

Equalizing Closeness Centralities via Edge Additions

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Abstract

Graph modification problems with the goal of optimizing some measure of a given node’s network position have a rich history in the algorithms literature. Less commonly explored are modification problems with the goal of *equalizing* positions, though this class of problems is well-motivated from the perspective of equalizing social capital, i.e., algorithmic fairness. In this work, we study how to add edges to make the closeness centralities of a given pair of nodes more equal. We formalize two versions of this problem: CLOSENESS RATIO IMPROVEMENT, which aims to maximize the ratio of closeness centralities between two specified nodes, and CLOSENESS GAP MINIMIZATION, which aims to minimize the absolute difference of centralities. We show that both problems are NP-hard, and for CLOSENESS RATIO IMPROVEMENT we present a quasilinear-time $\frac{6}{11}$ -approximation, complemented by a bicriteria inapproximability bound. In contrast, we show that CLOSENESS GAP MINIMIZATION admits no multiplicative approximation unless P = NP. We conclude with a discussion of open directions for this style of problem, including several natural generalizations.

1 Introduction

An important class of problems in the *algorithmic fairness* literature presents as input a network (a graph, possibly with some algorithmically relevant metadata) and asks for an efficiently computed set of *network interventions* meant to alter the social dynamics of the network. Generally, the motivating assumption is that real-world resources accrue to people in prominent positions within social networks, and that the distribution of this so-called *social capital* hinges on network structure itself [31]. For a broad survey, see [42]. Work in this direction has generally proceeded along two (occasionally overlapping) lines of inquiry.

The first has focused on studying classic combinatorial optimization problems with *fairness constraints*, and exploring the algorithmic cost of these constraints by either proving impossibility results or lifting algorithmic techniques used for the classic unconstrained version of the problem. Fair clustering has received special emphasis, e.g., [3, 5, 4, 1, 30, 46, 20], especially since the seminal work of Chierichetti *et al.* [15]. Often, relatively tractable combinatorial problems are significantly harder when the solution space is constrained to enforce fairness, e.g., [14], though there have been some algorithmic success stories involving bicriteria approximations, e.g. [8, 1], or dual parameterizations, e.g., [34]. Somewhat more successful has been the trend of encoding fairness via a modified objective function, rather than by imposing additional constraints on a classic objective, e.g., [29, 1]. With this perspective, a series of works on minimax correlation clustering (beginning with [40], most recently [20]) as well as modification with “degree constraints,” e.g., [34, 2], can be seen through a fairness lens, even if fairness was not the original motivation. This line of work has been primarily theoretical, although an important practical insight, seen in e.g., [8, 20, 16], is that

the *cost* of fairness is often low in the sense that fair solutions are also often high-quality when evaluated by traditional criteria.

A second line of inquiry is distinguished by an empirical focus on information access, combined with some theoretical analysis, often in the context of some non-trivial diffusion model such as Independent Cascade. This has given rise to fair variants of, e.g., the classic influence maximization problem [26, 44]. Such interventions could include selecting vertices at which to seed information (in which case the intervention is a modification of the metadata, rather than the underlying graph) or modifying the graph directly through e.g., edge addition, including formal or empirical analysis of the information spreading properties of proposed heuristics [7, 45, 26, 44, 36, 33]. While the majority of works in this literature have introduced and experimentally evaluated heuristics, recently a greater interest in the analysis of algorithms has emerged, e.g., [10, 41]. This network-focused fairness work builds on a broader literature in algorithmic fairness that has considered both fairness as addressed through added constraints (e.g., [32]) and via modified objective functions (e.g., [28]) where some emphasis has been given to understanding which classes of objective functions are most socially desirable from a fairness perspective, e.g., [47].

Both lines of work can be divided further into approaches which emphasize *individual fairness* by optimizing a maximin or minimax objective over all nodes, e.g., [26, 40], or by imposing a per-node fairness constraint, e.g., [34], and approaches which emphasize *group fairness* by optimizing various measures based on demographic node attributes, e.g., [22, 44].

Our work is inspired by elements of both research trends. We take a group fairness approach and directly optimize a fairness objective, but we do so in a purely graph-theoretic context, and our approach is closely aligned with a classic discrete algorithms perspective. Instead of considering a complex model of information flow, we use *centrality* to measure a node’s social capital, e.g., in the spirit of [13]. Though many notions of centrality exist, we focus on *closeness centrality*, which possesses the dual advantages of being simple to understand and exhibiting many qualities desirable in a centrality measure¹ [12]. Thus, closeness is a natural starting point for the goal of algorithmically equalizing node centralities.

Formally, we consider a simple, unweighted, undirected graph G with two specified vertices a, b and an integer k . Our task is to add at most k edges to G while maximizing the ratio of the centralities of a and b , i.e., the best possible objective value is 1. Here, the closeness centrality $\text{cc}_G(v)$ of a vertex v is the sum of v ’s shortest-path distances to all other vertices.²

Naturally, our work is closely related to the vast algorithmic literature on edge-modification in graphs; see e.g., the recent survey by Crespelle *et al.* [18]. Especially relevant is the body of work on adding edges to optimize a given node’s centrality [9, 17, 43, 6, 39], or even to optimize a group’s centrality [37]. Conceptually, a major distinguishing feature of our work is that the ratio objective is not amenable (see Section 2.1) to approaches appearing frequently in the edge-modification literature, e.g., submodularity analysis or reductions to clustering problems [2, 11, 21, 35, 38, 27]. Thus, among our technical contributions is a demonstration of how to reason about the structure of hard instances in a way that can be harnessed algorithmically, even when the objective function is a ratio.

Our Results: We study two optimization problems related to balancing the closeness centrality of a given pair of nodes. In CLOSENESS RATIO IMPROVEMENT, the goal is to maximize the ratio of the closeness centralities of two given vertices with a budgeted number of edge additions. We prove that this problem is NP-hard for any target ratio τ in the interval $(\frac{1}{2}, 1]$, and trivial when $\tau \leq \frac{1}{2}$.

¹In our context (undirected connected graphs), the “main problem” with closeness centrality identified by [12] (non-reachability) does not occur.

²Closeness is also commonly defined as the reciprocal of this quantity. In the context of our objective (a ratio of closeness centralities), there is no meaningful distinction between these definitions. The impossibility result we establish for a gap-variant (Corollary 1) also holds for both definitions.

before building on these insights to give a $\frac{6}{11}$ -approximation. Additionally, we show that it is NP-hard to approximate CLOSENESS RATIO IMPROVEMENT within any factor better than $\frac{5e}{5e+1-\varepsilon} \approx 0.932$, and we generalize this result to give a $\frac{5e^c}{5e^c+1-\varepsilon}$ approximation bound for any algorithm which is allowed to add ck edges, where $c \geq 1$ is a constant. In CLOSENESS GAP MINIMIZATION, the goal is to minimize the absolute difference between the closeness centralities of two nodes, again with a budgeted number of edge additions. We prove that this problem is NP-hard and does not admit any multiplicative approximation algorithm. We conclude by proposing two new problems that extend this framework to balancing closeness centralities across groups of nodes or the entire graph, rather than focusing only on a fixed pair of vertices.

2 Motivation and Problem Definition

We introduce the problem of CLOSENESS RATIO IMPROVEMENT on a social network where people are modeled as vertices and edges represent connections between people. Specifically, let $G = (V, E)$ be an undirected graph with vertices V and edges E . Our goal will be to add a budget of k edges to G to equalize the closeness centrality of two vertices. We denote a graph $G = (V, E)$ augmented with an edge set $S \subseteq V^2 \setminus E$ as $G + S = (V, E \cup S)$. Within G , the shortest path length between two vertices $u, v \in V$ is represented as $d_G(u, v)$. Finally, we define the closeness centrality of a vertex $v \in V$ in a graph $G = (V, E)$ as $cc_G(v) = \sum_{u \in V} d_G(u, v)$, the sum of the shortest paths to each other vertex in the graph. Note that having a smaller closeness centrality value means a node is closer to more vertices in the graph, and thus more important or central in the graph.

In the problem of CLOSENESS RATIO IMPROVEMENT, we are given a graph G and vertices a, b , and the goal is to add the k edges to G which will maximize the ratio of a and b 's closeness centralities. We define the closeness ratio of a and b as $cc\text{-ratio}_G(a, b) = \frac{\min(cc_{G+S}(a), cc_{G+S}(b))}{\max(cc_{G+S}(a), cc_{G+S}(b))}$, so that we can guarantee our ratio is never more than 1.

CLOSENESS RATIO IMPROVEMENT

Input: A graph $G = (V, E)$, vertices $a, b \in V$, and a positive integer $k \in \mathbb{N}$.

Task: Find a set of edges S of size at most k which maximizes $cc\text{-ratio}_{G+S}(a, b)$

A key aspect of the problem definition is the choice to consider the ratio of closeness centralities instead of their difference. Another natural formulation is the following:

CLOSENESS GAP MINIMIZATION

Input: A graph $G = (V, E)$, vertices $a, b \in V$, and a positive integer $k \in \mathbb{N}$.

Task: Find a set of edges S of size at most k which minimizes $|cc_{G+S}(a) - cc_{G+S}(b)|$.

Within the fairness literature, both the ratio and difference are commonly considered variants for equalization of various measures [28], but recent work supports the use of ratio by default unless domain-specific concerns indicate otherwise [47]. In the context of closeness centrality equalization, we additionally find justification for the use of the ratio instead of the difference as, unlike CLOSENESS RATIO IMPROVEMENT, CLOSENESS GAP MINIMIZATION is unlikely to admit any multiplicative approximation algorithm, as we show in Section 3 (Corollary 1).

2.1 Intuition-Building Examples

We examine three natural strategies for solving CLOSENESS RATIO IMPROVEMENT and demonstrate that none of them yield good approximations. Observe Figure 1, and note that for every constant d , for sufficiently large X , the distances from a and b to vertices in IS_X dominate their

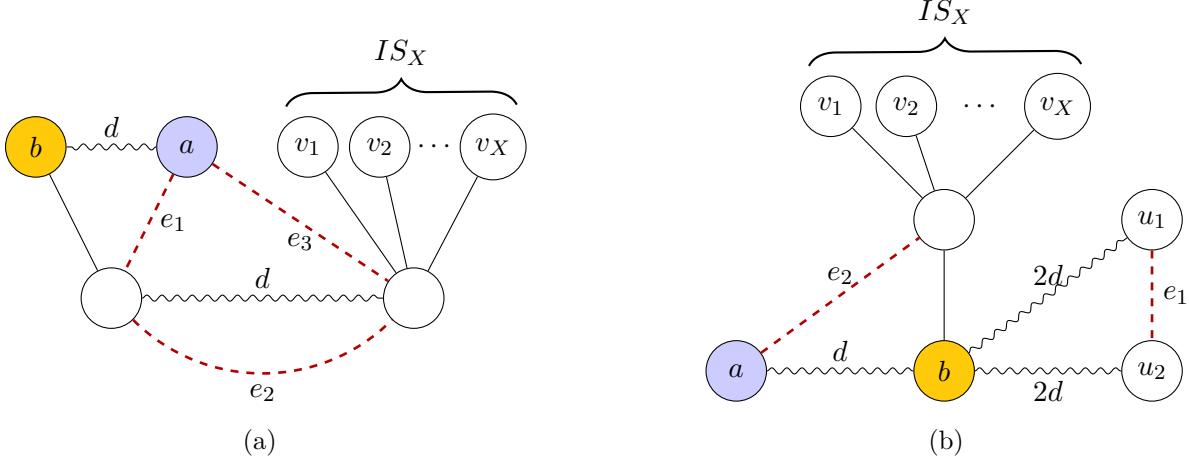


Figure 1: The constructions used for counterexamples in Section 2.1. Dashed edges are potential additions, squiggly lines are paths of length d , and IS_X denotes an independent set of size X .

closeness centralities. Hence, throughout the section we will simplify calculations by ignoring other asymptotically negligible distances.

Much of the existing work on centrality maximization relies on centrality objectives being monotone and submodular [17, 9, 37, 43], but our objective satisfies neither property. Observe that in Figure 1a, $\text{cc-ratio}_G(a, b) = \frac{d+2}{2d+2}$. Consider candidate edge additions e_1 and e_2 and the resulting ratios: $\text{cc-ratio}_{G+e_1}(a, b) = 1$, $\text{cc-ratio}_{G+e_2}(a, b) = \frac{3}{d+3}$, and $\text{cc-ratio}_{G+e_1+e_2}(a, b) = 1$. Certainly $\text{cc-ratio}_G(a, b) > \text{cc-ratio}_{G+e_2}(a, b)$ shows that our objective is not monotone. We also have that $\text{cc-ratio}_{G+e_1}(a, b) - \text{cc-ratio}_G(a, b) = 1 - \frac{d+2}{2d+2} = \frac{d}{2d+2}$, while $\text{cc-ratio}_{G+e_2+e_1}(a, b) - \text{cc-ratio}_{G+e_2}(a, b) = 1 - \frac{3}{d+3} = \frac{d}{d+3}$. The marginal gain from adding e_1 to the graph is larger after adding e_2 , and thus our objective is not submodular.

We also observe that the task of equalizing the centrality of two vertices is not necessarily the same as greedily improving the less central of the two (until they are equal). Again using Figure 1a, given that $\text{cc}_G(a) > \text{cc}_G(b)$, if we want to improve $\text{cc}_G(a)$ as much as possible with $k = 1$, we would add e_3 . But $\text{cc-ratio}_{G+e_3}(a, b) = \frac{2}{d+2}$ while $\text{cc-ratio}_{G+e_1}(a, b) = 1$, so this strategy does not lead to any constant-factor approximation.

Finally, we illustrate how our problem diverges from the well-researched problem of adding edges to a graph to minimize its diameter [21, 35, 2, 11, 27]. In Figure 1b, we claim that e_1 is a diameter-minimizing edge addition. Suppose otherwise; let pq be an edge addition achieving diameter strictly less than $3d$, and WLOG assume $d_G(a, p) < d_G(a, q)$. If $d_G(a, p) \geq d$, then $d_G(u_1, q) < 2d$ and $d_G(u_2, q) < 2d$, but no vertex satisfies both inequalities, so we may assume that $d_G(a, p) = d - x$ for some positive x . Now, WLOG assume that q lies on the path from b to u_2 , and let $d_G(u_2, q) = y$. We may assume that $x + y < d$, as otherwise $d_{G+pq}(u_1, u_2) \geq 3d$. But then it is simple to check that $d_{G+pq}(a, u_1) = 3d$. So, e_1 is diameter-minimizing, but $\text{cc-ratio}_{G+e_1}(a, b) = \frac{2}{d+2}$, while $\text{cc-ratio}_{G+e_2}(a, b) = 1$. It follows that minimizing the graph's diameter does not lead to any constant-factor approximation.

2.2 A Trivial $\frac{1}{2}$ -Approximation

We now present a simple observation: if $ab \in E$, then for all vertices v , $|d_G(a, v) - d_G(b, v)| \leq 1$. We will now show how to apply the triangle inequality to derive the following:

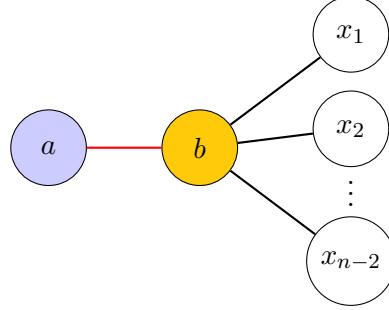


Figure 2: The worst possible closeness ratio when a and b are adjacent.

Observation 1. *If a and b are adjacent, then $\text{cc-ratio}_G(a, b) > \frac{1}{2}$.*

To see Observation 1, assume WLOG that $\text{cc}_G(b) \leq \text{cc}_G(a)$. By the triangle inequality, if a and b are adjacent, then for any vertex v , $d_G(a, v) \leq d_G(a, b) + d_G(b, v) = 1 + d_G(b, v)$. It follows that

$$\begin{aligned}\text{cc}_G(a) &= d_G(a, a) + d_G(a, b) + \sum_{v \in V \setminus \{a, b\}} d_G(a, v) \\ &= d_G(b, b) + d_G(a, b) + \sum_{v \in V \setminus \{a, b\}} d_G(a, v) \\ &\leq d_G(b, b) + d_G(a, b) + \sum_{v \in V \setminus \{a, b\}} (1 + d_G(b, v)) \\ &= d_G(b, b) + d_G(a, b) + \sum_{v \in V \setminus \{a, b\}} d_G(b, v) + n - 2 \\ &= \text{cc}_G(b) + n - 2.\end{aligned}$$

Furthermore, it is trivially true that $\text{cc}_G(b) \geq n - 1$. Thus, we may write $\text{cc}_G(b) = n - 1 + x$ for some non-negative x . Then, making use of Observation 2,

$$\text{cc-ratio}_G(a, b) = \frac{\text{cc}_G(b)}{\text{cc}_G(a)} \geq \frac{\text{cc}_G(b)}{\text{cc}_G(b) + (n - 2)} = \frac{(n - 1) + x}{(n - 1) + x + (n - 2)} \geq \frac{n - 1}{2(n - 1) - 1} > \frac{1}{2},$$

where in the second to last inequality we have made use of a fundamental observation:

Observation 2. *For all real numbers w, p, q with $0 \leq w < p < q$, $\frac{p+w}{q+w} \geq \frac{p}{q}$.*

Observation 2 is simple to verify via a proof by contradiction, and will be used frequently throughout this paper.

Recalling that no ratio greater than 1 is possible, a trivial $\frac{1}{2}$ -approximation for CLOSENESS RATIO IMPROVEMENT follows: add the edge ab if it is not already present. This is also *no better* than a $\frac{1}{2}$ -approximation. Given $k \geq n - 2$, an optimal algorithm achieves a ratio of 1 for the example in Figure 2, but the edge ab alone achieves only $\frac{1}{2} + o(1)$.

3 Lower Bounds

In this section, we establish the NP-hardness of CLOSENESS RATIO IMPROVEMENT by reducing from SET COVER. First, we formally define the decision variant.

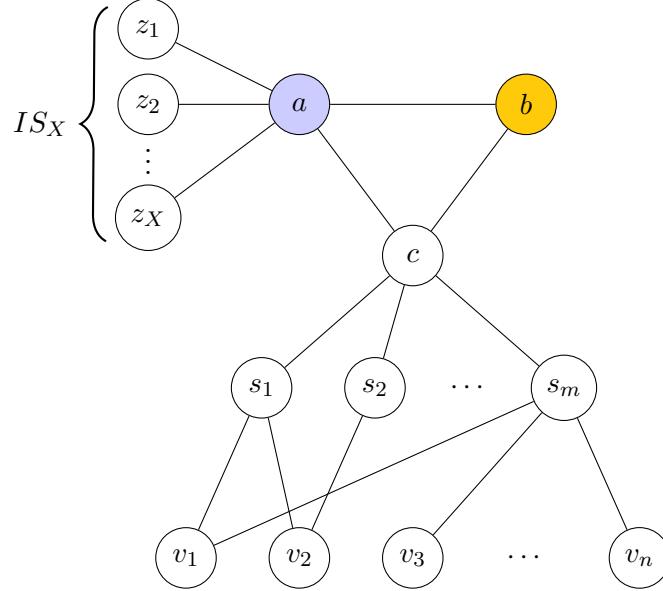


Figure 3: The construction given by Theorem 1. We denote set vertices by s_j and element vertices by v_i . Here, IS_X is an independent set of X vertices, with each vertex adjacent to a . We refer to the proof of Theorem 1 for a formal description of the construction and the accompanying analysis.

CLOSENESS RATIO IMPROVEMENT (DECISION VARIANT)

Input: A graph $G = (V, E)$, vertices $a, b \in V$, a budget $k \in \mathbb{N}$, and a target ratio $\tau \in (0, 1]$.

Question: Does there exist a set $T \subseteq V^2$ of size at most k such that $\text{cc-ratio}_{G+T}(a, b) \geq \tau$?

We begin by giving a polynomial and parameter-preserving reduction when $\tau = 1$, which we will then generalize to obtain stronger lower bounds.

Theorem 1. CLOSENESS RATIO IMPROVEMENT is NP-hard and W[2]-hard³ with respect to k , even when $\tau = 1$.

Proof. Let $(U, S_1, S_2, \dots, S_m, k)$ be an instance of SET COVER, where U denotes a universe of n elements and $\{S_j\}_{j=1}^m$ is a set family over U . We construct an instance $(G = (V, E), a, b, k', \tau = 1)$ of CLOSENESS RATIO IMPROVEMENT as follows; see Figure 3 for a visual depiction. We create three vertices a, b, c , which induce a triangle. We create n element vertices v_1, v_2, \dots, v_n corresponding to the elements u_1, u_2, \dots, u_n of U , ordered arbitrarily. For each set S_j , we create a set vertex s_j , as well as the edge cs_j and the edges $s_j v_i$ for all $u_i \in S_j$. Finally, we create $X = n + k$ vertices, and add an edge from each of these vertices to a . We also set $k' = k$. This completes the construction.

We claim that there exists a set cover of U of size at most k if and only if there is a set T of at most k edges such that $\text{cc-ratio}_{G+T}(a, b) = 1$. For the forward direction, suppose that there exists a cover $C \subseteq \{S_j\}_{j=1}^m$ of size k . Let $T = \{bs_j : S_j \in C\}$, i.e., the set of edges from b to the set vertices corresponding to sets in the cover C . Then for each element vertex v_i , $d_{G+T}(v_i, b) = 2$, and $d_{G+T}(v_i, a) = 3$. Also, b is adjacent to k set vertices, so $\sum_{j=1}^m d_{G+T}(s_j, b) = 2m - k$, while $\sum_{j=1}^m d_{G+T}(s_j, a) = 2m$. Then

$$\text{cc}_{G+T}(b) = 2X + 1 + 1 + 2m - k + 2n = 2(n + k) + 2m - k + 2n + 2 = 4n + 2m + k + 2.$$

³W[2]-hardness can be interpreted as very strong evidence that no fixed-parameter tractable algorithm exists for the parameter k . For an introduction to parameterized complexity, see the standard text [19].

Similarly, $\text{cc}_{G+T}(a) = X + 1 + 1 + 2m + 3n = 4n + 2m + k + 2$, so $\text{cc}_{G+T}(b) = \text{cc}_{G+T}(a)$, implying $\text{cc-ratio}_{G+T}(a, b) = 1 = \tau$, as desired.

For the reverse direction, assume that there is no collection of k sets which covers U . We must prove that there is no set $T \subseteq V^2 \setminus E$ of size at most k such that $\text{cc}_{G+T}(b) = \text{cc}_{G+T}(a)$. We accomplish this by showing that for every set T of size at most k , $\text{cc}_{G+T}(b) > \text{cc}_G(a)$. This is sufficient to complete the proof, since it follows from monotonicity that $\text{cc}_{G+T}(a) \leq \text{cc}_G(a)$. Toward this end, let T^* be a set of (at most) k edge additions such that for all sets $T \subseteq V^2 \setminus E$ of size at most k , $\text{cc}_{G+T^*}(b) \leq \text{cc}_{G+T}(b)$. We now state prove a useful structural lemma, which will allow us to assume that all edges in T^* connect b to set vertices.

Claim 1. *There exists a set $T \subseteq V^2 \setminus E$ of size at most k such that every edge in T is incident on b , all other endpoints of edges in T are set vertices, and $\text{cc}_{G+T}(b) \leq \text{cc}_{G+T^*}(b)$.*

Proof. We begin by creating a set T' as follows. For each edge in T^* that is incident on b , include it in T' . For each edge $xy \in T^*$ that is not incident on b , suppose without loss of generality that $d_{G+T^*\setminus\{xy\}}(b, x) \leq d_{G+T^*\setminus\{xy\}}(b, y)$, and then include the edge by in T' . Now $|T'| \leq k$ and all edges in T are incident on b .

Let $v \in V$, and let P be a shortest path from b to v in $G + T^*$. If P uses no edges from T^* , then $d_{G+T'}(b, v) \leq d_{G+T^*}(b, v)$. Otherwise, let xy be the last edge from T^* appearing along P (x may be b), so P has length $d_{G+T^*\setminus\{xy\}}(b, x) + 1 + d_G(y, v)$. Then $d_{G+T^*\setminus\{xy\}}(b, y) > d_{G+T^*\setminus\{xy\}}(b, x)$, as otherwise there exists a path from b to v in $G + T^*$ with length at most $d_{G+T^*\setminus\{xy\}}(b, y) + d_G(y, v)$ which is strictly less than the length of P . Then by construction, the edge by is in T' , so $d_{G+T'}(b, v) \leq d_{G+T^*}(b, v)$.

We now use a second exchange argument on T' to construct a set of at most k edges T such that every edge in T is incident on b , all other endpoints of edges in T are set vertices, and $\text{cc}_{G+T}(b) \leq \text{cc}_{G+T'}(b) \leq \text{cc}_{G+T^*}(b)$. Let bz_i be an edge (if one exists) in T' which is incident on a z_i in the independent set. Then $\text{cc}_{G+T'}(b) = \text{cc}_{G+T'\setminus\{bz_i\}}(b) - 1$. We may then replace bz_i with bs_j , where s_j is any set vertex not already an endpoint of an edge in T' , and achieve the same (or better) closeness centrality. Hence, we may assume that all edges in T' are incident on b and all other endpoints are either set or element vertices.

Suppose that there are $k' \leq k$ element vertices incident on edges in T' . If $k' = 0$ then by setting $T = T'$ we are done. We induct on k' , assuming that any T'' with $k' - 1$ edges incident on element vertices may be transformed into the desired set T . Let v_{i_1} be an element vertex which is an endpoint of an edge in T' , and let C' be the set vertices which are endpoints of edges in T' . Since by hypothesis there is no set cover of size k , there exists an element u_{i_2} which is not covered by the sets corresponding to C' (it is possible that $i_1 = i_2$). Let S_j be any set covering u_{i_2} , and set $T'' = (T' \setminus \{bv_{i_1}\}) \cup \{bs_j\}$. Then $d_{G+T''}(b, s_j) = d_{G+T'}(b, s_j) - 1$. If $i_1 = i_2$, then $d_{G+T''}(b, v_{i_1}) = d_{G+T'}(b, v_{i_1}) + 1$ and $d_{G+T''}(b, v) \leq d_{G+T'}(b, v)$ for all remaining vertices $v \notin \{v_{i_1}, s_j\}$. Otherwise, $d_{G+T''}(b, v_{i_1}) \leq d_{G+T'}(b, v_{i_1}) + 2$, $d_{G+T''}(b, v_{i_2}) = d_{G+T'}(b, v_{i_2}) - 1$, and $d_{G+T''}(b, v) \leq d_{G+T'}(b, v)$ for all remaining vertices $v \notin \{v_{i_1}, v_{i_2}, s_j\}$. In either case, summing shows that $\text{cc}_{G+T''}(b) \leq \text{cc}_{G+T'}(b)$. The result now follows from the inductive hypothesis. \square

Claim 1 allows us to give a useful lower bound on $\text{cc}_{G+T^*}(b)$. We use the fact that the sets associated with the endpoints of edges in T^* are (by hypothesis) *not* a cover of U to conclude that there exists at least one element vertex v_i with $d_{G+T^*}(b, v_i) = 3$. Then,

$$\begin{aligned} \text{cc}_{G+T^*}(b) &\geq 2X + 2 + 2m - k + 2(n - 1) + 3 = 2(n + k) + 2 + 2m - k + 2(n - 1) + 3 \\ &= 4n + 2m + k + 3 > 4n + 2m + k + 2 = \text{cc}_G(a), \end{aligned}$$

as desired. \square

Observe that a target ratio of 1 for CLOSENESS RATIO IMPROVEMENT corresponds exactly to a target gap of 0 in the context of CLOSENESS GAP MINIMIZATION. Thus, Theorem 1 also (conditionally) rules out any multiplicative approximation for the latter problem.

Corollary 1. CLOSENESS GAP MINIMIZATION is NP-hard, W[2]-hard with respect to k , and (unless P = NP) admits no polynomial-time multiplicative approximation algorithm.

Proof. Using the same construction as the proof of Theorem 1 and Figure 3, we claim that there exists a set cover of U of size at most k if and only if there is a set T of at most k edges such that $|\text{cc}_{G+T}(a) - \text{cc}_{G+T}(b)| = 0$.

Suppose there is a set cover C of U of size at most k , then let $T = \{bs_j : S_j \in C\}$. Now, $\text{cc}_{G+T}(a) = (n+k) + 2 + 2m + 3n = 4n + 2m + k + 2$ and $\text{cc}_{G+T}(b) = 2(n+k) + 2 + 2m - k + 2n = 4n + 2m + k + 2$, meaning $|\text{cc}_{G+T}(a) - \text{cc}_{G+T}(b)| = 0$. Suppose instead that there is no collection of k sets which covers U . We claim that for every set T of size at most k , $\text{cc}_{G+T}(b) > \text{cc}_G(a) \geq \text{cc}_{G+T}(a)$. Let T^* be a set of (at most) k edge additions such that for all sets $T \subseteq V^2 \setminus E$ of size at most k , $\text{cc}_{G+T^*}(b) \leq \text{cc}_{G+T}(b)$. Then by the same argument used in the proof of Theorem 1, $\text{cc}_{G+T^*}(b) \geq 2(n+k) + 2 + 2m - k + 2(n-1) + 3(1) = 4n + 2m + k + 3$, while $\text{cc}_{G+T^*}(a) \leq \text{cc}_G(a) = 4n + 2m + k + 2$. Consequently, $|\text{cc}_{G+T}(a) - \text{cc}_{G+T}(b)| \geq 1$ for all sets $T \subseteq V^2 \setminus E$ of size at most k .

Since CLOSENESS GAP MINIMIZATION is NP-hard at parameter value 0, it admits no polynomial-time multiplicative approximation algorithm. Suppose that such an algorithm \mathcal{A} did exist, achieving a gap at most $\alpha \cdot \text{opt}(y)$ for some α possibly a function of the input y . Then consider an instance of SET COVER x reduced (as we've described) to an instance y of CLOSENESS GAP MINIMIZATION. We've shown that if x has a set cover, then $\text{opt}(y) = 0$, so $\mathcal{A}(y) \leq \alpha \cdot 0 = 0$. If x has no set cover, then $\mathcal{A}(y) \geq \text{opt}(y) \geq 1$. Therefore, \mathcal{A} decides SET COVER in polynomial time. \square

In Section 2.2, we observed that CLOSENESS RATIO IMPROVEMENT is trivial when $\tau \leq \frac{1}{2}$. Meanwhile, by Theorem 1, the problem is hard when $\tau = 1$. The natural question is whether we can find the barrier between tractability and NP-hardness. We now resolve this question by showing that CLOSENESS RATIO IMPROVEMENT is NP-hard for every $\tau > \frac{1}{2}$. The construction is a generalization of that given by Theorem 1; we use the value of τ to determine an appropriate size for the independent set adjacent to a , as well as to determine an appropriate number of vertices corresponding to each element in the SET COVER instance.

Theorem 2. CLOSENESS RATIO IMPROVEMENT is NP-hard and W[2]-hard with respect to k , even when restricted to any constant $\tau > \frac{1}{2}$.

Proof. Let $\tau \in (\frac{1}{2}, 1)$, and note that Theorem 1 covers the case where $\tau = 1$. Consider an instance of SET COVER with m sets, n elements, and $k \in \mathbb{N}$. We first claim that we may assume $\frac{2m+4n+k}{1+2m+4n+k} \geq \tau^2$. Otherwise, we create an equivalent instance of SET COVER with $\lceil \frac{\tau^2}{4n(1-\tau^2)} \rceil$ copies of each element, giving each copy the same set memberships as the original element.⁴ Then, letting n' denote the

⁴Conceptually, one may also think of this procedure as the introduction of $\lceil \frac{\tau^2}{4n(1-\tau^2)} \rceil$ twins for each element vertex in our constructed CLOSENESS RATIO IMPROVEMENT instance.

number of elements in our equivalent instance, we have

$$\begin{aligned} n' &\geq \frac{\tau^2}{4(1-\tau^2)} \\ 4n'(1-\tau^2) &\geq \tau^2 > \tau^2 + 2m(\tau^2 - 1) + k(\tau^2 - 1) \\ \frac{2m + 4n' + k}{1 + 2m + 4n' + k} &\geq \tau^2, \end{aligned}$$

so our assumption is safe. We now repeat the same construction given by Theorem 1 and depicted in Figure 3. This time, we let the number X of vertices in the independent set adjacent to a be an integer in the interval given by the following lemma. We prove that a suitable value always exists.

Claim 2. *For every $m, n, k \in \mathbb{N}$ and $\tau \in (\frac{1}{2}, 1)$, there exists a natural number X in the interval $(\frac{2+2m+3n-\tau(2+2m-k+2n+1)}{2\tau-1}, \frac{2+2m+3n-\tau(2+2m-k+2n)}{2\tau-1}]$.*

Proof. Taking the difference of the bounds of the interval, we get that the length of the interval is $\frac{\tau}{2\tau-1}$. If $\tau < 1$, then $-\tau > -1$ and thus $\tau > 2\tau - 1$. Thus the length of the interval is greater than 1 when $\tau \in (\frac{1}{2}, 1)$, so there must exist some integer X within this interval. Furthermore, for $m, n, k \in \mathbb{N}$ and $\tau \in (\frac{1}{2}, 1)$, the bounds of this interval are both positive, so X must be a natural number, as desired. \square

If there exists a set cover C of size k , then we add k edges from b to the set vertices corresponding to C . In the analysis, some care must be taken to handle the case in which these edge additions cause b 's closeness centrality to become lower than that of a . For the other direction, we will use our selected parameter values to show that for every set T of k edge additions, $\text{cc}_{G+T}(b) > \frac{1}{\tau} \cdot \text{cc}_G(a) \geq \frac{1}{\tau} \cdot \text{cc}_{G+T}(a)$. The details are as follows.

For the forward direction, suppose that there exists a cover $C \subseteq \{S_j\}_{j=1}^m$ of size k . Again, let $T = \{bs_j : S_j \in C\}$, the set of edges from b to the set vertices corresponding to sets in the cover C . Now, $\text{cc}_{G+T}(a) = X + 2 + 2m + 3n$ and $\text{cc}_{G+T}(b) = 2X + 2 + 2m - k + 2n$. We want to show that $\text{cc-ratio}_{G+T}(a, b) \geq \tau$, but we must still ensure that $\text{cc-ratio}_{G+T}(a, b) \leq 1$, or else we violate our min/max definition of closeness ratio. Note that this was not a concern for $\tau = 1$, as we showed our ratio was exactly 1 when there was a set cover. Thus we consider the case where $\text{cc}_{G+T}(b) \geq \text{cc}_{G+T}(a)$ and the case where $\text{cc}_{G+T}(b) < \text{cc}_{G+T}(a)$, and show that in both cases $\text{cc-ratio}_{G+T}(a, b) \geq \tau$.

If $\text{cc}_{G+T}(b) \geq \text{cc}_{G+T}(a)$, then $\text{cc-ratio}_{G+T}(a, b) = \frac{X+2+2m+3n}{2X+2+2m-k+2n}$. Observe that this ratio is greater than $\frac{1}{2}$, and as $X \rightarrow \infty$, $\text{cc-ratio}_{G+T}(a, b) \rightarrow \frac{1}{2}$. Thus $\text{cc-ratio}_{G+T}(a, b)$ is a decreasing function of X and using Claim 2, we have

$$\begin{aligned} \text{cc-ratio}_{G+T}(a, b) &\geq \frac{\left(\frac{2+2m+3n-\tau(2+2m-k+2n)}{2\tau-1}\right) + 2 + 2m + 3n}{2\left(\frac{2+2m+3n-\tau(2+2m-k+2n)}{2\tau-1}\right) + 2 + 2m - k + 2n} \\ &= \frac{2 + 2m + 3n - \tau(2 + 2m - k + 2n) + (2\tau - 1)(2 + 2m + 3n)}{2(2 + 2m + 3n) - 2\tau(2 + 2m - k + 2n) + (2\tau - 1)(2 + 2m - k + 2n)} \\ &= \frac{2\tau(2 + 2m + 3n) - \tau(2 + 2m - k + 2n)}{2(2 + 2m + 3n) - 2 + 2m - k + 2n} = \tau. \end{aligned}$$

Alternatively, if $\text{cc}_{G+T}(b) < \text{cc}_{G+T}(a)$, then $\text{cc-ratio}_{G+T}(a, b) = \frac{2X+2+2m-k+2n}{X+2+2m+3n}$. Observe that this ratio is now an *increasing* function of X and using Claim 2 we have,

$$\begin{aligned} \text{cc-ratio}_{G+T}(a, b) &> \frac{2(\frac{2+2m+3n-\tau(2+2m-k+2n+1)}{2\tau-1}) + 2 + 2m - k + 2n}{(\frac{2+2m+3n-\tau(2+2m-k+2n+1)}{2\tau-1}) + 2 + 2m + 3n} \\ &= \frac{2(2+2m+3n) - 2\tau(2+2m-k+2n+1) + (2\tau-1)(2+2m-k+2n)}{(2+2m+3n) - \tau(2+2m-k+2n+1) + (2\tau-1)(2+2m+3n)} \\ &= \frac{2(2+2m+3n) - 2\tau - (2+2m-k+2n)}{2\tau(2+2m+3n) - \tau(2+2m-k+2n+1)} \\ &= \frac{4+4m+6n-2\tau-2-2m+k-2n}{4\tau+4m\tau+6n\tau-2\tau-2m\tau+k\tau-2n\tau-\tau} = \frac{(2-2\tau)+2m+4n+k}{\tau+2m\tau+4n\tau+k\tau}. \end{aligned}$$

As $\tau \in (\frac{1}{2}, 1)$, $2-2\tau > 0$, so

$$\frac{(2-2\tau)+2m+4n+k}{\tau+2m\tau+4n\tau+k\tau} > \frac{2m+4n+k}{\tau+2m\tau+4n\tau+k\tau} = \frac{1}{\tau} \left(\frac{2m+4n+k}{1+2m+4n+k} \right).$$

We have already shown that we may assume $\frac{2m+4n+k}{1+2m+4n+k} \geq \tau^2$. Thus we have that,

$$\text{cc-ratio}_{G+T}(a, b) > \frac{1}{\tau} \left(\frac{2m+4n+k}{1+2m+4n+k} \right) \geq \frac{1}{\tau} (\tau^2) = \tau.$$

Thus in either case, if there is a set cover of U of size k , we can add k edges to G to get a closeness ratio greater than or equal to τ .

For the reverse direction, suppose that there is no collection of k sets which covers U . We must prove that there is no set $T \subseteq V^2 \setminus E$ of size at most k such that $\text{cc-ratio}_{G+T}(a, b) \geq \tau$. Let T^* be a set of (at most) k edge additions such that for all sets $T \subseteq V^2 \setminus E$ of size at most k , $\text{cc}_{G+T^*}(b) \leq \text{cc}_{G+T}(b)$. Using the same analysis from the proof of Theorem 1, $\text{cc}_{G+T^*}(b) \geq 2X + 2 + 2m - k + 2n + 1$, while $\text{cc}_{G+T^*}(a) \leq X + 2 + 2m + 3n$. Note that our choice of X (Claim 2) guarantees that $\text{cc}_{G+T^*}(a) \leq \text{cc}_{G+T^*}(b)$, so we now have $\text{cc-ratio}_{G+T^*}(a, b) \leq \frac{X+2+2m+3n}{2X+2+2m-k+2n+1}$. This ratio is greater than $\frac{1}{2}$, and as $X \rightarrow \infty$, $\text{cc-ratio}_{G+T^*}(a, b) \rightarrow \frac{1}{2}$. Thus $\text{cc-ratio}_{G+T^*}(a, b)$ is a decreasing function of X and we have,

$$\begin{aligned} \text{cc-ratio}_{G+T^*}(a, b) &< \frac{(\frac{2+2m+3n-\tau(2+2m-k+2n+1)}{2\tau-1}) + 2 + 2m + 3n}{2(\frac{2+2m+3n-\tau(2+2m-k+2n+1)}{2\tau-1}) + 2 + 2m - k + 2n + 1} \\ &= \frac{(2+2m+3n) - \tau(2+2m-k+2n+1) + (2\tau-1)(2+2m+3n)}{2(2+2m+3n) - 2\tau(2+2m-k+2n+1) + (2\tau-1)(2+2m-k+2n+1)} \\ &= \frac{2\tau(2+2m+3n) - \tau(2+2m-k+2n+1)}{2(2+2m+3n) - (2+2m-k+2n+1)} = \tau. \end{aligned}$$

□

We conclude this section by giving a bicriteria inapproximability result for CLOSENESS RATIO IMPROVEMENT. We will once again reduce from SET COVER, but we make use of a stronger assumption due to [23, 25, 24]. Let $c \geq 1$ be a constant. Then given a family of m sets over n elements, it is NP-hard to distinguish between the existence of a set cover of size k , and the non-existence of any collection of ck sets which cover more than $(1 - (1 - \frac{1}{k})^{ck} + \delta)n$ elements, where

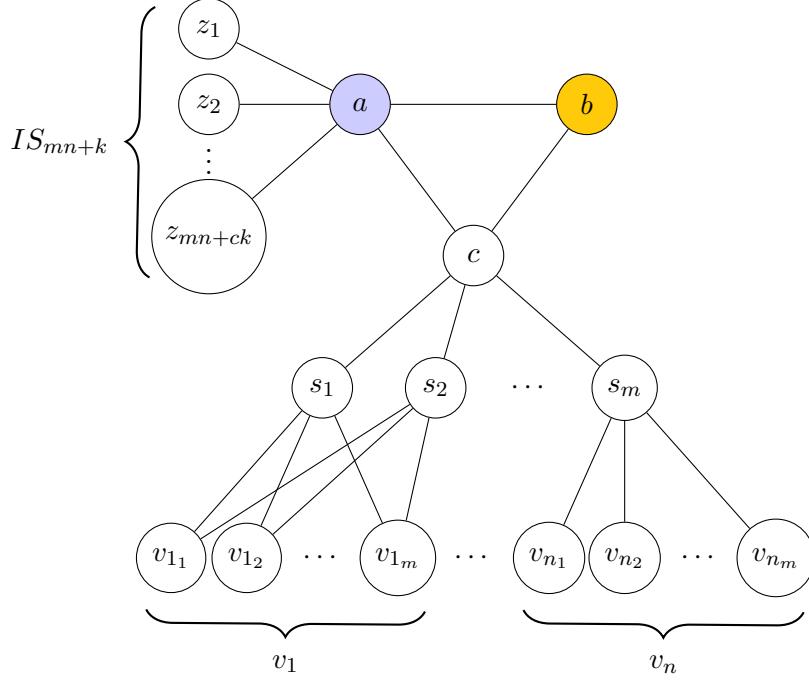


Figure 4: A depiction of the construction given by Theorem 3. Given an instance of SET COVER, set vertices s_j correspond to sets. Each element vertex v_i is duplicated into m twins. Here, IS_X denotes an independent set of size $mn + ck$, with each vertex adjacent to a . We refer to the proof of Theorem 3 for a formal description of the construction and the accompanying analysis.

$\delta > 0$ is any constant sufficiently small to keep the term in parenthesis less than 1. In particular, we may assume that if there is no set cover of size k , then every collection of ck sets covers at most $(1 - \frac{1}{e^c} + \delta)n$ elements, for arbitrarily small positive δ . The construction and analysis are similar to Theorem 1.

Theorem 3. *For every $\varepsilon > 0, c \geq 1$, unless $P = NP$ there is no polynomial-time algorithm which takes as input an instance $(G = (V, E), a, b, k)$ of CLOSENESS RATIO IMPROVEMENT with optimal objective value opt and returns a set $T \subseteq V^2 \setminus E$ of size at most ck while guaranteeing that $\text{cc-ratio}_{G+T}(a, b) \geq \frac{5e^c}{5e^c + 1 - \varepsilon} \cdot \text{opt}$.*

Proof. Let $\varepsilon > 0$ and $c \geq 1$. Set $\varepsilon' = \frac{\varepsilon}{2(5e^c - 1)}$, and $\delta = \frac{\varepsilon}{2e^c(1 - \varepsilon')}$. Let $(U, S_1, S_2, \dots, S_m, k)$ be an instance of SET COVER, where U denotes a universe of n elements and $\{S_j\}_{j=1}^m$ is a set family over U . We will assume that $n > \frac{ck}{\varepsilon'}$. This assumption is safe, as we may always assume that $k < n$ (otherwise finding a cover is trivial) and so we may always create an equivalent instance which satisfies our assumption by introducing $\lceil \frac{c}{\varepsilon'} \rceil$ copies of each element (where copies have the same set memberships as the original element). We construct an instance $(G = (V, E), a, b, k', \tau = 1)$ of CLOSENESS RATIO IMPROVEMENT as in Theorem 1, with two slight adjustments; see Figure 4 for a visual depiction. Specifically, for each element u_i , create m twin element vertices and let X , the size of the independent set, be $mn + ck$.

Suppose that there is a set cover C of U with size k . Again, let $T = \{bs_j : S_j \in C\}$, and now $\text{cc}_{G+T}(b) = 2(mn + ck) + 1(2) + 1(k) + 2(m - k) + 2(mn) = 4mn + 2m + (2c - 1)k + 2$. Similarly, $\text{cc}_{G+T}(a) = (mn + ck) + 1(2) + 2(m) + 3(mn) = 4mn + 2m + ck + 2$. Since, $c \geq 1$, we have

$\text{cc}_G(a) \leq \text{cc}_G(b)$. Then, by Observation 2 and our assumption regarding n , we have

$$\begin{aligned} \text{cc-ratio}_{G+T}(a, b) &= \frac{4mn + 2m + ck + 2}{4mn + 2m + (2c - 1)k + 2} > \frac{4mn + 2m + ck + 2}{4mn + 2m + 2ck + 2} \\ &> \frac{n}{n + ck} \\ &> \frac{n}{n} \cdot \frac{1}{1 + \varepsilon'} > 1 - \varepsilon', \end{aligned}$$

where the final inequality can be verified via a simple proof by contradiction, making use of the fact that $\varepsilon' > 0$. Thus, if a set cover of size k exists, we can always achieve a closeness ratio greater than $1 - \varepsilon'$.

Now suppose that the union of any ck sets covers at most $(1 - \frac{1}{e^c} + \delta)n$ elements. The idea of the proof is to give a lower bound on $\text{cc}_{G+T}(b)$, across all sets T of at most ck edge additions. By an analysis identical to Claim 1, we may assume that all edges in an optimal solution (of size up to ck) are incident on b , and that all other endpoints are set vertices. Thus, given an optimal solution T , at most ck set vertices are distance 1 from b , and at most $(1 - \frac{1}{e^c} + \delta)mn$ element vertices are distance 2 from b . In particular, there remain $(m - ck)$ set vertices with distance 2 from b , and $(\frac{1}{e^c} - \delta)mn$ element vertices with distance 3 from b . Then we have

$$\begin{aligned} \text{cc}_{G+T}(b) &\geq 2(mn + ck) + 2 + 2m - ck + 2(1 - \frac{1}{e^c} + \delta)(mn) + 3(\frac{1}{e^c} - \delta)(mn) \\ &= (4 + \frac{1}{e^c} - \delta)mn + 2m + ck + 2. \end{aligned}$$

Now we will analyze $\text{cc-ratio}_{G+T}(a, b)$. We note that by monotonicity, $\text{cc}_{G+T}(a) \leq \text{cc}_G(a)$, so we will use the latter quantity when deriving our bound. We also note that we may assume $ck + 2 < m$, as otherwise all covers of size ck may be checked in polynomial time. Substituting, we have

$$\begin{aligned} \text{cc-ratio}_{G+T}(a, b) &\leq \frac{4mn + 2m + ck + 2}{(4 + \frac{1}{e^c} - \delta)mn + 2m + ck + 2} \\ &\leq \frac{4mn + 2m + ck + 2 + (m - ck - 2)}{(4 + \frac{1}{e^c} - \delta)mn + 2m + ck + 2 + (m - ck - 2)} \quad (\text{using Observation 2}) \\ &= \frac{4mn + 3m}{(4 + \frac{1}{e^c} - \delta)mn + 3m} \\ &= \frac{4n + 3}{(4 + \frac{1}{e^c} - \delta)n + 3} \\ &\leq \frac{4n + 3 + n - 3}{(4 + \frac{1}{e^c} - \delta)n + 3 + n - 3} \quad (\text{using Observation 2}) \\ &\leq \frac{5n}{(5 + \frac{1}{e^c} - \delta)n} \\ &= \frac{5e^c}{5e^c + 1 - e^c\delta}. \end{aligned}$$

We have shown that if there is a set cover, a solution T of size at most k exists with $\text{cc-ratio}_{G+T}(a, b) > 1 - \varepsilon'$. Meanwhile, if the union of any ck sets covers at most $(1 - \frac{1}{e^c} + \delta)n$ elements, then $\text{cc-ratio}_{G+T}(a, b) \leq \frac{5e^c}{5e^c + 1 - e^c\delta}$ for every set T of at most ck edge additions. To complete the proof, it suffices to show that

$$\frac{5e^c}{5e^c + 1 - e^c\delta} \cdot (1 - \varepsilon') \geq \frac{5e^c}{5e^c + 1 - e^c\delta},$$

since then a polynomial-time $\frac{5e^c}{5e^c+1-\varepsilon}$ -approximation (using up to ck edges) could distinguish between the two relevant cases for any SET COVER instance. Simplifying the above, it suffices to show that

$$\begin{aligned} \frac{1 - \varepsilon'}{5e^c + 1 - \varepsilon} &\geq \frac{1}{5e^c + 1 - e^c\delta} \\ (1 - \varepsilon')(5e^c + 1 - e^c\delta) &\geq 5e^c + 1 - \varepsilon \\ \varepsilon &\geq \varepsilon'5e^c - \varepsilon' + e^c\delta - e^c\delta\varepsilon' \\ \varepsilon &\geq \varepsilon'(5e^c - 1) + \delta e^c(1 - \varepsilon'). \end{aligned}$$

The result now follows by substituting our selected values $\delta = \frac{\varepsilon}{2e^c(1-\varepsilon')}$ and $\varepsilon' = \frac{\varepsilon}{2(5e^c-1)}$. \square

4 A $\frac{6}{11}$ -Approximation

In this section we show that a surprisingly simple strategy can improve upon the $\frac{1}{2}$ -approximation yielded by adding the edge ab : we need only add edges from a to b 's neighborhood. The main intuition is that $\text{cc-ratio}_{G+ab}(a, b)$ is actually much larger than $\frac{1}{2}$ unless a significant portion of the graph is in b 's neighborhood but not a 's. Even in this case, unless the optimum objective value is very close to 1, $G + ab$ is already $\frac{6}{11}$ -approximate. These two insights impose significant structural restrictions on hard instances, and the achievement of this section is to show how to reason about these instances. We do so via careful arguments to bound the sizes of relevant vertex sets. Throughout the section, given a graph $G = (V, E)$ and vertices $a, b \in V$, we denote by $N_G(a)$ the open neighborhood of a in G , $N_G[a] = N_G(a) \cup \{a\}$ the closed neighborhood of a , and $A^p = N_G(a) \setminus N_G[b]$ the *private neighbors* of a . Given a vertex subset $O \subseteq V$, we write $\tilde{d}_G(a, O) = \frac{1}{|O|} \cdot \sum_{v \in O} d_G(a, v)$ for the mean distance from a to vertices in O . We now state our algorithm in Algorithm 1.

Algorithm 1:

Input : An instance $(G = (V, E), a, b, k)$ of CLOSENESS RATIO IMPROVEMENT, with $\text{cc}_G(b) < \text{cc}_G(a)$.

Output : A set $S \subseteq V^2 \setminus E$ of cardinality at most k .

```

1 if  $ab \notin E$  then
2    $S \leftarrow \emptyset$ 
3   if  $ab \notin E$  then
4      $S \leftarrow S \cup \{ab\}$ 
5     if  $\text{cc}_{G+S}(a) \leq \text{cc}_{G+S}(b)$  then
6       return  $S$  or  $\emptyset$ , whichever is better.
7   while  $|S| < k$  do
8      $u \leftarrow$  arbitrary vertex from  $N_{G+S}(b) \setminus N_{G+S}[a]$ 
9      $S \leftarrow S \cup \{au\}$ 
10    if  $\text{cc}_{G+S}(a) \leq \text{cc}_{G+S}(b)$  then
11      return  $S$  or  $S \setminus \{au\}$ , whichever is better.
12
13 return  $S$  or  $\emptyset$ , whichever is better.

```

To analyze Algorithm 1, we begin by proving that the pair of early termination conditions on lines 4 and 9 are safe. Recall that we assume $\text{cc}_G(b) < \text{cc}_G(a)$, WLOG. We show in Lemma 1 that either $\text{cc}_{G+ab}(b) < \text{cc}_{G+ab}(a)$ or one of G or $G + ab$ satisfies our desired approximation bound. In other words, the condition given on Line 4 of Algorithm 1 is safe.

Lemma 1 (The No Switching Lemma). *Let (G, a, b, k) be an instance of CLOSENESS RATIO IMPROVEMENT with $\text{cc}_G(b) < \text{cc}_G(a)$. Then at least one of the following is true:*

- (i) $\text{cc}_{G+ab}(b) < \text{cc}_{G+ab}(a)$,
- (ii) $\text{cc-ratio}_{G+ab}(a, b) \geq \frac{6}{11}$, or
- (iii) $\text{cc-ratio}_G(a, b) \geq \frac{6}{11}$.

Proof. Assume toward a contradiction that none of (i), (ii), or (iii) is true. Recall that $A^p = N_G(a) \setminus N_G[b] = N_{G+ab}(a) \setminus N_{G+ab}[b]$ denotes the private neighbors of a . Let $M = N_G(a) \cap N_G(b)$ denote the mutual neighbors of a and b . Let $R = V(G) \setminus (A^p \cup M \cup \{a, b\})$. We will begin by using the assumption that parts (i) and (ii) do not hold, i.e., $\text{cc}_{G+ab}(a) < \frac{6}{11} \cdot \text{cc}_{G+ab}(b)$, to show that A^p must be large.

Claim 3. $|A^p| > \frac{4(n-2)}{5}$.

Proof of Claim 3. Assume otherwise. Then there exist non-negative x_R, x_M, y_R, y_M such that $y_R + y_M = \frac{(n-2)}{5}$, $|A^p| = \frac{4(n-2)}{5} - x_R - x_M$, $|R| = y_R + x_R$, and $|M| = y_M + x_M$. Now we complete the proof by comparing $\text{cc}_{G+ab}(a)$ to an upper bound for $\text{cc}_{G+ab}(b)$. We begin with the former:

$$\begin{aligned} \text{cc}_{G+ab}(a) &= d_{G+ab}(a, a) + d_{G+ab}(a, b) + 1 \cdot |A^P| + 1 \cdot |M| + \tilde{d}_{G+ab}(a, R) \cdot |R| \\ &= 1 + \frac{4(n-2)}{5} - x_R - x_M + y_M + x_M + \tilde{d}_{G+ab}(a, R) \cdot (y_R + x_R). \end{aligned}$$

Now, we observe that $R \cap N_{G+ab}[a] = \emptyset$, so we may write $\tilde{d}_{G+ab}(a, R) = 2 + z$ for some non-negative z . Then we have

$$\begin{aligned} \text{cc}_{G+ab}(a) &= 1 + \frac{4(n-2)}{5} - x_R - x_M + y_M + x_M + 2y_R + 2x_R + zy_R + zx_R \\ &= 1 + \frac{4(n-2)}{5} + y_M + 2y_R + x_R + zy_R + zx_R \\ &= 1 + \frac{6(n-2)}{5} - y_M + x_R + zy_R + zx_R \quad (\text{using } y_R = \frac{n-2}{5} - y_M). \end{aligned}$$

Now we write an upper bound for $\text{cc}_{G+ab}(b)$, using that for every $v \in A^p$, $d_{G+ab}(b, v) = 2$.

$$\begin{aligned} \text{cc}_{G+ab}(b) &= d_{G+ab}(b, b) + d_{G+ab}(b, a) + 2 \cdot |A^P| + 1 \cdot |M| + \tilde{d}_{G+ab}(b, R) \cdot |R| \\ &= 1 + \frac{8(n-2)}{5} - 2x_M - 2x_R + y_M + x_M + \tilde{d}_{G+ab}(b, R)(y_R + x_R). \end{aligned}$$

By the triangle inequality, $\tilde{d}_{G+ab}(b, R) \leq \tilde{d}_{G+ab}(a, R) + 1 = 3 + z$, so

$$\begin{aligned} \text{cc}_{G+ab}(b) &\leq 1 + \frac{8(n-2)}{5} - 2x_M - 2x_R + y_M + x_M + 3y_R + 3x_R + zy_R + zx_R \\ &= 1 + \frac{8(n-2)}{5} + y_M + 3y_R + x_R + zy_R + zx_R - x_M \\ &\leq 1 + \frac{8(n-2)}{5} + y_M + 3y_R + x_R + zy_R + zx_R \\ &= 1 + \frac{11(n-2)}{5} - 2y_M + x_R + zy_R + zx_R \quad (\text{using } y_R = \frac{n-2}{5} - y_M). \end{aligned}$$

Now we will examine $\text{cc-ratio}_{G+ab}(a, b)$. We note that since $y_M \leq \frac{n-2}{5}$, we have $\frac{6(n-2)}{5} - y_M < \frac{11(n-2)}{5} - 2y_M$. Thus, we may apply Observation 2 as follows.

$$\frac{\text{cc}_{G+ab}(a)}{\text{cc}_{G+ab}(b)} \geq \frac{1 + \frac{6(n-2)}{5} - y_M + x_R + zy_R + zx_R}{1 + \frac{11(n-2)}{5} - 2y_M + x_R + zy_R + zx_R} \geq \frac{\frac{6(n-2)}{5} - y_M}{\frac{11(n-2)}{5} - 2y_M}.$$

Now we wish to eliminate the remaining y_M terms. We use another observation:

Observation 3. For all real numbers w, q, p with $0 \leq w < \frac{q}{2} < p < q$, $\frac{p-w}{q-2w} \geq \frac{p}{q}$.

Observation 3 is also simple to verify, and it allows us to complete the proof:

$$\text{cc-ratio}_{G+ab}(a, b) = \frac{\text{cc}_{G+ab}(a)}{\text{cc}_{G+ab}(b)} \geq \frac{\frac{6(n-2)}{5} - y_M}{\frac{11(n-2)}{5} - 2y_M} \geq \frac{\frac{6(n-2)}{5}}{\frac{11(n-2)}{5}} = \frac{6}{11},$$

contradicting the assumption that part (ii) of the lemma does not hold. \square

Now we will use Claim 3 and the assumption that part (iii) does not hold, i.e., $\text{cc}_G(b) < \frac{6}{11} \cdot \text{cc}_G(a)$, to show that $d_G(a, b) = 1$. This will complete the proof, as then $G + ab = G$, so by hypothesis $\text{cc}_{G+ab}(b) = \text{cc}_G(b) < \text{cc}_G(a) = \text{cc}_{G+ab}(a)$, contradicting the assumption that part (i) does not hold. In the following, let $O = M \cup R$, i.e, all of the vertices besides a, b , and a 's private neighbors. We will begin by giving a lower bound for $\text{cc}_G(b)$. Using the triangle inequality, we note that for each $v \in A^p$, $d_G(b, v) \geq d_G(a, b) - 1$. Then we have

$$\begin{aligned} \text{cc}_G(b) &= d_G(b, b) + d_G(b, a) + \tilde{d}_G(b, A^p) \cdot |A^p| + \tilde{d}_G(b, O) \cdot |O| \\ &\geq 1 + (d_G(a, b) - 1) \cdot |A^p| + \tilde{d}_G(b, O) \cdot |O| \end{aligned}$$

Now we write an upper bound for $\text{cc}_G(a)$. We again make use of the triangle inequality: for each $v \in O$, $d_G(a, v) \leq d_G(a, b) + d_G(b, v)$. Then $\tilde{d}_G(a, O) \leq d_G(a, b) + \tilde{d}_G(b, O)$, and thus

$$\begin{aligned} \text{cc}_G(a) &= d_G(a, a) + d_G(a, b) + |A^p| + \tilde{d}_G(a, O) \cdot |O| \\ &\leq 1 + |A^p| + (d_G(a, b) + \tilde{d}_G(b, O)) \cdot |O|. \end{aligned}$$

From the assumption that part (iii) does not hold, we now have the following,

$$\begin{aligned} \text{cc}_G(b) &< \frac{6}{11} \cdot \text{cc}_G(a) \\ 1 + (d_G(a, b) - 1) \cdot |A^p| + \tilde{d}_G(b, O) \cdot |O| &< \frac{6}{11} \left[1 + |A^p| + (d_G(a, b) + \tilde{d}_G(b, O)) \cdot |O| \right] \\ \frac{5}{11} + \left(d_G(a, b) - \frac{17}{11} \right) \cdot |A^p| &< \left[\frac{6}{11} d_G(a, b) - \frac{5}{11} \tilde{d}_G(b, O) \right] \cdot |O| \\ \left(d_G(a, b) - \frac{17}{11} \right) \cdot |A^p| &< \left[\frac{6}{11} d_G(a, b) - \frac{5}{11} \tilde{d}_G(b, O) \right] \cdot |O|. \end{aligned}$$

Recall that if $d_G(a, b) = 1$ then we are done, since this is what we are trying to prove. Then we may assume that $d_G(a, b) \geq 2$, which implies that the LHS of the above inequality is positive. It

follows that the RHS is also positive. We now use Claim 3 to substitute $\frac{4(n-2)}{5} < |A^p|$ and $|O| < \frac{n-2}{5}$:

$$\begin{aligned}
& \left(d_G(a, b) - \frac{17}{11} \right) \cdot \frac{4(n-2)}{5} < \left[\frac{6}{11} d_G(a, b) - \frac{5}{11} \tilde{d}_G(b, O) \right] \cdot \frac{n-2}{5} \\
& d_G(a, b) \cdot \frac{44(n-2)}{55} - \frac{68(n-2)}{55} < d_G(a, b) \cdot \frac{6(n-2)}{55} - \tilde{d}_G(b, O) \cdot \frac{5(n-2)}{55} \\
& d_G(a, b) \cdot \frac{38(n-2)}{55} < \frac{68(n-2)}{55} - \tilde{d}_G(b, O) \cdot \frac{5(n-2)}{55} \\
& d_G(a, b) \cdot \frac{38(n-2)}{55} < \frac{68(n-2)}{55} \\
& d_G(a, b) < \frac{68}{38} < 2,
\end{aligned}$$

so $d_G(a, b) = 1$, contradicting the assumption that part (i) does not hold, as desired. \square

Next we show that the termination condition given on Line 9 of Algorithm 1 is safe. This will enable us to assume not only that $cc_{G+ab}(b) < cc_{G+ab}(a)$ (by Lemma 1), but also that this inequality holds for the duration of the algorithm.

Lemma 2 (The Termination Lemma). *Let $G = (V, E)$, $a, b \in V$, $ab \in E$, $cc_G(b) < cc_G(a)$, $cc\text{-ratio}_G(a, b) < \frac{2}{3}$, and $u \in N_G(b)$. If $cc_{G+au}(a) \leq cc_{G+au}(b)$, then $cc\text{-ratio}_{G+au}(a, b) \geq \frac{2}{3}$.*

Proof. Consider a partition of the vertices V into five sets: $\{a\}$, $\{b\}$, other vertices closer to a than to b , closer to b than to a , and those equidistant from a and b . Formally, the latter three sets are $A_G := \{v \in V \setminus \{a, b\} : d_G(a, v) < d_G(b, v)\}$, $B_G := \{v \in V \setminus \{a, b\} : d_G(a, v) > d_G(b, v)\}$, and $C_G := \{v \in V \setminus \{a, b\} : d_G(a, v) = d_G(b, v)\}$.

Let A_{G+au} , B_{G+au} , and C_{G+au} be defined similarly. We will now use these definitions and the triangle inequality to show that B_G is large.

Claim 4. $|B_G| > \frac{n-2}{2}$.

Proof of Claim 4. We first use the definitions of A_G and C_G to write a lower bound for $cc_G(b)$.

$$\begin{aligned}
cc_G(b) &= d_G(b, b) + d_G(b, a) + \tilde{d}_G(b, B_G) \cdot |B_G| + \tilde{d}_G(b, A_G \cup C_G) \cdot |A_G \cup C_G| \\
&\geq 1 + \tilde{d}_G(b, B_G) \cdot |B_G| + \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G|.
\end{aligned}$$

Next, we will give an upper bound for $cc_G(a)$. Using the triangle inequality and the fact that $ab \in E(G)$ (by hypothesis), we observe that $\tilde{d}_G(a, B_G) \leq \tilde{d}_G(b, B_G) + 1$ (in fact equality holds, using the definition of B_G). Then we have

$$\begin{aligned}
cc_G(a) &= d_G(a, a) + d_G(a, b) + \tilde{d}_G(a, B_G) \cdot |B_G| + \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G| \\
&\leq 1 + (\tilde{d}_G(b, B_G) + 1) \cdot |B_G| + \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G|.
\end{aligned}$$

Now recall that, by hypothesis, $cc_G(b) < \frac{2}{3} \cdot cc_G(a)$. Then by substituting our bounds we have

$$\begin{aligned}
& 1 + \tilde{d}_G(b, B_G) \cdot |B_G| + \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G| \\
&< \frac{2}{3} \cdot \left[1 + (\tilde{d}_G(b, B_G) + 1) \cdot |B_G| + \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G| \right] \\
&= \frac{2}{3} + \frac{2}{3} \cdot \tilde{d}_G(b, B_G) \cdot |B_G| + \frac{2}{3} \cdot |B_G| + \frac{2}{3} \cdot \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G|.
\end{aligned}$$

Combining terms, we have

$$\frac{1}{3} + \frac{1}{3} \cdot \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G| < \left[\frac{2}{3} - \frac{1}{3} \cdot \tilde{d}_G(b, B_G) \right] \cdot |B_G|,$$

or, by dropping the $\frac{1}{3}$,

$$\frac{1}{3} \cdot \tilde{d}_G(a, A_G \cup C_G) \cdot |A_G \cup C_G| < \left[\frac{2}{3} - \frac{1}{3} \cdot \tilde{d}_G(b, B_G) \right] \cdot |B_G|.$$

Now, we complete the proof by assuming toward a contradiction that the claim is false, i.e., that $|B_G| \leq \frac{n-2}{2}$, and consequently $\frac{n-2}{2} \leq |A_G \cup C_G|$. Then we substitute these inequalities as follows:

$$\frac{1}{3} \cdot \tilde{d}_G(a, A_G \cup C_G) \cdot \frac{n-2}{2} < \left[\frac{2}{3} - \frac{1}{3} \cdot \tilde{d}_G(b, B_G) \right] \cdot \frac{n-2}{2},$$

which implies

$$\tilde{d}_G(a, A_G \cup C_G) + \tilde{d}_G(b, B_G) < 2.$$

This is a contradiction, since $a \notin A_G \cup C_G$ and $b \notin B_G$, implying that both summands on the LHS are at least 1. This completes the proof of the claim. \square

Now we complete the proof of the lemma by assuming toward a contradiction that $\text{cc}_{G+au}(a) < \frac{2}{3} \cdot \text{cc}_{G+au}(b)$. Then, similarly to Claim 4, $|A_{G+au}| > \frac{n-2}{2}$. To see why this is a contradiction, observe that since $u \in N_G(b)$, $A_{G+au} \subseteq A_G$ (in fact the sets are equal). Then we have $|A_G|, |B_G| > \frac{n-2}{2}$, but these sets are disjoint by definition, and their union has cardinality at most $n-2$, so we have a contradiction. \square

What remains is to analyze the behavior of the loop in Algorithm 1. We observe that if at any point $N_{G+S}(b) \setminus N_{G+S}(a) = \emptyset$, then $\text{cc}_{G+S}(a) \leq \text{cc}_{G+S}(b)$. We may therefore assume that the number of edge additions $k < |N_G(b) \setminus N_G(a)|$. Otherwise the condition on Line 9 will be triggered and, by Lemma 2, the returned solution will be a $\frac{2}{3}$ -approximation. This implies that the vertex u on Line 7 is well-defined.

To complete the analysis, we need to show that the set S returned on Line 11 yields a $\frac{6}{11}$ -approximation. We begin by showing a sufficient condition: a large value of $k + |N_G(a)|$.

Lemma 3. *Let $(G = (V, E), a, b, k)$ be an instance of CLOSENESS RATIO IMPROVEMENT with $\text{cc}_G(b) \leq \text{cc}_G(a)$. If $k + |N_G(a)| \geq \frac{n}{6}$, then Algorithm 1 returns a $\frac{6}{11}$ -approximation.*

Proof. Let S be the set of edge additions returned by Algorithm 1. By Lemma 2, we may assume that $\text{cc}_{G+S}(b) < \text{cc}_{G+S}(a)$. Thus, it is sufficient to show that $\text{cc}_{G+S}(a) \leq \frac{11}{6} \text{cc}_{G+S}(b)$.

Let $A = N_{G+S}(a) \setminus \{b\}$. Note that by Lemmas 1 and 2, we may assume that $|S| = k$, i.e., that the algorithm did not terminate early. It follows that $|A| \geq \frac{n}{6} - 1$. Setting $B = V \setminus (A \cup \{a, b\})$, we have $|B| \leq \frac{5n}{6} - 1$.

We now wish to find an upper bound for $\text{cc}_{G+S}(a)$ in terms of n and $\text{cc}_{G+S}(b)$. We begin with the following:

$$\begin{aligned} \text{cc}_{G+S}(a) &= d_{G+S}(a, a) + d_{G+S}(a, b) + \sum_{v \in A} d_{G+S}(a, v) + \sum_{v \in B} d_{G+S}(a, v) \\ &= d_{G+S}(b, b) + d_{G+S}(b, a) + \sum_{v \in A} d_{G+S}(a, v) + \sum_{v \in B} d_{G+S}(a, v). \end{aligned}$$

By Lemma 1, we may assume that $d_{G+S}(a, b) = 1$. Then for each $v \in B$, $d_{G+S}(a, v) \leq d_{G+S}(b, v) + 1$. We use this to make a substitution in the fourth summand of the RHS:

$$\text{cc}_{G+S}(a) \leq d_{G+S}(b, b) + d_{G+S}(b, a) + \sum_{v \in A} d_{G+S}(a, v) + \sum_{v \in B} (d_{G+S}(b, v) + 1).$$

Meanwhile, since $b \notin A$, we have $d_{G+S}(a, v) \leq d_{G+S}(b, v)$ for all $v \in A$. Then we may perform a substitution in the third summand to obtain the bound we desire.

$$\begin{aligned} \text{cc}_{G+S}(a) &\leq d_{G+S}(b, b) + d_{G+S}(b, a) + \sum_{v \in A} d_{G+S}(b, v) + \sum_{v \in B} (d_{G+S}(b, v) + 1) \\ &= \text{cc}_{G+S}(b) + |B| \\ &\leq \text{cc}_{G+S}(b) + \frac{5n}{6} - 1. \end{aligned}$$

We now complete the proof in two cases. In the first case, we assume that $\text{cc}_{G+S}(b) \geq n$. Then we may write $\text{cc}_{G+S}(b) = n + x$ for some $x \geq 0$. We now have

$$\begin{aligned} \text{cc-ratio}_{G+S}(a, b) &= \frac{\text{cc}_{G+S}(b)}{\text{cc}_{G+S}(a)} \geq \frac{\text{cc}_{G+S}(b)}{\text{cc}_{G+S}(b) + \frac{5n}{6} - 1} \geq \frac{\text{cc}_{G+S}(b)}{\text{cc}_{G+S}(b) + \frac{5n}{6}} \\ &= \frac{n+x}{n+x+\frac{5n}{6}} \\ &\geq \frac{n}{n+\frac{5n}{6}} \quad (\text{by Observation 2}) \\ &= \frac{6}{11}. \end{aligned}$$

In the other case, we have $\text{cc}_{G+S}(b) = n - 1$. Then b is adjacent to every other vertex in the graph, meaning that a has distance at most 2 to every vertex in B . Then

$$\begin{aligned} \text{cc}_{G+S}(a) &\leq d_{G+S}(a, a) + 1 \cdot ((n-1) - |B|) + 2 \cdot |B| \\ &= n + |B| - 1 \leq n + \frac{5n}{6} - 2 = \frac{11n}{6} - 2 < \frac{11n}{6} - \frac{11}{6} = \frac{11}{6} \cdot \text{cc}_{G+S}(b). \end{aligned}$$

This completes the proof. \square

By Lemma 2, if Algorithm 1 returns on Line 10, then it produces a $\frac{2}{3}$ -approximation. Similarly, by Lemma 1, if Algorithm 1 returns on Line 5, then it produces a $\frac{6}{11}$ -approximation. Thus, we may assume that Algorithm 1 returns on Line 11. By design, if G is already $\frac{6}{11}$ -approximate, then we are guaranteed to succeed. Recall that other than the edge ab , every edge added by Algorithm 1 is from a to the neighborhood of b . Thus, after the edge ab is added, the closeness centrality of b does not change, while that of a monotonically decreases. Then we may conclude that if $G + ab$ is $\frac{6}{11}$ -approximate, our algorithm returns a $\frac{6}{11}$ -approximation. It now suffices to show that if neither G nor $G + ab$ is $\frac{6}{11}$ -approximate, then the sufficient condition given by Lemma 3 must occur.

Lemma 4. *Let $(G = (V, E), a, b, k)$ be an instance of CLOSENESS RATIO IMPROVEMENT with $\text{cc}_G(b) < \text{cc}_G(a)$ and $\text{cc}_{G+ab}(b) < \text{cc}_{G+ab}(a)$. Let G^* be an optimal solution, and assume that $\max\{\text{cc-ratio}_G(a, b), \text{cc-ratio}_{G+ab}(a, b)\} < \frac{6}{11} \cdot \text{cc-ratio}_{G^*}(a, b)$. Then $k + |N_G(a)| \geq \frac{n}{6}$.*

Proof. Recall the notation $B^p = N_G(b) \setminus N_G[a]$ for the private neighbors of b . We begin by establishing some additional structure.

Claim 5. Under the hypotheses of the lemma, all of the following hold:

- (i) $\text{cc}_{G+ab}(b) < \frac{6(n-2)}{5}$,
- (ii) $\text{cc-ratio}_{G^*}(a, b) > \frac{11}{12}$, and
- (iii) $|B^p| > \frac{4(n-2)}{5}$.

Proof of Claim 5. Part (i) is a consequence of the observation that $\text{cc}_{G+ab}(a) \leq \text{cc}_{G+ab}(b) + n - 2$, which follows from the triangle inequality and the fact that $d_{G+ab}(a, b) = 1$. Thus, if $\text{cc}_{G+ab}(b) \geq \frac{6(n-2)}{5}$, then we may write $\text{cc}_{G+ab}(b) = \frac{6(n-2)}{5} + x$ for some non-negative x . Then

$$\text{cc-ratio}_{G+ab}(a, b) = \frac{\text{cc}_{G+ab}(b)}{\text{cc}_{G+ab}(a)} \geq \frac{\frac{6(n-2)}{5} + x}{\frac{11(n-2)}{5} + x} \geq \frac{6}{11},$$

where in the final inequality we have used Observation 2. This contradicts the hypothesis that $\text{cc-ratio}_{G+ab}(a, b) < \frac{6}{11} \text{cc-ratio}_{G^*}(a, b)$, since $\text{cc-ratio}_{G^*}(a, b) \leq 1$.

Part (ii) follows from the observation that $\text{cc-ratio}_{G+ab}(a, b) > \frac{1}{2}$ (recall Section 2.2). Then since $\text{cc-ratio}_{G+ab}(a, b) < \frac{6}{11} \text{cc-ratio}_{G^*}(a, b)$, we have $\text{cc-ratio}_{G^*}(a, b) > \frac{11}{6} \cdot \frac{1}{2} = \frac{11}{12}$.

To prove part (iii), it is sufficient to show that b has strictly more than $\frac{4(n-2)}{5}$ private neighbors in $G + ab$, since any private neighbor of b in $G + ab$ is also a private neighbor of b in G . The remainder of the argument is identical to the proof of Claim 3, except that the roles of a and b have been reversed. \square

Our strategy will now be to use Claim 5 to find upper bounds on $\text{cc}_{G^*}(b)$ and lower bounds on $\text{cc}_{G^*}(a)$, where the latter will be expressed as a function of k . We will then leverage the gap between these bounds, together with condition (ii) of Claim 5, to prove that $k + |N_G(a)|$ must be large. It is easiest to perform this analysis with a specific value of $d_{G^*}(a, b)$ in mind, since then (using the triangle inequality) we have some control over the ratios achieved on individual vertices. We begin by obtaining an upper bound on $\text{cc}_{G^*}(b)$.

Claim 6. $\text{cc}_{G^*}(b) < d_{G^*}(a, b) + \frac{4(n-2)}{5} + (d_{G^*}(a, b) + 1) \cdot \frac{n-2}{5}$.

Proof of Claim 6. Note that if $d_{G^*}(a, b) = 1$, then the claim follows from part (i) of Claim 5, using monotonicity to observe $\text{cc}_{G^*}(b) \leq \text{cc}_{G+ab}(b)$. Thus, we may assume that $d_{G^*}(a, b) \geq 2$. Let $O = V \setminus (B^p \cup \{a, b\})$. We claim it is sufficient to show that for each $v \in O$,

$$d_{G^*}(b, v) \leq d_{G+ab}(b, v) + d_{G^*}(a, b) - 1. \quad (1)$$

To see this, begin with a straightforward expression: $\text{cc}_{G^*}(b) = d_{G^*}(a, b) + |B^p| + \sum_{v \in O} d_{G^*}(b, v)$. Then using Inequality (1), we have

$$\text{cc}_{G^*}(b) \leq d_{G^*}(a, b) + |B^p| + \sum_{v \in O} d_{G+ab}(b, v) + (d_{G^*}(a, b) - 1) \cdot |O|.$$

By parts (i) and (iii) of Claim 5, $\sum_{v \in O} d_{G+ab}(b, v) < \frac{2(n-2)}{5}$, giving

$$\text{cc}_{G^*}(b) < d_{G^*}(a, b) + |B^p| + \frac{2(n-2)}{5} + (d_{G^*}(a, b) - 1) \cdot |O|.$$

Now, using part (iii) of Claim 5 we write $|B^p| = \frac{4(n-2)}{5} + x$ and $|O| = \frac{n-2}{5} - x$ for some positive x . Substituting, we have

$$\begin{aligned} \text{cc}_{G^*}(b) &< d_{G^*}(a, b) + \frac{4(n-2)}{5} + x + \frac{2(n-2)}{5} + (d_{G^*}(a, b) - 1) \cdot \left(\frac{n-2}{5} - x \right) \\ &= d_{G^*}(a, b) + \frac{4(n-2)}{5} + (d_{G^*}(a, b) + 1) \cdot \frac{n-2}{5} + (2 - d_{G^*}(a, b))x \\ &\leq d_{G^*}(a, b) + \frac{4(n-2)}{5} + (d_{G^*}(a, b) + 1) \cdot \frac{n-2}{5} \quad (\text{using } d_{G^*}(a, b) \geq 2). \end{aligned}$$

This proves that it suffices to show Inequality (1). Consider $v \in O$. Observe that if $d_G(b, v) \leq d_G(a, v)$, then $d_G(b, v) = d_{G+ab}(b, v)$. Then we simply observe that $d_{G^*}(b, v) \leq d_G(b, v) = d_{G+ab}(b, v) \leq d_{G+ab}(b, v) + d_{G^*}(a, b) - 1$, as desired. Thus, we may assume that $d_G(b, v) > d_G(a, v)$, in which case $d_{G+ab}(b, v) = 1 + d_{G+ab}(a, v)$. Then by the triangle inequality,

$$\begin{aligned} d_{G^*}(b, v) &\leq d_{G^*}(a, b) + d_{G^*}(a, v) \leq d_{G^*}(a, b) + d_G(a, v) = d_{G^*}(a, b) + d_{G+ab}(a, v) \\ &= d_{G^*}(a, b) + d_{G+ab}(b, v) - 1, \end{aligned}$$

which completes the proof. \square

We now combine Claims 5 and 6 to show that $d_{G^*}(a, b) \in \{1, 2\}$.

Claim 7. *There is no optimal solution G^* with $d_{G^*}(a, b) \geq 3$.*

Proof of Claim 7. By the triangle inequality, for each $v \in B^p$, $d_G(a, v) \geq d_{G^*}(a, b) - 1$. Moreover, for every vertex $v \in O = V \setminus (A^p \cup \{a, b\})$, trivially $d_{G^*}(a, v) \geq 1$. Using part (iii) of Claim 5, we write $|B^p| = \frac{4(n-2)}{5} + x$ and $|O| = \frac{n-2}{5} - x$, for some positive x . Then

$$\begin{aligned} \text{cc}_{G^*}(a) &\geq d_{G^*}(a, b) + (d_{G^*}(a, b) - 1) \cdot \left(\frac{4(n-2)}{5} + x \right) + \frac{n-2}{5} - x \\ &= d_{G^*}(a, b) + (d_{G^*}(a, b) - 1) \cdot \frac{4(n-2)}{5} + \frac{n-2}{5} + (d_{G^*}(a, b) - 2)x \\ &> d_{G^*}(a, b) + (d_{G^*}(a, b) - 1) \cdot \frac{4(n-2)}{5} + \frac{n-2}{5} \quad (\text{If } d_{G^*}(a, b) \leq 2 \text{ we're done.}) \\ &= d_{G^*}(a, b) + (4d_{G^*}(a, b) - 3) \cdot \frac{n-2}{5}. \end{aligned}$$

Meanwhile, by Claim 6,

$$\text{cc}_{G^*}(b) < d_{G^*}(a, b) + (d_{G^*}(a, b) + 5) \cdot \frac{n-2}{5}.$$

Finally, by part (ii) of Claim 5,

$$\text{cc}_{G^*}(a) \leq \frac{12}{11} \cdot \text{cc}_{G^*}(b).$$

Now we complete the proof by substituting our bounds and solving for $d_{G^*}(a, b)$.

$$\begin{aligned}
d_{G^*}(a, b) + (4d_{G^*}(a, b) - 3) \cdot \frac{n-2}{5} &< \frac{12}{11} \left[d_{G^*}(a, b) + (d_{G^*}(a, b) + 5) \cdot \frac{n-2}{5} \right] \\
\frac{32}{11} \cdot d_{G^*}(a, b) \cdot \frac{n-2}{5} &< \frac{1}{11} \cdot d_{G^*}(a, b) + \frac{93}{11} \cdot \frac{n-2}{5} \\
32 \cdot d_{G^*}(a, b) \cdot \frac{n-2}{5} &< d_{G^*}(a, b) + 93 \cdot \frac{n-2}{5} \\
d_{G^*}(a, b) &< \frac{93 \cdot \frac{n-2}{5}}{32 \cdot \frac{n-2}{5} - 1} \\
d_{G^*}(a, b) &< \frac{93 \cdot \frac{n-2}{5}}{32 \cdot \frac{n-2}{5} - 1 \cdot \frac{n-2}{5}} \quad (\text{whenever } n \geq 7) \\
d_{G^*}(a, b) &< \frac{93}{31} = 3.
\end{aligned}$$

To complete the proof, note that when $n < 7$ we may assume that $d_{G^*}(a, b) < 3$. Otherwise, it is impossible to satisfy part (iii) of Claim 5. \square

Thus, either $d_{G^*}(a, b) = 1$ or $d_{G^*}(a, b) = 2$. We handle the easy case first.

Claim 8. *If there exists an optimal G^* with $d_{G^*}(a, b) = 1$, then $|N_G(a)| + k > \frac{2n}{5} > \frac{n}{6}$.*

Proof of Claim 8. Assume otherwise, i.e., that $|N_G(a)| + k \leq \frac{2n}{5}$. Then $\text{cc}_{G^*}(a) \geq \frac{2n}{5} + 2 \left(\frac{3n}{5} - 1 \right) = \frac{8n}{5} - 2$. Meanwhile, since $d_{G^*}(a, b) = 1$, $\text{cc}_{G^*}(b) \leq \text{cc}_{G+ab}(b) < \frac{6(n-2)}{5}$, where the latter inequality is from part (i) of Claim 5. But then by part (ii) of Claim 5 we can obtain the following contradiction:

$$\begin{aligned}
\text{cc}_{G^*}(a) &< \frac{12}{11} \text{cc}_{G^*}(b) \\
\frac{8n}{5} - 2 &< \frac{12}{11} \left(\frac{6n}{5} - \frac{12}{5} \right) \\
\frac{88n}{55} - \frac{110}{55} &< \frac{72n}{55} - \frac{124}{55} \\
16n + 14 &< 0 \\
n &< 0.
\end{aligned}$$

\square

It follows that in the remainder of the proof, we may assume that $d_{G^*}(a, b) = 2$. Our goal is to show that if $\text{cc-ratio}_{G+ab}(a, b) < \frac{6}{11} \cdot \text{cc-ratio}_{G^*}(a, b)$, then $k + |N_G(a)| \geq \frac{n}{6}$. We begin by using Claim 6 to establish an upper bound for $\text{cc}_{G^*}(b)$: $\text{cc}_{G^*}(b) < \frac{7(n-2)}{5} + 2$.

Now we will find a lower bound for $\text{cc}_{G^*}(a)$. Once again we define $O = V \setminus (B^p \cup \{a, b\})$, and use part (iii) of Claim 5 to write $|B^p| = \frac{4(n-2)}{5} + x$ and $|O| = \frac{n-2}{5} - x$ for some positive x . We will also use the facts that at least $|B^p| - k$ private neighbors of b in G remain private neighbors of b in G^* , and that $\tilde{d}_{G^*}(a, O) \geq 1$.

$$\begin{aligned}
\text{cc}_{G^*}(a) &\geq d_{G^*}(a, b) + 2 \cdot (|B^p| - k) + k + \tilde{d}_{G^*}(a, O) \cdot |O| \\
&\geq d_{G^*}(a, b) + 2 \cdot |B^p| + 1 \cdot |O| - k \\
&= d_{G^*}(a, b) + \frac{8(n-2)}{5} + 2x + \frac{n-2}{5} - x - k \\
&= 2 + \frac{9(n-2)}{5} - k + x > 2 + \frac{9(n-2)}{5} - k.
\end{aligned}$$

We now complete the proof using part (ii) of Claim 5, which implies $\text{cc}_{G^*}(a) < \frac{12}{11} \cdot \text{cc}_{G^*}(b)$. By substituting our upper bound on $\text{cc}_{G^*}(b)$ and our lower bound on $\text{cc}_{G^*}(a)$, we have

$$2 + \frac{9(n-2)}{5} - k < \frac{12}{11} \left(\frac{7(n-2)}{5} + 2 \right),$$

which simplifies to

$$\begin{aligned} 2 + \frac{99(n-2)}{55} - k &< \frac{84(n-2)}{55} + \frac{24}{11} \\ \frac{15(n-2)}{55} - \frac{2}{11} &< k \\ \frac{3(n-2)}{11} - \frac{2}{11} &< k \\ \frac{3n}{11} - \frac{8}{11} &< k \\ \frac{n}{6} &< k \quad (\text{whenever } n > 3). \end{aligned}$$

We note that the desired condition is trivially true when $n \leq 3$, since we may assume that $k \geq 1$. Thus, the proof is complete. \square

Putting all the pieces together, we achieve the main result. Using BFS for centrality computations, a naive implementation of Algorithm 1 runs in $O(k(n+m))$ time, but it is simple to achieve quasilinear time. Instead of evaluating the condition on line 9 in each iteration of the loop, we check at the end of the loop whether $\text{cc}_{G+S}(a) \leq \text{cc}_{G+S}(b)$. If so, then (using the fact that $\text{cc}_{G+S}(a)$ monotonically decreases while $\text{cc}_{G+S}(b)$ stays constant) we determine the correct return value via a binary search over the $k-1$ previous states of S .

Theorem 4. *Algorithm 1 is a $\frac{6}{11}$ -approximation for CLOSENESS RATIO IMPROVEMENT, running in $O((n+m)\log k)$ time.*

5 Conclusion and Open Problems

In this paper, we used closeness centrality to represent social capital in networks and introduced two novel problems aimed at equalizing this capital between two specified nodes via edge additions. For CLOSENESS RATIO IMPROVEMENT, we proved NP-hardness for all target ratios $\tau \in (\frac{1}{2}, 1]$, while observing that the problem is trivially solvable $\tau \leq \frac{1}{2}$: just add the edge ab between the relevant vertices. We then showed how to quantify and harness the structure of non-trivial instances, i.e., instances where adding ab is not sufficient, leading to a scalable (quasilinear time) $\frac{6}{11}$ -approximation. Additionally, we proved the problem is inapproximable within any factor $> \frac{6e}{5e+1-\varepsilon} \approx .932$. For CLOSENESS GAP MINIMIZATION, we also showed NP-hardness and further ruled out any multiplicative approximation.

Motivated by the broader goal of promoting fairness across an entire network, we now propose two new problems that extend the scope of CLOSENESS RATIO IMPROVEMENT and CLOSENESS GAP MINIMIZATION. First, given two groups of vertices, the goal is to add at most k edges to the graph to maximize the ratio between the least central node in each group. In the following, we write $\text{cc-worst}_G(A)$ for $\arg \max_{a \in A} \text{cc}_G(a)$.

GROUP CLOSENESS RATIO IMPROVEMENT

Input: A graph $G = (V, E)$, two groups $A, B \subseteq V$, $A \cap B = \emptyset$, and $k \in \mathbb{N}$.

Task: Find $S \subseteq V^2 \setminus E$, $|S| \leq k$, maximizing $\text{cc-ratio}_{G+S}(\text{cc-worst}_{G+S}(A), \text{cc-worst}_{G+S}(B))$.

This is a generalization of CLOSENESS RATIO IMPROVEMENT, but it seems that new insights will be needed to tackle it. For example, if $|A|, |B|$ are unbounded with respect to k , then there is not a clear way to lift the trivial approach outlined in Section 2.2, which is also a crucial component of the algorithm in Section 4. Even when $|A|, |B|$ are bounded, it is not clear that useful generalizations of our Lemmas 1 and 2 can be obtained, and the lower bound on k required for a straightforward generalization of Lemma 3 is likely to be very large. One might also consider optimizing the ratio of mean or best centralities within each group; the minimax formulation presented above is common in the fairness literature, e.g., [7], usually with the justification that it focuses on aiding the most disadvantaged individuals.

We also propose the problem of adding edges to a graph to maximize the worst pairwise closeness ratio. A ratio close to 1 indicates that all nodes are similarly advantaged.

ALL-PAIRS CLOSENESS RATIO IMPROVEMENT

Input: A graph $G = (V, E)$ and $k \in \mathbb{N}$.

Task: Find $S \subseteq V^2 \setminus E$, $|S| \leq k$, maximizing $\frac{\min_{u \in V} cc_{G+S}(u)}{\max_{v \in V} cc_{G+S}(v)}$.

Finally, we suggest that while much algorithmic work has been devoted to graph modification with the intent of optimizing some characteristic of a given node, modification to *equalize* node characteristics is both well-motivated and relatively under-explored. For example, it may be interesting to consider variants of our problems that equalize betweenness or PageRank.

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