

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE



<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

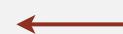
# Two classic sorting algorithms

---

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20<sup>th</sup> century in science and engineering.

Mergesort.



last lecture

- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

Quicksort.

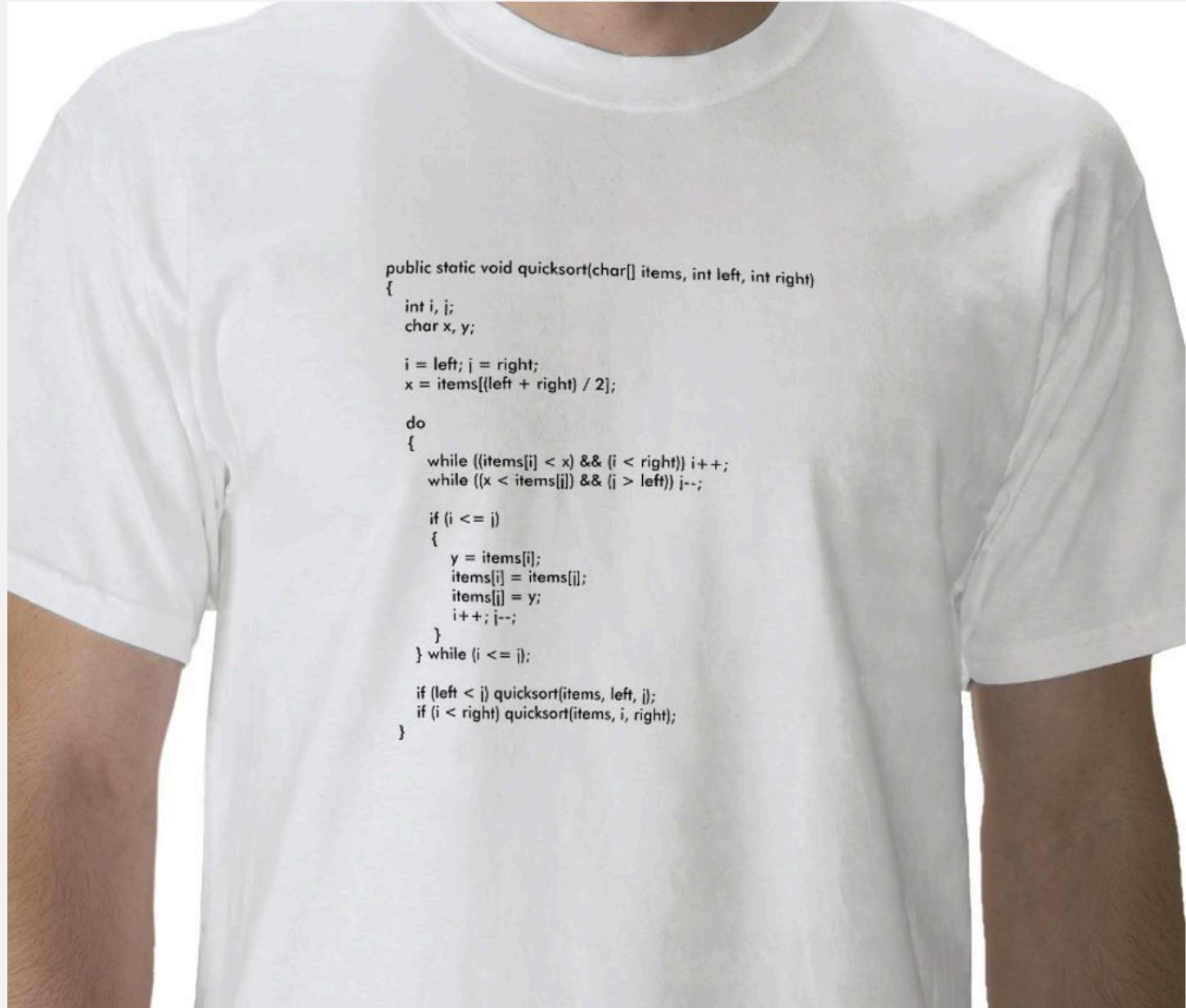


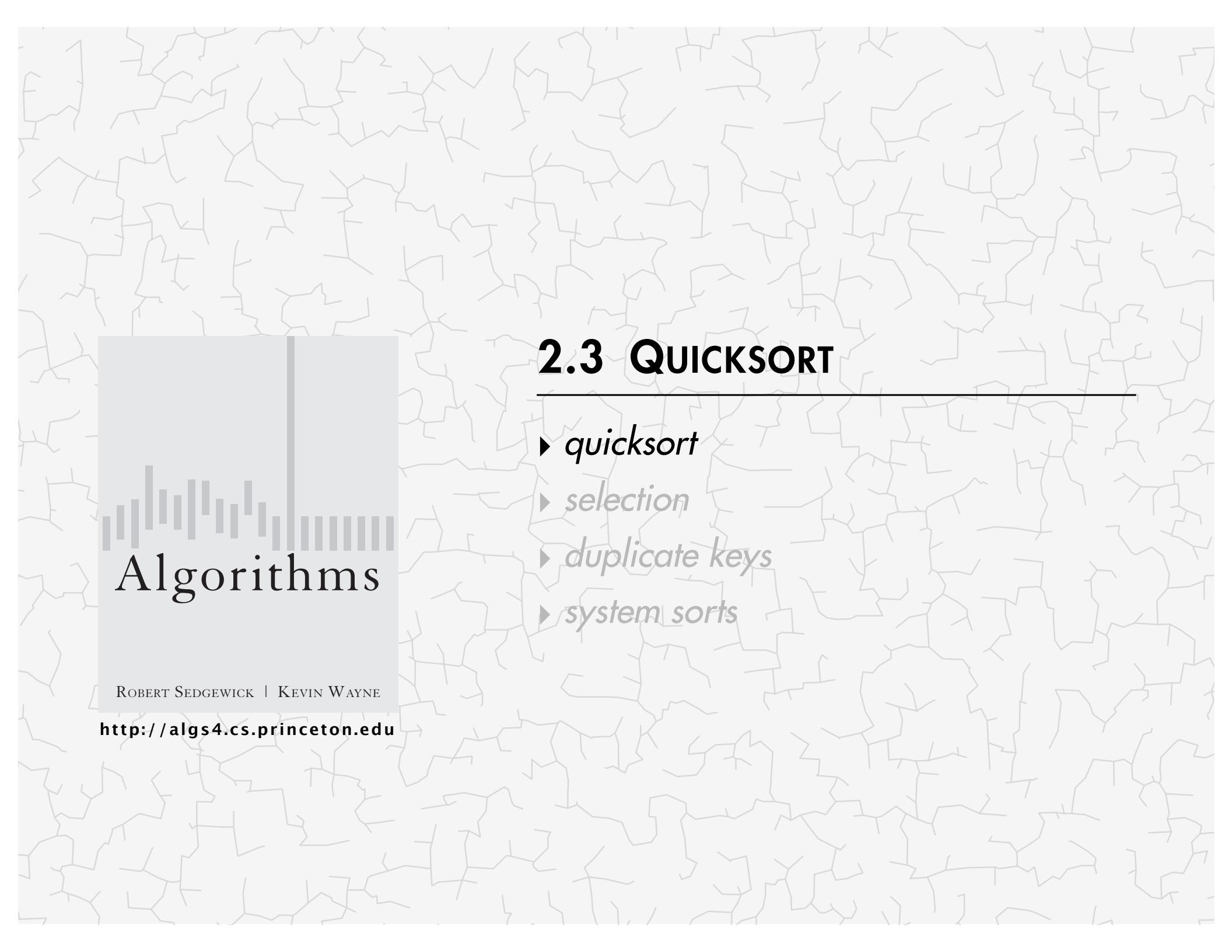
this lecture

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

# Quicksort t-shirt

---





# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

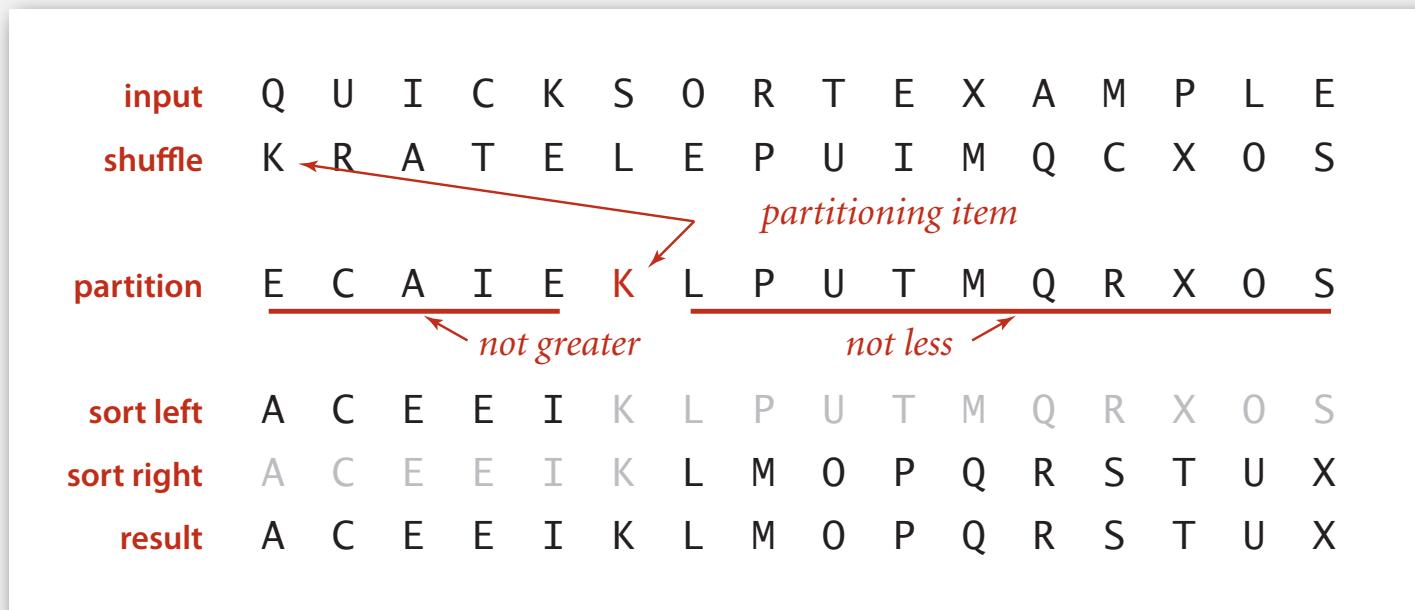
# Quicksort

Basic plan.

- **Shuffle** the array.
- **Partition** so that, for some  $j$ 
  - entry  $a[j]$  is in place
  - no larger entry to the left of  $j$
  - no smaller entry to the right of  $j$
- **Sort** each piece **recursively**.



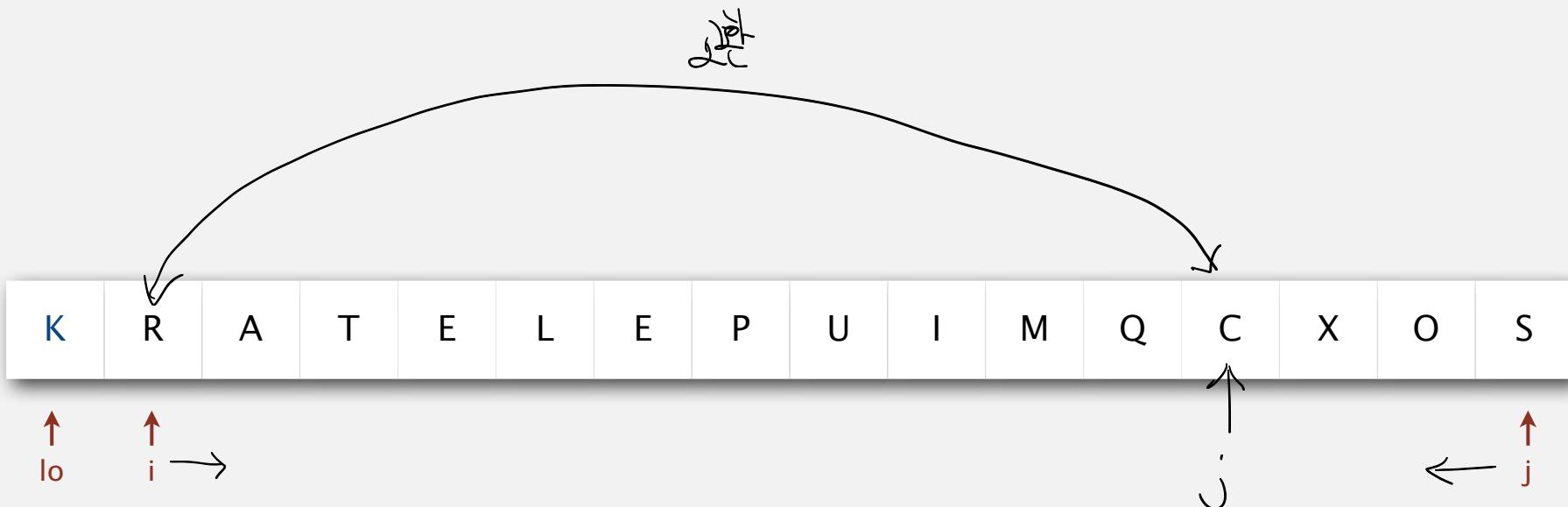
Sir Charles Antony Richard Hoare  
1980 Turing Award



# Quicksort partitioning demo

Repeat until  $i$  and  $j$  pointers cross.

- Scan  $i$  from left to right so long as  $(a[i] < a[lo])$ .
- Scan  $j$  from right to left so long as  $(a[j] > a[lo])$ .
- Exchange  $a[i]$  with  $a[j]$ .



# Quicksort partitioning demo

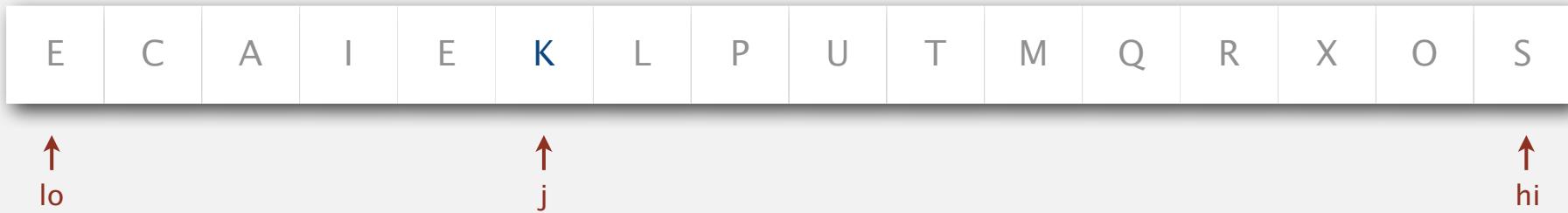
---

Repeat until  $i$  and  $j$  pointers cross.

- Scan  $i$  from left to right so long as  $(a[i] < a[lo])$ .
- Scan  $j$  from right to left so long as  $(a[j] > a[lo])$ .
- Exchange  $a[i]$  with  $a[j]$ .

When pointers cross.

- Exchange  $a[lo]$  with  $a[j]$ .



partitioned!

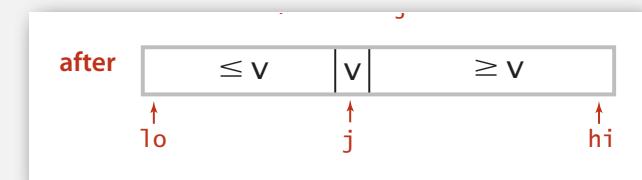
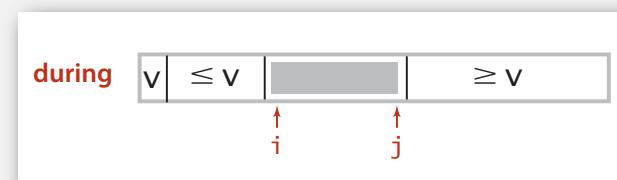
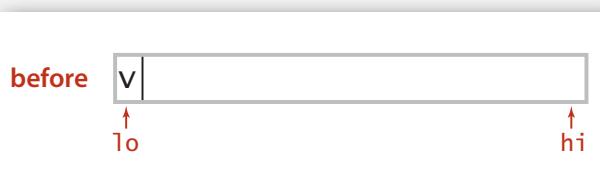
# Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
    int i = lo, j = hi+1;
    while (true)
    {
        while (less(a[++i], a[lo]))           find item on left to swap
            if (i == hi) break;

        while (less(a[lo], a[--j]))           find item on right to swap
            if (j == lo) break;

        if (i >= j) break;                  check if pointers cross
        exch(a, i, j);                   swap
    }

    exch(a, lo, j);                  swap with partitioning item
    return j;                        return index of item now known to be in place
}
```



# Quicksort: Java implementation

---

```
public class Quick
{
    private static int partition(Comparable[] a, int lo, int hi)
    { /* see previous slide */ }

    public static void sort(Comparable[] a)
    {
        StdRandom.shuffle(a);
        sort(a, 0, a.length - 1);
    }

    private static void sort(Comparable[] a, int lo, int hi)
    {
        if (hi <= lo) return;
        int j = partition(a, lo, hi);
        sort(a, lo, j-1);
        sort(a, j+1, hi);
    }
}
```

shuffle needed for  
performance guarantee  
(stay tuned)

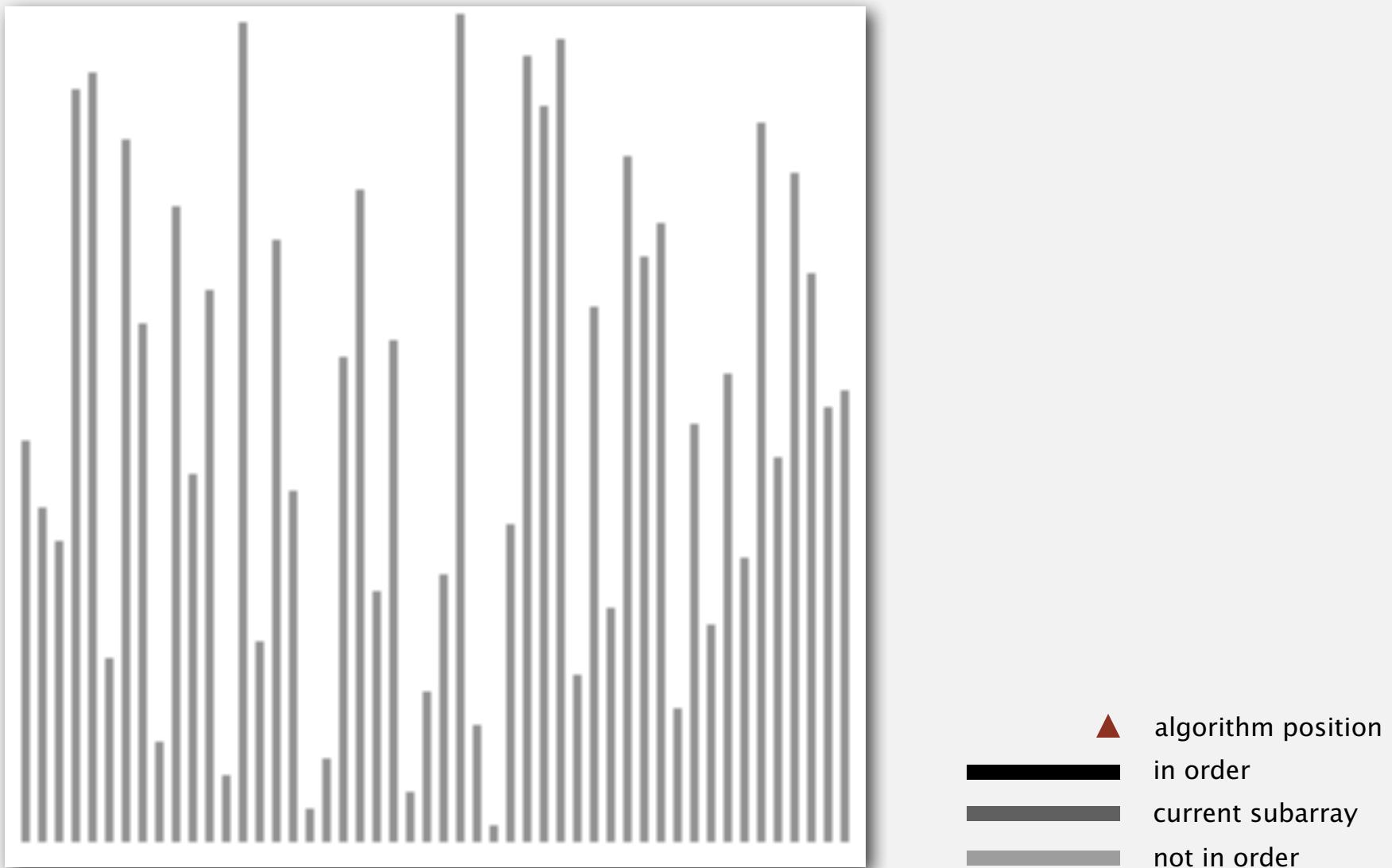
# Quicksort trace

| lo             | j  | hi | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |   |
|----------------|----|----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|---|
| initial values |    |    | Q | U | I | C | K | S | O | R | T | E | X  | A  | M  | P  | L  | E  |   |
| random shuffle |    |    | K | R | A | T | E | L | E | P | U | I | M  | Q  | C  | X  | 0  | S  |   |
| 0              | 5  | 15 | E | C | A | I | E | K | L | P | U | T | M  | Q  | R  | X  | 0  | S  |   |
| 0              | 3  | 4  | E | C | A | E | I | K | L | P | U | T | M  | Q  | R  | X  | 0  | S  |   |
| 0              | 2  | 2  | A | C | E | E | I | K | L | P | U | T | M  | Q  | R  | X  | 0  | S  |   |
| 0              | 0  | 1  | A | C | E | E | I | K | L | P | U | T | M  | Q  | R  | X  | 0  | S  |   |
| 1              | 1  | 1  | A | C | E | E | I | K | L | P | U | T | M  | Q  | R  | X  | 0  | S  |   |
| 4              | 4  | 4  | A | C | E | E | I | K | L | P | U | T | M  | Q  | R  | X  | 0  | S  |   |
| 6              | 6  | 15 | A | C | E | E | I | K | L | P | U | T | M  | Q  | R  | X  | 0  | S  |   |
| 7              | 9  | 15 | A | C | E | E | I | K | L | M | O | P | T  | Q  | R  | X  | U  | S  |   |
| 7              | 7  | 8  | A | C | E | E | I | K | L | M | M | O | P  | T  | Q  | R  | X  | U  | S |
| 8              | 8  | 8  | A | C | E | E | I | K | L | M | M | O | P  | T  | Q  | R  | X  | U  | S |
| 10             | 13 | 15 | A | C | E | E | I | K | L | M | M | O | P  | S  | Q  | R  | T  | U  | X |
| 10             | 12 | 12 | A | C | E | E | I | K | L | M | M | O | P  | R  | Q  | S  | T  | U  | X |
| 10             | 11 | 11 | A | C | E | E | I | K | L | M | M | O | P  | Q  | R  | S  | T  | U  | X |
| 10             | 10 | 10 | A | C | E | E | I | K | L | M | M | O | P  | Q  | R  | S  | T  | U  | X |
| 14             | 14 | 15 | A | C | E | E | I | K | L | M | M | O | P  | Q  | R  | S  | T  | U  | X |
| 15             | 15 | 15 | A | C | E | E | I | K | L | M | M | O | P  | Q  | R  | S  | T  | U  | X |
| result         |    |    | A | C | E | E | I | K | L | M | O | P | Q  | R  | S  | T  | U  | X  |   |

Quicksort trace (array contents after each partition)

# Quicksort animation

## 50 random items



<http://www.sorting-algorithms.com/quick-sort>

## Quicksort: implementation details

---

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Merge Sort에 비해 QuickSort의  
더 나은 공간(주 Space)을 사용하는 것

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The ( $j == lo$ ) test is redundant (why?), but the ( $i == hi$ ) test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on keys equal to the partitioning item's key.

# Quicksort: empirical analysis

---

## Running time estimates:

- Home PC executes  $10^8$  compares/second.
- Supercomputer executes  $10^{12}$  compares/second.

|          | insertion sort ( $N^2$ ) |           |           | mergesort ( $N \log N$ ) |          |         | quicksort ( $N \log N$ ) |         |         |
|----------|--------------------------|-----------|-----------|--------------------------|----------|---------|--------------------------|---------|---------|
| computer | thousand                 | million   | billion   | thousand                 | million  | billion | thousand                 | million | billion |
| home     | instant                  | 2.8 hours | 317 years | instant                  | 1 second | 18 min  | instant                  | 0.6 sec | 12 min  |
| super    | instant                  | 1 second  | 1 week    | instant                  | instant  | instant | instant                  | instant | instant |

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

# Quicksort: best-case analysis

장악하는 방법으로 나누는 경우

**Best case.** Number of compares is  $\sim N \lg N$ .

| lo | j  | hi | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | a[ ] |
|----|----|----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|------|
|    |    |    | H | A | C | B | F | E | G | D | L | I | K  | J  | N  | M  | O  |      |
|    |    |    | H | A | C | B | F | E | G | D | L | I | K  | J  | N  | M  | O  |      |
| 0  | 7  | 14 | D | A | C | B | F | E | G | H | L | I | K  | J  | N  | M  | O  |      |
| 0  | 3  | 6  | B | A | C | D | F | E | G | H | L | I | K  | J  | N  | M  | O  |      |
| 0  | 1  | 2  | A | B | C | D | F | E | G | H | L | I | K  | J  | N  | M  | O  |      |
| 0  | 0  | A  | B | C | D | F | E | G | H | L | I | K | J  | N  | M  | O  |    |      |
| 2  | 2  | A  | B | C | D | F | E | G | H | L | I | K | J  | N  | M  | O  |    |      |
| 4  | 5  | 6  | A | B | C | D | E | F | G | H | L | I | K  | J  | N  | M  | O  |      |
| 4  | 4  | A  | B | C | D | E | F | G | H | L | I | K | J  | N  | M  | O  |    |      |
| 6  | 6  | A  | B | C | D | E | F | G | H | L | I | K | J  | N  | M  | O  |    |      |
| 8  | 11 | 14 | A | B | C | D | E | F | G | H | J | I | K  | L  | N  | M  | O  |      |
| 8  | 9  | 10 | A | B | C | D | E | F | G | H | I | J | K  | L  | N  | M  | O  |      |
| 8  | 8  | A  | B | C | D | E | F | G | H | I | J | K | L  | N  | M  | O  |    |      |
| 10 | 10 | A  | B | C | D | E | F | G | H | I | J | K | L  | N  | M  | O  |    |      |
| 12 | 13 | 14 | A | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |      |
| 12 | 12 | A  | B | C | D | E | F | G | H | I | J | K | L  | M  | N  | O  |    |      |
| 14 | 14 | A  | B | C | D | E | F | G | H | I | J | K | L  | M  | N  | O  |    |      |
|    |    |    | A | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |      |

# Quicksort: worst-case analysis

Worst case. Number of compares is  $\sim \frac{1}{2} N^2$ .

|    |    |    | a[ ] |   |   |   |   |   |   |   |   |   |    |    |    |    |    |
|----|----|----|------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| lo | j  | hi | 0    | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|    |    |    | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
|    |    |    | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 0  | 0  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 1  | 1  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 2  | 2  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 3  | 3  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 4  | 4  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 5  | 5  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 6  | 6  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 7  | 7  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 8  | 8  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 9  | 9  | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 10 | 10 | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 11 | 11 | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 12 | 12 | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 13 | 13 | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
| 14 |    | 14 | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |
|    |    |    | A    | B | C | D | E | F | G | H | I | J | K  | L  | M  | N  | O  |

## Quicksort: average-case analysis

**Proposition.** The average number of compares  $C_N$  to quicksort an array of  $N$  distinct keys is  $\sim 2N \ln N$  (and the number of exchanges is  $\sim \frac{1}{3} N \ln N$ ).

Pf.  $C_N$  satisfies the recurrence  $C_0 = C_1 = 0$  and for  $N \geq 2$ :

$$C_N = (N+1) + \left( \frac{C_0 + C_{N-1}}{N} \right) + \left( \frac{C_1 + C_{N-2}}{N} \right) + \dots + \left( \frac{C_{N-1} + C_0}{N} \right)$$

partitioning      total prob      left      right

partitioning probability

- Multiply both sides by  $N$  and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \dots + C_{N-1})$$

- Subtract this from the same equation for  $N-1$ :

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by  $N(N+1)$ :

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

## Quicksort: average-case analysis

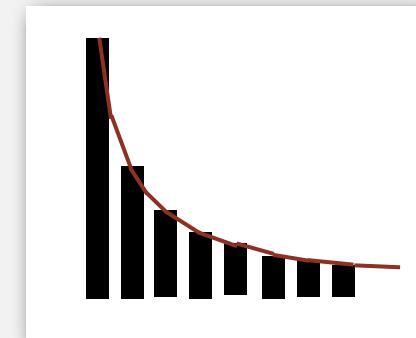
- Repeatedly apply above equation:

$$\begin{aligned}\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\ &= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \quad \leftarrow \text{substitute previous equation} \\ &= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\ &= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \dots + \frac{2}{N+1}\end{aligned}$$

← previous equation

- Approximate sum by an integral:

$$\begin{aligned}C_N &= 2(N+1) \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{N+1} \right) \\ &\sim 2(N+1) \int_3^{N+1} \frac{1}{x} dx\end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1) \ln N \approx \underline{1.39 N \lg N}$$

## Quicksort: summary of performance characteristics

**Worst case.** Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \dots + 1 \sim \frac{1}{2} N^2$ .
  - More likely that your computer is struck by lightning bolt.

한국전통서예  
한국서예

Average case. Number of compares is  $\sim 1.39 N \lg N$ .

- 39% more compares than mergesort.
  - **But** faster than mergesort in practice because of less data movement.

## Random shuffle.

- Probabilistic guarantee against worst case.
  - Basis for math model that can be validated with experiments.

**Caveat emptor.** Many textbook implementations go quadratic if array

- Is sorted or reverse sorted.
  - Has many duplicates (even if randomized!)

## Quicksort properties

Proposition. Quicksort is an **in-place** sorting algorithm.

Pf.

- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

Random Shuffle ↳  $\log n$

can guarantee logarithmic depth by recurring on smaller subarray before larger subarray

Proposition. Quicksort is **not stable**.

Pf.

↪ Stable ↳  $\log n$ 의 깊이  
같은값으로 주어 같은값을 출력.

| i | j | 0     | 1     | 2     | 3     |
|---|---|-------|-------|-------|-------|
|   |   | $B_1$ | $C_1$ | $C_2$ | $A_1$ |
| 1 | 3 | $B_1$ | $C_1$ | $C_2$ | $A_1$ |
| 1 | 3 | $B_1$ | $A_1$ | $C_2$ | $C_1$ |
| 0 | 1 | $A_1$ | $B_1$ | $C_2$ | $C_1$ |

## Quicksort: practical improvements – |

---

### Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for  $\approx 10$  items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

## Quicksort: practical improvements - 2

### Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

일반적으로,  
Median을 찾는데 서는 편리합니다.

~  $12/7 N \ln N$  compares (slightly fewer)  
~  $12/35 N \ln N$  exchanges (slightly more)

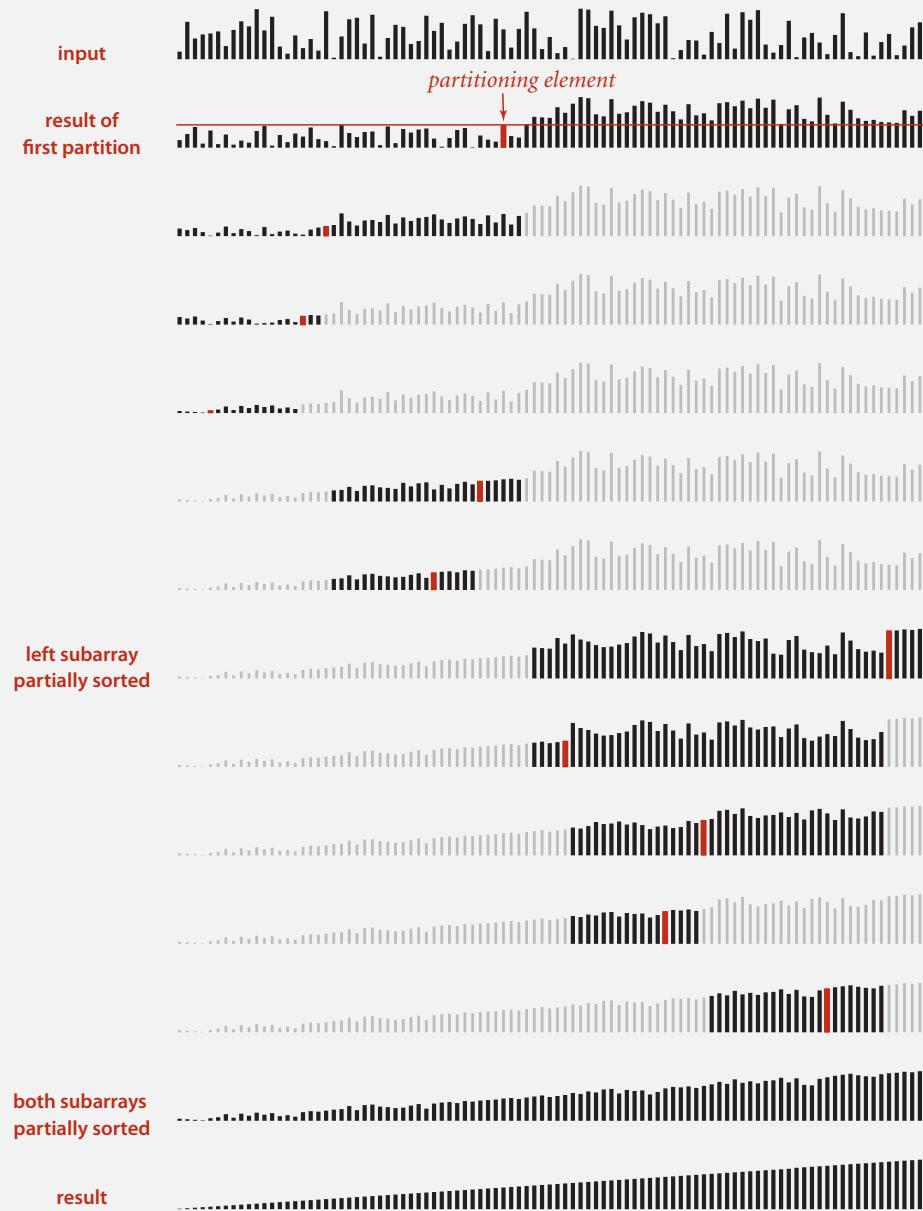
```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo) return;

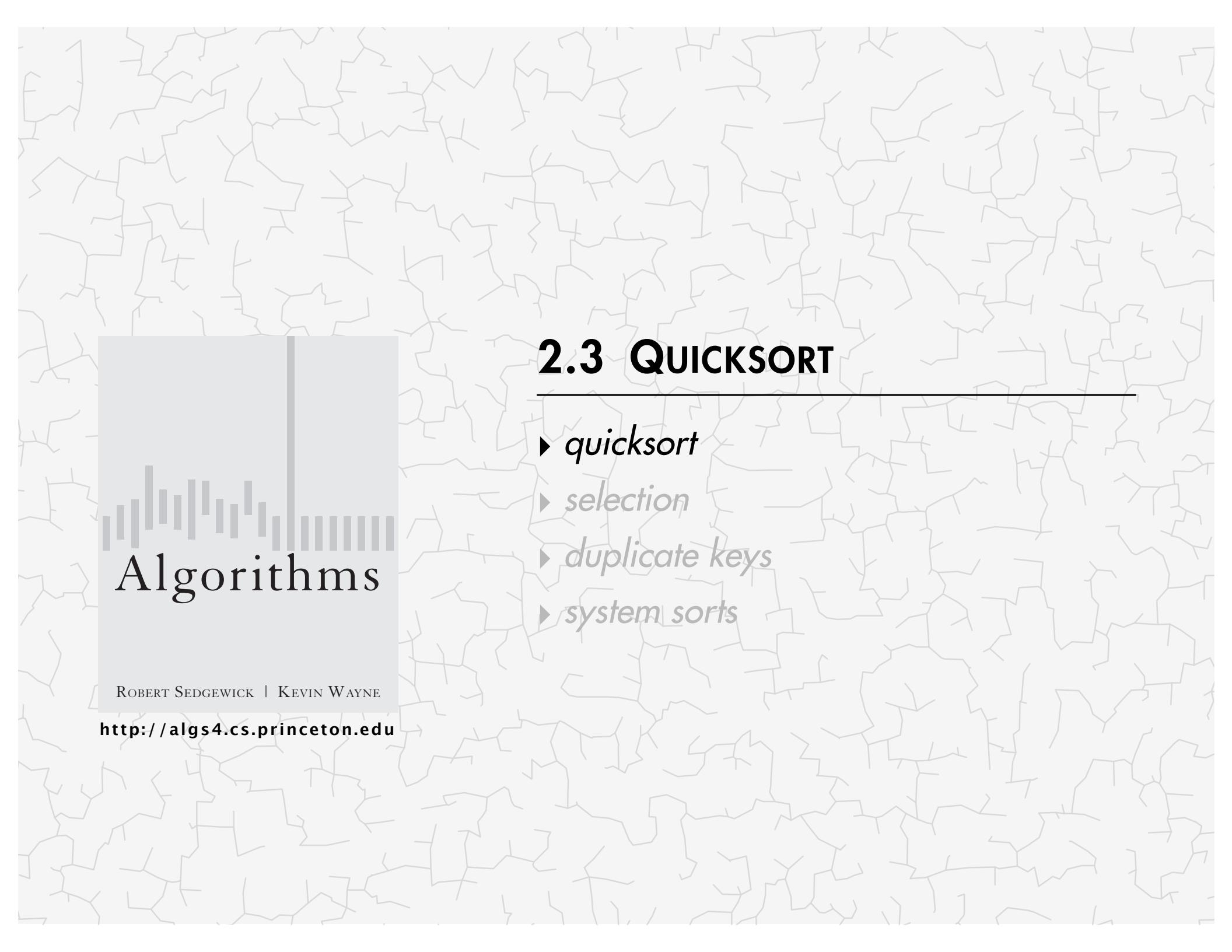
    int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
    swap(a, lo, m);

    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

# Quicksort with median-of-3 and cutoff to insertion sort: visualization

---





# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

# Selection

---

**Goal.** Given an array of  $N$  items, find a  $k^{\text{th}}$  smallest item.

**Ex.** Min ( $k = 0$ ), max ( $k = N - 1$ ), median ( $k = N/2$ ).

## Applications.

- Order statistics.
- Find the "top  $k$ ."

## Use theory as a guide.

- Easy  $N \log N$  upper bound. How? Quicksort  $\frac{k}{N}$  index  $\geq$ .
- Easy  $N$  upper bound for  $k = 1, 2, 3$ . How? Small sets  $\leq N$ .
- Easy  $N$  lower bound. Why?

## Which is true?

- $N \log N$  lower bound? ← is selection as hard as sorting?
- $N$  upper bound? ← is there a linear-time algorithm for each  $k$ ?

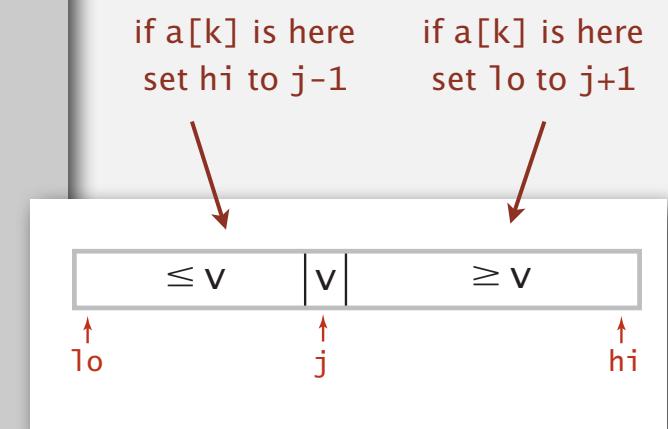
# Quick-select

Partition array so that:

- Entry  $a[j]$  is in place.
- No larger entry to the left of  $j$ .
- No smaller entry to the right of  $j$ .

Repeat in **one** subarray, depending on  $j$ ; finished when  $j$  equals  $k$ .

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
        int j = partition(a, lo, hi);
        if      (j < k) lo = j + 1;
        else if (j > k) hi = j - 1;
        else            return a[k];
    }
    return a[k];
}
```



## Quick-select: mathematical analysis

---

Proposition. Quick-select takes linear time on average.

Pf sketch.

- Intuitively, each partitioning step splits array approximately in half:  
 $N + N/2 + N/4 + \dots + 1 \sim 2N$  compares.
- Formal analysis similar to quicksort analysis yields:

$$C_N = 2N + 2k \ln(N/k) + 2(N-k) \ln(N/(N-k))$$

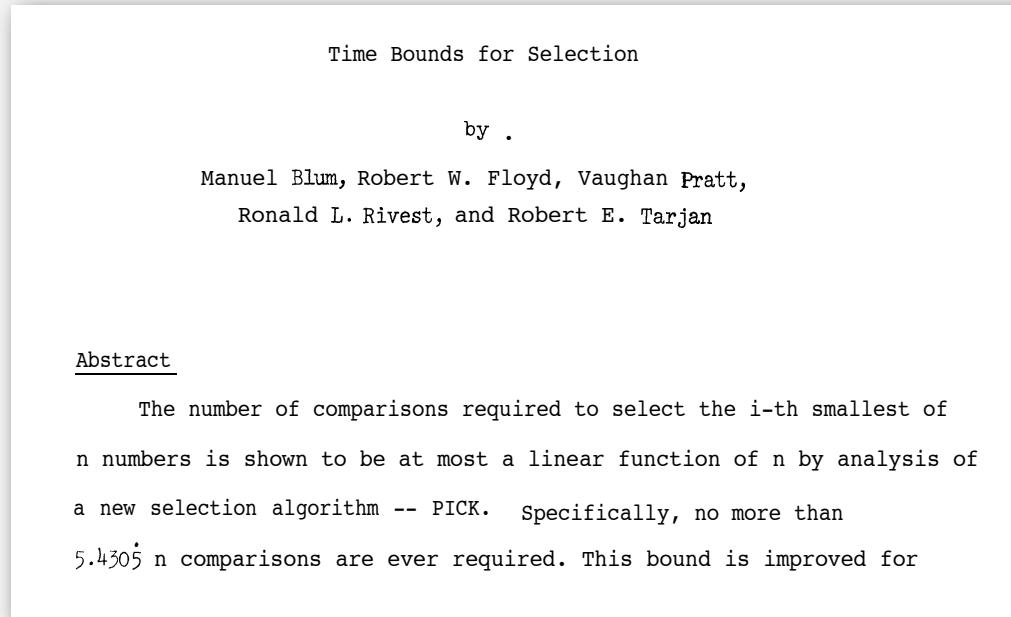
  
 $(2 + 2 \ln 2)N$  to find the median

Remark. Quick-select uses  $\sim \frac{1}{2}N^2$  compares in the worst case, but  
(as with quicksort) the random shuffle provides a probabilistic guarantee.

# Theoretical context for selection

---

**Proposition.** [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] Compare-based selection algorithm whose worst-case running time is linear.



**Remark.** But, constants are too high  $\Rightarrow$  not used in practice.

Use theory as a guide.

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select if you don't need a full sort.

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

Duplicate keys  $\rightsquigarrow$  grouping  $\nmid$  grouped sorting  $\Rightarrow$

---

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

Chicago 09:25:52  
Chicago 09:03:13  
Chicago 09:21:05  
Chicago 09:19:46  
Chicago 09:19:32  
Chicago 09:00:00  
Chicago 09:35:21  
Chicago 09:00:59  
Houston 09:01:10  
Houston 09:00:13  
Phoenix 09:37:44  
Phoenix 09:00:03  
Phoenix 09:14:25  
Seattle 09:10:25  
Seattle 09:36:14  
Seattle 09:22:43  
Seattle 09:10:11  
Seattle 09:22:54

↑  
key

# Duplicate keys

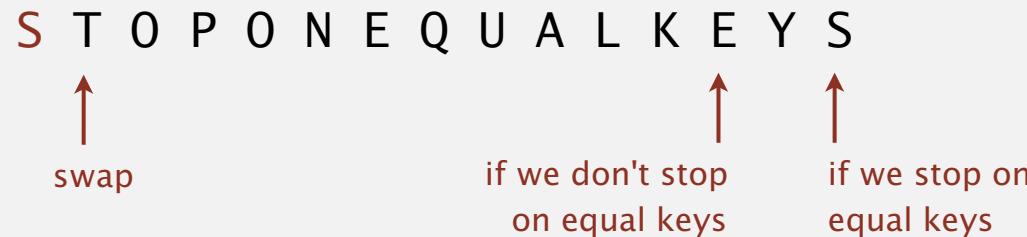
---

Mergesort with duplicate keys. Between  $\frac{1}{2}N\lg N$  and  $N\lg N$  compares.

## Quicksort with duplicate keys.

- Algorithm goes **quadratic** unless partitioning stops on equal keys!
- 1990s C user found this defect in `qsort()`.

several textbook and system  
implementation also have this defect



## Duplicate keys: the problem

---

Mistake. Put all items equal to the partitioning item on one side.

Consequence.  $\sim \frac{1}{2} N^2$  compares when all keys equal.

B A A B A B B **B** C C C      A A A A A A A A A A **A**

Recommended. Stop scans on items equal to the partitioning item.

Consequence.  $\sim N \lg N$  compares when all keys equal.

B A A B A **B** C C B C B      A A A A A **A** A A A A A A

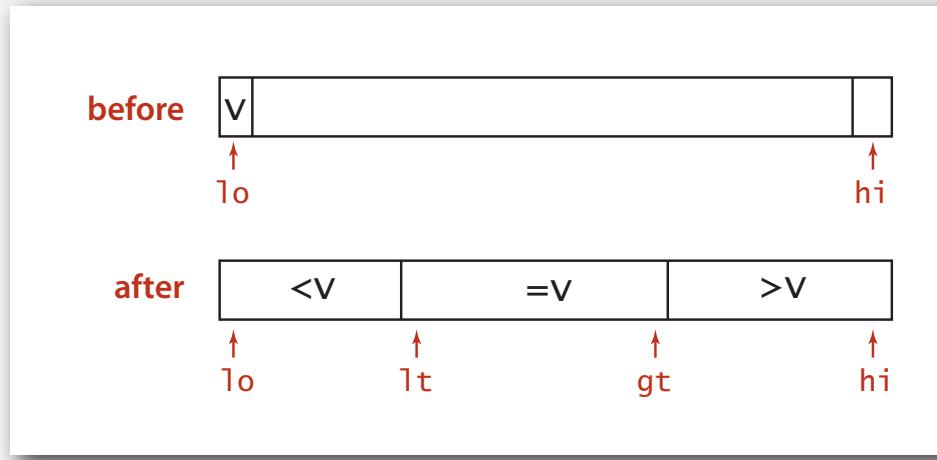
Desirable. Put all items equal to the partitioning item in place.

A A A **B** B B B C C C      **A** A A A A A A A A A A A A

## 3-way partitioning

**Goal.** Partition array into 3 parts so that:

- Entries between  $lt$  and  $gt$  equal to partition item  $v$ .
- No larger entries to left of  $lt$ .
- No smaller entries to right of  $gt$ .



**Dutch national flag problem.** [Edsger Dijkstra]

- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library `qsort()`.
- Now incorporated into `qsort()` and Java system `sort`.

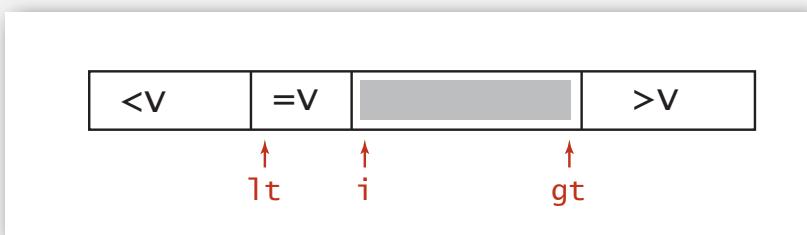
## Dijkstra 3-way partitioning demo

→ 중복된 값 | 같은 값 | 다른 값 | 초기값

- Let  $v$  be partitioning item  $a[lo]$ .
- Scan  $i$  from left to right.
  - $(a[i] < v)$ : exchange  $a[lt]$  with  $a[i]$ ; increment both  $lt$  and  $i$
  - $(a[i] > v)$ : exchange  $a[gt]$  with  $a[i]$ ; decrement  $gt$
  - $(a[i] == v)$ : increment  $i$

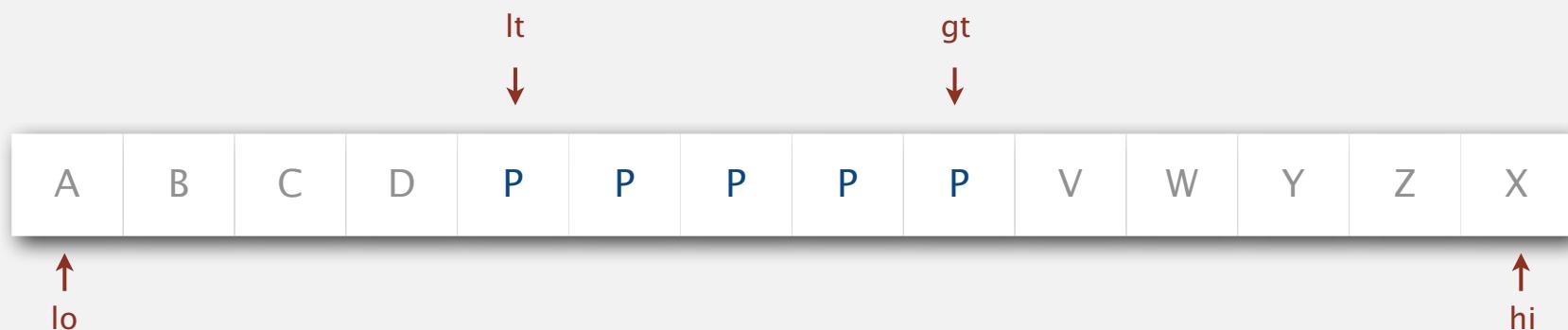


invariant

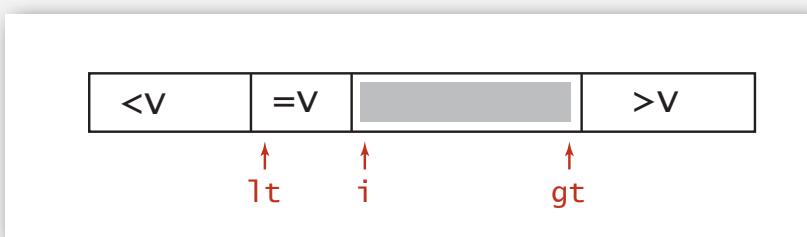


# Dijkstra 3-way partitioning demo

- Let  $v$  be partitioning item  $a[lo]$ .
- Scan  $i$  from left to right.
  - $(a[i] < v)$ : exchange  $a[lt]$  with  $a[i]$ ; increment both  $lt$  and  $i$
  - $(a[i] > v)$ : exchange  $a[gt]$  with  $a[i]$ ; decrement  $gt$
  - $(a[i] == v)$ : increment  $i$



invariant



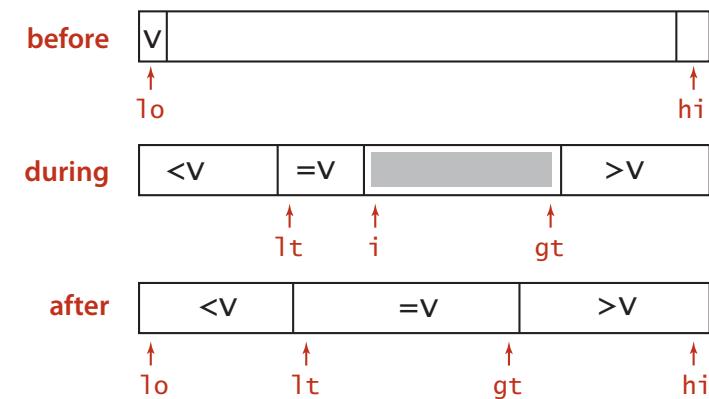
# Dijkstra's 3-way partitioning: trace

| lt | i | gt | a[]          |              |              |              |              |              |              |              |   |              |
|----|---|----|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|---|--------------|
|    |   |    | 0            | 1            | 2            | 3            | 4            | 5            | 6            | 7            | 8 | 9            |
| 0  | 0 | 11 | R            | B            | W            | W            | R            | W            | B            | R            | R | W            |
| 0  | 1 | 11 | R            | <del>B</del> | W            | W            | R            | W            | B            | R            | R | W            |
| 1  | 2 | 11 | <del>B</del> | R            | <del>W</del> | W            | R            | W            | B            | R            | R | W            |
| 1  | 2 | 10 | B            | R            | <del>R</del> | W            | R            | W            | B            | R            | R | W            |
| 1  | 3 | 10 | B            | R            | R            | <del>W</del> | R            | W            | B            | R            | R | W            |
| 1  | 3 | 9  | B            | R            | <del>R</del> | <del>B</del> | R            | W            | B            | R            | R | W            |
| 2  | 4 | 9  | B            | <del>B</del> | R            | R            | <del>R</del> | W            | B            | R            | R | W            |
| 2  | 5 | 9  | B            | B            | R            | R            | R            | <del>W</del> | <del>B</del> | R            | R | W            |
| 2  | 5 | 8  | B            | B            | R            | R            | R            | <del>W</del> | <del>B</del> | R            | R | W            |
| 2  | 5 | 7  | B            | B            | R            | R            | R            | <del>R</del> | <del>B</del> | R            | R | W            |
| 2  | 6 | 7  | B            | B            | R            | R            | R            | R            | <del>B</del> | R            | R | W            |
| 3  | 7 | 7  | B            | B            | B            | <del>R</del> | R            | R            | R            | <del>R</del> | R | W            |
| 3  | 8 | 7  | B            | B            | B            | <del>R</del> | <del>R</del> | R            | R            | R            | R | <del>W</del> |
| 3  | 8 | 7  | B            | B            | B            | <del>R</del> | <del>R</del> | <del>R</del> | R            | R            | R | W            |

3-way partitioning trace (array contents after each loop iteration)

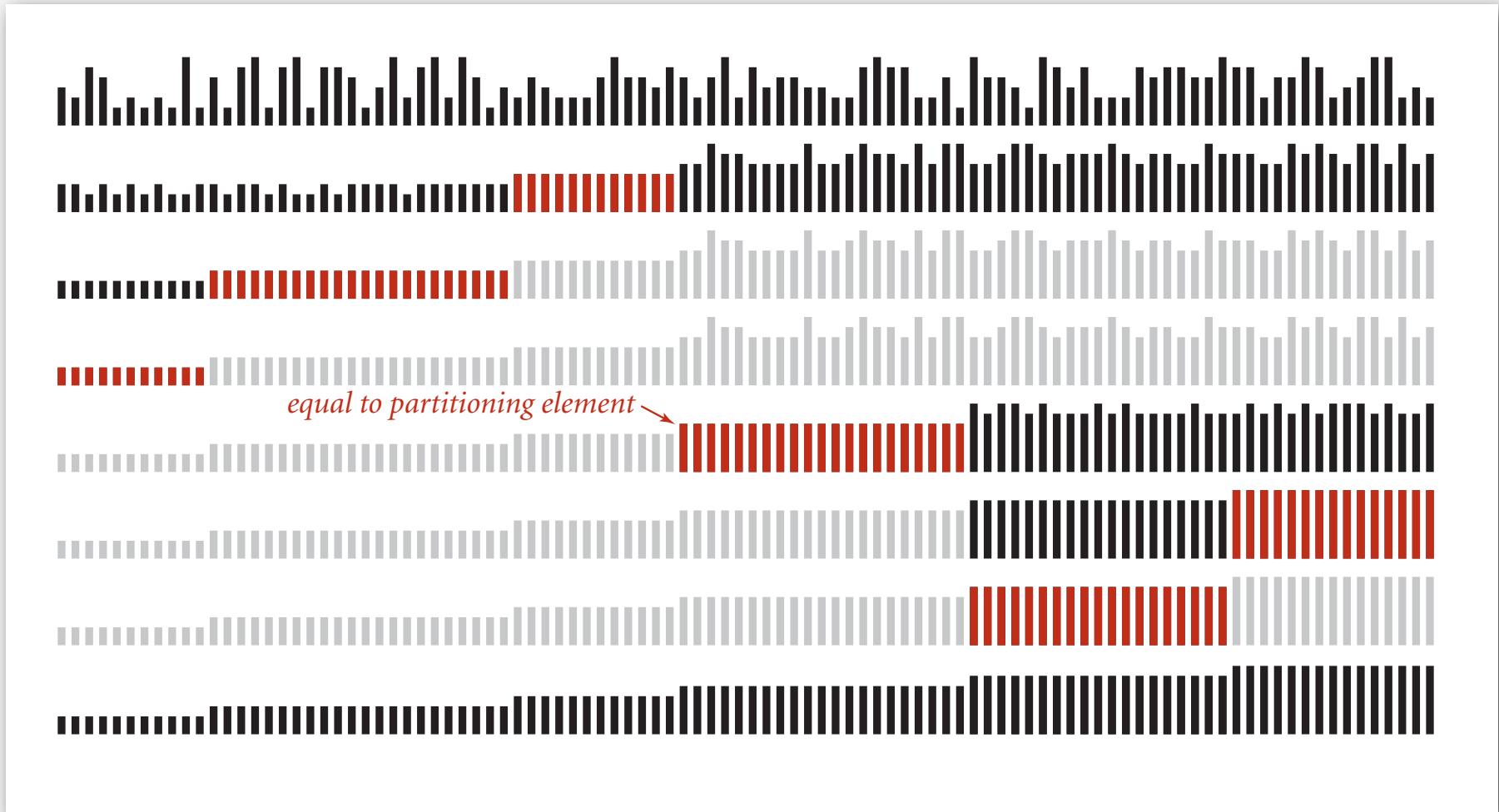
## 3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo) return;
    int lt = lo, gt = hi;
    Comparable v = a[lo];
    int i = lo;
    while (i <= gt)
    {
        int cmp = a[i].compareTo(v);
        if      (cmp < 0) exch(a, lt++, i++);
        else if (cmp > 0) exch(a, i, gt--);
        else                i++;
    }
    sort(a, lo, lt - 1);
    sort(a, gt + 1, hi);
}
```



## 3-way quicksort: visual trace

---



## Duplicate keys: lower bound

Sorting lower bound. If there are  $n$  distinct keys and the  $i^{\text{th}}$  one occurs  $x_i$  times, any compare-based sorting algorithm must use at least

$$\lg \left( \frac{N!}{x_1! x_2! \cdots x_n!} \right) \sim - \sum_{i=1}^n x_i \lg \frac{x_i}{N}$$

← *N lg N when all distinct;  
linear when only a constant number of distinct keys*

compares in the worst case.

Proposition. [Sedgewick-Bentley, 1997]

proportional to lower bound

Quicksort with 3-way partitioning is **entropy-optimal**.

Pf. [beyond scope of course]

Bottom line. Randomized quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

# Sorting applications

---

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library. obvious applications
- Display Google PageRank results.
- List RSS feed in reverse chronological order.
  
- Find the median.
- Identify statistical outliers. problems become easy once items are in sorted order
- Binary search in a database.
- Find duplicates in a mailing list.
  
- Data compression.
- Computer graphics. non-obvious applications
- Computational biology.
- Load balancing on a parallel computer.
  
- . . .

## Java system sorts

# Arrays.sort().

- Has different method for each primitive type.
  - Has a method for data types that implement Comparable.
  - Has a method that uses a Comparator.
  - Uses tuned quicksort for primitive types; tuned mergesort for objects.

```
import java.util.Arrays;  
  
public class StringSort  
{  
    public static void main(String[] args)  
    {  
        String[] a = StdIn.readStrings();  
        Arrays.sort(a);  
        for (int i = 0; i < N; i++)  
            StdOut.println(a[i]);  
    }  
}
```

**Q.** Why use different algorithms for primitive and reference types?

→ Maggescort + Stable 히브  $n_{f,g}$  을 포함

## War story (C qsort function)

---

AT&T Bell Labs (1991). Allan Wilks and Rick Becker discovered that a `qsort()` call that should have taken seconds was taking minutes.



At the time, almost all `qsort()` implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.



# Engineering a system sort

---

Basic algorithm = quicksort.

- Cutoff to insertion sort for small subarrays.
- Partitioning scheme: Bentley-McIlroy 3-way partitioning.
- Partitioning item.
  - small arrays: middle entry
  - medium arrays: median of 3
  - large arrays: Tukey's ninther [next slide]

Engineering a Sort Function

JON L. BENTLEY  
M. DOUGLAS McILROY  
*AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, U.S.A.*

**SUMMARY**

We recount the history of a new `qsort` function for a C library. Our function is clearer, faster and more robust than existing sorts. It chooses partitioning elements by a new sampling scheme; it partitions by a novel solution to Dijkstra's Dutch National Flag problem; and it swaps efficiently. Its behavior was assessed with timing and debugging testbeds, and with a program to certify performance. The design techniques apply in domains beyond sorting.

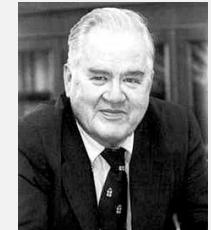
Now widely used. C, C++, Java 6, ....

# Tukey's ninther

---

Tukey's ninther. Median of the median of 3 samples, each of 3 entries.

- Approximates the median of 9.
- Uses at most 12 compares.



Q. Why use Tukey's ninther?

A. Better partitioning than random shuffle and less costly.

# Achilles heel in Bentley-McIlroy implementation (Java system sort)

Q. Based on all this research, Java's system sort is solid, **right?**

A. No: a killer input.

- Overflows function call stack in Java and crashes program.
- Would take quadratic time if it didn't crash first.

```
% more 250000.txt
```

```
0
```

```
218750
```

```
222662
```

```
11
```

```
166672
```

```
247070
```

```
83339
```

```
...
```



250,000 integers  
between 0 and 250,000

```
% java IntegerSort 250000 < 250000.txt
```

```
Exception in thread "main"
```

```
java.lang.StackOverflowError
```

```
at java.util.Arrays.sort1(Arrays.java:562)
```

```
at java.util.Arrays.sort1(Arrays.java:606)
```

```
at java.util.Arrays.sort1(Arrays.java:608)
```

```
at java.util.Arrays.sort1(Arrays.java:608)
```

```
at java.util.Arrays.sort1(Arrays.java:608)
```

```
...
```

Java's sorting library crashes, even if  
you give it as much stack space as Windows allows

# System sort: Which algorithm to use?

---

Many sorting algorithms to choose from:

## Internal sorts.

- Insertion sort, selection sort, bubblesort, shaker sort.
- Quicksort, mergesort, heapsort, samplesort, shellsort.
- Solitaire sort, red-black sort, splaysort, **Yaroslavskiy sort**, psort, ...

External sorts. Poly-phase mergesort, cascade-merge, oscillating sort.

String/radix sorts. Distribution, MSD, LSD, 3-way string quicksort.

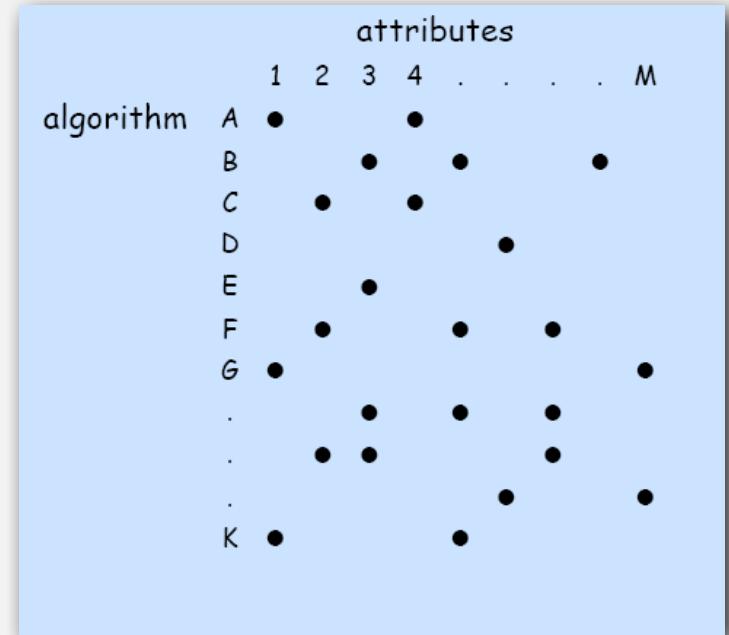
## Parallel sorts.

- Bitonic sort, Batcher even-odd sort.
- Smooth sort, cube sort, column sort.
- GPUsort.

# System sort: Which algorithm to use?

Applications have diverse attributes.

- Stable?
- Parallel?
- Deterministic?
- Keys all distinct?
- Multiple key types?
- Linked list or arrays?
- Large or small items?
- Is your array randomly ordered?
- Need guaranteed performance?



many more combinations of attributes than algorithms

Elementary sort may be method of choice for some combination.

Cannot cover **all** combinations of attributes.

Q. Is the system sort good enough?

A. Usually.

# Sorting summary

---

|             | inplace? | stable? | worst     | average    | best      | remarks   |
|-------------|----------|---------|-----------|------------|-----------|---|
| selection   | ✓        |         | $N^2 / 2$ | $N^2 / 2$  | $N^2 / 2$ | $N$ exchanges   |
| insertion   | ✓        | ✓       | $N^2 / 2$ | $N^2 / 4$  | $N$       | use for small $N$ or partially ordered                    |
| shell       | ✓        |         | ?         | ?          | $N$       | tight code, subquadratic                                  |
| merge       |          | ✓       | $N \lg N$ | $N \lg N$  | $N \lg N$ | $N \log N$ guarantee, stable                              |
| quick       | ✓        |         | $N^2 / 2$ | $2N \ln N$ | $N \lg N$ | $N \log N$ probabilistic guarantee<br>fastest in practice |
| 3-way quick | ✓        |         | $N^2 / 2$ | $2N \ln N$ | $N$       | improves quicksort in presence<br>of duplicate keys       |
| ???         | ✓        | ✓       | $N \lg N$ | $N \lg N$  | $N$       | holy sorting grail  |

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

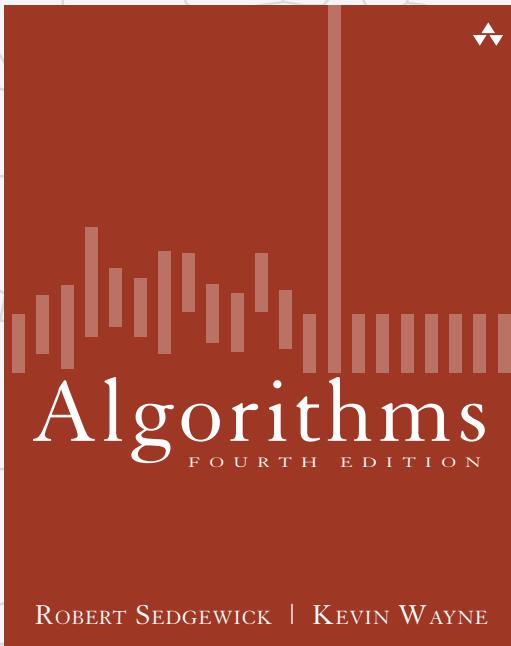
## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE



<http://algs4.cs.princeton.edu>

## 2.3 QUICKSORT

---

- ▶ *quicksort*
- ▶ *selection*
- ▶ *duplicate keys*
- ▶ *system sorts*