** Picture has millions to billions of pixels.



**Solution.** Build a grid graph.

- Vertex: pixel.
- Edge: between two adjacent gray pixels.
- Blob: all pixels connected to given pixel.



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.1 UNDIRECTED GRAPHS

---

- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *connected components*
- ▶ *challenges*

# Algorithms

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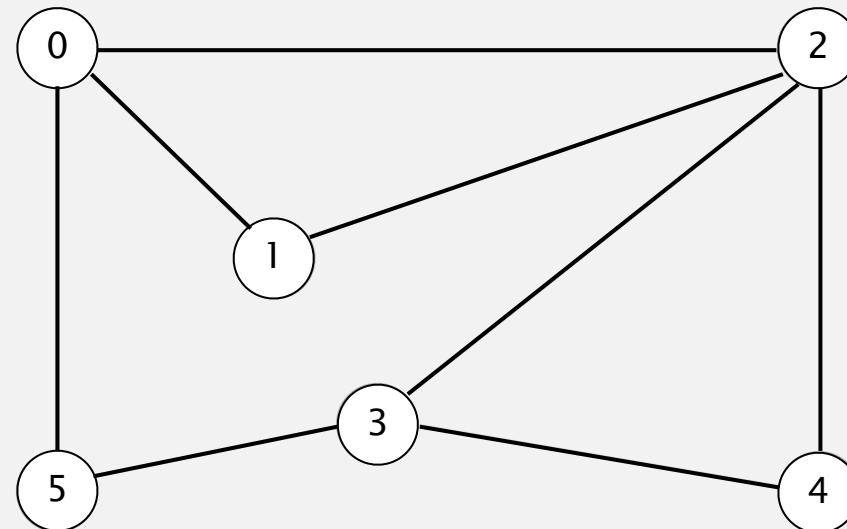
- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *connected components*
- ▶ *challenges*

## Breadth-first search demo

queue 는  $\{v\}$  초기화할 때 만족

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices adjacent to  $v$  and mark them.



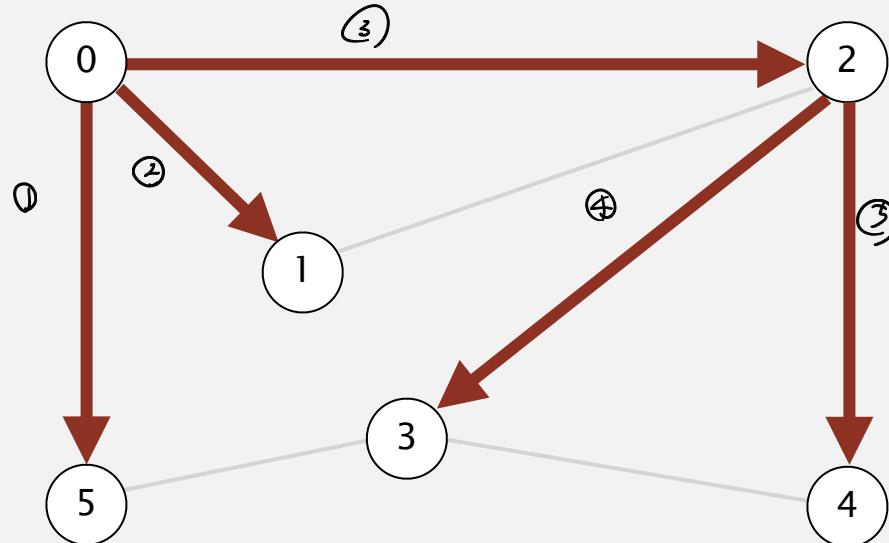
tinyCG.txt  
V → 6  
E ← 8  
0 5  
2 4  
2 3  
1 2  
0 1  
3 4  
3 5  
0 2

graph G

## Breadth-first search demo

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices adjacent to  $v$  and mark them.



DSol[ $\leftarrow$ ]  
↓

| $v$ | edgeTo[] | distTo[] |
|-----|----------|----------|
| 0   | -        | 0        |
| 1   | 0        | 1        |
| 2   | 0        | 1        |
| 3   | 2        | 2        |
| 4   | 2        | 2        |
| 5   | 0        | 1        |

done

## Breadth-first search

Depth-first search. Put unvisited vertices on a **stack**.

Breadth-first search. Put unvisited vertices on a **queue**.

\$\\$ \quad \text{Shortest path.} \quad \text{Find path from } s \text{ to } t \text{ that uses fewest number of edges.}

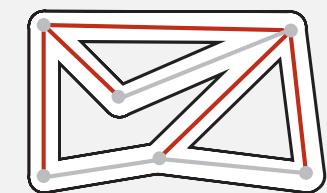
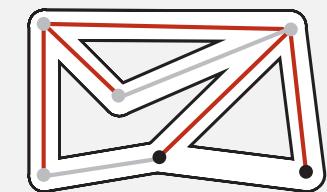
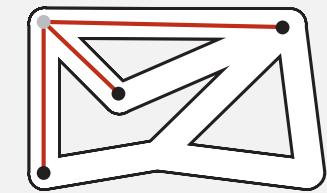
~~BFS CAT!~~

### BFS (from source vertex $s$ )

Put  $s$  onto a FIFO queue, and mark  $s$  as visited.

Repeat until the queue is empty:

- remove the least recently added vertex  $v$
- add each of  $v$ 's unvisited neighbors to the queue,  
and mark them as visited.



Intuition. BFS examines vertices in increasing distance from  $s$ .

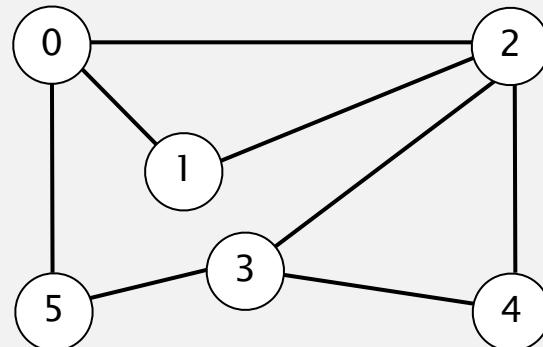
## Breadth-first search properties

---

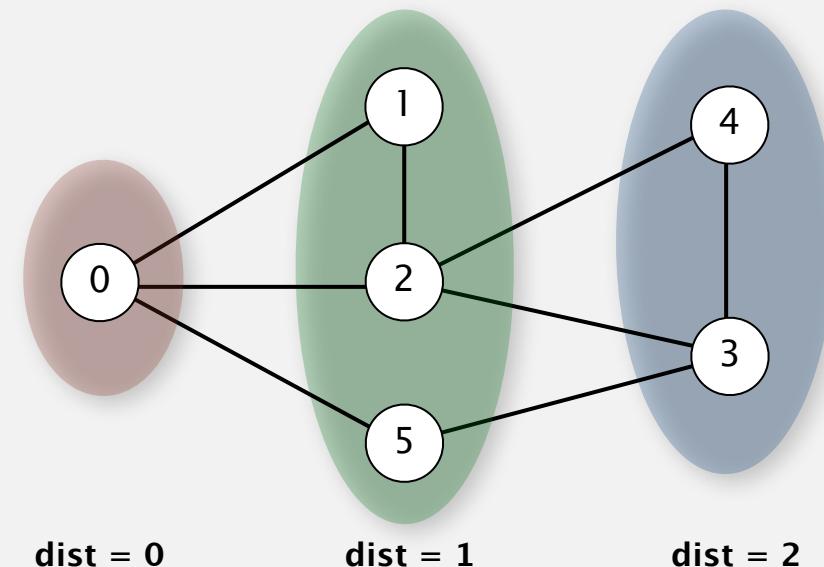
**Proposition.** BFS computes shortest paths (fewest number of edges) from  $s$  to all other vertices in a graph in time proportional to  $E + V$ .

**Pf. [correctness]** Queue always consists of zero or more vertices of distance  $k$  from  $s$ , followed by zero or more vertices of distance  $k + 1$ .

**Pf. [running time]** Each vertex connected to  $s$  is visited once.



graph



dist = 0

dist = 1

dist = 2

# Breadth-first search

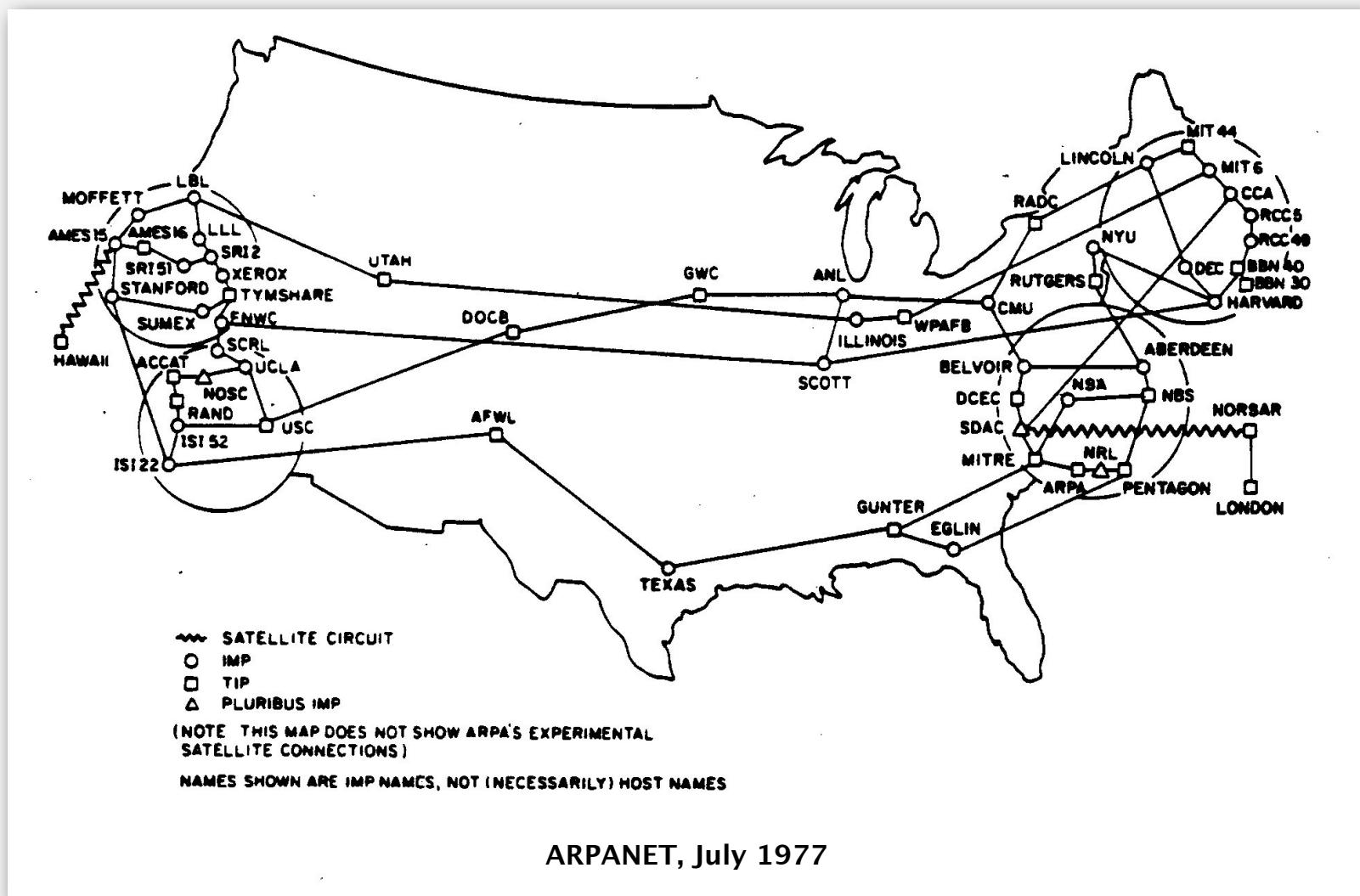
---

```
public class BreadthFirstPaths
{
    private boolean[] marked;
    private int[] edgeTo;
    ...

    private void bfs(Graph G, int s)
    {
        Queue<Integer> q = new Queue<Integer>();
        q.enqueue(s);
        marked[s] = true; ← ↘
        while (!q.isEmpty())
        {
            int v = q.dequeue();
            for (int w : G.adj(v))
            {
                if (!marked[w])
                {
                    q.enqueue(w);
                    marked[w] = true;
                    edgeTo[w] = v;
                }
            }
        }
    }
}
```

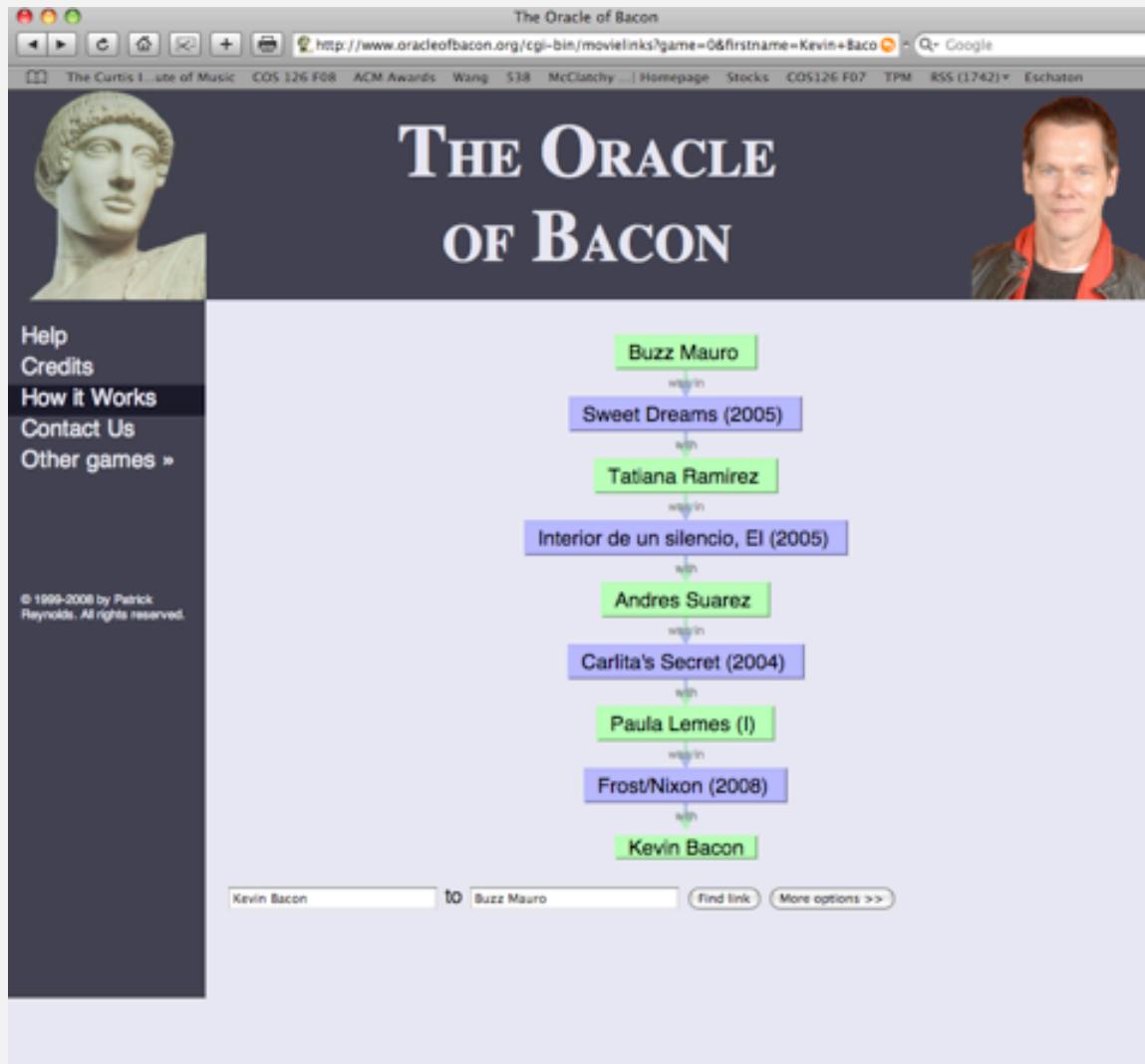
# Breadth-first search application: routing

Fewest number of hops in a communication network.



# Breadth-first search application: Kevin Bacon numbers

Kevin Bacon numbers.



<http://oracleofbacon.org>



Endless Games board game

New 2 Degrees

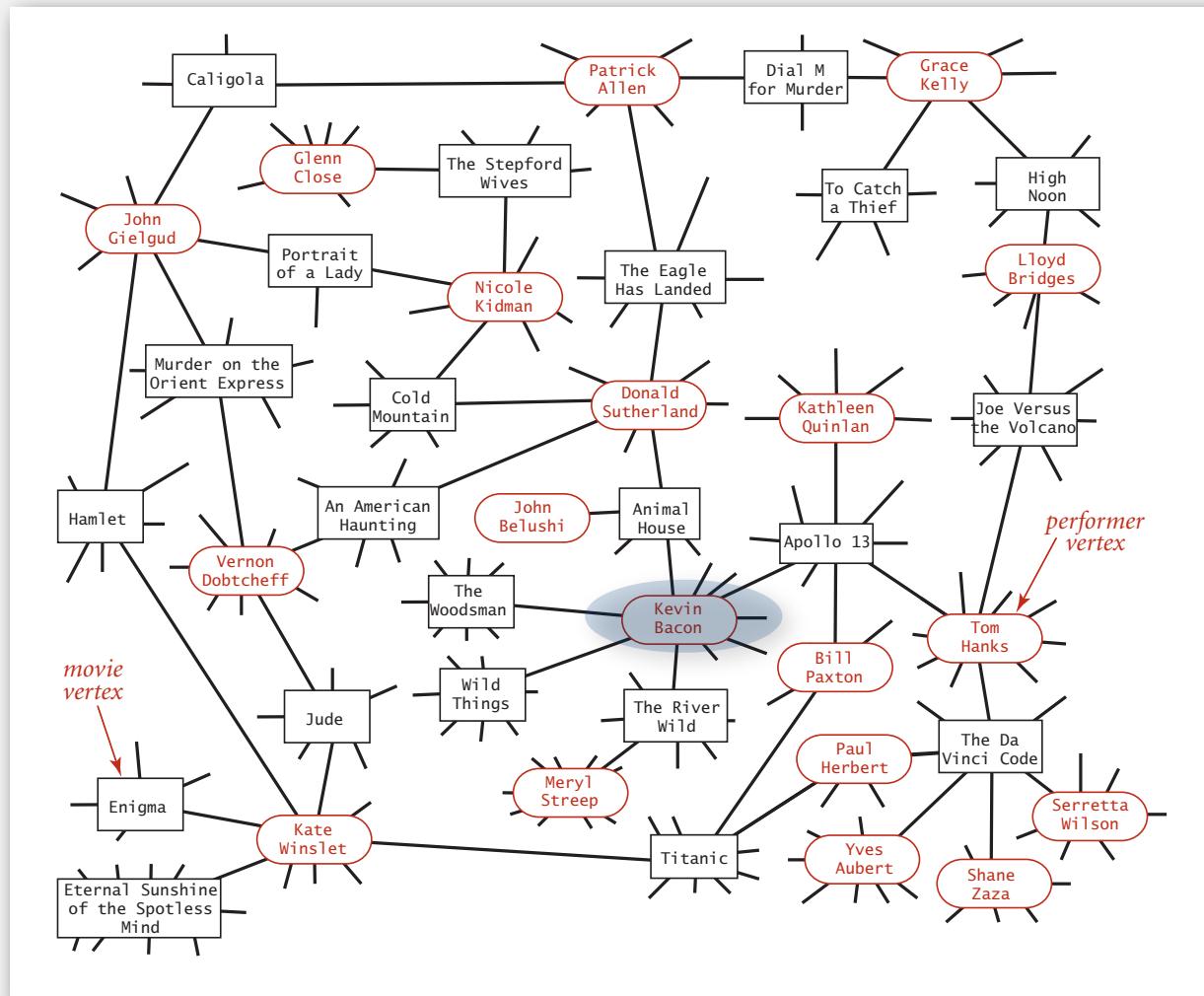
Uma Thurman acted in Be Cool (2005) with Scott Adsit who acted in The Informant! (2009) with Matt Damon

Lookup Trivia Guess Degrees Scoreboard

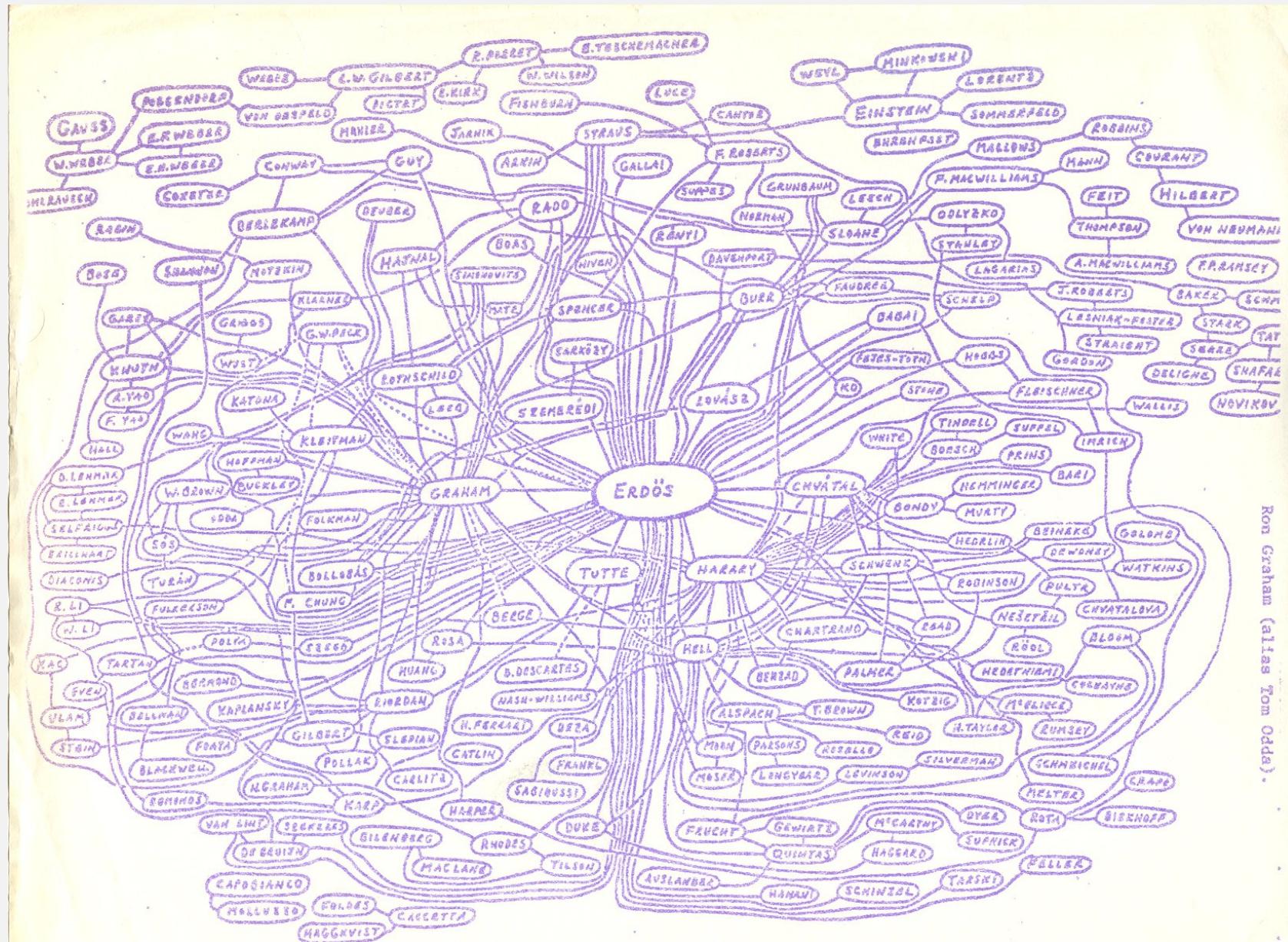
SixDegrees iPhone App

# Kevin Bacon graph

- Include one vertex for each performer **and** one for each movie.
- Connect a movie to all performers that appear in that movie.
- Compute shortest path from  $s = \text{Kevin Bacon}$ .



# Breadth-first search application: Erdős numbers



**hand-drawing of part of the Erdős graph by Ron Graham**

RUMI Gamanus (alias Lou Sava)

# Algorithms

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## 4.1 UNDIRECTED GRAPHS

---

- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *connected components*
- ▶ *challenges*

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- ▶ ***connected components***
- ▶ *challenges*

## Connectivity queries

---

**Def.** Vertices  $v$  and  $w$  are **connected** if there is a path between them.

**Goal.** Preprocess graph to answer queries of the form *is  $v$  connected to  $w$ ?* in **constant time**.

|                                 |   |
|---------------------------------|---|
| public class CC                 |   |
| CC(Graph G)                     | <i>find connected components in <math>G</math></i>      |
| boolean connected(int v, int w) | <i>are <math>v</math> and <math>w</math> connected?</i> |
| int count()                     | <i>number of connected components</i>                   |
| int id(int v)                   | <i>component identifier for <math>v</math></i>          |

Union-Find? Not quite. (Constant time  $\alpha(n)$ )

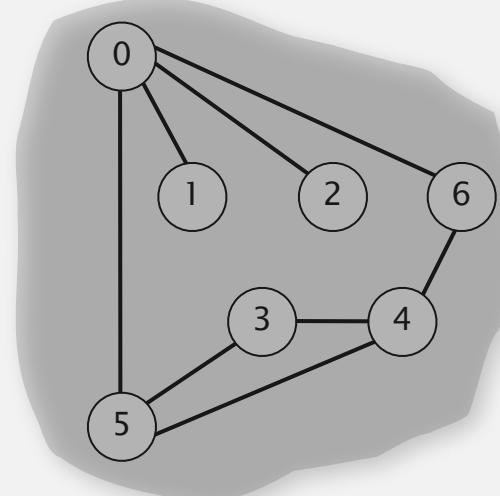
Depth-first search. Yes. [next few slides]

# Connected components

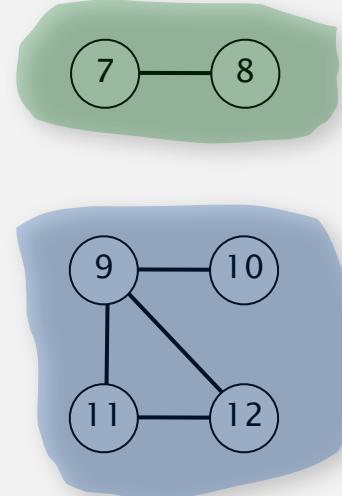
The relation "is connected to" is an **equivalence relation**:

- Reflexive:  $v$  is connected to  $v$ .
- Symmetric: if  $v$  is connected to  $w$ , then  $w$  is connected to  $v$ .
- Transitive: if  $v$  connected to  $w$  and  $w$  connected to  $x$ , then  $v$  connected to  $x$ .

**Def.** A **connected component** is a maximal set of connected vertices.



3 connected components



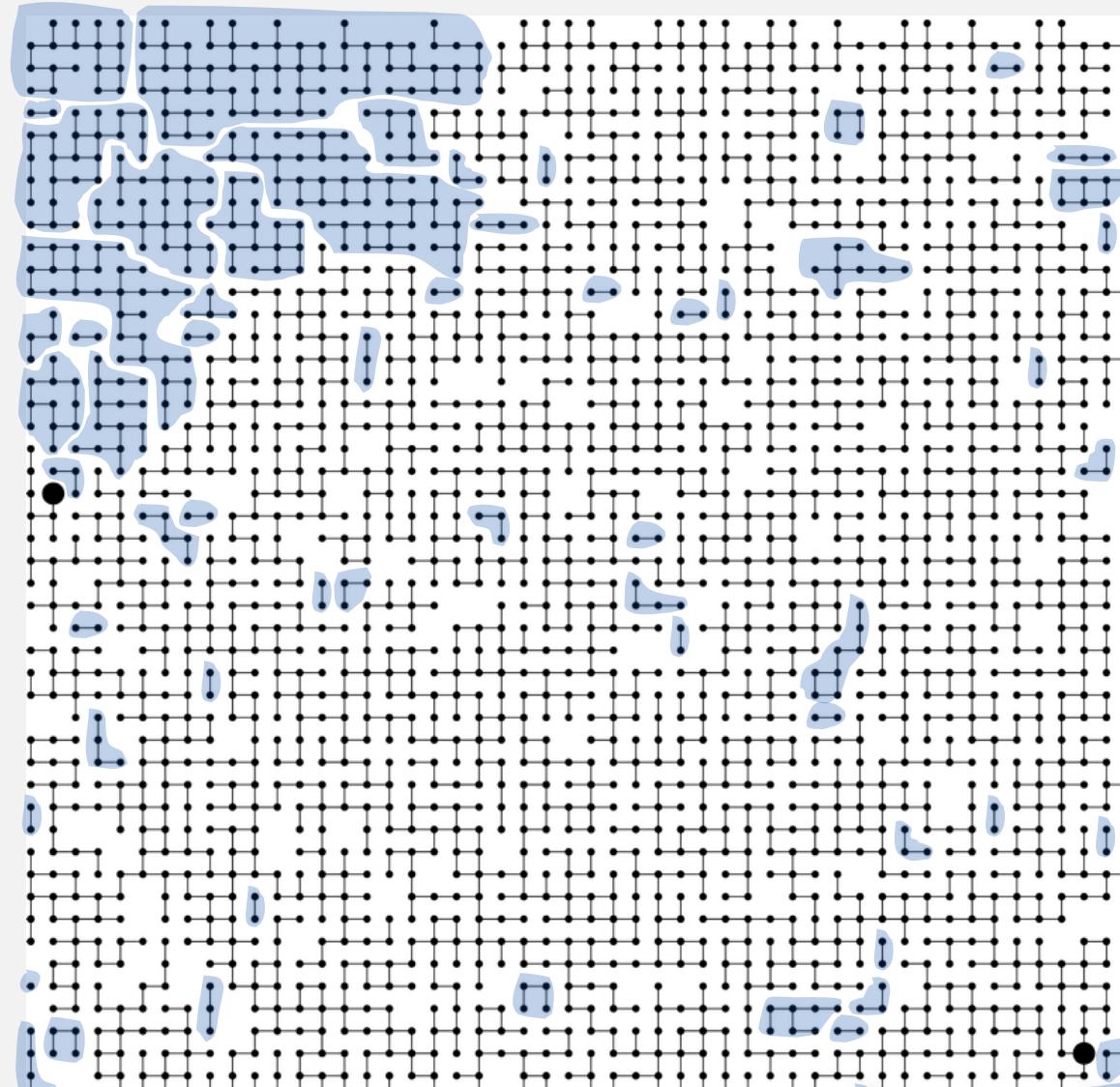
| <i>v</i> | <i>id[]</i> |
|----------|-------------|
| 0        | 0           |
| 1        | 0           |
| 2        | 0           |
| 3        | 0           |
| 4        | 0           |
| 5        | 0           |
| 6        | 0           |
| 7        | 1           |
| 8        | 1           |
| 9        | 2           |
| 10       | 2           |
| 11       | 2           |
| 12       | 2           |

**Remark.** Given connected components, can answer queries in constant time.

# Connected components

---

Def. A **connected component** is a maximal set of connected vertices.



63 connected components

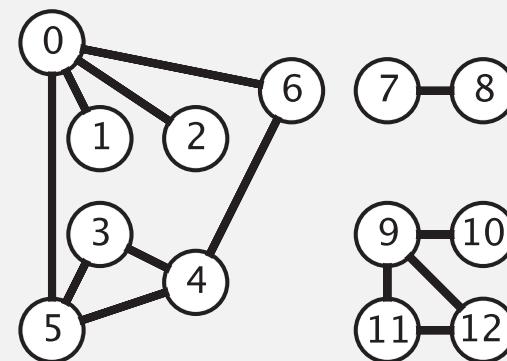
# Connected components

**Goal.** Partition vertices into connected components.

## Connected components

Initialize all vertices  $v$  as unmarked.

For each unmarked vertex  $v$ , run DFS to identify all vertices discovered as part of the same component.



*tinyG.txt*

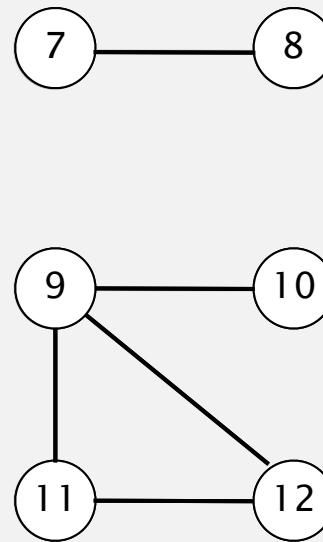
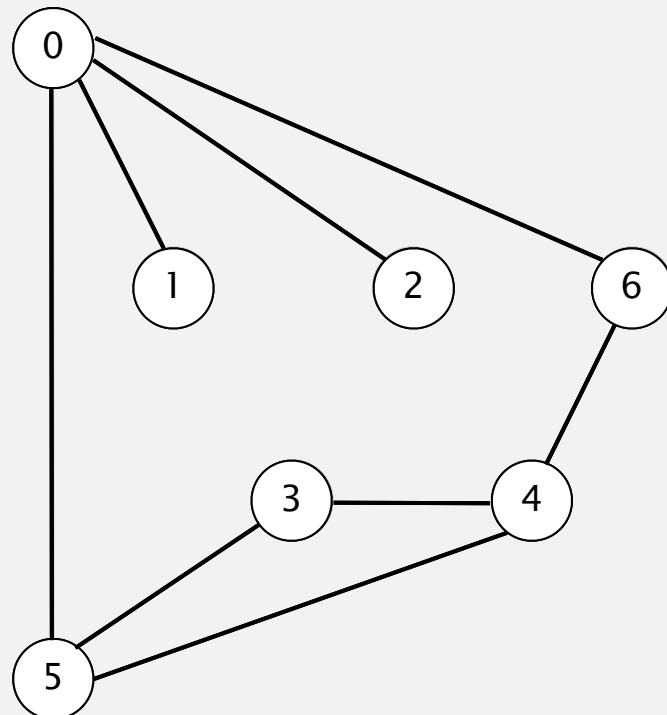
$V \rightarrow 13$        $E \leftarrow$

|       |    |
|-------|----|
| 13    | 13 |
| 0 5   |    |
| 4 3   |    |
| 0 1   |    |
| 9 12  |    |
| 6 4   |    |
| 5 4   |    |
| 0 2   |    |
| 11 12 |    |
| 9 10  |    |
| 0 6   |    |
| 7 8   |    |
| 9 11  |    |
| 5 3   |    |

# Connected components demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



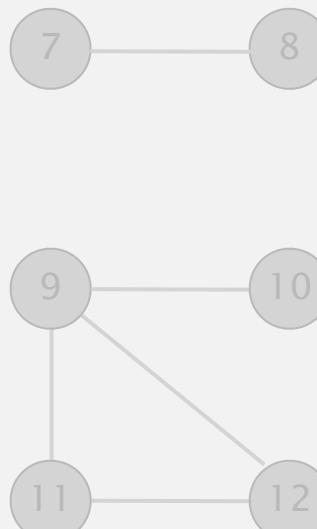
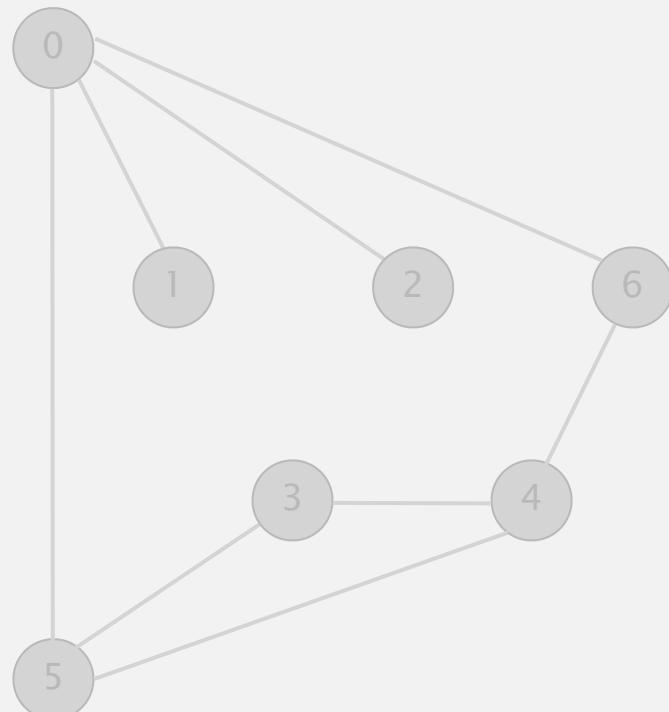
| $v$ | marked[] | id[] |
|-----|----------|------|
| 0   | F        | -    |
| 1   | F        | -    |
| 2   | F        | -    |
| 3   | F        | -    |
| 4   | F        | -    |
| 5   | F        | -    |
| 6   | F        | -    |
| 7   | F        | -    |
| 8   | F        | -    |
| 9   | F        | -    |
| 10  | F        | -    |
| 11  | F        | -    |
| 12  | F        | -    |

graph G

# Connected components demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



| $v$ | marked[] | id[] |
|-----|----------|------|
| 0   | T        | 0    |
| 1   | T        | 0    |
| 2   | T        | 0    |
| 3   | T        | 0    |
| 4   | T        | 0    |
| 5   | T        | 0    |
| 6   | T        | 0    |
| 7   | T        | 1    |
| 8   | T        | 1    |
| 9   | T        | 2    |
| 10  | T        | 2    |
| 11  | T        | 2    |
| 12  | T        | 2    |

Handwritten notes:

- unmarked vertex  $\rightarrow$   $\text{방문하지 않은 vertex}$
- not visited yet  $\rightarrow$   $\text{방문하지 않은 vertex}$
- visit  $\rightarrow$   $\text{방문하기}$
- id + 1  $\rightarrow$   $\text{id} + 1 \text{ 증가.}$

done

# Finding connected components with DFS

```
public class CC
{
    private boolean[] marked;
    private int[] id;
    private int count;

    public CC(Graph G)
    {
        marked = new boolean[G.V()];
        id = new int[G.V()];
        for (int v = 0; v < G.V(); v++)
        {
            if (!marked[v])
            {
                dfs(G, v);
                count++;
            }
        }
    }

    public int count()
    public int id(int v)
    private void dfs(Graph G, int v)
}
```

$\text{id}[v] = \text{id}$  of component containing  $v$   
number of components

run DFS from one vertex in  
each component

see next slide

## Finding connected components with DFS (continued)

```
public int count()
{   return count;   }

public int id(int v)
{   return id[v];   }

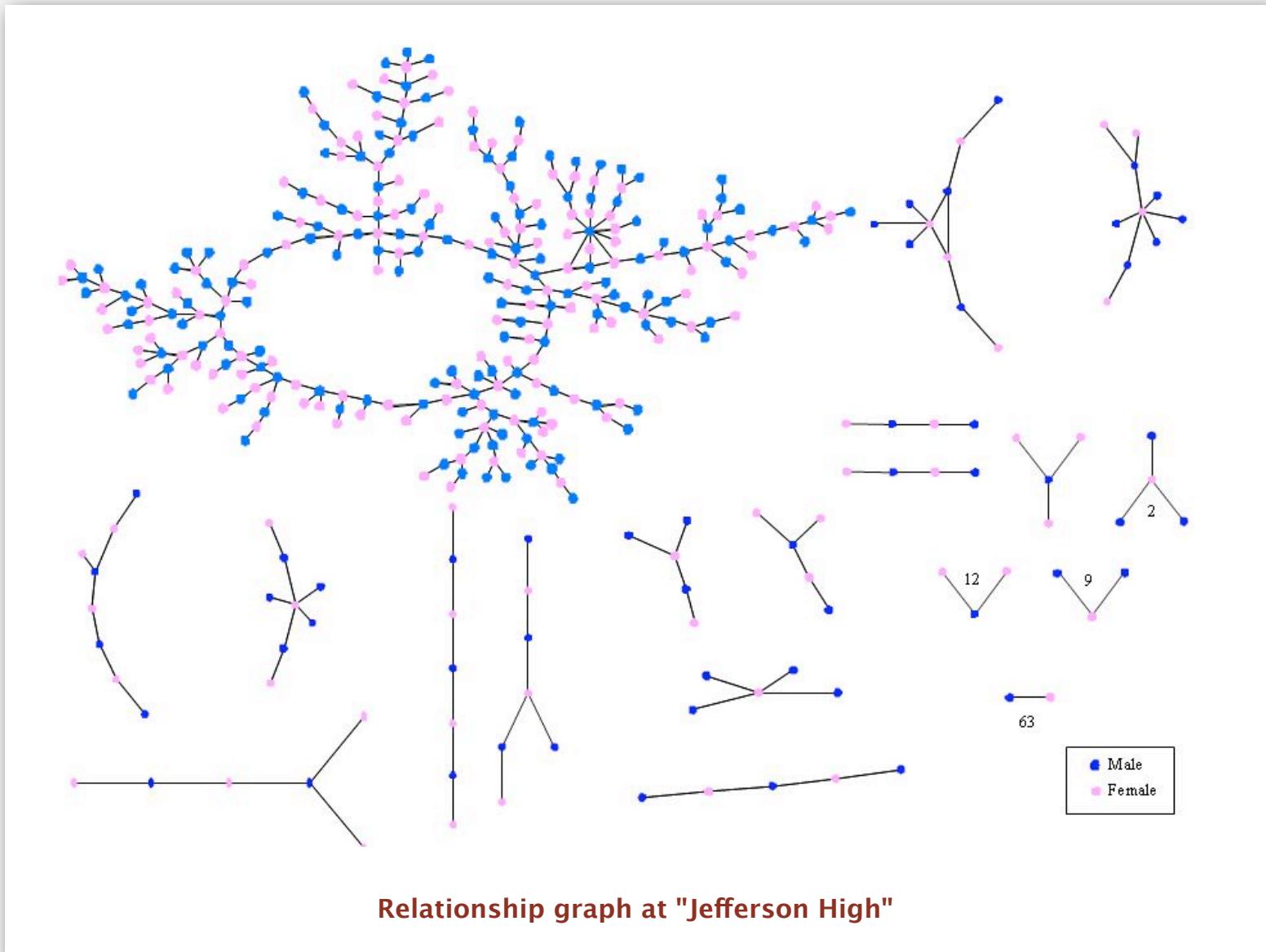
private void dfs(Graph G, int v) ← adjacent vertex → marking ← marking
{
    marked[v] = true;
    id[v] = count;
    for (int w : G.adj(v))
        if (!marked[w])
            dfs(G, w);
}
```

← number of components

← id of component containing v

all vertices discovered in  
same call of dfs have same id

## Connected components application: study spread of STDs



Peter Bearman, James Moody, and Katherine Stovel. Chains of affection: The structure of adolescent romantic and sexual networks. American Journal of Sociology, 110(1): 44–99, 2004.

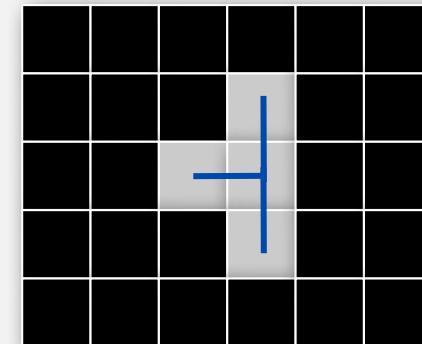
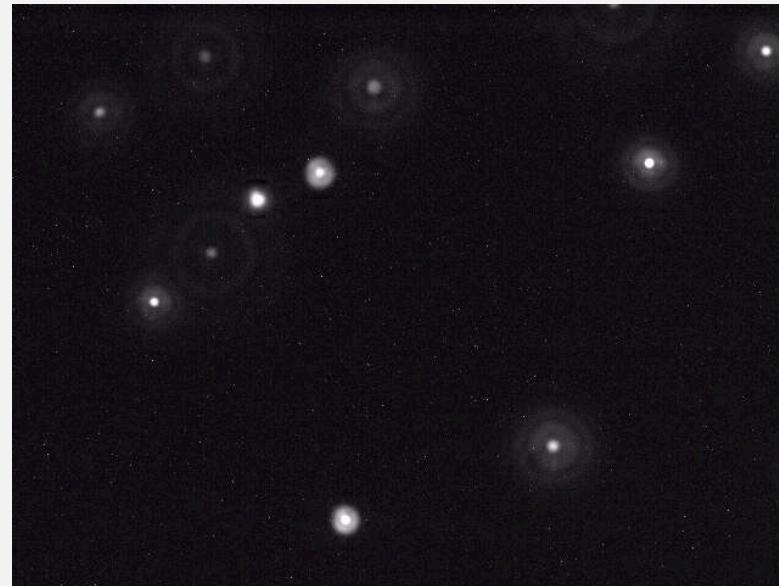
## Connected components application: particle detection

---

**Particle detection.** Given grayscale image of particles, identify "blobs."

- Vertex: pixel.
- Edge: between two adjacent pixels with grayscale value  $\geq 70$ .
- Blob: connected component of 20-30 pixels.

black = 0  
white = 255



**Particle tracking.** Track moving particles over time.

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- ▶ *breadth-first search*
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- ▶ *challenges*

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# Graph-processing challenge 1

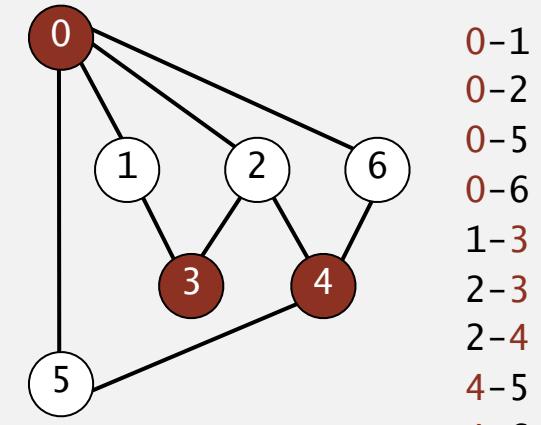
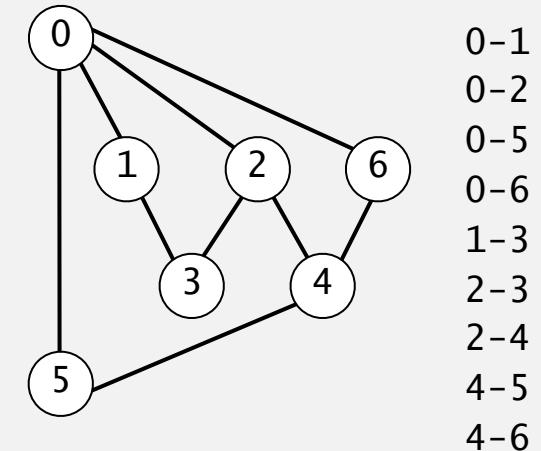
Problem. Is a graph bipartite?

→ vertices 2개의 그룹으로 나누고  
2개의 그룹이 연결하지 않도록 하는지

How difficult?

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

simple DFS-based solution  
(see textbook)



{ 0, 3, 4 }

# Bipartiteness application: is dating graph bipartite?

---

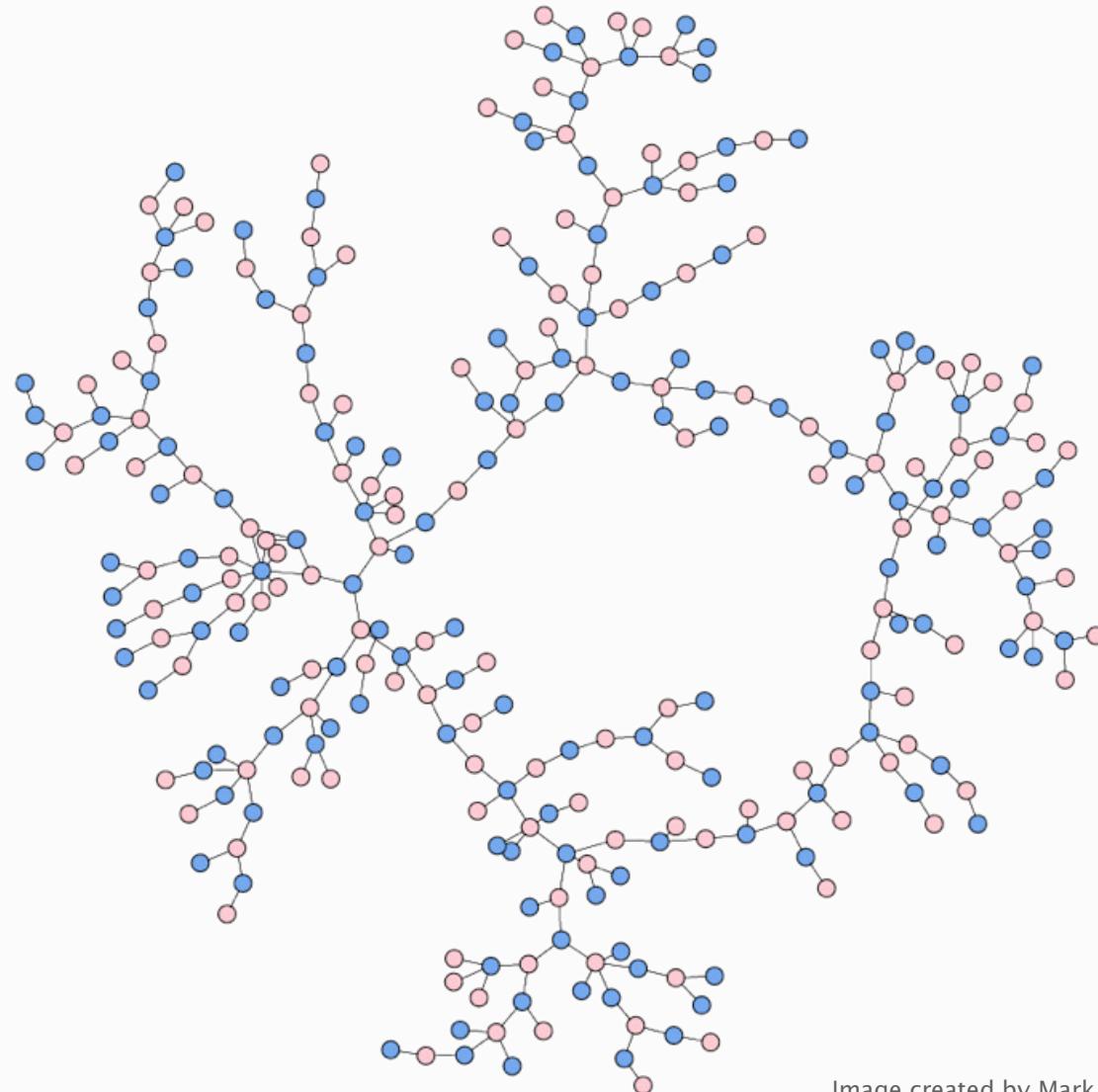


Image created by Mark Newman.

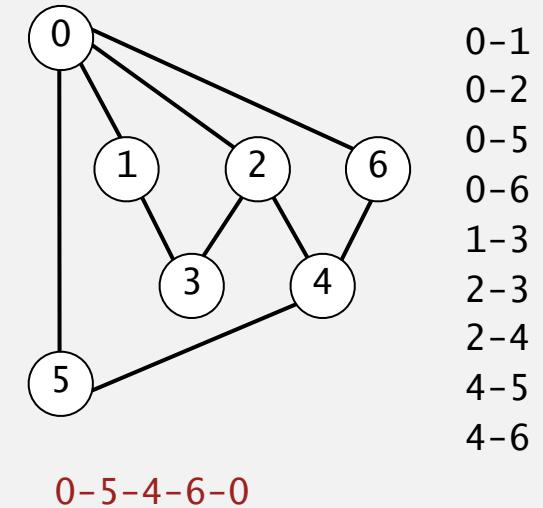
# Graph-processing challenge 2

**Problem.** Find a cycle.

**How difficult?**

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

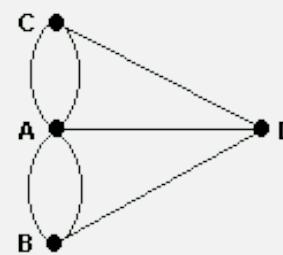
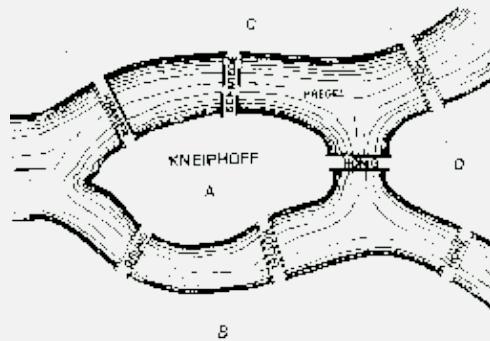
simple DFS-based solution  
(see textbook)



# Bridges of Königsberg

The Seven Bridges of Königsberg. [Leonhard Euler 1736]

*“ ... in Königsberg in Prussia, there is an island A, called the Kneiphof; the river which surrounds it is divided into two branches ... and these branches are crossed by seven bridges. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he could cross each bridge once and only once.”*



Euler tour. Is there a (general) cycle that uses each edge exactly once?

Answer. A connected graph is Eulerian iff all vertices have **even** degree.

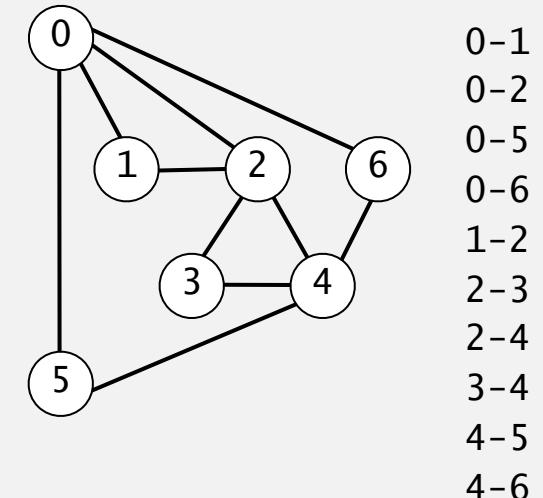
# Graph-processing challenge 3

**Problem.** Find a (general) cycle that uses every edge exactly once.

**How difficult?**

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

Eulerian tour  
(classic graph-processing problem)



0-1-2-3-4-2-0-6-4-5-0

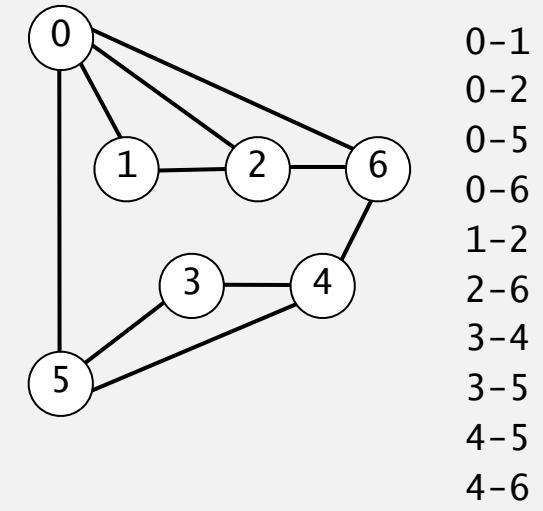
# Graph-processing challenge 4

**Problem.** Find a cycle that visits every vertex exactly once.

**How difficult?**

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- Hire an expert.
- ✓ • Intractable. ←  
• No one knows.  
• Impossible.

Hamiltonian cycle  
(classical NP-complete problem)



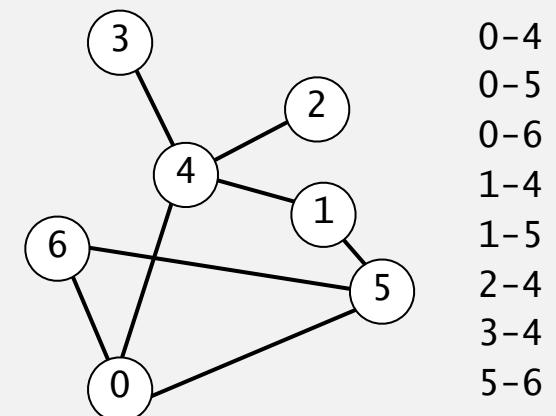
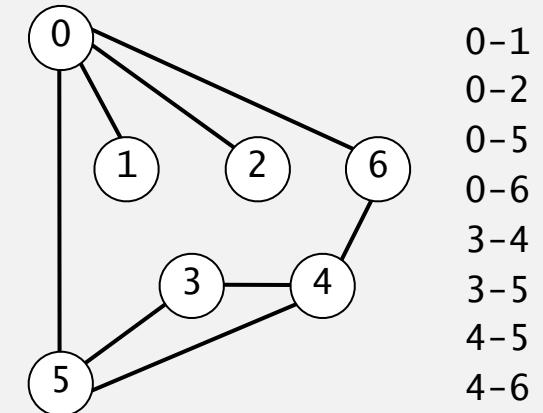
# Graph-processing challenge 5

**Problem.** Are two graphs identical except for vertex names?

**How difficult?**

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- ✓ • No one knows.
- Impossible.

graph isomorphism is  
longstanding open problem



$0 \leftrightarrow 4, 1 \leftrightarrow 3, 2 \leftrightarrow 2, 3 \leftrightarrow 6, 4 \leftrightarrow 5, 5 \leftrightarrow 0, 6 \leftrightarrow 1$

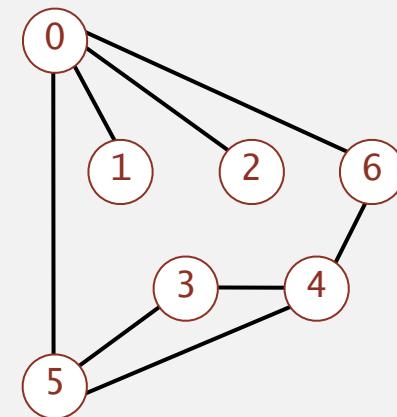
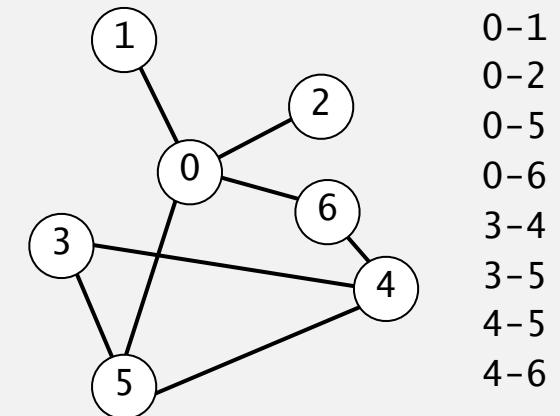
# Graph-processing challenge 6

**Problem.** Lay out a graph in the plane without crossing edges?

**How difficult?**

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- ✓ • Hire an expert.
- Intractable.
- No one knows.
- Impossible.

linear-time DFS-based planarity algorithm  
discovered by Tarjan in 1970s  
(too complicated for most practitioners)



# Algorithms

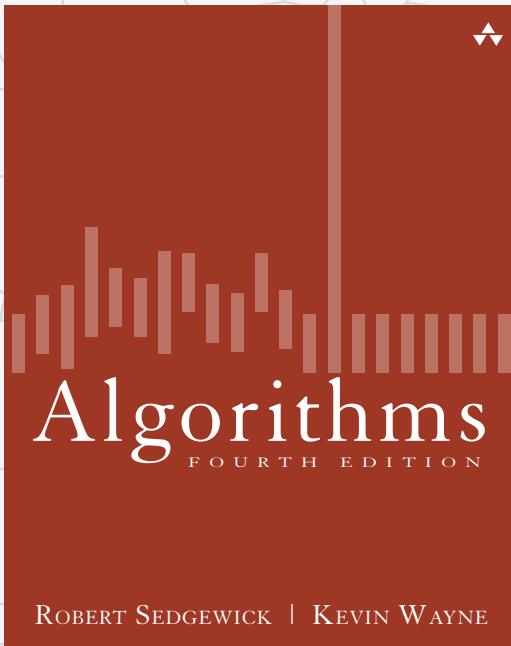
ROBERT SEDGEWICK | KEVIN WAYNE

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- ▶ *connected components*
- ▶ ***challenges***



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## 4.1 UNDIRECTED GRAPHS

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- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *connected components*
- ▶ *challenges*