AUCTIONS
AN INTRODUCTION*

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Abstract

This is a fairly detailed review of auction theory. It begins with basic results on private value auctions, with particular emphasis on the generality and limitations of the revenue equivalence of a large class of distinct auction rules. The basic framework is then gradually modified to admit, for example, risk aversion, a minimum price, entry fees and other fixed costs of bidding, multi–unit auctions, and bidder collusion. There follows an introduction to the theory of optimal auctions, and to common value auctions and the associated winner's curse problem. It closes with a sample of applications of auction theory in economics, such as the regulation of natural monopolies, the theory of oligopoly, and the government securities market.

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"Men prize the thing ungained more than it is."
Shakespeare *(Troilus and Cressida)*

1 Introduction

Suppose you inherit a unique and valuable good — say a painting by the late van Gogh or a copy of the 1776 edition of Adam Smith's *Wealth of Nations*. For some reason you are in desperate need for money, and decide to sell. You have a clear idea of your reservation price. Of course, you hope to earn more than that. You wonder: isn't there some way to reach the highest possible price? How should you go about it?

If you knew the potential buyers and their valuations of the object for sale, your pricing problem would have a simple solution. You would only need to call a nonnegotiable price equal to the highest valuation, and wait for the right customer to come and claim the item. It's as simple as that.

Information problem The trouble is that you, the seller, usually have only incomplete information about buyers' valuations. Therefore, you have to figure out a pricing scheme that performs well even under incomplete information. Your problem begins to be interesting.

Basic assumptions Suppose there are $N > 1$ potential buyers, each of whom knows the object for sale well enough to decide how much he is willing to pay. Denote buyers' valuations by $V := (V_1, \ldots, V_N)$. Since you, the seller, do not know V, you must think of it as a random variable, distributed by some joint probability distribution function (CDF) $\mathcal{F} : [\underline{v}_1, \bar{v}_1] \times \cdots \times [\underline{v}_N, \bar{v}_N] \to [0, 1]$, $\mathcal{F}(v_1, \ldots, v_N) := \Pr(V_1 \leq v_1, \ldots, V_N \leq v_N)$. Similarly, potential buyers know their own valuation but not that of others. Therefore, buyers also view valuations as a random variable, except their own.

Cournot monopoly approach Of course, you could stick to the "take–it–or–leave–it" pricing rule, even as you are subject to incomplete information about buyers' valuations. You would then call a nonnegotiable price p at or above your *reservation price* r, and hope that some customer has the right valuation and is willing to buy. Your pricing problem would then be reduced to a variation of the standard Cournot monopoly problem. The only unusual part would be that the trade–off between price and quantity is replaced by one between price and the probability of sale.

Alluding to the usual characterization of Cournot monopoly, you can even draw a "demand curve", with the nonnegotiable price p on the "price axis" and the probability that at least one bidder's valuation exceeds the listed price p

$$\pi(p) = 1 - G(p), \quad G(p) := \mathcal{F}(p, \ldots, p), \tag{1}$$

on the "quantity axis"(as in *Figure 1* below). And you would then pick that price p that maximizes the expected value of your gain from trade $\pi(p)(p-r)$.

Equivalently, you can state the decision problem in quantity coordinates, as is customary in the analysis of Cournot monopoly, where "quantity" is here represented by the probability of sale $q := 1 - G(p)$

$$\max_q [P(q) - r]q, \quad P(q) := G^{-1}(1-q). \tag{2}$$

Computing the first–order condition, the optimal price p^* can then be characterized in the familiar form as a relationship between "marginal revenue" (MR) and "marginal cost" (MC):[1]

$$\text{MR} := p - \frac{1 - G(p)}{G'(p)} = r =: \text{MC}. \tag{3}$$

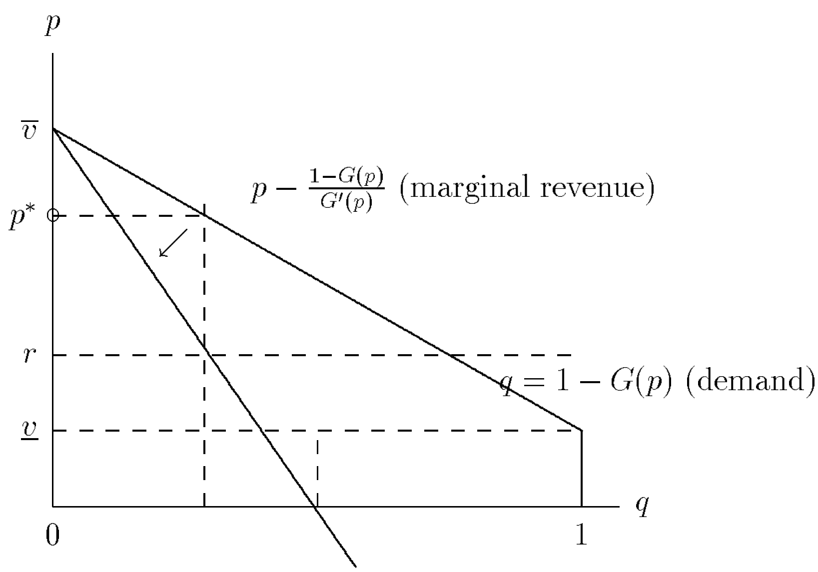

Figure 1: Optimal take–it–or–leave–it price

[1]As always, this condition applies only if the maximization problem is well–behaved. Well–behavedness requires that revenue is continuous and marginal revenue strict monotone increasing in p.

Example 1 *Suppose valuations V_i are independent and uniformly distributed over the $[0,1]$ interval, and the reservation price is $r = 0$. Then, for all $p \in [0,1]$, $G(p) = F(p)^N = p^N$. Hence, the optimal take–it–or–leave–it price is*

$$p^*(N) = \sqrt[N]{\frac{1}{N+1}}. \tag{4}$$

The resulting expected price paid (price asked times probability of success) is

$$\bar{p}(N) := p^*(N)(1 - G(p^*(N))) = p^*(N)\frac{N}{N+1}. \tag{5}$$

Both $p^*(N)$ and $\bar{p}(N)$ are strict monotone increasing in N, starting from $p^*(1) = 1/2$, $\bar{p}(1) = 1/4$, and approaching 1 as $N \to \infty$.[2]

Take–it–or–leave–it pricing is one way to go. But generally you can do better by setting up an auction. This brings us to the analysis of auctions, and the design of optimal auction rules. Interestingly, the Cournot monopoly price p^* will continue to play a role as optimal minimum price in the optimal auction setting, as you will learn in the section on optimal auctions.

Auctions – what, where, and why An auction is a bidding mechanism, described by a set of auction rules that specify how the winner is determined and how much he has to pay. In addition, auction rules may restrict participation and feasible bids and impose certain rules of behavior.

Auctions are widely used in many transactions — not just in the sale of art and wine. Every week, the Treasury auctions–off billions of dollars of bills, notes and Treasury bonds. Governments and private corporations solicit delivery–price offers of products ranging from office supplies to tires or construction jobs. The Department of the Interior sells the rights to drill oil and other natural resources on federally owned properties. And private firms sell products ranging from fresh flowers, fish and tobacco to diamonds and real estate. Altogether, auctions account for an enormous volume of transactions.

Essentially, auctions are used for three reasons:

- speed of sale,
- to reveal information about buyers' valuations,
- to prevent dishonest dealing between the seller's agent and the buyer.

[2] For a simple proof of monotonicity, work with $\ln \bar{p}$, which is obviously a monotone transformation of $\bar{p}$.

The latter agency role of auctions is particularly important if government agencies are involved in buying or selling. As a particularly extreme example just think of the current privatization programs in Eastern Europe. Clearly, if the agencies involved, such as the *Treuhandanstalt* that is in charge of privatizing the state run corporations of former communist East Germany, were free to negotiate the terms of sale, the lucky winner probably would be the one who made the largest bribe or political contribution. However, if assets are put up for auction, cheating the taxpayer is much more difficult and costly and hence less likely to succeed.

Popular auctions There are many different auction rules. Actually, the word itself is something of a misnomer. *Auctio* means increase, but not all auctions involve calling out higher and higher bids.

One distinguishes between *oral* and *written* auctions. In oral auctions, bidders hear each other's bids, and can make counteroffers; each bidder knows his rivals. In a written or closed–seal bid, bidders submit their bids simultaneously without revealing them to others; often bidders do not even know how many rival bidders participate.

The best known and most frequently used auction is the ascending price or English auction, followed by the first–price closed–seal bid or Dutch auction, and the second–price closed–seal bid (also known as Vickrey auction).[3]

Oral	seal–Bid
ascending price (English)	second-price
descending price (Dutch)	first-price

Table 1: The most popular auctions

In the ascending price or *English* auction, the auctioneer seeks increasing bids until all but the highest bidder(s) are eliminated. If the last bid is at or above the reserve price, the item is awarded or knocked down to the remaining bidder(s). If a tie bid occurs, the item is awarded for example by a chance rule. In one variation, used in Japan, the price is posted on a screen and raised continuously. Any bidder who wants to be active at the current price pushes a button. Once the button is released, the bidder has withdrawn altogether.

[3]Many variations of these basic auctions are reviewed in Cassady, R. [1967]. *Auctions and Auctioneering*, University of California Press.

Instead of letting the price rise, the descending price or *Dutch* auction follows a descending pattern. The auctioneer begins by asking a certain price, and gradually lowers it until some bidder(s) shout "Mine" to claim the item. Frequently, the auctioneer uses a mechanical device called the "Dutch clock". This clock is started, and the asking price ticks down until someone calls out. The clock then stops and the buyer pays the indicated price. Again, provisions are made to deal with tie bids.[4]

Finally, in a written auction bidders are invited to submit a closed–seal bid, with the understanding that the item is awarded to the highest bidder. Under the first–price rule the winner actually pays as much as his own bid, whereas under the second–price rule the price is equal to the second highest bid.

Of course, this terminology is not always used consistently in the academic, and in the financial literature. For example, in the financial community a multiple–unit, single–price auction is termed a Dutch auction, and a multiple–unit closed–seal bid auction is termed an English auction (except by the English, who call it an American auction). Whereas in the academic literature, the labels English and Dutch would be exactly reversed. In order to avoid confusion, keep in mind that here we adhere to the academic usage of these terms, and use English as equivalent to ascending and Dutch as equivalent to descending price.

Early history of auctions Probably the earliest report on auctions is found in Herodotus in his account of the bidding for men and wives in Babylon around 500 B.C. These auctions were unique, since bidding sometimes started at a negative price.[5] In ancient Rome, auctions were used in commercial trade, to liquidate property, and to sell plundered war booty. A most notable auction was held in 193 A.D. when the Praetorian Guard put the whole empire up for auction. After killing the previous emperor, the guards announced that they would appoint the highest bidder as the next emperor. Didius Julianus outbid his competitors, but after two months was beheaded by Septimius Severus who seized power — a terminal case of the winner's curse?

[4] An interesting Dutch auction in disguise is "time discounting". In many cities in the U.S. discounters use this method to sell cloth. Each item is sold at the price on the tag minus a discount that depends on how many weeks the item was on the shelf. As time passes, the final price goes down at the rate of say 10 % per week, until the listed bottom price is reached.

[5] See Shubik, M. [1983]. "Auctions, bidding, and markets: an historical sketch", in: Stark, R. (ed.). *Auctions, Bidding, and Contracting: Uses and Theory.* New York University Press, p. 39.

2 Private value auctions

We begin with the most thoroughly researched auction model, the *symmetric independent private values (SIPV)* model with risk neutral agents. In that model:

- a single indivisible object is put up for sale to one of several bidders *(unit auction)*;
- both the seller and all bidders are *risk neutral*;
- each bidder knows his valuation — no one else does (*private values*);
- the unknown valuations are independent random variables, drawn from some continuous probability distribution (*continuity/independence*);
- all bidders are indistinguishable; therefore, the probability distribution of each bidder's valuation is the same *(symmetry)*;
- the seller's reservation price is normalized to $r = 0$.

We will model bidders' behavior as a non–cooperative game under incomplete information. One of our goals will be to rank the four common auctions. To what extent does institutional detail matter? Can the seller get a higher average price by an oral or a written auction? Is competition between bidders fiercer if the winner pays a price equal to the second highest rather than the highest bid? And which auction, if any, is strategically easier to handle?

2.1 Some basic results on Dutch and English auctions

The comparison between the four standard auction rules is considerably facilitated by the fact that the Dutch auction is equivalent to the first–price and the English to the second–price closed–seal bid. Therefore, the comparison of the four standard auctions can be reduced to comparing the Dutch and the English auction. Later you will learn the far more surprising result that even these two auctions are payoff equivalent.

The strategic equivalency between the Dutch auction and the first–price seal bid is immediately obvious. In either case, a bidder has to decide how much he should bid or at what price he should shout "Mine" to claim the item. Therefore, the strategy space is the same, and so are payoff functions, and hence equilibrium outcomes.

Similarly, the English auction and the second–price closed–seal bid are equivalent, but for different reasons, and only as long as we stick to the private values framework. Unlike in a seal bid, in an English auction bidders can respond to rivals' bids. Therefore, the two auction games are not strategically equivalent. However, in both cases bidders have the obvious dominant strategy to bid an amount, or set a limit, equal to the own true valuation. Therefore, the equilibrium outcomes are the same, as long as bidders' valuations are not affected by observing rivals' bidding behavior, which always holds true in the private values framework.[6]

The essential property of the English auction and the second–price closesd–seal bid is that the price paid by the winner is exclusively determined by rivals' bids. Bidders are thus put in the position of price takers who should accept any price up to their own valuation. The "truth–revealing" strategy $b(v) = v$ is a dominant strategy. The immediate implication is that both auctions forms have the same equilibrium outcome.

2.2 Revenue equivalence theorem

Using elementary reasoning we have already established the payoff equivalence of Dutch and first–price and of English and second–price auctions. Remarkably, payoff equivalence is a much more general feature of auction games. Indeed, *all* auctions that award the item to the highest bidder are payoff equivalent. The four standard auctions are a case in point. We now turn to proof of this surprising result, and then explore modifications of the *SIPV* framework.

We mention that the older literature conjectured that Dutch auctions lead to a higher average price than English auctions, because, as Cassady put it, in the Dutch auction "... each potential buyer tends to bid his highest demand price, whereas a bidder in the English system need advance a rival's offer by only one increment."[7] Both the reasoning and the conclusion are wrong.

Since bidders' valuations are independent random variables, their joint distribution function $\mathcal{F}$ is just the product of their separate distribution functions $F : [0, \bar{v}] \to [0, 1]$, that are in turn identical, due to the assumed symmetry. Therefore,

$$\mathcal{F}(v_1, v_2, \ldots, v_n) = F(v_1)F(v_2)\cdots F(v_n). \tag{6}$$

A bidder's strategy is a mapping from valuations to bids, $b : [0, \bar{v}] \to \mathbb{R}_+$. A vector of strategies is an equilibrium, if it has the mutual best response property. Since all bidders are alike, whatever is an optimal bidding strategy for one bidder should also be an optimal strategy for any other bidder. This suggests that the equilibrium should be *symmetric*.

[6] Obviously, if rivals' bids signal something about the underlying common value of the item, each bidders' estimated value of the item is updated in the course of an English auction. No such updating can occur under a seal bid, where bidders place their bids before observing rivals' behavior. Therefore, if the item has a *common* value component, the English auction and the second–price closed–seal bid do not have the same equilibrium outcome.

[7] Cassady, R. [1967]. *Auctions and Auctioneering*, University of California Press, p. 260.

The key insight that leads to nice and simple proofs is that, given rivals' strategies, each bidder's decision problem can be viewed as one of choosing through his bid $b(v)$ a *probability of winning*, ρ, and an *expected payment* $\mathcal{E}$.[8] This way, the choice of strategies is looked at as a *self–selection* of $(\rho, \mathcal{E})$. Due to the independence assumption, both ρ and $\mathcal{E}$ depend only on the bid $b(v)$ but not *directly* on the bidder's own valuation v.

In all results reviewed in this section we characterize symmetric equilibria. We assume the SIPV framework, and consider the set of auctions that adhere to the principle of selecting the highest bidder as winner, and leave the number of active bidders the same.[9] No other assumptions are made. Therefore, our analysis applies to a myriad of auction rules, not just the four common auctions.

The first useful result is:

Lemma 1 (Monotonicity) *The equilibrium bid strategy b^* is monotone increasing in v.*

Proof Given rivals' strategies, the expected utility of a bidder with valuation v and bidding strategy b is: $U(b(v), v) := \rho(b(v))v - \mathcal{E}(b(v))$. Define the *indirect utility* function $U^* : [0, \bar{v}] \to \mathbb{R}$

$$U^*(v) := \rho(b^*(v))v - \mathcal{E}(b^*(v)). \tag{7}$$

Since U^* is a maximum value function, the Envelope Theorem applies, and one has

$$U^{*'}(v) = \rho(b^*(v)). \tag{8}$$

Obviously, U^* is convex in v.[10] Therefore,

$$0 \leq U^{*''}(v) = \rho'(b^*(v))b^{*'}(v), \tag{9}$$

and hence $b^{*'}(v) \geq 0$, by $\rho'(b^*(v)) > 0$. Invoking that the optimal bid is never the same for different valuations, the monotonicity is even strict. □

An immediate implication of strict monotonicity, combined with symmetry, is:

[8] Notice, in this general account it is not assumed that it is only the winner who has to make a payment.

[9] The auction rule may affect the number of active bidders if payment is not restricted to the winner. For example, if the seller requires an entry fee, he will loose all those whose valuation is below that fee. This point is taken up on page 19.

[10] The proof is similar to the proof of concavity of the expenditure function in the price vector, that you have learned in your micro course. Make sure that you also understand the intuition of this result. Notice, that the bidder could always leave the bid unchanged when v changes, as a fall back. Therefore, U^* must be at or above its gradient hyperplane, which proves convexity.

Lemma 2 (Pareto optimality) *The bidder with the highest valuation wins the auction.*

This brings us to the amazing revenue equivalence result:

Proposition 1 (Revenue equivalence) *Assume the SIPV framework, and $U^*(0) = 0$. Then, all auctions that select the highest bidder as winner and that have the same number of bidders give rise to the same expected payoffs of the seller and bidders.*

Proof Integrating (8) one obtains, using the fact that $U^*(0) = 0$

$$U^*(v) = \int_0^v \rho(b^*(\tilde{v}))d\tilde{v}. \tag{10}$$

Since each of the admitted auctions is Pareto efficient, $\rho(b^*(v))$ must be equal to the probability that all other bidders' valuations are below v. Therefore, by (10), each bidder's payoff is the same under all admitted auctions. Moreover, the total surplus generated by trade must also be the same, due to the Pareto optimality of the outcome.[11] Therefore, the seller's payoff must also be the same. □

As you can see from the ingredients to the proof of this result, revenue equivalence applies not only to the four standard auctions, but to all auctions that select the highest bidder as winner. In particular, revenue equivalence also applies to third– and higher–price auctions, that are discussed below. However, the proof of revenue equivalence assumes the *SIPV* framework. It remains to be seen whether all of the *SIPV* ingredients are crucial.

Remark 1 *Assuming the SIPV framework, revenue equivalence holds for all auctions that award the item to the highest bidder. This prerequisite seems to apply to most auction rules. However, it is already violated when the seller sets a minimum price above his reservation value.*

2.3 The case of uniformly distributed valuations

Before we go into the general account of equilibrium bids, and then proceed with various modifications of the *SIPV* model, take a look at the following extensive example in which it is particularly easy to compute equilibrium strategies and equilibrium outcomes.

[11] The overall surplus or gain from trade is equal to $\max\{E[V_{(N)}] - r, 0\}$. The seller's payoff is the difference between total surplus and bidders' payoffs.

Example 2 *Suppose all bidders' valuations are uniformly distributed on the* $[0,1]$ *interval, as in example 1. Then, the unique equilibrium bidding strategies are*

$$\begin{aligned} b^*(v_i) &= v_i \quad \textit{(English auction)} & (11) \\ b^*(v_i) &= (1-\frac{1}{N})v_i \quad \textit{(Dutch auction)}. & (12) \end{aligned}$$

In both auctions, the expected price is

$$\bar{p} = \frac{N-1}{N+1}. \tag{13}$$

Therefore, more bidders mean fiercer competition between buyers.

We now explain in detail how these results come about, using the symmetry of equilibrium and the monotonicity of equilibrium strategies.

Dutch auction If bidder i bids the amount $b = b^*(v)$, he wins with probability[12]

$$\begin{aligned} \rho(b) &= \Pr\{b^*(V_{(N-1)}) < b\} \\ &= \Pr\{V_{(N-1)} < \sigma(b)\} \\ &= F(\sigma(b))^{N-1} & (14) \end{aligned}$$

where $\sigma(b)$ is the inverse of $b^*(v_i)$, which indicates the valuation that leads to bidding the amount b if strategy b^* is played, and $V_{(i)}$ is the i–th order statistic of the sample of N random valuations.

In equilibrium the bid b must be a best–response to rivals' bids. Therefore, b must be a maximizer of the expected gain $\rho(b)[v_i - b]$, leading to the first–order condition

$$\rho'(b)(v-b) - \rho(b) = 0. \tag{15}$$

Using the fact that $v \equiv \sigma(b^*(v))$, and inserting (14) one can restate (15) in the following form

$$(N-1)f(\sigma(b))\,(\sigma(b)-b)\,\sigma'(b) - F(\sigma(b)) = 0, \tag{16}$$

which simplifies to, due to $F(\sigma(b)) = \sigma(b)$ (uniform distribution)

$$(N-1)(\sigma(b)-b)\sigma'(b) - \sigma(b) = 0. \tag{17}$$

[12] To be careful, the condition should read as follows: $\rho(b) = \Pr\{b^*(V_{(N-1)} \leq b\}$; however, since the $V's$ are continuous random variables, there is no probability mass on single points, and therefore you can safely replace the strict by the weak inequality.

This differential equation has the obvious solution $\sigma(b) = \frac{N}{N-1}b$.[13] Finally, solve this for b, and you have the equilibrium bid strategy

$$b^*(v) = (1 - \frac{1}{N})v, \tag{18}$$

as asserted.[14] Notice that the margin of profit that each player allows himself in his bid decreases as the number of bidders increases.

Based on this result you can now determine the expected price $\bar{p}(N)$, by the following consideration: In a Dutch auction, the random price paid is equal to the highest bid, which can be written as $b^*(V_{(N)})$, where $V_{(N)}$ denotes the highest order statistic of the entire sample of N valuations. Therefore, by a known result on order statistics:[15]

$$\begin{aligned} \bar{p}(N) &= E[b^*(V_{(N)})] \\ &= \frac{N-1}{N} E[V_{(N)}] \quad (\text{since } b^*(v) = \tfrac{N-1}{N} v) \\ &= \frac{N-1}{N+1}, \end{aligned} \tag{19}$$

as asserted.

English auction Recall, in an English auction each bidder bids his true valuation.[16] The bidder with the highest valuation wins, and pays a price equal to the second highest order statistic $V_{(N-1)}$ of the given sample of N valuations. Therefore, the expected price is

$$\bar{p}(N) = E[V_{(N-1)}] = \frac{N-1}{N+1}. \tag{20}$$

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[13] Why is it obvious? Suppose the solution is linear; then you will easily find the solution, which also proves that there is a linear solution.

[14] This begs the question whether the equilibrium is unique. The proof is straightforward for $N = 2$. In this case, one obtains from (17) $\sigma d\sigma = \sigma db + b d\sigma = d(\sigma b)$. Therefore, $\sigma b = \frac{1}{2}\sigma^2 + \sigma(0)$. Since $b(0) = 0$ one has $\sigma(0) = 0$, and thus $\sigma(b) = 2bv$. Hence, the unique equilibrium strategy is $b^*(v) = \frac{1}{2}v$. In order to generalize the proof to $N > 2$, apply a transformation of variables, from (σ, b) to (z, b), where $z := \sigma/b$. Then one can separate variables and uniquely solve the differential equation by integration.

[15] Let $V_{(r)}$ be the r–th order statistic of a given sample of N independent and identically distributed random variables V_i with distribution function $F(v) = v$ on the support $[0, 1]$. Then, $E[V_{(r)}] = \frac{r}{N+1}$ and $\mathrm{Var}\,(V_{(r)}) = \frac{r(N-r+1)}{(N+1)^2(N+2)}$. See rules Kendall, S. M. and A. Stuart [1977]. *The Advanced Theory of Statistics*, Vol. I (Distribution Theory), Charles Griffin & Co., pp 268 and p. 348.

[16] Notice, however, that the English auction is plagued by a multiplicity problem. For example, the strategy combination $b_1(v) = 1$, $b_i(v) = 0, i = 2, \ldots, N$, is also an equilibrium, even though it prescribes rather pathological behavior. However, the equilibrium analyzed in the main text is the only one that is not “dominated”.

Comparison Since the two expected prices are the same, this example illustrates the general principle of revenue equivalency. Notice, however, that *actual* payoffs tend to differ, even in their risk characteristics. Indeed, the Dutch auction leads to a lower price risk. Therefore, risk averse sellers should prefer the Dutch over the English auction.

To see this clearly, compute the variance of the random price P, in the English (E) and the Dutch (D) auction

$$\begin{aligned}\operatorname{Var}(P)_E &= \operatorname{Var}(V_{(N-1)}) \\ &= \frac{2(N-1)}{(N+1)^2(N+2)} \qquad (21)\\ \operatorname{Var}(P)_D &= \operatorname{Var}\left(b^*(V_{(N)})\right) \\ &= \frac{(N-1)^2}{N^2}\operatorname{Var}(V_{(N)}) \\ &= \frac{(N-1)^2}{N(N+1)^2(N+2)}. \qquad (22)\end{aligned}$$

Since $\frac{N-1}{N} < \frac{N}{N} < 2$, you see immediately that the English auction gives rise to a higher variance of price, as asserted.

Third– and higher price auctions* Consider a generalization of the second–price auction: the "third–price" auction, where the highest bidder wins but pays only as much as the third highest bid, and more generally, the "k–th price" auction, where the winner pays the k–th highest bid. If you follow the procedure used above to derive the equilibrium bid function under the first price auction, you will find the following solution[17]

$$b^*(v) = \frac{N-1}{N-k+1}v, \quad \text{for} \quad N > 2,\ k = 1, \dots, N. \qquad (23)$$

Third– and higher price auctions have three striking properties: 1) bids are higher than the own valuation; 2) equilibrium bids diminish as the number of bidders is increased (since $\frac{N-1}{N-k+1} > \frac{N}{N-k+2}$); and 3), the variance of the random price P

$$\operatorname{Var}(P) = \operatorname{Var}\left(b^*(V_{(N-k+1)})\right) = \frac{k(N-1)}{(N-k+1)(N+1)^2(N+2)^2}. \qquad (24)$$

[17] Hints: Suppose the equilibrium bid function of the k–price auction is linear in the own valuation, $b^*(v) = \alpha_k v$. Revenue equivalency applies also to the k–price auction. Therefore, the expected price must be the same as under the English auction: $E[V_{(N-1)}] = E[b^*(V_{(N-k+1)})] = \alpha_k E[V_{(N-k+1)}]$. Compute the two expected values, and you have the asserted bid function (23). Once you have a candidate for solution, you only need to confirm that it does indeed satisfy the mutual best response requirement.

increases in k.

Once you have figured out why it pays to "speculate", and bid higher than the own valuation, it is easy to interpret the second property. Just keep in mind that a rational bidder may get "burned", and suffer a loss, because the k–th highest bid is above the own valuation. As the number of bidders is increased, it becomes more likely that the k–th highest bid is in close vicinity to the own valuation. Therefore, it makes sense to bid more conservatively when the number of bidders is increased, even though this may seem somewhat puzzling, at first glance.

Finally, the third property suggests that a risk averse seller should always prefer lower order k–price auctions, and therefore should most prefer the Dutch auction. The English auction is always appealing because of its overwhelming strategic simplicity. Third– and higher–price auctions are strategically just as complicated as the Dutch auction, and in addition expose the seller to unnecessary price risk. Therefore, we conclude that third– and higher–price auctions are strictly dominated by Dutch auctions.

Nobody has ever, to our knowledge, applied a third– or higher–price auction. So is this just an intellectual curiosity, useful only to challenge your intuition and technical skills?

Experimental economists have exposed inexperienced subjects to third–price seal bids, and examined their response to a higher number of bidders. Amazingly, the majority of bidders reduced their bid, just as the theory recommends[18]. The authors take great pride in this result; they take it as evidence that inexperienced subjects are not as unsophisticated as is often suspected by critics of the experimental approach. The particular irony is that if one asks trained theorists to make a guess, they tend to come up with the wrong hunch concerning the relationship between the number of bidders and equilibrium bids, until they actually sit down to confirm the computations.

"Charity" or "all pay" auctions* As a last example consider an auction that is frequently used in charities, which is why we call it "charity" or "all pay" auction, for lack of a better name. Its peculiar feature is that every bidder is required to actually pay his bid. Just like in a standard auctions, the item is awarded to the highest bidder. But unlike in standard auctions, each bidder must pay his bid, even if he is not awarded the item.

The charity or all pay auction sticks to the principle of awarding the item to the highest bidder. Therefore, revenue equivalence applies. But we want to confirm it directly, and compute equilibrium bids, as a further exercise.

[18]See Kagel, J. H. and D. Levin, [1988]. "Independent private value auctions: bidder behavior in first–, second–, and third–price auctions with varying numbers of bidders", University of Houston, Working Paper.

Using a procedure similar to the above, the equilibrium bid function is[19]

$$b^*(v) = \frac{N-1}{N} v^N. \tag{25}$$

This is in line with the observation that bidders often bid only nominal amounts of money for relatively valuable items.

In order to determine the expected price, notice that the seller collects the following expected payment from each bidder:

$$E\left[b^*(V)\right] = \frac{N-1}{N} \int_0^1 v^N dv = \frac{N-1}{N(N+1)}. \tag{26}$$

Take the sum over all bidders, and one obtains $E[Nb^*(V)] = \frac{N-1}{N+1}$, which confirms the asserted revenue equivalence.

The crucial feature of charity auctions is that they make winners and losers pay. We mention that this is also a feature of the not so charitable seller optimal auction, if bidders are risk averse.[20]

2.4 Generalization*

In many non–cooperative games it is difficult if not impossible to find a closed–form solution of the Nash equilibrium, unless one works with particular functional specifications. The *SIPV* auction game is an exception. Indeed, the English and the Dutch auction games have a unique symmetric equilibrium, for all probability distributions of private valuations.

Proposition 2 (Equilibrium bidding rules) *Suppose the distribution function $F(v)$ is continuous. Then, the Dutch auction has a unique equilibrium*

$$b^*(v) = v - \frac{\int_r^v F(\tilde{v})^{N-1} d\tilde{v}}{F(v)^{N-1}}, \tag{27}$$

where r denotes the seller's reserve price.

Of course, the English auction has several equilibria; but, $b(v) \equiv v$ is the only one that is not (weakly) dominated.

Proof The bidding rule for the English auction follows from the fact that truth–telling is a dominant strategy. To prove the bidding rule for the Dutch

[19]The probability of winning is just like under the Dutch auction (see eq. (14)). However, since all bids must actually be paid, the equilibrium bid must be a maximizer of $\rho(b)v - b$; hence it must solve the differential equation $\rho' v = 1$. From these two building blocks it is easy to compute the equilibrium bid function.

[20]See Matthews, S. [1983]. "Selling to risk averse buyers with unobservable tastes", *Journal of Economic Theory*, 30, 370-400.

auction, you have to solve the differential equation (16). The details are spelled out for example in Milgrom and Weber.[21] □

2.5 Robustness

Assuming the *SIPV* framework with risk neutral agents we have derived the amazing result that *all* auctions that award the item to the highest bidder give rise to the same payoffs. In other words, if all participants are risk neutral, many different auction forms — including the four standard auctions — are nothing but irrelevant institutional detail.

This result is in striking contrast to the apparent popularity of the English auction. Has the *SIPV* model missed something of crucial importance, or have we missed something at work even in this framework?

A strong argument in favor of the English auction is its strategic simplicity. In this auction bidders need not engage in complicated strategic considerations. Even an unsophisticated bidder should be able to work out that setting the limit equal to one's true valuation is the best one can do. Not so under the Dutch auction. This auction game has no dominant strategy equilibrium. As a result, understanding the game is conceptually more demanding, not to speak of computational problems.

However, if the auction cannot be oral, say because it would be too costly to bring bidders together at the same time and place, there is a very simple yet strong reason why one tends to use a first–price closed–seal bid, even though it is strategically much more complicated than the second–price or Vickrey auction. A second–price auction can usually be manipulated by soliciting *phantom* bids in close vicinity of the highest submitted bid. This suggests that second–price closed–seal bids should only be observed if the seller is a public agency, that is not profit oriented.

There are several more good reasons why the seller should favor the English auction. But these come up only as we modify the *SIPV* model. We now turn to this task. In addition we will give a few examples of auctions that do not adhere to the principle of awarding the item to the highest bidder, and on this ground fail to be payoff equivalent to the four common auctions.

2.5.1 Removing risk neutrality

The simplest variation of the *SIPV* model with risk averse agents is obtained by introducing risk averse bidders. Obviously, this modification does not affect the equilibrium strategy under the English auction. Therefore, the expected price in this auction is unaffected as well.

[21] Milgrom, P. and R. J. Weber [1982]. "A theory of auctions and competitive bidding", *Econometrica*, 50: 1089-1122

However, strategies and payoffs change under the Dutch auction. In this auction form bidders always take a chance and "shade" their bid below their valuation.[22] Therefore, the more risk averse a bidder, the more reluctant he will be to bid far below his true valuation, and consequently risk averse bidders tend to bid more conservatively. As a result, they raise the seller's payoff and, alas, reduce their own.[23]

Proposition 3 *Consider the SIPV model with risk averse bidders. The seller's payoff is higher and the bidders' payoff lower under the Dutch than under the English auction.*

2.5.2 Removing symmetry

The symmetry assumption is central to the revenue equivalence result. For if bidders are characterized by different probability distributions of valuations, under the Dutch auction (and the equivalent first–price seal bid) it is no longer assured that the object is awarded to the bidder with the highest valuation. But precisely this property was needed in the proof of Proposition 1.

Example 3 *Suppose there are two bidders, A and B, characterized by their random valuations, with support $(\underline{a}, \bar{a})$ and $(\underline{b}, \bar{b})$ with $\bar{a} < \underline{b}$. Consider the Dutch auction. Obviously, B could always win and pocket a gain by bidding the amount $\bar{a}$. But, B can do even better by shadeing the bid further below $\bar{a}$. But then the low valuation bidder A wins with positive probability, violating Pareto–optimality.*

An even more important implication is that Pareto efficiency is only assured by second–price bids. If a public authority uses auctions as an allocation device, efficiency should be the primary objective. This suggests that public authorities should adopt second–price bids.

A rigorous proof of existence and uniqueness of the equilibrium solution with heterogenous bidders is in Plum.[24] If you are interested in a careful mathematical study of existence and uniqueness of auction games, this contribution is a highly recommended source.

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[22] Recall, a bidder can always do better than bid his true valuation (which would give him zero payoff). In fact, in example 2 the equilibrium strategy was $b^*(v) = \frac{N-1}{N} v$, which implies shadeing the bid all the way down to $b^*(v) = \frac{1}{2} v$ if the number of bidders is $N = 2$.

[23] For a formal proof using stochastic dominance see Maskin, E. and J. G. Riley [1985]. "Auction theory with private values", *American Economic Review*, 75: 150-155.

[24] Plum, M. [1992]. "Characterization and computation of Nash–equilibria for auctions with incomplete information", *International Journal of Game Theory*, 20: 393-418.

2.5.3 Introducing a minimum price or participation fee

Revenue equivalence is a fairly general property of auction games. It applies to all auctions that adhere to the principle of awarding the item to the highest bidder. This restriction sounds pretty mild. Is there any common auction where it is violated? And is ever desirable to deviate from it?

Many seemingly innocuous modifications of standard auction rules deviate from the principle of awarding the item to the highest bidder, and therefore violate revenue equivalence. For example, suppose the seller sets a minimum price above his reservation price. Then he deviates from this principle, because he only sells to the highest bidder if that bid exceeds the minimum price. Of course, as we indicated in our comparison of examples 1 and 2, such deviations can be profitable for the seller. Indeed, the theory of optimal auctions tells us that the seller should always set such a minimum price above his reservation price, and thus leave the domain of revenue equivalence. More about this in the section on optimal auctions.

Another modification of standard auctions that violates revenue equivalence occurs when a "participation fee" is added. Adding such a fee to the English auction has also an interesting effect on the number of bidders who actually participate. This makes two good reasons for briefly elaborating on this modification.

Example 4 (Participation fee) *Suppose the seller requires a participation fee $c \in [0, 1]$ from each bidder, in an English auction. Let there be two bidders, with uniformly distributed valuations, on the support $[0, 1]$.*

Then, a bidder participates if and only if his valuation is at or above the critical level $\hat{v} := \sqrt{c}$. The expected revenue maximizing entry fee is $c^* = \frac{1}{4}$, and the maximum expected revenue $\frac{5}{12}$. Therefore, for $N = 2$, the participation fee augmented English auction is more profitable than the standard English auction and than the Cournot monopoly approach (see examples 1 and 2).

The proof is sketched as follows: 1) In order to compute $\hat{v}$, consider the marginal bidder, with valuation $\hat{v}$. His cost of participation, c, must be exactly matched by the expected gain from bidding

$$\Pr\{\text{winning}\}\hat{v} - c = 0. \tag{28}$$

Therefore,

$$\hat{v} = \sqrt{c}, \tag{29}$$

as asserted.

2) Notice, the seller collects the entry fee from both bidders if and only if $V_{(1)} \geq \hat{v}$, and in that case also collects a price equal to $V_{(1)}$. Whereas if $V_{(1)} < \hat{v} \leq V_{(2)}$, only one bidder participates, the entry fee goes all the way

down to zero, and the seller's only earning is the one entry fee paid the only one active bidder. Therefore, the seller's expected profit is equal to

$$\Pi := \Pr\{V_{(1)} \geq \hat{v}\} \left(E[V_{(1)} \mid V_{(1)} \geq \hat{v}] + 2c \right) + \Pr\{V_{(1)} < \hat{v} \leq V_{(2)}\} c. \tag{30}$$

Of course,

$$\Pr\{V_{(1)} \geq \hat{v}\} = (1 - \hat{v})^2, \tag{31}$$

and

$$\Pr\{V_{(1)} < \hat{v} \leq V_{(2)}\} = 2\hat{v}(1 - \hat{v}). \tag{32}$$

Hence, by all of the above,

$$\Pi = \frac{(1 - \sqrt{c})(4c + \sqrt{c} + 1)}{3}. \tag{33}$$

3) Evidently, Π is strictly concave in c, and $c^* = \frac{1}{4}$ solves the first order condition $1 - 2\sqrt{c}$, which completes the proof.

2.5.4 Introducing "numbers uncertainty"

In many auctions bidders are uncertain about the number of participants ("numbers uncertainty"). This seems almost compelling in closed–seal bids, where bidders do not convene in one location, but make bids each in their own office.

Notice, however, that the seller can easily eliminate numbers uncertainty, if he wishes to do so. He only needs to solicit *contingent bids*, where each bidder makes a whole list of bids, each contingent on a different number of participating bidders. Therefore, the seller has a choice. The question is: is it in the interest of the seller to leave bidders subject to numbers uncertainty?

McAfee, Preston and McMillan[25] explored this issue. They showed that if bidders are risk averse and have constant or decreasing absolute risk aversion, numbers uncertainty leads to more aggressive bidding in a first–price closed–seal bid. Since numbers uncertainty has obviously no effect on bidding strategies under the three other auction rules, one can conclude from the revenue equivalence proposition 1 that numbers uncertainty favors the first–price closed–seal bid. Incidentally, this result is used in experiments to test whether bidders are risk averse.[26]

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[25] See McAfee, R., R. Preston and J. McMillan [1987]. "Auctions with a stochastic number of bidders", *Journal of Economic Theory*, 43: 1-19.

[26] See Dyer, D., J. Kagel and D. Levin [1989]. "Resolving uncertainty about the number of bidders in independent private–value auctions: an experimental analysis", *Rand Journal of Economics*, 20: 268 ff.

2.5.5 Endogenous quantity

On the impact of endogenous quantity on the ranking of the four common auctions see Hansen[27]. Hansen shows that first–price auctions lead to a higher expected price; therefore, revenue equivalence breaks down in this case. An important application deals with industrial procurement contracts in private industry and government. Hansen's result explains why procurement is usually in the form of a first–price closed–seal bid. For a fuller analysis of this matter, see example 10 on p. 42, below.

2.5.6 Multi–unit auctions

Suppose the seller offers more than one unit of a good. Then everything generalizes quite easily, as long as each buyer demands at most one unit. But revenue equivalence fails if the demand is price elastic.

Consider the inelastic demand case. First of all we need to generalize the definition of first and second–price auctions. Suppose x units are put up for sale. In a second–price auction, a uniform price is set at such a level that all but x buyers drop out. Each remaining bidder receives one unit and pays that price. In turn, in a first price auction, the x highest bidders are awarded the x items, and they pay their own bid.

Just like in the single unit case, one can show that all auction rules are revenue equivalent, provided they adhere to the principle of awarding each item to the highest bidder.[28]

However, this does not generalize to the case when bidders' demand is price elastic. But in view of the previous section, this should not come as much of a surprise.

The theory of multi–unit auctions is not very well developed. Until recently, analysts of applied multi–unit auctions avoided to model the auction as a non cooperative game, and instead analyzed bidders' optimal strategy, assuming a given probability distribution of being awarded the demanded items.[29] However, Maskin and Riley[30] have generalized the theory of optimal auctions to include the multi–unit case. And Spindt and Stolz[31] showed that as the num-

[27]Hansen, R. G. [1988], "Auctions with endogenous quantity", *Rand Journal of Economics*, 19, 44-58.

[28]See Harris, M. and A. Raviv [1981]. "A theory of monopoly pricing schemes with demand uncertainty", *American Economic Review*, 71, 347-365.

[29]See for example the analysis of treasury bill auctions by Scott J. and C. Wolf [1978]. "The efficient diversification of bids in treasury bill auctions", *Review of Economics and Statistics*, 60, 280-287. Treasury bill auctions are explained on page 43.

[30]Maskin, E. and J. G. Riley [1989]. "Optimal multi–unit auctions", in: Hahn, F. (ed.). *The Economics of Missing Markets, Information, and Games*, Clarendon Press, 312-335.

[31]Spindt, P. A. and R. W. Stolz [19989]. "The expected stop–out price in a discriminatory auction", *Economics Letters*, 31, 133–137.

ber of bidders or the quantity put up for auction is increased, the expected stop–out price (the lowest price served) goes up, which generalizes well–known properties of single–unit auctions.

The most important applications of multi–unit auctions are in financial markets. For example, the U.S. Treasury sells marketable bills, notes, and bonds in more than 150 regular auctions per year, using a closed–seal bid, multiple–price auction mechanism. Some of the institutional details of the government securities market are sketched in section 6.3 below.

2.5.7 Removing independence: correlated beliefs

Finally, remove independence from the *SIPV* model, and replace it by the assumption of a positive correlation between bidders' valuations. Loosely speaking, positive correlation means that a high own valuation makes it more likely that rival bidders' valuations are high as well.

The main consequence of correlation is that it makes bidders more conservative in a Dutch auction. Of course, correlation does not affect bidding in an English auction, where truth–telling is always the dominant strategy. Therefore, the introduction of correlation reduces the expected price under the Dutch auction, but has no effect on the expected price under the English auction. In other words, correlation entails that the seller should prefer the English to the Dutch auction.

Proposition 4 *In the symmetric private values model with positively correlated valuations, the seller's payoff is higher under the English than under the Dutch auction, and that of potential buyers is lower.*[32]

The introduction of correlation has a number of comparative static implications with interesting public policy implications. In particular, the release of information about the item's value should raise bids in a Dutch auction. Incidentally, this comparative static property was used by Kagel, Harstadt and Levin[33] to test the predictions of auction theory in experimental settings.

3 Auction rings

So far we have analyzed auctions in a non–cooperative framework. But what if bidders collude and form an *auction ring*? Surely, bidders can gain a lot

[32]The formal proof is in Milgrom, P. and R. J. Weber [1982]. "A theory of auctions and competitive bidding", *Econometrica*, 50: 1089-1122.

[33]Kagel, J., R. M. Harstad and D. Levin [1987]. "Information impact and allocation rules in auctions with affiliated private values: a laboratory study", *Econometrica*, 55: 1275-1304.

by collusive agreements that exclude mutual competition. But can they be expected to achieve a stable and reliable agreement? And if so, which auctions are more susceptible to collusion than others?

To rank different auctions in the face of collusion, suppose all potential bidders have come to a collusive agreement. They have selected their *designated winner*, presumably the one with the highest valuation, recommended him to follow a particular strategy, and committed others to abstain from bidding. However, suppose the ring faces an enforcement problem because ring members cannot be prevented from breaching the agreement. Therefore, the agreement has to be self–enforcing. Does it pass this test at least in some auctions?

Consider the Dutch auction (or first–price seal bid). Here, the designated winner will be recommended to place a bid roughly equal to the seller's reserve price, whereas all other ring members are asked to abstain from bidding. But then each of those asked to abstain can gain by placing a slightly higher bid, in violation of the ring agreement. Therefore, the agreement is not self–enforcing.

Not so under the English auction (or second–price closed–seal bid). Here the designated bidder is recommended to bid up to his own valuation, and everyone else to abstain from bidding. Now, no one can gain by breaching the agreement, because no one will ever exceed the designated bidder's limit.

Therefore:[34]

Proposition 5 *Collusive agreements between potential bidders are self–enforcing in an English, but not in a Dutch auction.*[35]

These results give a clear indication that the English auction (or second–price closed–seal bid) is particularly susceptible to auction rings, and that the seller should choose a Dutch in lieu of an English auction if he has to deal with an auction ring that is unable to enforce agreements.

Even if auction rings can write enforceable agreements, the ring faces the problem of how to select the designated winner, and avoid strategic behavior of ring members. This is usually done by running a pre–auction. But can one set it up in such a way that it always selects the highest valuation ring member?

In a pre–auction, every ring member is asked to place a bid, and the highest bidder is chosen as the ring's sole bidder at the subsequent auction. But if the bid at the pre–auction only affects the chance of becoming designated

[34]This point was made by Robinson, M. [1985]. "Collusion and the choice of auction", *Rand Journal of Economics*, 16: 141-145.

[35]To be more precise, under an English auction the bidding game has two equilibria: one where everybody sticks to the cartel agreement and one where it is breached. The latter can, however, be ruled out on the ground of payoff dominance.

winner, at no cost, each ring member has every reason to exaggerate his valuation. Therefore, the pre–auction problem can only be solved if one makes the designated winner share his alleged gain from trade.

Graham and Marshall[36] proposed a simple scheme that resolves the pre–auction problem by an appropriate side–payment arrangement. Essentially, this scheme uses an English auction (or second–price closed–seal bid) to select the designated winner. If the winner of the pre–auction also wins the auction, he is required to pay the other ring members the difference between the price paid and the second highest bid from the pre–auction bid. This way it is assured that truth–telling is an optimal strategy at the pre–auction, excluding strategic behavior.

Notice, however, that this solution of the pre–auction problem works only if the ring can enforce the agreed upon side–payments. In the absence of enforcement mechanisms, auction rings are plagued by strategic manipulation, so that no stable ring agreement may get off the ground.

Finally, we mention that the seller may fight a suspected auction ring by "pulling bids off the chandelier", that is by introducing imaginary bids into the proceedings. In 1985, the New York City Department of Consumer Affairs proposed to outlaw imaginary bids, after it had become known that *Christie's* had reported the sale of several paintings that had in fact not been sold. This proposed regulation was strongly opposed by most New York based auction houses, precisely on the ground that it would deprive them of one of their most potent weapons to fight rings of bidders.

4 Optimal auctions

Among all possible auctions, which one maximizes the seller's expected profit? At first glance, this question seems unmanageable. There is a myriad of possible auction rules, limited only by one's imagination. Therefore, whatever your favorite auction rule, how can you ever be sure that someone will not come along and find a better one?

On the other hand, you may ask, what is the issue? Didn't we show that virtually all auction rules are payoff equivalent?

4.1 Application of the "revelation principle"

The breakthrough that made the problem of optimal auction design tractable is due to Myerson.[37] He observed that one may restrict attention, without

[36]Graham, D. A. and R. C. Marshall [1987]. "Collusive bidder behavior at single–object second–price and English auctions", *Journal of Political Economy*, 95: 1217–1239.

[37]Myerson, R. B. [1981]. "Optimal auction design", *Mathematics of Operations Research*, 6: 58-73, and Myerson, R. B. [1983]. "The basic theory of optimal auctions",

loss of generality, to incentive compatible direct auctions. This fundamental insight applies the famous *revelation principle*, which is also due to Myerson,[38] that has been useful in many areas of modern economics, from the public good problem to the theory of optimal labor contracts.

Example 5 *Auctions that award the item strictly to the highest bidder are not optimal. To show this, consider the assumptions of example 2, and let* $N = 2$*. By revenue equivalence, all auctions that select the highest bidder as winner are payoff equivalent to the Dutch and English auction. Therefore, the expected price is* $\bar{p}_1 = 1/3$*. Compare this to the Cournot–seller from example 1 that sets the take–it–or–leave–it price* $p^* = (1/3)^{\frac{1}{2}}$*, and earns the expected price* $\bar{p}_2 = p^* 2/3$*. Evidently, the Cournot approach is more profitable. The immediate implication is that by imposing a minimum bid equal to* p^**, the Dutch and English auction can be improved and be made even more profitable than the simple Cournot approach. Of course, by raising the minimum bid above the seller's reservation price the sale fails with positive probability. Therefore, the hybrid auction violates Pareto optimality.*

Direct auctions An auction is called *direct* if each bidder is only asked to report his valuation to the seller, and the auction rules select the winner and bidders' payments. Closed seal–bids are direct auctions — English and Dutch auctions are indirect.

Incentive compatibility A direct auction is *incentive compatible* if honest reporting of valuations is a non–cooperative Nash equilibrium. Many direct auctions are not incentive compatible. For example, in a first–price closed–seal bid every bidder bids less than his valuation. Therefore, first–price seal bids are direct but not incentive compatible. In turn, in a second–price closed–seal bid truth–telling is a dominant strategy. Therefore, second–price closed–seal bids are direct as well as incentive compatible.

Since we consider optimality from the point of view of the seller, incentive compatibility requires only that buyers reveal their true valuations, but we do not require that the seller reports his true reservation price, before bids are solicited. Again, the second–price closed–seal bid is a good illustration. Obviously, it is incentive compatible in the sense that all buyers report their true valuation. But, as example 5 indicates, it is not truth–revealing with regard to the sellers reservation price, since the seller would always quote a minimum bid above his true reservation price.

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in: Engelbrecht-Wiggans, R. and M. Shubik and J. Stark [1983], *Auctions, Bidding, and Contracting*, New York University Press, 149-163.

[38] Myerson, R. B. [1979]. "Incentive compatibility and the bargaining problem", *Econometrica*, 47: 61-73.

Revelation principle The revelation principle says that *for any equilibrium of any auction game, there exists an equivalent incentive compatible direct auction.* Therefore, the auction that is optimal among the incentive compatible direct auctions is also optimal for all types of auctions.

To prove the revelation principle, consider an equilibrium of some arbitrarily chosen auction game. We will show that the following procedure describes an equivalent incentive compatible direct auction.

1. Ask each bidder to report his valuation.
2. Compute each bidder's optimal bidding strategy in the given equilibrium of the assumed auction game.
3. Select the winner and collect payments exactly as in the given equilibrium of the assumed auction game.

Evidently, if all bidders are honest this direct auction is equivalent to the given equilibrium of the assumed auction game. Finally, this direct auction is also incentive compatible. Because if it were profitable for any bidder to lie in the direct auction, it would also be profitable for him to lie to himself in executing his equilibrium strategy in the original auction game. It's as simple as that.

Example 6 *Consider the Dutch auction equilibrium from example 2. The equivalent direct incentive compatible auction is described by the* $2N$ *probability and expected payment functions*

$$\rho_i(v_1, \dots, v_N) = \begin{cases} 1 & \textit{if} \quad v_i > v_j, \forall j \neq i \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathcal{E}_i(v_1, \dots, v_N) = \begin{cases} \frac{N-1}{N} v_i & \textit{if} \quad v_i > v_j, \forall j \neq i \\ 0 & \textit{otherwise.} \end{cases}$$

Feasible direct auctions A direct auction is described by a set of $2N$ *outcome functions*, ρ_i, $\mathcal{E}_i$, defined on the support of valuations. Thereby, $\rho_i(v_1, \dots, v_N)$ denotes the $i'th$ bidder's probability of winning, and $\mathcal{E}_i(v_1, \dots, v_N)$ his expected payment (notice: the bidder may have to make a payment even if he does not win the auction[39]). To be *feasible*, these outcome functions have to satisfy the following three conditions.

[39]For example, in the so called *All Pay or Charity Auction* each bidder makes an unconditional payment, and the item is awarded to the one who makes the highest payment (go back to example on page 15).

First, because there is only one object to allocate, the probabilities $\rho_i(v_1, \ldots, v_N)$ must sum to no more than 1, for all valuations (notice: the sale may fail, with positive probability)

$$\sum_{i=1}^{N} \rho_i(v_1, \ldots, v_N) \leq 1 \quad \textit{(budget constraint)}. \tag{34}$$

Second, the direct auction has to be incentive compatible. That is, if a bidder's true valuation is v_i, reporting the truth must be at least as good as reporting any other valuation $\hat{v}_i \neq v_i$. The utility of an agent whose valuation is v_i but who reports the valuation $\hat{v}_i$ is

$$\begin{aligned} U(\hat{v}_i \mid v_i) : \;=\; & \mathrm{E}\left[\rho_i(V_1, \ldots, \hat{v}_i, \ldots, V_N) v_i\right] \\ & - \mathrm{E}\left[\mathcal{E}_i(V_1, \ldots, \hat{v}_i, \ldots, V_N)\right]. \end{aligned} \tag{35}$$

(The $V's$ are random variables, v_i and $\hat{v}_i$ particular realizations.) Therefore incentive compatibility requires that, for all $v_i, \hat{v}_i \in [\underline{v}_i, \bar{v}_i]$

$$U(v_i \mid v_i) \geq U(\hat{v}_i \mid v_i) \quad \textit{(incentive compatibility)}. \tag{36}$$

Third, participation must be voluntary. Therefore, each bidder must be offered a nonnegative expected utility, whatever his valuation

$$U(v_i \mid v_i) \geq 0, \quad \forall v_i \in [\underline{v}_i, \bar{v}_i] \quad \textit{(participation constraint)}. \tag{37}$$

The programming problem In view of the revelation principle, and the above feasibility considerations, the optimal auction design problem is reduced to the choice of $2N$ outcome functions $(\rho_i, \mathcal{E}_i)$, $i = 1, \ldots, N$, so that the seller's expected profit is maximized

$$\max_{\{\rho_i, \mathcal{E}_i\}} \sum_{i=1}^{N} \mathrm{E}\left[\mathcal{E}_i(V_1, \ldots, V_N)\right] - \sum_{i=1}^{N} \mathrm{E}\left[\rho_i(V_1, \ldots, V_N)\right] r, \tag{38}$$

subject to the budget (34), the incentive compatibility (36), and the participation constraints (37). What looked like an unmanageable mechanism design problem has been reduced to a straightforward optimal control problem.

In the following we review and interpret the solution. If you want to follow the mathematical proof you have to look up Myerson's[40] beautiful contribution.

[40] Myerson, R. B. [1981]. "Optimal auction design", *Mathematics of Operations Research*, 6: 58-73.

Solution In order to describe the solution, the so called "priority levels" associated with a bid play the pivotal role. Essentially, these priority levels resemble and play the same role as individual customers' marginal revenue in the analysis of monopolistic price discrimination.

The "priority level" associated with the bid v_i is defined as

$$\gamma_i(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}. \tag{39}$$

We assume that priority levels are monotone increasing in the $v_i's$.[41]

Going back to the Cournot monopoly problem under incomplete information, reviewed in the introduction (see eq. (3)), $\gamma_i(v_i)$ can be interpreted as the marginal revenue from offering the item for sale to buyer i at a take–it–or–leave–it price equal to $p = v_i$. Therefore, the monotonicity assumption means that marginal revenue is diminishing in "quantity", where quantity is here the probability of sale.[42] Of course, the priority level is always less than the underlying valuation v_i.

With these preliminaries, the optimal auction is summarized by the following rules. The first set of rules describes how one should pick the winner, and the second how much the winner shall pay.

Selection of winner

1. Ask each bidder to report his valuation and compute the associated priority levels.

2. Award the item to the bidder with the highest priority level, unless it is below the reservation price r.

3. Keep the item if all priority levels are below the reservation price r (even though the highest valuation may exceed r).

Pricing rule To describe the optimal pricing rule we need to generalize the second–price auction rule. Suppose i has the valuation v_i which leads to the highest priority level $\gamma_i(v_i) \geq r$. Now ask: by how much could i have

[41] Notice $\frac{1-F_i(v_i)}{f_i(v_i)}$ is the inverse of the so–called *hazard rate*. Therefore, the assumed monotonicity of priority levels is satisfied whenever the hazard rate is not decreasing. A sufficient condition is obviously the log–concavity of the complementary distribution function $\bar{F}(v) := 1 - F(v)$. ($\bar{F}$ is called log–concave, if $\ln \bar{F}$ is concave.) This condition holds, for example, if the distribution function F is uniform, normal, logistic, chi-squared, exponential, Laplace. See Bagnoli M. und T. Bergstrom [1989]. "Log–concave probability and its applications", University of Michigan, Working Paper.

[42] Myerson [1981] also covers the case when priority levels are not monotone increasing.

reported a lower valuation, and would still have won the auction under the above rules?

Let z_i be the lowest reported valuation that would still lead to winning

$$z_i := \min\{\tilde{v} \mid \gamma_i(\tilde{v}) \geq r,\ \gamma_i(\tilde{v}) \geq \gamma_j(v_j),\ \forall j \neq i\}. \tag{40}$$

Then, the optimal auction rule says that the winner shall pay z_i. This completes the description of the auction rule.

Interpretation Essentially, the optimal auction rule combines the idea of a "second–price" (or Vickrey) auction with that of "third degree" monopolistic price discrimination.[43] As a first step, customers are ranked by their marginal revenue $v_i - \frac{1-F_i(v_i)}{f(v_i)}$, evaluated at the reported valuation. Since only one indivisible unit is up for sale, only one customer can win. The optimal rule selects the customer who ranks highest in the marginal revenue hierarchy, unless it falls short of the seller's reservation price r. In this sense, the optimal selection of winner rule employs the techniques of third degree price discrimination.

But the winner is neither asked to pay his marginal revenue nor his reported valuation. Instead, the optimal price is equal to the lowest valuation that would still make the winner's marginal revenue equal or higher than that of all rivals. In this particular sense, the optimal auction rule employs the idea of a "second–price" auction.

Illustrations As an illustration, take a look at the following two examples. The first one shows that the optimal auction coincides with the second–price closed–seal bid (or English auction), supplemented by a minimum price above the seller's reservation price, if customers are indistinguishable, so that there is no basis for third degree price discrimination. The second example then brings out the discrimination aspect, assuming that bidders are viewed as different.

In both examples, the seller deviates from the principle of selling to the highest bidder (which underlies revenue equivalence), and the equilibrium outcome fails to be Pareto optimal, with positive probability. This indicates that the four standard auctions are not optimal. However, the optimal auction itself poses a credibility problem that will be discussed towards the end of this section.

[43]Recall, in the theory of *Monopoly* one distinguishes three kinds of price discrimination: perfect or *first degree* price discrimination, imperfect or *second–degree* price discrimination by means of self–selection devices (offering different price–quantity packages), and finally *third degree* price discrimination based on different signals about demand (for example, different national markets).

Example 7 (Identical bidders — SIPV model) *Consider the independent private values model, assume $r \in [0,1)$, and let each bidders' valuation be uniformly distributed on the support $[0,1]$. Then, the priority levels are*

$$\gamma_i(v_i) = v_i - \frac{1 - v_i}{1} = 2v_i - 1.$$

By the above optimal auction rules, the item is sold if and only if the highest valuation exceeds $\frac{1}{2} + \frac{r}{2}$. The winner pays either $\frac{1}{2} + \frac{r}{2}$ or the second highest bid, whichever is higher. The optimal auction is thus equivalent to a modified "second–price" closed–seal bid, where the seller sets a minimum price equal to $\frac{1}{2} + \frac{r}{2}$. Because the minimum price $\frac{1}{2} + \frac{r}{2}$ is higher than the reservation price r, the seller must sometimes keep the object even if some buyer's valuation exceeds the reservation price. This inefficient outcome occurs with probability $(\frac{1+r}{2})^N > 0$. Therefore, the optimal auction does not assure Pareto optimality.

If there are only two bidders, and the reservation price is zero, $r = 0$, as in many of our examples, the optimal minimum price is 1/2, and the expected revenue is equal to $\frac{5}{8}$.[44] Compare this to the Cournot approach, the English auction, and the participation fee augmented English auction (see examples 1, 2, 4).

Example 8 (Differing bidders) *Assume there are only two bidders, A and B, and suppose $r = 0$. Both bidders' valuation is a uniform random variable, but the supports differ. Let A's support be the $[0,1]$ and B's the $[0,2]$ interval. Then, their priority levels are*

$$\begin{aligned} \gamma_A(v_A) &= 2v_A - 1 \\ \gamma_B(v_B) &= 2v_B - 2. \end{aligned}$$

Obviously, if the two happen to have the same valuation, bidder A wins. In fact, bidder B can only win if his valuation exceeds that of A by 1/2. The optimal auction rule discriminates against bidder B; it thus "handicaps" him, as a way to encourage him to bid more aggressively. For if bidder B were not discriminated against, he would never report a valuation greater than 1, and hence always pay less than 1 in case of winning. Whereas under the optimal auction rule he may have to pay up to 3/2 to win.

4.2 Illustration: two bidders and two valuations*

This brief introduction to optimal auctions stressed essential properties, and left out proofs and computations. This may leave you somewhat unsatisfied.

[44] The expected revenue is: $\Pr\{V_{(1)} \geq \frac{1}{2}\} E[V_{(1)} \mid V_{(1)} \geq \frac{1}{2}] + \Pr\{V_{(1)} < \frac{1}{2} \leq V_{(2)}\}\frac{1}{2}$, with $E[V_{(1)} \mid V_{(1)} \geq \frac{1}{2}] = \frac{3}{2}$, and $\Pr\{V_{(1)} \geq \frac{1}{2}\} = \frac{1}{4}$, $\Pr\{V_{(1)} < \frac{1}{2} \leq V_{(2)}\} = \frac{1}{2}$.

Therefore, we now add a full scale example with two identical bidders and two valuations.[45] Its main purpose is to exemplify the computation of optimal auctions in a particularly simple case where the price discrimination aspect of optimal auctions cannot play any role.

Suppose the two bidders are identical, in the sense that they have either a low (v_l) or a high valuation (v_h). These valuations occur with the probabilities $\Pr\{V = v_l\} = \pi$, $\Pr\{V = v_h\} = 1 - \pi$.

A direct auction is completely described by the probabilities of obtaining the item ρ_h, ρ_l, and bidders' expected payments $\mathcal{E}_h$, $\mathcal{E}_l$.

Optimization problem The optimal auction maximizes the expected gain from each bidder

$$\max_{\rho_h,\rho_l,\mathcal{E}_h,\mathcal{E}_l} G := (1-\pi)\mathcal{E}_h + \pi\mathcal{E}_l, \tag{41}$$

subject to the "incentive compatibility"

$$\begin{aligned} \rho_h v_h - \mathcal{E}_h &\geq \rho_l v_h - \mathcal{E}_l & (42)\\ \rho_l v_l - \mathcal{E}_l &\geq \rho_h v_l - \mathcal{E}_h. & (43) \end{aligned}$$

and "participation" constraints

$$\begin{aligned} \rho_h v_h - \mathcal{E}_h &\geq 0 & (44)\\ \rho_l v_l - \mathcal{E}_l &\geq 0. & (45) \end{aligned}$$

An immediate implication of incentive compatibility is the monotonicity property

$$\rho_h \geq \rho_l, \quad \mathcal{E}_h \geq \mathcal{E}_l. \tag{46}$$

Restrictions due to symmetry Having stated the problem, we now make it more "user friendly". The key observation is that symmetry implies three further restrictions on the probabilities ρ_h, ρ_l.

From the seller's perspective each bidder wins the item with probability $(1-\pi)\rho_h + \pi\rho_l$; in a symmetric equilibrium this probability cannot exceed $\frac{1}{2}$. Therefore,

$$(1-\pi)\rho_h + \pi\rho_l \leq \frac{1}{2}. \tag{47}$$

Also, in a symmetric equilibrium type h must lose with probability $\frac{1}{2}$ or more whenever he faces a type h rival. Therefore, $1 - \rho_h \geq \frac{1}{2}(1-\pi)$, or equivalently

$$\rho_h \leq \pi + \frac{1}{2}(1-\pi) = \frac{1}{2}(\pi + 1). \tag{48}$$

[45]The example is borrowed from K. Binmore [1992]. *Fun and Games. A Text on Game Theory*. D. C. Heath and Company, pp. 532–536.

And similarly, type l must lose with probability $\frac{1}{2}$ or more when he faces a type l rival, $1 - \rho_l \geq \frac{1}{2}\pi$, or equivalently

$$\rho_l \leq 1 - \frac{1}{2}\pi. \tag{49}$$

Combining these restrictions, the set of feasible probabilities ρ_h, ρ_l is illustrated in Figure 3. Similarly, the set of feasible expected payments $\mathcal{E}_h, \mathcal{E}_l$ is illustrated in Figure 2. In both diagrams, the dashed lines represent the seller's indifference curves.

Why some constraints do not bind As you can see immediately from Figure 2, only two constraints are binding: the truth–telling condition concerning type h, and the participation constraint concerning type l. Therefore, you can ignore inequalities (43), (44), and replace the inequalities (42), (45) by equalities.

User friendly optimization problem Using these results, you can now eliminate the expected payment variables, and find the optimal auction as the solution of the following linear programming problem

$$\max_{\rho_h, \rho_l} G := (1 - \pi)v_h(\rho_h - \rho_l) + \rho_l v_l, \tag{50}$$

subject to (47)–(49).

The only variables are ρ_h, ρ_l.

Solution The solution is easily characterized by the two diagrams. Take a look at the seller's indifference curves (the dashed lines) in Figure 3. Evidently, the marginal rate of substitution can be positive or negative, depending on the prior probability π. In particular, if type l is sufficiently likely ($\pi \geq \frac{v_h - v_l}{v_h}$), the marginal rate of substitution is negative (the dashed indifference curve slopes downwards), whereas if it is sufficiently unlikely ($\pi \leq \frac{v_h - v_l}{v_h}$), the marginal rate of substitution is positive (the dashed indifference curve slopes upwards). Therefore, the solution is at one of two corners of the set of feasible $\rho' s$, depending upon the size of the prior probability π, as summarized in Table 2:

4.3 Discussion: endogenizing bidder participation*

We close this introduction to optimal auctions with a slightly technical note on an interesting side issue that may have already plagued some readers. This has to do with the need to endogenize the number of bidders, and the robustness of results with regard to a more meaningful handling of bidder participation.

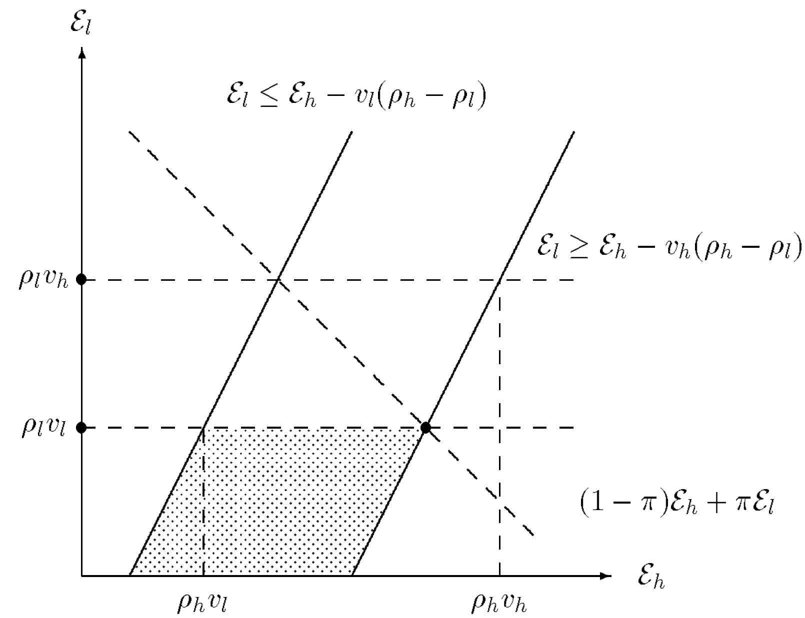

Figure 2: Optimal auction: optimal $\mathcal{E}$'s

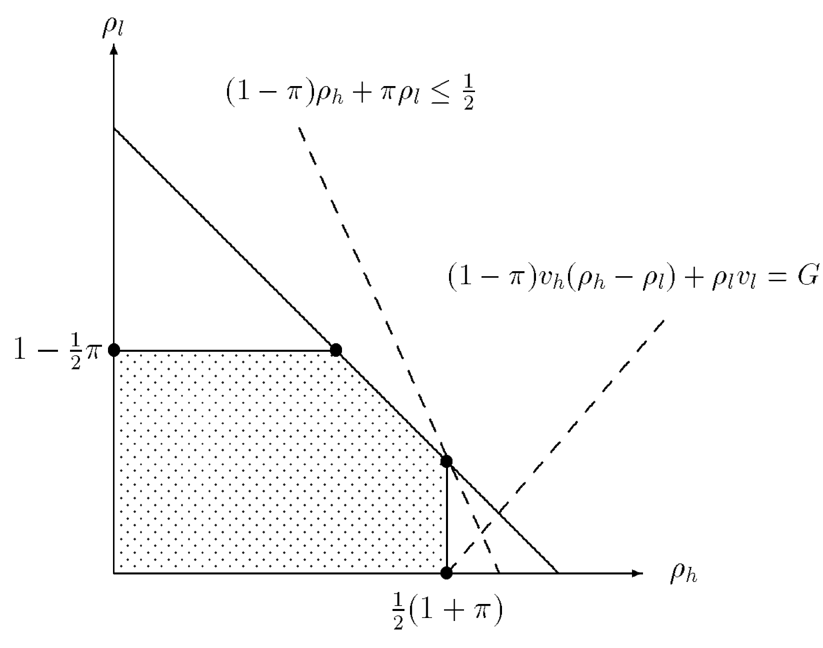

Figure 3: Optimal auction: optimal ρ's

π	ρ_l	$\mathcal{E}_l$	ρ_h	$\mathcal{E}_h$
$\pi \leq \frac{v_h - v_l}{v_h}$	0	0	$\frac{1}{2}(1+\pi)$	$\frac{1}{2}(\pi+1)v_h$
$\pi > \frac{v_h - v_l}{v_h}$	$\frac{1}{2}\pi$	$\frac{1}{2}\pi v_l$	$\frac{1}{2}(1+\pi)$	$\frac{1}{2}(v_h + \pi v_l)$

Table 2: Optimal auction

Recall, in the above optimal auction program it was assumed that all potential bidders do actually participate whenever they can be sure that they do not suffer a loss from it. This seems perfectly innocuous. But it has the unappealing implication that bidders participate even if they are sure to go empty handed. For example, in the symmetric case, the optimal auction entailed a minimum price above the seller's reservation value. When that minimum price is announced, all potential bidders with valuations below this minimum price can be sure that bidding is absolutely pointless for them. Therefore, it would make all the sense in the world to assume that they withdraw from the auction and do not participate — contrary to what is assumed in the above optimal auction model.[46] This way, the number of active bidders becomes endogenous. And the question comes up: does this endogenizing of the number of active bidders have an impact upon the optimal reserve price, that is not accounted for in the alleged optimal auction?

Remarkably, the answer is no. In the *SIPV* framework *the optimal minimum price is independent of the number of bidders.*

Proposition 6 *Consider a second–price auction, with a minimum price at or above the seller's reservation price. Then, the optimal minimum price p^* is independent of the number of bidders; it is implicitly defined as the solution of*

$$p^* = \frac{1 - F(p^*)}{f(p^*)}, \tag{51}$$

which gives $p^* = \frac{1}{2}$ if F is the uniform distribution with support $[0, 1]$.

Proof The two order statistics $V_{(N)}, V_{(N-1)}$ have the following joint density

[46]This withdrawal would necessarily occur if bidding is costly, no matter how small the cost. Of course, in the limiting case of costless bidding, it is also part of an equilibrium configuration to have everybody participate, even those who are sure to have no chance of winning.

function[47]

$$g(x,y) = \begin{cases} N(N-1)F(x)^{N-2}f(x)f(y) & \text{for} \quad y \geq x \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

Let p denote the minimum price. In an English auction the item is sold iff $V_{(N)} \geq p$, and the actual price paid is equal to $V_{(N-1)}$ iff $V_{(N-1)} \geq p$, and equal to p iff $V_{(N-1)} < p \leq V_{(N)}$. Therefore, the optimal minimum price p^* solves

$$\max_p \int_p^{\bar{v}} \int_p^y x\, g(x,y)dxdy + p \int_p^{\bar{v}} \int_0^p g(x,y)dxdy. \tag{53}$$

The associated first-order condition is

$$p^* \int_0^{p^*} g(x,p^*)dx \quad = \quad \int_{p^*}^{\bar{v}} \int_0^{p^*} g(x,y)dxdy. \tag{54}$$

Its left-hand-side (LHS) can be rearranged as follows[48]

$$\begin{aligned} \text{LHS} \quad &:= \quad p^* N\, f(p^*) \int_0^{p^*} (N-1)F(x)^{N-2} f(x)dx \\ &= \quad p^*\, Nf(p^*)F(p^*)^{N-1}. \end{aligned}$$

Similarly, its right–hand–side (RHS) is

$$\begin{aligned} \text{RHS} \quad &:= \quad N \int_{p^*}^{\bar{v}} f(y) \int_0^{p^*} (N-1)F(x)^{N-2} f(x)dxdy \\ &= \quad N\, F(p^*)^{N-1}\, [1 - F(p^*)]. \end{aligned}$$

Therefore, condition (54) simplifies to

$$p^*\, f(p^*) = 1 - F(p^*). \tag{55}$$

We conclude that p^* is independent of N for all $N > 1$, as asserted. In particular, $p^* = 1/2$, for all N, if valuations are uniformly distributed on $[0, 1]$. □

Therefore, even if we correct the usual optimal auction program, and take into account that raising the reserve price has an adverse effect on the expected number of bidders, the optimal reserve price will remain the same.

[47]The joint density function of the two order statistics $V_{(r)}$, $V_{(s)}$, $1 \leq r < s \leq N$, is

$$f_{rs}(x,y) = \frac{N!}{(r-1)!((s-r-1)!(N-s)!} F(x)^{r-1} f(x)[F(y) - F(x)]^{s-r-1} f(y)[1-F(y)]^{N-s},$$

if $x \leq y$, and equal to zero otherwise. See David, H. A. [1970]. *Order Statistics*, Wiley, p. 9. Set $r = N-1$, $s = N$, and you have the asserted joint density function.

[48]Notice, $\frac{d}{dx} F(x)^{N-1} = (N-1)F(x)^{N-2} f(x)$.

5 Common value auctions and the winner's curse

In a famous experiment, Bazerman and Samuelson[49] filled jars with coins, and auctioned them off to MBA students at Boston university. Each jar had a value of \$8, which was not made known to bidders. The auction was conducted as a first–price closed–seal bid. A large number of identical runs were performed. Altogether, the *average bid* was \$5.13; however, the *average winning bid* was \$10.01. Therefore, the average winner suffered a loss of \$2.01; alas, the winners were the ones who lost wealth ("winner's curse").

What makes this auction different is just one crucial feature: the object for sale has an *unknown common* value rather than *known private* values. The result is typical for common value auction experiments. So what drives the winner's curse; and how can bidders avoid falling prey to it?

Most auctions involve some *common value* element. Even if bidders at an art auction purchase primarily for their own pleasure, they are usually also concerned about the eventual resale. And even though construction companies tend to have private values, for example due to different capacity utilizations, construction costs are also affected by common events, such as unpredictable weather conditions, the unknown difficulty of specific tasks, and random input quality or factor prices. This suggests that a satisfactory theory of auctions should cover private as well as unknown common value components.

For simplicity, we now focus on the other extreme, the pure common value auction. In a pure common value auction, the item for sale has the same value for each bidder (just like the jar in the above experiment contains the same number of coins for each bidder). At the time of the bidding, this common value is unknown. Bidders may have some imperfect estimate (everyone has his own rule of thumb to guess the number of coins in the jar); but the item's true value is only observed after the auction has taken place.

To sketch the cause of the winner's curse, suppose all bidders obtain an unbiased estimate of the item's value. Also assume bids are an increasing function of this estimate. Then, the auction will select the one bidder as winner who received the most optimistic estimate. But this entails that the average winning estimate is higher than the item's value.

To play the auction right, this *adverse selection* bias must already be accounted for at the bidding stage, by shadeing the bid. Failure to follow this advice will result in winning bids that earn less than average profits or even losses.

[49] Bazerman, M. and W. Samuelson [1983]. "I won the auction but don't want the prize", *Journal of Conflict Resolution*, 27:618-634.

A simple framework Consider the following simplifying assumptions:[50]

(A1) The common value V is drawn randomly from a uniform distribution on the domain $(\underline{v}, \bar{v})$.

(A2) Before bidding, each bidder receives a private signal S_i, drawn randomly from a uniform distribution on $(V - \epsilon, V + \epsilon)$. The unknown common value determines the location of the signals' support. A lower ϵ indicates greater signal precision.

(A3) The auction is first–price, like a Dutch auction or a closed–seal bid.[51]

As an illustration you may think of oil companies interested in the drilling rights to a particular site that is worth the same to all bidders. Each bidder obtains an estimate of the site's value from its experts, and then uses this information in making a bid.

Computing the right expected value Just like in the private values framework, each bidder has to determine the item's expected value, and then strategically "shade" his bid, taking a bet on rival bidders' valuations. Without shadeing the bid, there is no chance to gain from bidding. The difference is however that it is a bit more tricky to determine the right expected value, in particular since this computation should already account for the built–in adverse selection bias.

For signal values from the interval $s_i \in [v - \epsilon, v + \epsilon]$, the *expected value* of the item conditional on the signal s_i is[52]

$$E[V|S_i = s_i] = s_i. \tag{56}$$

This estimate is unbiased in the sense that the average estimate is equal to the site's true value.

Nevertheless, bids should not be based on this expected value. Instead, a bidder should anticipate that he would revise his estimate whenever he actually

[50] Assumptions (A1) and (A2) were widely used by John Kagel and Dan Levin in their various experiments on common value auctions. See for example Kagel, J. H. and D. Levin [1986]. "The winner's curse and public information in common value auctions", *American Economic Review*, 76:894-920.

[51] Second–price common value auctions are easier to analyze, but less instructive. A nice and simple proof of their equilibrium properties, based entirely on the principle of "iterated dominance", is in Harstad, R. M. and D. Levin [1985]. "A class of dominance solvable common–value auctions", *Review of Economic Studies*, 52:525-528. The procedure proposed in this paper requires, however, that the highest signal is a *sufficient statistic* of the entire signal vector.

[52] Here and elsewhere we ignore signals "near" corners, that is $v \notin (\underline{v}, \underline{v} + \epsilon)$, and $v \notin (\bar{v} - \epsilon, \bar{v})$.

wins the auction. As in many other contexts, the clue to rational behavior is in thinking one step ahead.

In a symmetric equilibrium, a bidder wins the auction if he actually received the highest signal.[53] Therefore, when a bidder learns that he won the auction, he knows that his signal s_i was the largest received by all bidders. Using this information, he should value the item by its expected value conditional on having the highest signal, denoted by $E[V|S_{max} = s_i]$, $S_{max} := \max\{S_1, \ldots, S_N\}$.

Evidently, the *updated expected value* is lower

$$\begin{aligned} E[V|S_{max} = s_i] &= s_i - \epsilon \frac{N-1}{N+1} \\ &< E[V|S_i = s_i]. \end{aligned} \tag{57}$$

Essentially, a bidder should realize that if he wins, it is likely that the signal he received was unusually high, relative to those received by rival bidders.

Notice, the adverse selection bias, measured by the difference between the two expected values, is increasing in N and ϵ. Therefore, raising the number of bidders or lowering the precision of signals gives rise to a higher winner's curse, if bidder are subject to judgmental failure, and base their bid on $E[V|S_i = s_i]$ rather than on $E[V|S_{max} = s_i]$.

Equilibrium bids The symmetric Nash equilibrium of the common value auction game was found by Wilson[54] and later generalized by Milgrom and Weber[55] to cover auctions with a combination of private and common value elements.

In the present framework, the symmetric equilibrium bid function is[56]

$$b(s_i) = s_i - \epsilon + \phi(s_i), \tag{58}$$

where

$$\phi(s_i) := \left(\frac{2\epsilon}{N+1}\right) e^{-\frac{N}{2\epsilon}[s_i - (\underline{v} + \epsilon)]} \tag{59}$$

The term $\phi(s_i)$ diminishes rapidly as s_i increases beyond $\underline{v} + \epsilon$. Ignoring it, the bid function is approximately equal to $b(s_i) = s_i - \epsilon$, and the expected profit of the high bidder is positive and equal to $2\epsilon/(N+1)$.

[53] This standard property follows from the monotonicity of the equilibrium bid function; in the present context, it is confirmed by the equilibrium bid function stated below.

[54] Wilson, R. B. [1977]. "A bidding model of perfect competition", *Review of Economic Studies*, 44:511-518.

[55] Milgrom, P. and R. J. Weber [1982]. "A theory of auctions and competitive bidding", *Econometrica*, 50:1089-1122.

[56] Again, we restrict the analysis to signal values from the interval $s_i \in [\underline{v} + \epsilon, \bar{v} - \epsilon]$.

Conclusions The main lesson to be learned from this introduction to common value auctions is that bidders should shade their bids, for two different reasons. First, because without shadeing bids there can be no profits in a first price auction. Second, because the auction always selects the one bidder as winner who received the most optimistic estimate of the item's value, and thus induces an adverse selection bias. Without shadeing the bid to account for this adverse selection bias, the winning bidder regrets his bid and falls prey to the winner's curse.

The discount associated with the first reason for shadeing bids will decrease with the number of bidders, because signal values are more congested when there are more bidders. In turn, the discount associated with the adverse selection bias increases with the number of bidders, because the adverse selection bias becomes more severe as the number of bidders is increased.

Altogether, increasing the number of bidders has two conflicting effects on equilibrium bids. On the one hand, competitive considerations require more aggressive bidding. On the other hand, accounting for the adverse selection bias requires greater discounts. For low numbers of bidders the competitive effect prevails, and the bid function is increasing in N; but, eventually the adverse selection effect takes over, and the equilibrium bid function decreases in N.

Altogether it is clear that common value auctions are more difficult to play, and that unsophisticated bidders may be susceptible to the winner's curse. Of course, the winner's curse cannot occur if bidders are rational, and properly account for the adverse selection bias. The winner's curse is strictly a matter of judgmental failure; rational bidders do not fall prey to it.

The experimental evidence demonstrates that it is often difficult to avoid the winner's curse. Even experienced subjects who are given plenty of time and opportunity to learn often fail to bid below the updated expected value. And most subjects fail to bid more conservatively when the number of bidders is increased.

Similarly, many real life decision problems are plagued by persistent winner's curse effects. For example, in the field of book publishing, Dessauer[57] reports that "... most of the auctioned books are not earning their advances." Capen, Clapp and Campbell[58] claim that the winner's curse is responsible for the low profits of oil and gas corporations on drilling rights in the Gulf of Mexico during the 1960s. And Bhagat, Shleifer and Vishny[59] observe that most of

[57]See Dessauer, J. P. [1981]. *Book Publishing*, Bowker Publ.

[58]See Capen, E. C. and R. V. Clapp and W. M. Campbell [1971]. "Competitive bidding in high–risk situations", *Journal of Petroleum Technology*, 23:641-653.

[59]See Bhagat S. and A. Shleifer and R. W. Vishny [1990]. "Hostile takeovers in the 1980's: the return to corporate specialization", *Brookings Papers on Economic Activities*, Special Issue on Microeconomics, 1-84.

the major corporate takeovers that made the financial news during the 1980's have tended to actually reduce bidding shareholder's wealth.[60]

6 Further applications

We close with a few remarks on the use of the auction selling mechanism in financial markets, in dealing with the natural monopoly problem, and in oligopoly theory. Each application poses some unique problem. This indicates that auction theory is still a rich mine of unresolved research problems.

6.1 Auctions and oligopoly

Recall the analysis of price competition, and the Bertrand paradox (*"two is enough for competition")* in the theory of oligopoly. There one usually elaborates on various proposed resolutions of the Bertrand paradox that typically have to do with capacity constrained price competition, and that culminate in a defense of the Cournot model. This resolution is successful, though a bit complicated — but unfortunately not quite robust with regard to the assumed rationing rule.[61]

A much simpler resolution of the Bertrand paradox can be found by introducing incomplete information. This explanation requires a marriage of oligopoly and auction theory, which is why we sketch it here, in two examples.

Example 9 (Another resolution of the Bertrand paradox) *Consider a simple Bertrand oligopoly game, with inelastic demand, and $N \geq 2$ "identical" firms. Each firm has constant unit costs c, and unlimited capacity. Unlike in the standard Bertrand model, each firm knows only its own cost, but not those of others. Rivals' unit costs are viewed as identical, and independent random variables, described by a uniform distribution with the support $[0, 1]$.*

Since the lowest price wins the market, the Bertrand game is a Dutch auction, with the understanding that we are dealing here with a "buyers' auction" in lieu of the "seller's auction" considered before. Since costs are independent and identically distributed, we can employ the apparatus and results of the SIPV auction model. Strategies are price functions that map each firm's unit cost c into a unit price $p(c)$.

[60] See also Roll, R. [1986]. "The hubris hypothesis of corporate takeovers", *Journal of Business*, 59:197-216.

[61] See the ingenious defense of the Cournot model by Kreps, D. and J. A. Scheinkman [1983]. "Quantity precommitment and Bertrand competition yield Cournot outcomes", *Bell Journal of Economics*, 14, 326-337.

The unique equilibrium price strategy is characterized by the mark–up rule

$$p^*(c) = \frac{1}{N} + \frac{(N-1)}{N}c. \tag{60}$$

The equilibrium price is $p^*(C_{(1)})$*, the equilibrium expected price*[62]

$$\bar{p}(N) := E[p^*(C_{(1)}] = E[C_{(2)}] = \frac{2}{N+1}, \tag{61}$$

and each firm's equilibrium expected profit

$$\bar{\pi}(c,N) = \frac{(1-c)^N}{N}. \tag{62}$$

Evidently, more competition leads to more aggressive pricing, beginning with the monopoly price $p^*(1) = 1$*, and approaching the competitive pricing rule* $p = c$ *as the number of oligopolists becomes very large. Similarly, the expected price rule and profit are diminishing in* N*, beginning with* $\bar{p}(1) = 1$*,* $\bar{\pi}(1) = 1 - c$*, with* $\lim_{N\to\infty} \bar{p}(N) = 0$ *and* $\lim_{N\to\infty} \bar{\pi}(N) = 0$*. Hence, numbers matter. "Two is not enough for competition".*

In order to prove these results all we need to do is to use a transformation of random variables, and then apply previous results on Dutch auctions. For this purpose, define

$$b := 1 - p \quad \text{and} \quad v := 1 - c. \tag{63}$$

Obviously, this transformation maps the price competition game into a Dutch auction, where the highest bid b (the lowest p) wins, and $b, v \in [0,1]$ since $p, c \in [0,1]$. By example 2, we know that $b(v) := \frac{N-1}{N}v$. Therefore,

$$\begin{aligned} p^* &= 1 - b \\ &= 1 - \frac{N-1}{N}v \\ &= 1 - \frac{N-1}{N}(1-c) \\ &= \frac{1}{N} + \frac{N-1}{N}c, \end{aligned} \tag{64}$$

as asserted.

[62] Notice, $E[p^*(C_{(1)}] = E[C_{(2)}]$ is the familiar "revenue equivalence", which holds for all distribution functions. (If the market were "second price", each firm would set $p^*(c) \equiv c$, the lowest price would win, and trade would occur at the second lowest price; therefore, the expected price at which trade occurs would be equal to $E[C_{(2)}]$.

Remark 2 (Why numbers matter) *By revenue equivalence one has*

$$\bar{p}(N) := E[p^*(C_{(1)}] = E[C_{(2)}]. \tag{65}$$

Therefore, the "two is enough for competition" property of the Bertrand paradox survives in terms of expected values. Still, numbers matter, essentially because $E[C_{(1)}]$ and $E[C_{(2)}]$ move closer together, as the size of the sample N is increased. The gap between these statistics determines the aggressiveness of pricing, because the price function has the form $p^*(c) = (1-\beta) + \beta c$, with

$$\beta := \frac{N-1}{N} = \frac{1 - E[C_{(2)}]}{1 - E[C_{(1)}]}. \tag{66}$$

Example 10 (Extension to price elastic demand*) *Suppose demand is a decreasing function of price, as is usually assumed in oligopoly theory. Specifically, assume the simplest possible inverse demand function $P(X) := 1 - X$. Otherwise, maintain the assumptions of the previous example. Then, the equilibrium is characterized by the following mark–up rule $p^* : [0,1] \to [0,1]$*

$$p^*(c) = \frac{1 + Nc}{N+1}. \tag{67}$$

Obviously, $p^*(1) = 1$ and $p^*(0) = \frac{1}{N+1}$.

Compare this to the equilibrium mark–up rule in the inelastic demand framework analyzed in the previous example, which was $p^*(c) = \frac{1+(N-1)c}{N}$. Evidently, pricing is more aggressive if demand responds to price. Of course, if the auction were second–price, bidding would not be affected, since $p^*(c) = c$ is a dominant strategy in this case.

We conclude: if demand is a decreasing function of price, a first–price auction leads to higher expected consumer surplus. This may explain why first–price closed–seal bids are the common auction form in industrial and government procurement.

This is just the beginning of a promising marriage of oligopoly and auction theory. Among the many interesting extensions, an exciting issue concerns the analysis of the repeated play Bertrand game. The latter is particularly interesting, because it leads to a dynamic price theory, where agents successively learn about rivals' costs, which makes the information structure itself endogenous.

6.2 Natural gas and electric power auctions

As a next example, recall the natural monopoly problem. In many practical applications, it has often turned out that the natural monopoly characteristic applies only to a small part of an industry's activity. In these cases, vertical disintegration may help to reduce the natural monopoly problem to a minimum.

The public utility industry (gas and electric power) has traditionally been viewed as a prime example of natural monopoly. However, the presence of economies of scale in the production of energy has always been contested. An unambiguous natural monopoly exists only in the transportation and local distribution of energy. This suggests that the natural monopoly problem in public utilities can best be handled when one disintegrates production, transportation, and local distribution of energy.

In the U.K., the electric power industry has been reshaped in recent years by the privatization of production, combined with the introduction of electric power auctions. These auctions are run on a daily basis by the *National Grid Company.* At 10 a.m. every day, the suppliers of electric power make a bid for each of their generators to be operated on the following day. By 3 p.m., the National Grid Company has finalized a plan of action for the following day, in the form of merit order, ranked by bids, from low to high. Depending upon the random demand on the following day, the spot price of electric power is then determined in such a way that demand is matched by supply, according to the merit order determined on the previous day.

Similar institutional innovations have been explored, dealing with the transportation of natural gas in long–distance pipelines, and in the allocation of airport landing rights in the U.S.. Interestingly, these innovations are often evaluated in laboratory experiments, before they are put to a real life test.[63]

6.3 Treasury bill auctions

Each week the U.S. Treasury uses a discriminatory auction to sell Treasury bills (T–bills).[64] On Tuesday the Treasury announces the amount of 91–day and 182–day bills it wishes to sell on the following Monday and invites bids for specified quantities. On Thursday, the bills are issued to the successful bidders. Altogether, in fiscal year 1991 the Treasury sold over \$ 1.7 trillion of marketable Treasury securities (bills, notes, and bonds).

Prior to the early 1970's, the most common method of selling T–bills was that of a *subscription offering*. The Treasury fixed an interest rate on the securities to be sold, and then sold them at a fixed price. A major deficiency

[63]See for example Rassenti, S. J. and V. L. Smith and R. L. Bulfin [1982], "A combinatorial auction mechanism for airport time slot allocation", *Bell Journal of Economics*, 13, 402-417, and McCabe, K. and S. Rassenti and V. Smith [1989], "Designing "smart" computer–assisted markets: an experimental auction for gas networks", *Journal of Political Economy*, 99, 259-283.

[64]For more details see Tucker, J. F. [1985]. "Buying Treasury securities at Federal Reserve banks", *Federal Reserve Bank of Richmond,* Feb. 1985.

of this method of sale was that market yields could change between the announcement and the deadline for subscriptions.

The increased market volatility in the 1970's made fixed–price offerings too risky for the Treasury. Subsequently, the Treasury switched to an auction technique in which the coupon rate was still preset by the Treasury, and bids were made on the basis of price. The remaining problem with this method was that presetting the coupon rate still required forecasting interest rates, with the risk that the auction price could deviate substantially from the par value of the securities.

In 1974 the Treasury switched to auction coupon issues on a yield basis. Thereby, bids were accepted on the basis of an annual percentage yield, with the coupon rate based the weighted average yield of accepted competitive tenders received in the auction. This freed the Treasury from having to preset the coupon rate.

Another sale method was used in several auction of long–term bonds in early 1970's. This was the closed–seal bid, uniform–price auction method.[65] Here, the coupon rate was preset by the Treasury, and bids were accepted in terms of price. All successful bidders were awarded securities at the lowest price of accepted bids.

In the currently used auction technique, two kinds of bids can be submitted: *competitive* and *noncompetitive.* Competitive bidders are typically financial intermediaries who buy large quantities. A competitive bidder indicates the number of bills he wishes to buy, and the price he is willing to pay. Multiple bids are permitted. Noncompetitive bidders are typically small or inexperienced bidders. Their bids indicate the number of bills they wish to purchase (up to \$ 1,000,000), with the understanding that the price will be equal to the quantity weighted average of all accepted competitive bids.

When all bids have been made, the Treasury sets aside the bills requested by noncompetitive bidders. The remainder is allocated among the competitive bidders, beginning with the highest bidder, until the total quantity is placed. The price to be paid by noncompetitive bidders can then be calculated.

T–bill auctions are unique discriminatory auctions because of the distinct treatment of competitive and noncompetitive bids. Compared to the standard discriminatory auction, there is an additional element of uncertainty, because competitive bidders do not know the exact amount of bills auctioned to them.

Another distinct feature is the presence of a secondary after–auction market, and of pre–auction trading on a "when–issued" basis, which serves a "price–discovering" purpose, and where dealers typically engage in short–sales.

[65] In the financial community, a single–price, multiple–unit auction is often called "Dutch" (except by the English, who call it "American"). Be aware of the fact that in the academic literature one would never call it Dutch. Because of its second–price quality one would perhaps call it "English".

In many countries, central banks use similar discriminatory auctions to sell *short term repurchase agreements* to provide banks with short term liquidity.[66] Several years ago, the German Bundesbank switched from a uniform–price auction to a multiple–price, discriminatory auction. The uniform–price auction had induced small banks to place very high bids, because this gave them sure access to liquidity without the risk of having a significant impact on the price to be paid. Subsequently, large banks complained that this procedure put them at a competitive disadvantage, and the Bundesbank responded by switching to a discriminatory auction. However, this problem could also have been solved without changing auction procedures simply by introducing a finer grid of feasible bids.

In recent years, the currently used discriminatory auction procedures became the subject of considerable public debate. Many observers claimed that discriminatory auctions invite strategic manipulations, and perhaps paradoxically, lead to unnecessarily low revenues. Based on these observations, several prominent economists proposed to replace the discriminatory by uniform–price auction procedures.

The recent policy debate was triggered by an inquiry into illegal bidding practices by one of the major security dealers, *Salomon Brothers*. Apparently, during the early 1990's, Salomon Brothers repeatedly succeeded to "corner" the market, by buying up to 95 % of a security issue. This was in violation of U.S. regulations that do not allow a bidder to acquire more than 35 % of an issue.[67] Such "cornering" of the market tends to be profitable because many security dealers engage in short–sales during the time when an issue is announced and the time it is actually issued. This makes them vulnerable to a short squeeze.

Typically, a short squeeze develops during the "when–issued" period before a security is auctioned and settled. During this time, dealers already sell the soon–to–be–available securities and thus incur an obligation to deliver at the issue date. Of course, dealers must later cover this position either by buying back the security at some point in the "when–issued" market, or in the auction, or in the post–auction secondary market or any combination of these. If those dealers who are short do not bid aggressively enough in the auction, they may have difficulties to cover their positions in the secondary market.

[66] The procedures used by the Federal Reserve are described in *The Federal Reserve System: Purposes and Functions*. Board of Governors of the Federal Reserve System, Washington D.C.

[67] Salomon Brothers circumvented the law by placing unauthorized bids in the names of customers and employees, at several Treasury auctions. When these practices leaked, the Treasury security market suffered a substantial loss of confidence. For a detailed assessment of this crisis and some recommended policy changes consult the *Joint Report on the Government Securities Market* [1992], U. S. Government Printing Office, Washington, D.C.

Are discriminatory auctions the best choice? Or should central banks and the Treasury go back to single–price auction procedures? This is still an exciting and important research issue. Already during the early 1960's Milton Friedman[68] made a strong case in favor of a uniform–price, closed–seal bid. He asserted that this would end cornering attempts by eliminating gains from market manipulation. And, perhaps paradoxically, he also claimed that total revenue would go up by surrendering the possibility to price–discriminate. Essentially, both claims are based on the expectation that the switch in auction rules would completely unify the primary and secondary markets, and induce bidders to reveal their true willingness to pay.[69]

References

[1] O. Ashenfelter. How auctions work for wine and art. *Journal of Economic Perspectives*, 3:23–36, 1989.

[2] M. Bagnoli and T. Bergstrom. Log–concave probability and its applications. Working paper, University of Michigan, 1989.

[3] M. H. Bazerman and W. F. Samuelson. I won the auction but don't want the prize. *Journal of Conflict Resolution*, 27:618–634, 1983.

[4] S. Bhagat, A. Shleifer, and R. W. Vishny. Hostile takeovers in the 1980's: the return to corporate specialization. *Brookings Papers on Economic Activity*, Special Issue on Microeconomics:1–84, 1990.

[5] S. Bikchandari and C.-F. Huang. Auctions with resale markets: An exploratory model of Treasury bill markets. *Journal of Financial Studies*, 2:311–339, 1989.

[6] K. Binmore. *Fun and Games. A Text on Game Theory.* D. C. Heath and Company, 1992.

[7] W. J. Boyes and S. K. Happel. Auctions as an allocation mechanism in academia: the case of faculty offices. *Journal of Economic Perspectives*, 3:37–40, 1989.

[68] See Friedman, M. [1964]. "Comment on collusion in the auction market for Treasury bills", *Journal of Political Economy, 72: 513-514*, and more recently Friedman, M. [1991]. "How to sell government securities", *Wall Street Journal*, August 28.

[69] For a discussion of Friedman's ideas, see Goldstein, H. [1962]. "The Friedman proposal for auctioning Treasury bills", *Journal of Political Economy*, 70: 386-392, and Bikchandari, S. and C.-F. Huang [1989]. "Auctions with resale markets: an exploratory model of Treasury bill markets", *The Review of Financial Studies*, 2: 311-339.

[8] N. F. Brady, R. C. Breeden, and A. Greenspan. *Joint Report on the Government Securities Market.* U. S. Government Printing Office, 1992.

[9] J. Bulow and J. Roberts. The simple economics of optimal auctions. *Journal of Political Economy*, 97:1060–1090, 1989.

[10] E. C. Capen, R. V. Clapp, and W. M. Campbell. Competitive bidding in high–risk situations. *Journal of Petroleum Technology*, 23:641–653, 1971.

[11] R. Cassady. *Auctions and Auctioneering.* University of California Press, 1967.

[12] J. Cox, V. Smith, and J. Walker. Theory and behavior of multiple unit discriminative auctions. *Journal of Finance*, 39:983–1010, 1984.

[13] H. A. David. *Order Statistics.* Wiley, 1970.

[14] J. P. Dessauer. *Book Publishing.* Bowker, N.Y., 1981.

[15] D. Dyer, J. Kagel, and D. Levin. Resolving uncertainty about the number of bidders in independent private–value auctions: an experimental analysis. *Rand Journal of Economics*, 20:268 ff., 1989.

[16] M. Friedman. Comment on 'Collusion in the auction market for treasury bills'. *Journal of Political Economy*, 72:513–514, 1964.

[17] H. Goldstein. The Friedman proposal for auctioning Treasury bills. *Journal of Political Economy*, 70:386–392, 1962.

[18] D. A. Graham and R. C. Marshall. Collusive bidder behavior at single–object second–price and English auctions. *Journal of Political Economy*, 95:1217–1239, 1987.

[19] R. G. Hansen. Auctions with endogenous quantity. *Rand Journal of Economics*, 19:44–58, 1988.

[20] M. Harris and A. Raviv. A theory of monopoly pricing schemes with demand uncertainty. *American Economic Review*, 71:347–365, 1981.

[21] R. M. Harstad and D. Levin. A class of dominance solvable common–value auctions. *Review of Economic Studies*, 52:525–528, 1985.

[22] J. Kagel, R. M. Harstad, and D. Levin. Information impact and allocation rules in auctions with affiliated private values: a laboratory study. *Econometrica*, 55:1275–1304, 1987.

[23] J. H. Kagel and D. Levin. The winner's curse and public information in common value auctions. *American Economic Review*, 76:894–920, 1986.

[24] J. H. Kagel and D. Levin. Independent private value auctions: bidder behavior in first–, second–, and third–price auctions with varying numbers of bidders. *Economic Journal*, 103:868–879, 1993.

[25] S. M. Kendall and A. Stuart. *The Advanced Theory of Statistics*, volume I. Charles Griffin & Co., 1977.

[26] D. M. Kreps and J. A. Scheinkman. Quantity precommitment and Bertrand competition yield Cournot outcomes. *Bell Journal of Economics*, 14:326–337, 1983.

[27] E. Maskin and J. G. Riley. Auction theory with private values. *American Economic Review*, 75:150–155, 1985.

[28] E. Maskin and J. G. Riley. Optimal multi–unit auctions. In F. Hahn, editor, *The Economics of Missing Markets, Information, and Games*, pages 312–335. Clarendon Press, 1989.

[29] S. Matthews. Selling to risk averse buyers with unobservable tastes. *Journal of Economic Theory*, 30:370–400, 1983.

[30] R. McAfee, R. Preston, and J. McMillan. Auctions with a stochastic number of bidders. *Journal of Economic Theory*, 43:1–19, 1987.

[31] K. McCabe, S. Rassenti, and V. Smith. Designing "smart" computer–assisted markets: an experimental auction for gas networks. *Journal of Political Economy*, 99:259–283, 1989.

[32] P. Milgrom. Auctions and bidding: a primer. *Journal of Economic Perspectives*, 3:3–22, 1989.

[33] P. Milgrom and R. J. Weber. A theory of auctions and competitive bidding. *Econometrica*, 50:1089–1122, 1982.

[34] P. R. Milgrom. Auction theory. In T. F. Bewley, editor, *Advances in Economic Theory*, pages 1–32. Econometric Society Monographs No. 12, Cambridge University Press, Cambridge, 1987.

[35] R. B. Myerson. Incentive compatibility and the bargaining problem. *Econometrica*, 47:61–73, 1979.

[36] R. B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6:58–73, 1981.

[37] R. B. Myerson. The basic theory of optimal auctions. In R. Engelbrecht-Wiggans, M. Shubik, and J. Stark, editors, *Auctions, Bidding, and Contracting*, pages 149–163. New York University Press, New York, 1983.

[38] M. Plum. Characterization and computation of Nash–equilibria for auctions with incomplete information. *International Journal of Game Theory*, 20:393–418, 1992.

[39] S. J. Rassenti, V. L. Smith, and R. L. Bulfin. A combinatorial auction mechanism for airport time slot allocation. *Bell Journal of Economics*, 13:402–417, 1982.

[40] J. G. Riley. Expected revenue from open and sealed bid auctions. *Journal of Economic Perspectives*, 3:41–50, 1989.

[41] M. Robinson. Collusion and the choice of auction. *Rand Journal of Economics*, 16:141–145, 1985.

[42] R. Roll. The hubris hypothesis of corporate takeovers. *Journal of Business*, 59:197–216, 1986.

[43] M. H. Rothkopf, T. J. Teisberg, and E. P. Kahn. Why are Vickrey auctions rare? *Journal of Political Economy*, 98:94–109, 1990.

[44] J. Scott and C. Wolf. The efficient diversification of bids in Treasury bill auctions. *Review of Economics and Statistics*, 60:280–287, 1978.

[45] M. Shubik. Auctions, bidding, and markets: an historical sketch. In R. Engelbrecht-Wiggans, M. Shubik, and J. Stark, editors, *Auctions, Bidding, and Contracting*, pages 165–191. New York University Press, New York, 1983.

[46] P. A. Spindt and R. W. Stolz. The expected stop–out price in a discriminatory auction. *Economics Letters*, 31:133–137, 1989.

[47] J. F. Tucker. Buying treasury securities at federal reserve banks. *Federal Reserve Bank of Richmond*, 1985.

[48] W. Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16:8–37, 1961.

[49] R. J. Weber. Multiple–object auctions. In R. Engelbrecht-Wiggans, M. Shubik, and J. Stark, editors, *Auctions, Bidding, and Contracting*, pages 165–191. New York University Press, New York, 1983.

[50] R. B. Wilson. A bidding model of perfect competition. *Review of Economic Studies*, 44:511–518, 1977.