

THE GEOMETRY OF CONJUGATION IN AFFINE COXETER GROUPS

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ABSTRACT. We develop new and precise geometric descriptions of the conjugacy class $[x]$ and coconjugation set $C(x, x') = \{y \in \overline{W} \mid yxy^{-1} = x'\}$ for all elements x, x' of any affine Coxeter group \overline{W} . The centralizer of x in \overline{W} is the special case $C(x, x)$. The key structure in our description of the conjugacy class $[x]$ is the mod-set $\text{MOD}_{\overline{W}}(w) = (w - I)R^\vee$, where w is the finite part of x and R^\vee is the coroot lattice. The coconjugation set $C(x, x')$ is then described by $\text{MOD}_{\overline{W}}(w')$ together with the fix-set of w' , where w' is the finite part of x' . For any element w of the associated finite Weyl group W , the mod-set of w is contained in the classical move-set $\text{Mov}(w) = \text{Im}(w - I)$. We prove that the rank of $\text{MOD}_{\overline{W}}(w)$ equals the dimension of $\text{Mov}(w)$, and then further investigate type-by-type the surprisingly subtle structure of the \mathbb{Z} -module $\text{MOD}_{\overline{W}}(w)$. As corollaries, we determine exactly when $\text{MOD}_{\overline{W}}(w) = \text{Mov}(w) \cap R^\vee$, in which case our closed-form descriptions of conjugacy classes and coconjugation sets are as simple as possible.

1. INTRODUCTION

The study of conjugacy classes and centralizers in Coxeter groups has a long history, going back to classical work of Frobenius, Schur, Specht [Spe37], and Young [You30]. Carter [Car72] gave the first systematic study of conjugacy classes in all finite Weyl groups, and Carter [Car72] and Springer [Spr74] studied centralizers of Coxeter elements in these groups. In his thesis from 1994, Krammer showed that there exists a polynomial-time algorithm for the conjugacy problem in all finite rank Coxeter groups (see [Kra09]). Geck and Pfeiffer gave precise descriptions of conjugacy classes in all finite Coxeter groups and characterized their minimal length elements (see [GP93] and [GP00, Chapter 3], as well as an independent proof by He and Nie [HN12]). This characterization of minimal length elements was then established for all twisted Coxeter groups by Geck, Kim, and Pfeiffer [GKP00], all affine Coxeter groups by He and Nie [HN14], and all infinite Coxeter groups by Marquis [Mar21]. Motivated by the representation theory of Hecke algebras, much of this work dating back to [GP93] has focused on obtaining minimal length representatives through the operation of cyclic shifts; that is, conjugation by one simple reflection at a time.

In this work, we take a distinctive new approach and develop precise geometric descriptions of the entire *conjugacy class*

$$[x] = \{yxy^{-1} \mid y \in \overline{W}\}$$

and *coconjugation set*

$$C(x, x') = \{y \in \overline{W} \mid yxy^{-1} = x'\}$$

of arbitrary elements x, x' in any affine Coxeter group \overline{W} . Note that $C(x, x)$ is the centralizer of x in \overline{W} .

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In our companion paper [MST24], we provide closed geometric descriptions of conjugacy classes and coconjugation sets for all split subgroups of the full isometry group of Euclidean space. Since every affine Coxeter group \overline{W} splits as a semidirect product $\overline{W} = T \rtimes W$, where $T = \{t^\lambda \mid \lambda \in R^\vee\}$ is the group of translations by elements of the associated coroot lattice R^\vee , and W is the associated finite Weyl group, all the statements in [MST24] hold true for \overline{W} .

As we explain further below, a key player in the results of [MST24] is the *mod-set* of an element $x \in \overline{W}$, which we introduce as:

$$\text{MOD}_{\overline{W}}(x) = (\mathbf{I} - x)R^\vee = (x - \mathbf{I})R^\vee.$$

The mod-set is a \overline{W} -adapted version of the classical *move-set* $\text{Mov}(x) = \text{Im}(x - \mathbf{I}) = (x - \mathbf{I})\mathbb{R}^n$. For $x = t^\lambda w \in \overline{W}$, with $\lambda \in R^\vee$ and $w \in W$, we have $\text{MOD}_{\overline{W}}(x) = \lambda + \text{MOD}_{\overline{W}}(w)$ (see [MST24, Lemma 2.2]), and hence it is enough to study mod-sets for elements of the associated finite Weyl group W . We view $\text{MOD}_{\overline{W}}(w)$ as a (free) \mathbb{Z} -submodule of R^\vee , and note that $\text{MOD}_{\overline{W}}(w)$ is thus a submodule of $\text{Mov}(w) \cap R^\vee$ (see [MST24, Lemma 2.3]).

In this paper, we give explicit algebraic descriptions of $\text{MOD}_{\overline{W}}(w)$ for all finite Weyl groups W and all $w \in W$. In particular, we determine exactly when $\text{MOD}_{\overline{W}}(w)$ equals $\text{Mov}(w) \cap R^\vee$, in which case we say that w *fills its move-set*. Combined with the results of [MST24], we thus obtain the precise geometry of all conjugacy classes and coconjugation sets in all affine Coxeter groups.

1.1. Geometric description of (co)conjugation. The work in this paper refines and builds upon the results of [MST24]. We now briefly review the main contributions of [MST24] in the setting of an affine Coxeter group $\overline{W} = T \rtimes W$. Throughout, we assume that \overline{W} is irreducible. Recall that any $x \in \overline{W}$ can be expressed uniquely as the product of a *translation part* t^λ , where $\lambda \in R^\vee$, and a *spherical part* $w \in W$. For $w \in W$, the *fix-set* $\text{Fix}(w) = \text{Ker}(w - \mathbf{I})$ is the orthogonal complement of the move-set $\text{Mov}(w) = \text{Im}(w - \mathbf{I})$.

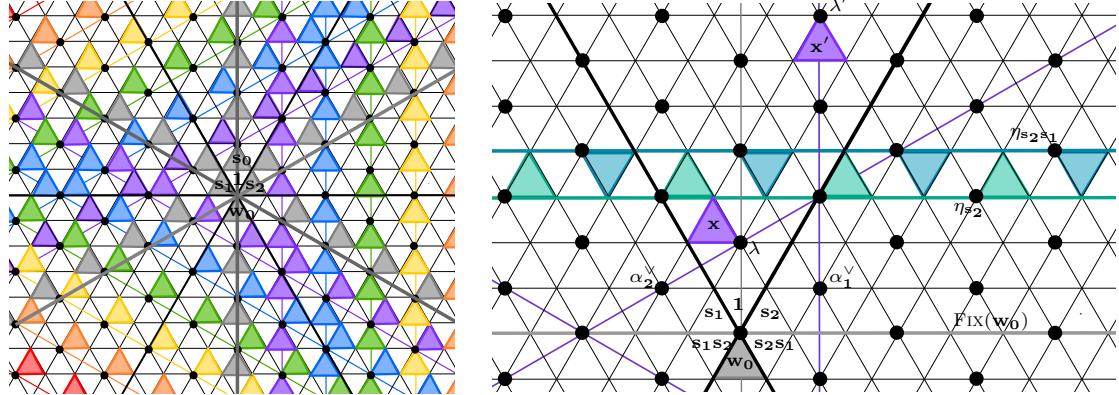


FIGURE 1. On the left, some conjugacy classes $[t^\lambda w]$ in type \tilde{A}_2 , where w is a reflection. On the right, the coconjugation set $C(x, x')$. See Example 1.1 for details.

Two of the main findings of [MST24] can now be stated informally as follows:

- For any $x = t^\lambda w \in \overline{W}$, the conjugacy class $[x]$ is obtained by first translating x by all elements of $\text{MOD}_{\overline{W}}(w) \subseteq \text{Mov}(w)$, then conjugating the so-obtained collection $t^{\text{MOD}_{\overline{W}}(w)}x$ by all elements of W .
- For any $x = t^\lambda w$ and $x' = t^{\lambda'} w'$ in \overline{W} , the coconjugation set $C(x, x')$ has a closed-form description involving $\text{MOD}_{\overline{W}}(w')$, and its shape is described by translates of $\text{Fix}(w')$.

We give formal statements of these and other results from [MST24] in Section 2.4, and illustrate them via the following examples.

Example 1.1. Let \overline{W} be the affine Coxeter group of type \tilde{A}_2 . Then $\overline{W} = T \rtimes W$ where $W \cong S_3$ is the Weyl group of type A_2 , generated by simple reflections s_1 and s_2 , and the group \overline{W} is generated by $\{s_1, s_2\}$ together with the affine simple reflection s_0 . The action of \overline{W} induces the tessellation of \mathbb{R}^2 by equilateral triangles, as depicted in Figure 1. There is a natural bijection between the elements of \overline{W} and the tiles in this tessellation, and we identify each x in \overline{W} with its corresponding triangle. The coroot lattice R^\vee is the set of heavy dots in this figure.

On the left of Figure 1, each set of triangles shaded in the same color is a conjugacy class $[t^\lambda w]$, where $w \in W$ is a reflection; that is, $w \in \{s_1, s_2, w_0\}$, where $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ is the longest element of W . For each reflection $w \in W$, the move-set $\text{Mov}(w)$ is the heavy gray line orthogonal to its fixed hyperplane, and the mod-set $\text{MOD}_{\overline{W}}(w)$ is the set of coroot lattice elements on this gray line. In other words, each reflection $w \in W$ fills its move-set. The other, colored lines on the left of Figure 1 are move-sets $\text{Mov}(t^\lambda w)$ for certain $\lambda \notin \text{MOD}_{\overline{W}}(w)$. Each conjugacy class $[t^\lambda w]$ is thus a triple of “lines” of triangles (of the same color). The remaining conjugacy classes $[t^\lambda w]$ for $w \in W$ a reflection are obtained by reflecting this picture in the vertical gray line $\text{MOD}_{\overline{W}}(w_0)$.

The right of Figure 1 depicts the coconjugation set $C(x, x')$, where $x = t^\lambda s_1$ and $x' = t^{\lambda'} w_0$. To describe $C(x, x')$, we first determine the $u \in W$ such that $u s_1 u^{-1} = w_0$ and $\lambda' - u\lambda \in \text{MOD}_{\overline{W}}(w_0)$; this gives $u \in \{s_2, s_2 s_1\}$. Then for each such u , we translate the horizontal gray line $\text{FIX}(w_0)$ by a particular solution $\eta_u \in R^\vee$ to the equation $\lambda' - u\lambda = (I - w_0)\eta$. The elements of $C(x, x')$ are the triangles $t^\mu u$ along these translates, as depicted in teal and aqua on the right of Figure 1.

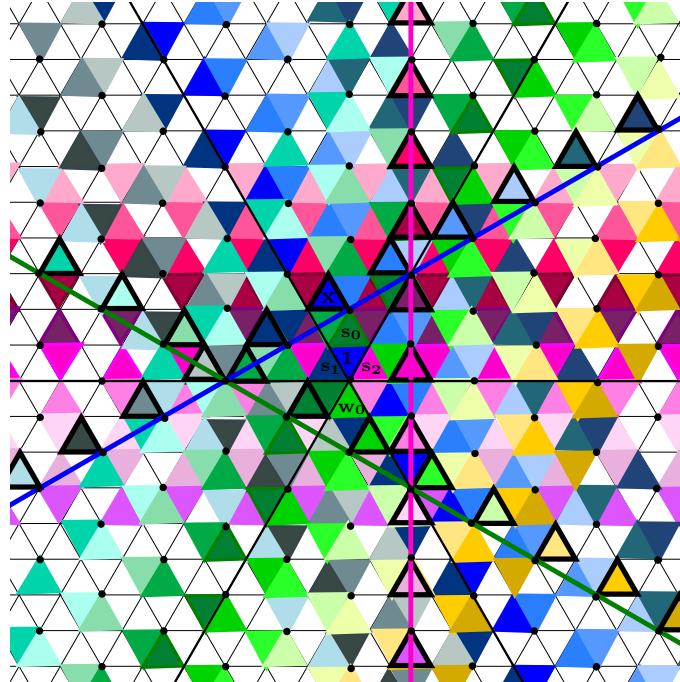


FIGURE 2. Coconjugation sets $C(x, x')$ in type \tilde{A}_2 ; see Example 1.1 for details.

In Figure 2, which is adapted from [MST23, Figure 1.1], the elements of the conjugacy class $[x]$ are outlined in black, for $x = t^\lambda s_1$ as shown in purple on the right of Figure 1. Then for each of the depicted elements $x' \in [x]$, the coconjugation set $C(x, x')$ is the collection of triangles of the same color as x' . We see that as $x' = t^{\lambda'} w'$ varies through $[x]$, the elements of $C(x, x')$ vary through translates of $\text{FIX}(w')$. In particular, the centralizer of x is the “band” of blue triangles running from top left to bottom right which contains x and the identity element 1. Note that x can be expressed as $x = s_0 s_1 s_2$, and hence is a Coxeter element of \overline{W} .

Example 1.2. Now let $\overline{W} = T \rtimes W$ be of type \tilde{C}_2 , so that W is of type C_2 . Again, W is generated by simple reflections s_1 and s_2 , and \overline{W} by $\{s_0, s_1, s_2\}$. In Figure 3, which depicts some of the conjugacy classes $[t^\lambda w]$ for $w \in W$ a reflection, we again see “lines” of conjugates. On the left of this figure, the reflections s_1 and $s_2 s_1 s_2$ fill their move-sets, as in type \tilde{A}_2 . However, on the right, the conjugacy classes leave “gaps” along the colored lines which are the move-sets. This is due to the fact that the reflections s_2 and $s_1 s_2 s_1$ do not fill their move-sets. More precisely, for $w \in \{s_2, s_1 s_2 s_1\}$, the mod-set $\text{MOD}_{\overline{W}}(w)$ is an index 2 submodule of $\text{MOV}(w) \cap R^\vee$.

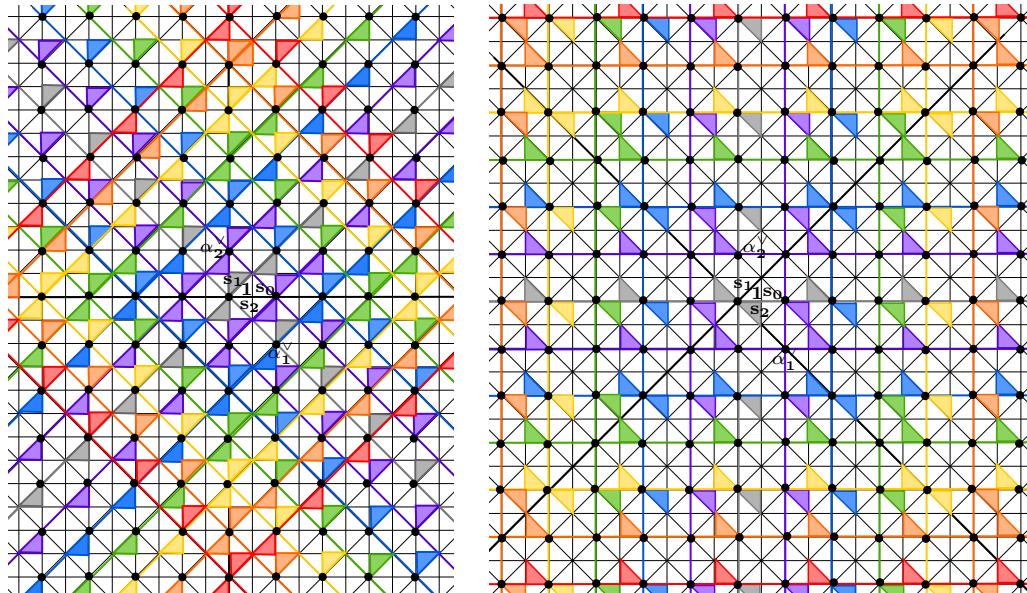


FIGURE 3. The conjugacy classes $[t^\lambda w]$ in type \tilde{C}_2 for $w \in \{s_1, s_2 s_1 s_2\}$ fill their move-sets (on the left), but do not fill their move-sets for $w \in \{s_2, s_1 s_2 s_1\}$ (on the right); see Example 1.2 for details.

Remark 1.3. We call each “line” of same-color triangles on the left of Figure 1 and in both parts of Figure 3 a *component* of the corresponding conjugacy class. The “global” action of \overline{W} on itself by conjugation can then be described as follows: the conjugation action of W permutes the components of any conjugacy class, while that of the translation subgroup T induces a transitive action by translations within each component.

We also observe that linearization induces a natural surjection from the components of any $[t^\lambda w]$ to the components of $[w]$. Although this surjection is sometimes a bijection, as on the left of Figure 1, in general it is not injective, as seen in Figure 3. There is moreover a natural bijection between the components of $[w]$ (the “lines” of gray triangles in these figures) and

the elements of the conjugacy class of w in \overline{W} . We prove the statements sketched in this remark for arbitrary \overline{W} in [MST24].

Remark 1.4. The components of a conjugacy class and the conjugation action of \overline{W} on itself, as described in the first paragraph of Remark 1.3, are closely related to the minimal length conjugacy class representatives found in [HN14] and (the affine case of) [Mar21]. To explain this, let $\overline{W} = T \rtimes W$ be an irreducible affine Coxeter group, so that W is generated by the simple reflections $\{s_1, \dots, s_n\}$, and \overline{W} has Coxeter generating set $\overline{S} = \{s_0, s_1, \dots, s_n\}$, where s_0 is the affine simple reflection. Then elements $x, x' \in \overline{W}$ are related by a *cyclic shift* if there is a reduced word $s_{i_1} \cdots s_{i_k}$ for x , with $s_{i_j} \in \overline{S}$ for $1 \leq j \leq k$, such that $x' = s_{i_2} \cdots s_{i_k} s_{i_1}$ or $x' = s_{i_k} s_{i_1} \cdots s_{i_{k-1}}$. The characterization mentioned in the first paragraph of this introduction is equivalent to the statement that any element of a conjugacy class can be transformed into a minimal length element via a sequence of cyclic shifts (see the introduction to [Mar21]).

Now each cyclic shift corresponds to conjugation by an element $s_i \in \overline{S}$. If $1 \leq i \leq n$, then conjugation by s_i permutes the set of components of the conjugacy class $[x]$. If $i = 0$, then $s_0 = t^{\theta^\vee} s_\theta$, where s_θ is the reflection corresponding to the highest root θ and θ^\vee is the corresponding coroot. Conjugation by s_θ again permutes the set of components of $[x]$, but conjugation by t^{θ^\vee} induces a translation within each component. That is, t^{θ^\vee} induces a shift along each of the “lines” of triangles in Figures 1 and 3 above. The effect of iterating certain cyclic shifts on any given $x \in \overline{W}$ is thus to spiral in gradually towards the minimal length elements of its conjugacy class, sometimes by switching components, and sometimes by translating within the current component.

Remark 1.5. For $x \in \overline{W}$, the coconjugation sets $C(x, x')$ include as a special case $C(x, x)$, the centralizer of x . If x is a Coxeter element in an arbitrary irreducible Coxeter group (of finite rank), then the centralizer of x is just the cyclic subgroup $\langle x \rangle$; this was established by Carter [Car72] for finite Weyl groups, by Blokhina [Blo89] for infinite simply-laced or affine Coxeter groups, and recently by Hollenbach and Wegener [HW22] in full generality. For instance, as noted in the last paragraph of Example 1.1, the element $x = t^{\alpha_1^\vee + \alpha_2^\vee} s_1$ in Figure 2 is a Coxeter element in \overline{W} of type \tilde{A}_2 . Hence the elements of $C(x, x) = \langle x \rangle$ are certain alcoves lying along certain translates of the hyperplane $\text{Fix}(s_1)$.

Remark 1.6. Let x be an arbitrary infinite-order element of an irreducible affine Coxeter group \overline{W} . Marquis [Mar23, Theorem D] describes the algebraic structure of the conjugacy class and centralizer of x in terms of an associated point $\eta = \eta_x$ in the visual boundary $\partial \overline{W}$ of \overline{W} ; for \overline{W} of rank $n+1$, this boundary can be identified with the sphere \mathbb{S}^{n-1} . The point $\eta = \eta_x \in \partial \overline{W}$ is defined to be the endpoint of some (hence any) *axis* for x , where an axis is a line in \mathbb{R}^n on which x acts by translations. Marquis then associates several groups and a *transversal complex* Σ^η to the point $\eta \in \partial \overline{W}$, and uses these to give algebraic descriptions of the conjugacy class $[x]$ and the centralizer $C(x, x)$.

For example, for the glide-reflection $x = t^{\alpha_1^\vee + \alpha_2^\vee} s_1$ depicted in Figure 2, the point $\eta_x \in \partial \overline{W}$ is the top left endpoint of the line which runs halfway between $\text{Fix}(s_1)$ and $(\alpha_1^\vee + \alpha_2^\vee) + \text{Fix}(s_1)$ (compare [Mar23, Figure 1]). The associated group \overline{W}^η (in Marquis’ notation) is the reflection subgroup of \overline{W} of type \tilde{A}_1 generated by s_1 and $s_0 s_2 s_0$, and Σ^η is the corresponding Coxeter complex. We observe that, in this example, the mod-set of the spherical part of x , namely $\mathbb{Z}\alpha_1^\vee$, can then be identified with the translation subgroup of \overline{W}^η ; it would be interesting to relate the mod-set to the group \overline{W}^η and its action upon Σ^η in general. The centralizer of x in Figure 2 is then described in terms of the elements of \overline{W} which fix η and other data (see part (6) of [Mar23, Theorem D]).

Remark 1.7. Our results in [MST24] include an algorithm to solve the conjugacy problem and compute coconjugation sets in all split crystallographic groups, and hence in all affine Coxeter groups. As discussed in [MST24, Section 4], we have not yet implemented this algorithm nor investigated its complexity, and we expect both of these tasks to be substantial.

We note that Krammer [Kra09, Section 4.2] gives an algorithm for the conjugacy problem in affine Coxeter groups which uses their semidirect product structure and has linear runtime. It would be desirable to further compare his algorithm to ours.

1.2. Mod-sets and move-sets. We now highlight the results of this paper in greater detail. Our first main theorem establishes a close relationship between the \mathbb{Z} -module $\text{MOD}_{\overline{W}}(w) = (\mathbf{I} - w)R^\vee$, the subspace $\text{Mov}(w) = \text{Im}(\mathbf{I} - w)$, and reflection length, for arbitrary $w \in W$. Recall that the *reflection length* of $w \in W$, which we denote by $\ell_R(w)$, is the minimal integer k such that w is a product of k reflections in W . The following result appears as Corollary 3.6; see Section 3 for more details and precise definitions.

Theorem 1.8. *Let $\overline{W} = T \rtimes W$ be an affine Coxeter group. Then for all $w \in W$,*

$$\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(w)) = \dim_{\mathbb{R}}(\text{Mov}(w)) = \ell_R(w).$$

The first equality in this theorem tells us that the move-set is the “enveloping subspace” of the mod-set. That is, the geometry of conjugacy classes in \overline{W} is coarsely described by the linear subspaces comprising the move-sets of the elements of W (as seen in Figures 1 and 3).

Corollary 1.9. *Let $\overline{W} = T \rtimes W$ be an affine Coxeter group. Then for all $w \in W$,*

- (1) $\text{MOD}_{\overline{W}}(w)$ is a finite-index submodule of $\text{Mov}(w) \cap R^\vee$; and
- (2) $\text{MOD}_{\overline{W}}(w) = \text{Mov}(w) \cap R^\vee$ if and only if $R^\vee/\text{MOD}_{\overline{W}}(w)$ is torsion-free.

Part (1) here says that any “gaps” between elements in the same component of a conjugacy class are bounded and constant within conjugacy classes (see the right of Figure 3). Part (2) then gives a criterion for an element to fill its move-set. See Section 1.3 for additional discussion of Corollary 1.9 and its consequences.

In fact, we prove the first equality in Theorem 1.8 as well as Corollary 1.9 for all split crystallographic groups which are contained in affine Coxeter groups. See Theorem 3.7 for the precise statement, and Remark 1.13 below regarding the relationship between crystallographic groups and affine Coxeter groups.

Remark 1.11. The tesselation of Euclidean space induced by the action of \overline{W} is sometimes known as the Coxeter complex of \overline{W} . Now any diagram automorphism τ of the affine Coxeter group \overline{W} induces a (non-type-preserving) automorphism of the Coxeter complex. Hence if $x, y \in \overline{W}$ and $y = \tau(x)$, we can obtain the conjugacy class of y from that of x by applying the corresponding automorphism of the Coxeter complex to $[x]$.

For example, in type \tilde{A}_2 , any permutation of the affine Coxeter generating set $\overline{S} = \{s_0, s_1, s_2\}$ induces a diagram automorphism of \overline{W} . Swapping s_1 and s_2 induces a reflection in the vertical gray line $\text{Mov}(w_0)$ on the left of Figure 1; as already mentioned in Example 1.1, this reflection yields the remaining conjugacy classes of the form $[t^\lambda w]$ where w is a reflection in W . A cyclic permutation of the elements of \overline{S} induces a diagram automorphism which cyclically permutes the 3 conjugacy classes depicted in Figure 4.

In type \tilde{C}_2 , the only nontrivial diagram automorphism swaps s_0 and s_2 , and this induces a reflection of the Coxeter complex (in a line from top left to bottom right which is not depicted in Figure 3). On the left of Figure 3, this reflection preserves all conjugacy classes, while on the right, it yields the remaining conjugacy classes of the form $[t^\lambda w]$ where $w \in \{s_2, s_1 s_2 s_1\}$.

Example 1.10. Suppose $\overline{W} = T \rtimes W$ is of type \tilde{A}_2 . The rotations in W are the elements $w \in \{s_1s_2, s_2s_1\}$, and for these w , we have $\text{Mov}(w) = \mathbb{R}^2$, therefore $\text{Mov}(w) \cap R^\vee = R^\vee$. On the other hand, we have $\text{MOD}_{\overline{W}}(w) = \{c_1\alpha_1^\vee + c_2\alpha_2^\vee \mid c_1 + c_2 \equiv 0 \pmod{3}\}$, where $c_1, c_2 \in \mathbb{Z}$ and $\{\alpha_1^\vee, \alpha_2^\vee\}$ are the simple coroots. Hence $\text{MOD}_{\overline{W}}(w)$ has index 3 in $\text{Mov}(w) \cap R^\vee$. In particular, w does not fill its move-set.

In Figure 4, the set $\text{MOD}_{\overline{W}}(w)$ for $w \in W$ a rotation is shown by the gray dots, and the 2 other cosets of $\text{MOD}_{\overline{W}}(w)$ in R^\vee are depicted by light and dark purple dots, respectively. Note that $\text{MOD}_{\overline{W}}(w)$ is W -invariant, while for any $\lambda \notin \text{MOD}_{\overline{W}}(w)$, the W -orbit of $\lambda + \text{MOD}_{\overline{W}}(w)$ has 2 elements, corresponding to the 2 shades of purple dots in this figure. The conjugacy class $[t^\lambda w]$ for any $\lambda \in \text{MOD}_{\overline{W}}(w)$ is the set of gray triangles, while the 2 conjugacy classes $[t^\lambda w]$ for $\lambda \notin \text{MOD}_{\overline{W}}(w)$ are shown as light and dark pink triangles, respectively.

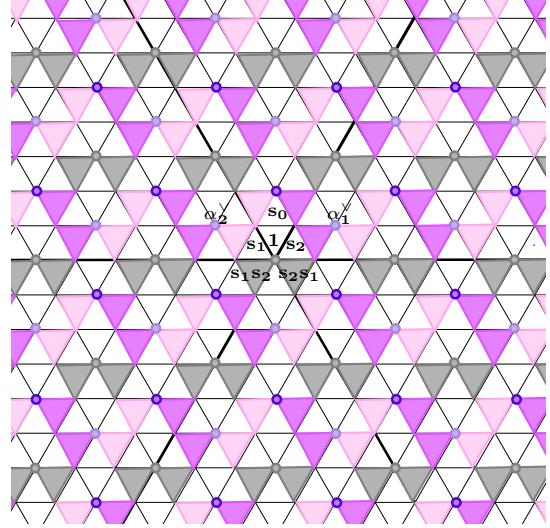


FIGURE 4. Conjugacy classes $[t^\lambda w]$ in type \tilde{A}_2 , where w is a rotation; see Example 1.10 for details.

Our proof of Theorem 1.8 relies upon a result of Carter [Car72, Lemma 3] which characterizes minimal length reflection presentations in the finite Weyl group W . We recall this statement as Theorem 2.2 below. Carter's proof of this result yields that the dimension of $\text{Mov}(w)$ equals the reflection length of $w \in W$ (see Corollary 2.3). With some care, we adapt Carter's argument to work over \mathbb{Z} , and hence determine that $\text{MOD}_{\overline{W}}(w)$ has rank equal to $\ell_R(w)$. We then extend the first equality in Theorem 1.8 to all split crystallographic subgroups of \overline{W} , and prove Corollary 1.9 in this level of generality. We note that our proof of Theorem 1.8 is type- and rank-free, essentially because Carter's proof of Theorem 2.2 is.

Remark 1.12. After publishing the first version of this work on the arXiv, Dermenjian and Evetts directed us to their recent preprint [DE23], in which they study conjugacy class growth for finitely generated virtually abelian groups. Lemma 4.5 in [DE23] provides a characterization of conjugacy classes similar to Theorem 2.12 below using an algebraic approach involving commutator subgroups in place of Mod-sets. There is also a connection between the proof of Theorem 4.11 in [DE23] and our proof of Theorem 1.8.

Remark 1.13. It seems to be folklore that every (split) crystallographic group is contained in an affine Coxeter group in dimension $n = 2, 3$, but that this no longer holds for $n \geq 4$. We sketch how to obtain these facts from the literature in Appendix A. We do not know if the conclusions of Theorem 3.7 are true for split crystallographic groups which are not contained in affine Coxeter groups.

1.3. Structure of mod-sets. To complete the description of conjugacy classes in all affine Coxeter groups, we provide explicit descriptions of the mod-sets $\text{MOD}_{\overline{W}}(w) = (\mathbf{I} - w)R^\vee$ for all $w \in W$. The examples given above already show the delicate behavior of these \mathbb{Z} -modules, and our detailed results are thus necessarily type-by-type. The principal work of this paper is in establishing the theorems below.

Throughout, we follow the conventions of Bourbaki [Bou02] (see Table 9). Suppose $\overline{W} = T \rtimes W$ is of rank n . Then in all types, W is generated (as a group) by the simple reflections $\{s_1, \dots, s_n\}$, and the corresponding coroot lattice R^\vee is generated (as a \mathbb{Z} -module) by the simple coroots $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$. Define $[n] = \{1, \dots, n\}$. To simplify notation, in this section we write $\text{MOD}(w)$ for $\text{MOD}_{\overline{W}}(w)$.

For all $u, w \in W$, we have $\text{MOD}(uwu^{-1}) = u\text{MOD}(w)$, by [MST24, Lemma 2.1]. Hence it suffices to consider the mod-sets for a single element in each conjugacy class of the finite group W . We do so using the minimal length conjugacy class representatives determined by Geck and Pfeiffer in [GP00, Chapter 3]. In the classical types, we recall and give a “visual” rephrasing of the construction of these representatives. In types B and C , this involves the normal form forests of du Cloux (see [BB05, Section 3.4]).

We first consider W the finite Weyl group of type A_n for $n \geq 1$. The conjugacy classes of W are parameterized by weakly decreasing compositions (that is, partitions) $\beta = (\beta_1, \dots, \beta_p)$ of $n+1$; see [GP00, Proposition 3.4.1], for instance. Each such β determines a subset $J_\beta \subseteq [n]$, such that $n \in J_\beta$ if and only if $\beta_p \geq 2$. The corresponding representative w_β is the product of the simple reflections $\{s_j \mid j \in J_\beta\}$ in increasing order. The set I_β records the partial sums of the parts of β , and $I_\beta - 1$ subtracts 1 from each partial sum. The following result appears as Theorem 4.2; see Section 4 for more details and precise definitions.

Theorem 1.14. *Suppose W is of type A_n with $n \geq 1$. Let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$ with corresponding conjugacy class representative $w_\beta \in W$.*

(1) *The module $\text{MOD}(w_\beta) = (\mathbf{I} - w_\beta)R^\vee$ equals*

$$\left\{ \sum_{i=1}^n c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, \quad c_i = 0 \text{ for } i \in [n] \setminus J_\beta, \quad \sum_{i=1}^n c_i \equiv 0 \pmod{\gcd(\beta_k)} \right\}.$$

(2) *If $\beta_p = 1$ then the module $\text{MOD}(w_\beta) = (\mathbf{I} - w_\beta)R^\vee$ has a \mathbb{Z} -basis given by*

$$\{\alpha_j^\vee \mid j \in J_\beta\},$$

and if $\beta_p \geq 2$ then the module $\text{MOD}(w_\beta) = (\mathbf{I} - w_\beta)R^\vee$ has a \mathbb{Z} -basis given by

$$\{\alpha_i^\vee - \alpha_{i+1}^\vee \mid i \in J_\beta \setminus (I_\beta - 1)\} \cup \{\alpha_i^\vee - \alpha_{i+2}^\vee \mid i \in (I_\beta - 1) \setminus \{n\}\} \cup \{\gcd(\beta_k) \alpha_n^\vee\}.$$

(3) *If $\beta = (1, \dots, 1)$, so that w_β is trivial, then $(\mathbf{I} - w_\beta)$ has Smith normal form $\text{diag}(0^n)$. For w_β nontrivial and any $w \in [w_\beta]$, the Smith normal form of $(\mathbf{I} - w)$ equals*

$$S_\beta = \text{diag}(1^{n-p}, \gcd(\beta_k), 0^{p-1}).$$

(4) *For any partition β and any $w \in [w_\beta]$, we have*

$$R^\vee / \text{MOD}(w) \cong (\mathbb{Z}/\gcd(\beta_k)\mathbb{Z}) \oplus \mathbb{Z}^{p-1}.$$

From part (2) of Corollary 1.9 and part (4) of Theorem 1.14, we characterize those elements of W of type A_n which fill their move-sets, as follows.

Corollary 1.15. *Suppose W is of type A_n with $n \geq 1$. Let $w \in W$, and let the conjugacy class of w in W be indexed by the partition $\beta = (\beta_1, \dots, \beta_p)$. The following are equivalent:*

- (1) *w fills its move-set; that is, $\text{MOD}(w) = \text{Mov}(w) \cap R^\vee$; and*
- (2) *$\gcd(\beta_k) = 1$.*

In particular, if $\beta_p = 1$, equivalently the conjugacy class of w in W is represented by w_β contained in the type A_{n-1} subsystem generated by $\{s_1, \dots, s_{n-1}\}$, then w fills its move-set.

Example 1.16. For W of type A_2 , the conjugacy class $\{s_1, s_2, s_1s_2s_1\}$ of reflections corresponds to the partition $\beta = (2, 1)$, and is represented by $w_\beta = s_1$. As seen in Example 1.1, we have $\text{MOD}(s_1) = \mathbb{Z}\alpha_1^\vee$. The conjugacy class $\{s_1s_2, s_2s_1\}$ of rotations corresponds to the partition $\beta = (3)$, and is represented by $w_\beta = s_1s_2$. As seen in Example 1.10, we have $\text{MOD}(s_1s_2) = \{c_1\alpha_1^\vee + c_2\alpha_2^\vee \mid c_1, c_2 \in \mathbb{Z}, c_1 + c_2 \equiv 0 \pmod{3}\}$. Our \mathbb{Z} -basis for $\text{MOD}(s_1s_2)$ is given by $\{\alpha_1^\vee - \alpha_2^\vee, 3\alpha_2^\vee\}$.

We next consider W of type C_n , for $n \geq 2$. As will be seen later in this introduction, the statements in type B_n are much more delicate than in type C_n , which is why we present type C_n first. Following [GP00, Proposition 3.4.7], the conjugacy classes of W are parameterized by ordered pairs of compositions (β, γ) such that β is weakly decreasing, γ is weakly increasing, and $|\beta| + |\gamma| = n$. Write $|\beta| = m$. Then each such pair determines subsets $J_\beta \subseteq [m-1]$ and $I_\gamma, J_\gamma \subseteq [n]$, such that $I_\gamma = \emptyset$ if and only if $|\gamma| = 0$. The corresponding representative is given by $w_{\beta, \gamma} = w_\beta w_\gamma$, where the element w_β is contained in the type A_{m-1} subsystem of W generated by $\{s_1, \dots, s_{m-1}\}$, according to our conventions (see Table 9), and is constructed exactly as sketched in the paragraph above Theorem 1.14. The element w_γ is contained in the type C_{n-m} subsystem generated by $\{s_{m+1}, \dots, s_n\}$ and is constructed from γ by a more complicated process, so that w_γ is nontrivial if and only if $|\gamma| \geq 1$; see Proposition 5.1. The following result appears as Theorem 5.2; see Section 5 for more details and precise definitions.

Theorem 1.17. Suppose W is of type C_n with $n \geq 2$. Let (β, γ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$, with corresponding conjugacy class representative $w_{\beta, \gamma} \in W$. Write $m = |\beta|$, so that $0 \leq m \leq n$ and $|\gamma| = n - m$.

(1) The module $\text{MOD}(w_{\beta, \gamma}) = (I - w_{\beta, \gamma})R^\vee$ equals

$$\left\{ \sum_{i=1}^n c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_i = 0 \text{ for } i \in [m] \setminus J_\beta, c_i \equiv 0 \pmod{2} \text{ for } i \in I_\gamma \right\}.$$

(2) The module $\text{MOD}(w_{\beta, \gamma})$ has \mathbb{Z} -basis given by

$$\{\alpha_i^\vee \mid i \in J_\beta\} \cup \{\alpha_i^\vee \mid i \in J_\gamma \setminus I_\gamma\} \cup \{2\alpha_i^\vee \mid i \in I_\gamma\}.$$

(3) For any $w \in [w_{\beta, \gamma}]$, the Smith normal form of $(I - w)$ equals

$$S_{\beta, \gamma} = \text{diag}(1^{n-p-q}, 2^q, 0^p).$$

(4) For any $w \in [w_{\beta, \gamma}]$, the quotient of R^\vee by the mod-set is

$$R^\vee / \text{MOD}(w) \cong (\mathbb{Z}/2\mathbb{Z})^q \oplus \mathbb{Z}^p.$$

We note that, in contrast to our results in type A_n (see Theorem 1.14), neither the value of $\gcd(\beta_k)$ nor the indexing set I_β appears in our statements in type C_n .

From part (2) of Corollary 1.9 and part (4) of Theorem 1.17, we characterize those elements of W of type C_n which fill their move-sets, as follows.

Corollary 1.18. Suppose W is of type C_n with $n \geq 2$. Let $w \in W$, and let the conjugacy class of w in W be indexed by the pair of compositions (β, γ) . The following are equivalent:

- (1) w fills its move-set; that is, $\text{MOD}(w) = \text{Mov}(w) \cap R^\vee$;
- (2) the conjugacy class of w in W is represented by the element w_β of the type A_{n-1} subsystem on $\{s_1, \dots, s_{n-1}\}$; and
- (3) $|\gamma| = 0$.

Example 1.19. In type C_2 , the conjugacy class $\{s_1, s_2s_1s_2\}$ corresponds to the pair of compositions (β, γ) with $\beta = (2)$ and $|\gamma| = 0$, and is represented by $w_{\beta, \gamma} = w_\beta = s_1$. Here, $J_\beta = \{1\}$ and $I_\gamma = J_\gamma = \emptyset$. As seen on the left of Figure 3, we have $\text{MOD}(s_1) = \mathbb{Z}\alpha_1^\vee$, and $w_{\beta, \gamma}$ fills its move-set. The conjugacy class $\{s_2, s_1s_2s_1\}$ corresponds to $\beta = (1)$ and $\gamma = (1)$, and is represented by $w_{\beta, \gamma} = w_\gamma = s_2$. Here, $J_\beta = \emptyset$ and $J_\gamma = I_\gamma = \{2\}$, so that $J_\gamma \setminus I_\gamma = \emptyset$. As seen on the right of Figure 3, we have $\text{MOD}(s_2) = 2\mathbb{Z}\alpha_2^\vee$, and $w_{\beta, \gamma}$ does not fill its move-set.

Next, suppose W is of type B_n for $n \geq 2$. The minimal length representatives of conjugacy classes $w_{\beta, \gamma}$ are identical to those in type C_n , and the Cartan matrices in types B_n and C_n differ only by exchanging the $(n-1, n)$ and $(n, n-1)$ entries. However, this small change alters the results on mod-sets considerably. The following result appears as Theorem 6.1; see Section 6 for more details and precise definitions. We write $\gcd(\beta_k, 2) = \gcd(\beta_1, \dots, \beta_p, 2)$, and if $\beta_p \geq 2$, we write $\gcd(\beta_k, \beta_p - 2) = \gcd(\beta_1, \dots, \beta_p, \beta_p - 2)$.

Theorem 1.20. *Suppose W is of type B_n with $n \geq 2$. Let (β, γ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$, with corresponding conjugacy class representative $w_{\beta, \gamma} \in W$. For any $w \in [w_{\beta, \gamma}]$, the Smith normal form of $(I-w)$ is as follows:*

- (1) If $\gcd(\beta_k, 2) = 2$ (including $|\beta| = 0$), $|\gamma| \geq 1$, and all parts of γ have the same parity, then

$$S_{\beta, \gamma} = \text{diag}(1^{n-q-p}, 2^q, 0^p).$$

- (2) If $\gcd(\beta_k, 2) = 2$ (including $|\beta| = 0$), $|\gamma| \geq 1$, and γ has a change in parity, then

$$S_{\beta, \gamma} = \text{diag}(1^{n-q-p+1}, 2^{q-2}, 4, 0^p).$$

- (3) If $\gcd(\beta_k, 2) = 1$ and $|\gamma| \geq 1$, then

$$S_{\beta, \gamma} = \text{diag}(1^{n-q-p+1}, 2^{q-1}, 0^p).$$

- (4) If $|\gamma| = 0$, $\beta_p \geq 2$, and $\gcd(\beta_k, \beta_p - 2) \geq 2$, then

$$S_{\beta, \gamma} = \text{diag}(1^{n-p-1}, \gcd(\beta_k, \beta_p - 2), 0^p).$$

- (5) If $|\gamma| = 0$, and either $\beta_p = 1$, or $\beta_p \geq 2$ and $\gcd(\beta_k, \beta_p - 2) = 1$, then

$$S_{\beta, \gamma} = \text{diag}(1^{n-p}, 0^p).$$

The Smith normal form is canonical and fully characterizes the isomorphism type of $R^\vee/\text{MOD}(w)$ for any $w \in [w_{\beta, \gamma}]$. Hence, the cases required to state Theorem 1.20 indicate the delicate nature of the results in type B ; compare the uniform statement for the Smith normal form in type C given by part 3 of Theorem 1.17. Many further cases and lengthy descriptions appear when finding a basis for $\text{MOD}(w_{\beta, \gamma})$, so we refer the reader directly to Theorem 6.2 for our results on \mathbb{Z} -bases for $\text{MOD}(w_{\beta, \gamma})$ when either $|\beta| = 0$ or $|\gamma| = 0$. We provide many illustrative examples in type B_n , including sketches of how to find a basis for all pairs (β, γ) , but leave the general proof of our results in type B_n to the reader, since much of the argument is similar to that in types A_n and C_n .

From part (2) of Corollary 1.9 and Theorem 1.20, we characterize those elements of W of type B_n which fill their move-sets, as follows.

Corollary 1.21. *Suppose W is of type B_n with $n \geq 2$. Let $w \in W$, and let the conjugacy class of w in W be indexed by the pair of compositions (β, γ) . The following are equivalent:*

- (1) w fills its move-set; that is, $\text{MOD}(w) = \text{Mov}(w) \cap R^\vee$; and
- (2) (a) β has at least one odd part, and γ has exactly one part; or
(b) $\beta_p \geq 2$, $\gcd(\beta_k, \beta_p - 2) = 1$, and $|\gamma| = 0$; or

(c) $\beta_p = 1$ and $|\gamma| = 0$.

We remark that for $|\beta| = m$ with $0 \leq m < n$, the composition γ having exactly one part is equivalent to w_γ being the Coxeter element $w_\gamma = s_n s_{n-1} \dots s_{m+1}$ in the type B_{n-m} subsystem generated by $\{s_{m+1}, \dots, s_n\}$.

Example 1.22. In type B_2 , as in type C_2 (see Example 1.19), the conjugacy class $\{s_1, s_2 s_1 s_2\}$ corresponds to $\beta = (2)$ and $|\gamma| = 0$, and is represented by $w_{\beta, \gamma} = w_\beta = s_1$. By case (4) of Theorem 1.20, the Smith normal form for $I - s_1$ is $\text{diag}(2, 0)$. Thus $R^\vee/\text{MOD}_{\overline{W}}(s_1) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, and s_1 does not fill its move-set in type B_2 . The conjugacy class $\{s_2, s_1 s_2 s_1\}$ corresponds to $\beta = (1)$ and $\gamma = (1)$, and is represented by $w_{\beta, \gamma} = w_\gamma = s_2$. We have $S_{\beta, \gamma} = \text{diag}(1, 0)$ by case (3) of Theorem 1.20, and so $R^\vee/\text{MOD}_{\overline{W}}(s_2) \cong \mathbb{Z}$, and s_2 fills its move-set in type B_2 . Compare Example 1.19 in type C_2 , where these situations are reversed.

Now suppose W is of type D_n for $n \geq 4$. Following [GP00, Section 3.4], the conjugacy classes of W are parameterized by ordered pairs of compositions (β, δ) such that β is weakly decreasing, δ is weakly increasing and has an even number of parts, and $|\beta| + |\delta| = n$. In the special case $(\beta, 0)$ where all parts of β are even, there are two distinct conjugacy classes, parameterized by β^+ and β^- . Let $|\beta| = m$. Then the conjugacy class representatives are given by $w_{\beta, \delta} = w_\beta w_\delta$, where w_β is contained in the type A_{m-1} subsystem generated by $\{s_1, \dots, s_{m-1}\}$ and w_δ is contained in the subsystem generated by $\{s_{m+1}, \dots, s_n\}$ (which could have various types). The following result appears as Theorem 7.1; see Section 7 for more details and precise definitions. As in type B , we write $\gcd(\beta_k, 2)$ for $\gcd(\beta_1, \dots, \beta_p, 2)$.

Theorem 1.23. Suppose W is of type D_n with $n \geq 4$. Let (β, δ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\delta = (\delta_1, \dots, \delta_{2r})$ is weakly increasing with an even number of parts, and $|\beta| + |\delta| = n$. Let $w_{\beta^\pm, \delta} \in W$ be the corresponding conjugacy class representative in W .

For any $w \in [w_{\beta^\pm, \delta}]$, the Smith normal form of $(I - w)$ is as follows:

- (1) If $\gcd(\beta_k, 2) = 2$ (including the case that $|\beta| = 0$), $|\delta| \geq 2$, and all parts of δ have the same parity, then

$$S_{\beta, \delta} = \text{diag}(1^{n-2r-p}, 2^{2r}, 0^p).$$

- (2) If $\gcd(\beta_k, 2) = 2$ (including the case that $|\beta| = 0$), $|\delta| \geq 2$, and δ has a change in parity, then

$$S_{\beta, \delta} = \text{diag}(1^{n-2r-p+1}, 2^{2r-2}, 4, 0^p).$$

- (3) If $\gcd(\beta_k, 2) = 1$ and $|\delta| \geq 2$, then

$$S_{\beta, \delta} = \text{diag}(1^{n-2r-p+1}, 2^{2r-1}, 0^p).$$

- (4) If $|\delta| = 0$ and $\beta = (1, \dots, 1)$, so that $w_{\beta, \delta}$ is trivial, then $S_{\beta, \delta} = \text{diag}(0^n)$. If $|\delta| = 0$ and $\beta \neq (1, \dots, 1)$, then

$$S_{\beta^\pm, \delta} = \text{diag}(1^{n-p-1}, \gcd(\beta_k, 2), 0^p).$$

For bases in type D_n , there are even more cases than in type B_n , and so we refer the reader directly to Theorems 7.2 and 7.3. Roughly speaking, this proliferation of cases is due to the structural differences in type D_n between n odd and n even, the various possibilities for the interaction of a type A_{m-1} subsystem with the node of valence 3 in the Dynkin diagram, and the various types possible for the subsystem generated by $\{s_{m+1}, \dots, s_n\}$. From part (2) of Corollary 1.9 and Theorem 1.23, we obtain the following result.

Corollary 1.24. Suppose W is of type D_n with $n \geq 4$. Let $w \in W$, and let the conjugacy class of w in W be indexed by the pair of compositions (β, δ) . The following are equivalent:

- (1) w fills its move-set; that is, $\text{MOD}(w) = \text{Mov}(w) \cap R^\vee$; and
- (2) β has at least one odd part, and $|\delta| = 0$.

In particular, if the conjugacy class of w in W is not represented by the element w_β of the type A_{n-1} subsystem on $\{s_1, \dots, s_{n-1}\}$, then w does not fill its move-set.

Despite the vast differences in the characterizations of mod-sets in types A , B , C , and D , note that the condition for $w \in W$ to fill its move-set is almost identical across types.

In all classical types, the results outlined in this section are obtained by the following process. After presenting the conjugacy class representatives from [GP00, Chapter 3], we determine the matrix for each such representative, with respect to the \mathbb{Z} -basis $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ for R^\vee . We then formulate algorithms involving column and/or row operations over \mathbb{Z} to obtain an explicit description of the corresponding mod-set, to find a basis for the mod-set, and to determine the Smith normal form. From the Smith normal form, one can write down the isomorphism type of the quotient of R^\vee by the mod-set, and so complete the proofs.

In order to aid readability, for the classical types we have included many examples of our algorithms in the body of this work. We caution that, in the classical types, the conventions of [Bou02], which we follow in this paper, differ from those used in [GP00, Chapter 3]. In Table 9 we have set out, side-by-side for each type, the conventions from [Bou02] and [GP00, Chapter 3].

In Section 8, we provide a complete system of minimal length representatives for all conjugacy classes of W of exceptional type. We use the tables in Appendix B of [GP00] for the cuspidal classes, and then iterate over all proper parabolic subgroups of W , choosing representatives for the cuspidal classes in each proper parabolic using the algorithms from [GP00, Chapter 3]. For each representative $w \in W$, we present the Smith normal form S_w for $(I - w)$, which is canonical and fully characterizes the isomorphism type of the quotient $R^\vee/\text{MOD}(w)$, and answers the question of whether or not w fills its move-set. In the exceptional types, the Smith normal form S_w was calculated using the `smith_form()` command in Sage [Sag24].

1.4. Discussion of related work. Our geometric approach to describing conjugacy classes and coconjugation sets is quite distinct from what has appeared in the literature. We review here some previous results on conjugacy classes and centralizers in Coxeter groups, in addition to those works already discussed in the first paragraph of the introduction and Remark 1.4. We do not know of any previous work on coconjugation sets in Coxeter groups, per se.

If G is a simple algebraic group, then any element $g \in G$ admits a generalized Jordan decomposition $g = su$, where s is semisimple and u is unipotent. The conjugacy classes of G can thus be understood by studying the semisimple and unipotent conjugacy classes within G ; see the surveys [Spa82, SS70, LS12]. Work of Lusztig cleverly exchanges the study of the unipotent conjugacy classes for the conjugacy classes of the finite Weyl group [Lus11]. Lusztig's map from conjugacy classes in W to unipotent classes in G is injective when restricted to the elliptic conjugacy classes in W , and Adams, He, and Nie generalize Lusztig's result, proving that the straight conjugacy classes in \overline{W} play the analogous role in the affine setting [AHN21]. These conjugacy classes in affine Weyl groups also enjoy a wide range of applications, playing a central role in the character theory of affine Hecke algebras, homological properties of affine Deligne-Lusztig varieties, and geometric representations of p -adic groups via an action of the cocenter of the Hecke algebra; see [HN12, HN14, HN15, CH17].

Much of the literature on centralizers, a special case of coconjugation sets, is in the setting of either finite Coxeter groups, or arbitrary Coxeter groups, the latter often motivated by the still-open isomorphism problem for Coxeter groups (discussed in [SRS24], for example). Much of this work also focuses on obtaining an algebraic description of centralizers of certain

“special” elements. For example, as discussed in Remark 1.5, the centralizer of a Coxeter element w is equal to the cyclic group $\langle w \rangle$. Richardson [Ric82] investigated centralizers of involutions in arbitrary Coxeter groups, and centralizers of reflections in arbitrary Coxeter groups have been studied by Howlett [How80], Brink [Bri96], and Allcock [All13]. For W an arbitrary finite Coxeter group, the final statement in this direction seems to be the work of Konvalinka, Pfeiffer, and Röver [KPR11], which gives an algebraic description of the centralizer of any $w \in W$; see Remark [?] for a brief discussion of Marquis’ work in the setting of arbitrary Coxeter groups.

1.5. Structure of the paper. In Section 2 we fix notation and recall some background, and then restate the main definitions and results of [MST24] in the setting of affine Coxeter groups. Section 3 establishes Theorem 1.8 and Corollary 1.9, concerning the relationship between mod-sets and move-sets. Our main results in types A_n , C_n , B_n , and D_n are presented in Sections 4, 5, 6, and 7, respectively, and we discuss the exceptional types in Section 8. Appendix A briefly considers the relationship between split crystallographic groups and affine Coxeter groups, and Appendix B describes several relevant conventions for labeling the nodes of Dynkin diagrams.

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2. PRELIMINARIES

We fix notation and collect some useful results on affine Coxeter systems in Section 2.1, then discuss move- and fix-sets in Section 2.2. We discuss cuspidal and Coxeter elements of the finite Coxeter group W in Section 2.3. In Section 2.4, we state the main definitions and results of [MST24] in the setting of affine Coxeter groups.

2.1. Affine Coxeter systems, roots, and coroots. This section reviews definitions and notation on affine Coxeter systems and their associated root systems and coroot lattices. We assume the reader is familiar with this material at the level of the reference [Hum72].

Let $(\overline{W}, \overline{S})$ be an irreducible affine Coxeter system of rank n . Let V be the associated n -dimensional real vector space on which \overline{W} acts by affine transformations, which we can identify with n -dimensional Euclidean space \mathbb{R}^n . Denote the origin of V by 0 . Then $(\overline{W}, \overline{S})$ has associated spherical Coxeter system (W, S) such that W is the stabilizer in \overline{W} of 0 and $S = \{s_1, \dots, s_n\}$ is the set of elements of $\overline{S} = \{s_0, s_1, \dots, s_n\}$ which fix 0 . We call $\{s_1, \dots, s_n\}$ the *simple reflections*. Denote the index set for S by $[n] := \{1, 2, \dots, n\}$. We typically use the letters x, y for elements of \overline{W} , and u, w for elements of W , and we write w_0 for the longest element of the spherical Coxeter group W .

The vector space V admits an ordered basis $\Delta = (\alpha_i)_{i \in [n]}$, and a symmetric bilinear form $B(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m(i,j)}$, where $m(i, j)$ is the (i, j) -entry of the associated Coxeter matrix. Given any $i \in [n]$, define a linear functional $\alpha_i^\vee \in V^*$ by $\langle \alpha_i^\vee, v \rangle := 2B(\alpha_i, v)$ for any $v \in V$,

where $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{Z}$ denotes the evaluation pairing. The ordered set $\Delta^\vee = (\alpha_i^\vee)_{i \in [n]}$ is then a basis for the dual space V^* . The elements of Δ are called the *simple roots*, and the elements of Δ^\vee are the *simple coroots*.

The action of \overline{W} on V induces an action of \overline{W} on V^* , as follows. Recall that, using the conventions in [Bou02], every generator $s_i \in S$ acts as a linear reflection:

$$(2.1.1) \quad s_i(\alpha_j^\vee) = \alpha_j^\vee - \langle \alpha_j^\vee, \alpha_i \rangle \alpha_i^\vee = \alpha_j^\vee - c_{ij} \alpha_i^\vee,$$

where c_{ij} equals the (i, j) -entry of the Cartan matrix. The fix-set of the reflection s_i is the linear hyperplane

$$\mathcal{H}_{\alpha_i} = \{z \in V^* \mid \langle z, \alpha_i \rangle = 0\}.$$

We sometimes write $s_i = s_{\alpha_i}$ for $i \in [n]$.

The vectors $\Phi = \{w\alpha_i \mid w \in W, i \in [n]\} \subset V$ are called *roots*, and $\Phi^\vee = \{w\alpha_i^\vee \mid w \in W, i \in [n]\} \subset V^*$ are the corresponding *coroots*. For any root $\beta = w\alpha_i$, the corresponding coroot β^\vee is given by $\langle \beta^\vee, v \rangle := 2B(\beta, v)$ for $v \in V$, and we write s_β for the linear reflection $ws_{\alpha_i}w^{-1} = s_{w\alpha_i}$. For $\beta = w\alpha_i$, the reflection s_β has fix-set the linear hyperplane

$$\mathcal{H}_\beta = \{z \in V^* \mid \langle z, \beta \rangle = 0\} = w\mathcal{H}_{\alpha_i}.$$

There is a distinguished lattice in V^* , which is preserved by the actions of \overline{W} and W , called the *coroot lattice* $R^\vee = \bigoplus \mathbb{Z}\alpha_i^\vee$. Denote by t^λ the translation in V^* by the coroot $\lambda \in R^\vee$. The set of all such translations $T = \{t^\lambda \mid \lambda \in R^\vee\}$ is the translation subgroup of the affine Coxeter group \overline{W} . For any $\lambda, \mu \in R^\vee$, we have $t^\lambda t^\mu = t^{\lambda+\mu} = t^{\mu+\lambda} = t^\mu t^\lambda$, and given any $w \in W$, we have $wt^\lambda w^{-1} = t^{w\lambda}$.

We may also canonically identify V and its dual V^* , and we use this identification freely throughout. Then for any root $\beta \in \Phi$, the corresponding coroot β^\vee is identified with the vector $2\frac{\beta}{(\beta, \beta)}$, where (\cdot, \cdot) is the standard inner product on \mathbb{R}^n .

2.2. Move-sets and fix-sets. The *move-set* and *fix-set* of elements of \overline{W} play an important role in our study of conjugation. They are defined as follows.

Definition 2.1. For any $x \in \overline{W}$:

- (1) the *move-set* of x is $\text{Mov}(x) = \{p \in V^* \mid q + p = x(q), q \in V^*\}$; and
- (2) the *fix-set* of x is $\text{Fix}(x) = \{p \in V^* \mid x(p) = p\}$.

For example, the reflection $s_\beta \in W$ has $\text{Mov}(s_\beta) = \mathbb{R}\beta^\vee$ and $\text{Fix}(s_\beta) = \mathcal{H}_\beta$. Note that for $i \in [n]$ we have $\text{Mov}(s_i) = \mathbb{R}\alpha_i^\vee$ and $\text{Fix}(s_i) = \mathcal{H}_{\alpha_i}$.

For all elements $w \in W$, the sets

$$\text{Mov}(w) = \text{Im}(w - \text{I}) \quad \text{and} \quad \text{Fix}(w) = \text{Ker}(w - \text{I})$$

are sub-vector spaces of V^* and in fact orthogonal complements, i.e. $V^* = \text{Mov}(w) \oplus \text{Fix}(w)$. In addition, by straightforward computation, one has

$$\text{Mov}(uwu^{-1}) = u\text{Mov}(w) \text{ for all } u, w \in W.$$

For move-sets of elements of \overline{W} , we have, by Proposition 1.21 of [LMPS19], that

$$\text{Mov}(x) = \lambda + \text{Mov}(w),$$

where $x = t^\lambda w \in \overline{W}$ with $\lambda \in R^\vee$ and $w \in W$.

We write \mathcal{R} for the set of all reflections in W . That is,

$$\mathcal{R} = \{s_\beta \mid \beta \in \Phi\} = \{ws_iw^{-1} \mid w \in W, i \in [n]\}.$$

Hence for each $r = ws_iw^{-1} \in \mathcal{R}$, we have $\text{Mov}(r) = \mathbb{R}(w\alpha_i^\vee) = w\text{Mov}(s_i)$.

The *reflection length* of an element $w \in W$, denoted $\ell_R(w)$, is the minimal integer k such that $w = r_1 \cdots r_k$ where each r_i is a reflection in W . We will use the following fundamental result of Carter [Car72].

Theorem 2.2 (Lemma 3 of [Car72]). *Let $s_{\beta_1}, \dots, s_{\beta_k} \in \mathcal{R}$ and let $w = s_{\beta_1} \cdots s_{\beta_k} \in W$. Then $\ell_R(w) = k$ if and only if the set of roots $\{\beta_1, \dots, \beta_k\} \subset \Phi$ is linearly independent over \mathbb{R} .*

The next result recalls part of the statement of [LMPS19, Lemma 1.26].

Corollary 2.3. *Let $w \in W$. Suppose $\ell_R(w) = k$, with $w = s_{\beta_1} \cdots s_{\beta_k}$ where each $s_{\beta_i} \in \mathcal{R}$. Then $\text{Mov}(w)$ has an \mathbb{R} -basis given by $\{\beta_1^\vee, \dots, \beta_k^\vee\}$, and hence $\dim_{\mathbb{R}}(\text{Mov}(w)) = \ell_R(w)$.*

Proof. As explained at the start of the proof of [BM15, Proposition 5.1], since each s_{β_i} moves points just in the β_i^\vee -direction, we have $\text{Mov}(w) \subseteq \text{SPAN}_{\mathbb{R}}\{\beta_1^\vee, \dots, \beta_k^\vee\}$. Now by one direction of Theorem 2.2, since $\ell_R(w) = k$, then the set $\{\beta_1, \dots, \beta_k\}$ is linearly independent over \mathbb{R} . Moreover, Carter's proof of the other direction of Theorem 2.2 shows that when the set $\{\beta_1, \dots, \beta_k\}$, and thus $\{\beta_1^\vee, \dots, \beta_k^\vee\}$, is linearly independent over \mathbb{R} , the subspace $(w - I)\mathbb{R}^n$ contains each of β_1, \dots, β_k , and thus also $\beta_1^\vee, \dots, \beta_k^\vee$, in turn. Since $\text{Mov}(w) = (w - I)\mathbb{R}^n$, this completes the proof. \square

Remark 2.4. In [LMPS19, Lemma 1.26], the claims in Corollary 2.3 are stated to be proved via [BM15, Lemma 6.4], which considers move-sets for elements of the full isometry group of Euclidean space. Tracing [BM15, Lemma 6.4] back, we arrive at [BM15, Proposition 5.1]. However, we do not understand the proof of the following statement within [BM15, Proposition 5.1]: if the roots corresponding to the reflections r_i are linearly independent, then $\dim_{\mathbb{R}}(\text{Mov}(w)) = k$. Since Corollary 2.3 is important for our results, we have included here a proof which relies upon Theorem 2.2 instead, for the sake of completeness.

2.3. Cuspidal and Coxeter elements. In this section we recall some background on cuspidal and Coxeter elements of W , mostly following Chapter 3 of [GP00].

Given any subset $J \subseteq [n]$, we denote by W_J the *standard parabolic subgroup* of W generated by $S_J = \{s_j \mid j \in J\}$. Recall that (W_J, S_J) is a spherical Coxeter system. A *parabolic subgroup* of W is then any subgroup which is conjugate to a standard parabolic subgroup. For $w \in W$, the *parabolic closure* of w , denoted $\text{Pc}(w)$, is the smallest parabolic subgroup of W which contains w . The relationship between conjugation in W and parabolic closures is given by the next lemma, whose proof is an easy exercise.

Lemma 2.5. *For all $u, w \in W$, we have $u \text{Pc}(w)u^{-1} = \text{Pc}(uwu^{-1})$.*

For any $w \in W$, we write $[w]_W$ for its conjugacy class in W .

Definition 2.6 (Cuspidal). A conjugacy class \mathcal{C} of W is called *cuspidal* if $\mathcal{C} \cap W_J = \emptyset$ for all proper subsets $J \subsetneq [n]$. An element $w \in W$ is called *cuspidal* if its conjugacy class $[w]_W$ is cuspidal.

These definitions can also be phrased in terms of parabolic closures, as follows: a conjugacy class \mathcal{C} of W is *cuspidal* if $\text{Pc}(w) = W$ for some, hence by Lemma 2.5 any, $w \in \mathcal{C}$, and $w \in W$ is *cuspidal* if $\text{Pc}(w) = W$. If $w \in W$ is an element of a parabolic subgroup W_J , we will sometimes describe w as being *cuspidal in W_J* if $\text{Pc}(w) = W_J$.

Let $w \in W$ and let \mathcal{C} be a W -conjugacy class. We write $\text{Supp}(w)$ for the *support* of w in S ; that is, the set of simple reflections appearing in some, and hence any, reduced expression for w (see Corollary 1.2.3 of [GP00]). We denote by \mathcal{C}_{\min} the set of minimal length elements of \mathcal{C} .

Proposition 2.7. *Let $w \in W$ have W -conjugacy class \mathcal{C} . The following are equivalent:*

- (1) \mathcal{C} is cuspidal.
- (2) $\text{FIX}(w) = \{0\}$.
- (3) $\text{MOV}(w) = V^*$.
- (4) $\text{Supp}(w) = S$ for all $w \in \mathcal{C}_{\min}$.
- (5) There exists an element $w \in \mathcal{C}_{\min}$ such that $\text{Supp}(w) = S$.

Proof. The equivalence of items (1), (4), and (5) is Proposition 3.1.12 of [GP00]. Now suppose \mathcal{C} is cuspidal. Then as remarked on p. 77 of [GP00], the fix-set of w in V , and thus in V^* , is trivial, so (2) holds. Therefore as $\text{FIX}(w)$ and $\text{MOV}(w)$ are orthogonal complements in V^* , we obtain (3). \square

Definition 2.8 (Coxeter elements). A *Coxeter element* of W is a product of all elements of S , in any given order.

Theorem 3.1.4 and Proposition 3.1.6 of [GP00] imply the following statement.

Proposition 2.9. *The Coxeter elements of W are contained in a single conjugacy class, which is cuspidal.*

In type A_n , by Example 3.1.16 of [GP00], the unique cuspidal conjugacy class of W is the one containing all Coxeter elements. However in general there exist cuspidal elements which are neither Coxeter nor in the conjugacy class containing the Coxeter elements, such as $w_0 = s_1 s_2 s_1 s_2$ in type C_2 , equivalently type B_2 .

2.4. Summary of previous results on (co)conjugation. In this section we give formal statements of the main definitions and results of [MST24] on Euclidean isometry groups, in the setting of affine Coxeter groups. We also recall some relevant results from [MST23].

We first restate a key definition from the introduction above.

Definition 2.10 (Mod-set, see Definition 1.1 of [MST24]). For any $x \in \overline{W}$, the *mod-set* of x (with respect to \overline{W}) is given by:

$$\text{MOD}_{\overline{W}}(x) = (x - I)R^\vee = (I - x)R^\vee.$$

The next result gathers some first properties of mod-sets. Parts (1), (2), and (3) in the next statement are special cases of Lemmas 2.1, 2.2, and 2.3 of [MST24], respectively.

Lemma 2.11 (Properties of mod-sets). *Let $\lambda \in R^\vee$ and $w \in W$. Then:*

- (1) For all $u \in W$, $u\text{MOD}_{\overline{W}}(w) = \text{MOD}_{\overline{W}}(uwu^{-1})$;
- (2) $\text{MOD}_{\overline{W}}(t^\lambda w) = \lambda + \text{MOD}_{\overline{W}}(w)$; and
- (3) $\text{MOD}_{\overline{W}}(w) \subseteq \text{Mov}(w) \cap R^\vee$.

The following theorem describes conjugacy classes in \overline{W} in terms of mod-sets. For any $x \in \overline{W}$, we write $[x] = \{yxy^{-1} \mid y \in \overline{W}\}$ for its conjugacy class in \overline{W} .

Theorem 2.12 (Closed form of conjugacy classes, see Theorem 1.2 of [MST24]). *Let $x = t^\lambda w \in \overline{W}$, where $\lambda \in R^\vee$ and $w \in W$. Then the conjugacy class of x in \overline{W} satisfies*

$$(2.4.1) \quad [x] = \bigcup_{u \in W} u \left(t^{\text{MOD}_{\overline{W}}(w)} x \right) u^{-1}$$

and also

$$(2.4.2) \quad [x] = \bigcup_{u \in W} t^{u(\lambda + \text{MOD}_{\overline{W}}(w))} uwu^{-1} = \bigcup_{u \in W} t^{u\text{MOD}_{\overline{W}}(x)} uwu^{-1}.$$

For any $x \in \overline{W}$ the class $[x]$ is thus obtained by first translating x by all elements of $\text{MOD}_{\overline{W}}(w)$, and then conjugating the so-obtained collection $t^{\text{MOD}_{\overline{W}}(w)}x$ by all elements of W . Alternatively, for each $u \in W$, we translate the u -conjugate of the spherical part w of x by the set $t^{u(\lambda+\text{MOD}_{\overline{W}}(w))} = t^{u\lambda}t^{u\text{MOD}_{\overline{W}}(w)}$.

Let us emphasize here that the conjugacy class of every element $x = t^\lambda w \in \overline{W}$ is determined by finite data: the conjugacy class of the spherical part w in the finite Coxeter group W , together with the collection of shifted mod-sets $t^{u(\lambda+\text{MOD}_{\overline{W}}(w))}$, where $u \in W$ varies.

Corollary 2.13 (Conjugacy classes and move-sets, see Corollary 1.3 of [MST24]). *Let $x = t^\lambda w \in \overline{W}$, where $\lambda \in R^\vee$ and $w \in W$. Then,*

$$(2.4.3) \quad [x] \subseteq \bigcup_{u \in W} u \left(t^{\text{Mov}(w) \cap R^\vee} x \right) u^{-1}$$

and also

$$(2.4.4) \quad [x] \subseteq \bigcup_{u \in W} t^{u(\text{Mov}(x) \cap R^\vee)} u w u^{-1}.$$

The second main definition of [MST24] is motivated by these containments.

Definition 2.14 (Filling, see Definition 1.4 of [MST24]). *Let $x \in \overline{W}$. We say that x fills its move-set, or that filling occurs for x , if*

$$\text{MOD}_{\overline{W}}(x) = \text{Mov}(x) \cap R^\vee.$$

In [MST23] we studied the filling condition for simple reflections, using direct computations. We will use the next result to study the filling property for arbitrary elements of \overline{W} . Its proof (in [MST24]) is similar to that of [MST23, Lemma 5.8].

Proposition 2.15 (Conjugacy classes and filling, see Proposition 2.4 of [MST24]). *For all $x = t^\lambda w \in \overline{W}$, where $\lambda \in R^\vee$ and $w \in W$, the following are equivalent:*

- (1) x fills its move-set;
- (2) w fills its move-set;
- (3) $[x] = \bigcup_{u \in W} u \left(t^{\text{Mov}(w) \cap R^\vee} x \right) u^{-1}$; and
- (4) $[x] = \bigcup_{u \in W} t^{u(\text{Mov}(x) \cap R^\vee)} u w u^{-1}$.

A component of a conjugacy class $[x]$ is defined in [MST24, Definition 2.5] to be a subset of $[x]$ of the form $u(t^{\text{MOD}_{\overline{W}}(w)}x)u^{-1}$ with $u \in W$; that is, a set appearing in the union given by (2.4.1) above. We write $\text{Comp}(x)$ for the set of components of $[x]$. By definition, the group W acts transitively by conjugation on $\text{Comp}(x)$. The following result is discussed in the introduction in Remark 1.3.

Theorem 2.16 (Components, see Theorem 1.11 of [MST24]). *Let $x = t^\lambda w \in \overline{W}$, where $\lambda \in R^\vee$ and $w \in W$. Then:*

- (1) *The conjugation action of T induces a transitive action by translations on the elements of each component of $[x]$.*
- (2) *Linearization induces a natural surjection from $\text{Comp}(x)$ to $\text{Comp}(w)$.*
- (3) *There is a natural bijection between $\text{Comp}(w)$ and $[w]_W$.*

For $x, x' \in \overline{W}$ we define the coconjugation set

$$\text{C}(x, x') = \{y \in \overline{W} \mid yxy^{-1} = x'\},$$

and for $w, w' \in W$ we define the *spherical coconjugation set*

$$C_W(w, w') = \{u \in W \mid uwu^{-1} = w'\}.$$

Our geometric description of $C(x, x')$ involves the following subset of the corresponding spherical coconjugation set.

Definition 2.17 (see Definition 1.12 of [MST24]). Let $x = t^\lambda w$ and $x' = t^{\lambda'} w'$ be elements of \overline{W} , where $\lambda, \lambda' \in R^\vee$ and $w, w' \in W$. The *translation-compatible part* of the coconjugation set $C_W(w, w')$ is the set

$$C_W^{\lambda, \lambda'}(w, w') = \{u \in C_W(w, w') \mid \lambda' - u\lambda \in \text{MOD}_{\overline{W}}(w')\}.$$

The final main result of [MST24] describes coconjugation sets as follows.

Theorem 2.18 (Coconjugation, see Theorem 1.13 of [MST24]). *Let $x = t^\lambda w$ and $x' = t^{\lambda'} w'$ be elements of \overline{W} , where $\lambda, \lambda' \in R^\vee$ and $w, w' \in W$. Then*

$$(2.4.5) \quad C(x, x') \neq \emptyset \iff C_W^{\lambda, \lambda'}(w, w') \neq \emptyset.$$

Moreover, if these sets are nonempty, then

$$(2.4.6) \quad C(x, x') = \bigsqcup_{u \in C_W^{\lambda, \lambda'}(w, w')} t^{\eta_u + (\text{FIX}(w') \cap R^\vee)} u$$

where for each u , the element $\eta_u \in R^\vee$ is a solution to the equation

$$(2.4.7) \quad \lambda' - u\lambda = (\mathbf{I} - w')\eta.$$

In the special case that $\text{FIX}(w) = \{0\}$, we have that

$$\eta_u = (\mathbf{I} - w')^{-1}(\lambda' - u\lambda)$$

is the unique solution to (2.4.7), and $C(x, x')$ is in bijection with $C_W^{\lambda, \lambda'}(w, w')$.

3. MOD-SETS AND MOVE-SETS

In this section we prove Theorem 1.8 and Corollary 1.9 of the introduction. We also prove the generalization Theorem 3.7 below, which applies to both affine Coxeter groups and their split crystallographic subgroups.

The bulk of this section is devoted to the proof of Theorem 1.8. This says that for $w \in W$,

$$\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(w)) = \dim_{\mathbb{R}}(\text{Mov}(w)) = \ell_R(w),$$

where $\ell_R(w)$ is the reflection length of w (see Section 2.1). To prove these equalities, we establish some results on intersections of hyperplanes in Section 3.1, then use these to complete the proof of Theorem 1.8 in Section 3.2. We then state and prove Theorem 3.7 in Section 3.3.

3.1. Hyperplane intersections. In this section we establish two technical results on hyperplane intersections that we will use in our proof of Theorem 1.8. These two results are obvious if working over \mathbb{R} , but require some care over \mathbb{Z} .

Recall from Section 2.1 that for any root $\alpha \in \Phi$, we denote by \mathcal{H}_α the (linear) hyperplane which is the fix-set of the reflection s_α .

Lemma 3.1. *Let $\{\beta_1, \dots, \beta_k\} \subset \Phi$ be a set of roots such that the set of corresponding coroots $\{\beta_1^\vee, \dots, \beta_k^\vee\}$ is linearly independent over \mathbb{Z} . Then the submodule of R^\vee given by*

$$(3.1.1) \quad \left(\bigcap_{i=1}^k \mathcal{H}_{\beta_i} \right) \cap R^\vee$$

has rank $(n - k)$ as a \mathbb{Z} -module.

Proof. To simplify notation, for $1 \leq i \leq k$ write \mathcal{H}_i for \mathcal{H}_{β_i} , and write M for the intersection given in (3.1.1). Then $M = \bigcap_{i=1}^k (\mathcal{H}_i \cap R^\vee)$, and for $1 \leq i \leq k$, we have

$$\mathcal{H}_i \cap R^\vee = \{\mu \in R^\vee \mid \langle \mu, \beta_i \rangle = 0\}.$$

Write $\mu = \sum_{i \in [n]} m_i \alpha_i^\vee$ for unknown $m_i \in \mathbb{Z}$, and expand $\beta_i = \sum_{j \in [n]} b_{ij} \alpha_j$ where $b_{ij} \in \mathbb{Z}$. Since the evaluation pairing on $V^* \times V$ is bilinear, the condition $\langle \mu, \beta_i \rangle = 0$ then becomes a single equation in the n variables m_i . Moreover, since $\langle \alpha_i^\vee, \alpha_j \rangle \in \mathbb{Z}$ for all $1 \leq i, j \leq n$, and all $b_{ij} \in \mathbb{Z}$, the equation $\langle \mu, \beta_i \rangle = 0$ has \mathbb{Z} -coefficients. We may thus identify the points of M with the set of solutions over \mathbb{Z} to the corresponding system of k homogeneous linear equations in n variables, with coefficients in \mathbb{Z} .

Now the set $\{\beta_1^\vee, \dots, \beta_k^\vee\}$ is linearly independent over \mathbb{Q} , since by assumption it is linearly independent over \mathbb{Z} . Hence over the field \mathbb{Q} , the set of solutions to this system of homogeneous equations is a subspace of dimension $(n - k)$. Therefore the \mathbb{Z} -rank of the submodule M of R^\vee is at most $(n - k)$.

If $\text{rk}_{\mathbb{Z}}(M) = m < n - k$, let $\{\eta_1^\vee, \dots, \eta_m^\vee\}$ be a \mathbb{Z} -basis for M . Then the \mathbb{Q} -span of $\{\eta_1^\vee, \dots, \eta_m^\vee\}$ is a proper subspace of the set of solutions over \mathbb{Q} , so there is a $\lambda \in R^\vee$ which is a solution over \mathbb{Q} with $\lambda \notin \text{SPAN}_{\mathbb{Q}}\{\eta_1^\vee, \dots, \eta_m^\vee\}$. Now $\lambda \in \text{SPAN}_{\mathbb{Q}}\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$, so for large enough N we have $N\lambda \in \text{SPAN}_{\mathbb{Z}}\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$. That is, for large enough N we have $N\lambda \in R^\vee$. And since the system of equations defining M is homogeneous, then $N\lambda$ is also a solution to this system. Hence $N\lambda \in \text{SPAN}_{\mathbb{Z}}\{\eta_1^\vee, \dots, \eta_m^\vee\}$. But then $\lambda \in \text{SPAN}_{\mathbb{Q}}\{\eta_1^\vee, \dots, \eta_m^\vee\}$, a contradiction. We conclude that $\text{rk}_{\mathbb{Z}}(M) = n - k$. \square

Corollary 3.2. *Let $\{\beta_1, \dots, \beta_k\} \subset \Phi$ be a set of roots such that the set of corresponding coroots $\{\beta_1^\vee, \dots, \beta_k^\vee\}$ is linearly independent over \mathbb{Z} . Then for all $1 \leq j \leq k$ there exists $\mu_j \in R^\vee$ such that*

$$\mu_j \in \left(\bigcap_{i=j+1}^k \mathcal{H}_{\beta_i} \right) \quad \text{but} \quad \mu_j \notin \mathcal{H}_{\beta_j}.$$

Proof. By Lemma 3.1, the intersection $\left(\bigcap_{i=j+1}^k \mathcal{H}_{\beta_i} \right) \cap R^\vee$ has \mathbb{Z} -rank $n - (k - j) = n - k + j$, while its submodule $\left(\bigcap_{i=j}^k \mathcal{H}_{\beta_i} \right) \cap R^\vee$ has \mathbb{Z} -rank $n - (k - (j - 1)) = n - k + j - 1$. The result follows. \square

3.2. Rank vs dimension for affine Coxeter groups. In this section we establish Theorem 1.8 of the introduction. We consider reflections in W , then arbitrary elements of W .

Lemma 3.3. *For all reflections $r = s_\alpha \in W$, the mod-set $\text{MOD}_{\overline{W}}(r)$ is generated as a \mathbb{Z} -module by some nonzero integer multiple of α^\vee .*

Proof. Since $\text{MOD}_{\overline{W}}(s_\alpha) \subset \text{Mov}(s_\alpha) = \mathbb{R}\alpha^\vee$, we have $\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(s_\alpha)) \leq 1$. Now by the case $j = k = 1$ of Corollary 3.2, there is a $\mu \in R^\vee$ such that $\mu \notin \mathcal{H}_\alpha = \text{FIX}(s_\alpha)$. Then

$(I - s_\alpha)\mu \neq 0$, and

$$(I - s_\alpha)\mu \in \text{MOD}_{\overline{W}}(s_\alpha) \subseteq (\text{Mov}(s_\alpha) \cap R^\vee) = (\mathbb{R}\alpha^\vee \cap R^\vee) = \mathbb{Z}\alpha^\vee.$$

Thus $\text{MOD}_{\overline{W}}(s_\alpha)$ contains some nonzero integer multiple of α^\vee . The result follows. \square

Corollary 3.4. *For all reflections $r \in W$, we have*

$$\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(r)) = \dim_R(\text{Mov}(r)) = \ell_{\mathcal{R}}(r) = 1.$$

We now consider arbitrary elements of W . We make essential use of one direction of Theorem 2.2, and generalize Carter's proof of the other direction of that result.

Proposition 3.5. *Let $w \in W$. Then $\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(w)) = \ell_{\mathcal{R}}(w)$.*

Proof. Let $w = r_1 \cdots r_k$ be a minimal length reflection factorization of w , so that $\ell_{\mathcal{R}}(w) = k$, and for $1 \leq i \leq k$ let $\beta_i \in \Phi$ be a root such that $r_i = s_{\beta_i}$. To simplify notation, write \mathcal{H}_i for \mathcal{H}_{β_i} . Then by Theorem 2.2, the set $\{\beta_1, \dots, \beta_k\}$ is linearly independent over \mathbb{R} . Hence the set $\{\beta_1^\vee, \dots, \beta_k^\vee\}$ is linearly independent over \mathbb{R} , and so $\{\beta_1^\vee, \dots, \beta_k^\vee\} \subset \Phi^\vee$ is linearly independent over \mathbb{Z} .

By Lemma 3.3, each r_i only moves elements of R^\vee by integer multiples of β_i^\vee , and so the motion of any $\mu \in R^\vee$ under w is a \mathbb{Z} -linear combination of the β_i^\vee . Thus $\text{MOD}_{\overline{W}}(w) \subseteq \text{SPAN}_{\mathbb{Z}}\{\beta_1^\vee, \dots, \beta_k^\vee\}$ and so $\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(w)) \leq k$.

We now use Corollary 3.2 to prove by induction that $\text{MOD}_{\overline{W}}(w)$ contains some nonzero integer multiple of each of $\beta_1^\vee, \dots, \beta_k^\vee$. Since the β_i^\vee are linearly independent over \mathbb{Z} , this establishes $\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(w)) = k$.

First, by the case $j = 1$ of Corollary 3.2, there is a $\mu_1 \in R^\vee$ such that $\mu_1 \in (\cap_{i=2}^k \mathcal{H}_i)$ but $\mu_1 \notin \mathcal{H}_1$. Then since r_2, \dots, r_k fix μ_1 while r_1 does not, by Lemma 3.3 we have $w\mu_1 = r_1\mu_1 = \mu_1 - c_1\beta_1^\vee$ for some $c_1 \in \mathbb{Z}$ with $c_1 \neq 0$. It follows that $\text{MOD}_{\overline{W}}(w)$ contains a nonzero integer multiple of β_1^\vee , namely $c_1\beta_1^\vee$.

Assume inductively that $\text{MOD}_{\overline{W}}(w)$ contains $c_1\beta_1^\vee, \dots, c_{j-1}\beta_{j-1}^\vee$ where c_1, \dots, c_{j-1} are nonzero integers. Define $c = \prod_{i=1}^{j-1} c_i$, so that $c \in \mathbb{Z}$ and $c \neq 0$. By Corollary 3.2 there is a $\mu_j \in R^\vee$ such that $\mu_j \in (\cap_{i=j+1}^k \mathcal{H}_i)$ but $\mu_j \notin \mathcal{H}_j$. Hence $c\mu_j \in R^\vee$, $c\mu_j \in (\cap_{i=j+1}^k \mathcal{H}_i)$, but $c\mu_j \notin \mathcal{H}_j$. Now

$$w(c\mu_j) = cr_1 \cdots r_j\mu_j = cr_1 \cdots r_{j-1}(\mu_j - c_j\beta_j^\vee) = c\mu_j - (cc_j)\beta_j^\vee - \sum_{i=1}^{j-1} c_{ij}\beta_i^\vee,$$

for some $c_j \in \mathbb{Z}$ with $c_j \neq 0$ and some $c_{ij} \in \mathbb{Z}$ divisible by c_i , for $1 \leq i \leq j-1$. Then by subtracting suitable multiples of $c_1\beta_1^\vee, \dots, c_{j-1}\beta_{j-1}^\vee$, we obtain that $\text{MOD}_{\overline{W}}(w)$ contains a nonzero scalar multiple of β_j^\vee , namely $cc_j\beta_j^\vee$. This completes the proof. \square

Corollary 3.6. *For all $w \in W$, we have*

$$\text{rk}_{\mathbb{Z}}(\text{MOD}_{\overline{W}}(w)) = \dim_{\mathbb{R}}(\text{Mov}(w)) = \ell_{\mathcal{R}}(w).$$

Proof. This follows from Corollary 2.3 and Proposition 3.5. \square

3.3. Mod-sets and move-sets for split crystallographic groups. We now state and prove Theorem 3.7. This generalizes the first equality in Theorem 1.8, and both parts of Corollary 1.9.

In order to state Theorem 3.7, recall that an n -dimensional crystallographic group is a discrete, cocompact group of isometries of n -dimensional Euclidean space \mathbb{R}^n . We define two n -dimensional crystallographic groups to be *equivalent*, denoted \sim , if they are conjugate under

some affine transformation of \mathbb{R}^n (the classification of crystallographic groups is usually up to this relation). Then by abuse of terminology, we say that an n -dimensional crystallographic group H is *contained in an affine Coxeter group* if there is an n -dimensional crystallographic group H' , and an affine Coxeter group \overline{W} (not necessarily irreducible) which is also an n -dimensional crystallographic group, such that $H \sim H'$ and $H' \leq \overline{W}$.

It is classical that the full isometry group G of n -dimensional Euclidean space splits as $G = \mathbb{R}^n \rtimes O(n)$, where \mathbb{R}^n is the additive group of all translations and $O(n)$ is the orthogonal group. An n -dimensional crystallographic group H is *split* if it respects this splitting, that is, $H = T_H \rtimes H_0$ where T_H is the translation subgroup of H and $H_0 = H \cap O(n)$. The set $L_H = \{\lambda : t^\lambda \in T_H\}$ is a cocompact lattice in \mathbb{R}^n , and so L_H can be viewed as a free \mathbb{Z} -module of rank n .

Theorem 3.7 describes the relationships between the mod-set $\text{MOD}_H(h_0)$ for $h_0 \in H_0$, the move-set $\text{Mov}(h_0)$ for $h_0 \in H_0$, and the lattice L_H , in the case that H is contained in an affine Coxeter group.

Theorem 3.7 (Mod-sets and move-sets). *Let $H = T_H \rtimes H_0$ be a split crystallographic group. Assume that H is contained in an affine Coxeter group $W = T \rtimes \overline{W}$. Then for all $h_0 \in H_0$:*

- (1) $\text{rk}_{\mathbb{Z}}(\text{MOD}_H(h_0)) = \dim_{\mathbb{R}}(\text{Mov}(h_0))$;
- (2) $\text{MOD}_H(h_0)$ is a finite-index submodule of $\text{Mov}(h_0) \cap L_H$; and
- (3) $\text{MOD}_H(h_0) = \text{Mov}(h_0) \cap L_H$ if and only if $L_H/\text{MOD}_H(h_0)$ is torsion-free.

The next result establishes part (1) of this theorem.

Corollary 3.8. *Let $H = T_H \rtimes H_0$ be a split crystallographic group which is contained in an affine Coxeter group $\overline{W} = T \rtimes W$. Then for every $h_0 \in H_0$, we have*

$$\text{rk}_{\mathbb{Z}}(\text{MOD}_H(h_0)) = \dim_{\mathbb{R}}(\text{Mov}(h_0)).$$

Proof. We may assume without loss of generality that H is a subgroup of \overline{W} , and so in particular $H_0 \leq W$ and $L_H \subseteq R^\vee$.

Let $h_0 \in H_0$. Then $h_0 \in W$, so we may write h_0 as a minimal length product of reflections in W , say $h_0 = r_1 \cdots r_k$ with $r_i = s_{\beta_i}$ for $1 \leq i \leq k$. Since $L_H \subseteq R^\vee$, the same argument as in the proof of Proposition 3.5 then tells us that $\text{MOD}_H(h_0) \subseteq \text{SPAN}_{\mathbb{Z}}\{\beta_1^\vee, \dots, \beta_k^\vee\}$, and hence $\text{rk}_{\mathbb{Z}}(\text{MOD}_H(h_0)) \leq k$.

Choose $\mu_1, \dots, \mu_k \in R^\vee$ as in the statement of Corollary 3.2. Now the lattices L_H and R^\vee are both free \mathbb{Z} -modules of rank n , so L_H must be a finite-index submodule of R^\vee . Thus there is an integer, say d , such that $d\mu_i \in L_H$ for all $1 \leq i \leq k$. Then since each \mathcal{H}_{β_i} is a subspace, we have

$$d\mu_j \in \left(\bigcap_{i=j+1}^k \mathcal{H}_{\beta_i} \right) \quad \text{but} \quad d\mu_j \notin \mathcal{H}_{\beta_j}.$$

Now by a similar argument to that given in the proof of Proposition 3.5, but replacing each μ_j by $d\mu_j$, we see that $\text{MOD}_H(h_0)$ contains some nonzero integer multiple of each of $\beta_1^\vee, \dots, \beta_k^\vee$, and so it has \mathbb{Z} -rank equal to k . \square

We now prove the remainder of Theorem 3.7. In particular, the next result establishes Corollary 1.9.

Corollary 3.9. *Let $H = T_H \rtimes H_0$ be a split crystallographic group which is contained in an affine Coxeter group. Then for all $h_0 \in H$:*

- (1) $\text{MOD}_H(h_0)$ is a finite-index submodule of $\text{Mov}(h_0) \cap L_H$; and

(2) $\text{MOD}_H(h_0) = \text{Mov}(h_0) \cap L_H$ if and only if $L_H/\text{MOD}_H(h_0)$ is torsion-free.

Proof. By Corollary 3.8, we have $\text{rk}_{\mathbb{Z}}(\text{MOD}_H(h_0)) = \dim_{\mathbb{R}}(\text{Mov}(h_0)) = k$, say, and so $\text{rk}_{\mathbb{Z}}(\text{Mov}(h_0) \cap L_H) \leq \dim_{\mathbb{R}}(\text{Mov}(h_0)) = k$. Hence $\text{Mov}(h_0) \cap L_H$ and its submodule $\text{MOD}_H(h_0)$ are both free \mathbb{Z} -modules of rank k , and (1) follows.

For (2), given (1) it suffices to show that the quotient $L_H/(\text{Mov}(h_0) \cap L_H)$ is torsion-free. If $\text{Mov}(h_0) = \mathbb{R}^n$ then $\text{Mov}(h_0) \cap L_H = L_H$ so this quotient is trivial and we are done. Assume now that $\text{Mov}(h_0)$ is a proper subspace of \mathbb{R}^n , and suppose $\lambda \in L_H$ represents a torsion element of $L_H/(\text{Mov}(h_0) \cap L_H)$. Then $\lambda \notin \text{Mov}(h_0)$ but there is some $c \in \mathbb{Z}$ with $c \neq 0$ so that $c\lambda \in \text{Mov}(h_0)$. This is a contradiction, as $\text{Mov}(h_0)$ is a subspace of \mathbb{R}^n . \square

In light of the results in this section and the main results obtained in [MST24] (as restated in Section 2.4), it has become clear that the mod-sets are the crucial objects ruling the shape of conjugacy classes in affine Coxeter groups, and are critical to the structure of their coconjugation sets as well. The remainder of the paper, starting with Section 4, is hence devoted to the study of precisely these \mathbb{Z} -modules.

4. TYPE A MOD-SETS

In this section, we give an explicit description of all mod-sets in type A , stated as Theorem 4.2. These results will also be used in Sections 5, 6, and 7, since types B , C , and D all have type A subsystems. Let W be the finite Weyl group of type A_n , with $n \geq 1$. In Section 4.1, we recall the complete system of minimal length representatives for the conjugacy classes of W , which we rephrase in Proposition 4.1. We then give several key examples in Section 4.2, before proving our results in type A_n in Section 4.3.

Throughout this paper, we order the nodes of the Dynkin diagram for W of type A_n increasing from left to right, as in both [Bou02] and [GP00]. See Table 9 in Appendix B for a direct comparison of choices of labeling.

4.1. Conjugacy class representatives and mod-sets in type A . Recall from [GP00, Prop. 3.4.1], for example, that the conjugacy classes of W of type A_n are parameterized by partitions of $n+1$; that is, weakly decreasing sequences $\beta = (\beta_1, \dots, \beta_p)$ of nonnegative integers such that $|\beta| = \sum \beta_i = n+1$. For each such β , we will define a standard parabolic subgroup W_β of W , and a Coxeter (equivalently cuspidal, in type A) element w_β of W_β , so that the set of all such w_β forms a complete system of minimal length representatives for the conjugacy classes of W .

We first provide a summary of our algorithm for obtaining a minimal length conjugacy class representative w_β associated to a partition β . The following result is classical, though we emphasize a visualization, as this perspective will be helpful in the other classical types.

Proposition 4.1. *Let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$. Define an element w_β of the finite Weyl group W of type A_n as follows:*

- (1) *subtract 1 from each part of β ,*
- (2) *take the connected subdiagrams of the Dynkin diagram with $\beta_1 - 1$, $\beta_2 - 1$, \dots , $\beta_p - 1$ nodes, respectively, going from left to right, omitting only a single node between any two such nonempty subdiagrams, and*
- (3) *multiply together the simple reflections indexed by the nodes of these subdiagrams in increasing order to obtain w_β .*

Then the set of all such w_β forms a complete system of minimal length representatives for the conjugacy classes of W .

The following figure illustrates the construction in Proposition 4.1; see Section 4.2 for additional examples, including Example 4.6 for more details on Figure 5.

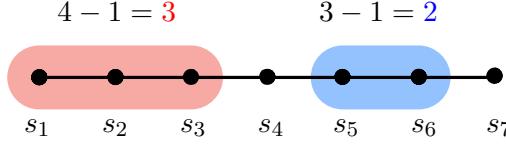


FIGURE 5. Constructing $w_\beta = (s_1s_2s_3)(s_5s_6)$ for $\beta = (4, 3, 1)$ in type A_7 .

We now establish the notation needed to formally state Theorem 4.2 characterizing all mod-sets in type A . Fix a partition $\beta = (\beta_1, \dots, \beta_p)$ of $n+1$. We define a strictly increasing subsequence of $0, 1, \dots, n$ as follows. Let $j_1^\beta = 0$, and for $2 \leq k \leq p$, define

$$j_k^\beta = \sum_{i=1}^{k-1} \beta_i = \beta_1 + \dots + \beta_{k-1}.$$

In other words, for $p \geq 2$ and any $1 \leq k < p$, we have $j_{k+1}^\beta = j_k^\beta + \beta_k$. Hence in particular, $0 = j_1^\beta < j_2^\beta < \dots < j_p^\beta \leq n$.

We next define subsets J_k^β of $[n]$, so that $[n]$ will be the disjoint union of the J_k^β with the set of integers $\{j_k^\beta \mid 2 \leq k \leq p\}$. For $k \in [p]$, if $\beta_k = 1$ put $J_k^\beta = \emptyset$, and if $\beta_k \geq 2$ define J_k^β to be the subinterval of $[n]$ given by

$$(4.1.1) \quad J_k^\beta = \{j_k^\beta + 1, j_k^\beta + 2, \dots, j_k^\beta + (\beta_k - 1)\}.$$

Thus in particular, if $\beta_1 \geq 2$ then J_1^β is the initial subinterval $J_1^\beta = \{1, 2, \dots, \beta_1 - 1\}$, and if $p \geq 2$ and $\beta_p \geq 2$ then J_p^β is the terminal subinterval $J_p^\beta = \{\beta_1 + \dots + \beta_{p-1} + 1, \dots, n\}$. Notice that the J_k^β are pairwise disjoint, with each J_k^β containing $\beta_k - 1$ elements, and that if J_k^β and J_{k+1}^β are both nonempty, then the unique element of $[n]$ which lies strictly between these subintervals is the integer $j_{k+1}^\beta = j_k^\beta + \beta_k$. When $\beta_k \geq 2$, we may identify the elements of the nonempty interval J_k^β with the nodes of the corresponding (connected) subdiagram of type A_{β_k-1} of the Dynkin diagram, as illustrated in Figure 5. We also define the union $J_\beta = \sqcup_{k=1}^p J_k^\beta \subseteq [n]$. Then if $p = 1$ we have $J_\beta = J_1^\beta = [n]$, while for all $p \geq 2$, we have

$$[n] \setminus J_\beta = \{j_k^\beta \mid 2 \leq k \leq p\} = \{\beta_1, \beta_1 + \beta_2, \dots, \beta_1 + \dots + \beta_{p-1}\}.$$

Next, for each $k \in [p]$, write $W_{J_k^\beta}$ for the (possibly trivial) standard parabolic subgroup of W generated by the simple reflections $\{s_j \mid j \in J_k^\beta\}$. Then each nontrivial $W_{J_k^\beta}$ is of type A_{β_k-1} , and the $W_{J_k^\beta}$ pairwise commute. Hence the (possibly trivial) standard parabolic subgroup of W generated by the simple reflections $\{s_j \mid j \in J_\beta\}$ has the form

$$W_\beta = W_{J_\beta} = W_{J_1^\beta} \times \dots \times W_{J_p^\beta}.$$

Now if $\beta_k = 1$ let w_k^β be the trivial element of W , and if $\beta_k \geq 2$ let w_k^β be the Coxeter element of $W_{J_k^\beta}$ given by:

$$(4.1.2) \quad w_k^\beta = s_{j_k^\beta+1} s_{j_k^\beta+2} \cdots s_{j_k^\beta+(\beta_k-1)} \in W_{J_k^\beta}.$$

That is, w_k^β is the product of the simple reflections in $W_{J_k^\beta}$, in increasing order. (In the language of [GP00, Sec. 3.4.2], the element w_k^β is the positive block $b_{j_k^\beta, \beta_k}^+$ of length β_k starting at j_k^β .)

Finally, the partition $\beta = (\beta_1, \dots, \beta_p)$ corresponds to the product

$$w_\beta = w_1^\beta \cdots w_p^\beta \in W_\beta.$$

That is, w_β is the Coxeter element of W_β which is the product of its simple reflections, in increasing order. Note that $\text{Supp}(w_\beta) = \{s_j \mid j \in J_\beta\}$, so that $S \setminus \text{Supp}(w_\beta)$ is indexed by $[n] \setminus J_\beta = \{j_k^\beta \mid 2 \leq k \leq p\}$.

To simplify notation in the theorem below, given any partition $\beta = (\beta_1, \dots, \beta_p)$ of $n+1$, we define the following p -element subsets of $[n+1]$:

$$I_\beta = \{j_k^\beta \mid 2 \leq k \leq p\} \cup \{n+1\} \quad \text{and} \quad I_\beta - 1 = \{i-1 \mid i \in I_\beta\}.$$

With this notation established, we can now state our results describing mod-sets in type A .

Theorem 4.2. *Suppose W is of type A_n with $n \geq 1$. Let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$ with corresponding conjugacy class representative $w_\beta \in W$.*

(1) *The module $\text{MOD}(w_\beta) = (\mathbf{I} - w_\beta)R^\vee$ equals*

$$\left\{ \sum_{i=1}^n c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, \quad c_i = 0 \text{ for } i \in [n] \setminus J_\beta, \quad \sum_{i=1}^n c_i \equiv 0 \pmod{\gcd(\beta_k)} \right\}.$$

(2) *If $\beta_p = 1$ then the module $\text{MOD}(w_\beta) = (\mathbf{I} - w_\beta)R^\vee$ has a \mathbb{Z} -basis given by*

$$\{\alpha_j^\vee \mid j \in J_\beta\},$$

and if $\beta_p \geq 2$ then the module $\text{MOD}(w_\beta) = (\mathbf{I} - w_\beta)R^\vee$ has a \mathbb{Z} -basis given by

$$\{\alpha_i^\vee - \alpha_{i+1}^\vee \mid i \in J_\beta \setminus (I_\beta - 1)\} \cup \{\alpha_i^\vee - \alpha_{i+2}^\vee \mid i \in (I_\beta - 1) \setminus \{n\}\} \cup \{\gcd(\beta_k) \alpha_n^\vee\}.$$

(3) *If $\beta = (1, \dots, 1)$ so that w_β is trivial, then $(\mathbf{I} - w_\beta)$ has Smith normal form $\text{diag}(0^n)$.*

For w_β nontrivial and any $w \in [w_\beta]$, the Smith normal form of $(\mathbf{I} - w)$ equals

$$S_\beta = \text{diag}(1^{n-p}, \gcd(\beta_k), 0^{p-1}).$$

(4) *For any partition β and any $w \in [w_\beta]$, we have*

$$R^\vee / \text{MOD}(w) \cong (\mathbb{Z}/\gcd(\beta_k)\mathbb{Z}) \oplus \mathbb{Z}^{p-1}.$$

We illustrate this theorem with several examples in Section 4.2, and provide the proof in Section 4.3.

4.2. Examples in type A . In this section, we describe four examples which illustrate our general results and proof techniques in type A . We consider the partitions:

- $\beta = (4)$ in Example 4.3;
- $\beta = (6, 4)$ in Example 4.4;
- $\beta = (4, 1)$ in Example 4.5; and
- $\beta = (4, 3, 1)$ in Example 4.6.

For each of these examples, we give an explicit description of the \mathbb{Z} -module $\text{MOD}(w_\beta) = (\mathbf{I} - w_\beta)R^\vee$, construct a \mathbb{Z} -basis for $\text{MOD}(w_\beta)$, find the Smith normal form for $(\mathbf{I} - w_\beta)$, and determine the isomorphism class of the quotient $R^\vee / \text{MOD}(w_\beta)$. We continue all notation from Section 4.1.

Example 4.3. Let W be of type A_3 , and consider the partition $\beta = (\beta_1) = (4)$ of $n + 1 = 4$. Then $J_\beta = J_1^\beta = \{1, 2, 3\}$, so w_β is the Coxeter element $w_\beta = s_1s_2s_3$ of W . We will show that $(I - w_\beta)$ has Smith normal form $(1, 1, 4)$, and hence $R^\vee/\text{MOD}(w_\beta) \cong \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/\beta_1\mathbb{Z}$.

We work with the \mathbb{Z} -basis $\Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ for R^\vee and compute using (2.1.1) that

$$\begin{aligned} w_\beta(\alpha_1^\vee) &= s_1s_2(\alpha_1^\vee) = s_1(\alpha_1^\vee + \alpha_2^\vee) = -\alpha_1^\vee + (\alpha_1^\vee + \alpha_2^\vee) = \alpha_2^\vee \\ w_\beta(\alpha_2^\vee) &= s_1s_2s_3(\alpha_2^\vee) = s_1s_2(\alpha_2^\vee + \alpha_3^\vee) = s_1(\alpha_3^\vee) = \alpha_3^\vee \\ w_\beta(\alpha_3^\vee) &= s_1s_2s_3(\alpha_3^\vee) = s_1s_2(-\alpha_3^\vee) = s_1(-\alpha_2^\vee - \alpha_3^\vee) = -\alpha_1^\vee - \alpha_2^\vee - \alpha_3^\vee. \end{aligned}$$

Hence, with respect to the basis Δ^\vee , the matrix for w_β is as on the left, and the matrix M_β for $(I - w_\beta)$ is as on the right:

$$w_\beta = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \quad M_\beta = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

We now solve the \mathbb{Z} -linear system $M_\beta \mathbf{x} = \mathbf{c}$ by performing certain row operations on M_β . We replace row 1 of M_β by the sum of its rows 1, 2, and 3, to obtain the matrix

$$M'_\beta = \begin{pmatrix} 0 & 0 & 4 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Tracking the effect of these row operations on \mathbf{c} , and noticing that for $i = 2, 3$, the leading entry in row i of M'_β is a -1 at position $(i, i-1)$, we see that the corresponding system of linear equations has solution over \mathbb{Z} if and only if

$$c_1 + c_2 + c_3 \equiv 0 \pmod{4}.$$

Hence for the Coxeter element $w_\beta = s_1s_2s_3$ in W of type A_3 ,

$$\text{MOD}(w_\beta) = M_\beta R^\vee = \left\{ \sum_{i=1}^3 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z} \text{ and } c_1 + c_2 + c_3 \equiv 0 \pmod{4} \right\}.$$

Next, we obtain the Smith normal form S_β for M_β . To do this, we first, for $i = 2, 3$, replace row i of M_β by the sum of rows $1, \dots, i$ of M_β , to obtain the upper-triangular matrix

$$T_\beta = \begin{pmatrix} \boxed{1} & 0 & \circled{1} \\ 0 & \boxed{1} & \circled{2} \\ 0 & 0 & 4 \end{pmatrix}.$$

Here, we have boxed the pivot entries 1 in rows 1 and 2, and we have circled the non-zero entries (in column 3) which we will clear at the next step. Using the boxed pivots, for $i = 1, 2$ we subtract i times column i from column 3, to obtain

$$S_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

which is in Smith normal form. That is, for $w_\beta = s_1s_2s_3$ in W of type A_3 , the matrix M_β has Smith normal form $\text{diag}(1, 1, 4)$. Hence

$$R^\vee/\text{MOD}(w_\beta) = R^\vee/M_\beta R^\vee \cong \mathbb{Z}/4\mathbb{Z}.$$

Finally, to obtain a \mathbb{Z} -basis for $\text{MOD}(w_\beta)$, we go back to M_β and perform column operations. We replace column 3 of M_β by the sum of column 3 and $-i$ times column i for $i = 1, 2$, to obtain the matrix

$$B_\beta = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 4 \end{pmatrix}.$$

From here, we see that a \mathbb{Z} -basis for $\text{MOD}(w_\beta)$ is given by the set

$$\{\alpha_1^\vee - \alpha_2^\vee, \alpha_2^\vee - \alpha_3^\vee, 4\alpha_3^\vee\}.$$

In this example, $J_\beta = \{1, 2, 3\}$ and $I_\beta = \{4\}$, in which case $I_\beta - 1 = \{3\}$ and $(I_\beta - 1) \setminus \{3\} = \emptyset$. Therefore, this basis matches the one given in Theorem 4.2.

Example 4.4. Let W be of type A_9 , and consider the partition $\beta = (\beta_1, \beta_2) = (6, 4)$ of $n + 1 = 10$. Then $J_1^\beta = \{1, 2, 3, 4, 5\}$ and $J_2^\beta = \{7, 8, 9\}$, so $J_\beta = \{1, \dots, 9\} \setminus \{6\}$ and $w_\beta = w_1^\beta w_2^\beta = (s_1 s_2 s_3 s_4 s_5)(s_7 s_8 s_9)$ is Coxeter in $W_\beta = W_{J_1^\beta} \times W_{J_2^\beta}$. We will show that $(I - w_\beta)$ has Smith normal form $(1^7, 2, 0)$, and hence $R^\vee / \text{MOD}(w_\beta) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. That is, the torsion in this quotient is determined by $2 = \gcd(6, 4) = \gcd(\beta_1, \beta_2)$, while its free rank equals $1 = p - 1$, where $p = 2$ is the number of parts of β .

We work with the \mathbb{Z} -basis $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_9^\vee\}$ for R^\vee . Now $w_1^\beta = s_1 s_2 s_3 s_4 s_5$ fixes α_i^\vee for $7 \leq i \leq 9$, and $w_2^\beta = s_7 s_8 s_9$ fixes α_i^\vee for $1 \leq i \leq 5$, so we can compute similarly to Example 4.3 that $w_\beta(\alpha_i^\vee) = \alpha_{i+1}^\vee$ for $i \in \{1, 2, 3, 4\} \cup \{7, 8\}$, while

$$w_\beta(\alpha_5^\vee) = - \sum_{i=1}^5 \alpha_i^\vee \quad \text{and} \quad w_\beta(\alpha_9^\vee) = - \sum_{i=7}^9 \alpha_i^\vee.$$

By inserting $s_6 s_6$ at the end of w_1^β , we also compute using (2.1.1) that

$$\begin{aligned} w_\beta(\alpha_6^\vee) &= (s_1 s_2 s_3 s_4 s_5)(s_7 s_8 s_9)(\alpha_6^\vee) \\ &= (s_1 s_2 s_3 s_4 s_5)s_7(\alpha_6^\vee) \\ &= s_1 s_2 s_3 s_4 s_5(\alpha_6^\vee + \alpha_7^\vee) \\ &= s_1 s_2 s_3 s_4 s_5 s_6(s_6 \alpha_6^\vee) + \alpha_7^\vee \\ &= -s_1 s_2 s_3 s_4 s_5 s_6(\alpha_6^\vee) + \alpha_7^\vee \\ &= - \left(- \sum_{i=1}^6 \alpha_i^\vee \right) + \alpha_7^\vee \\ &= \sum_{i=1}^7 \alpha_i^\vee. \end{aligned}$$

Hence the matrices for w_β and $M_\beta = I - w_\beta$ with respect to the basis Δ^\vee are:

$$w_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix} \quad M_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 \end{pmatrix}.$$

Here, omitted entries are all 0s.

We now solve the \mathbb{Z} -linear system $M_\beta \mathbf{x} = \mathbf{c}$ using row operations. We replace row 1 of M_β by the sum of its rows $1, \dots, 9$ to obtain the matrix

$$M'_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 6 & -6 & 0 & 0 & 4 \\ -1 & 1 & 0 & 0 & 1 & -1 & & & \\ 0 & -1 & 1 & 0 & 1 & -1 & & & \\ 0 & 0 & -1 & 1 & 1 & -1 & & & \\ 0 & 0 & 0 & -1 & 2 & -1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 & & & \\ 0 & -1 & 1 & 1 & 1 & -1 & & & \\ 0 & 0 & -1 & 2 & 0 & 2 \end{pmatrix}.$$

Tracking the effect of these row operations on \mathbf{c} , and noticing that for $i = 2, 3, 4, 5, 7, 8, 9$, the leading entry in row i of M'_β is its $(i, i-1)$ entry, which equals -1 , while row 6 of M'_β is all 0s, we see that the corresponding system of linear equations has solution over \mathbb{Z} if and only if $c_6 = 0$ and there are $x_5, x_6, x_9 \in \mathbb{Z}$ such that

$$6x_5 - 6x_6 + 4x_9 = \sum_{i=1}^9 c_i.$$

Since $\gcd(6, -6, 4) = 2$, by Bezout's Theorem this equation has integer solution if and only if $\sum_{i=1}^9 c_i \equiv 0 \pmod{2}$. Hence

$$\text{MOD}(w_\beta) = M_\beta R^\vee = \left\{ \sum_{i=1}^9 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_6 = 0, \sum_{i=1}^n c_i \equiv 0 \pmod{2} \right\}.$$

We next obtain the Smith normal form S_β for M_β , using a combination of column and row operations. Starting with the matrix M_β , we add columns 5, 7, and 8 to column 6, to obtain

$$\hat{M}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 & 0 & 0 \end{pmatrix}.$$

Next, for $i_1 = 2, 3, 4, 5$ we replace row i_1 of \hat{M}_β by the sum of its rows $1, \dots, i_1$, and for $i_2 = 8, 9$ we replace row i_2 of \hat{M}_β by the sum of its rows $7, \dots, i_2$. This yields the matrix

$$T_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

Here, we have boxed the pivot entries 1 in columns $i_1 = 1, 2, 3, 4$ and $i_2 = 7, 8$, and circled the non-zero entries in columns 5 and 9 which we will clear, using these pivots and column operations, at the next step. This results in the matrix

$$T'_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

We now add row 5 to row 9, which clears the circled -1 and puts a 6 in position $(9, 5)$, and then subtract 6 times column 6 from column 5, which clears the circled 6, to obtain

$$T''_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where we have boxed the pivot entries 1 in columns 1, 2, 3, 4, 6, 7, 8. By applying Bezout's Theorem to the last row, whose nonzero entries are $\beta_1 = 6$ and $\beta_2 = 4$, we obtain that the Smith normal form S_β for M_β is $\text{diag}(1^7, 2, 0)$, where $2 = \gcd(6, 4)$. Therefore, the quotient $R^\vee/\text{MOD}(w_\beta) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.

Finally, to obtain a \mathbb{Z} -basis for $\text{MOD}(w_\beta)$, we go back to M_β above and perform the following column operations. First replace column 6 by the sum of columns 5 and 6. Then, apply the same column operations as used to obtain B_β in Example 4.3, here on the upper 5×5 block and the lower 3×3 block, respectively. Then, replace column 5 by -6 times the

sum of columns 6, 7, 8 to obtain

$$B'_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & -1 \\ & & & & 0 & 1 \\ & & & & 0 & 0 \\ & & & & 6 & 0 \end{pmatrix}.$$

Since $\gcd(6, 4) = 2$, by Bezout's Theorem, there exist column operations on columns 5 and 9 which first replaces column 9 with $\gcd(6, 4) = 2$ and then clears the 6 from column 5, resulting in

$$B_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ & & & & 0 \\ & & & & -1 & 1 & 0 & 0 \\ & & & & 0 & -1 & 1 & 0 \\ & & & & 0 & 0 & -1 & 1 & 0 \\ & & & & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Therefore, one \mathbb{Z} -basis for $\text{MOD}(w_\beta)$ is given by

$$\{\alpha_i^\vee - \alpha_{i+1}^\vee \mid i = 1, 2, 3, 4, 7, 8\} \cup \{\alpha_5^\vee - \alpha_7^\vee\} \cup \{2\alpha_9^\vee\}.$$

In this example, $J_\beta = \{1, 2, 3, 4, 5, 7, 8, 9\}$ and $I_\beta = \{6, 10\}$, in which case $I_\beta - 1 = \{5, 9\}$ and $(I_\beta - 1) \setminus \{9\} = \{5\}$. Therefore, this basis matches the one given in Theorem 4.2.

Example 4.5. Let W be of type A_4 , and consider the partition $\beta = (\beta_1, \beta_2) = (4, 1)$ of $n + 1 = 5$. We now have $J_1^\beta = \{1, 2, 3\}$ and $J_2^\beta = \emptyset$, so $J_\beta = J_1^\beta = \{1, 2, 3\}$ and $w_\beta = s_1s_2s_3$ is Coxeter in $W_\beta = W_{J_1^\beta}$. We will show that $(I - w_\beta)$ has Smith normal form $(1, 1, 1, 0) = (1^{n-p+1}, 0^{p-1})$, and hence $R^\vee/\text{MOD}(w_\beta) \cong \mathbb{Z} = \mathbb{Z}^{p-1}$ is free of rank $p - 1$, where $p = 2$ is the number of parts of β .

We work with the \mathbb{Z} -basis $\Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee\}$ for R^\vee , and similarly to Example 4.3 compute that

$$w_\beta(\alpha_1^\vee) = \alpha_2^\vee, \quad w_\beta(\alpha_2^\vee) = \alpha_3^\vee, \quad \text{and} \quad w_\beta(\alpha_3^\vee) = -\alpha_1^\vee - \alpha_2^\vee - \alpha_3^\vee.$$

This time we also need

$$w_\beta(\alpha_4^\vee) = s_1s_2s_3(\alpha_4^\vee) = s_1s_2s_3s_4(s_4\alpha_4^\vee) = -s_1s_2s_3s_4(\alpha_4^\vee) = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee.$$

Hence the matrices for w_β and $M_\beta = I - w_\beta$ with respect to the basis Δ^\vee are given by

$$w_\beta = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_\beta = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now solve the \mathbb{Z} -linear system $M_\beta \mathbf{x} = \mathbf{c}$. We replace row 1 of M_β by the sum of its rows 1, 2, and 3 to obtain the matrix

$$M'_\beta = \begin{pmatrix} 0 & 0 & 4 & -3 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that the corresponding system of linear equations has solution over \mathbb{Z} if and only if $c_4 = 0$ and there are $x_3, x_4 \in \mathbb{Z}$ such that

$$4x_3 - 3x_4 = c_1 + c_2 + c_3.$$

However since $\gcd(4, -3) = 1$, by Bezout's Theorem this equation has integer solution for all $c_1, c_2, c_3 \in \mathbb{Z}$. Hence for $w_\beta = s_1 s_2 s_3$ in W of type A_4 , we have

$$\text{MOD}(w_\beta) = M_\beta R^\vee = \left\{ \sum_{i=1}^3 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z} \right\} = \left\{ \sum_{i=1}^4 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z} \text{ and } c_4 = 0 \right\}.$$

It is immediate that a \mathbb{Z} -basis for $\text{MOD}(w_\beta)$ is given by $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\} = \{\alpha_j^\vee : j \in J_\beta\}$.

Finally, in this case we can use just column operations to obtain the Smith normal form. Starting with M_β , for $i = 1, 2, 3$ we replace column i of M_β by the sum of its columns $i, \dots, 4$, to obtain the upper-triangular matrix

$$T_\beta = \begin{pmatrix} \boxed{1} & 0 & 0 & \circled{-1} \\ 0 & \boxed{1} & 0 & \circled{-1} \\ 0 & 0 & \boxed{1} & \circled{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, we have boxed the pivot entries 1, and we have circled the non-zero entries which we will clear, using column operations, at the next step. This yields Smith normal form

$$S_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $R^\vee / \text{MOD}(w_\beta) = R^\vee / M_\beta R^\vee \cong \mathbb{Z}$.

Our final example in type A is as follows.

Example 4.6. Let W be of type A_7 , and consider the partition $\beta = (\beta_1, \beta_2, \beta_3) = (4, 3, 1)$ of $n+1 = 8$. We now have $J_1^\beta = \{1, 2, 3\}$, $J_2^\beta = \{5, 6\}$, and $J_3^\beta = \emptyset$, so $w_\beta = (s_1 s_2 s_3)(s_5 s_6)$. We will show that $(I - w_\beta)$ has Smith normal form $(1^5, 0^2) = (1^{n-p+1}, 0^{p-1})$, hence $R^\vee / \text{MOD}(w_\beta) \cong \mathbb{Z}^2 = \mathbb{Z}^{p-1}$, where $p = 3$ is the number of parts of β .

We work with the \mathbb{Z} -basis $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_7^\vee\}$ for R^\vee , and similarly to previous examples compute that the matrices for w_β and $M_\beta = I - w_\beta$ with respect to the basis Δ^\vee are

$$w_\beta = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & -1 & 1 \\ & & 0 & 1 & -1 & 1 \\ & & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_\beta = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & -1 & 1 & 1 & -1 \\ & & 0 & -1 & 2 & -1 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now solve the \mathbb{Z} -linear system $M_\beta \mathbf{x} = \mathbf{c}$, using row operations. We replace row 1 of M_β by the sum of its rows 1 through 7 to obtain the matrix

$$M'_\beta = \begin{pmatrix} 0 & 0 & 4 & -4 & 0 & 3 & -2 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & -1 & 1 & 1 & -1 \\ & & 0 & -1 & 2 & -1 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\gcd(4, -4, 3, -2) = \gcd(3, -2) = 1$, while rows 4 and 7 are all 0s, we see that

$$\text{MOD}(w_\beta) = M_\beta R^\vee = \left\{ \sum_{i=1}^7 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z} \text{ and } c_4 = c_7 = 0 \right\}.$$

It is immediate that a \mathbb{Z} -basis for $\text{MOD}(w_\beta)$ is given by $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_5^\vee, \alpha_6^\vee\} = \{\alpha_j^\vee \mid j \in J_\beta\}$.

Finally, we obtain the Smith normal form S_β for M_β , using just column operations. To do this, starting with M_β , for $i = 4, 5, 6$ we replace column i by the sum of columns $i, \dots, 7$, to obtain

$$T_\beta = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & \boxed{1} & 0 & \circled{-1} \\ & & 0 & 0 & \boxed{1} & \circled{-1} \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now for $i = 1, 2, 3$ we replace column i of T_β by the sum of columns $i, \dots, 4$. This yields

$$T'_\beta = \begin{pmatrix} \boxed{1} & 0 & 0 & \circled{-1} \\ 0 & \boxed{1} & & \circled{-1} \\ 0 & 0 & \boxed{1} & \circled{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & \boxed{1} & 0 & \circled{-1} \\ & & 0 & 0 & \boxed{1} & \circled{-1} \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We then use column operations with the boxed pivots to clear the circled entries, and so obtain that the Smith normal form is $S_\beta = (1^5, 0^2)$, and hence $R^\vee/\text{MOD}(w_\beta) \cong \mathbb{Z}^2$.

4.3. Proofs in type A. In order to prove Theorem 4.2, in Section 4.3.1 we define a collection of auxiliary matrices and record some related \mathbb{Z} -linear-algebraic results. We then determine the matrix for w_β , and hence for $M_\beta = I - w_\beta$, with respect to Δ^\vee in Section 4.3.2, and complete the proof of Theorem 4.2 in Section 4.3.3.

4.3.1. Matrix definitions and results in type A. In this section, we define a collection of useful matrices, and record some results related to these matrices.

For $r \geq 1$, we write I_r or sometimes just I for the $r \times r$ identity matrix, and 0_r for the $r \times r$ zero matrix. For any matrix M , we write $C_i(M)$ for the i th column of M and $R_i(M)$ for the i th row of M .

We next define 1×1 matrices $W_1 = [-1]$ and $M_1 = I_1 - W_1 = [2]$, and for $r \geq 2$, define

$$(4.3.1) \quad W_r = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & -1 \\ 1 & 0 & \ddots & \ddots & 0 & -1 \\ 0 & 1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & 0 & -1 \\ \vdots & & \ddots & 1 & 0 & -1 \\ 0 & \dots & \dots & 0 & 1 & -1 \end{pmatrix}, \quad M_r = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 1 \\ -1 & 1 & 0 & \ddots & 0 & 1 \\ 0 & -1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 1 & 0 & 1 \\ \vdots & & \ddots & -1 & 1 & 1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

That is, the (r, r) -entry of $M_r = I - W_r$ is 2, its (i, i) - and (i, r) -entries are 1 for $1 \leq i \leq r - 1$, its $(i + 1, i)$ -entry is -1 for $1 \leq i \leq r - 1$, and all other entries of M_r are 0. We also define $(r + 1) \times (r + 1)$ upper block-triangular matrices $W_{r,1}$ and $M_{r,1}$ by

$$(4.3.2) \quad W_{r,1} = \begin{pmatrix} & & & & & 1 \\ & W_r & & & & \\ & & & & & \\ 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & -1 & 1 \\ 1 & 0 & \ddots & \ddots & 0 & -1 & 1 \\ 0 & 1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & 0 & -1 & 1 \\ \vdots & & \ddots & 1 & 0 & -1 & 1 \\ 0 & \dots & \dots & 0 & 1 & -1 & 1 \\ 0 & \dots & \dots & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$(4.3.3) \quad M_{r,1} = I - W_{r,1} = \begin{pmatrix} & & & & & -1 \\ & M_r & & & & \\ & & & & & \\ 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 1 & -1 \\ -1 & 1 & 0 & \ddots & 0 & 1 & -1 \\ 0 & -1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & 1 & 0 & 1 & -1 \\ \vdots & & \ddots & -1 & 1 & 1 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next, define M'_r to be the matrix obtained from M_r by replacing $\mathcal{R}_1(M_r)$ by the sum $\sum_{i=1}^r \mathcal{R}_i(M_r)$, so that $M'_1 = M_1 = [2]$ and for $r \geq 2$, we have

$$M'_r = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & r+1 \\ -1 & 1 & 0 & \ddots & 0 & 1 \\ 0 & -1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 1 & 0 & 1 \\ \vdots & & \ddots & -1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

That is, the matrix M'_r is the same as M_r in rows 2 through r , and the only nonzero entry in the first row of M'_r is the $(r+1)$ in position $(1, r)$. Similarly, we define $M'_{r,1}$ to be the matrix obtained from $M_{r,1}$ by replacing $\mathcal{R}_1(M_{r,1})$ by the sum $\sum_{i=1}^r \mathcal{R}_i(M_{r,1})$, so that

$$M'_{r,1} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & r+1 & -r \\ -1 & 1 & 0 & \ddots & 0 & 1 & -1 \\ 0 & -1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & 1 & 0 & 1 & -1 \\ \vdots & & \ddots & -1 & 1 & 1 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

That is, the matrix $M'_{r,1}$ is the same as $M_{r,1}$ in rows 2 through $r+1$, and the only nonzero entries in the first row of $M'_{r,1}$ are the $(r+1)$ in position $(1, r)$, and the $-r$ in position $(1, r+1)$. We use M'_r and $M'_{r,1}$ to establish the following result.

Lemma 4.7. *For all $r \geq 1$:*

- (1) *The \mathbb{Z} -linear system $M_r \mathbf{x} = \mathbf{c}$ has solution $\mathbf{x} \in \mathbb{Z}^r$ if and only if*

$$\sum_{i=1}^r c_i \equiv 0 \pmod{r+1}.$$

- (2) *The \mathbb{Z} -linear system $M_{r,1} \mathbf{x} = \mathbf{c}$ has solution $\mathbf{x} \in \mathbb{Z}^{r+1}$ if and only if $c_{r+1} = 0$.*

Proof. Since $M_1 = [2]$, the result (1) is obvious for $r = 1$. For $r \geq 2$, write $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{Z}^r$. Then, tracking the effect of the row operations used to obtain M'_r from M_r on \mathbf{c} , we see that the system $M_r \mathbf{x} = \mathbf{c}$ has solution $\mathbf{x} \in \mathbb{Z}^r$ if and only if the system $M'_r \mathbf{x} = \mathbf{c}'$ has solution $\mathbf{x} \in \mathbb{Z}^r$, where $\mathbf{c}' = (\sum_{i=1}^r c_i, c_2, \dots, c_r)$. Now for $2 \leq i \leq r$, row i of M'_r has leading entry given by the -1 in position $(i, i-1)$. Hence $M'_r \mathbf{x} = \mathbf{c}'$ has solution $\mathbf{x} \in \mathbb{Z}^r$ if and only if the equation $(r+1)x_r = \sum_{i=1}^r c_i$, coming from the first row of M'_r and first entry of \mathbf{c}' , has solution $x_r \in \mathbb{Z}$. This is equivalent to $\sum_{i=1}^r c_i \equiv 0 \pmod{r+1}$.

For (2), notice that for $2 \leq i \leq r$, the leading entry in row i of $M'_{r,1}$ is the -1 in position $(i, i-1)$, while row $r+1$ is all 0s. Therefore $M_{r,1} \mathbf{x} = \mathbf{c}$ has solution $\mathbf{x} \in \mathbb{Z}^{r+1}$ if and only if there are $x_r, x_{r+1} \in \mathbb{Z}$ such that $(r+1)x_r - rx_r = \sum_{i=1}^r c_i$ (from the first row) and $c_{r+1} = 0$. But $\gcd(r+1, -r) = 1$, so by Bezout's Theorem, the equation $(r+1)x_r - rx_r = c$ in fact has a solution $x_r, x_{r+1} \in \mathbb{Z}$ for any $c \in \mathbb{Z}$. Thus the only condition for $M_{r,1} \mathbf{x} = \mathbf{c}$ to have solution over \mathbb{Z} is that $c_{r+1} = 0$. \square

We now consider the coroot lattice R^\vee as a free \mathbb{Z} -module with \mathbb{Z} -basis $\Delta^\vee = \{\alpha_i^\vee\}$. When taking the subsystem of type A_r for $1 \leq r < n$ determined by the first r nodes of the Dynkin diagram, we denote the coroot lattice by R_r^\vee and the corresponding basis by Δ_r^\vee .

The following is then an immediate corollary of Lemma 4.7.

Corollary 4.8. *For all $r \geq 1$, we have*

$$M_r R_r^\vee = (\mathbf{I} - W_r) R_r^\vee = \left\{ \sum_{i=1}^r c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, \sum_{i=1}^r c_i \equiv 0 \pmod{r+1} \right\}$$

and

$$M_{r,1} R_{r+1}^\vee = (\mathbf{I} - W_{r,1}) R_{r+1}^\vee = \left\{ \sum_{i=1}^{r+1} c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_{r+1} = 0 \right\}.$$

We next define T_r to be the matrix obtained from M_r by replacing $\mathcal{R}_i(M_r)$ by the sum $\sum_{j=1}^i \mathcal{R}_j(M_r)$ for $2 \leq i \leq r$, so that $T_1 = M_1 = [2]$ and for $r \geq 2$, we have

$$(4.3.4) \quad T_r = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 2 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 1 & 0 & r-2 \\ \vdots & \vdots & \ddots & 0 & 1 & r-1 \\ 0 & 0 & \cdots & 0 & 0 & r+1 \end{pmatrix}.$$

The definition of $T_{r,1}$ is qualitatively different to that of T_r : for $1 \leq i \leq r$, we replace $\mathcal{C}_i(M_{r,1})$ by the sum $\sum_{i=1}^{r+1} \mathcal{C}_i(M_{r,1})$, to obtain

$$(4.3.5) \quad T_{r,1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 1 & 0 & -1 \\ \vdots & \vdots & \ddots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

We then define S_r to be the matrix obtained from T_r by replacing $\mathcal{C}_r(T_r)$ by the difference $\mathcal{C}_r(T_r) - \sum_{i=1}^{r-1} i \mathcal{C}_i(T_r)$. Then S_r is the diagonal matrix $S_r = \text{diag}(1^{r-1}, r+1)$. We also define $S_{r,1}$ to be the matrix obtained from $T_{r,1}$ by replacing $\mathcal{C}_{r+1}(T_{r,1})$ by the difference $\mathcal{C}_{r+1}(T_{r,1}) + \sum_{i=1}^r \mathcal{C}_i(T_{r,1})$. Then $S_{r,1}$ is the diagonal matrix $S_{r,1} = \text{diag}(1^r, 0)$. Since S_r (respectively, $S_{r,1}$) is in Smith normal form, and has been obtained from M_r (respectively, $M_{r,1}$) by \mathbb{Z} -linear row and/or column operations, we obtain:

Lemma 4.9. *For all $r \geq 1$, we have*

- (1) $M_r = \mathbf{I} - W_r$ has Smith normal form $S_r = \text{diag}(1^{r-1}, r+1)$; and
- (2) $M_{r,1} = \mathbf{I} - W_{r,1}$ has Smith normal form $S_{r,1} = \text{diag}(1^r, 0)$.

Corollary 4.10. *For all $r \geq 1$, we have*

- (1) $R_r^\vee / (\mathbf{I} - W_r) R_r^\vee \cong \mathbb{Z}/(r+1)\mathbb{Z}$; and
- (2) $R_{r+1}^\vee / (\mathbf{I} - W_{r,1}) R_{r+1}^\vee \cong \mathbb{Z}$.

Finally, we define B_r to be the matrix obtained from M_r by replacing $\mathcal{C}_r(M_r)$ by the sum $-\sum_{j=i}^{r-1} j\mathcal{C}_j(M_r) + \mathcal{C}_r(M_r)$. Then

$$(4.3.6) \quad B_r = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ -1 & 1 & 0 & \ddots & 0 & 0 \\ 0 & -1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 1 & 0 & 0 \\ \vdots & & \ddots & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & r+1 \end{pmatrix}.$$

In other words, B_r and M_r differ only in column r , which has a single nonzero entry $r+1$ in the last entry. Since M_r and B_r have the same column-space, while $M_{r,1}$ and $S_{r,1}$ have the same column-space (since we used only column operations to obtain $S_{r,1}$ from $M_{r,1}$), it follows that:

Lemma 4.11. *For all $r \geq 1$, we have*

- (1) $M_r R_r^\vee = (I - W_r) R_r^\vee$ has a \mathbb{Z} -basis given by $\{\alpha_i^\vee - \alpha_{i+1}^\vee \mid 1 \leq i \leq r-1\} \cup \{(r+1)\alpha_r^\vee\}$.
- (2) $M_{r,1} R_{r+1}^\vee = (I - W_{r,1}) R_{r+1}^\vee$ has a \mathbb{Z} -basis given by $\{\alpha_i^\vee \mid 1 \leq i \leq r\} = \Delta_{r+1}^\vee \setminus \{\alpha_{r+1}^\vee\}$.

4.3.2. The matrices for w_β and M_β . In this section, we determine the matrices for w_β and $M_\beta = I - w_\beta$ with respect to the basis Δ^\vee for R^\vee in type A_n for $n \geq 1$, in Corollaries 4.16 and 4.17, respectively. This result will involve the auxiliary matrices W_r , $W_{r,1}$, M_r , and $M_{r,1}$ defined in Section 4.3.1.

We begin with the following observation.

Lemma 4.12. *Let W be of type A_n , for $n \geq 1$. If $\beta = (n+1)$, then w_β is the Coxeter element $s_1 s_2 \dots s_n$ of W , and the matrix for w_β with respect to Δ^\vee is W_n from (4.3.1).*

Proof. This is obtained by straightforward generalization of the corresponding computations in Example 4.3. \square

We next consider the individual parts $\beta_k \geq 2$ of the partition β . These determine blocks of size $(\beta_k - 1) \times (\beta_k - 1)$ in the matrix for w_β , as described by the next result.

Lemma 4.13. *Let W be of type A_n , for $n \geq 1$, and let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$. Then for all $k \in [p]$ such that $\beta_k \geq 2$:*

- (1) *the $(\beta_k - 1) \times (\beta_k - 1)$ square submatrix of w_β with entries $(w_\beta)_{ij}$ for $i, j \in J_k^\beta$ is equal to the matrix W_r from (4.3.1) with $r = \beta_k - 1$; and*
- (2) *for all $j \in J_k^\beta$ and $i \in [n] \setminus J_k^\beta$, the (i, j) -entry of w_β equals 0.*

Proof. Recall that the element w_k^β is the product, in increasing order, of the simple reflections in the type A_{β_k-1} subsystem indexed by $J_k^\beta = \{j_k^\beta + 1, \dots, j_k^\beta + (\beta_k - 1)\}$. Hence w_k^β fixes α_i^\vee for all $1 \leq i \leq j_k^\beta - 1$ and all $j_k^\beta + \beta_k + 1 \leq i \leq n$. The result is then obtained by shifting the indexing and applying Lemma 4.12. \square

For partitions β which have at least two parts equal to 1, we have the following.

Lemma 4.14. *Let W be of type A_n for $n \geq 1$, and let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$. Suppose that the last $m \geq 2$ parts of β are equal to 1. Then:*

- (1) *the $(m-1) \times (m-1)$ square submatrix of w_β with entries $(w_\beta)_{ij}$ for $n-m+2 \leq i, j \leq n$ is equal to I_{m-1} ; and*

(2) for all $1 \leq i \leq n - m + 1$ and $n - m + 2 \leq j \leq n$, the (i, j) -entry of w_β equals 0.

Proof. We have that $\sum_{k=1}^{p-m} \beta_k = (n+1) - m = (n-m) + 1$ is the sum of the parts of β which are ≥ 2 . Hence w_β is a product of a subset of the simple reflections $\{s_1, \dots, s_{n-m}\}$, and so w_β fixes α_i^\vee for $n - m + 2 \leq i \leq n$. The result follows. \square

To describe the remaining columns of w_β ; that is, the columns not in the blocks given by the previous two lemmas, we use the following.

Lemma 4.15. *Let W be of type A_n for $n \geq 1$, and let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$.*

(1) *If $2 \leq k \leq p$ is such that $\beta_{k-1} \geq 2$ and $\beta_k \geq 2$, then*

$$w_\beta \left(\alpha_{j_k}^\vee \right) = \sum_{i=j_{k-1}^\beta + 1}^{j_k^\beta + 1} \alpha_i^\vee.$$

(2) *If $2 \leq k \leq p$ is such that $\beta_{k-1} \geq 2$ and $\beta_k = 1$, then*

$$w_\beta \left(\alpha_{j_k}^\vee \right) = \sum_{i=j_{k-1}^\beta + 1}^{j_k^\beta} \alpha_i^\vee.$$

Proof. To simplify notation, we write j_k for j_k^β . We will prove (2) first. In the notation of (4.1.2), we compute using (2.1.1) that

$$\begin{aligned} w_{k-1}^\beta \left(\alpha_{j_k}^\vee \right) &= s_{j_{k-1}+1} \dots s_{j_k-1} \left(\alpha_{j_k}^\vee \right) \\ &= s_{j_{k-1}+1} \dots s_{j_k-1} s_{j_k} \left(s_{j_k} \alpha_{j_k}^\vee \right) \\ &= - \left(- \sum_{i=j_{k-1}+1}^{j_k} \alpha_i^\vee \right) \\ &= \sum_{i=j_{k-1}+1}^{j_k} \alpha_i^\vee. \end{aligned}$$

Since $w_\beta = w_1^\beta \dots w_{k-1}^\beta$ in this case, and the product $w_1^\beta \dots w_{k-2}^\beta$ fixes α_i^\vee for all $j_{k-1}^\beta + 1 \leq i \leq n$, we obtain that $w_\beta \left(\alpha_{j_k}^\vee \right) = w_{k-1}^\beta \left(\alpha_{j_k}^\vee \right)$ has the required formula.

Now for (1), since w_ℓ^β fixes $\alpha_{j_k}^\vee$ for all $\ell \notin \{k-1, k\}$ and w_{k-1}^β and w_k^β commute, we obtain using the proof of (2) that

$$\begin{aligned} w_\beta(\alpha_{j_k}^\vee) &= w_k^\beta w_{k-1}^\beta(\alpha_{j_k}^\vee) \\ &= s_{j_k+1} \dots s_{j_k+\beta_k-1} \left(\sum_{i=j_{k-1}+1}^{j_k} \alpha_i^\vee \right) \\ &= \left(\sum_{i=j_{k-1}+1}^{j_k-1} \alpha_i^\vee \right) + s_{j_k+1}(\alpha_{j_k}^\vee) \\ &= \left(\sum_{i=j_{k-1}+1}^{j_k-1} \alpha_i^\vee \right) + \alpha_{j_k}^\vee + \alpha_{j_k+1}^\vee \end{aligned}$$

as required. \square

Applying Lemmas 4.13, 4.14, and 4.15, we can now describe the matrix for w_β in type A_n . In the next two results, (1) generalizes Examples 4.3 and 4.4, and (2) generalizes Examples 4.5 and 4.6. The matrix for w_β is as follows.

Corollary 4.16. *Let W be of type A_n for $n \geq 1$, and let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$.*

- (1) *If $\beta_p \geq 2$, then the matrix for w_β with respect to the basis Δ^\vee for R^\vee satisfies:*
 - (a) *for all $k \in [p]$, the $(\beta_k - 1) \times (\beta_k - 1)$ square submatrix of w_β with entries $(w_\beta)_{ij}$ for $i, j \in J_k^\beta$ is equal to W_{β_k-1} .*
 - (b) *for all $2 \leq k \leq p$ and all $j_{k-1}^\beta + 1 \leq i \leq j_k^\beta + 1$, the (i, j_k^β) -entry of w_β is equal to 1.*
 - (c) *all other entries of w_β are zero.*
- (2) *If the last $m \geq 1$ parts of the partition β are equal to 1, then the matrix for w_β with respect to the basis Δ^\vee for R^\vee satisfies:*
 - (a) *for all $k \in [p]$ such that $\beta_k \geq 2$, the $\beta_k \times \beta_k$ square submatrix of w_β with entries $(w_\beta)_{ij}$ for $i, j \in J_k^\beta \cup \{j_k^\beta\}$ is equal to $W_{\beta_k-1, 1}$.*
 - (b) *for all $2 \leq k \leq p$ such that $\beta_k \geq 2$, the $(j_k^\beta + 1, j_k^\beta)$ -entry of w_β is equal to 1.*
 - (c) *the $(m-1) \times (m-1)$ square submatrix of w_β with entries $(w_\beta)_{ij}$ for $n-m+2 \leq i, j \leq n$ is equal to I_{m-1} .*
 - (d) *all other entries of w_β are zero.*

Therefore, we immediately obtain a formula for the matrix $M_\beta = I - w_\beta$ as follows.

Corollary 4.17. *Let W be of type A_n for $n \geq 1$, and let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$. Let M_β be the matrix for $I - w_\beta$ with respect to the basis Δ^\vee for R^\vee .*

- (1) *If $\beta_p \geq 2$, then M_β satisfies:*
 - (a) *for all $k \in [p]$, the $(\beta_k - 1) \times (\beta_k - 1)$ square submatrix of w_β with entries $(w_\beta)_{ij}$ for $i, j \in J_k^\beta$ is equal to M_{β_k-1} .*
 - (b) *for all $2 \leq k \leq p$ and all $i \in J_{k-1}^\beta \cup \{j_k^\beta + 1\}$, the (i, j_k^β) -entry of w_β is equal to -1.*
 - (c) *all other entries of M_β are zero.*
- (2) *If the last $m \geq 1$ parts of the partition β are equal to 1, then M_β satisfies:*

- (a) for all $k \in [p]$ such that $\beta_k \geq 2$, the $\beta_k \times \beta_k$ square submatrix of w_β with entries $(w_\beta)_{ij}$ for $i, j \in J_k^\beta \cup \{j_k^\beta\}$ is equal to $M_{\beta_k-1,1}$.
- (b) for all $2 \leq k \leq p$ such that $\beta_k \geq 2$, the $(j_k^\beta + 1, j_k^\beta)$ -entry of w_β is equal to -1 .
- (c) all other entries of M_β are zero.

Equipped with explicit formulas for the matrix M_β from Corollary 4.17, we are now prepared to prove our main theorem in type A.

4.3.3. Proof of Theorem 4.2. We now complete the proof of Theorem 4.2. We will use the description of the matrix $M_\beta = I - w_\beta$ from Corollary 4.17, as well as all of the auxiliary matrices and results concerning them from Section 4.3.1.

Let W be of type A_n with $n \geq 1$ and let $\beta = (\beta_1, \dots, \beta_p)$ be a partition of $n+1$. Note that the partition $\beta = (1, \dots, 1)$ of $n+1$ corresponds to w_β being the trivial element of W . The statement of Theorem 4.2(3) in this case is obvious, so we assume from now on that $\beta_1 \geq 2$.

We next consider the case that $\beta_p \geq 2$. If $p=1$ then $\beta = (n+1)$, so $\gcd(\beta_k) = \beta_1 = n+1$, and Theorem 4.2 in this case is immediate from Lemma 4.12 and the results for the matrix W_n in Section 4.3.1.

We now assume that $p \geq 2$. Note that $M_\beta = I - w_\beta$ is as described in Corollary 4.17(1). We proceed to generalize Examples 4.3 and 4.4, and we use the same notation as developed in those examples here in the general case.

To determine $\text{MOD}(w_\beta) = M_\beta R^\vee$, we replace row 1 of M_β by the sum of its rows $1, \dots, n$, to obtain a matrix M'_β . Then for $2 \leq k \leq p$, the $(1, j_k^\beta - 1)$ -entry of M'_β is equal to β_k and the $(1, j_k^\beta)$ -entry of M'_β is equal to $-\beta_k$. Also the $(1, n)$ -entry of M'_β is equal to β_p , and all other entries in row 1 of M'_β are 0. Rows $2, \dots, n$ of M'_β are the same as in M_β , and hence M'_β has row i all 0s for $i \in [n] \setminus J_\beta$, while for $i \in J_\beta$ with $i \neq 1$, the leading entry in row i of M'_β is its $(i, i-1)$ entry, which equals -1 . Since $\gcd(\beta_1, -\beta_1, \dots, \beta_{p-1}, -\beta_{p-1}, \beta_p) = \gcd(\beta_k)$, it follows that

$$\text{MOD}(w_\beta) = \left\{ \sum_{i=1}^n c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_i = 0 \text{ for all } i \in [n] \setminus J_\beta, \sum_{i=1}^n c_i \equiv 0 \pmod{\gcd(\beta_k)} \right\}.$$

We next obtain the Smith normal form S_β for M_β . To do this, starting with the matrix M_β , for all $2 \leq k \leq p$, we add to column j_k^β of M_β its columns with indexes $i \in \{j_k^\beta - 1\} \cup \{j_k^\beta + 1, j_k^\beta + 2, \dots, j_k^\beta + \beta_k\}$. That is, we add to column j_k^β the column with the highest index in J_{k-1}^β , and the columns with all indexes in J_k^β except for the highest. Let \hat{M}_β be the so-obtained matrix. Then \hat{M}_β is the same as M_β except that for $2 \leq k \leq p$, column j_k^β is now all 0s except for an entry 1 in row $j_k^\beta - 1$ (the highest index in J_{k-1}^β), and an entry -1 in row $j_k^\beta + (\beta_k - 1)$ (the highest index in J_k^β).

Next, for all $k \in [p]$, we carry out row operations on the rows of \hat{M}_β which contain the submatrix M_{β_k-1} , so as to replace M_{β_k-1} by the matrix T_{β_k-1} defined in (4.3.4). Call the resulting matrix T_β . For each $k \in [p]$, and each $i_k \in J_k^\beta$ with i_k not the highest index in J_k^β , the matrix T_β has a pivot entry 1 in its (i_k, i_k) -position. At the next step, for $k \in [p]$ we use these pivots to clear all of the nonzero entries in column i_k , where i_k is the highest index in J_k^β , other than the (i_k, i_k) -entry (which equals β_k). Call the resulting matrix T'_β . We now, for $2 \leq k \leq p$, replace the row of T'_β with index the highest in J_k^β by the sum of its rows with indices the highest in $J_1^\beta, \dots, J_{k-1}^\beta, J_k^\beta$. Finally, we use column operations to clear all

entries in rows $1, \dots, n-1$ which are not already pivot 1s. The resulting matrix T''_β satisfies the following:

- (1) for all $1 \leq k < p$, the $(\beta_k - 1) \times (\beta_k - 1)$ square submatrix of T''_β with entries $(T''_\beta)_{ij}$ for $i, j \in J_k^\beta$ is I_{β_k-1} ;
- (2) the $(\beta_p - 1) \times (\beta_p - 1)$ square submatrix of T''_β with entries $(T''_\beta)_{ij}$ for $i, j \in J_p^\beta$ is $\text{diag}(1^{\beta_p-1}, \beta_p)$;
- (3) for all $2 \leq k \leq p$, the $(j_k^\beta - 1, j_k^\beta)$ -entry is 1;
- (4) for all $2 \leq k \leq p$, the $(n, j_k^\beta - 1)$ -entry is β_{k-1} ;
- (5) the (n, n) -entry is β_p ; and
- (6) all other entries are 0.

By applying Bezout's Theorem to the last row, where the nonzero entries are β_1, \dots, β_p , we obtain that the Smith normal form S_β for M_β is $\text{diag}(1^{n-p}, \gcd(\beta_k), 0^{p-1})$.

Finally, to obtain a \mathbb{Z} -basis for $\text{MOD}(w_\beta)$, we perform the following column operations on M_β . For $2 \leq k \leq p$, first add column $(j_k^\beta - 1)$ to column j_k^β , which clears all but two entries in column j_k^β ; namely the 1 in row $(j_k^\beta - 1)$ and -1 in row $(j_k^\beta + 1)$. Then use the column operations which transform each submatrix M_{β_k-1} into the matrix B_{β_k-1} defined in (4.3.6), which places a unique nonzero entry β_k on the diagonal in column $(j_k^\beta - 1)$, for $2 \leq k \leq p$. Now for $2 \leq k \leq p$, add to column $(j_k^\beta - 1)$ the sum of $-\beta_k$ times all columns strictly to its right, excluding the other columns numbered $(j_\ell^\beta - 1)$ for $k < \ell \leq p$. The result moves the nonzero entries β_k in columns $(j_k^\beta - 1)$ all down to row n , as seen in B'_β from Example 4.4. To produce the matrix B_β from which the claimed \mathbb{Z} -basis for $\text{MOD}(w_\beta)$ can easily be read, we apply Bezout's Theorem on successive pairs of columns with a nonzero entry in row n , clearing out all but $\gcd(\beta_k)$ in position (n, n) .

To complete the proof of Theorem 4.2, we suppose that the last $m \geq 1$ parts of $\beta = (\beta_1, \dots, \beta_p)$ are equal to 1. Thus we are generalizing Examples 4.5 and 4.6. Notice that in this case, $\gcd(\beta_k) = 1$. Let $M_\beta = I - w_\beta$ be as described in Corollary 4.17(2).

To determine $\text{MOD}(w_\beta)$, we replace row 1 of M_β by the sum of its rows 1 through $n-m = \sum_{k=1}^{p-m} \beta_k$, to obtain a matrix M'_β in which row i is zero for all $i \in [n] \setminus J_\beta$, while for $2 \leq i \leq n-m$ with $i \in J_\beta$, the first entry in row i of M'_β is its $(i, i-1)$ -entry, which equals -1 . Now the first row of M'_β has nonzero entries as follows: for $2 \leq k \leq p-m+1$, the $(1, j_k^\beta - 1)$ -entry is β_{k-1} ; for $2 \leq k \leq p-m$, the $(1, j_k^\beta)$ -entry is $-\beta_{k-1}$; and the $(1, n-m+1)$ -entry is $-(\beta_{p-m} - 1)$. Since $\gcd(\beta_1, -\beta_1, \beta_2, \dots, \beta_{p-m}, -(\beta_{p-m} - 1)) = \gcd(\beta_{p-m}, -(\beta_{p-m} - 1)) = 1$, it follows that

$$\text{MOD}(w_\beta) = \left\{ \sum_{i=1}^n c_i \alpha_i^\vee \mid c_i \in \mathbb{Z} \text{ and } c_i = 0 \text{ for all } i \in [n] \setminus J_\beta \right\}.$$

with a \mathbb{Z} -basis given by $\{\alpha_j^\vee \mid j \in J_\beta\}$.

For the Smith normal form, we start with M_β and for $i = j_{p-m-1}^\beta, \dots, n-m$, we replace column i of M_β by the sum of its columns $i, \dots, n-m+1$. The resulting matrix has the matrix $T_{\beta_{p-m-1}, 1}$ from (4.3.5) replacing $M_{\beta_{p-m-1}, 1}$ from (4.3.3), while column j_{p-m-1}^β of the resulting matrix now has all 0s except for the -1 s in rows $i \in J_{p-m-1}^\beta$. We then inductively repeat this process, moving from right to left, to obtain an upper-triangular matrix T_β which

has blocks $T_{\beta_1-1,1}, T_{\beta_2-1,1}, \dots, T_{\beta_{p-m}-1,1}$ going down the diagonal, up to row $n-m+1$, and then all-0 blocks (if $m \geq 2$).

Each of the $T_{\beta_k-1,1}$ can then be replaced by its Smith normal form $S_{\beta_k-1,1}$. Hence for $1 \leq k \leq p-m$, the k th block contributes $\beta_k - 1$ entries equal to 1 and one entry equal to 0 to the Smith normal form S_β . Now

$$\sum_{k=1}^{p-m} (\beta_k - 1) = \left(\sum_{k=1}^{p-m} \beta_k \right) - (p-m) = \left(\sum_{k=1}^p \beta_k \right) - m - (p-m) = n + 1 - p.$$

Therefore S_β has $n-p+1$ diagonal entries equal to 1, and all remaining entries 0. Hence $S_\beta = (1^{n-p+1}, 0^{p-1}) = (1^{n-p}, \gcd(\beta_k), 0^{p-1})$, and thus $R^\vee/\text{MOD}(w_\beta) \cong \mathbb{Z}^{p-1}$.

This completes the proof of Theorem 4.2. \square

5. TYPE C MOD-SETS

In this section we give an explicit description of all mod-sets in type C , as stated in Theorem 5.2. Although the finite Weyl groups in types B and C are identical, the mod-sets in these two types differ substantially. We treat type C first, since the results are considerably more straightforward in type C ; compare Theorems 6.1 and 6.2 in type B .

Let W be the finite Weyl group of type C_n , with $n \geq 2$. In Section 5.1, we review the complete system of minimal length representatives for the conjugacy classes of W provided in [GP00, Ch. 3], which we rephrase in Proposition 5.1. We then give several key examples in Section 5.2, and prove our results in type C_n in Section 5.3.

Throughout this section, we order the nodes of the Dynkin diagram increasing from left to right, as in both Bourbaki [Bou02] and Sage [Sag24], so that the first $n-1$ nodes form a type A_{n-1} subsystem, and the special node is indexed by n on the right. We note that this is the reverse of the ordering of nodes used in [GP00]. We also note that in Sage [Sag24] the Cartan matrices in types B_n and C_n are reversed with respect to the Dynkin diagrams in these types. See Appendix B for a direct comparison of these conventions.

5.1. Conjugacy class representatives and mod-sets in type C . Following [GP00, Proposition 3.4.7], we first explain how the conjugacy classes of W of type C_n are parameterized by ordered pairs of compositions (β, γ) such that β is weakly decreasing, γ is weakly increasing, and $|\beta| + |\gamma| = n \geq 2$. (In particular, note that at most one of $|\beta| = 0$ and $|\gamma| = 0$ is permitted.) For each such pair (β, γ) , we will define standard parabolic subgroups W_β and W_γ of W , and an element $w_{\beta, \gamma} = w_\beta \cdot w_\gamma$ with w_β cuspidal in W_β and w_γ cuspidal in W_γ , so that the set of all such $w_{\beta, \gamma}$ forms a complete system of minimal length representatives for the conjugacy classes of W .

We first provide a summary of our algorithm for obtaining a minimal length conjugacy class representative $w_{\beta, \gamma}$ associated to the pair of compositions (β, γ) . The following result is contained in [GP00, Proposition 3.4.7], though we present a visual algorithm for quickly constructing these elements; see Figure 6 for an illustration of Proposition 5.1(2).

Proposition 5.1. *Let (β, γ) be a pair of compositions such that β is weakly decreasing, γ is weakly increasing, and $|\beta| + |\gamma| = n \geq 2$. Define an element $w_{\beta, \gamma}$ of the finite Weyl group W of type C_n as follows:*

- (1) *Let $|\beta| = m$, and define w_β as in Proposition 4.1, using the type A_{m-1} subsystem formed by the first nodes of the Dynkin diagram; note that w_β is trivial if $m \in \{0, 1\}$.*
- (2) *Let $|\gamma| = n-m$, and write $\gamma = (\gamma_1, \dots, \gamma_q)$. If $|\gamma| = 0$, then w_γ is trivial. Otherwise, define w_γ as follows:*

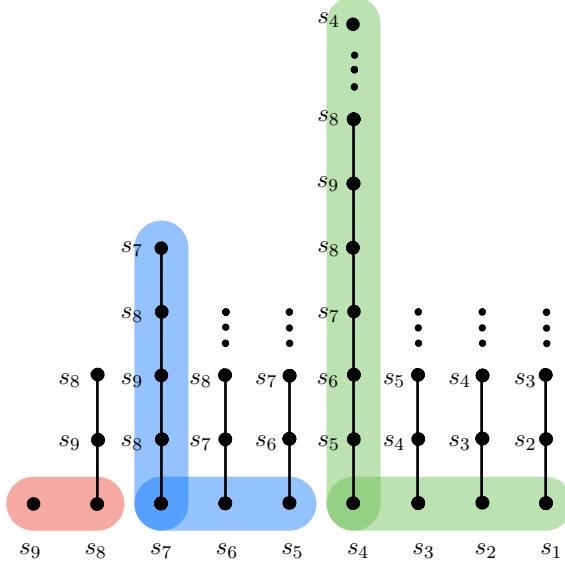


FIGURE 6. $w_\gamma = (\textcolor{red}{s_9 s_8})(\textcolor{blue}{s_7 s_8 s_9 s_8 s_7 \cdot s_6 s_5})(\textcolor{green}{s_4 s_5 s_6 s_7 s_8 s_9 s_8 s_7 s_6 s_5 s_4 \cdot s_3 s_2 s_1})$ for the composition $\gamma = (2, 3, 4)$ in type C_9 .

- (a) Draw $|\gamma|$ trees, rooted at s_{n-i} for $i \in \{0, 1, \dots, n-m-1\}$ from left to right.
- (b) Label the vertices of the tree rooted at s_{n-i} vertically by the palindrome $s_{n-i}, s_{n-i+1}, \dots, s_{n-1}, s_n, s_{n-1}, \dots, s_{n-i+1}, s_{n-i}$.
- (c) For each part γ_i of the composition γ from $i = 1, \dots, q$, shade the entire tree to the right of the previous shaded area, together with the next consecutive γ_i roots.
- (d) Multiply together the simple reflections indexed by the shaded subgraphs, reading down trees and then left to right within shaded areas, to obtain w_γ .

Then the set of all $w_{\beta, \gamma} = w_\beta \cdot w_\gamma$ forms a complete system of minimal length representatives for the conjugacy classes of W .

Figure 6 illustrates the construction of w_γ in Proposition 5.1; see Section 5.2 for additional examples, including Example 5.5 for more details on Figure 6. We remark that the underlying collection of trees in Figure 6 is a normal form forest for the Coxeter group W of type C_9 , in the sense of du Cloux (see [BB05, Section 3.4]), using the ordering on generators which goes from right to left on the Dynkin diagram.

We now establish the notation needed to formally state Theorem 5.2 characterizing all mod-sets in type C_n . If either $|\beta| = 0$ or $|\beta| = 1$, we define both w_β and W_β to be trivial. If β is a partition of m with $2 \leq m \leq n$, then we define both w_β and W_β as in Section 4.1, so that the w_β are the minimal length conjugacy class representatives in the type A_{m-1} subsystem of W indexed by the first $m-1$ nodes of the Dynkin diagram, as stated in Proposition 5.1.

Now let $\gamma = (\gamma_1, \dots, \gamma_q)$ be any weakly increasing composition of $n-m$. If $|\gamma| = 0$, we define both w_γ and W_γ to be trivial. It thus remains to define w_γ and W_γ for $0 \leq m < n$. For this, following [GP00, Section 3.4] but using the opposite ordering, we define $t_0 = s_n$, and for $1 \leq i \leq n-1$, define t_i to be the conjugate

$$(5.1.1) \quad t_i = s_{n-i} t_{i-1} s_{n-i}.$$

That is,

$$t_1 = s_{n-1}s_n s_{n-1}, \quad t_2 = s_{n-2}s_{n-1}s_n s_{n-1}s_{n-2}, \quad \dots, \quad t_{n-1} = s_1 \dots s_{n-1}s_n s_{n-1} \dots s_1.$$

Note that the elements t_i provide the labeling of the rooted trees in 2(2b) of Proposition 5.1. We also define a strictly increasing subsequence of $\{0, 1, \dots, n-m\}$ in the same way as we did for the partition β in Section 4.1. That is, let $j_1^\gamma = 0$, and for all $2 \leq k \leq q$, define

$$j_k^\gamma = \sum_{i=1}^{k-1} \gamma_i = \gamma_1 + \dots + \gamma_{k-1}.$$

It will be convenient to additionally define $j_{q+1}^\gamma = \sum_{i=1}^q \gamma_i = n-m$.

Now we define subintervals J_k^γ of the interval $[n]$ by putting

$$J_k^\gamma = \{n - j_{k+1}^\gamma + 1, n - j_{k+1}^\gamma + 2, \dots, n\} = \left\{ n - \left(\sum_{i=1}^k \gamma_i \right) + 1, \dots, n \right\}$$

for all $k \in [q]$; for example, $J_1^\gamma = \{n - \gamma_1 + 1, \dots, n\}$. In the language of Proposition 5.1, the set J_k^γ indexes the support of the shaded diagram corresponding to part γ_k in step 2(2c). Note that the J_k^γ are *not* defined the same way as for β . In particular, these sets are all nonempty, with $n \in J_k^\gamma$ for all $k \in [q]$, and they satisfy the following strict inclusions

$$J_1^\gamma \subsetneq J_2^\gamma \subsetneq \dots \subsetneq J_q^\gamma.$$

Observe also that for all $k \in [q]$, the set J_k^γ has $\sum_{i=1}^k \gamma_i = j_{k+1}^\gamma$ elements. Therefore, $J_1^\gamma = \{n\}$ if and only if $\gamma_1 = 1$, and $J_q^\gamma = \{n - (n-m) + 1, \dots, n\} = \{m+1, \dots, n\}$, and hence $|J_q^\gamma| = |\gamma| = n-m$ for all γ . We may thus identify the elements of each J_k^γ with the last $j_{k+1}^\gamma \geq 1$ nodes of the Dynkin diagram for W ; that is, with the nodes of its (connected and nonempty) subdiagram of type $C_{j_{k+1}^\gamma}$. To simplify notation, we also define

$$J_\gamma = J_q^\gamma = \bigcup_{k=1}^q J_k^\gamma.$$

Next, for each $k \in [q]$, we write $W_{J_k^\gamma}$ for the nontrivial standard parabolic subgroup of W generated by the simple reflections $\{s_j \mid j \in J_k^\gamma\}$. Then each $W_{J_k^\gamma}$ is of type $C_{j_{k+1}^\gamma}$, and we have the strict inclusions

$$W_{J_1^\gamma} \subsetneq W_{J_2^\gamma} \subsetneq \dots \subsetneq W_{J_q^\gamma}.$$

We write W_γ for the nontrivial standard parabolic subgroup of W generated by the simple reflections $\{s_j \mid j \in J_\gamma\}$, so that $W_\gamma = W_{J_q^\gamma} = \bigcup_{k=1}^q W_{J_k^\gamma}$.

Now for all $k \in [q]$, we define an element $w_k^\gamma \in W_{J_k^\gamma}$ by

$$(5.1.2) \quad w_k^\gamma = t_{j_k^\gamma} s_{n-(j_k^\gamma+1)} s_{n-(j_k^\gamma+2)} \cdots s_{n-j_{k+1}^\gamma+1}.$$

Thus as $j_1^\gamma = 0$ and $t_0 = s_n$, we have

$$(5.1.3) \quad w_1^\gamma = s_n s_{n-1} \cdots s_{n-\gamma_1+1}$$

while for all $2 \leq k \leq q$, by expanding out the expression (5.1.1) for $i = j_k^\gamma$, we get

$$(5.1.4) \quad w_k^\gamma = s_{n-j_k^\gamma} s_{n-j_k^\gamma+1} \cdots s_{n-1} \cdot s_n s_{n-1} \cdots s_{n-j_{k+1}^\gamma+1}.$$

That is, the support of w_k^γ is the set J_k^γ , and w_k^γ is the product of the simple reflections s_j for $j \in \{n - j_k^\gamma, \dots, n-1\} \subsetneq J_k^\gamma$ in increasing order, followed by the product of the simple reflections s_j for all $j \in J_k^\gamma$ in decreasing order. For the example in Figure 6, we have $w_1^\gamma = s_9 s_8$, whereas $w_2^\gamma = s_7 s_8 \cdot s_9 s_8 s_7 s_6 s_5$ and $w_3^\gamma = s_4 s_5 s_6 s_7 s_8 \cdot s_9 s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_1$. In

particular, each w_k^γ has support J_k^γ , and is a nontrivial element of $W_{J_k^\gamma}$. (In the language of [GP00, Section 3.4.2], the element w_k^γ is the negative block $b_{j_k^\gamma, \gamma_k}^-$ of length γ_k starting at j_k^γ .)

Finally, the composition $\gamma = (\gamma_1, \dots, \gamma_q)$ corresponds to the product

$$w_\gamma = w_1^\gamma \cdots w_q^\gamma \in W_\gamma.$$

By Proposition 3.4.6 of [GP00], the element w_γ is cuspidal in the parabolic subgroup W_γ of type C_{n-m} , generated by the simple reflections indexed by the last $|\gamma| = n - m$ nodes of the Dynkin diagram. By Proposition 3.4.7 of [GP00], the set of $w_{\beta, \gamma} = w_\beta \cdot w_\gamma$ for all distinct pairs of compositions (β, γ) such that β is weakly decreasing, γ is weakly increasing, and $|\beta| + |\gamma| = n$, forms a complete system of minimal length representatives of the conjugacy classes of W in type C_n , as recorded in Proposition 5.1.

To simplify notation in the theorem below, for any weakly increasing composition $\gamma = (\gamma_1, \dots, \gamma_q)$ with $|\gamma| \leq n$, we define the following q -element subset of $[n]$:

$$I_\gamma = \{n - j_k^\gamma \mid 1 \leq k \leq q\} = \left\{n, n - \gamma_1, \dots, n - \sum_{k=1}^{q-1} \gamma_k\right\} = \{n - (|\gamma| - \gamma_q), \dots, n - \gamma_1, n\}.$$

With this notation established, we now state our results describing mod-sets in type C .

Theorem 5.2. *Suppose W is of type C_n with $n \geq 2$. Let (β, γ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$, with corresponding conjugacy class representative $w_{\beta, \gamma} \in W$. Write $m = |\beta|$, so that $0 \leq m \leq n$ and $|\gamma| = n - m$.*

(1) *The module $\text{MOD}(w_{\beta, \gamma}) = (\mathbf{I} - w_{\beta, \gamma})R^\vee$ equals*

$$\left\{ \sum_{i=1}^n c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_i = 0 \text{ for } i \in [m] \setminus J_\beta, c_i \equiv 0 \pmod{2} \text{ for } i \in I_\gamma \right\}.$$

(2) *The module $\text{MOD}(w_{\beta, \gamma})$ has \mathbb{Z} -basis given by*

$$\{\alpha_i^\vee \mid i \in J_\beta\} \cup \{\alpha_i^\vee \mid i \in J_\gamma \setminus I_\gamma\} \cup \{2\alpha_i^\vee \mid i \in I_\gamma\}.$$

(3) *For any $w \in [w_{\beta, \gamma}]$, the Smith normal form of $(\mathbf{I} - w)$ equals*

$$S_{\beta, \gamma} = \text{diag}(1^{n-p-q}, 2^q, 0^p).$$

(4) *For any $w \in [w_{\beta, \gamma}]$, the quotient of R^\vee by the mod-set is*

$$R^\vee / \text{MOD}(w) \cong (\mathbb{Z}/2\mathbb{Z})^q \oplus \mathbb{Z}^p.$$

We illustrate this theorem with several examples in Section 5.2, and provide a detailed proof in Section 5.3.

5.2. Examples in type C . In this section, we present a sequence of examples to illustrate our general results and proof techniques in type C . We mostly restrict to the case $|\beta| = 0$, since the element w_β is in a type A subsystem, and the contribution of w_β in type C can thus be explained quickly via our results from Section 4. The examples we consider are:

- $|\beta| = 0$ and $\gamma = (4)$ in Example 5.3;
- $|\beta| = 0$ and $\gamma = (3, 4)$ in Example 5.4;
- $|\beta| = 0$ and $\gamma = (2, 3, 4)$ in Example 5.5;
- $|\beta| = 0$ and $\gamma = (1, 3)$ in Example 5.6;
- $\beta = (1)$ and $\gamma = (3)$ in Example 5.7;
- $\beta = (4)$ and $\gamma = (3)$ in Example 5.8; and

- $\beta = (3, 3)$ and $\gamma = (3)$ in Example 5.9.

For each of these examples, we give an explicit description of the \mathbb{Z} -module $\text{MOD}(w_{\beta,\gamma}) = (\mathbf{I} - w_{\beta,\gamma})R^\vee$, construct a \mathbb{Z} -basis for $\text{MOD}(w_{\beta,\gamma})$, and find the Smith normal form for $(\mathbf{I} - w_{\beta,\gamma})$ and the isomorphism class of the quotient $R^\vee/\text{MOD}(w_{\beta,\gamma})$. We continue all notation from Section 5.1.

Example 5.3. Let W be of type C_4 and suppose $|\beta| = 0$ and $\gamma = (4)$, so that $w_{\beta,\gamma} = w_\gamma$. We will see that the Smith normal form for $\mathbf{I} - w_\gamma$ is $\text{diag}(1, 1, 1, 2)$, so that $R^\vee/\text{MOD}(w_\gamma) \cong \mathbb{Z}/2\mathbb{Z}$ is torsion of rank equal to the number of parts of γ .

We have that $w_\gamma = s_4s_3s_2s_1$ is Coxeter in W . Observing that the subexpression $s_3s_2s_1$ is Coxeter in the type A_3 subsystem indexed by the first 3 nodes, we may, after reversing the labelings, use our computations for $w_\beta = s_1s_2s_3$ in Example 4.3 above (which considered the partition (4) in type A_3). Since $s_4(\alpha_3^\vee) = \alpha_3^\vee + 2\alpha_4^\vee$ in type C_4 by (2.1.1), we thus obtain

$$\begin{aligned} w_\gamma(\alpha_1^\vee) &= s_4(-\alpha_1^\vee - \alpha_2^\vee - \alpha_3^\vee) = -\alpha_1^\vee - \alpha_2^\vee - \alpha_3^\vee - 2\alpha_4^\vee \\ w_\gamma(\alpha_2^\vee) &= s_4(\alpha_1^\vee) = \alpha_1^\vee \\ w_\gamma(\alpha_3^\vee) &= s_4(\alpha_2^\vee) = \alpha_2^\vee. \end{aligned}$$

We also compute

$$w_\gamma(\alpha_4^\vee) = s_4s_3(\alpha_4^\vee) = s_4(\alpha_3^\vee + \alpha_4^\vee) = \alpha_3^\vee + 2\alpha_4^\vee - \alpha_4^\vee = \alpha_3^\vee + \alpha_4^\vee.$$

Hence the matrices for w_γ and $N_\gamma = \mathbf{I} - w_\gamma$ with respect to the basis Δ^\vee are given by

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N_\gamma = \begin{pmatrix} 2 & \boxed{-1} & 0 & 0 \\ 1 & 1 & \boxed{-1} & 0 \\ 1 & 0 & 1 & \boxed{-1} \\ \textcircled{2} & 0 & 0 & 0 \end{pmatrix}.$$

To determine $\text{MOD}(w_\gamma)$, notice that in rows $i = 1, 2, 3$ of N_γ , the last nonzero entry is the boxed -1 in the $(i, i+1)$ -position. Since the only nonzero entry in row 4 is the 2 in column 1, it follows that the system $N_\gamma \mathbf{x} = \mathbf{c}$ has solution over \mathbb{Z} if and only if $c_4 \equiv 0 \pmod{2}$. Thus

$$\text{MOD}(w_\gamma) = N_\gamma R^\vee = \left\{ \sum_{i=1}^4 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z} \text{ and } c_4 \equiv 0 \pmod{2} \right\}.$$

It is immediate that a \mathbb{Z} -basis for $\text{MOD}(w_{\beta,\gamma})$ is given by $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, 2\alpha_4^\vee\}$.

To determine the Smith normal form S_γ for N_γ , we carry out the following column operations. Replace $\mathcal{C}_1(N_\gamma)$ by $\sum_{i=1}^4 i\mathcal{C}_i(N_\gamma)$, and for $i = 2, 3$ replace $\mathcal{C}_i(N_\gamma)$ by $\sum_{j=i}^4 \mathcal{C}_j(N_\gamma)$. This gives the matrix

$$T_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $S_\gamma = \text{diag}(1, 1, 1, 2)$, and $R^\vee/\text{MOD}(w_\gamma) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 5.4. Let W be of type C_7 and suppose $|\beta| = 0$ and $\gamma = (3, 4)$, so that $w_{\beta,\gamma} = w_\gamma$. We will see that the Smith normal form for $\mathbf{I} - w_\gamma$ is $\text{diag}(1^5, 2^2)$, so that $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is torsion of rank equal to the number of parts of γ .

We have $w_\gamma = w_1^\gamma w_2^\gamma = (s_7s_6s_5)(s_4s_5s_6s_7s_6s_5s_4s_3s_2s_1)$. We first determine the matrices for w_1^γ and w_2^γ , with respect to the basis Δ^\vee , then multiply these together to obtain the matrix

for w_γ . Now $w_1^\gamma = s_7s_6s_5$ fixes α_i^\vee for $1 \leq i \leq 3$. Since w_1^γ is Coxeter in the type C_3 subsystem indexed by the last 3 nodes, we obtain using similar arguments to those in Example 5.3 that

$$w_1^\gamma(\alpha_5^\vee) = -\alpha_5^\vee - \alpha_6^\vee - 2\alpha_7^\vee, \quad w_\gamma(\alpha_6^\vee) = \alpha_5^\vee, \quad \text{and} \quad w_\gamma(\alpha_7^\vee) = \alpha_6^\vee + \alpha_7^\vee.$$

We also have, by inserting s_4s_4 and working in the type C_4 subsystem on nodes 4, 5, 6, 7, that

$$w_1^\gamma(\alpha_4^\vee) = s_7s_6s_5s_4(s_4\alpha_4^\vee) = s_7s_6s_5s_4(-\alpha_4^\vee) = \alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee + 2\alpha_7^\vee.$$

Hence the matrix for w_1^γ is block lower-triangular of the form

$$w_1^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 1 \end{pmatrix}.$$

We now consider w_2^γ . Since this ends with the Coxeter element $s_7s_6s_5s_4s_3s_2s_1$, by (2.1.1) we have

$$\begin{aligned} w_2^\gamma(\alpha_1^\vee) &= s_4s_5s_6(-\alpha_1^\vee - \cdots - \alpha_6^\vee - 2\alpha_7^\vee) \\ w_2^\gamma(\alpha_i^\vee) &= s_4s_5s_6(\alpha_{i-1}^\vee) \quad \text{for } 2 \leq i \leq 6 \\ w_2^\gamma(\alpha_7^\vee) &= s_4s_5s_6(\alpha_6^\vee + \alpha_7^\vee). \end{aligned}$$

Next we compute

$$\begin{aligned} s_4s_5s_6(\alpha_i^\vee) &= \alpha_i^\vee \quad \text{for } i = 1, 2 \\ s_4s_5s_6(\alpha_3^\vee) &= \alpha_3^\vee + \alpha_4^\vee \\ s_4s_5s_6(\alpha_i^\vee) &= \alpha_{i+1}^\vee \quad \text{for } i = 4, 5 \\ s_4s_5s_6(\alpha_6^\vee) &= -\alpha_4^\vee - \alpha_5^\vee - \alpha_6^\vee \\ s_4s_5s_6(\alpha_7^\vee) &= \alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee + \alpha_7^\vee. \end{aligned}$$

Putting these two collections of equations together, we compute that $w_2^\gamma(\alpha_1^\vee)$ equals

$$(-\alpha_1^\vee - \alpha_2^\vee) - (\alpha_3^\vee + \alpha_4^\vee) - (\alpha_5^\vee + \alpha_6^\vee) + (\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee) - 2(\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee + \alpha_7^\vee)$$

which simplifies to

$$w_2^\gamma(\alpha_1^\vee) = -\alpha_1^\vee - \alpha_2^\vee - \alpha_3^\vee - 2(\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee + \alpha_7^\vee),$$

while

$$\begin{aligned} w_2^\gamma(\alpha_i^\vee) &= \alpha_{i-1}^\vee \quad \text{for } i = 2, 3 \\ w_2^\gamma(\alpha_4^\vee) &= \alpha_3^\vee + \alpha_4^\vee \\ w_2^\gamma(\alpha_i^\vee) &= \alpha_{(i-1)+1}^\vee = \alpha_i^\vee \quad \text{for } i = 5, 6 \\ w_2^\gamma(\alpha_7^\vee) &= -(\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee) + (\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee + \alpha_7^\vee) = \alpha_7^\vee. \end{aligned}$$

Hence the matrix for w_2^γ is block lower-triangular of the form

$$w_2^\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To compute the matrix product $w_\gamma = w_1^\gamma w_2^\gamma$, we observe that w_1^γ and w_2^γ are both block lower-triangular, with I_4 in the top left-hand corner of w_1^γ (respectively, the bottom right-hand corner of w_2^γ). Hence w_γ is block lower-triangular, with its first 4 rows the same as in w_2^γ , and its last 4 columns the same as in w_1^γ . For the remaining 3×3 block in the lower left-hand corner, we observe that for $i = 5, 6, 7$, the entries in row i of w_1^γ sum to 1, and these rows are all 0 in columns 1, 2, 3. Also, column 1 of w_2^γ has all entries in rows 4, 5, 6, 7 equal to -2 , and columns 2 and 3 of w_2^γ are all 0s in rows 4, 5, 6, 7. Hence for $i = 5, 6, 7$, the $(i, 1)$ -entry of w_γ is -2 , and the $(i, 2)$ - and $(i, 3)$ -entries are 0. That is,

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 & -1 & 1 & 0 \\ -2 & 0 & 0 & 1 & -1 & 0 & 1 \\ -2 & 0 & 0 & 2 & -2 & 0 & 1 \end{pmatrix}.$$

Therefore $N_\gamma = I - w_\gamma$ is given by

$$N_\gamma = \begin{pmatrix} 2 & \boxed{-1} & 0 & 0 \\ 1 & 1 & \boxed{-1} & 0 \\ 1 & 0 & 1 & \boxed{-1} \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 2 & \boxed{-1} & 0 \\ 2 & 0 & 0 & -1 & 1 & 1 & \boxed{-1} \\ 2 & 0 & 0 & -2 & 2 & 0 & 0 \end{pmatrix}.$$

For $\text{MOD}(w_\gamma)$, we observe just from N_γ that for $i = 1, 2, 3, 5, 6$, the last nonzero entry in row i is the boxed -1 in position $(i, i+1)$, while for $i = 4, 7$, all nonzero entries in row i are ± 2 . Therefore

$$\text{MOD}(w_\gamma) = \left\{ \sum_{i=1}^7 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_i \equiv 0 \pmod{2} \text{ for } i \in \{4, 7\} \right\},$$

and it is clear that a \mathbb{Z} -basis for $\text{MOD}(w_\gamma)$ is given by $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, 2\alpha_4^\vee, \alpha_5^\vee, \alpha_6^\vee, 2\alpha_7^\vee\}$.

For the Smith normal form, we carry out column operations on N_γ . We replace $\mathcal{C}_5(N_\gamma)$ by $\mathcal{C}_5(N_\gamma) + 2\mathcal{C}_6(N_\gamma) + 3\mathcal{C}_7(N_\gamma)$, and replace $\mathcal{C}_6(N_\gamma)$ by $\mathcal{C}_6(N_\gamma) + \mathcal{C}_7(N_\gamma)$, to obtain

$$T_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ \textcircled{2} & 0 & 0 & \textcircled{-1} \\ \textcircled{2} & 0 & 0 & \textcircled{-1} \\ \textcircled{2} & 0 & 0 & \textcircled{-2} \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 \\ & & 0 & -1 \\ & & 0 & 0 \\ & & 2 & 0 \end{pmatrix}.$$

We can now use the boxed pivot entries in columns 5, 6, and 7 of T_γ to clear all of the circled entries, so that the resulting matrix is block-diagonal. We then replace $\mathcal{C}_1(T_\gamma)$ by $\mathcal{C}_1(T_\gamma) + 2\mathcal{C}_2(T_\gamma) + 3\mathcal{C}_3(T_\gamma) + 4\mathcal{C}_4(T_\gamma)$, and for $i = 2, 3, 4$ replace $\mathcal{C}_i(T_\gamma)$ by $\sum_{j=i}^4 \mathcal{C}_j(T_\gamma)$, to obtain

$$T'_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 \\ & & 0 & -1 \\ & & 0 & 0 \\ & & 2 & 0 \end{pmatrix}.$$

Thus $S_\gamma = \text{diag}(1^5, 2^2)$, and so $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Example 5.5. Let W be of type C_9 and suppose $|\beta| = 0$ and $\gamma = (2, 3, 4)$, so that $w_{\beta, \gamma} = w_\gamma$. We will see that the Smith normal form for $I - w_\gamma$ is $\text{diag}(1^6, 2^3)$, so that $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})^3$ is torsion of rank equal to the number of parts of γ .

We have $w_\gamma = w_1^\gamma w_2^\gamma w_3^\gamma = (s_9 s_8)(s_7 s_8 s_9 s_8 s_7 s_6 s_5)(s_4 \cdots s_8 s_9 s_8 \cdots s_4 s_3 s_2 s_1)$; see Figure 6. Let γ' be the composition $\gamma' = (2, 3)$ of 5. Then since the product $w_1^\gamma w_2^\gamma$ has support given by the type C_5 subsystem indexed by $J_2^\gamma = \{s_5, \dots, s_9\}$, we see that the matrix for $w_1^\gamma w_2^\gamma$ with respect to $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_9^\vee\}$ has a 5×5 block in its bottom right-hand corner which is equal to the matrix for $w_1^{\gamma'} w_2^{\gamma'}$ with respect to the subsystem $\Delta_5^\vee = \{\alpha_1^\vee, \dots, \alpha_5^\vee\}$, and is all 0s above this block. Moreover, $w_1^\gamma w_2^\gamma$ fixes α_i^\vee for $i = 1, 2, 3$. Now as

$$w_1^\gamma w_2^\gamma(\alpha_4^\vee) = (s_9 s_8)(s_7 s_8 s_9 s_8 s_7 s_6 s_5)s_4 s_4(\alpha_4^\vee) = -(s_9 s_8)(s_7 s_8 s_9 s_8 s_7 s_6 s_5 s_4)(\alpha_4^\vee),$$

by considering the composition $\gamma'' = (2, 4)$ of 6 and then shifting indexes, we see that column 4 of the matrix for $w_1^\gamma w_2^\gamma$ has entries $[0, 0, 0, 1, 1, 1, 2, 2, 2]$. Therefore the matrix for $w_1^\gamma w_2^\gamma$ is block lower-triangular of the form

$$w_1^\gamma w_2^\gamma = \begin{pmatrix} I_3 & & & \\ & 1 & & \\ & 1 & -1 & 1 & 0 \\ & 1 & -1 & 0 & 1 \\ & 2 & -2 & 0 & 1 \\ & 2 & -2 & 0 & 1 & -1 & 1 \\ & 2 & -2 & 0 & 2 & -2 & 1 \end{pmatrix}.$$

Now w_3^γ is block lower-triangular of the form

$$w_3^\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 & I_5 \end{pmatrix}$$

where for $5 \leq i \leq 9$ and $2 \leq j \leq 4$, the $(i, 1)$ -entry of w_3^γ is -2 and the (i, j) -entry is 0 . Therefore multiplying these together, we obtain that the matrix for w_γ is as on the left, and hence that the matrix for $N_\gamma = I - w_\gamma$ is as on the right:

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & -1 & 1 & 0 \\ -2 & 0 & 0 & 1 & -1 & 0 \\ -2 & 0 & 0 & 2 & -2 & 0 & 1 \\ -2 & 0 & 0 & 2 & -2 & 0 & 1 & -1 & 1 \\ -2 & 0 & 0 & 2 & -2 & 0 & 2 & -2 & 1 \end{pmatrix} \quad N_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 2 & -1 & 0 \\ 2 & 0 & 0 & -1 & 1 & 1 & -1 \\ 2 & 0 & 0 & -2 & 2 & 0 & 0 \\ 2 & 0 & 0 & -2 & 2 & 0 & -1 & 2 & -1 \\ 2 & 0 & 0 & -2 & 2 & 0 & -2 & 2 & 0 \end{pmatrix}.$$

By similar arguments to the previous examples, we see that

$$\text{MOD}(w_\gamma) = \left\{ \sum_{i=1}^9 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_i \equiv 0 \pmod{2} \text{ for } i \in \{4, 7, 9\} \right\}$$

and that $\text{MOD}(w_\gamma)$ has a \mathbb{Z} -basis given by $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, 2\alpha_4^\vee, \alpha_5^\vee, \alpha_6^\vee, 2\alpha_7^\vee, \alpha_8^\vee, 2\alpha_9^\vee\}$.

For the Smith normal form, we replace $\mathcal{C}_8(N_\gamma)$ by $\mathcal{C}_8(N_\gamma) + 2\mathcal{C}_9(N_\gamma)$, to obtain

$$T_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 2 & -1 & 0 \\ 2 & 0 & 0 & -1 & 1 & 1 & -1 \\ 2 & 0 & 0 & -2 & 2 & 0 & 0 \\ \textcircled{2} & 0 & 0 & \textcircled{-2} & \textcircled{2} & 0 & \textcircled{-1} & 0 & \boxed{-1} \\ \textcircled{2} & 0 & 0 & \textcircled{-2} & \textcircled{2} & 0 & \textcircled{-2} & \boxed{2} & 0 \end{pmatrix}$$

We then use the boxed pivots in columns 8 and 9 to clear all of the nonzero entries in rows 8 and 9 (these entries have been circled). Now we carry out similar column operations to those

in Example 5.4 on the 7×7 block in the top left of T_γ , to eventually obtain

$$T'_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 & & & \\ 0 & 0 & -1 & 0 & & & \\ 0 & 0 & 0 & -1 & & & \\ 2 & 0 & 0 & 0 & & & \\ & & & & 0 & -1 & 0 \\ & & & & 0 & 0 & -1 \\ & & & & 2 & 0 & 0 & & 0 & -1 \\ & & & & & 2 & 0 & & & 2 \end{pmatrix}.$$

Thus $S_\gamma = \text{diag}(1^6, 2^3)$, and so $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})^3$.

Example 5.6. Let W be of type C_4 and suppose $|\beta| = 0$ and $\gamma = (1, 3)$, so that $w_{\beta, \gamma} = w_\gamma$. We will see that the Smith normal form for $I - w_\gamma$ is $\text{diag}(1, 1, 2, 2)$, so that $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is torsion of rank equal to the number of parts of γ . In particular, γ having a part equal to 1 does *not* change the final result, unlike the situation for β in type A ; compare Example 4.5.

Here we have $w_\gamma = w_1^\gamma w_2^\gamma = (s_4)(s_3 s_4 s_3 s_2 s_1)$. In this case, since $w_1^\gamma = s_4$ we compute immediately that the matrix for w_1^γ is as on the left, while using similar arguments to those in Example 5.4, we obtain that w_2^γ is as on the right:

$$w_1^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \quad w_2^\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 0 & 2 & 1 \end{pmatrix}.$$

Therefore the matrices for w_γ and $N_\gamma = I - w_\gamma$ are:

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 0 & 2 & -1 \end{pmatrix} \quad N_\gamma = \begin{pmatrix} 2 & \boxed{-1} & 0 & 0 \\ 1 & 1 & \boxed{-1} & 0 \\ \boxed{2} & 0 & 0 & 0 \\ \circled{2} & 0 & \circled{-2} & \boxed{2} \end{pmatrix}.$$

Hence

$$\text{MOD}(w_\gamma) = \left\{ \sum_{i=1}^4 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_i \equiv 0 \pmod{2} \text{ for } i \in \{3, 4\} \right\}$$

has a \mathbb{Z} -basis given by $\{\alpha_1^\vee, \alpha_2^\vee, 2\alpha_3^\vee, 2\alpha_4^\vee\}$.

To obtain the Smith normal form, we used the boxed pivot in column 4 of N_γ to clear the circled entries in row 4, and then carry out column operations as in previous examples on the 3×3 block in the top left, to obtain the block-diagonal matrix

$$T_\gamma = \begin{pmatrix} 0 & -1 & 0 & & \\ 0 & 0 & -1 & & \\ 2 & 0 & 0 & & \\ & & & & 2 \end{pmatrix}.$$

Thus $S_\gamma = \text{diag}(1, 1, 2, 2)$, and so $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Our final three examples in type C illustrate the case when the compositions β and γ are both nonempty.

Example 5.7. Let W be of type C_4 and suppose $\beta = (1)$ and $\gamma = (3)$, so that w_β is trivial and hence $w_{\beta,\gamma} = w_\gamma$. We will see that the Smith normal form for $I - w_\gamma$ is $\text{diag}(1, 1, 2, 0)$, so that $R^\vee/\text{MOD}(w_\gamma) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ has torsion part of rank equal to the number of parts of γ , and free part of rank equal to the number of parts of β .

We have that $w_\gamma = s_4s_3s_2$. This is Coxeter in the type C_3 subsystem indexed by the last 3 nodes, and so by considering $w_\gamma(\alpha_1^\vee) = -w_\gamma s_1(\alpha_1^\vee)$, similarly to Example 5.4 above, we obtain that the matrices for w_γ and $N_\gamma = I - w_\gamma$ with respect to the basis Δ^\vee are

$$w_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 2 & -2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N_\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & \boxed{-1} & 0 \\ -1 & 1 & 1 & \boxed{-1} \\ -2 & 2 & 0 & 0 \end{pmatrix}.$$

To determine $\text{MOD}(w_\gamma)$, notice that row 1 of N_γ is all 0s, and that in rows $i = 2, 3$ of N_γ , the last nonzero entry is the boxed -1 in the $(i, i+1)$ -position. Since the only nonzero entries in row 4 are the ± 2 in columns 1 and 2, it follows that the system $N_\gamma \mathbf{x} = \mathbf{c}$ has solution over \mathbb{Z} if and only if $c_1 = 0$ and $c_4 \equiv 0 \pmod{2}$. Thus

$$\text{MOD}(w_\gamma) = N_\gamma R^\vee = \left\{ \sum_{i=1}^4 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_1 = 0, \text{ and } c_4 \equiv 0 \pmod{2} \right\}.$$

It is immediate that a \mathbb{Z} -basis for $\text{MOD}(w_{\beta,\gamma})$ is given by $\{\alpha_2^\vee, \alpha_3^\vee, 2\alpha_4^\vee\}$.

To determine the Smith normal form S_γ for N_γ , we carry out the following column operations. Replace $\mathcal{C}_2(N_\gamma)$ by $\sum_{i=2}^4 (i-1)\mathcal{C}_i(N_\gamma)$, and for $i = 3, 4$ replace $\mathcal{C}_i(N_\gamma)$ by $\sum_{j=i}^4 \mathcal{C}_j(N_\gamma)$. This gives the matrix

$$T_\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & \boxed{-1} & 0 \\ -1 & 0 & 0 & \boxed{-1} \\ -2 & \boxed{2} & 0 & 0 \end{pmatrix}.$$

Column operations using the boxed pivots can now be used to clear column 1. Hence $S_\gamma = \text{diag}(1, 1, 2, 0)$, and $R^\vee/\text{MOD}(w_\gamma) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.

Example 5.8. Let W be of type C_7 and suppose $\beta = (4)$ and $\gamma = (3)$. We will see that the Smith normal form for $I - w_\gamma$ is $\text{diag}(1^5, 2, 0)$, so that $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$ has torsion part of rank equal to the number of parts of γ , and free part of rank equal to the number of parts of β . In particular, $\gcd(\beta_k) = 4$ plays no role in this description, in contrast to type A ; compare Example 4.4. Note also that, since $\beta = (4)$ has only one part, the indexing set $(I_\beta - 1) \setminus \{3\}$ is empty here (as in Example 4.3).

We have $w_{\beta,\gamma} = w_\beta w_\gamma = (s_1s_2s_3)(s_7s_6s_5)$. So w_β fixes α_i^\vee for $i = 5, 6, 7$. Now let β' be the partition of $4+1 = 5$ given by $\beta' = (4, 1)$, and observe that, in type A_4 , we have $w_{\beta'} = s_1s_2s_3$. Thus we can apply results from Section 4 to obtain that the matrix for w_β , with respect to the type C_7 basis of simple coroots Δ^\vee , is block-diagonal of the form shown on the left:

$$w_\beta = \begin{pmatrix} w_{\beta'} & \\ & I_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{while} \quad w_\gamma = \begin{pmatrix} I_3 & & & \\ & 1 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 \\ & 1 & -1 & 0 & 1 \\ & 2 & -2 & 0 & 1 \end{pmatrix}$$

by considering $w_\gamma(\alpha_4^\vee) = -w_\gamma s_4(\alpha_4^\vee)$, as in Example 5.4 above. Therefore $w_{\beta,\gamma} = w_\beta w_\gamma$ and $M_{\beta,\gamma} = \mathbf{I} - w_\beta w_\gamma$ have matrices given by

$$w_{\beta,\gamma} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & 1 & -1 & 1 & 0 \\ & & & 1 & -1 & 0 & 1 \\ & & & 2 & -2 & 0 & 1 \end{pmatrix} \quad M_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 2 & -1 & 0 \\ & & & -1 & 1 & 1 & -1 \\ & & & -2 & 2 & 0 & 0 \end{pmatrix}.$$

Now from the shape of $M_{\beta,\gamma}$, we see that $M_{\beta,\gamma}\mathbf{x} = \mathbf{c}$ has solution over \mathbb{Z} if and only if $c_7 \equiv 0 \pmod{2}$ and the equations represented by the 4×4 matrix $M'_\beta = \mathbf{I} - w_{\beta'}$ in the top left of $M_{\beta,\gamma}$ have solution over \mathbb{Z} . Hence as the partition β' has last part equal to 1, we deduce from results of Section 4 that

$$\text{MOD}(w_{\beta,\gamma}) = \left\{ \sum_{i=1}^7 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_4 = 0, c_7 \equiv 0 \pmod{2} \right\},$$

which has a \mathbb{Z} -basis given by $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\} \cup \{\alpha_5^\vee, \alpha_6^\vee, 2\alpha_7^\vee\}$.

To find the Smith normal form $S_{\beta,\gamma}$, we start with $M_{\beta,\gamma}$, and first use column operations on columns 5, 6, 7 to obtain

$$T_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & (-1) & 0 & \boxed{-1} & 0 \\ & & & (-1) & 0 & 0 & \boxed{-1} \\ & & & (-2) & \boxed{2} & 0 & 0 \end{pmatrix}.$$

We then use the boxed pivots to clear the circled entries in rows 5, 6, 7. Now we can apply the same column operations as in Section 4 to the submatrix $M_{\beta'}$, to obtain

$$T'_{\beta,\gamma} = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & \boxed{-1} & 0 \\ & & & 0 & 0 & 0 & \boxed{-1} \\ & & & 0 & \boxed{2} & 0 & 0 \end{pmatrix}.$$

Therefore $S_{\beta,\gamma} = \text{diag}(1^5, 2, 0)$, and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$.

Example 5.9. Let W be of type C_9 and suppose $\beta = (3, 3)$ and $\gamma = (3)$. We will see that the Smith normal form for $\mathbf{I} - w_\gamma$ is $\text{diag}(1^6, 2, 0^2)$, so that $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}^2$ has torsion part of rank equal to the number of parts of γ , and free part of rank equal to the number of parts of β . In particular, $\gcd(\beta_k) = 3$ and the indexing set $(I_\beta - 1) \setminus \{5\} = \{2\}$ play no role in this description, in contrast to type A ; compare Example 4.4.

We have $w_{\beta,\gamma} = w_\beta w_\gamma = (s_1 s_2)(s_4 s_5)(s_9 s_8 s_7)$, and so w_β fixes α_i^\vee for $i = 7, 8, 9$. Now let β' be the partition of $6 + 1 = 7$ given by $\beta' = (3, 3, 1)$, and observe that, in type A_6 , we also have $w_{\beta'} = (s_1 s_2)(s_4 s_5)$. Thus we can apply results from Section 4 to obtain that the matrix for w_β , with respect to the type C_9 basis of simple coroots Δ^\vee , is block-diagonal of the form shown on the left:

$$w_\beta = \begin{pmatrix} w_{\beta'} & \\ & I_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 & & & & & & \\ 1 & -1 & 1 & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & & \\ & & & 1 & 0 & -1 & 1 & & \\ & & & & 1 & -1 & 1 & & \\ & & & & & 0 & 0 & 1 & \\ & & & & & & & & I_3 \end{pmatrix}, \quad \text{and} \quad w_\gamma = \begin{pmatrix} I_5 & & & & \\ & 1 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 \\ & 1 & -1 & 0 & 1 \\ & 2 & -2 & 0 & 1 \end{pmatrix}$$

by considering $w_\gamma(\alpha_6^\vee) = -w_\gamma s_6(\alpha_6^\vee)$, similarly to Example 5.4 above. Therefore $M_{\beta,\gamma} = I - w_\beta w_\gamma$ has matrix given by

$$M_{\beta,\gamma} = \begin{pmatrix} 1 & 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & & & -1 & 1 & 1 & -1 & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & -1 & 2 & -1 & 0 \\ & & & & & & & -1 & 1 & 1 & -1 \\ & & & & & & & & -2 & 2 & 0 & 0 \end{pmatrix}.$$

Now from the shape of $M_{\beta,\gamma}$, we see that $M_{\beta,\gamma}\mathbf{x} = \mathbf{c}$ has solution over \mathbb{Z} if and only if $c_9 \equiv 0 \pmod{2}$ and the equations represented by the 6×6 matrix $M'_\beta = I - w_{\beta'}$ in the top left of $M_{\beta,\gamma}$ have solution over \mathbb{Z} . Hence, as the partition β' has last part equal to 1, we deduce from results of Section 4 that

$$\text{MOD}(w_{\beta,\gamma}) = \left\{ \sum_{i=1}^9 c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_3 = 0, c_6 = 0, c_9 \equiv 0 \pmod{2} \right\},$$

which has a \mathbb{Z} -basis given by $\{\alpha_1^\vee, \alpha_2^\vee\} \cup \{\alpha_4^\vee, \alpha_5^\vee\} \cup \{\alpha_7^\vee, \alpha_8^\vee, 2\alpha_9^\vee\}$.

To find the Smith normal form $S_{\beta,\gamma}$, we start with $M_{\beta,\gamma}$, and first use column operations on columns 7, 8, 9 to obtain

$$T'_{\beta,\gamma} = \begin{pmatrix} 1 & 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & & & -1 & 1 & 1 & -1 & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & -1 & 0 & -1 & 0 \\ & & & & & & & -1 & 0 & 0 & -1 \\ & & & & & & & & -2 & 2 & 0 & 0 \end{pmatrix}.$$

We then use the boxed pivots to clear the circled entries in rows 7, 8, 9. Now we can apply the same column operations as in Section 4 to the submatrix $M_{\beta'}$, to obtain

$$T_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

Therefore $S_{\beta,\gamma} = \text{diag}(1^6, 2, 0^2)$, and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}^2$.

5.3. Proofs in type C. In order to prove Theorem 5.2, we will use many of our definitions and results from Section 4. We also define several additional auxiliary matrices and record some results for them in Section 5.3.1. We then determine the matrix for $w_{\beta,\gamma}$, and hence for $M_{\beta,\gamma} = \mathbf{I} - w_{\beta,\gamma}$, with respect to Δ^\vee in Section 5.3.2, and complete the proof of Theorem 5.2 in Section 5.3.3.

5.3.1. Matrix definitions and results in type C. In this section, we define several additional matrices which we will use in type C , and record some results for these.

We define 1×1 matrices $V_1 = [-1]$ and $N_1 = \mathbf{I} - V_1 = [2]$. Then for all $r \geq 2$ we define $r \times r$ matrices

$$(5.3.1) \quad V_r = \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ -1 & 0 & \cdots & \cdots & 0 & 1 \\ -2 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad N_r = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \ddots & \ddots & & \vdots \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & -1 & 0 \\ 1 & 0 & \cdots & 0 & 1 & -1 \\ 2 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

That is, the $(1, 1)$ - and $(r, 1)$ -entry of $N_r = \mathbf{I} - V_r$ is 2, the $(i, 1)$ - and (i, i) -entries are 1 for $2 \leq i \leq r-1$, the $(i, i+1)$ -entry is -1 for $1 \leq i \leq r-1$, and all other entries of N_r are 0.

Lemma 5.10. *For all $r \geq 1$, the \mathbb{Z} -linear system $N_r \mathbf{x} = \mathbf{c}$ has solution $\mathbf{x} \in \mathbb{Z}^r$ if and only if $c_r \equiv 0 \pmod{2}$.*

Proof. This is clear for $r = 1$. When $r \geq 2$, for $1 \leq i \leq r-1$, the last nonzero entry in row i of N_r is the -1 in position $(i, i+1)$, while the only nonzero entry in row r of N_r is the 2 in column 1. The result follows. \square

Consider the coroot lattice R^\vee as a free \mathbb{Z} -module with \mathbb{Z} -basis $\Delta^\vee = \{\alpha_i^\vee\}$. In this section, when taking a subsystem of type C_r for some $1 \leq r < n$, we denote the coroot lattice by R_r^\vee with corresponding basis $\Delta_r^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$, where r indexes the special node.

Corollary 5.11. *For all $r \geq 1$, we have*

$$N_r R_r^\vee = (\mathbf{I} - V_r) R_r^\vee = \left\{ \sum_{i=1}^r c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_r \equiv 0 \pmod{2} \right\},$$

with a \mathbb{Z} -basis given by $\{\alpha_1^\vee, \dots, \alpha_{r-1}^\vee, 2\alpha_r^\vee\}$.

Now define U_r to be the matrix obtained from N_r by replacing $\mathcal{C}_1(N_r)$ by $\sum_{i=1}^r i\mathcal{C}_i(N_r)$, and replacing $\mathcal{C}_i(N_r)$ for $2 \leq i \leq r-1$ by $\sum_{j=i}^r \mathcal{C}_j(N_r)$. Thus

$$(5.3.2) \quad U_r = \begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & -1 \\ 2 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

For $2 \leq i \leq r$, multiply $\mathcal{C}_i(U_r)$ by -1 , and then permute columns to obtain the Smith normal form.

Lemma 5.12. *The matrix N_r has Smith normal form $\text{diag}(1^{r-1}, 2)$.*

Corollary 5.13. $R_r^\vee/N_r R_r^\vee = R_r^\vee/(\mathbf{I} - V_r) R_r^\vee \cong \mathbb{Z}/2\mathbb{Z}$.

We also define several additional matrices, which will appear as blocks in subsequent arguments, as follows. For $k \in \mathbb{Z}$ and $m, n \geq 1$ we define $m \times n$ matrices

$$L_{m,n}(k) = \begin{pmatrix} k & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ k & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad R_{m,n}(k) = \begin{pmatrix} 0 & \cdots & 0 & k \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & k \end{pmatrix}.$$

That is, $L_{m,n}(k)$ (respectively, $R_{m,n}(k)$) is all 0s except for its first (respectively, last) column, which is all ks . Then for $k, k' \in \mathbb{Z}$, $m \geq 1$, and $n \geq 2$, we define

$$N_{m,n}(k; k') = L_{m,n}(k) + R_{m,n}(k') = \begin{pmatrix} k & 0 & \cdots & 0 & k' \\ \vdots & \vdots & & \vdots & \vdots \\ k & 0 & \cdots & 0 & k' \end{pmatrix}.$$

For $k \in \mathbb{Z}$, $m \geq 2$, and $n \geq 1$, we define

$$R_{m,n}(1, 2) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 2 \end{pmatrix} \quad \text{and} \quad R_{m,n}(-1, -2) = \begin{pmatrix} 0 & \cdots & 0 & -1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & -2 \end{pmatrix}.$$

That is, $R_{m,n}(1, 2)$ is all 0s except for its last column, where the (m, n) -entry is 2 and all other entries are 1, while $R_{m,n}(-1, -2) = -R(1, 2)$. Finally, for $k \in \mathbb{Z}$, $m \geq 2$, and $n \geq 1$, we define

$$\begin{aligned} N_{m,n}(k; 1, 2) &= L_{m,n}(k) + R_{m,n}(1, 2) \\ N_{m,n}(k; -1, -2) &= L_{m,n}(k) + R_{m,n}(-1, -2). \end{aligned}$$

To simplify notation, we will omit the subscript m, n when the size of these matrices is clear.

5.3.2. The matrices for $w_{\beta,\gamma}$ and $M_{\beta,\gamma}$. In this section we determine the matrices for $w_{\beta,\gamma}$ and hence $M_{\beta,\gamma} = \mathbf{I} - w_{\beta,\gamma}$, with respect to the basis Δ^\vee for R^\vee in type C_n with $n \geq 2$. Most of this section is devoted to finding the matrix for w_γ . The matrix for w_β then has a relatively straightforward description, and it is not difficult to multiply these matrices together to obtain the matrix for $w_{\beta,\gamma} = w_\beta \cdot w_\gamma$.

To construct the matrix for w_γ , we first record the following special cases.

Lemma 5.14. *Let W be of type C_n , for $n \geq 2$.*

- (1) *If $\gamma = (n)$, then the matrix for w_γ with respect to Δ^\vee is V_n .*
- (2) *If $\gamma = (1, \dots, 1)$ with $|\gamma| = n$, then the matrix for w_γ with respect to Δ^\vee is $-\mathbf{I}_n$.*

Proof. The proof of (1) is a straightforward generalization of the corresponding computations in Example 5.3. For (2), we have $w_\gamma = t_0 t_1 \dots t_{n-1}$ in the notation of (5.1.1). Then w_γ is the longest element in W (see, for instance, Table 1 of [BKOP14]). Since we are in type C_n , we then have $w_\gamma(\alpha_i^\vee) = -\alpha_i^\vee$ for all $i \in [n]$, as claimed. \square

We next find matrices for w_1^γ and w_q^γ , as defined in (5.1.3) and (5.1.4), respectively, for $|\gamma| = n$.

Lemma 5.15. *Let W be of type C_n for $n \geq 2$, and let $\gamma = (\gamma_1, \dots, \gamma_q)$ be a weakly increasing composition of n .*

- (1) *If $2 \leq \gamma_1 < n$, then the matrix for w_1^γ with respect to Δ^\vee is block lower-triangular of the form*

$$w_1^\gamma = \begin{pmatrix} \mathbf{I}_{n-\gamma_1} & \\ R(1, 2) & V_{\gamma_1} \end{pmatrix}.$$

- (2) *If $2 \leq \gamma_q < n$, then the matrix for w_q^γ with respect to Δ^\vee is block lower-triangular of the form*

$$w_q^\gamma = \begin{pmatrix} V_{\gamma_q} & \\ L(-2) & \mathbf{I}_{n-\gamma_q} \end{pmatrix}.$$

Proof. Parts (1) and (2) in this statement are obtained using the same methods as in the computation of w_1^γ and w_2^γ from Example 5.4, respectively. \square

We then obtain the matrix for $w_\gamma = w_1^\gamma \dots w_q^\gamma$ when $|\gamma| = n$ and $\gamma_1 \geq 2$ as follows.

Corollary 5.16. *Let W be of type C_n for $n \geq 2$, and let $\gamma = (\gamma_1, \dots, \gamma_q)$ be a weakly increasing composition of n . If $\gamma_1 \geq 2$, then the matrix for w_γ with respect to Δ^\vee is block lower-triangular of the form*

$$w_\gamma = \begin{pmatrix} V_{\gamma_q} & & & & \\ N(-2; 1, 2) & V_{\gamma_{q-1}} & & & \\ N(-2; 2) & N(-2; 1, 2) & V_{\gamma_{q-2}} & & \\ \vdots & \ddots & \ddots & \ddots & \\ N(-2; 2) & \cdots & N(-2; 2) & N(-2; 1, 2) & V_{\gamma_1} \end{pmatrix}.$$

Proof. This statement is obtained by generalizing the argument in Example 5.5, by induction on the number of parts of γ , and using the matrices given by Lemma 5.15. \square

We now consider the case $|\gamma| = n$ and $\gamma_1 = 1$.

Lemma 5.17. *Let W be of type C_n for $n \geq 2$, and let $\gamma = (\gamma_1, \dots, \gamma_q)$ be a weakly increasing composition of n . If the first m parts of γ are equal to 1, with $1 \leq m < n$, then the matrix for $w_1^\gamma \cdots w_m^\gamma$ with respect to Δ^\vee is block lower-triangular of the form*

$$w_1^\gamma \cdots w_m^\gamma = \begin{pmatrix} \mathbf{I}_{n-m} & \\ R(2) & -\mathbf{I}_m \end{pmatrix}.$$

Proof. If $m = 1$ we have $w_1^\gamma = t_0 = s_n$, and the result is immediate (see Example 5.6 for the case $n = 4$). Now if $m \geq 2$, we have $w_1^\gamma \cdots w_m^\gamma = t_0 \cdots t_{m-1}$. This is the longest element in the type C_m subsystem indexed by the last m nodes, which fixes α_i^\vee for $1 \leq i \leq n-m-1$. Hence by Lemma 5.14(2), it now suffices to prove that the matrix for $w_1^\gamma \cdots w_m^\gamma$ has $(n-m, n-m)$ -entry equal to 1, and $(i, n-m)$ -entry equal to 2 for $n-m+1 \leq i \leq n$. For this, we first obtain using similar arguments to those used to prove Corollary 5.16 that

$$w_m^\gamma(\alpha_{n-m}^\vee) = (s_{n-m+1} \cdots s_{n-1})(s_n s_{n-1} \cdots s_{n-m+1})(\alpha_{n-m}^\vee) = \alpha_{n-m}^\vee + \sum_{i=n-m+1}^n 2\alpha_i^\vee.$$

Using this result and induction on $m \geq 2$, we then get via (2.1.1) that

$$\begin{aligned} w_1^\gamma \cdots w_m^\gamma(\alpha_{n-m}^\vee) &= w_1^\gamma \cdots w_{m-1}^\gamma \left(\alpha_{n-m}^\vee + \sum_{i=n-m+1}^n 2\alpha_i^\vee \right) \\ &= \alpha_{n-m}^\vee + 2 \left(\alpha_{n-m+1}^\vee + \sum_{i=n-m+2}^n 2\alpha_i^\vee \right) + 2 \left(\sum_{i=n-m+2}^n -\alpha_i^\vee \right) \end{aligned}$$

which simplifies to give the required result. \square

Combining Lemma 5.17 and Corollary 5.16, we thus obtain the following uniform description of the matrix for w_γ when $|\gamma| = n$.

Corollary 5.18. *Let W be of type C_n for $n \geq 2$, and let $\gamma = (\gamma_1, \dots, \gamma_q)$ be a weakly increasing composition of n . Define $0 \leq m \leq q$ such that $\gamma_k = 1$ for all $1 \leq k \leq m$ and $\gamma_k \geq 2$ for all $m+1 \leq k \leq q$. Then the matrix for w_γ with respect to Δ^\vee is block lower-triangular of the form*

$$w_\gamma = \begin{pmatrix} V_{\gamma_q} & & & & & \\ N(-2; 1, 2) & V_{\gamma_{q-1}} & & & & \\ N(-2; 2) & N(-2; 1, 2) & V_{\gamma_{q-2}} & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ N(-2; 2) & \cdots & N(-2; 2) & N(-2; 1, 2) & V_{\gamma_{m+1}} & \\ N(-2; 2) & \cdots & N(-2; 2) & N(-2; 2) & N(-2; 2) & -\mathbf{I}_m \end{pmatrix}.$$

We can now put together the above results to describe the matrix for w_γ when $1 \leq |\gamma| \leq n$ (when $|\gamma| = 0$, the element w_γ is trivial). The proof of the next result generalizes the computation of the matrix for w_γ from Example 5.8.

Corollary 5.19. *Let W be of type C_n for $n \geq 2$, and let $\gamma = (\gamma_1, \dots, \gamma_q)$ be a weakly increasing composition such that $1 \leq |\gamma| \leq n$. Let $0 \leq m \leq q$ be such that $\gamma_k = 1$ for all $1 \leq k \leq m$ and $\gamma_k \geq 2$ for all $m+1 \leq k \leq q$. Then the matrix for w_γ with respect to Δ^\vee is*

block lower-triangular of the form

$$w_\gamma = \begin{pmatrix} I_{n-|\gamma|} & & & & \\ R(1,2) & V_{\gamma_q} & & & \\ R(2) & N(-2;1,2) & V_{\gamma_{q-1}} & & \\ \vdots & N(-2;2) & N(-2;1,2) & V_{\gamma_{q-2}} & \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ R(2) & N(-2;2) & \cdots & N(-2;2) & N(-2;1,2) & V_{\gamma_{m+1}} \\ R(2) & N(-2;2) & \cdots & N(-2;2) & N(-2;2) & N(-2;2) & -I_m \end{pmatrix}.$$

We now turn to the matrix for w_β in type C . The formula is given by the next result, which generalizes the computation of the matrix for w_β in Example 5.8.

Lemma 5.20. *Let W be of type C_n for $n \geq 2$, and let $\beta = (\beta_1, \dots, \beta_p)$ be a partition with $|\beta| \leq n$.*

- (1) *If $|\beta| \in \{0, 1\}$ then $w_\beta = I_n$.*
- (2) *If $|\beta| \geq 2$, let β' be the partition of $|\beta| + 1$ given by $\beta' = (\beta_1, \dots, \beta_p, 1)$, and let $w_{\beta'}$ be the matrix for the corresponding conjugacy class representative in type $A_{|\beta'|}$. Then the matrix for $w_\beta \in W$ with respect to Δ^\vee is the block-diagonal matrix $w_\beta = \begin{pmatrix} w_{\beta'} & \\ & I_{n-|\beta|} \end{pmatrix}$.*

Proof. If $|\beta| \in \{0, 1\}$ then by definition w_β is trivial, so the result in (1) is clear.

Assume now that $|\beta| = m \geq 2$, and recall from Section 5.1 that w_β is the product, in increasing order, of certain simple reflections in the type A_{m-1} subsystem indexed by the first $m-1$ nodes. Hence w_β fixes α_i^\vee for all $m+1 \leq i \leq n$, which explains the block $I_{n-|\beta|}$.

Now since the last part of β' equals 1, the expression for w_β in terms of the simple reflections in type A_{m-1} is equal to the expression for $w_{\beta'}$ in terms of the simple reflections in type A_m . Moreover, for all $2 \leq m \leq n$, the formula $s_{m-1}(\alpha_m^\vee) = \alpha_{m-1}^\vee + \alpha_m^\vee$ holds in type C_n as well as in type A_n by (2.1.1), since the Cartan matrix entries $c_{m-1,m} = -1$ for all $2 \leq m \leq n$ for both types. Hence for $1 \leq i \leq m$, the expression for $w_\beta(\alpha_i^\vee)$ in terms of the simple coroots $\{\alpha_1^\vee, \dots, \alpha_m^\vee\}$ in type C_n is equal to the expression for $w_{\beta'}(\alpha_i^\vee)$ in terms of the simple coroots $\{\alpha_1^\vee, \dots, \alpha_m^\vee\}$ in type A_n . The result in (2) follows. \square

The identity matrix blocks of w_β and w_γ mean that we can now easily determine the matrix for $w_{\beta,\gamma} = w_\beta \cdot w_\gamma$ as follows.

Corollary 5.21. *Let W be of type C_n for $n \geq 2$. Let (β, γ) be a pair of compositions so that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$. Let β' be the partition of $|\beta| + 1$ given by $\beta' = (\beta_1, \dots, \beta_p, 1)$.*

- (1) *If $|\beta| = n$, then the matrix for $w_{\beta,\gamma} = w_\beta$ with respect to Δ^\vee in type C_n the same as the matrix for $w_{\beta'}$ with respect to Δ^\vee in type A_n .*
- (2) *If $|\gamma| = n$, then the matrix for $w_{\beta,\gamma} = w_\gamma$ with respect to Δ^\vee in type C_n is given by Corollary 5.19 above.*
- (3) *Otherwise, the matrix for $w_{\beta,\gamma} = w_\beta \cdot w_\gamma$ with respect to Δ^\vee in type C_n is the block lower-triangular matrix obtained by replacing the $I_{n-|\gamma|}$ block in the top left of the matrix in Corollary 5.19 by the matrix $w_{\beta'}$.*

Finally, we are able to determine the matrix for $M_{\beta,\gamma} = I - w_{\beta,\gamma}$. Recall N_r from (5.3.1).

Corollary 5.22. *Let W be of type C_n for $n \geq 2$. Let (β, γ) be a pair of compositions so that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$. Let β' be the partition of $|\beta| + 1$ given by $\beta' = (\beta_1, \dots, \beta_p, 1)$, and let $0 \leq m \leq q$ be such that $\gamma_k = 1$ for all $1 \leq k \leq m$ and $\gamma_k \geq 2$ for all $m + 1 \leq k \leq q$.*

- (1) *If $|\beta| = n$, then the matrix for $M_{\beta, \gamma} = I - w_{\beta}$ with respect to Δ^\vee is equal to the matrix $M_{\beta'}$ from Corollary 4.17.*
- (2) *If $|\gamma| = n$, then the matrix for $M_{\beta, \gamma} = I - w_{\gamma}$ with respect to Δ^\vee is block lower-triangular of the form*

$$M_{\beta, \gamma} = \begin{pmatrix} N_{\gamma_q} & & & & \\ N(2; -1, -2) & N_{\gamma_{q-1}} & & & \\ N(2; -2) & N(2; -1, -2) & N_{\gamma_{q-2}} & & \\ \vdots & \ddots & \ddots & \ddots & \\ N(2; -2) & \cdots & N(2; -2) & N(2; -1, -2) & N_{\gamma_{m+1}} \\ N(2; -2) & \cdots & N(2; -2) & N(2; -2) & N(2; -2) & 2I_m \end{pmatrix}.$$

- (3) *Otherwise, the matrix for $M_{\beta, \gamma} = I - w_{\beta, \gamma}$ with respect to Δ^\vee is block lower-triangular of the form*

$$M_{\beta, \gamma} = \begin{pmatrix} M_{\beta'} & & & & \\ R(-1, -2) & N_{\gamma_q} & & & \\ R(-2) & N(2; -1, -2) & N_{\gamma_{q-1}} & & \\ \vdots & N(2; -2) & N(2; -1, -2) & N_{\gamma_{q-2}} & \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ R(-2) & N(2; -2) & \cdots & N(2; -2) & N(2; -1, -2) & N_{\gamma_{m+1}} \\ R(-2) & N(2; -2) & \cdots & N(2; -2) & N(2; -2) & N(2; -2) & 2I_m \end{pmatrix}.$$

5.3.3. Proof of Theorem 5.2. In this section we complete the proof of Theorem 5.2. Let W be of type C_n with $n \geq 2$, and let (β, γ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$. Let $M_{\beta, \gamma} = I - w_{\beta, \gamma}$ be the matrix given by Corollary 5.22.

We first assume that $|\beta| = n$. Then $I_\gamma = J_\gamma = \emptyset$ and $q = 0$. Now since $M_{\beta, \gamma} = M_{\beta'}$ where $\beta' = (\beta_1, \dots, \beta_p, 1)$ has last part equal to 1, the result follows from Theorem 4.2 in this case.

We next assume that $|\gamma| = n$, in which case $J_\beta = \emptyset$. By Corollary 5.22(2), for all $i \in I_\gamma$, the nonzero entries in row i of $M_{\beta, \gamma}$ are ± 2 , while for all $i \in [n] \setminus I_\gamma$, the last nonzero entry in row i is the -1 in the $(i, i+1)$ -position. Hence, if $|\beta| = 0$, then

$$\text{MOD}(w_{\beta, \gamma}) = \left\{ \sum_{i=1}^n c_i \alpha_i^\vee \mid c_i \in \mathbb{Z}, c_i \equiv 0 \pmod{2} \text{ for } i \in I_\gamma \right\},$$

which clearly has a \mathbb{Z} -basis given by $\{\alpha_i^\vee \mid i \in [n] \setminus I_\gamma\} \cup \{2\alpha_i^\vee \mid i \in I_\gamma\}$.

For the Smith normal form when $|\gamma| = n$, we first use the pivot entries in the $2I_m$ block in the lower right to clear all nonzero entries in the last m rows of $M_{\beta, \gamma}$. We then use column operations to convert $N_{\gamma_{m+1}}$ to the matrix $U_{\gamma_{m+1}}$ defined in (5.3.2). Since $U_{\gamma_{m+1}}$ has pivot entries -1 in all but its last row, and a pivot entry 2 in its last row, we can then use column operations to clear all nonzero entries to the left of $U_{\gamma_{m+1}}$. We then repeat this process

moving up one block at a time, to obtain the block-diagonal matrix

$$U_{\beta,\gamma} = \begin{pmatrix} U_{\gamma_q} & & & \\ & \ddots & & \\ & & U_{\gamma_{m+1}} & \\ & & & 2I_m \end{pmatrix}.$$

Now $2I_m$ is in Smith normal form already, while for $m+1 \leq k \leq q$, the matrix U_{γ_k} has Smith normal form $\text{diag}(1^{\gamma_k-1}, 2)$ by Lemma 5.12. We thus obtain that $M_{\beta,\gamma}$ has Smith normal form $S_{\beta,\gamma} = \text{diag}(1^{n-q}, 2^q)$, and hence $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong (\mathbb{Z}/2\mathbb{Z})^q$. This completes the proof of Theorem 5.2 in the case $|\gamma| = n$.

The remaining case is when $1 \leq |\beta| \leq n-1$ and $1 \leq |\gamma| \leq n-1$, which corresponds to case (3) of Corollary 5.22. Applying the same argument as for $|\gamma| = n$, we initially see that $M_{\beta,\gamma}\mathbf{x} = \mathbf{c}$ has solution over \mathbb{Z} if and only if $c_i \equiv 0 \pmod{2}$ for all $i \in I_\gamma$ and the $|\beta|$ equations represented by the matrix $M_{\beta'}$ in the top left of $M_{\beta,\gamma}$ have solution over \mathbb{Z} . Now as the partition β' has last part equal to 1, these $|\beta|$ equations have solution over \mathbb{Z} if and only if $c_i = 0$ for all $i \in \{1, \dots, |\beta|\} \setminus J_\beta$ by part (1) of Theorem 4.2. This establishes the description of $\text{MOD}(w_{\beta,\gamma})$ from Theorem 5.2(1), and the \mathbb{Z} -basis given in (2) is immediate.

For the Smith normal form in this final case, we first apply to the $|\gamma| \times |\gamma|$ block in the bottom right-hand corner of $M_{\beta,\gamma}$ the same sequence of column operations as in the case $|\gamma| = n$. This converts $M_{\beta,\gamma}$ to the block lower-triangular matrix

$$U_{\beta,\gamma} = \begin{pmatrix} M_{\beta'} & & & & \\ R(-1, -2) & U_{\gamma_q} & & & \\ R(-2) & & U_{\gamma_{q-1}} & & \\ \vdots & & & U_{\gamma_{q-2}} & \\ \vdots & & & & \ddots \\ R(-2) & & & & U_{\gamma_{m+1}} \\ R(-2) & & & & & 2I_m \end{pmatrix}.$$

We can then use the pivot entries in the U_{γ_k} and $2I_m$ to clear the nonzero entries in the corresponding matrices $R(-1, -2)$ and $R(-2)$. We now have the block-diagonal matrix

$$U'_{\beta,\gamma} = \begin{pmatrix} M_{\beta'} & & & & \\ & U_{\gamma_q} & & & \\ & & \ddots & & \\ & & & U_{\gamma_{m+1}} & \\ & & & & 2I_m \end{pmatrix}.$$

Since the partition β' has $p+1$ parts with last part equal to 1, by Theorem 4.2 the Smith normal form for $M_{\beta'}$ is $\text{diag}(1^{|\beta|-(p+1)}, 1, 0^{(p+1)-1}) = \text{diag}(1^{|\beta|-p}, 0^p)$. Now from the case $|\gamma| = n$, the $|\gamma| \times |\gamma|$ matrix in the bottom right has Smith normal form $\text{diag}(1^{|\gamma|-q}, 2^q)$. Putting this together with the equation $|\beta| + |\gamma| = n$, we obtain that $M_{\beta,\gamma}$ has Smith normal form $S_{\beta,\gamma} = (1^{n-p-q}, 2^q, 0^p)$. Therefore $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong (\mathbb{Z}/2\mathbb{Z})^q \oplus \mathbb{Z}^p$.

This completes the proof of Theorem 5.2. \square

6. TYPE *B* MOD-SETS

In this section, we give a description of all mod-sets in type *B*. Since the minimal length representatives of conjugacy classes in W are identical in types B_n and C_n , we start by

directly introducing the general notation required for the main results stated in Section 6.1. We then give several key examples in Section 6.2, designed to fully illustrate the proofs of our results on mod-sets.

Let W be the finite Weyl group of type B_n , with $n \geq 2$. Throughout this section, we order the nodes of the Dynkin diagram increasing from left to right, as in both Bourbaki [Bou02] and Sage [Sag24], so that the first $n - 1$ nodes form a type A_{n-1} subsystem, and the special node is indexed by n on the right. We note that our labeling is the reverse of the ordering of nodes used in [GP00], and also that in Sage [Sag24] the Cartan matrices in types B_n and C_n are reversed with respect to the Dynkin diagrams; see Appendix B for a direct comparison of these conventions.

6.1. Conjugacy class representatives and mod-sets in type B . Following [GP00, Proposition 3.4.7] as reviewed in Section 5.1, the conjugacy classes of W of type B_n are parameterized by ordered pairs of compositions (β, γ) such that β is weakly decreasing, γ is weakly increasing, and $|\beta| + |\gamma| = n \geq 2$. For each such pair (β, γ) , recall from Section 5.1 the standard parabolic subgroups W_β and W_γ of W , and the element $w_{\beta, \gamma} = w_\beta \cdot w_\gamma$ with w_β cuspidal in W_β and w_γ cuspidal in W_γ such that $w_{\beta, \gamma}$ form a complete system of minimal length representatives for the conjugacy classes of W .

We now establish the notation needed to formally state our results on mod-sets in type B . Throughout this section, we let

$$\beta = (\beta_1, \dots, \beta_p) \quad \text{and} \quad \gamma = (\gamma_1, \dots, \gamma_q)$$

be a pair of compositions such that β is weakly decreasing, γ is weakly increasing, and $|\beta| + |\gamma| = n$, with corresponding conjugacy class representative $w_{\beta, \gamma} \in W$, constructed in Section 5.1. Write $|\beta| = m$ so that $|\gamma| = n - m$. Recall the subsets J_β and J_γ which index the support of w_β and w_γ , as well as the sequences (j_k^β) and (j_k^γ) of partial sums of the compositions β and γ , which both satisfy $j_1^\beta = j_1^\gamma = 0$.

For the weakly increasing composition γ , we define the following q -element subset of $[n]$ exactly as in type C_n :

$$I_\gamma = \left\{ n - j_k^\gamma \mid 1 \leq k \leq q \right\} = \left\{ n, n - \gamma_1, \dots, n - \sum_{k=1}^{q-1} \gamma_k \right\} = \{n - (|\gamma| - \gamma_q), \dots, n - \gamma_1, n\}.$$

In type B_n , unlike type C_n , we also have the following subset depending on the partition β :

$$I_\beta = \{j_k^\beta \mid 2 \leq k \leq p\} \cup \{m\}.$$

Given any subset $I \subseteq [n]$, we denote by $I - 1 = \{i - 1 \mid i \in I\}$.

To simplify notation in the statements below, we further define $\gamma_{1,0} = 0$ and $\gamma_{k,k-1} = (\gamma_k + \gamma_{k-1}) \bmod (2)$ for $2 \leq k \leq q$. That is, $\gamma_{k,k-1} = 0$ if the parts γ_k and γ_{k-1} have the same parity, and otherwise $\gamma_{k,k-1} = 1$. In particular, note that all parts of γ have the same parity if and only if $\gamma_{k,k-1} = 0$ for all $2 \leq k \leq q$. We define $\gcd(\beta_k, 2) = \gcd(\beta_1, \dots, \beta_k, 2)$, so that $\gcd(\beta_k, 2) = 1$ if and only if β has at least one odd part. Finally, if β has last part $\beta_p \geq 2$, we write $\gcd(\beta_k, \beta_p - 2) = \gcd(\beta_1, \dots, \beta_p, \beta_p - 2)$, noting that this integer equals $\gcd(\beta_k)$ if $\beta_p = 2$.

With this notation established, we now state our results describing mod-sets in type B . Theorem 6.1 provides the Smith normal form of $(I - w_{\beta, \gamma})$ in the type B_n basis Δ^\vee , from which the isomorphism type of the quotient $R^\vee/\text{MOD}(w_{\beta, \gamma})$ is immediate. Note that even though the Cartan matrices in types B_n and C_n differ only by exchanging the $(n-1, n)$ - and

$(n, n-1)$ -entries, the results on mod-sets are rather different; compare Theorems 6.1 and 6.2 in type B below to their (more straightforward) type C counterpart Theorem 5.2.

Theorem 6.1. *Suppose W is of type B_n with $n \geq 2$. Let (β, γ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$, with corresponding conjugacy class representative $w_{\beta, \gamma} \in W$.*

Then for any $w \in [w_{\beta, \gamma}]$, the Smith normal form of $(I - w)$ is as follows:

- (1) *If $\gcd(\beta_k, 2) = 2$ (including $|\beta| = 0$), $|\gamma| \geq 1$, and all parts of γ have the same parity, then*

$$S_{\beta, \gamma} = \text{diag}(1^{n-q-p}, 2^q, 0^p).$$

- (2) *If $\gcd(\beta_k, 2) = 2$ (including $|\beta| = 0$), $|\gamma| \geq 1$, and γ has a change in parity, then*

$$S_{\beta, \gamma} = \text{diag}(1^{n-q-p+1}, 2^{q-2}, 4, 0^p).$$

- (3) *If $\gcd(\beta_k, 2) = 1$ and $|\gamma| \geq 1$, then*

$$S_{\beta, \gamma} = \text{diag}(1^{n-q-p+1}, 2^{q-1}, 0^p).$$

- (4) *If $|\gamma| = 0$, $\beta_p \geq 2$, and $\gcd(\beta_k, \beta_p - 2) \geq 2$, then*

$$S_{\beta, \gamma} = \text{diag}(1^{n-p-1}, \gcd(\beta_k, \beta_p - 2), 0^p).$$

- (5) *If $|\gamma| = 0$, and either $\beta_p = 1$, or $\beta_p \geq 2$ and $\gcd(\beta_k, \beta_p - 2) = 1$, then*

$$S_{\beta, \gamma} = \text{diag}(1^{n-p}, 0^p).$$

We also give a basis for $\text{MOD}(w_{\beta, \gamma})$ in Theorem 6.2 in the cases where either $|\beta| = 0$ or $|\gamma| = 0$. Note that $|\beta| = 0$ if and only if $w_{\beta, \gamma} = w_\gamma$ is cuspidal in W .

Theorem 6.2. *Suppose W is of type B_n with $n \geq 2$. Let (β, γ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\gamma = (\gamma_1, \dots, \gamma_q)$ is weakly increasing, and $|\beta| + |\gamma| = n$, with corresponding conjugacy class representative $w_{\beta, \gamma} \in W$.*

If either $|\beta| = 0$ or $|\gamma| = 0$, then the module $\text{MOD}(w_{\beta, \gamma})$ has a \mathbb{Z} -basis as follows:

- (1) *If $|\beta| = 0$ and $\gamma_1 \geq 3$, then $\text{MOD}(w_{\beta, \gamma})$ has \mathbb{Z} -basis given by*

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k^\gamma}^\vee + \gamma_{k,k-1}\alpha_n^\vee \mid 2 \leq k \leq q \right\} \cup \left\{ 2\alpha_{n-1}^\vee, (\gamma_1 \bmod 2)\alpha_{n-1}^\vee + \alpha_n^\vee \right\} \\ & \quad \cup \left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\gamma \cup (I_\gamma - 1) \right\} \\ & \quad \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\gamma - 1), i \neq n-1 \right\}. \end{aligned}$$

- (2) *If $|\beta| = 0$ and $\gamma_1 = 2$, then $\text{MOD}(w_{\beta, \gamma})$ has \mathbb{Z} -basis given by*

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k^\gamma}^\vee + \gamma_{k,k-1}\alpha_{n-1}^\vee \mid 2 \leq k \leq q \right\} \cup \left\{ 2\alpha_{n-1}^\vee, (\gamma_1 \bmod 2)\alpha_{n-1}^\vee + \alpha_n^\vee \right\} \\ & \quad \cup \left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\gamma \cup (I_\gamma - 1) \right\} \\ & \quad \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\gamma - 1), i \neq n-1 \right\}. \end{aligned}$$

- (3) *If $|\beta| = 0$, $\gamma_1 = 1$, and $\gamma_2 \geq 2$, then $\text{MOD}(w_{\beta, \gamma})$ has \mathbb{Z} -basis given by*

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k^\gamma}^\vee + \gamma_{k,k-1}\alpha_n^\vee \mid 1 \leq k \leq q \right\} \\ & \quad \cup \left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\gamma \cup (I_\gamma - 1) \right\} \\ & \quad \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\gamma - 1), i \neq n-1 \right\}. \end{aligned}$$

- (4) If $|\beta| = 0$ and $\gamma_k = 1$ for all $1 \leq k \leq \ell < n$ for some $\ell \geq 2$, while $\gamma_{\ell+1} \geq 2$, then $\text{MOD}(w_{\beta,\gamma})$ has \mathbb{Z} -basis given by

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k}^\vee + \gamma_{k,k-1}\alpha_n^\vee \mid 1 \leq k \leq q \right\} \cup \{-\alpha_{n-\ell-1}^\vee + \alpha_n^\vee\} \\ & \quad \cup \{-\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\gamma \cup (I_\gamma - 1)\} \\ & \quad \cup \{-\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\gamma - 1), i < n - \ell - 1\}. \end{aligned}$$

- (5) If $|\beta| = 0$ and $\gamma = (1, \dots, 1)$, then $\text{MOD}(w_{\beta,\gamma})$ has \mathbb{Z} -basis given by

$$\{2\alpha_i^\vee \mid i \in [n]\}.$$

- (6) If $|\gamma| = 0$, $\beta_p \geq 2$, and $\gcd(\beta_k, \beta_p - 2) \geq 2$, then $\text{MOD}(w_{\beta,\gamma})$ has \mathbb{Z} -basis given by

$$\{\alpha_i^\vee - \alpha_{i+1}^\vee \mid i \in J_\beta \setminus (I_\beta - 1)\} \cup$$

$$\{\alpha_i^\vee - \alpha_{i+2}^\vee \mid i \in (I_\beta - 1) \setminus \{n - 1\}\} \cup \{\gcd(\beta_k, \beta_p - 2)\alpha_{n-1}^\vee\}.$$

- (7) If $|\gamma| = 0$, and either $\beta_p = 1$, or $\beta_p \geq 2$ and $\gcd(\beta_k, \beta_p - 2) = 1$, then $\text{MOD}(w_{\beta,\gamma})$ has \mathbb{Z} -basis given by

$$\{\alpha_j^\vee \mid j \in J_\beta\}.$$

As seen above, the two special cases considered in Theorem 6.2 already involve several subcases and delicate statements. Thus for the remaining “mixed” case, where $|\beta| \geq 1$ and $|\gamma| \geq 1$, we have opted not to give even lengthier general statements, but instead to provide illustrative examples of how to identify convenient bases. In all cases, a set of equations explicitly describing $\text{MOD}(w_{\beta,\gamma})$ can easily be deduced once the (relatively sparse) basis is known.

6.2. Examples in type B. In this section, we illustrate both Theorems 6.1 and 6.2 with a sequence of examples, which can then be generalized in a straightforward manner to prove these results, as well as to find a basis in all cases. In Section 6.2.1, we consider the same pairs of compositions as in Section 5.2:

- $|\beta| = 0$ and $\gamma = (4)$ in Example 6.3;
- $|\beta| = 0$ and $\gamma = (3, 4)$ in Example 6.4;
- $|\beta| = 0$ and $\gamma = (2, 3, 4)$ in Example 6.5;
- $|\beta| = 0$ and $\gamma = (1, 3)$ in Example 6.6;
- $\beta = (1)$ and $\gamma = (3)$ in Example 6.7;
- $\beta = (4)$ and $\gamma = (3)$ in Example 6.8; and
- $\beta = (3, 3)$ and $\gamma = (3)$ in Example 6.9.

In Section 6.2.2, we then also consider:

- $|\beta| = 0$ and $\gamma = (1, \dots, 1)$ in Example 6.10;
- $|\beta| = 0$ and $\gamma = (1, 1, 1, 4)$ in Example 6.11;
- $\beta = (1)$ and $\gamma = (3, 4)$ in Example 6.12;
- $\beta = (5)$ and $\gamma = (3)$ in Example 6.13;
- $\beta = (4)$ and $|\gamma| = 0$ in Example 6.14;
- $\beta = (3, 3)$ and $|\gamma| = 0$ in Example 6.15; and
- $\beta = (4, 1)$ and $|\gamma| = 0$ in Example 6.16.

For each of these examples, we construct a \mathbb{Z} -basis for $\text{MOD}(w_{\beta,\gamma})$ and find the Smith normal form for $(I - w_{\beta,\gamma})$ in type B.

6.2.1. *Examples from type C, reconsidered in type B.* In this section, we illustrate our results in type B by reconsidering the same examples as in Section 5.2.

Example 6.3. This example illustrates part (1) of Theorem 6.1 and part (1) of Theorem 6.2. Let W be of type B_4 and suppose $|\beta| = 0$ and $\gamma = (4)$, so that $w_{\beta,\gamma} = w_\gamma = s_4s_3s_2s_1$ is Coxeter in W . The matrices for w_γ and $N_\gamma = I - w_\gamma$ with respect to the type B_4 basis Δ^\vee are given by

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & \textcircled{2} \\ \textcircled{-1} & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the matrix for w_γ differs from that of Example 5.3 in only the circled entries, due to the slight difference in the Cartan matrices.

To determine $\text{MOD}(w_\gamma)$, we use similar column operations to those we used to obtain B_β from M_β in Example 4.3. In particular, replace $\mathcal{C}_1(N_\gamma)$ by $\sum_{i=1}^3 i\mathcal{C}_i(N_\gamma)$ to give us B'_γ :

$$B'_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \textcircled{4} & 0 & 1 & \boxed{-2} \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow B_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\gcd(\beta_k, 2) = 2$ in this case, we can use the boxed -2 in column 4 to clear out the circled entry in column 1, giving us B_γ . The corresponding \mathbb{Z} -basis for $\text{MOD}(w_{\beta,\gamma})$ is then given by $\{-\alpha_1^\vee + \alpha_2^\vee, -\alpha_2^\vee + \alpha_3^\vee, 2\alpha_3^\vee, \alpha_4^\vee\}$, matching part (1) of Theorem 6.2.

To determine the Smith normal form S_γ for N_γ , it is clear that following several row operations on B_γ , we obtain

$$T_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $S_\gamma = \text{diag}(1, 1, 1, 2)$, and $R^\vee/\text{MOD}(w_\gamma) \cong \mathbb{Z}/2\mathbb{Z}$, confirming part (1) of Theorem 6.1.

Example 6.4. This example illustrates part (2) of Theorem 6.1 and part (1) of Theorem 6.2, in the case that γ has more than one part. Let W be of type B_7 and suppose $|\beta| = 0$ and $\gamma = (3, 4)$, so that $w_{\beta,\gamma} = w_\gamma = w_1^\gamma w_2^\gamma = (s_7s_6s_5)(s_4s_5s_6s_7s_6s_5s_4s_3s_2s_1)$. The matrices for w_γ and $N_\gamma = I - w_\gamma$ with respect to the type B_7 basis Δ^\vee are given by

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ \textcircled{-1} & 0 & 0 & \textcircled{1} \end{pmatrix}, \quad N_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 \\ 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Note that the matrix for w_γ differs from that of Example 5.4 in only the circled entries, due to the slight difference in the Cartan matrices.

For $\text{MOD}(w_\gamma)$, we carry out column operations on N_γ similar to those in the previous example within blocks. First add column 5 to column 4. We then replace $\mathcal{C}_1(N_\gamma)$ by $\mathcal{C}_1(N_\gamma) + 2\mathcal{C}_2(N_\gamma) + 3\mathcal{C}_3(N_\gamma)$ and then replace $\mathcal{C}_5(N_\gamma)$ by $\mathcal{C}_5(N_\gamma) + 2\mathcal{C}_6(N_\gamma)$ to give us B'_γ :

$$B'_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \textcircled{4} & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ \textcircled{2} & 0 & 0 & 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & 3 & 1 & -2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow B''_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ \textcircled{8} & 0 & 0 & 0 & \textcircled{3} & 1 & \boxed{-2} \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using columns 4 and 6 to clear the circled entries in column 1, we obtain B''_γ above. Using the boxed pivot -2 in the last column, we can then reduce all other entries in row $n-1$ modulo 2 to obtain B_γ

$$B_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

It may be checked that the $(n-1, 1)$ -entry of B_γ equals $\gamma_2(\text{mod } 2)$, and the $(n-1, 5)$ -entry of B_γ equals $\gamma_1(\text{mod } 2)$, so that these entries differ if and only if $\gamma_{2,1} = 1$. From here, the \mathbb{Z} -basis $\{2\alpha_4^\vee + \alpha_7^\vee\} \cup \{2\alpha_6^\vee, \alpha_6^\vee + \alpha_7^\vee\} \cup \{-\alpha_1^\vee + \alpha_2^\vee, -\alpha_2^\vee + \alpha_3^\vee, -\alpha_5^\vee + \alpha_6^\vee\} \cup \{-\alpha_3^\vee + \alpha_5^\vee\}$ can be read off, matching part (1) of Theorem 6.2.

For the Smith normal form, continuing from B_γ , we can add rows 1 to 2, 2 to 3, 3 to 5, and 5 to 6 successively to obtain T'_γ :

$$T'_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Extracting the 3×3 submatrix containing the nonzero entries of columns 1, 5, and 7, we then have

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & -4 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & -4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Inserting these columns back into T'_γ then gives us the matrix T_γ :

$$T_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $S_\gamma = \text{diag}(1^5, 4)$, and so $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/4\mathbb{Z})$, confirming part (2) of Theorem 6.1.

Example 6.5. This example illustrates part (2) of Theorem 6.1 and part (2) of Theorem 6.2. Let W be of type B_9 and suppose $|\beta| = 0$ and $\gamma = (2, 3, 4)$, so that $w_{\beta, \gamma} = w_\gamma = w_1^\gamma w_2^\gamma w_3^\gamma = (s_9 s_8)(s_7 s_8 s_9 s_8 s_7 s_6 s_5)(s_4 \cdots s_8 s_9 s_8 \cdots s_4 s_3 s_2 s_1)$. The matrices for w_γ and $N_\gamma = I - w_\gamma$ with respect to the type B_9 basis Δ^\vee are given by

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 1 & -1 & 1 & 0 \\ -2 & 0 & 0 & 1 & -1 & 0 & 1 \\ -2 & 0 & 0 & 2 & -2 & 0 & 1 \\ -2 & 0 & 0 & 2 & -2 & 0 & 1 & -1 & \textcircled{2} \\ (-1) & 0 & 0 & \textcircled{1} & (-1) & 0 & \textcircled{1} & (-1) & 1 \end{pmatrix},$$

$$N_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 2 & -1 & 0 \\ 2 & 0 & 0 & -1 & 1 & 1 & -1 \\ 2 & 0 & 0 & -2 & 2 & 0 & 0 \\ 2 & 0 & 0 & -2 & 2 & 0 & -1 & 2 & -2 \\ 1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Note that the matrix for w_γ differs from that of Example 5.5 in only the circled entries, due to the slight difference in the Cartan matrices.

For $\text{MOD}(w_\gamma)$, we carry out column operations on N_γ similar to those in the previous two examples. Start by adding columns 5 and 8 to columns 4 and 7, respectively. Clearing entries

within the first columns of the blocks on the diagonal, we obtain B'_γ :

$$B'_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \textcircled{4} & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ \textcircled{2} & 0 & 0 & 1 & 0 & -1 & 0 \\ \textcircled{2} & 0 & 0 & 0 & \textcircled{3} & 1 & -1 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \boxed{1} & 0 \end{pmatrix}.$$

Notice that we now have a boxed pivot entry 1 in the last row, since $\delta_1 = 2$. We use this to clear the last row, and also clear the circled entries using columns to their right with first entry -1 , as in the previous examples, and we get

$$B''_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ \textcircled{2} & 0 & 0 & 0 & 2 & 0 & 0 \\ 10 & 0 & 0 & 0 & 5 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We then clear the circled entry using column 5, and carry out several final column operations to obtain B_γ :

$$B_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

from which the \mathbb{Z} -basis $\{2\alpha_4^\vee + \alpha_8^\vee, 2\alpha_7^\vee + \alpha_8^\vee\} \cup \{2\alpha_8^\vee, \alpha_9^\vee\} \cup \{-\alpha_1^\vee + \alpha_2^\vee, -\alpha_2^\vee + \alpha_3^\vee, -\alpha_5^\vee + \alpha_6^\vee\} \cup \{-\alpha_3^\vee + \alpha_5^\vee, -\alpha_6^\vee + \alpha_8^\vee\}$ can be read off, matching part (2) of Theorem 6.2.

For the Smith normal form, continuing from B_γ , we can add rows 1 to 2, 2 to 3, 3 to 5, 5 to 6, and 6 to 8 successively to obtain T'_γ :

$$T'_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

Extracting the 3×3 submatrix containing the nonzero entries of columns 1, 5, and 8, that is, the columns of T'_γ which have 2s, we have

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -2 & -4 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 4 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Inserting these columns back into T'_γ then gives us the matrix T_γ :

$$T_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus $S_\gamma = \text{diag}(1^6, 2, 4)$, and so $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$, confirming part (2) of Theorem 6.1.

Example 6.6. This example illustrates part (1) of Theorem 6.1 and part (3) of Theorem 6.2. Let W be of type B_4 and suppose $|\beta| = 0$ and $\gamma = (1, 3)$, so that $w_{\beta, \gamma} = w_\gamma = w_1^\gamma w_2^\gamma = (s_4)(s_3 s_4 s_3 s_2 s_1)$. The matrices for w_γ and $N_\gamma = I - w_\gamma$ with respect to the type B_4 basis Δ^\vee are given by

$$w_\gamma = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ \textcircled{-1} & 0 & \textcircled{1} & -1 \end{pmatrix}, \quad N_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 \end{pmatrix}.$$

Note that the matrix for w_γ differs from that of Example 5.6 in only the circled entries, due to the slight difference in the Cartan matrices.

Proceeding using similar column operations to the previous examples, we obtain

$$B_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

from which the \mathbb{Z} -basis $\{2\alpha_3^\vee, 2\alpha_4^\vee\} \cup \{-\alpha_1^\vee + \alpha_2^\vee\} \cup \{-\alpha_2^\vee + \alpha_4^\vee\}$ can be read off. Here, we have $I_\gamma = \{3, 4\}$ so $I_\gamma - 1 = \{2, 3\}$, and this matches part (3) of Theorem 6.2.

To obtain the Smith normal form, continue by doing successive row operations on B_γ as in the previous examples, to yield

$$T_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

Thus $S_\gamma = \text{diag}(1, 1, 2, 2)$, and so $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})^2$, confirming part (1) of Theorem 6.1.

Our final examples in this section illustrate the case when the compositions β and γ are both nonempty.

Example 6.7. This example illustrates part (3) of Theorem 6.1, and shows how to obtain a basis for $\text{MOD}(w_{\beta,\gamma})$. Let W be of type B_4 and suppose $\beta = (1)$ and $\gamma = (3)$, so that $w_{\beta,\gamma} = w_\gamma = s_4s_3s_2$. The matrices for $w_{\beta,\gamma}$ and $M_{\beta,\gamma} = \mathbf{I} - w_{\beta,\gamma}$ with respect to the type B_4 basis Δ^\vee are given by

$$w_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & \textcircled{2} \\ \textcircled{1} & \textcircled{-1} & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & 1 & 1 & -2 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that the matrix for $w_{\beta,\gamma}$ differs from that of Example 5.7 in only the circled entries, due to the slight difference in the Cartan matrices.

To find $\text{MOD}(w_{\beta,\gamma})$, we now add column 2 to column 1 of $M_{\beta,\gamma}$, to obtain $B'_{\beta,\gamma}$:

$$B'_{\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \boxed{1} & 2 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We then use the boxed pivot entry 1 in the first column of $B'_{\beta,\gamma}$ to clear the remainder of row 2, and then use the resulting pivot entry 1 in column 3 to clear the remainder of row 3, to obtain $B_{\beta,\gamma}$:

$$B_{\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & \boxed{1} & 0 & 0 \end{pmatrix}.$$

The resulting \mathbb{Z} -basis is $\{\alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee\}$. We also have $S_{\beta,\gamma} = \text{diag}(1^3, 0)$, and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong \mathbb{Z}$, confirming part (3) of Theorem 6.1.

Example 6.8. This example illustrates part (1) of Theorem 6.1 in the case $|\beta| \neq 0$, and shows how to obtain a basis for $\text{MOD}(w_{\beta,\gamma})$. Let W be of type B_7 and suppose $\beta = (4)$ and $\gamma = (3)$ so that $w_{\beta,\gamma} = w_\beta w_\gamma = (s_1s_2s_3)(s_7s_6s_5)$. The matrices for $w_{\beta,\gamma}$ and $M_{\beta,\gamma} = \mathbf{I} - w_{\beta,\gamma}$

with respect to the type B_7 basis Δ^\vee are given by

$$w_{\beta,\gamma} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & 1 & -1 & 1 & 0 \\ & & & 1 & -1 & 0 & \textcircled{2} \\ & & & \textcircled{1} & \textcircled{-1} & 0 & 1 \end{pmatrix} \quad M_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 2 & -1 & 0 \\ & & & -1 & 1 & 1 & -2 \\ & & & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that the matrix for $w_{\beta,\gamma}$ differs from that of Example 5.8 in only the circled entries, due to the slight difference in the Cartan matrices.

We start by adding columns 3 and 5 to column 4, and then proceed with column operations as in previous type A examples, to obtain

$$B'_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 2 & -1 & 0 \\ & & & 0 & 1 & 1 & -2 \\ & & & 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow B''_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 1 & 2 & -1 & 0 \\ & & & -4 & 0 & 1 & 1 & -2 \\ & & & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using columns 4 and 6, we can move the $\beta_k = 4$ down into the row with the -2 in the last column, giving $B''_{\beta,\gamma}$. At this point, we replace the -4 by $\gcd(\beta_k, 2) \in \{1, 2\}$.

Note that the lower right-hand 3×3 matrix is the same as for $\gamma = (3)$ in type B_3 , so that the basis coming from the last 3 columns is the same as if we were working in type B_3 . Finally, we use $\gcd(\beta_k, 2)$ to clear the final column, removing $2\alpha_6^\vee$ from the basis for the lower right 3×3 minor. The resulting \mathbb{Z} -basis from the matrix $B_{\beta,\gamma}$ below

$$B_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 1 & 0 & -1 & 0 \\ & & & 2 & 0 & 1 & 1 & 0 \\ & & & 0 & 1 & 0 & 0 \end{pmatrix}.$$

is read off as $\{\alpha_1^\vee - \alpha_2^\vee, \alpha_2^\vee - \alpha_3^\vee, \alpha_5^\vee - \alpha_6^\vee\} \cup \{\alpha_3^\vee + \alpha_5^\vee\} \cup \{\gcd(\beta_k, 2)\alpha_6^\vee\} \cup \{\alpha_6^\vee + \alpha_7^\vee\}$.

To find the Smith normal form, continue by doing successive row operations on $B_{\beta,\gamma}$ as in previous examples, to yield

$$T_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & -1 & 0 \\ & & & 2 & 0 & 0 & 0 & 0 \\ & & & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Therefore $S_{\beta,\gamma} = \text{diag}(1^5, 2, 0)$, and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, confirming part (1) of Theorem 6.1.

Example 6.9. This example illustrates part (3) of Theorem 6.1, and shows how to obtain a basis for $\text{MOD}(w_{\beta,\gamma})$. Let W be of type B_9 and suppose $\beta = (3, 3)$ and $\gamma = (3)$. The matrix $M_{\beta,\gamma} = \mathbf{I} - w_{\beta,\gamma}$ with respect to the type B_9 basis Δ^\vee is given by

$$M_{\beta,\gamma} = \begin{pmatrix} 1 & 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & & & -1 & 1 & 1 & -1 & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & -1 & 2 & -1 & 0 \\ & & & & & & & -1 & 1 & 1 & -2 \\ & & & & & & & & -1 & 1 & 0 & 0 \end{pmatrix}.$$

We add columns 5 and 7 to column 6, carry out column operations on the first 5 columns as done in type A , and then move $\gcd(\beta_k) = 3$ down to row $n-1$, giving us $B'_{\beta,\gamma}$:

$$B'_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & & & & & & \\ -1 & 0 & 1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & & & -1 & 1 & 0 & 0 & & \\ & & & & -1 & 0 & 1 & & \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 1 & 2 & -1 & 0 \\ & & & & & & & -3 & 0 & 1 & 1 & -2 \\ & & & & & & & & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now since $\gcd(\beta_k, 2) = 1$ we get a pivot 1 in entry $(n-1, m-1)$. We can use this pivot to successively clear entries in the last $\gamma_1 = 3$ rows, and then clear entries in successive rows, using only column operations, and moving up the matrix, resulting in $B_{\beta,\gamma}$:

$$B_{\beta,\gamma} = \begin{pmatrix} \boxed{1} & 0 & 0 & & & & & & \\ 0 & 0 & \boxed{1} & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & & & 0 & \boxed{1} & 0 & 0 & & \\ & & & & 0 & 0 & \boxed{1} & & \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & \boxed{1} & 0 \\ & & & & & & & \boxed{1} & 0 & 0 & 0 \\ & & & & & & & & 0 & \boxed{1} & 0 & 0 \end{pmatrix}.$$

Therefore $S_{\beta,\gamma} = \text{diag}(1^7, 0^2)$, and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong \mathbb{Z}^2$, confirming part (3) of Theorem 6.1.

6.2.2. Additional examples in type B . We now give some further examples in type B , to illustrate the differences in behavior compared to type C . See the introduction to Section 6.2 for the list of further examples we consider in type B .

Example 6.10. This example proves a special case of part (1) of Theorem 6.1, and part (5) of Theorem 6.2. Let W be of type B_n with $n \geq 2$, and let $|\beta| = 0$ and $\gamma = (1, \dots, 1)$. For

this case, we have that

$$w_{\beta,\gamma} = w_\gamma = s_n(s_{n-1}s_n s_{n-1}) \cdots (s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1),$$

which equals the longest element w_0 of W ; see Table 1 in [BKOP14], for example. Thus w_0 is central in W and acts as $-\mathbf{I}$ on R^\vee . Hence, in this case

$$\text{MOD}(w_{\beta,\gamma}) = (\mathbf{I} - w_\gamma)R^\vee = (2\mathbf{I})R^\vee = 2R^\vee,$$

which has \mathbb{Z} -basis $\{2\alpha_i^\vee \mid i \in [n]\}$. The Smith normal form for $(\mathbf{I} - w_{\beta,\gamma})$ is also clearly $S_{\beta,\gamma} = (2^n)$.

Example 6.11. This example illustrates part (2) of Theorem 6.1 and part (4) of Theorem 6.2. Let W be of type B_7 , and let $|\beta| = 0$ and $\gamma = (1, 1, 1, 4)$. The matrix for $N_\gamma = \mathbf{I} - w_\gamma$ with respect to the type B_7 basis Δ^\vee is given by

$$N_\gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & & & -2 & 2 & 0 & 0 \\ 2 & & & -2 & 0 & 2 & 0 \\ 1 & & & -1 & 0 & 0 & 2 \end{pmatrix}.$$

We carry out column operations on the first 4 columns as in type C , and reduce entries in the last $\ell = 3$ rows modulo 2, to obtain B_γ :

$$B_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & 0 & 2 & 0 & 0 \\ 0 & & & 0 & 0 & 2 & 0 \\ 1 & & & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Thus a \mathbb{Z} -basis for $\text{MOD}(w_{\beta,\gamma})$ is given by the union of $\{2\alpha_4^\vee + \alpha_7^\vee, 2\alpha_5^\vee, 2\alpha_6^\vee, 2\alpha_7^\vee\}$, $\{-\alpha_3^\vee + \alpha_7^\vee\}$, and $\{-\alpha_1^\vee + \alpha_2^\vee, -\alpha_2^\vee + \alpha_3^\vee\}$. Here, we have $I_\gamma = \{7, 6, 5, 4\}$, so $I_\gamma - 1 = \{6, 5, 4, 3\}$, and we have $n - \ell - 1 = 3$. Hence this matches part (4) of Theorem 6.2. Note that the set of vectors $\{-\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\gamma - 1), i < n - \ell - 1\}$ is empty in this example, since γ has only one part which is strictly greater than 1.

To find the Smith normal form, we continue from B_γ , now using some row operations. We successively add row 1 to row 2, row 2 to row 3, and row 3 to row 7, to obtain

$$T_\gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & 0 & 2 & 0 & 0 \\ 0 & & & 0 & 0 & 2 & 0 \\ 1 & & & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The entry 1 in the bottom left of T_γ is due to the parity change in γ . We can now extract a suitable minor containing this entry, and using row and column operations verify that $S_\gamma = \text{diag}(1^4, 2^2, 4)$, confirming part (2) of Theorem 6.1.

Example 6.12. This example illustrates part (3) of Theorem 6.1 in the case $\beta = (1)$, and shows how to obtain a basis for $\text{MOD}(w_{\beta,\gamma})$, for γ with more than one part. Let W be of type B_8 and suppose $\beta = (1)$ and $\gamma = (3, 4)$, so that $w_{\beta,\gamma} = w_\gamma = w_1^\gamma w_2^\gamma = (s_8 s_7 s_6)(s_5 s_6 s_7 s_8 s_7 s_6 s_5 s_4 s_3 s_2)$. The matrix for $N_\gamma = \mathbf{I} - w_\gamma$ with respect to the type B_8 basis Δ^\vee is given by

$$N_\gamma = \begin{pmatrix} 0 & & & & & & & \\ -1 & 2 & -1 & 0 & 0 & & & \\ -1 & 1 & 1 & -1 & 0 & & & \\ -1 & 1 & 0 & 1 & -1 & & & \\ -2 & 2 & 0 & 0 & 0 & & & \\ & & & & & -1 & 2 & -1 & 0 \\ & & & & & -1 & 1 & 1 & -2 \\ & & & & & -1 & 1 & 0 & 0 \end{pmatrix}.$$

For $\text{MOD}(w_\gamma)$, we first add column 2 to column 1, which creates a pivot entry 1 in the first column. We then use this pivot to clear entries in the 4×4 block corresponding to $\gamma_2 = 4$, which gives B'_γ :

$$B'_\gamma = \begin{pmatrix} 0 & & & & & & & \\ \boxed{1} & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & \boxed{1} & 0 & 0 & & & \\ 0 & 0 & 0 & \boxed{1} & 0 & & & \\ 0 & \boxed{2} & 0 & 0 & 0 & & & \\ & & & & & -1 & 2 & -1 & 0 \\ & & & & & -1 & 1 & 1 & -2 \\ & & & & & -1 & 1 & 0 & 0 \end{pmatrix}.$$

We then add column 6 to column 5, and similarly clear entries in the 3×3 block corresponding to $\gamma_1 = 3$, to give B_γ :

$$B_\gamma = \begin{pmatrix} 0 & & & & & & & \\ \boxed{1} & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & \boxed{1} & 0 & 0 & & & \\ 0 & 0 & 0 & \boxed{1} & 0 & & & \\ 0 & \boxed{2} & 0 & 0 & 0 & & & \\ & & & & & \boxed{1} & 0 & 0 & 0 \\ & & & & & 0 & 0 & \boxed{1} & 0 \\ & & & & & 0 & \boxed{1} & 0 & 0 \end{pmatrix}.$$

The resulting \mathbb{Z} -basis is $\{\alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee, 2\alpha_5^\vee\} \cup \{\alpha_6^\vee, \alpha_7^\vee, \alpha_8^\vee\}$. We have $S_\gamma = \text{diag}(1^6, 2, 0)$, and so $R^\vee/\text{MOD}(w_\gamma) \cong (\mathbb{Z}/2\mathbb{Z})$, confirming part (3) of Theorem 6.1.

Example 6.13. This example illustrates part (3) of Theorem 6.1 in the case $|\beta| \geq 2$. Let W be of type B_8 and suppose $\beta = (5)$ and $\gamma = (3)$ so that $w_{\beta,\gamma} = w_\beta w_\gamma = (s_1 s_2 s_3 s_4)(s_8 s_7 s_6)$.

The matrix for $M_{\beta,\gamma} = \mathbf{I} - w_{\beta,\gamma}$ with respect to the type B_8 basis Δ^\vee is given by

$$M_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 2 \\ & & & -1 & 1 \\ & & & -1 & 1 & 1 & -2 \\ & & & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Then similarly to Example 6.8, we carry out column operations on $M_{\beta,\gamma}$ to obtain $B''_{\beta,\gamma}$:

$$B''_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 1 & 2 & -1 & 0 \\ & & & -5 & 0 & 1 & 1 & -2 \\ & & & 0 & 1 & 0 & 0 \end{pmatrix}.$$

At this point, we replace the -5 by $\gcd(\beta_k, 2) \in \{1, 2\}$. But now, since $\gcd(\beta_k, 2) = 1$, we create a pivot entry 1 in row $n-1$. We can now successively clear entries in the last $\gamma_1 = 3$ rows, and so obtain a pivot entry 1 in row 4 of the matrix $B'_{\beta,\gamma}$:

$$B'_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & \boxed{1} & 0 \\ & & & \boxed{1} & 0 & 0 & 0 & 0 \\ & & & 0 & \boxed{1} & 0 & 0 & 0 \end{pmatrix}.$$

We can then clear entries in the first 4 rows, to get $B_{\beta,\gamma}$:

$$B_{\beta,\gamma} = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & \boxed{1} & 0 \\ & & & \boxed{1} & 0 & 0 & 0 & 0 \\ & & & 0 & \boxed{1} & 0 & 0 & 0 \end{pmatrix}.$$

Therefore $S_{\beta,\gamma} = \text{diag}(1^7, 0)$, and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong \mathbb{Z}$, confirming part (3) of Theorem 6.1.

Example 6.14. This example illustrates part (4) of Theorem 6.1, and part (6) of Theorem 6.2.

Let W be of type B_4 and suppose $\beta = (4)$ and $|\gamma| = 0$, so that $w_{\beta,\gamma} = w_\beta = s_1s_2s_3$. The matrix for $M_{\beta,\gamma} = I - w_{\beta,\gamma}$ with respect to the type B_4 basis Δ^\vee is given by

$$M_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 1 & -2 \\ -1 & 1 & 1 & -2 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the last row of this matrix is all 0s, while the $(n-1) \times (n-1)$ minor in the top left is the same as the matrix M_β in type A_{n-1} .

We start by adding column 3 to column 4, then multiply column 4 by -1 , to obtain $B'_{\beta,\gamma}$:

$$B'_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that the number of 1s in column n of $B'_{\beta,\gamma}$ is equal to $(\beta_p - 2) \geq 0$. (This is also true when $\beta_p = 2$.) We then add to both columns $n-1$ and n suitable linear combinations of columns 1 and 2, to obtain $B''_{\beta,\gamma}$:

$$B''_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that in $B''_{\beta,\gamma}$, the $(n-1, n-1)$ -entry is $\gcd(\beta_k)$, and the $(n-1, n)$ -entry is $(\beta_p - 2)$. We can thus replace the $(n-1, n-1)$ -entry of $B''_{\beta,\gamma}$ by $\gcd(\beta_k, \beta_p - 2)$ and clear its $(n-1, n)$ -entry. In this example, since $\gcd(\beta_k, \beta_p - 2) = 2$, this gives us:

$$B_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From here, the basis $\{\alpha_1^\vee - \alpha_2^\vee, \alpha_2^\vee - \alpha_3^\vee, 2\alpha_{n-1}^\vee\}$ can be read off, which matches part (6) of Theorem 6.2. Note that in this example, the set of basis vectors $\{\alpha_i^\vee - \alpha_{i+2}^\vee \mid i \in (I_\beta - 1) \setminus \{n-1\}\}$ is empty, since β has only one part.

To find the Smith normal form, we successively add row 1 to row 2, then row 2 to row 3, to directly obtain $S_{\beta,\gamma}$:

$$S_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In other words, $S_{\beta,\gamma} = \text{diag}(1^2, 2, 0)$, and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, confirming part (4) of Theorem 6.1.

Example 6.15. This example illustrates part (5) of Theorem 6.1 and part (7) of Theorem 6.2 in the case where $\beta_p \geq 2$. (The case $\beta_p = 1$ is discussed in Example 6.16 below.)

Let W be of type B_6 and suppose $\beta = (3, 3)$ and $|\gamma| = 0$ so that $w_{\beta,\gamma} = w_\beta = (s_1s_2)(s_4s_5)$. The matrix $M_{\beta,\gamma} = I - w_{\beta,\gamma}$ with respect to the type B_6 basis Δ^\vee is given by

$$M_{\beta,\gamma} = \begin{pmatrix} 1 & 1 & -1 & & & \\ -1 & 2 & -1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & -1 & 1 & 1 & -2 \\ & & 0 & -1 & 2 & -2 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We add column 2 to column 3 and then proceed as in type A and the previous example to obtain:

$$B'_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & & & \\ -1 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & -1 & 1 & 0 & 0 \\ & & 0 & -1 & 3 & 1 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that in $B'_{\beta,\gamma}$, the $(n-1, n-1)$ -entry is $\gcd(\beta_k)$, and the $(n-1, n)$ -entry is $(\beta_p - 2) \geq 0$. We can thus replace the $(n-1, n-1)$ -entry by $\gcd(\beta_k, \beta_p - 2)$ and clear the $(n-1, n)$ -entry. In this example, since $\gcd(\beta_k, \beta_p - 2) = 1$, we get a pivot 1 in the $(n-1, n-1)$ -entry. We can use this resulting pivot to successively clear entries moving up the matrix, resulting in $B_{\beta,\gamma}$:

$$B_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From here, we read off the basis $\{\alpha_1^\vee, \alpha_2^\vee\} \cup \{\alpha_4^\vee, \alpha_5^\vee\}$, confirming part (7) of Theorem 6.2 in this case. The Smith normal form is also now clearly $S_{\beta,\gamma} = \text{diag}(1^4, 0^2)$ and so $R^\vee/\text{MOD}(w_{\beta,\gamma}) \cong \mathbb{Z}^2$, confirming part (5) of Theorem 6.1 in this case.

Example 6.16. This example illustrates part (5) of Theorem 6.1 and part (7) of Theorem 6.2, in the case that β has last part $\beta_p = 1$.

Let W be of type B_5 and suppose $\beta = (4, 1)$ and $|\gamma| = 0$, so that $w_{\beta,\gamma} = w_\beta = s_1s_2s_3$. Note that since $\beta_p = 1$, the element w_β has support in the type A_{n-2} subsystem indexed by the first $n-2$ nodes, and so in particular w_β fixes α_n^\vee . Moreover, letting $\beta' = (4, 1, 1)$ be the partition obtained by adding last part 1 to β , the element w_β acts on α_{n-1}^\vee exactly as does the element $w_{\beta'}$ in type A_{n-1} . Hence the matrix for $M_{\beta,\gamma} = I - w_{\beta,\gamma}$ with respect to the type B_5 basis Δ^\vee is given by

$$M_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that if we denote by ℓ the number of parts of β which are equal to 1, then the last $\ell+1$ rows of this matrix are all 0s.

We then proceed as in type A_{n-1} in the $(n-1) \times (n-1)$ submatrix in the top left (compare Example 4.5, which considers $\beta = (4, 1)$ in type A_4 , as well as Example 4.6, which considers $\beta = (4, 3, 1)$ in type A_7). Since $\beta_p = 1$, there is a pivot 1 in column $n-1$, and we thus obtain

$$B_{\beta,\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(If instead $\gcd(\beta_k, \beta_p - 2) = 1$, the pivot 1 is created in the same row of column $n-2$.) From here, we read off the basis $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$, confirming part (7) of Theorem 6.2 in the case $\beta_p = 1$. In addition, clearly $S_{\beta,\gamma} = \text{diag}(1^3, 0^2)$, confirming part (5) of Theorem 6.1 in the case $\beta_p = 1$.

7. TYPE D MOD-SETS

In this section we give a description of all mod-sets in type D . Let W be the finite Weyl group of type D_n , for $n \geq 4$. In Section 7.1, we review the complete system of minimal length representatives for the conjugacy classes of W provided in [GP00, Chapter 3]. Our results are stated in Section 7.2, and we then give examples in Section 7.3, designed to fully illustrate the proofs of our results on mod-sets.

Throughout this section, we label the nodes of the Dynkin diagram by $[n]$ as in both Bourbaki [Bou02] and Sage [Sag24], so that the first $n-1$ nodes form a type A_{n-1} subsystem, and the unique node of valence 3 is indexed by $(n-2)$. We remark that this is different from the indexing used in [GP00], where the nodes of the Dynkin diagram in type D_n are labeled by $\{0, \dots, n-1\}$, and the unique node of valence 3 is indexed by 2. See Table 9 in Appendix B for a direct comparison of these choices of labeling.

7.1. Conjugacy class representatives in type D . Following [GP00], we first explain how the conjugacy classes of W of type D_n are parameterized by ordered pairs of compositions (β, δ) such that β is weakly decreasing, δ is weakly increasing and has an even number of parts, and $|\beta| + |\gamma| = n \geq 4$. In the special case $(\beta, 0)$ where all parts of β are even, there are two distinct conjugacy classes, denoted by β^+ and β^- , or β^\pm to include both cases. For each pair (β, δ) , we will define standard parabolic subgroups W_β and W_δ of W , and an element $w_{\beta,\delta} = w_\beta \cdot w_\delta$ with w_β cuspidal in W_β and w_δ cuspidal in W_δ . The set of all $w_{\beta^\pm,\delta}$ forms a complete system of minimal length representatives for the conjugacy classes of W .

If $|\beta| = 0$ or $|\beta| = 1$, we define w_β and W_β to both be trivial. If β is a partition of m with $2 \leq m < n$, then we define both w_β and W_β as in Section 4.1, so that the w_β are the minimal length conjugacy class representatives in the type A_{m-1} subsystem of W indexed by the first $m-1$ nodes of the Dynkin diagram. If $|\beta| = n$ and β has at least one odd part, then w_β is again defined as in Section 4.1. If $|\beta| = n$ and β has all parts even, then w_{β^+} is defined to be the minimal length conjugacy class representative w_β in the type A_{n-1} subsystem of W indexed by the first $n-1$ nodes of the Dynkin diagram, as in Section 4.1. If $|\beta| = n$ and β has all parts even, the element w_{β^-} is defined by replacing the element s_{n-1} in w_{β^+} by s_n .

Now let $\delta = (\delta_1, \dots, \delta_{2r})$ be any weakly increasing composition of $n-m$, with an even number of parts. If $|\delta| = 0$, we define w_δ and W_δ to both be trivial. Note that since δ must have an even number of parts, the case $|\delta| = 1$ does not occur. It thus remains to define w_δ and W_δ for $2 \leq m < n$.

Following [GP00, Section 3.4] but using different labeling conventions and notation, we first define elements

$$u_0 = s_n \quad \text{and} \quad u_1 = s_n s_{n-1},$$

and then for $2 \leq j \leq n-1$, inductively define

$$u_j = s_{n-j} u_{j-1} s_{n-j}.$$

We also define a strictly increasing subsequence of $\{0, 1, \dots, n\}$ in the same way as we did for the compositions β and γ in Sections 4.1 and 5.1, respectively. That is, let $j_1^\delta = 0$ and for all $2 \leq k \leq 2r+1$, define

$$(7.1.1) \quad j_k^\delta = \sum_{i=1}^{k-1} \delta_i = \delta_1 + \cdots + \delta_{k-1}.$$

In particular, note that $j_2^\delta = \delta_1$ and $j_{2r+1}^\delta = n - m$. Notice also that if $\delta_1 = \cdots = \delta_k = 1$ for some $k \geq 1$, then $j_{k+1}^\delta = k$. Now for all $2 \leq k \leq 2r$ such that $\delta_k \geq 2$, define the element

$$(7.1.2) \quad v_k^\delta = u_{j_k^\delta} s_{n-(j_k^\delta+1)} s_{n-(j_k^\delta+2)} \cdots s_{n-(j_k^\delta+\delta_k-1)},$$

analogous to (5.1.2). In particular, v_k^δ ends with a product of $\delta_k - 1$ consecutive simple reflections ordered by decreasing index. Considering the initial $u_{j_k^\delta}$ as a single term, the expression for v_k^δ has $\delta_k \geq 2$ terms. Note also that if the last part $\delta_{2r} \geq 2$, then the element v_{2r}^δ has final simple reflection equal to $s_{n-|\delta|+1}$.

We can now define the element w_δ for $2 \leq m < n$, which has several cases. We first consider the cases where at least one part of δ equals 1. In the special case that all $2r$ parts of δ are equal to 1 (and hence $|\delta| = n - m = 2r$), we define

$$(7.1.3) \quad w_\delta = u_1 u_2 \cdots u_{2r-1}.$$

Next, if $\delta_1 = \cdots = \delta_\ell = 1$ while $\delta_{\ell+1} \geq 2$, we define

$$(7.1.4) \quad w_\delta = (u_1 \cdots u_{\ell-1}) v_{\ell+1}^\delta v_{\ell+2}^\delta \cdots v_{2r}^\delta,$$

where the product $(u_1 \cdots u_{\ell-1})$ is trivial if $\ell = 1$. Note that $v_{\ell+1}^\delta$ begins with u_ℓ , and hence in all cases where $\delta_1 = \cdots = \delta_\ell = 1$ for some $\ell \geq 1$, the element w_δ begins with the product $u_1 \cdots u_\ell$. We next consider the cases where δ does not contain any parts equal to 1. Define

$$(7.1.5) \quad w_\delta = (u_0 s_{n-2} s_{n-3} \cdots s_{n-(\delta_1-1)}) v_2^\delta v_3^\delta \cdots v_{2r}^\delta,$$

where the product $s_{n-2} s_{n-3} \cdots s_{n-(\delta_1-1)}$ is trivial if $\delta_1 = 2$, so that the parenthetical expression in the formula for w_δ contains $\delta_1 - 1$ terms, including $u_0 = s_n$.

By [GP00, Prop. 3.4.11], the element w_δ is cuspidal in the parabolic subgroup W_δ of type D_{n-m} , generated by the simple reflections indexed by the last $|\delta| = n - m$ nodes of the Dynkin diagram. By [GP00, Prop. 3.4.12], the set of $w_{\beta^\pm, \delta} = w_{\beta^\pm} \cdot w_\delta$ for all distinct pairs of compositions (β, δ) such that β is weakly decreasing, δ is weakly increasing with an even number of parts, and $|\beta| + |\delta| = n$ forms a complete system of minimal length representatives of the conjugacy classes of W in type D_n .

7.2. Results on mod-sets in type D . In this section, we state our results in type D . It turns out that there are quite a few cases in type D , and so we state three separate theorems. Throughout this section, we assume W is of type D_n with $n \geq 4$, and we let

$$\beta = (\beta_1, \dots, \beta_p) \quad \text{and} \quad \delta = (\delta_1, \dots, \delta_{2r})$$

be a pair of compositions such that β is weakly decreasing, δ is weakly increasing with an even number of parts, and $|\beta| + |\delta| = n$, with corresponding conjugacy class representative $w_{\beta, \delta} \in W$, constructed as in Section 7.1. In the special case $|\beta| = n$ and all parts of β are even, recall that we have two cases β^\pm .

Theorem 7.1 provides the Smith normal form of $(I - w_{\beta, \delta})$, from which the isomorphism type of the quotient $R^\vee/\text{MOD}(w_{\beta, \delta})$ is immediate. In Theorem 7.2, we give a basis for $\text{MOD}(w_{\beta, \delta})$ in the case that $|\beta| = 0$, equivalently whenever $w_{\beta, \delta} = w_\delta$ is cuspidal in W . In Theorem 7.3, we then give a basis for $\text{MOD}(w_{\beta, \delta})$ when $|\delta| = 0$. As in type B_n , for the remaining noncuspidal cases, where $|\beta| \geq 1$ and $|\delta| \geq 2$, we provide illustrative examples of how to identify convenient bases. In all cases, a set of equations explicitly describing $\text{MOD}(w_{\beta, \delta})$ can easily be deduced once the (relatively sparse) basis is known.

Throughout this section, we write $\gcd(\beta_k, 2) = \gcd(\beta_1, \dots, \beta_p, 2)$. In particular, $\gcd(\beta_k, 2) = 2$ when all parts of β are even (including the case that $|\beta| = 0$), and otherwise $\gcd(\beta_k, 2) = 1$. We now state our results on the Smith normal form for $(I - w_{\beta, \delta})$ in type D .

Theorem 7.1. *Suppose W is of type D_n with $n \geq 4$. Let (β, δ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $\delta = (\delta_1, \dots, \delta_{2r})$ is weakly increasing with an even number of parts, and $|\beta| + |\delta| = n$. Let $w_{\beta^\pm, \delta} \in W$ be the corresponding conjugacy class representative in W .*

Then for any $w \in [w_{\beta^\pm, \delta}]$, the Smith normal form of $(I - w)$ is as follows:

- (1) *If $\gcd(\beta_k, 2) = 2$ (including the case that $|\beta| = 0$), $|\delta| \geq 2$, and all parts of δ have the same parity, then*

$$S_{\beta, \delta} = \text{diag}(1^{n-2r-p}, 2^{2r}, 0^p).$$

- (2) *If $\gcd(\beta_k, 2) = 2$ (including the case that $|\beta| = 0$), $|\delta| \geq 2$, and δ has a change in parity, then*

$$S_{\beta, \delta} = \text{diag}(1^{n-2r-p+1}, 2^{2r-2}, 4, 0^p).$$

- (3) *If $\gcd(\beta_k, 2) = 1$ and $|\delta| \geq 2$, then*

$$S_{\beta, \delta} = \text{diag}(1^{n-2r-p+1}, 2^{2r-1}, 0^p).$$

- (4) *If $|\delta| = 0$ and $\beta = (1, \dots, 1)$, so that $w_{\beta, \delta}$ is trivial, then $S_{\beta, \delta} = \text{diag}(0^n)$. If $|\delta| = 0$ and $\beta \neq (1, \dots, 1)$, then*

$$S_{\beta^\pm, \delta} = \text{diag}(1^{n-p-1}, \gcd(\beta_k, 2), 0^p).$$

Since Smith normal form is canonical, the delicate nature of the results in type D is evident already from the many cases necessarily appearing in Theorem 7.1 above.

The next theorem gives a \mathbb{Z} -basis for $\text{MOD}(w_{\beta, \delta})$ when $|\beta| = 0$, equivalently whenever $w_{\beta, \delta} = w_\delta$ is cuspidal in W . Before stating this result, we require some additional notation.

Given $\beta = (\beta_1, \dots, \beta_p)$ with $|\beta| = m$, recall from (4.1.1) the pairwise disjoint subintervals J_k^β of $[m-1]$, for $1 \leq k \leq p$. These subintervals are constructed such that $J_\beta = \sqcup_{k=1}^p J_k^\beta$ indexes the simple reflections whose product equals w_β , when written in increasing order. In type D_n , as in type B_n , we also have the following subsets

$$I_\beta = \{j_k^\beta \mid 2 \leq k \leq p\} \cup \{m\} \quad \text{and} \quad I_\beta - 1 = \{i-1 \mid i \in I_\beta\}.$$

Given a weakly increasing composition $\delta = (\delta_1, \dots, \delta_{2r})$ with $|\delta| = n - m$, recall from (7.1.1) that $j_1^\delta = 0$ and $j_k^\delta = \sum_{i=1}^{k-1} \delta_i$ for $2 \leq k \leq 2r$. Define the following $2r$ -element subset of $[n]$ associated to δ :

$$I_\delta = \left\{ n - j_k^\delta \mid 1 \leq k \leq 2r \right\} = \{m + \delta_{2r}, m + \delta_{2r} + \delta_{2r-1}, \dots, n - \delta_1, n\}.$$

In addition, we denote by $I_\delta - 1 = \{i - 1 \mid i \in I_\delta\}$.

To simplify notation in the statements below, we further define $\delta_{1,0} = 0$ and $\delta_{k,k-1} = (\delta_k + \delta_{k-1}) \bmod 2$ for $2 \leq k \leq 2r$. That is, $\delta_{k,k-1} = 0$ if the parts δ_k and δ_{k-1} have the same parity, and otherwise $\delta_{k,k-1} = 1$. In particular, note that all parts of δ have the same parity if and only if $\delta_{k,k-1} = 0$ for all $2 \leq k \leq 2r$.

With this additional notation established, we now state our results providing bases for the mod-sets in type D .

Theorem 7.2. *Suppose W is of type D_n with $n \geq 4$. Let (β, δ) be a pair of compositions such that $|\beta| = 0$, $\delta = (\delta_1, \dots, \delta_{2r})$ is weakly increasing with an even number of parts, and $|\delta| = n$. Let $w_{\beta,\delta} = w_\delta$ be the corresponding conjugacy class representative in W , which is cuspidal in W . Then:*

(1) *If $\delta = (1, \dots, 1)$, then $\text{MOD}(w_{\beta,\delta}) = 2R^\vee$ and has \mathbb{Z} -basis given by*

$$\{2\alpha_i^\vee \mid i \in [n]\}.$$

(2) *If $\delta_k = 1$ for all $1 \leq k \leq \ell < n$ for some $\ell \geq 2$, while $\delta_{\ell+1} \geq 2$, then $\text{MOD}(w_{\beta,\delta})$ has \mathbb{Z} -basis given by*

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k^\delta}^\vee + \delta_{k,k-1}(\alpha_{n-1}^\vee + \alpha_n^\vee) \mid 1 \leq k \leq 2r \right\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \in [n], i \notin I_\delta \cup (I_\delta - 1) \cup \{n - \ell - 1\} \right\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\delta - 1), i \leq n - (\delta_{\ell+1} + \ell) \right\} \\ & \cup \left\{ -\alpha_{n-\ell-1}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee \right\}. \end{aligned}$$

(3) *If $\delta_1 = 1$ and $\delta_2 \geq 2$, then $\text{MOD}(w_{\beta,\delta})$ has \mathbb{Z} -basis given by*

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k^\delta}^\vee + \delta_{k,k-1}\alpha_{n-2}^\vee \mid 1 \leq k \leq 2r \right\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \in [n], i \notin I_\delta \cup (I_\delta - 1) \cup \{n - 1, n\} \right\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\delta - 1), i \leq n - (\delta_2 + 1) \right\} \\ & \cup \left\{ -\alpha_{n-2}^\vee - \alpha_{n-1}^\vee + \alpha_n^\vee \right\}. \end{aligned}$$

(4) *If $\delta_1 = 2$, then $\text{MOD}(w_{\beta,\delta})$ has \mathbb{Z} -basis given by*

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k^\delta}^\vee + \delta_{k,k-1}\alpha_n^\vee \mid 1 \leq k \leq 2r \right\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\delta \cup (I_\delta - 1) \text{ or } i = n - 1 \right\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\delta - 1), i \neq n - 1 \right\}. \end{aligned}$$

(5) *If $\delta_1 \geq 3$, then $\text{MOD}(w_{\beta,\delta})$ has \mathbb{Z} -basis given by*

$$\begin{aligned} & \left\{ 2\alpha_{n-j_k^\delta}^\vee + \delta_{k,k-1}\alpha_{n-1}^\vee \mid 2 \leq k \leq 2r \right\} \cup \{2\alpha_{n-1}^\vee, ((\delta_1 - 1) \bmod 2)\alpha_{n-1}^\vee + \alpha_n^\vee\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\delta \cup (I_\delta - 1) \right\} \\ & \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\delta - 1), i \neq n - 1 \right\}. \end{aligned}$$

Finally, we give a \mathbb{Z} -basis for $\text{MOD}(w_{\beta^\pm, \delta})$ in the case that $|\delta| = 0$.

Theorem 7.3. Suppose W is of type D_n with $n \geq 4$. Let (β, δ) be a pair of compositions such that $\beta = (\beta_1, \dots, \beta_p)$ is weakly decreasing, $|\beta| = n$, and $|\delta| = 0$. Let $w_{\beta^\pm, \delta} = w_{\beta^\pm}$ be the corresponding conjugacy class representative in W , which is not cuspidal in W . Then:

(1) If $\beta_p = 1$, then $\text{MOD}(w_{\beta, \delta})$ has \mathbb{Z} -basis given by

$$\{\alpha_j^\vee \mid j \in J_\beta\}.$$

(2) If $\beta_p \geq 2$, then:

(a) Both $\text{MOD}(w_{\beta, \delta})$ and $\text{MOD}(w_{\beta^+, \delta})$ have \mathbb{Z} -bases given by

$$\begin{aligned} & \{\alpha_i^\vee - \alpha_{i+1}^\vee \mid i \in J_\beta \setminus (I_\beta - 1)\} \cup \\ & \{\alpha_i^\vee - \alpha_{i+2}^\vee \mid i \in (I_\beta - 1) \setminus \{n-1\}\} \cup \{\gcd(\beta_k, 2)\alpha_{n-1}^\vee\}. \end{aligned}$$

(b) Replace every α_{n-1}^\vee by α_n^\vee in the basis for $\text{MOD}(w_{\beta^+, \delta})$ to obtain a basis for $\text{MOD}(w_{\beta^-, \delta})$.

As seen in Theorems 7.2 and 7.3, the two special cases considered here already involve many subcases and delicate statements. Thus for the remaining “mixed” case, where $|\beta| \geq 1$ and $|\delta| \geq 2$, we have opted not to give even lengthier general statements, but instead to provide illustrative examples of how to identify convenient bases. We illustrate each of these type D_n results with several examples in Section 7.3, which can then be generalized in a straightforward manner to prove each of these three theorems.

7.3. Examples in type D . In this section, we present a sequence of examples which can be generalized to obtain the proofs of the theorems stated in Section 7.2. The collection of examples is chosen to fully illustrate every case that arises in Theorems 7.1, 7.2, and 7.3.

In Section 7.3.1, we restrict to the case $|\beta| = 0$, equivalently $w_{\beta, \delta} = w_\delta$ is cuspidal. These examples illustrate the proofs of parts (1) and (2) of Theorem 7.1 in the case $|\beta| = 0$, and the proof of Theorem 7.2. The cuspidal examples we consider are:

- $|\beta| = 0$ and $\delta = (1, \dots, 1)$ with $n \geq 4$ even in Example 7.4;
- $|\beta| = 0$ and $\delta = (1, 1, 1, 2, 2, 3)$ in Example 7.5;
- $|\beta| = 0$ and $\delta = (1, 3, 3, 4)$ in Example 7.6;
- $|\beta| = 0$ and $\delta = (2, 3, 3, 3)$ in Example 7.7; and
- $|\beta| = 0$ and $\delta = (3, 3, 3, 3)$ in Example 7.8.

In Section 7.3.2, we consider examples with $\beta_p = 1$ and $|\delta| \geq 2$, to illustrate the proof of part (3) of Theorem 7.1 in this case. The examples with $\beta_p = 1$ we consider are:

- $\beta = (1)$ and $\delta = (1, \dots, 1)$ with $n \geq 5$ odd in Example 7.9;
- $\beta = (1)$ and $\delta = (1, 1, 1, 2)$ in Example 7.10;
- $\beta = (1)$ and $\delta = (1, 3)$ in Example 7.11;
- $\beta = (1)$ and $\delta = (2, 2)$ in Example 7.12;
- $\beta = (1)$ and $\delta = (3, 3)$ in Example 7.13; and
- $\beta = (3, 1)$ and $\delta = (1, 2)$ in Example 7.14.

In Section 7.3.3, we consider examples with $\beta_p \geq 2$ and $|\delta| \geq 2$. These examples illustrate parts (1), (2), and (3) of Theorem 7.1 in this case. The examples we consider in this section are:

- $\beta = (4)$ and $\delta = (1, 1, 1, 2)$ in Example 7.15;
- $\beta = (4)$ and $\delta = (1, 2)$ in Example 7.16;
- $\beta = (4)$ and $\delta = (2, 2)$ in Example 7.17;
- $\beta = (4)$ and $\delta = (3, 3)$ in Example 7.18; and
- $\beta = (3)$ and $\delta = (3, 3)$ in Example 7.19.

Finally, in Section 7.3.4, we give examples in the case where $|\delta| = 0$, illustrating the proof of part (4) of Theorem 7.1, and the proof of Theorem 7.3. The examples with $|\delta| = 0$ we consider are:

- $\beta = (3, 1)$ and $|\delta| = 0$ in Example 7.20;
- $\beta^\pm = (2, 2)$ and $|\delta| = 0$ in Example 7.21; and
- $\beta^\pm = (4)$ and $|\delta| = 0$ in Example 7.22.

For each of the examples listed above, we construct a \mathbb{Z} -basis for $\text{MOD}(w_{\beta, \delta})$ and find the Smith normal form for $(I - w_{\beta, \delta})$. From this information, it is then possible to give an explicit description of the \mathbb{Z} -module $\text{MOD}(w_{\beta, \delta}) = (I - w_{\beta, \delta})R^\vee$, and to easily obtain the isomorphism class of the quotient $R^\vee / \text{MOD}(w_{\beta, \delta})$.

7.3.1. Cuspidal examples. We first consider the situation where $|\beta| = 0$, equivalently $w_{\beta, \delta} = w_\delta$ is cuspidal in W . These examples illustrate the proofs of parts (1) and (2) of Theorem 7.1 in the case $|\beta| = 0$, and the proof of Theorem 7.2; see the introduction to Section 7.3 for the list of examples.

Example 7.4. This example proves part (1) of Theorem 7.1 in the special case $|\beta| = 0$ and $\delta = (1, \dots, 1)$, and part (1) of Theorem 7.2. Let W be of type D_n with $n = 2r \geq 4$ even. Let $|\beta| = 0$ and $\delta = (1, \dots, 1)$. For this case, recall from (7.1.3) that

$$w_{\beta, \delta} = w_\delta = u_1 u_2 \dots u_{n-1},$$

which equals the longest element w_0 of W ; see Table 1 in [BKOP14], for example. In type D_n with n even, note that w_0 is central in W and acts as $-I$ on R^\vee . Hence, in this case

$$\text{MOD}(w_{\beta, \delta}) = (I - w_\delta)R^\vee = (2I)R^\vee = 2R^\vee,$$

which has a \mathbb{Z} -basis $\{2\alpha_i^\vee \mid i \in [n]\}$. The Smith normal form for $(I - w_{\beta, \delta})$ is also clearly $S_{\beta, \delta} = (2^n)$, confirming part (1) of Theorem 7.1 and part (1) of Theorem 7.2 in this case.

Example 7.5. This example illustrates part (2) of Theorem 7.1 in the case $|\beta| = 0$, and part (2) of Theorem 7.2. Let W be of type D_{10} , and let $|\beta| = 0$ and $\delta = (1, 1, 1, 2, 2, 3)$. By (7.1.4) with $\ell = 3$, we have

$$w_{\beta, \delta} = w_\delta = u_1 u_2 v_4^\delta v_5^\delta v_6^\delta = u_1 u_2 (u_3 s_6)(u_5 s_4)(u_7 s_2 s_1).$$

Applying (2.1.1), the matrix for $I - w_\delta$, with respect to the basis Δ^\vee , is thus given by

$$P_\delta = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & 0 & 0 \\ 2 & -1 & 2 & -1 \\ 2 & -2 & 2 & 0 \\ 2 & -2 & 2 & -1 & 2 & -1 \\ 2 & -2 & 2 & -2 & 2 & 0 \\ 2 & -2 & 2 & -2 & 2 & -2 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 2 \end{pmatrix},$$

where we have omitted most 0 entries for readability. Notice that P_δ is block lower-triangular with diagonal blocks of sizes 3, 2, 2, 1, 1, 1 from left to right, obtained from the parts of δ in reverse order.

More specifically, the diagonal blocks of size 1 have entry equal to 2, with all 0s underneath. The diagonal blocks of size ≥ 2 have first column 2, 1, ..., 1, 2, diagonal entries 2, 1, ..., 1, 0,

above-diagonal entries -1 , and all other entries 0 . Underneath the diagonal blocks of size ≥ 2 , the last two rows have 1 s in the leftmost column and -1 s in the rightmost column, and then 2 s fill the remaining entries of the leftmost column. The rightmost column below the $\delta_{\ell+1}$ block is filled with -2 s. For the blocks of size δ_k with $k \geq \ell + 2$, place -1 s in the first $(\delta_{\ell+1} - 1)$ entries of the rightmost column, then put -2 s in the remaining entries of the rightmost column. (In this example, $\delta_{\ell+1} - 1 = \delta_4 - 1 = 2 - 1 = 1$, so we have one -1 at the top of the rightmost column below the leftmost blocks of sizes 3 and 2 , but not in the δ_4 block itself.) All other entries equal 0 . One can check via (2.1.1) that this description in fact characterizes P_δ for all δ with $|\delta| = n$ and $\delta_1 = \dots = \delta_\ell = 1$ for some $2 \leq \ell < n$ such that $\delta_{\ell+1} \geq 2$; that is, the case considered in part (2) of Theorem 7.2.

We now carry out a series of column operations on P_δ to obtain a matrix from which a basis for $\text{MOD}(w_\delta)$ can be read off; see the matrix Q_δ in (7.3.1) below. We start by carrying out column operations on columns 1 and 3; that is, the first and last columns of the leftmost block of P_δ . We replace $\mathcal{C}_1(P_\delta)$ by $\mathcal{C}_1(P_\delta) + 2\mathcal{C}_2(P_\delta) + 3\mathcal{C}_3(P_\delta)$, and then replace $\mathcal{C}_3(P_\delta)$ by $\mathcal{C}_3(P_\delta) + \mathcal{C}_4(P_\delta)$, to obtain

$$P'_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ (-1) & 1 & (2) & -1 \\ -4 & 2 & 0 \\ -4 & 2 & -1 & 2 & -1 \\ -4 & 2 & (-2) & 2 & 0 \\ -4 & 2 & (-2) & 2 & -2 & 2 \\ -2 & 1 & (-1) & 1 & -1 & 2 \\ -2 & 1 & (-1) & 1 & -1 & 2 \end{pmatrix}.$$

We now clear the circled entry in the first column of P'_δ using its second block, by replacing $\mathcal{C}_1(P'_\delta)$ by $\mathcal{C}_1(P'_\delta) - \mathcal{C}_5(P'_\delta)$. We then clear the circled entries in columns 4 and 5 of P'_δ by carrying out column operations on these columns (i.e. the first and last columns of the second block) similar to those we did on columns 1 and 3 in the previous step. Namely, we replace $\mathcal{C}_4(P'_\delta)$ by $\mathcal{C}_4(P'_\delta) + 2\mathcal{C}_5(P'_\delta)$, and then replace $\mathcal{C}_5(P'_\delta)$ by $\mathcal{C}_5(P'_\delta) + \mathcal{C}_6(P'_\delta)$. This yields

$$P''_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 2 & 0 \\ (-3) & 0 & 1 & (2) & -1 \\ -2 & -2 & 2 & 0 \\ -2 & -2 & 2 & -2 & 2 \\ -1 & -1 & 1 & -1 & 0 & 2 \\ -1 & -1 & 1 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

We now clear the circled entry in the first column of P''_δ using its third block, by replacing $\mathcal{C}_1(P''_\delta)$ by $\mathcal{C}_1(P''_\delta) - 3\mathcal{C}_7(P''_\delta)$. In the third block, which is the last block of size ≥ 2 , we clear

the circled entry of P''_δ by replacing $\mathcal{C}_6(P'_\delta)$ by $\mathcal{C}_6(P'_\delta) + 2\mathcal{C}_7(P'_\delta)$, to obtain

$$P'''_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \textcircled{-4} & & \mathbf{2} & 0 \\ 0 & 0 & 1 & 0 & -1 \\ \textcircled{-2} & & \textcircled{-2} & \mathbf{2} & 0 \\ 4 & -2 & -2 & -2 & \boxed{2} \\ 2 & -1 & -1 & -1 & \boxed{2} \\ 2 & -1 & -1 & -1 & \boxed{2} \end{pmatrix}.$$

From here on, we will freely work mod 2 in the (i, j) entries for $8 \leq i \leq 10$, since there are boxed pivot entries 2 in the (j, j) -entry of P'''_δ for all $8 \leq j \leq 10$. In particular, this allows us to clear all entries in row 8 other than the $(8, 8)$ -entry, since these are all even. We can also clear any even entries which occur in rows 9 and 10 throughout the remainder of this argument.

Our next goal is to clear the circled entries of P''_δ . For this, we use the 2s in the bottom left corners of the second and third blocks, shown in bold above, using the following column operations. First add column 6 to column 4 and column 1, and then add twice the resulting column 4 to column 1, to obtain:

$$(7.3.1) \quad Q_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & 0 & -1 \\ & 2 & 0 \\ & 1 & 0 & -1 \\ & 2 & 0 \\ & & 2 \\ 1 & & 1 & 2 \\ 1 & & 1 & 2 \end{pmatrix}.$$

Since we have only used column operations so far, a \mathbb{Z} -basis for $\text{MOD}(w_\delta)$ can be read off from the columns of Q_δ , as follows. The columns with 2s in them yield

$$\{2\alpha_3^\vee + (\alpha_9^\vee + \alpha_{10}^\vee), \quad 2\alpha_5^\vee, \quad 2\alpha_7^\vee + (\alpha_9^\vee + \alpha_{10}^\vee), \quad 2\alpha_8^\vee, \quad 2\alpha_9^\vee, \quad 2\alpha_{10}^\vee\}$$

and then the remaining columns give the basis elements

$$\{-\alpha_1^\vee + \alpha_2^\vee\} \cup \{-\alpha_2^\vee + \alpha_4^\vee, -\alpha_4^\vee + \alpha_6^\vee\} \cup \{-\alpha_6^\vee + \alpha_9^\vee + \alpha_{10}^\vee\}.$$

Now recall that $\delta = (1, 1, 1, 2, 2, 3)$, so the sequence $(j_k^\delta) = (0, 1, 2, 3, 5, 7)$ and the sequence $(\delta_{k,k-1}) = (0, 0, 0, 1, 0, 1)$. Hence, the first set in this basis equals

$$\{2\alpha_{n-j_k^\delta}^\vee + \delta_{k,k-1}(\alpha_{n-1}^\vee + \alpha_n^\vee) \mid 1 \leq k \leq 2r\}$$

with $n = 10$ and $2r = 6$. We also have $I_\delta = \{3, 5, 7, 8, 9, 10\}$ hence $I_\delta - 1 = \{2, 4, 6, 7, 8, 9\}$, and the first $\ell = 3$ parts of δ equal 1 while $\delta_{\ell+1} \geq 2$, so

$$\{-\alpha_1^\vee + \alpha_2^\vee\} = \{-\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \in [n], i \notin I_\delta \cup (I_\delta - 1) \cup \{n - \ell - 1\}\}$$

and

$$\{-\alpha_2^\vee + \alpha_4^\vee, -\alpha_4^\vee + \alpha_6^\vee\} = \{-\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in I_\delta - 1, i \leq n - \ell - 2\},$$

while $\{-\alpha_6^\vee + \alpha_9^\vee + \alpha_{10}^\vee\} = \{-\alpha_{n-\ell-1}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee\}$, confirming part (2) of Theorem 7.2.

To find the Smith normal form for P_δ , we continue from Q_δ , now using some row operations. Working from left to right, we use the -1 s above the diagonal to successively clear all of the 1 s underneath them. This gives

$$Q'_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 2 & 0 & 0 \\ & & & 0 & -1 \\ & & & 2 & 0 \\ & & & & 0 & -1 \\ & & & & 2 & 0 \\ & & & & & 2 \\ 1 & & & & 1 & & 2 \\ 1 & & & & 1 & & 2 \end{pmatrix}.$$

From here we extract the 6×6 minor of Q'_δ consisting of all rows and columns containing 2 s. We then carry out a series of row and then column operations, as follows. First subtract row 6 from row 5, then subtract twice row 6 from row 1, and then add row 3 to row 1 to obtain the second matrix. Then use the boxed pivots to clear out the remaining entries in the same row, via column operations:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & -2 \\ \boxed{1} & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

After scaling and rearrangement, this implies that P_δ has Smith normal form $\text{diag}(1^5, 2^4, 4)$, confirming part (2) of Theorem 7.1 in this case.

Example 7.6. This example illustrates part (2) of Theorem 7.1 in the case $|\beta| = 0$, and part (3) of Theorem 7.2. Let W be of type D_{11} and let $|\beta| = 0$ and $\delta = (1, 3, 3, 4)$. By (7.1.4) with $\ell = 1$, we have

$$w_{\beta, \delta} = w_\delta = v_2^\delta v_3^\delta v_4^\delta = (u_1 s_9 s_8)(u_4 s_6 s_5)(u_7 s_3 s_2 s_1).$$

Applying (2.1.1), the matrix for $I - w_\delta$ with respect to the basis Δ^\vee is given by

$$P_\delta = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ 2 & & -1 & 2 & -1 & 0 \\ 2 & & -1 & 1 & 1 & -1 \\ 2 & & -2 & 2 & 0 & 0 \\ 2 & & -2 & 2 & & -1 & 2 & -1 & 0 & 0 \\ 2 & & -2 & 2 & & -1 & 1 & 1 & -1 & -1 \\ 1 & & -1 & 1 & & -1 & 1 & 0 & 1 & -1 \\ 1 & & -1 & 1 & & -1 & 1 & 0 & -1 & 1 \end{pmatrix}.$$

Notice that P_δ is block lower-triangular with diagonal blocks of sizes 4, 3, and $3 + 1$ going from left to right. That is, the sizes of the diagonal blocks of P_δ are the parts of δ in reverse order, except for the first two parts of δ , which correspond to a single block of size $\delta_2 + \delta_1 = \delta_2 + 1$ on the right. The diagonal blocks and entries underneath them corresponding to the parts δ_k for $k \geq 3$ are the same as described in Example 7.5 above. The rightmost diagonal block of size $\delta_2 + 1$ has first column 2, 1, ..., 1, diagonal entries 2, 1, ..., 1, above-diagonal entries all -1 , an additional -1 in entries $(n, n-1)$ and $(n-2, n)$ of P_δ , and all other entries 0. One can check via (2.1.1) that this description in fact characterizes P_δ for all $|\delta| = n$ with $\delta_1 = 1$ and $\delta_2 \geq 2$; that is, the case considered in part (3) of Theorem 7.2.

We now carry out a series of column operations on P_δ to obtain a matrix from which a basis for $\text{MOD}(w_\delta)$ can be read off; see the matrix Q_δ in (7.3.2) below. We start with column operations on columns 1 and 4 similar to those done on columns 1 and 3 in the previous example. More specifically, we replace $\mathcal{C}_1(P_\delta)$ by $\mathcal{C}_1(P_\delta) + 2\mathcal{C}_2(P_\delta) + 3\mathcal{C}_3(P_\delta) + 4\mathcal{C}_4(P_\delta)$, and then replace $\mathcal{C}_4(P_\delta)$ by $\mathcal{C}_4(P_\delta) + \mathcal{C}_5(P_\delta)$, to obtain

$$P'_\delta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 \\ (-2) & & 1 & (2) & -1 & 0 \\ (-2) & & (1) & 1 & -1 \\ -6 & & 2 & 0 & 0 \\ -6 & & 2 & -1 & 2 & -1 & 0 & 0 \\ -6 & & 2 & (1) & 1 & 1 & -1 & -1 \\ -3 & & 1 & (-1) & 1 & 0 & 1 & -1 \\ -3 & & 1 & (-1) & 1 & 0 & -1 & 1 \end{pmatrix}.$$

We now clear the circled entries in column 1 using the second block, specifically by replacing $\mathcal{C}_1(P'_\delta)$ by $\mathcal{C}_1(P'_\delta) - 2(\mathcal{C}_6(P'_\delta) + 2\mathcal{C}_7(P'_\delta))$. Similar to how we began with the leftmost block, we now also clear the circled entries in columns 5 and 7 by replacing $\mathcal{C}_5(P'_\delta)$ by $\mathcal{C}_5(P'_\delta) + 2\mathcal{C}_6(P'_\delta) + 3\mathcal{C}_7(P'_\delta)$, and then replacing $\mathcal{C}_7(P'_\delta)$ by $\mathcal{C}_7(P'_\delta) + \mathcal{C}_8(P'_\delta)$.

In the last 4 columns, corresponding to the parts $\delta_2 + \delta_1 = 3 + 1$ of δ , the procedure is a little different. We replace $\mathcal{C}_8(P'_\delta)$ by $\mathcal{C}_8(P'_\delta) + 2\mathcal{C}_9(P'_\delta) + 3[\mathcal{C}_{10}(P'_\delta) + \mathcal{C}_{11}(P'_\delta)]$, and then add

column 10 to column 11. Altogether, this yields

$$P''_\delta = \left(\begin{array}{ccccccccc} 0 & -1 & 0 & 0 & & & & & \\ 0 & 1 & -1 & 0 & & & & & \\ 0 & 0 & 1 & -1 & & & & & \\ 2 & 0 & 0 & 0 & & & & & \\ 0 & & 1 & 0 & -1 & 0 & & & \\ 0 & & 0 & 1 & -1 & & & & \\ -6 & & 2 & 0 & 0 & & & & \\ \textcircled{-2} & & \textcircled{-1} & & & 1 & 0 & -1 & 0 & 0 \\ -2 & & -1 & & & & -3 & 1 & -1 & \boxed{-2} \\ \textcircled{1} & & \textcircled{-2} & & & & 1 & 0 & 1 & 0 \\ \textcircled{1} & & \textcircled{-2} & & & & 1 & 0 & -1 & 0 \end{array} \right).$$

From here on, we will freely work mod 2 in row 9, using the boxed pivot entry to clear all entries as much as possible. We now clear the circled entries in columns 1 and 5 of P''_δ by adding multiples of columns 8 and 9. We then add column 10 to column 8, to obtain

$$P'''_\delta = \left(\begin{array}{ccccccccc} 0 & -1 & 0 & 0 & & & & & \\ 0 & 1 & -1 & 0 & & & & & \\ 0 & 0 & 1 & -1 & & & & & \\ 2 & 0 & 0 & 0 & & & & & \\ & & 1 & 0 & -1 & 0 & & & \\ & & 0 & 1 & -1 & & & & \\ -6 & & \boxed{2} & 0 & 0 & & & & \\ & & & & & 1 & 0 & -1 & 0 & 0 \\ 1 & & & & & 0 & 1 & 1 & \textcircled{-2} \\ 0 & & & & & 2 & 0 & 1 & 0 \\ 0 & & & & & & & -1 & 0 \end{array} \right).$$

We then clear the circled entry in column 1 of P'''_δ using the boxed entry in column 5. In the rightmost block, we first use column 10 to clear the circled entry in column 11, then use column 8 to clear the resulting (10, 11)-entry mod 2, and lastly multiply columns 10 and 11 by -1 , to obtain

$$(7.3.2) \quad Q_\delta = \left(\begin{array}{ccccccccc} 0 & -1 & 0 & 0 & & & & & \\ 0 & 1 & -1 & 0 & & & & & \\ 0 & 0 & 1 & -1 & & & & & \\ 2 & 0 & 0 & 0 & & & & & \\ & & 1 & 0 & -1 & 0 & & & \\ & & 0 & 1 & -1 & & & & \\ & & 2 & 0 & 0 & & & & \\ & & & & & 1 & 0 & -1 & 0 & \\ 1 & & & & & 0 & 1 & -1 & & \\ 0 & & & & & 2 & 0 & -1 & & \\ 0 & & & & & & & 1 & 2 \end{array} \right).$$

Since we have only used column operations, the following basis for $\text{MOD}(w_\delta)$ can be read off from the columns of Q_δ :

$$\{2\alpha_4^\vee + \alpha_9^\vee, \quad 2\alpha_7^\vee, \quad 2\alpha_{10}^\vee, \quad 2\alpha_{11}^\vee\}$$

together with

$$\{-\alpha_i^\vee + \alpha_{i+1}^\vee \mid i = 1, 2, 5, 8\} \cup \{-\alpha_i^\vee + \alpha_{i+2}^\vee \mid i = 3, 6\} \cup \{-\alpha_9^\vee - \alpha_{10}^\vee + \alpha_{11}^\vee\},$$

confirming part (3) of Theorem 7.2.

To find the Smith normal form for P_δ , we continue from Q_δ , now using some row operations. Working from left to right, we use the -1 s in columns 2, 3, 4, 6, 7, and 9 to successively clear the 1s underneath them, similar to how we obtained Q'_δ in Example 7.5. From there we extract the 5×5 minor of Q_δ consisting of all columns containing 2s, in addition to column $n-1=10$, together with the rows containing the nonzero entries of these columns, resulting in the lefthand matrix below. We then carry out a series of row and then column operations, as follows:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \boxed{1} & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

On this minor, we first add row 5 to rows 3 and 4, and then use columns 3 and 4 to clear the entries in column 5, to obtain the second matrix. Then use the boxed entry in column 1 to clear the 2 in column 5. Finally, use the same boxed entry in row 3 to clear the 2 in row 1, to obtain the final matrix. After scaling and rearrangement, this implies that P_δ has Smith normal form $\text{diag}(1^8, 2^2, 4)$, confirming part (2) of Theorem 7.1 in this case.

Example 7.7. This example illustrates part (2) of Theorem 7.1 in the case $|\beta| = 0$, and part (4) of Theorem 7.2. Let W be of type D_{11} and let $|\beta| = 0$ and $\delta = (2, 3, 3, 3)$. By (7.1.5), we have

$$w_{\beta, \delta} = w_\delta = u_0 v_2^\delta v_3^\delta v_4^\delta = u_0(u_2 s_8 s_7)(u_5 s_5 s_4)(u_8 s_2 s_1).$$

Applying (2.1.1), the matrix for $I - w_\delta$ with respect to the basis Δ^\vee is given by

$$P_\delta = \begin{pmatrix} 2 & -1 & 0 & & & & & & \\ 1 & 1 & -1 & & & & & & \\ 2 & 0 & 0 & & & & & & \\ 2 & & -1 & 2 & -1 & 0 & & & \\ 2 & & -1 & 1 & 1 & -1 & & & \\ 2 & & -2 & 2 & 0 & 0 & & & \\ 2 & & -2 & 2 & & -1 & 2 & -1 & 0 \\ 2 & & -2 & 2 & & -1 & 1 & 1 & -1 \\ 2 & & -2 & 2 & & -2 & 2 & 0 & 0 \\ 1 & & -1 & 1 & & -1 & 1 & 0 & 1 & -1 \\ 1 & & -1 & 1 & & -1 & 1 & -1 & 1 & 1 \end{pmatrix}.$$

Notice that P_δ is block lower-triangular with diagonal blocks of sizes 3, 3, 3, 2 going from left to right, obtained from the parts of δ in reverse order. For $k \geq 3$, both the diagonal blocks and the entries underneath them corresponding to the parts δ_k are the same as described in Example 7.5 above. The diagonal block corresponding to δ_2 is the same as in Example 7.5 above, and the diagonal block for $\delta_1 = 2$ is the 2×2 matrix in the lower right depicted in P_δ .

Finally, in the 2 rows underneath the block for δ_2 , the leftmost column has entries 1, 1, the rightmost column has entries 0, -1, and all other entries equal 0. One can check via (2.1.1) that this description in fact characterizes P_δ for all $|\delta| = n$ with $\delta_1 = 2$; that is, the case considered in part (4) of Theorem 7.2.

We now carry out a series of column operations on P_δ to obtain a matrix from which a basis for $\text{MOD}(w_\delta)$ can be read off; see the matrix Q_δ in (7.3.3) below. We start with column operations on columns 1, 3, 4, 6, and 7 modeled upon those in the two previous examples, clearing each of these columns as much as possible, to obtain

$$P'_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ & 1 & 0 & -1 & 0 \\ & 0 & 1 & -1 \\ & 2 & 0 & 0 \\ & & 1 & 0 & -1 & 0 \\ & & 0 & 1 & -1 \\ & & 2 & 0 & 0 \\ & & & 0 & 1 & -1 \\ \textcircled{-20} & \textcircled{-4} & \textcircled{-3} & -1 & 1 & 1 \end{pmatrix}.$$

The process of clearing the bottom row of P'_δ is somewhat unique to the case of $\delta_2 = 2$. We first add column 10 to column 9, and then add column 11 to column 10, resulting in the boxed entry below, which is used to clear the circled entries in the bottom row of $P'_\delta \bmod 2$, giving us

$$(7.3.3) \quad Q_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ & 1 & 0 & -1 & 0 \\ & 0 & 1 & -1 \\ & 2 & 0 & 0 \\ & & 1 & 0 & -1 & 0 \\ & & 0 & 1 & -1 \\ & & 2 & 0 & 0 \\ & & & 1 & 0 & -1 \\ & & & 1 & \boxed{2} & 1 \end{pmatrix}.$$

Since we have only used column operations, the following basis for $\text{MOD}(w_\delta)$ can be read off from the columns of Q_δ . The columns with 2s yield

$$\{2\alpha_3^\vee, 2\alpha_6^\vee, 2\alpha_9^\vee + \alpha_{11}^\vee, 2\alpha_{11}^\vee\},$$

and the remaining columns give us

$$\{-\alpha_i^\vee + \alpha_{i+1}^\vee \mid i = 1, 4, 7, 10\} \cup \{-\alpha_i^\vee + \alpha_{i+2}^\vee \mid i = 2, 5, 8\}.$$

In particular, since $\delta = (2, 3, 3, 3)$, then we obtain the sequences $(j_k^\delta) = (0, 2, 5, 8)$ and $(\delta_{k,k-1}) = (0, 1, 0, 0)$. Hence, the first set in this basis equals

$$\left\{ 2\alpha_{n-j_k^\delta}^\vee + \delta_{k,k-1}\alpha_n^\vee \mid 1 \leq k \leq 2r \right\}$$

with $n = 11$ and $2r = 4$. We also have $I_\delta = \{3, 6, 9, 11\}$ and hence $I_\delta - 1 = \{2, 5, 8, 10\}$, and so the second pair of sets in the basis obtained from Q_δ coincides with

$$\{-\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\delta \cup (I_\delta - 1) \text{ or } i = n - 1\} \cup \{-\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\delta - 1), i \neq n - 1\},$$

confirming part (4) of Theorem 7.2.

To find the Smith normal form for P_δ , we continue from Q_δ . Working from left to right, we use the -1 s above the diagonal to successively clear the 1 s underneath them. This yields

$$Q'_\delta = \begin{pmatrix} 0 & -1 & 0 & & & & \\ 0 & 0 & -1 & & & & \\ 2 & 0 & 0 & & & & \\ & & & 0 & -1 & 0 & \\ & & & 0 & 0 & -1 & \\ & & & 2 & 0 & 0 & \\ & & & & & & 0 & -1 & 0 \\ & & & & & & 0 & 0 & -1 \\ & & & & & & 2 & 0 & 0 \\ & & & & & & & & 0 & -1 \\ & & & & & & & & 1 & \\ & & & & & & & & & 2 & 0 \end{pmatrix}.$$

From here, noting that $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$ via one row and one column operation, it is easy to see that P_δ has Smith normal form $\text{diag}(1^8, 2^2, 4)$, confirming part (2) of Theorem 7.1 in this case.

Example 7.8. This example illustrates part (1) of Theorem 7.1 in the case $|\beta| = 0$, and part (5) of Theorem 7.2. Let W be of type D_{12} and let $|\beta| = 0$ and $\delta = (3, 3, 3, 3)$. By (7.1.5), we have

$$w_{\beta, \delta} = w_\delta = (u_0 s_{10}) v_2^\delta v_3^\delta v_4^\delta = (u_0 s_{10})(u_3 s_8 s_7)(u_6 s_5 s_4)(u_9 s_2 s_1).$$

Applying (2.1.1), the matrix for $I - w_\delta$ with respect to the basis Δ^\vee is given by

$$P_\delta = \begin{pmatrix} 2 & -1 & 0 & & & & & & \\ 1 & 1 & -1 & & & & & & \\ 2 & 0 & 0 & & & & & & \\ 2 & -1 & 2 & -1 & 0 & & & & \\ 2 & -1 & 1 & 1 & -1 & & & & \\ 2 & -2 & 2 & 0 & 0 & & & & \\ 2 & -2 & 2 & & -1 & 2 & -1 & 0 & \\ 2 & -2 & 2 & & -1 & 1 & 1 & -1 & \\ 2 & -2 & 2 & & -2 & 2 & 0 & 0 & \\ 2 & -2 & 2 & & -2 & 2 & & -1 & 2 & -1 & -1 \\ 1 & -1 & 1 & & -1 & 1 & & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & & -1 & 1 & & -1 & 1 & 0 & 0 \end{pmatrix}.$$

As in most previous examples, the sizes of the diagonal blocks of P_δ are the parts of δ in reverse order. The diagonal blocks and entries underneath them corresponding to the parts δ_k for $k \geq 3$ are the same as described in Example 7.5 above. The diagonal block corresponding to δ_2 is also the same as in Example 7.5. Underneath the δ_2 block, the leftmost column has entries $2, \dots, 2, 1, 1$ like other blocks to its left; however, the rightmost column has entries $-1, \dots, -1, 0, -1$, and all other entries equal 0. The diagonal block for δ_1 is the $\delta_1 \times \delta_1$ matrix which has first column $(2, 1^{\delta_1-3}, 0, 1)$, diagonal entries $2, 1, \dots, 1, 0$, above-diagonal

entries -1 , and an additional -1 entry in the $(n-2, n)$ position. One can check via (2.1.1) that this description characterizes P_δ for all $|\delta| = n$ with $\delta_1 \geq 3$; that is, the case considered in part (5) of Theorem 7.2.

We now carry out a series of column operations on P_δ to obtain a matrix Q_δ from which a basis for $\text{MOD}(w_\delta)$ can be read off; see (7.3.4) below. We start with column operations on columns 1, 3, 4, 6, 7, and 9 modeled upon those in previous examples, clearing each of these columns as much as possible, to obtain

$$P'_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ & 1 & 0 & -1 & 0 \\ & 0 & 1 & -1 \\ & 2 & 0 & 0 \\ & & 1 & 0 & -1 & 0 \\ & & 0 & 1 & -1 \\ & & 2 & 0 & 0 \\ & & & 1 & 2 & -1 & -1 \\ -32 & & -8 & & & 0 & 1 & -1 \\ -6 & & -6 & -2 & & 1 & 0 & 0 \end{pmatrix}.$$

After the initial steps of replacing $\mathcal{C}_1(P_\delta)$ by $\mathcal{C}_1(P_\delta) + 2\mathcal{C}_2(P_\delta) + 3\mathcal{C}_3(P_\delta)$ and $\mathcal{C}_3(P_\delta)$ by $\mathcal{C}_3(P_\delta) + \mathcal{C}_4(P_\delta)$ and so on, we point out that all subsequent column operations clearing columns 1, 4, and 7 below the diagonal result in even numbers in these columns, due to the absence of any change of parity in δ . The result is that we are able to completely clear columns 1, 4, and 7 in this example; compare P'_δ in Example 7.7, where the odd number in column 7 results from the change in parity between the first two parts of $\delta = (2, 3, 3, 3)$.

The process of clearing the bottom $\delta_1 - 1$ rows in columns 1, 4, and 7 now involves several column operations unique to the case of $\delta_1 \geq 3$. In the last block, we change its first column to $0, \dots, 0, \delta_1 - 1, 1$ by replacing its first column, which has index $n - \delta_1 + 1$, by the sum $\mathcal{C}_{n-\delta_1+1}(P_\delta) + 2\mathcal{C}_{n-\delta_1+2}(P_\delta) + 3\mathcal{C}_{n-\delta_1+3}(P_\delta) + \dots + (\delta_1 - 1)\mathcal{C}_{n-1}(P_\delta)$. We then subtract column $n - 1$ from column n . The resulting boxed entries in the δ_1 block can be used to clear the remaining entries mod 2 in row $n - 1$, and completely in row n , to obtain

$$(7.3.4) \quad Q_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \\ & 1 & 0 & -1 & 0 \\ & 0 & 1 & -1 \\ & 2 & 0 & 0 \\ & & 1 & 0 & -1 & 0 \\ & & 0 & 1 & -1 \\ & & 2 & 0 & 0 \\ & & & 1 & 0 & -1 & 0 \\ & & & 0 & 1 & \boxed{-2} \\ & & & \boxed{1} & 0 & 0 \end{pmatrix}.$$

Since we have only used column operations, the following basis for $\text{MOD}(w_\delta)$ can be read off from the columns of Q_δ . The columns with 2s, together with column $n - \delta_1 + 1$, yields

$$\{2\alpha_3^\vee, 2\alpha_6^\vee, 2\alpha_9^\vee\} \cup \{2\alpha_{11}^\vee, \alpha_{12}^\vee\},$$

and the remaining columns give us

$$\{-\alpha_i^\vee + \alpha_{i+1}^\vee \mid i = 1, 4, 7, 10\} \cup \{-\alpha_i^\vee + \alpha_{i+2}^\vee \mid i = 2, 5, 8\}.$$

Since $\delta = (3, 3, 3, 3)$ has no parity changes, then $(\delta_{k,k-1}) = (0, 0, 0, 0)$, and $(j_k^\delta) = (0, 3, 6, 9)$. Therefore, the first sets agree with

$$\left\{ 2\alpha_{n-j_k^\delta}^\vee + \delta_{k,k-1}\alpha_{n-1}^\vee \mid 2 \leq k \leq 2r \right\} \cup \left\{ 2\alpha_{n-1}^\vee, ((\delta_1 - 1) \bmod 2)\alpha_{n-1}^\vee + \alpha_n^\vee \right\}$$

for $n = 12$ and $2r = 4$. In this example, $I_\delta = \{3, 6, 9, 12\}$ and so $I_\delta - 1 = \{2, 5, 8, 11\}$. The second pair of sets in the basis obtained from Q_δ thus coincides with

$$\left\{ -\alpha_i^\vee + \alpha_{i+1}^\vee \mid i \notin I_\delta \cup (I_\delta - 1) \right\} \cup \left\{ -\alpha_i^\vee + \alpha_{i+2}^\vee \mid i \in (I_\delta - 1), i \neq n-1 \right\},$$

confirming part (5) of Theorem 7.2.

To find the Smith normal form for P_δ from Q_δ , we work left to right as in previous examples, using row operations with the -1 s above the diagonal to successively clear the 1 s below, giving us

$$Q'_\delta = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 2 & 0 & 0 \\ & 0 & -1 & 0 \\ & 0 & 0 & -1 \\ & 2 & 0 & 0 \\ & & 0 & -1 & 0 \\ & & 0 & 0 & -1 \\ & & 2 & 0 & 0 \\ & & & 0 & -1 & 0 \\ & & & 0 & 0 & -2 \\ & & & 1 & 0 & 0 \end{pmatrix}.$$

Therefore, P_δ has Smith normal form $\text{diag}(1^6, 2^4)$, confirming part (1) of Theorem 7.1 in this case.

7.3.2. Noncuspidal examples with $\beta_p = 1$. We now consider the situation where $\beta_p = 1$. This includes the special case $\beta = (1)$. Recall from Section 7.1 that when $|\beta| = 1$, we have w_β trivial and $J_\beta = \emptyset$ (as in the case $|\beta| = 0$). The following examples illustrate the proof of part (3) of Theorem 7.1, and how to obtain a basis in this case; see the introduction to Section 7.3 for the list of examples.

Example 7.9. Let W be of type D_n with $n \geq 5$ odd, and let $\beta = (1)$ and $\delta = (1, \dots, 1)$ with $n-1$ parts. We prove this special case of part (3) of Theorem 7.1, and find a basis. Recall from (7.1.3) that

$$w_{\beta, \delta} = w_\delta = u_1 u_2 \dots u_{n-2},$$

which equals the longest element in the type D_{n-1} subsystem indexed by the last $n-1$ nodes; see Table 1 in [BKOP14], for example. Therefore, w_δ acts as $(-\mathbf{I})$ on the submodule of R^\vee spanned by $\{\alpha_i^\vee \mid 2 \leq i \leq n\}$.

From (2.1.1), the matrix for $I - w_\delta$ with respect to Δ^\vee is therefore

$$P_\delta = \begin{pmatrix} 0 & & & \\ -2 & 2 & & \\ \vdots & & \ddots & \\ -2 & & & 2 \\ -1 & & & 2 \\ -1 & & & 2 \end{pmatrix}.$$

That is, first row of this matrix is all 0s, the $(n-1) \times (n-1)$ minor in the bottom right is $2I$, and the first column is $0, -2, \dots, -2, -1, -1$.

We now add columns 2 through n to column 1 to obtain P'_δ , and then use two further column operations to clear the circled entry and obtain Q_δ below:

$$P'_\delta = \begin{pmatrix} 0 & & & \\ 0 & 2 & & \\ \vdots & & \ddots & \\ 0 & & 2 & \\ 1 & & 2 & \\ 1 & & & \textcircled{2} \end{pmatrix} \rightarrow Q_\delta = \begin{pmatrix} 0 & 2 & & \\ 0 & 2 & & \\ \vdots & & \ddots & \\ 0 & & 2 & \\ 1 & & 2 & \\ 1 & & & 0 \end{pmatrix}.$$

A basis for $\text{MOD}(w_{\beta, \delta})$ can then be read off from Q_δ ; namely $\{2\alpha_2^\vee, \dots, 2\alpha_{n-1}^\vee\} \cup \{\alpha_{n-1}^\vee + \alpha_n\}$.

Now on the matrix Q_δ , we subtract row n from row $n-1$ to obtain

$$Q'_\delta = \begin{pmatrix} 0 & & & \\ 0 & 2 & & \\ \vdots & & \ddots & \\ 0 & & 2 & \\ 0 & & 2 & \\ 1 & & & 0 \end{pmatrix}.$$

The Smith normal form $(1, 2^{n-2}, 0)$ can be seen immediately from Q'_δ , which establishes this case of part (3) of Theorem 7.1.

Example 7.10. Let W be of type D_6 and let $\beta = (1)$ and $\delta = (1, 1, 1, 2)$. This example illustrates part (3) of Theorem 7.1 and how to find a basis, in the case that $\beta = (1)$, $\delta_1 = \delta_2 = 1$, and $\delta \neq (1, \dots, 1)$. By (7.1.4) with $\ell = 3$,

$$w_{\beta, \delta} = w_\delta = u_1 u_2 v_4^\delta = u_1 u_2 (u_3 s_2).$$

Applying (2.1.1), the matrix for $I - w_\delta$ with respect to Δ^\vee is

$$P_\delta = \begin{pmatrix} 0 & 0 & 0 & \\ -1 & 2 & -1 & \\ -2 & 2 & 0 & \\ -2 & 2 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -1 & 1 & -1 & 2 \end{pmatrix}.$$

Notice that the first row of this matrix is all 0s, while the $(n-1) \times (n-1)$ minor in the bottom right is the same as the matrix for the cuspidal element w_δ in the subsystem of type D_{n-1} indexed by the last $n-1$ nodes. The first column of P_δ is the negative of its second column, except in the $(2, 1)$ -entry, which equals -1 . The same holds whenever $\beta = (1)$.

We now add column 2 to column 1, so that the $(2, 1)$ -entry becomes the boxed pivot entry 1. We then carry out the same column operations on the $(n - 1) \times (n - 1)$ minor in the bottom right as we did in the cuspidal case (see Example 7.5), to obtain

$$P'_\delta = \begin{pmatrix} 0 & & & \\ \boxed{1} & 0 & \circled{-1} & \\ 2 & 0 & & \\ 0 & 0 & 2 & \\ \circled{1} & \boxed{1} & 2 & \\ \circled{1} & \boxed{1} & 2 & \end{pmatrix}.$$

Now, working from left to right, we can apply column operations which use the boxed entries to clear the circled entries of P'_δ to obtain the matrix P''_δ

$$P''_\delta = \begin{pmatrix} 0 & & & \\ 1 & 0 & 0 & \\ 2 & 0 & & \\ 0 & 0 & 2 & \\ 1 & & 2 & \\ 1 & & 2 & \end{pmatrix} \rightarrow Q_\delta = \begin{pmatrix} 0 & & & \\ 1 & 0 & 0 & \\ 2 & 0 & & \\ 0 & 0 & 2 & \\ 1 & & 2 & \\ 1 & & 0 & \end{pmatrix}$$

Finally, use columns 3 and 5 to clear column $n = 6$, which yields the matrix Q_δ , from which we can read off a basis for $\text{MOD}(w_{\beta, \delta})$, namely $\{\alpha_2^\vee, 2\alpha_3^\vee\} \cup \{2\alpha_4^\vee, 2\alpha_5^\vee\} \cup \{\alpha_5^\vee + \alpha_6\}$.

From here, observe that $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ via one row operation. Therefore, P_δ clearly has Smith normal form $\text{diag}(1^2, 2^3, 0)$, confirming part (3) of Theorem 7.1.

Example 7.11. Let W be of type D_5 and let $\beta = (1)$ and $\delta = (1, 3)$. This example illustrates part (3) of Theorem 7.1, and how to find a basis, in the case $\beta = (1)$, $\delta_1 = 1$, and $\delta_2 \geq 2$. By (7.1.4) with $\ell = 1$, we have

$$w_{\beta, \delta} = w_\delta = v_2^\delta = u_1 s_3 s_2.$$

Applying (2.1.1), the matrix for $I - w_\delta$ with respect to Δ^\vee is

$$P_\delta = \begin{pmatrix} 0 & & & & \\ -1 & 2 & -1 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 \end{pmatrix}.$$

We now add column 2 to column 1, so that the $(2, 1)$ -entry becomes a pivot entry 1. We then carry out the same column operations on the $(n - 1) \times (n - 1)$ minor in the bottom right as we did in the cuspidal case (see Example 7.6), to obtain

$$P'_\delta = \begin{pmatrix} 0 & & & & \\ \boxed{1} & 0 & \circled{-1} & 0 & \\ 0 & \boxed{1} & \circled{-1} & & \\ 2 & 0 & -1 & & \\ & & 1 & 2 & \end{pmatrix}.$$

Compare the bottom right 4×4 minor of Q_δ in (7.3.2). Working from left to right, we can apply column operations which use the boxed entries to clear the circled entries of P'_δ , yielding

$$P''_\delta = \begin{pmatrix} 0 & & & \\ 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & 2 & 0 & -1 \\ & & 1 & \boxed{2} \end{pmatrix}.$$

We now add twice column 4 to column 2, and then used the boxed pivot entry 2 in column n of P''_δ to clear the resulting $(n, 2)$ -entry. This gives us

$$Q_\delta = \begin{pmatrix} 0 & & & \\ \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ & & \circled{1} & \boxed{2} \end{pmatrix}.$$

A basis for $\text{MOD}(w_{\beta, \delta})$ given by $\{\alpha_2^\vee, \alpha_3^\vee\} \cup \{-\alpha_4^\vee + \alpha_5^\vee\} \cup \{2\alpha_5^\vee\}$ can now be read off Q_δ .

To obtain the Smith normal form of P_δ , we add row 4 to row 5, which clears the circled entry of Q_δ and gives us

$$Q'_\delta = \begin{pmatrix} 0 & & & \\ \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ & & & \boxed{2} \end{pmatrix}.$$

Therefore, P_δ has Smith normal form $\text{diag}(1^3, 2^1, 0)$, confirming part (3) of Theorem 7.1.

Example 7.12. Let W be of type D_5 and let $\beta = (1)$ and $\delta = (2, 2)$. This example illustrates part (3) of Theorem 7.1 and shows how to obtain a basis, in the case that $\beta = (1)$ and δ has first part $\delta_1 = 2$. By (7.1.5), we have

$$w_{\beta, \delta} = w_\delta = u_0 v_2^\delta = u_0 u_2 s_2,$$

and so by (2.1.1) the matrix for $I - w_\delta$ with respect to Δ^\vee is

$$P_\delta = \begin{pmatrix} 0 & & & & \\ -1 & 2 & -1 & & \\ -2 & 2 & 0 & & \\ -1 & 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{pmatrix}.$$

We now add column 2 to column 1, so that the $(2, 1)$ -entry becomes a pivot entry 1. We then carry out the same column operations on the $(n-1) \times (n-1)$ minor in the bottom right as we did in the cuspidal case (see Example 7.7), to obtain

$$P'_\delta = \begin{pmatrix} 0 & & & & \\ \boxed{1} & 0 & \circled{-1} & & \\ & 2 & 0 & & \\ & \boxed{1} & 0 & \circled{-1} & \\ & & \circled{2} & \boxed{1} & \end{pmatrix} \rightarrow Q_\delta = \begin{pmatrix} 0 & & & \\ 1 & 0 & 0 & \\ & 2 & 0 & \\ & & 1 & 0 & 0 \\ & & & 0 & 1 \end{pmatrix}.$$

Now, working from left to right, we can apply further column operations which use the boxed entries of P'_δ to clear the circled entries, yielding Q_δ . A basis for $\text{MOD}(w_{\beta,\delta})$ given by $\{\alpha_2^\vee, 2\alpha_3^\vee\} \cup \{\alpha_4^\vee, \alpha_5^\vee\}$ can now be read off from Q_δ .

Moreover, we can easily see from Q_δ that P_δ has Smith normal form $\text{diag}(1^3, 2^1, 0)$, confirming part (3) of Theorem 7.1.

Example 7.13. Let W be of type D_7 and let $\beta = (1)$ and $\delta = (3, 3)$. This example illustrates part (3) of Theorem 7.1 and how to find a basis, in the case that $\beta = (1)$ and δ has first part $\delta_1 \geq 3$. By (7.1.5), we have

$$w_{\beta,\delta} = w_\delta = (u_0 s_5) v_2^\delta = (u_0 s_5)(u_3 s_3 s_2).$$

By (2.1.1), the matrix for $I - w_\delta$ with respect to Δ^\vee is

$$P_\delta = \begin{pmatrix} 0 & & & & & & \\ -1 & 2 & -1 & 0 & & & \\ -1 & 1 & 1 & -1 & & & \\ -2 & 2 & 0 & 0 & & & \\ -2 & 2 & & -1 & 2 & -1 & -1 \\ -1 & 1 & & 0 & 0 & 1 & -1 \\ -1 & 1 & & -1 & 1 & 0 & 0 \end{pmatrix}.$$

We now add column 2 to column 1, so that the $(2, 1)$ -entry becomes a pivot entry 1. We then carry out the same column operations on the $(n-1) \times (n-1)$ minor in the bottom right as we did in the cuspidal case (see Example 7.8), to obtain P'_δ , which agrees with the lower right 6×6 minor in (7.3.4).

$$P'_\delta = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & \circled{-1} & 0 & & & \\ 0 & 1 & \circled{-1} & & & & \\ 2 & 0 & 0 & & & & \\ & 1 & 0 & \circled{-1} & 0 & & \\ 0 & 1 & \circled{-2} & & & & \\ 1 & 0 & 0 & & & & \end{pmatrix} \rightarrow Q_\delta = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & & & \\ 2 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & & & \\ 1 & 0 & 0 & & & & \end{pmatrix}$$

Now, working from left to right, we can apply column operations which use the boxed entries to clear the circled entries of P'_δ , yielding Q_δ above. A basis for $\text{MOD}(w_{\beta,\delta})$ given by $\{\alpha_2^\vee, \alpha_3^\vee, 2\alpha_4^\vee\} \cup \{\alpha_5^\vee, \alpha_6^\vee, \alpha_7^\vee\}$ can now be read off from Q_δ .

Moreover, we can easily see from Q_δ that P_δ has Smith normal form $\text{diag}(1^5, 2^1, 0)$, confirming part (3) of Theorem 7.1.

Example 7.14. This example illustrates part (3) of Theorem 7.1 in the case that $\beta_p = 1$ and β has more than one part, and shows how to obtain a basis.

Let W be of type D_7 , and let $\beta = (3, 1)$ and $\delta = (1, 2)$. By Proposition 4.1 and Equation (7.1.4), we have

$$w_{\beta,\delta} = (s_1 s_2)(u_1 s_5).$$

Let $|\beta| = m \leq n-2$ (so that $m = 4$ in this example). Then since $\beta_p = 1$, the element w_β has support in the type A_{m-2} subsystem indexed by the first $m-2$ nodes, so in particular, w_β fixes α_i^\vee for $m \leq i \leq n$. Moreover, letting $\beta' = (3, 1, 1)$ be the partition obtained by adding

last part 1 to β , the element w_β acts on α_{m-1}^\vee exactly as does the element $w_{\beta'}$ in type A_{m-1} . Thus applying (2.1.1), the matrix for $I - w_{\beta,\delta}$ with respect to Δ^\vee is given by $P_{\beta,\delta}$ below:

$$P_{\beta,\delta} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 2 & -1 & -1 \\ & & & -1 & 1 & 1 & -1 \\ & & & -1 & 1 & -1 & 1 \end{pmatrix}.$$

Notice that the $(m-1) \times (m-1)$ minor in the top left (here $m-1=3$) is the same as the matrix M_β from case (2) of Corollary 4.17 in type A_{m-1} , since the last part of β equals 1. Also, the $(n-m+1) \times (n-m+1)$ minor in the bottom right (here $n-m+1=4$) is analogous to the matrix P_δ for the case $\beta=(1)$ considered in Example 7.11. We can thus carry out column operations on columns 1 through $m-1$ as in type A , and on columns m through n as in the case $\beta=(1)$, to obtain

$$Q_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 2 & 0 \\ & & & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that, since the last part of β equals 1, we have $\gcd(\beta_k, 2) = 1$ in this case. Hence in columns 1 through $m-1$, we obtain diagonal entries $(1^{m-p}, 0^{p-1})$. In columns m through n , by the case $\beta=(1)$ we obtain diagonal entries $(1^{n-m+1-2r}, 2^{2r-1}, 0)$. Putting this together gives us $S_{\beta,\delta} = \text{diag}(1^{n-2r-p+1}, 2^{2r-1}, 0^p)$, which confirms part (3) of Theorem 7.1. The basis read off from $Q_{\beta,\delta}$ yields $\{\alpha_j^\vee \mid j \in J_\beta\}$ together with the basis obtained from the last $n-m+1$ columns as in the case $\beta=(1)$.

7.3.3. Noncuspidal examples with $\beta_p \geq 2$. We continue by considering some examples in which $\beta_p \geq 2$, to illustrate parts (1), (2), and (3) of Theorem 7.1 in this case. See the start of Section 7.3 for the full list of examples.

Example 7.15. This example illustrates part (2) of Theorem 7.1 in the case $\beta_p \geq 2$ and $\delta_1 = \delta_2 = 1$. Let W be of type D_9 and let $\beta = (4)$ and $\delta = (1, 1, 1, 2)$. By (7.1.4) with $\ell = 3$,

$$w_{\beta,\delta} = w_\beta w_\delta = (s_1 s_2 s_3) u_1 u_2 v_4^\delta = (s_1 s_2 s_3) u_1 u_2 (u_3 s_5),$$

and the matrix for $I - w_{\beta,\delta}$ with respect to Δ^\vee is given by

$$P_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 2 & -1 \\ & & & -2 & 2 & 0 \\ & & & -2 & 2 & -2 & 2 \\ & & & -1 & 1 & -1 & 2 \\ & & & -1 & 1 & -1 & & 2 \end{pmatrix}.$$

Let $m = |\beta|$ and define $\beta' = (4, 1)$; that is, β' is the partition of $m + 1$ obtained by adjoining a last part 1 to β . Then the $m \times m$ minor in the top left (here $m = 4$) is the matrix $M_{\beta'}$ from case (2) of Corollary 4.17, since the last part of β' equals 1. Also, the $(n - m + 1) \times (n - m + 1)$ minor in the bottom right (here $n - m + 1 = 6$) is the matrix P_δ for the case $\beta = (1)$ and $\delta = (1, 1, 1, 2)$ considered in Example 7.10.

To find a basis for $\text{MOD}(w_{\beta, \delta})$, we first add column $m - 1 = 3$ and column $m + 1 = 5$ to column $m = 4$, then carry out column operations in the first $m - 1$ columns as in type A to obtain $(m - 1, m - 1)$ -entry equal to $\gcd(\beta_k) = 4$. We also carry out column operations on the last $(n - m)$ columns (here $n - m = 5$) as done in the case $|\beta| = 0$ to obtain the matrix Q_δ (compare Example 7.5). This gives us $P'_{\beta, \delta}$:

$$P'_{\beta, \delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & -1 \\ & & & 0 & 2 & 0 \\ & & & 0 & 0 & 0 & 2 \\ & & & 0 & 1 & 1 & 2 \\ & & & 0 & 1 & 1 & 2 \end{pmatrix}.$$

We now subtract $\gcd(\beta_k) = 4$ times column m from column $m - 1$. Then, using a suitable linear combination of the columns to the right of column m which do not have 2s, we move $(-\gcd(\beta_k)) = -4$ down to the last two entries in column $m - 1$, as seen in $P''_{\beta, \delta}$:

$$P''_{\beta, \delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & -1 \\ & & & 0 & 2 & 0 \\ & & & 0 & 0 & 0 & 2 \\ & & & -4 & 0 & 1 & 1 & \boxed{2} \\ & & & -4 & 0 & 1 & 1 & \boxed{2} \end{pmatrix}.$$

Then, using the boxed entries 2 in the last two columns of $P''_{\beta, \delta}$, we can clear column $m - 1$ and replace both of these boxed 2s by $\gcd(\beta_k, 2) \in \{1, 2\}$. In this example, since $\gcd(\beta_k, 2) = 2$, we obtain

$$Q_{\beta, \delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & -1 \\ & & & 0 & 2 & 0 \\ & & & 0 & 0 & 0 & 2 \\ & & & 0 & 1 & 1 & 2 \\ & & & 0 & 1 & 1 & 2 \end{pmatrix}.$$

Notice that the $(n - m) \times (n - m)$ minor in the bottom right (here $n - m = 5$) does not change at this step in the case $\gcd(\beta_k, 2) = 2$. Since we have only used column operations so

far, and the nonzero columns of $Q_{\beta,\delta}$ are linearly independent over \mathbb{Z} , a basis for $\text{MOD}(w_{\beta,\delta})$ can be read off from the columns of $Q_{\beta,\delta}$.

To find the Smith normal form, we will carry out some row operations on $Q_{\beta,\delta}$. First, in the $(m-1) \times (m-1)$ minor in the top left (here $m-1 = 3$), we use successive row operations to clear the -1 s under 1s. We then subtract row $m-1 = 3$ from row $m+1 = 5$, to obtain

$$Q'_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & -1 \\ & & & 0 & 2 & 0 \\ & & & 0 & 0 & 0 & 2 \\ & & & 0 & 1 & 1 & 2 \\ & & & 0 & 1 & 1 & & 2 \end{pmatrix}.$$

After some rearrangement in columns 1 through $m=4$, we obtain diagonal entries $(1^{m-p}, 0^p) = (1^3, 0)$, since $\beta' = (4, 1)$ has last part equal to 1. We can then continue to find the Smith normal form for the $(n-m) \times (n-m)$ minor in the bottom right (here $n-m = 5$) as in the case $|\beta| = 0$ (compare Example 7.5). This will have diagonal entries $(1^{(n-m)-2r+1}, 2^{2r-2}, 4) = (1^2, 2^2, 4)$, since $\delta = (1, 1, 1, 2)$ has a change of parity in this example. Putting this together, we obtain $S_{\beta,\delta} = (1^{n-2r-p+1}, 2^{2r-2}, 4, 0^p) = (1^5, 2^2, 4, 0)$, confirming part (2) of Theorem 7.1.

Example 7.16. This example illustrates part (2) of Theorem 7.1 in the case $\beta_p \geq 2$, $\delta_1 = 1$, and $\delta_2 \geq 2$. Let W be of type D_7 and let $\beta = (4)$ and $\delta = (1, 2)$. By (7.1.4), we have

$$w_{\beta,\delta} = w_\beta w_\delta = (s_1 s_2 s_3)(u_1 s_5),$$

and the matrix for $I - w_{\beta,\delta}$ with respect to Δ^\vee is given by

$$P_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 2 & -1 & -1 \\ & & & -1 & 1 & 1 & -1 \\ & & & -1 & 1 & -1 & 1 \end{pmatrix}.$$

Let $|\beta| = m$. As in the previous example, we add columns $m-1 = 3$ and $m+1 = 5$ to column $m = 4$, carry out column operations on the first $m-1$ columns as in type A , and subtract $\gcd(\beta_k) = 4$ times column m from column $m-1$. This has the effect of moving $(-\gcd(\beta_k)) = -4$ down into row $m+1$.

The next step is particular to the case $\delta_1 = 1$ and $\delta_2 \geq 2$. We add to column $n - \delta_2 = 5$ twice the sum of columns $n-1$ and n , and then add column $n-1$ to column n (this step is analogous to that used to obtain the matrix P''_δ in Example 7.6). This creates a pivot -2 in

the $(n - 2, n)$ -entry, as well as clearing much of column $n - \delta_2 = 5$, as shown in $P'_{\beta, \delta}$:

$$P'_{\beta, \delta} = \begin{pmatrix} 1 & 0 & 0 & & & & \\ -1 & 1 & 0 & & & & \\ 0 & -1 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -4 & 1 & 0 & -1 \\ & & & & 1 & 1 & 0 \\ & & & & 1 & -1 & 0 \end{pmatrix}.$$

In this particular example, we have $m + 1 = n - 2 = 5$, and so we can immediately use this boxed -2 to clear column $m - 1 = 3$ and replace the $(n - 2, n)$ -entry by $\gcd(\beta_k, 2) \in \{1, 2\}$. (In general for $\delta_1 = 1$ and $\delta_2 \geq 2$, we can use a suitable linear combination of the columns to the right of column $m = 4$ which have nonzero entries 1 and -1 to move $(-\gcd(\beta_k)) = -4$ down to row $n - 2$. We can then use the boxed -2 to clear column $m - 1$ and replace the $(n - 2, n)$ -entry by $\gcd(\beta_k, 2) \in \{1, 2\}$.) In this example, since $\gcd(\beta_k, 2) = 2$, we obtain

$$P''_{\beta, \delta} = \begin{pmatrix} 1 & 0 & 0 & & & & \\ -1 & 1 & 0 & & & & \\ 0 & -1 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & -1 & 2 \\ & & & & 1 & 1 & 0 \\ & & & & 1 & -1 & 0 \end{pmatrix}.$$

Notice that the $(n - m) \times (n - m)$ minor in the bottom right (here $n - m = 3$) does not change at this step in the case $\gcd(\beta_k, 2) = 2$.

In order to write down a basis in the case $\delta_1 = 1$ and $\delta_2 \geq 2$, we will take the vectors corresponding to the nonzero columns of index 1 through $m = 4$ of $P''_{\beta, \delta}$ without further changes, but carry out additional column operations on the $(n - m) \times (n - m)$ minor in the bottom right (here $n - m = 3$). If $|\delta| \geq 4$, we carry out column operations on this minor to obtain a basis as in the case $|\beta| = 0$ (compare Example 7.6). In the special case $\delta = (1, 2)$ considered in this example, a basis can be read off from the matrix $Q_{\beta, \delta}$ below, which is obtained from $P''_{\beta, \delta}$ by column operations on the last $n - m = 3$ columns:

$$Q_{\beta, \delta} = \begin{pmatrix} 1 & 0 & 0 & & & & \\ -1 & 1 & 0 & & & & \\ 0 & -1 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 & 2 \\ & & & & 2 & 1 & 0 \\ & & & & 0 & -1 & 0 \end{pmatrix}.$$

To find the Smith normal form, as in the previous example, we carry out row operations on the first $m - 1 = 3$ rows of $Q_{\beta, \delta}$, and then subtract row $m - 1 = 3$ from row $m + 1 = 5$,

to obtain $Q'_{\beta,\delta}$ below:

$$Q'_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow Q''_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then Smith normal form for the entire matrix $P_{\beta,\delta}$ can be found by combining the Smith normal form for the $m \times m$ minor in the top left (here $m = 4$), namely $(1^{m-p}, 0^p) = (1^3, 0)$, with the Smith normal form for the $(n-m) \times (n-m)$ minor in the bottom right (here $n-m = 3$). (If $|\delta| \geq 4$, then the Smith normal form from the case $|\beta| = 0$ can be used for this $(n-m) \times (n-m)$ minor; compare Example 7.6.)

In the special case $\delta = (1, 2)$ considered in this example, the diagonal entries in the Smith normal form are $(1, 1, 4)$, by using row operations to obtain $Q''_{\beta,\delta}$, and then noting similarly to Example 7.7 that $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$ via one row and one column operation. The resulting Smith normal form for the entire matrix is thus $(1^5, 4, 0)$, confirming part (2) of Theorem 7.1.

Example 7.17. This example illustrates part (1) of Theorem 7.1 in the case $\beta_p \geq 2$ and $\delta_1 = 2$. Let W be of type D_8 and let $\beta = (4)$ and $\delta = (2, 2)$. By (7.1.5), we have

$$w_{\beta,\delta} = w_\beta w_\delta = (s_1 s_2 s_3)(u_0 u_2 s_5).$$

Here, the matrix for $I - w_{\beta,\delta}$ with respect to Δ^\vee is given by

$$P_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 2 & 0 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{pmatrix}.$$

Let $|\beta| = m$. As in the previous two examples, we add columns $m-1 = 3$ and $m+1 = 5$ to column $m = 4$, and then carry out column operations on columns 1 through $m-1 = 3$ as in type A. Now, on columns $m+1 = 5$ through n , we carry out the same column operations as in the case $|\beta| = 0$ (compare Example 7.7), to obtain:

$$P'_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & \boxed{2} & 1 \end{pmatrix}.$$

In this case, since $\delta_1 = 2$, we obtain the boxed pivot 2 in the $(n, n-1)$ -entry. We can now use further column operations, as in the previous examples, to move $(-\gcd(\beta_k)) = -4$ down to

row n , and then clear the $(n, m-1)$ -entry and replace the boxed pivot by $\gcd(\beta_k, 2) \in \{1, 2\}$. In this example, since $\gcd(\beta_k, 2) = 2$, we obtain:

$$Q_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & -1 \\ & & 0 & 2 & 0 \\ & & 0 & 0 & 1 & 0 & -1 \\ & & 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

Notice that the $(n-m) \times (n-m)$ minor in the bottom right (here $n-m=4$) does not change at this step in the case $\gcd(\beta_k, 2) = 2$. Since we have only used column operations so far, and the nonzero columns of $Q_{\beta,\delta}$ are linearly independent over \mathbb{Z} , a basis for $\text{MOD}(w_{\beta,\delta})$ can be read off from the columns of $Q_{\beta,\delta}$.

From $Q_{\beta,\delta}$, the Smith normal form is obtained similarly to the previous two examples. In this example, since all parts of $\delta = (2, 2)$ have the same parity, the $(n-m) \times (n-m)$ minor in the bottom right (here $n-m=4$) has Smith normal form $\text{diag}(1^2, 2^2)$. Hence the Smith normal form of the entire matrix $P_{\beta,\delta}$ is given by $S_{\beta,\delta} = \text{diag}(1^5, 2^2, 0)$, which confirms part (1) of Theorem 7.1.

Example 7.18. This example illustrates part (1) of Theorem 7.1 in the case $\beta_p \geq 2$ and $\delta_1 \geq 3$. Let W be of type D_{10} and let $\beta = (4)$ and $\delta = (3, 3)$. By (7.1.5), we have

$$w_{\beta,\delta} = w_{\beta}w_{\delta} = (s_1s_2s_3)(u_0s_8 \cdot u_3s_6s_5).$$

Here, the matrix for $I-w_{\delta}$ with respect to Δ^{\vee} is given by

$$P_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & -1 & 2 & -1 & 0 \\ & & -1 & 1 & 1 & -1 \\ & & -2 & 2 & 0 & 0 \\ & & -2 & 2 & -1 & 2 & -1 & -1 \\ & & -1 & 1 & 0 & 0 & 1 & -1 \\ & & -1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Let $|\beta| = m$. As in the previous three examples, we add columns $m-1 = 3$ and $m+1 = 5$ to column $m = 4$, then carry out column operations on columns 1 through $m-1 = 3$ as in type A . On columns $m+1 = 5$ through n , we then carry out the same column operations as

in the case $|\beta| = 0$ (compare Example 7.8), to obtain:

$$P'_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & -4 & 1 & 0 & -1 & 0 \\ & & & 0 & 1 & -1 \\ & & 2 & 0 & 0 \\ & & & 1 & 0 & -1 & 0 \\ & & & 0 & 1 & \boxed{-2} \\ & & & 1 & 0 & 0 \end{pmatrix}.$$

In this case, since $\delta_1 \geq 3$, we obtain the boxed pivot -2 in the $(n-1, n)$ -entry. We can then use further column operations to move $(-\gcd(\beta_k)) = -4$ down to row $n-1$, and then clear the $(n-1, m-1)$ -entry and replace the boxed pivot by $\gcd(\beta_k, 2) \in \{1, 2\}$. In this example, since $\gcd(\beta_k, 2) = 2$, we obtain:

$$Q_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & -1 & 0 \\ & & 0 & 1 & -1 \\ & & 2 & 0 & 0 \\ & & & 1 & 0 & -1 & 0 \\ & & & 0 & 1 & 2 \\ & & & 1 & 0 & 0 \end{pmatrix}.$$

Notice that the $(n-m) \times (n-m)$ minor in the bottom right (here $n-m=6$) does not change at this step in the case $\gcd(\beta_k, 2) = 2$. Since we have only used column operations so far, and the nonzero columns of $Q_{\beta,\delta}$ are linearly independent over \mathbb{Z} , a basis for $\text{MOD}(w_{\beta,\delta})$ can be read off from the columns of $Q_{\beta,\delta}$.

From $Q_{\beta,\delta}$, the Smith normal form is obtained similarly to the previous three examples. We note that in this example, since all parts of $\delta = (3, 3)$ have the same parity, the $(n-m) \times (n-m)$ minor in the bottom right has Smith normal form $\text{diag}(1^4, 2^2)$. Hence the Smith normal form of the entire matrix $P_{\beta,\delta}$ is given by $S_{\beta,\delta} = \text{diag}(1^7, 2^2, 0)$, which confirms part (1) of Theorem 7.1.

Example 7.19. This example illustrates part (3) of Theorem 7.1, which considers the case $\gcd(\beta_k, 2) = 1$ and $|\delta| \geq 2$, when $\beta_p \geq 2$ and $\delta_1 \geq 3$. The argument is similar for $\delta_1 \in \{1, 2\}$.

Let W be of type D_9 and let $\beta = (3)$ and $\delta = (3, 3)$. By (7.1.5), we have

$$w_{\beta,\delta} = w_\beta w_\delta = (s_1 s_2)(u_0 s_7 \cdot u_3 s_5 s_4).$$

Here, the matrix for $I - w_\delta$ with respect to Δ^\vee is given by

$$P_{\beta,\delta} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & -1 & 2 & -1 & 0 \\ & & -1 & 1 & 1 & -1 \\ & & -2 & 2 & 0 & 0 \\ & & -2 & 2 & -1 & 2 & -1 & -1 \\ & & -1 & 1 & 0 & 0 & 1 & -1 \\ & & -1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

We proceed similarly to the previous example, with $|\beta| = m = 3$ here. We move $(-\gcd(\beta_k)) = -3$ down to row $n-1$, and then clear the $(n-1, m-1)$ -entry and replace the $(n-1, n)$ -entry by $\gcd(\beta_k, 2) \in \{1, 2\}$. Now, since $\gcd(\beta_k, 2) = 1$ in this example, we obtain a pivot entry 1 in column n :

$$P'_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & -1 & 0 \\ & & 0 & 1 & -1 \\ & & 2 & 0 & 0 \\ & & & 1 & 0 & -1 & 0 \\ & & & 0 & 1 & \boxed{1} \\ & & & 1 & 0 & 0 \end{pmatrix}.$$

We can now use column operations and this boxed pivot entry 1 to clear many entries, moving up the matrix, to obtain:

$$Q_{\beta,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & -1 & 0 \\ & & 0 & 0 & -1 \\ & & 2 & 0 & 0 \\ & & & 0 & -1 & 0 \\ & & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \end{pmatrix},$$

from which a basis for $\text{MOD}(w_{\beta,\delta})$ can be easily read off. In addition, the Smith normal form is clearly given by $S_{\beta,\delta} = \text{diag}(1^{n-2r-p+1}, 2^{2r-1}, 0^p) = \text{diag}(1^7, 2, 0)$, confirming part (3) of Theorem 7.1.

7.3.4. Noncuspidal examples with $|\delta| = 0$. We conclude by considering the situation where $|\delta| = 0$ and $|\beta| = n$. If β has at least one odd part, then $w_{\beta,\delta} = w_\beta$ is cuspidal in the type A_{n-1} subsystem indexed by the first $n-1$ nodes. If all parts of β are even, then $w_{\beta,\delta} = w_{\beta^+}$ is also cuspidal in the type A_{n-1} subsystem indexed by the first $n-1$ nodes, while $w_{\beta,\delta} = w_{\beta^-}$ is obtained from w_{β^+} by replacing s_{n-1} by s_n , hence w_{β^-} is cuspidal in the type A_{n-1} subsystem indexed by $\{s_1, \dots, s_{n-2}, s_n\}$.

The following examples illustrate the proofs of part (4) of Theorem 7.1 and parts (1) and (2) of Theorem 7.3; see the introduction to Section 7.3 for the list of examples we consider.

Example 7.20. This example illustrates part (4) of Theorem 7.1, and part (1) of Theorem 7.3. Let W be of type D_4 , and let $\beta = (3, 1)$ and $|\delta| = 0$. By Proposition 4.1, we have

$$w_{\beta,\delta} = w_\beta = s_1 s_2.$$

Applying (2.1.1), the matrix for $I - w_\beta$ with respect to Δ^\vee is given by P_β below:

$$P_\beta = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow Q_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the last row of P_β is all 0s, while the $(n-1) \times (n-1)$ minor in the top left (for $n=4$) is the same as the matrix M_β from case (2) of Corollary 4.17 in type A_{n-1} , since the last part of β equals 1. Also, the last two columns of P_β are identical.

Since the last two columns of P_β are identical, we subtract column $(n-1)$ from column n , and then work as in type A_{n-1} in the top left minor, to obtain Q_β using only column operations. The resulting basis for $\text{MOD}(w_{\beta,\delta}) = \text{MOD}(w_\beta)$ is given then by part (1) of Theorem 7.3, using the first case of part (2) of Theorem 4.2.

The Smith normal form for $(I - w_{\beta,\delta}) = (I - w_\beta)$ will also be the same as that corresponding to β in type A_{n-1} , with one extra 0. By part (3) of Theorem 4.2, and using the fact that $\gcd(\beta_k, 2) = 1$, we thus have $S_{\beta,\delta} = (1, 1, 0, 0)$, confirming part (4) of Theorem 7.1.

Example 7.21. This example illustrates part (4) of Theorem 7.1, and part (2) of Theorem 7.3. Let W be of type D_4 , and let $\beta = (2, 2)$ and $|\delta| = 0$. Note that all parts of β are even, so there are two cases β^\pm to consider. By Proposition 4.1, we have

$$w_{\beta^\pm, \delta} = w_{\beta^\pm} = s_1 s_3.$$

Applying (2.1.1), the matrix for $I - w_{\beta^\pm}$ with respect to Δ^\vee is given by P_{β^\pm} below:

$$P_{\beta^\pm} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow Q_{\beta^\pm} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the last column and last row of P_{β^\pm} are all 0s, while the $(n-1) \times (n-1)$ minor in the top left (for $n=4$) is same as the matrix M_β from case (1) of Corollary 4.17 in type A_{n-1} .

Since the last row and last column of P_{β^\pm} are all 0s, we can work in the $(n-1) \times (n-1)$ minor in the top left exactly as in type A_{n-1} , to obtain Q_{β^\pm} using only column operations. More precisely, following the same column operations as used to obtain the matrix B'_β in Example 4.4, we first add column 1 to column 2. Since here each block M_{β_k} is already in the form B_{β_k} from Example 4.3, we proceed to move the part $\beta_1 = 2$ to the bottom nonzero row, by subtracting twice column 2 from column 1. The pivot $\beta_2 = 2$ in the third column can then clear the column 1, giving Q_{β^\pm} above.

The resulting basis for $\text{MOD}(w_{\beta^\pm, \delta}) = \text{MOD}(w_{\beta^\pm})$ is therefore

$$\{\alpha_1^\vee - \alpha_3^\vee, 2\alpha_3^\vee\}.$$

Since $J_\beta = \{1, 3\}$ and $I_\beta = \{2, 4\}$, then $J_\beta \setminus (I_\beta - 1) = \emptyset$, and so this basis matches the second case of part (2) of Theorem 4.2 with $n=3$, equivalently part (2) of Theorem 7.3 with $n=4$.

The Smith normal form for $(I - w_{\beta^+, \delta}) = (I - w_{\beta^+})$ will then be the same as that corresponding to β in type A_{n-1} , with one extra 0. By part (3) of Theorem 4.2, we thus have $S_{\beta^+, \delta} = (1, 2, 0, 0)$, confirming part (4) of Theorem 7.1.

We now consider the case of β^- . We switch the s_3 for s_4 in w_{β^+} and so obtain

$$w_{\beta^-, \delta} = w_{\beta^-} = s_1 s_4.$$

The matrix for $I - w_{\beta^-}$ with respect to Δ^\vee can then be obtained from P_{β^+} by exchanging both rows and columns $n-1$ and n :

$$P_{\beta^-} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \rightarrow Q_{\beta^-} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

Since P_{β^-} is equivalent to P_{β^+} up to row and column operations, then clearly $S_{\beta^-, \delta} = S_{\beta^+, \delta} = (1, 2, 0, 0)$, confirming part (4) of Theorem 7.1.

Similarly, using the same column operations as performed on P_{β^+} , we obtain Q_{β^-} , which differs only from Q_{β^+} by exchanging rows $n-1$ and n , confirming the description for $\text{MOD}(w_{\beta^-, \delta}) = \text{MOD}(w_{\beta^-})$ provided by part (2) of Theorem 7.3.

Example 7.22. This example illustrates part (4) of Theorem 7.1, and part (2) of Theorem 7.3. Let W be of type D_4 and let $\beta = (4)$ and $|\delta| = 0$. Note that all parts of β are even, so there are two cases β^\pm to consider. By Proposition 4.1, we have

$$w_{\beta^+, \delta} = w_{\beta^+} = s_1 s_2 s_3,$$

and applying (2.1.1) says that the matrix for $I - w_{\beta^+}$ with respect to Δ^\vee is

$$P_{\beta^+} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the last row of this matrix is all 0s, while the $(n-1) \times (n-1)$ minor in the top left (for $n=4$) is the same as the matrix M_β from case (1) of Corollary 4.17 in type A_{n-1} . Also, the last column of P_{β^+} is the negative of its second-last column, except for the $(n-1, n)$ -entry, which equals 0.

Upon adding column $n-1$ to column n we obtain P'_{β^+} below:

$$P'_{\beta^+} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow P''_{\beta^+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & \boxed{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow Q_{\beta^+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can now work on the first $(n-1)$ rows and columns of P'_{β^+} as in type A_{n-1} , without changing column n , to obtain P''_{β^+} . The boxed 2 in row 3 can be used to clear column 3 entirely in this case, since here $\gcd(\beta_k, 2) = 2$. The resulting basis for $\text{MOD}(w_{\beta^+, \delta}) = \text{MOD}(w_{\beta^+})$ is given by part (2) of Theorem 7.3, noting that $J_\beta = \{1, 2, 3\}$ and $I_\beta = \{4\}$.

Finally, the Smith normal form for $(I - w_{\beta^+, \delta})$ is the same as in type A_{n-1} , except that $\gcd(\beta_k, 2)$ replaces $\gcd(\beta_k)$ due to the pivot 2 in column 4, and we have one extra 0. By part (3) of Theorem 4.2, we thus have $S_{\beta^+, \delta} = (1, 1, 2, 0)$, confirming part (4) of Theorem 7.1.

The argument for $w_{\beta^-} = s_1 s_2 s_4$ is the same as in the previous example.

8. EXCEPTIONAL TYPES

In this section, we record an implicit description of all mod-sets in the exceptional types, for the sake of completeness. We provide a complete system of minimal length representatives for all conjugacy classes of W , where W is the finite Weyl group of type E_6, E_7, E_8, F_4 , or G_2 . For each representative $w \in W$, we present the Smith normal form S_w for $(\mathbf{I} - w)$ with respect to Δ^\vee , which is both canonical and fully characterizes the isomorphism type of the quotient $R^\vee/\text{MOD}(w)$. In the exceptional types, the Smith normal form S_w was calculated using the `smith_form()` command in Sage [Sag24].

For all exceptional types, we index the nodes of the Dynkin diagram using the common labeling from Bourbaki [Bou02], Geck and Pfeiffer [GP00], and Sage [Sag24]; see Table 9 in Appendix B. For brevity, we denote the product $s_i s_j$ by s_{ij} in our tables.

8.1. Type E_6 . To obtain a system of minimal length representatives for the 25 conjugacy classes in W of type E_6 , we use Table B.4 in [GP00] to identify the 5 cuspidal classes. We then iterate over all proper parabolic subgroups of W and choose representatives for the cuspidal conjugacy classes in each parabolic, using the classification in [GP00] for types A and D . See Table 1 for a summary of our results in type E_6 .

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\dots)$
1	1	(0,0,0,0,0,0)
s_1	A_1	(1,0,0,0,0,0)
s_{12}	$A_1 \times A_1$	(1,1,0,0,0,0)
s_{13}	A_2	(1,1,0,0,0,0)
s_{125}	$A_1 \times A_1 \times A_1$	(1,1,1,0,0,0)
s_{132}	$A_2 \times A_1$	(1,1,1,0,0,0)
s_{134}	A_3	(1,1,1,0,0,0)
s_{3456}	A_4	(1,1,1,1,0,0)
s_{3256}	$A_1 \times A_1 \times A_2$	(1,1,1,1,0,0)
s_{1346}	$A_3 \times A_1$	(1,1,1,1,0,0)
s_{1356}	$A_2 \times A_2$	(1,1,1,3,0,0)
s_{13456}	A_5	(1,1,1,1,3,0)
s_{12456}	$A_4 \times A_1$	(1,1,1,1,1,0)
s_{13256}	$A_2 \times A_2 \times A_1$	(1,1,1,1,3,0)
s_{2345}	D_4	(1,1,1,1,0,0)
s_{242345}	D_4	(1,1,1,1,0,0)
$s_{234234542345}$	D_4	(1,1,2,2,0,0)
s_{23456}	D_5	(1,1,1,1,1,0)
$s_{2423456}$	D_5	(1,1,1,1,1,0)
$s_{2342345423456}$	D_5	(1,1,1,2,2,0)
s_{123456}	E_6	(1,1,1,1,1,3)
$s_{12342345246}$	E_6	(1,1,1,1,1,3)
$s_{12342345423456}$	E_6	(1,1,1,1,2,6)
$s_{123142314542314565423456}$	E_6	(1,1,1,3,3,3)

TABLE 1. Smith normal form S_w for $(\mathbf{I} - w)$ with $w \in W$ of type E_6 .

8.2. Type E_7 . To obtain a system of minimal length representatives for the 60 conjugacy classes in W of type E_7 , we use Table B.5 in [GP00] to identify the 12 cuspidal classes. We then iterate over all proper parabolic subgroups of W and choose representatives for the cuspidal conjugacy classes in each parabolic, using the classification in [GP00] for types A, D , and E_6 . See Tables 2 and 3 for a summary of our results in type E_7 .

8.3. Type E_8 . To obtain a system of minimal length representatives for the 112 conjugacy classes in W of type E_8 , we use Table B.6 in [GP00] to identify the 30 cuspidal classes. We then iterate over all proper parabolic subgroups of W and choose representatives for the cuspidal conjugacy classes in each parabolic, using the classification in [GP00] for types A, D, E_6 , and E_7 . See Tables 4, 5, and 6 for a summary of our results in type E_8 .

8.4. Type F_4 . To obtain a system of minimal length representatives for the 25 conjugacy classes in W of type F_4 , we use Table B.3 in [GP00] to identify the 9 cuspidal classes. We then iterate over all proper parabolic subgroups of W and choose representatives for the cuspidal conjugacy classes in each parabolic, using the classification in [GP00] for types A, B , and C . See Table 7 for a summary of our results in type F_4 .

8.5. Type G_2 . To obtain a system of minimal length representatives for the 6 conjugacy classes in W of type G_2 , we calculate the conjugacy classes directly by hand, due to the small size of the group. See Table 8 for a summary of our results in type G_2 .

APPENDIX A. CRYSTALLOGRAPHIC GROUPS AND AFFINE COXETER GROUPS

In this appendix we briefly discuss the relationship between split n -dimensional crystallographic groups $H = T_H \rtimes H_0$ and affine Coxeter groups. For any such H , the *holohedry group* of the associated lattice L_H is its setwise stabilizer in $O(n)$. Thus as L_H is H_0 -invariant, H_0 is a subgroup of the holohedry group of L_H .

In dimension 2, it can be verified from the classification of wallpaper groups that every (split) wallpaper group is contained in an affine Coxeter group. More precisely, up to affine equivalence every 2-dimensional lattice L_H is either a square lattice or a hexagonal lattice. These lattices have holohedry groups the finite Weyl groups of types C_2 and G_2 , respectively. Hence every 2-dimensional split crystallographic group H is contained in an affine Coxeter group of type \tilde{C}_2 or \tilde{G}_2 .

Now let $H = T_H \rtimes H_0$ be a split 3-dimensional crystallographic group. Then the lattice L_H is, up to affine equivalence, a special case of one of the four “maximal lattices” given by [Gri15, Figure 2]. From the information in [Gri15], it can be checked that the holohedry group of each of these four “maximal lattices” is a finite Weyl group of type either C_3 or $A_1 \times G_2$, and that each such lattice is, up to affine equivalence, the coroot lattice of the same type as the corresponding Weyl group. It follows that in dimension 3, any split crystallographic H is contained in an affine Coxeter group of type either \tilde{C}_3 or $\tilde{A}_1 \times \tilde{G}_2$.

In dimension 4, however, there exist split crystallographic groups $H = T_H \rtimes H_0$ such that the reflections in the holohedry group of L_H generate a proper subgroup of H_0 (see, for example, [Hor06]). Since any finite Weyl group preserving L_H must be generated by reflections which preserve L_H , it follows that H is not contained in any affine Coxeter group. Then for all dimensions $n \geq 5$, one can take direct products to obtain examples which are also not contained in affine Coxeter groups.

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\dots)$
1	1	(0,0,0,0,0,0)
s_1	A_1	(1,0,0,0,0,0)
s_{12}	$A_1 \times A_1$	(1,1,0,0,0,0)
s_{13}	A_2	(1,1,0,0,0,0)
s_{132}	$A_2 \times A_1$	(1,1,1,0,0,0)
s_{134}	A_3	(1,1,1,0,0,0)
s_{125}	$A_1 \times A_1 \times A_1$	(1,1,1,0,0,0)
s_{257}	$A_1 \times A_1 \times A_1$	(1,1,2,0,0,0)
s_{4567}	A_4	(1,1,1,1,0,0)
s_{3567}	$A_3 \times A_1$	(1,1,1,1,0,0)
s_{2567}	$A_3 \times A_1$	(1,1,1,2,0,0)
s_{3467}	$A_2 \times A_2$	(1,1,1,1,0,0)
s_{2367}	$A_2 \times A_1 \times A_1$	(1,1,1,1,0,0)
s_{2357}	$A_1 \times A_1 \times A_1 \times A_1$	(1,1,1,2,0,0)
s_{34567}	A_5	(1,1,1,1,1,0)
s_{24567}	A_5	(1,1,1,1,2,0)
s_{14567}	$A_4 \times A_1$	(1,1,1,1,1,0)
s_{24367}	$A_3 \times A_2$	(1,1,1,1,1,0)
s_{12467}	$A_2 \times A_2 \times A_1$	(1,1,1,1,1,0)
s_{23567}	$A_3 \times A_1 \times A_1$	(1,1,1,1,2,0)
s_{13257}	$A_2 \times A_1 \times A_1 \times A_1$	(1,1,1,1,2,0)
s_{134567}	A_6	(1,1,1,1,1,1)
s_{124567}	$A_5 \times A_1$	(1,1,1,1,1,2)
s_{134267}	$A_4 \times A_2$	(1,1,1,1,1,1)
s_{132567}	$A_3 \times A_2 \times A_1$	(1,1,1,1,1,2)
s_{2345}	D_4	(1,1,1,1,0,0)
s_{242345}	D_4	(1,1,1,1,0,0)
$s_{234234542345}$	D_4	(1,1,2,2,0,0)
s_{23457}	$D_4 \times A_1$	(1,1,1,1,2,0)
$s_{2423457}$	$D_4 \times A_1$	(1,1,1,1,2,0)
$s_{2342345423457}$	$D_4 \times A_1$	(1,1,2,2,2,0)
s_{23456}	D_5	(1,1,1,1,1,0)
$s_{2423456}$	D_5	(1,1,1,1,1,0)
$s_{2342345423456}$	D_5	(1,1,1,2,2,0)
s_{254317}	$D_5 \times A_1$	(1,1,1,1,1,2)
$s_{24254317}$	$D_5 \times A_1$	(1,1,1,1,1,2)
$s_{25425434254317}$	$D_5 \times A_1$	(1,1,1,2,2,2)
s_{234567}	D_6	(1,1,1,1,1,2)
$s_{24234567}$	D_6	(1,1,1,1,1,2)
$s_{2454234567}$	D_6	(1,1,1,1,1,2)
$s_{23423454234567}$	D_6	(1,1,1,2,2,2)
$s_{2342345654234567}$	D_6	(1,1,1,1,2,4)
$s_{234234542345654234567654234567}$	D_6	(1,2,2,2,2,2)
s_{123456}	E_6	(1,1,1,1,1,1)
$s_{12342546}$	E_6	(1,1,1,1,1,1)
$s_{123142345465}$	E_6	(1,1,1,1,1,1)
$s_{12342345423456}$	E_6	(1,1,1,1,2,2)
$s_{123142314542314565423456}$	E_6	(1,1,1,1,3,3)

TABLE 2. Smith normal form S_w for $(\mathbf{I} - w)$ with $w \in W$ non-cuspidal in type E_7 .

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\dots)$
$s_{1234567}$	E_7	(1,1,1,1,1,1,2)
$s_{123425467}$	E_7	(1,1,1,1,1,1,2)
$s_{12342546576}$	E_7	(1,1,1,1,1,1,2)
$s_{1234254234567}$	E_7	(1,1,1,1,1,1,2)
$s_{123423454234567}$	E_7	(1,1,1,1,2,2,2)
$s_{12342345423456576}$	E_7	(1,1,1,1,1,2,4)
$s_{123142314354234654765}$	E_7	(1,1,1,1,1,1,2)
$s_{12314231435423143546576}$	E_7	(1,1,1,1,2,2,2)
$s_{1231423145423145654234567}$	E_7	(1,1,1,1,1,3,6)
$s_{1234234542345654234567654234567}$	E_7	(1,1,2,2,2,2,2)
$s_{123142345423145654234567654234567}$	E_7	(1,1,1,1,2,4,4)
$w_0 = s_{765432456713456245341324567134562453413245624534132453413241321}$	E_7	(2,2,2,2,2,2,2)

TABLE 3. Smith normal form S_w for $(I - w)$ with $w \in W$ cuspidal in type E_7 .

APPENDIX B. DYNKIN DIAGRAM LABELING CONVENTIONS

We gather in Table 9 several useful conventions for labeling the nodes of the Dynkin diagrams for finite Weyl groups. In the second column of this table, we give the labelings used in Plates I–IX of Bourbaki [Bou02], which are the same as the labelings in Sage [Sag24]. These are the conventions we follow throughout this work. Note, however, that in Sage [Sag24], the Cartan matrices in types B_n and C_n are swapped relative to the Dynkin diagrams, so that to carry out calculations in type B_n as labeled below, one should call type C_n , and vice versa.

In the third column, for the classical types we give the labelings used by Geck and Pfeiffer in Chapter 3 of [GP00], since their characterization of the conjugacy classes in these types uses their conventions in Chapter 3. Note that the labeling used in Chapter 3 of [GP00] is different to that given in Table 1.2 of [GP00] in types B_n , C_n , and D_n , however, due to the embedding of the Weyl group of type D_n as a normal subgroup of index 2 in that of type B_n . For the exceptional types, we give the common labelings used in both Table 1.2 and Appendix B of [GP00], which agree with those in Bourbaki [Bou02] and Sage [Sag24].

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\dots)$
1	1	(0,0,0,0,0,0,0)
s_1	A_1	(1,0,0,0,0,0,0)
s_{13}	A_2	(1,1,0,0,0,0,0)
s_{14}	$A_1 \times A_1$	(1,1,0,0,0,0,0)
s_{134}	A_3	(1,1,1,0,0,0,0)
s_{135}	$A_2 \times A_1$	(1,1,1,0,0,0,0)
s_{146}	$A_1 \times A_1 \times A_1$	(1,1,1,0,0,0,0)
s_{1345}	A_4	(1,1,1,1,0,0,0)
s_{1346}	$A_3 \times A_1$	(1,1,1,1,0,0,0)
s_{1356}	$A_2 \times A_2$	(1,1,1,1,0,0,0)
s_{1357}	$A_2 \times A_1 \times A_1$	(1,1,1,1,0,0,0)
s_{1468}	$A_1 \times A_1 \times A_1 \times A_1$	(1,1,1,1,0,0,0)
s_{13456}	A_5	(1,1,1,1,1,0,0)
s_{13457}	$A_4 \times A_1$	(1,1,1,1,1,0,0)
s_{13467}	$A_3 \times A_2$	(1,1,1,1,1,0,0)
s_{13468}	$A_3 \times A_1 \times A_1$	(1,1,1,1,1,0,0)
s_{13568}	$A_2 \times A_2 \times A_1$	(1,1,1,1,1,0,0)
s_{13257}	$A_2 \times A_1 \times A_1 \times A_1$	(1,1,1,1,1,0,0)
s_{134567}	A_6	(1,1,1,1,1,1,0)
s_{134568}	$A_5 \times A_1$	(1,1,1,1,1,1,0)
s_{134578}	$A_4 \times A_2$	(1,1,1,1,1,1,0)
s_{567812}	$A_4 \times A_1 \times A_1$	(1,1,1,1,1,1,0)
s_{134678}	$A_3 \times A_3$	(1,1,1,1,1,1,0)
s_{678124}	$A_3 \times A_2 \times A_1$	(1,1,1,1,1,1,0)
s_{135628}	$A_2 \times A_2 \times A_1 \times A_1$	(1,1,1,1,1,1,0)
$s_{1345678}$	A_7	(1,1,1,1,1,1,1)
$s_{1245678}$	$A_6 \times A_1$	(1,1,1,1,1,1,1)
$s_{1342678}$	$A_4 \times A_3$	(1,1,1,1,1,1,1)
$s_{1325678}$	$A_4 \times A_2 \times A_1$	(1,1,1,1,1,1,1)
s_{2345}	D_4	(1,1,1,1,0,0,0)
s_{242345}	D_4	(1,1,1,1,0,0,0)
$s_{234234542345}$	D_4	(1,1,2,2,0,0,0)
s_{23457}	$D_4 \times A_1$	(1,1,1,1,1,0,0)
$s_{2423457}$	$D_4 \times A_1$	(1,1,1,1,1,0,0)
$s_{2342345423457}$	$D_4 \times A_1$	(1,1,1,2,2,0,0)
s_{234578}	$D_4 \times A_2$	(1,1,1,1,1,1,0)
$s_{24234578}$	$D_4 \times A_2$	(1,1,1,1,1,1,0)
$s_{23423454234578}$	$D_4 \times A_2$	(1,1,1,1,2,2,0)
s_{23456}	D_5	(1,1,1,1,1,0,0)
$s_{2423456}$	D_5	(1,1,1,1,1,0,0)
$s_{2342345423456}$	D_5	(1,1,1,2,2,0,0)

TABLE 4. Smith normal form S_w for $(\mathbf{I} - w)$ with $w \in W$ non-cuspidal in type E_8 (continued in Table 5).

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\cdots)$
s_{234568}	$D_5 \times A_1$	(1,1,1,1,1,1,0,0)
$s_{24234568}$	$D_5 \times A_1$	(1,1,1,1,1,1,0,0)
$s_{23423454234568}$	$D_5 \times A_1$	(1,1,1,1,2,2,0,0)
$s_{2543178}$	$D_5 \times A_2$	(1,1,1,1,1,1,1,0)
$s_{242543178}$	$D_5 \times A_2$	(1,1,1,1,1,1,1,0)
$s_{254254342543178}$	$D_5 \times A_2$	(1,1,1,1,1,2,2,0)
s_{234567}	D_6	(1,1,1,1,1,1,0,0)
$s_{24234567}$	D_6	(1,1,1,1,1,1,0,0)
$s_{2454234567}$	D_6	(1,1,1,1,1,1,0,0)
$s_{23423454234567}$	D_6	(1,1,1,1,2,2,0,0)
$s_{2342345654234567}$	D_6	(1,1,1,1,2,2,0,0)
$s_{234234542345654234567654234567}$	D_6	(1,1,2,2,2,2,0,0)
$s_{2345678}$	D_7	(1,1,1,1,1,1,1,0)
$s_{242345678}$	D_7	(1,1,1,1,1,1,1,0)
$s_{24542345678}$	D_7	(1,1,1,1,1,1,1,0)
$s_{234234542345678}$	D_7	(1,1,1,1,1,2,2,0)
$s_{23423456542345678}$	D_7	(1,1,1,1,1,2,2,0)
$s_{234542345676542345678}$	D_7	(1,1,1,1,1,2,2,0)
$s_{2342345423456542345676542345678}$	D_7	(1,1,1,2,2,2,2,0)
s_{123456}	E_6	(1,1,1,1,1,1,0,0)
$s_{12342546}$	E_6	(1,1,1,1,1,1,0,0)
$s_{123142345465}$	E_6	(1,1,1,1,1,1,0,0)
$s_{12342345423456}$	E_6	(1,1,1,1,2,2,0,0)
$s_{123142314542314565423456}$	E_6	(1,1,1,1,3,3,0,0)
$s_{1234568}$	$E_6 \times A_1$	(1,1,1,1,1,1,1,0)
$s_{123425468}$	$E_6 \times A_1$	(1,1,1,1,1,1,1,0)
$s_{1231423454658}$	$E_6 \times A_1$	(1,1,1,1,1,1,1,0)
$s_{123423454234568}$	$E_6 \times A_1$	(1,1,1,1,1,2,2,0)
$s_{1231423145423145654234568}$	$E_6 \times A_1$	(1,1,1,1,1,3,3,0)
$s_{1234567}$	E_7	(1,1,1,1,1,1,1,0)
$s_{123425467}$	E_7	(1,1,1,1,1,1,1,0)
$s_{12342546576}$	E_7	(1,1,1,1,1,1,1,0)
$s_{1234254234567}$	E_7	(1,1,1,1,1,1,1,0)
$s_{123423454234567}$	E_7	(1,1,1,1,1,2,2,0)
$s_{12342345423456576}$	E_7	(1,1,1,1,1,2,2,0)
$s_{123142314354234654765}$	E_7	(1,1,1,1,1,1,1,0)
$s_{12314231435423143546576}$	E_7	(1,1,1,1,1,2,2,0)
$s_{1231423145423145654234567}$	E_7	(1,1,1,1,1,3,3,0)
$s_{1234234542345654234567654234567}$	E_7	(1,1,1,2,2,2,2,0)
$s_{123142345423145654234567654234567}$	E_7	(1,1,1,1,4,4,0)
$s_{765432456713456245341324567134562453413245624534132453413241321}$	E_7	(1,2,2,2,2,2,2,0)

TABLE 5. Smith normal form S_w for $(\mathbf{I} - w)$ with $w \in W$ non-cuspidal in type E_8 (continued from Table 4).

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\dots)$
$s_{12345678}$	E_8	(1,1,1,1,1,1,1,1)
$s_{1234254678}$	E_8	(1,1,1,1,1,1,1,1)
$s_{123425465478}$	E_8	(1,1,1,1,1,1,1,1)
$s_{12342542345678}$	E_8	(1,1,1,1,1,1,1,1)
$s_{1231423454657658}$	E_8	(1,1,1,1,1,1,1,1)
$s_{1234234542345678}$	E_8	(1,1,1,1,1,1,2,2)
$s_{123423454234565768}$	E_8	(1,1,1,1,1,1,2,2)
$s_{12314234542365476548}$	E_8	(1,1,1,1,1,1,1,1)
$s_{1234234542345654765876}$	E_8	(1,1,1,1,1,1,2,2)
$s_{1231423454316542345678}$	E_8	(1,1,1,1,1,1,1,1)
$s_{123142314542345654765876}$	E_8	(1,1,1,1,1,1,1,1)
$s_{123142314354316542345678}$	E_8	(1,1,1,1,1,1,2,2)
$s_{12314231454231456542345678}$	E_8	(1,1,1,1,1,1,3,3)
$s_{12342354234654276542345678}$	E_8	(1,1,1,1,1,1,2,2)
$s_{1231423145423145654234567687}$	E_8	(1,1,1,1,1,1,2,2)
$s_{123142314354231465423476548765}$	E_8	(1,1,1,1,1,1,2,2)
$s_{12342345423456542345676542345678}$	E_8	(1,1,1,1,2,2,2,2)
$s_{1231423454231456542345676542345678}$	E_8	(1,1,1,1,1,1,4,4)
$s_{1231423143542314565423145676542345678765}$	E_8	(1,1,1,1,1,1,1,1)
$s_{123142314354231435426542314567654234567876}$	E_8	(1,1,1,1,1,1,2,2)
$s_{12314231435423456542314567654231435465768765}$	E_8	(1,1,1,1,2,2,2,2)
$s_{12314231454231436542314354265431765423456787}$	E_8	(1,1,1,1,1,1,3,3)
$s_{1231423145423145654234567654234567876542345678}$	E_8	(1,1,1,1,1,1,6,6)
$s_{1231423154231654317654231435426543176542345678}$	E_8	(1,1,1,1,1,1,4,4)
$s_{123142314542314565423145676542314567876542345678}$	E_8	(1,1,1,1,1,1,5,5)
$s_{123142314354231435426542345765423143548765423143542654765876}$	E_8	(1,1,1,1,2,2,2,2)
$s_{1231423143542314354265423143542654317654231435426543176542345678}$	E_8	(1,1,2,2,2,2,2,2)
$s_{123142314354231435426542314354265431765423143542654317876542345678$	E_8	(1,1,1,1,2,2,4,4)
$s_{1231423143542314356542314354267654231435426543178765423}$	E_8	(1,1,1,1,3,3,3,3)
$\cdot s_{435426543176542345678765}$		
$w_0 = s_{876543245671345624534132456787654324567134562453413245678765}$	E_8	(2,2,2,2,2,2,2,2)
$\cdot s_{432456713456245341324567134562453413245624534132453413241321}$		

TABLE 6. Smith normal form S_w for $(\mathbf{I} - w)$ with $w \in W$ cuspidal in type E_8 .

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\dots)$
1	1	(0,0,0,0)
s_1	A_1	(1,0,0,0)
s_3	A_1	(1,0,0,0)
s_{12}	A_2	(1,1,0,0)
s_{34}	A_2	(1,1,0,0)
s_{13}	$A_1 \times A_1$	(1,1,0,0)
s_{134}	$A_1 \times A_2$	(1,1,1,0)
s_{124}	$A_2 \times A_1$	(1,1,1,0)
s_{23}	B_2	(1,1,0,0)
s_{3232}	B_2	(1,2,0,0)
s_{123}	B_3	(1,1,1,0)
s_{32321}	B_3	(1,1,2,0)
$s_{323212321}$	B_3	(1,2,2,0)
s_{234}	C_3	(1,1,1,0)
s_{23234}	C_3	(1,1,2,0)
$s_{232343234}$	C_3	(1,2,2,0)
s_{1234}	F_4	(1,1,1,1)
s_{123234}	F_4	(1,1,1,2)
$s_{12132343}$	F_4	(1,1,1,1)
$s_{1232343234}$	F_4	(1,1,2,2)
$s_{1213213234}$	F_4	(1,1,2,2)
$s_{121321343234}$	F_4	(1,1,2,2)
$s_{12132132343234}$	F_4	(1,1,2,4)
$s_{1213213432132343}$	F_4	(1,1,3,3)
$w_0 = s_{323212321432132343213234}$	F_4	(2,2,2,2)

TABLE 7. Smith normal form S_w for $(I - w)$ with $w \in W$ of type F_4 .

Representative $w \in [w]_W$	Type of $\text{Pc}(w)$	$S_w = \text{diag}(\dots)$
1	1	(0,0)
s_1	A_1	(1,0)
s_2	A_1	(1,0)
s_{12}	G_2	(1,1)
s_{1212}	G_2	(1,3)
$w_0 = s_{121212}$	G_2	(2,2)

TABLE 8. Smith normal form S_w for $(I - w)$ with $w \in W$ of type G_2 .

Type	Bourbaki [Bou02] & Sage [Sag24]	Geck and Pfeiffer [GP00]
$A_n, n \geq 1$		
$B_n, n \geq 2$		
$C_n, n \geq 2$		
$D_n, n \geq 4$		
E_6		
E_7		
E_8		
F_4		
G_2		

TABLE 9. Dynkin diagram labeling conventions

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