Mean Distance between Random Points in a Unit Sphere

We solve this by looking at a more general case first, f(R), which is the mean distance between random points in a sphere of radius R, and we declare that the value of f(1) is some unknown constant k.

$$f(1) = k$$

Since the radius R is a linear scaling factor for the uniform distribution of random points, f(R) should grow linearly with R.

$$f(R) = Rk$$

Furthermore, the derivate is k, which happens to also be f(1).

$$f'(R) = k = f(1)$$

We introduce a new free variable δR whose domain is $(0, R)$, and two new functions $g(R, \delta R)$ and $h(R, \delta R)$.

- $g(R, \delta R)$ is the average distance between any two random points in the sphere of radius R, where exactly one of the points more than $R - \delta R$ from the center of the sphere.
- $h(R, \delta R)$ is the average distance between any two random points in the sphere of radius R, where both of the points is more that $R - \delta R$ from the center of the sphere.

Using these two functions and our new variable δR, we define $f(R)$ recusively as the weighted average of each of three averages: one for the case where both points are less than $R - \delta R$ from the center, one for the case where one point is further away than that, and a third case where both points are further away than that. Each weight corresponds to the likelihood that points drawn uniformly from the sphere satify its respective condition.

$$\begin{aligned} f(R) = f(R - \delta R) & \qquad \left[\left(\frac{R - \delta R}{R}\right)^3\right]^2 && (1) \\ + g(R, \delta R) & \qquad 2\left(\frac{R - \delta R}{R}\right)^3 \left[1 - \left(\frac{R - \delta R}{R}\right)^3\right] && (2) \\ + h(R, \delta R) & \qquad \left[1 - \left(\frac{R - \delta R}{R}\right)^3\right]^2 && (3) \end{aligned}$$

Subtract $f(R-\delta R)$ from both sides.

$$f(R)-f(R-\delta R)=\left[\left(\frac{R-\delta R}{R}\right)^6-1\right]f(R-\delta R) \tag{4}$$
$$+2\left[\left(\frac{R-\delta R}{R}\right)^3-\left(\frac{R-\delta R}{R}\right)^6\right]g(R,\delta R) \tag{5}$$
$$+\left[1-2\left(\frac{R-\delta R}{R}\right)^3+\left(\frac{R-\delta R}{R}\right)^6\right]h(R,\delta R) \tag{6}$$

Divide both sides by δR

$$\frac{f(R)-f(R-\delta R)}{\delta R}=\left[\frac{(R-\delta R)^6}{R^6\delta R}-\frac{1}{\delta R}\right]f(R-\delta R) \tag{7}$$
$$+2\left[\frac{(R-\delta R)^3}{R^3\delta R}-\frac{(R-\delta R)^6}{R^6\delta R}\right]g(R,\delta R) \tag{8}$$
$$+\left[\frac{1}{\delta R}-2\frac{(R-\delta R)^3}{R^3\delta R}+\frac{(R-\delta R)^6}{R^6\delta R}\right]h(R,\delta R) \tag{9}$$

Partial polynomial expansion and like term consolidation…

$$\begin{aligned}
\frac{f(R)-f(R-\delta R)}{\delta R} &= \left[\frac{R^6}{R^6\delta R} - \frac{6R^5\delta R}{R^6\delta R} + \delta R(...) - \frac{1}{\delta R}\right] f(R-\delta R) && (10)\\
&+ 2\left[\frac{R^3}{R^3\delta R} - \frac{3R^2\delta R}{R^3\delta R} + \delta R(...) - \frac{R^6}{R^6\delta R} + \frac{6R^5\delta R}{R^6\delta R} + \delta R(...)\right] g(R,\delta R) && (11)\\
&+ \left[\frac{1}{\delta R} - 2\frac{R^3}{R^3\delta R} - \frac{6R^2\delta R}{R^3\delta R} + \delta R(...) + \frac{R^6}{R^6\delta R} + \frac{6R^5\delta R}{R^6\delta R} + \delta R(...)\right] h(R,\delta R) && (12)\\
&= \left[\frac{1}{\delta R} - \frac{6}{R} + \delta R(...) - \frac{1}{\delta R}\right] f(R-\delta R) && (13)\\
&+ 2\left[\frac{1}{\delta R} - \frac{3}{R} + \delta R(...) - \frac{1}{\delta R} + \frac{6}{R} + \delta R(...)\right] g(R,\delta R) && (14)\\
&+ \left[\frac{1}{\delta R} - \frac{2}{\delta R} + \frac{6}{R} + \delta R(...) + \frac{1}{\delta R} - \frac{6}{R} + \delta R(...)\right] h(R,\delta R) && (15)\\
&= \left[-\frac{6}{R} + \delta R(...)\right] f(R-\delta R) + 2\left[\frac{3}{R} + \delta R(...)\right] g(R,\delta R) + \left[\delta R(...)\right] h(R,\delta R) && (16)
\end{aligned}$$

The limit as $\delta R \to 0$ of the expression on the left defines the derivate of f(R).

$$\lim_{\delta R\to 0} \frac{f(R)-f(R-\delta R)}{\delta R} = f'(R)$$

$$f'(R) = \lim_{\delta R\to 0} \left[-\frac{6}{R} + \delta R(...)\right] f(R-\delta R) + 2\left[\frac{3}{R} + \delta R(...)\right] g(R,\delta R) + \left[\delta R(...)\right] h(R,\delta R)$$

Taking the limit, we see that all of the unexpanded polynomial terms are all of the form of $C\delta R^n f(R,\delta R)$, where C is some constant and f is bound between $[0, 2R]$. So those terms all go to zero with δR. The entire coefficient of the function h goes to 0 also.

$$f'(R) = -\frac{6}{R} f(R) + \frac{6}{R} \lim_{\delta R\to 0} g(R,\delta R)$$

Substituting the already established $f'(R) = k$ and $f(R) = kR\ldots$

$$k = -\frac{6}{R} kR + \frac{6}{R} \lim_{\delta R\to 0} g(R,\delta R)$$

$$k = -6k + \frac{6}{R} \lim_{\delta R\to 0} g(R,\delta R)$$

$$7k = \frac{6}{R} \lim_{\delta R \to 0} g(R, \delta R)$$

$$k = \frac{6}{7R} \lim_{\delta R \to 0} g(R, \delta R)$$

Now that we have an equation of k in terms of R and $g(R, \delta R)$, we define the function g at the limit of $\delta R \to 0$. Since we are taking $\delta R \to 0$, $g(R, \delta R)$ will be only the average distance between random points where exactly one of the points is on the surface of the sphere. By symmetry, we can choose an arbitrary point on the surface of the sphere and take the average distance from it to every other point in the sphere as representative of the total average.

By using polar coordinates, and placing the sphere with its origin at the (cartesian) point $(0, 0, R)$, and choosing the point $(0, 0, 0)$ as the representative point on the sphere, we need only integrate over ρ, times the Jacobian determinate $\rho^2 sin\theta$, and divide by the volume of the sphere to get the average distance from that representative point.

$$\lim_{\delta R \to 0} g(R, \delta R) = \frac{3}{4\pi R^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2Rcos(\theta)} \rho \, \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

Substituting $f(1)$ back in for k, and applying our definition of $\lim_{\delta R \to 0} g(R, \delta R)$, we have...

$$f(1) = \frac{6}{7R} \frac{3}{4\pi R^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2Rcos(\theta)} \rho \, \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi \tag{17}$$

$$= \frac{9}{14\pi R^4} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2Rcos(\theta)} \rho^3 \sin\theta \, d\rho \, d\theta \, d\phi \tag{18}$$

$$= \frac{9}{14\pi R^4} \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^4}{4} \sin\theta \Big|_0^{2Rcos(\theta)} d\theta \, d\phi \tag{19}$$

$$= \frac{9}{14\pi R^4} \int_0^{2\pi} \int_0^{\pi/2} 4R^4 cos^4\theta \sin\theta \, d\theta \, d\phi \tag{20}$$

$$= \frac{18}{7\pi} \int_0^{2\pi} \int_0^{\pi/2} cos^4\theta \sin\theta \, d\theta \, d\phi \tag{21}$$

Integration by substitution with: $u = cos\theta, du = -sin\theta \, d\theta$

$$
\begin{aligned}
f(1) &= \frac{18}{7\pi} \int_0^{2\pi} - \int_{cos(0)}^{cos(\pi/2)} u^4 \, du \, d\phi && (22) \\
&= \frac{18}{7\pi} \int_0^{2\pi} - \frac{u^5}{5} \bigg|_1^0 \, d\phi && (23) \\
&= \frac{18}{7\pi} \int_0^{2\pi} \frac{1}{5} \, d\phi && (24) \\
&= \frac{18}{35\pi} \int_0^{2\pi} 1 \, d\phi && (25) \\
&= \frac{18}{35\pi} 2\pi && (26) \\
&= \frac{36}{35} && (27)
\end{aligned}
$$