

# TRIVIAL KAZHDAN–LUSZTIG POLYNOMIALS AND CUBULATION OF THE BRUHAT GRAPH

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**ABSTRACT.** For  $(W, S)$  an arbitrary Coxeter system and any  $y \in W$ , we investigate the relationship between the condition that the Kazhdan–Lusztig polynomial  $P_{x,y}$  is trivial for all  $x \leq y$ , and the condition that the Bruhat graph for the interval  $[1, y]$  can be cubulated, meaning roughly that this graph can be spanned by a product of subintervals of  $\mathbb{Z}$ . In one direction, we combine results of Carrell–Peterson and Elias–Williamson to prove that if  $[1, y]$  can be cubulated, then  $P_{x,y} = 1$  for all  $x \leq y$ . We then investigate the converse of this statement. For  $(W, S)$  finite and  $w_0$  the longest element in  $W$ , so that  $P_{x,w_0} = 1$  for all  $x \in W$ , we construct cubulations of  $[1, w_0]$  in types  $A$  and  $B/C$ . However, in some exceptional types, we determine elements  $y \in W$  such that  $P_{1,y} = 1$  but  $[1, y]$  cannot be cubulated. We then prove that if there are infinitely many  $y \in W$  such that  $[1, y]$  can be cubulated, then  $(W, S)$  must be of type  $\tilde{A}_n$  for some  $n \geq 1$ . Finally, for  $(W, S)$  of type  $\tilde{A}_2$ , we exhibit a cubulation of  $[1, y]$  for each of the infinitely many  $y \in W$  such that  $P_{x,y} = 1$  for all  $x \leq y$ .

## 1. INTRODUCTION

In a landmark paper [KL79], Kazhdan and Lusztig constructed certain representations of the Hecke algebra  $\mathcal{H} = \mathcal{H}(W)$  associated to an arbitrary Coxeter group  $W$ , to study the representation theory of  $W$ . They introduced a family of polynomials  $P_{x,y}(q) = P_{x,y}$  defined for all  $x, y \in W$ , which are now called *Kazhdan–Lusztig polynomials*. These polynomials are the coefficients for the change-of-basis between the standard and canonical bases for  $\mathcal{H}$ .

Kazhdan–Lusztig polynomials quickly found many important applications in geometric and combinatorial representation theory and related areas. For example, when  $W$  is a finite Weyl group, the  $P_{x,y}$  are the Poincaré polynomials for the intersection cohomology of Schubert varieties [KL80], and are also  $q$ -analogues of the multiplicities for Verma modules [BB81, BK81].

Despite the importance of these polynomials, however, their combinatorial structure largely remains mysterious. For example, the Combinatorial Invariance Conjecture, due independently to Lusztig and Dyer [Dye87], is known only for several special cases (see [Bre04] for a survey). This conjecture has recently been the subject of investigation by machine learning models trained to predict Kazhdan–Lusztig polynomials from Bruhat intervals (see the *Nature* paper [DVB<sup>+</sup>21], as well as [BBD<sup>+</sup>22] in the mathematical literature).

**1.1. Results on trivial Kazhdan–Lusztig polynomials and cubulations.** Let  $(W, S)$  be any Coxeter system with  $S$  finite. For  $x, y \in W$ , we say that  $P_{x,y}$  is *trivial* if  $P_{x,y}$  is the constant polynomial  $P_{x,y} = 1$ . The goal of this paper is to investigate when Kazhdan–Lusztig polynomials are trivial using new techniques, from graph theory and geometric group theory.

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More precisely, we investigate the relationship between the condition  $P_{x,y} = 1$  for all  $x \leq y$ , and the condition that the Bruhat graph  $[1, y]_{\mathcal{B}}$  for the interval  $[1, y]$  can be *cubulated*, meaning that it has a spanning subgraph which is a *cubical lattice*; see Section 3.2 for the precise definitions, and Figure 1 for an example.

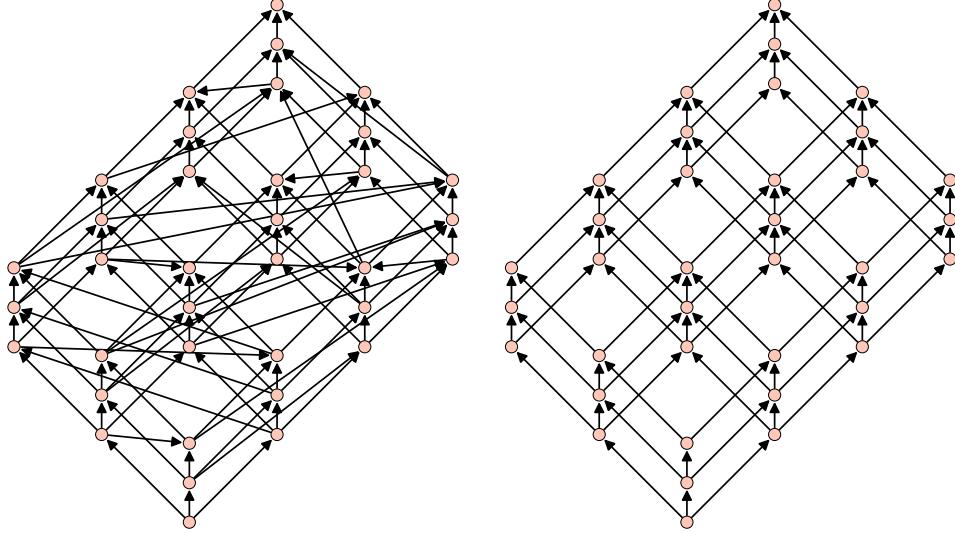


FIGURE 1. The Hasse diagram for the interval  $[1, s_1s_0s_2s_1s_0s_2s_0]$  in type  $\tilde{A}_2$  is depicted on the left. This graph is spanned by the cubical lattice  $\mathcal{C}(2, 2, 3)$  on the right, and hence the Bruhat graph for this interval can be cubulated.

Using the rank function for a cubical lattice, we calculate the Poincaré polynomial of the Bruhat interval  $[1, y]$  in the case that  $[1, y]_{\mathcal{B}}$  can be cubulated; see Proposition 5.3. Combining this with results of Carrell–Peterson [Car94] and Elias–Williamson [EW14], we obtain:

**Theorem 1.1.** *Let  $(W, S)$  be an irreducible Coxeter system with  $S$  finite, and let  $y \in W$ . If  $[1, y]_{\mathcal{B}}$  can be cubulated, then  $P_{x,y} = 1$  for all  $x \leq y$ .*

The majority of this paper then investigates the converse to Theorem 1.1. As explained further below, the converse is false in general; however, it does hold in certain contexts. For example, in the special cases where either  $y$  uses every simple reflection at most once, or  $y$  is an arbitrary element in type  $\tilde{A}_1$  or type  $I_2(m)$  for  $m \geq 3$ , then the graph  $[1, y]_{\mathcal{B}}$  can be cubulated and  $P_{x,y} = 1$  for all  $x \leq y$ ; see Section 6.1 for details.

If  $(W, S)$  is finite with longest element  $w_0$ , then  $P_{x,w_0} = 1$  for all  $x \in W$ ; see Exercise 7.14 in [Hum90]. We give a constructive proof of the following result in Section 6.2, using normal form forests for finite Coxeter systems.

**Theorem 1.2.** *Suppose  $(W, S)$  is finite. Then  $[1, w_0]_{\mathcal{B}}$  can be cubulated in the following types:*

- (1)  $A_n$  for  $n \geq 1$ ; and
- (2)  $B_n/C_n$  for  $n \geq 2$ .

*That is, in types A and B/C, the converse to Theorem 1.1 holds for  $y = w_0$ .*

In addition, via a computation by the first author available at [Bis24], we have verified the converse to Theorem 1.1 for all elements  $y \in W$  in the following types:  $A_3$ ,  $A_4$ ,  $B_3$ ,  $B_4$ ,  $D_4$ , and  $H_3$ . However, for the element  $y = w_0$ , we have found using this same code that the graph  $[1, w_0]_{\mathcal{B}}$  cannot be cubulated in types  $E_6$ ,  $F_4$ , or  $H_4$ . We prove in Proposition 6.5 that if the converse to Theorem 1.1 fails for some Coxeter system  $(W, S)$ , then it also fails for all Coxeter systems containing  $(W, S)$  as a subsystem. We thus obtain the following “poison subsystem” result in Section 6.3.

**Theorem 1.3.** *If  $(W, S)$  has a subsystem of type  $E_6$ ,  $F_4$ , or  $H_4$ , then there exists an element  $y \in W$  such that  $P_{x,y} = 1$  for all  $x \leq y$ , but  $[1, y]_{\mathcal{B}}$  cannot be cubulated. In particular, the converse to Theorem 1.1 does not hold in types  $E_7$  or  $E_8$ .*

Note that Theorem 1.3 puts strong restrictions on the simply-laced Coxeter systems for which the converse to Theorem 1.1 might hold; namely, their Dynkin diagrams cannot contain any  $E_6$  subdiagram. We leave type  $D$  as largely an open question: the code available at [Bis24] found a cubulation of  $[1, w_0]_{\mathcal{B}}$  in type  $D_5$ , but did not return any results in type  $D_6$  after running for more than four months on a cluster at the University of Sydney.

A Coxeter system  $(W, S)$  is said to be *minimal nonspherical* if  $W$  is infinite, but every proper parabolic subgroup of  $W$  is finite. For example, any irreducible affine Coxeter system is minimal nonspherical. All other minimal nonspherical Coxeter systems are reflection groups of hyperbolic space; see Remark 7.19. Utilizing a careful study of volume growth and Poincaré series in Coxeter groups, we obtain the following statement in Section 7.

**Theorem 1.4.** *Let  $(W, S)$  be a minimal nonspherical Coxeter system. If there are infinitely many distinct elements  $y \in W$  such that  $[1, y]_{\mathcal{B}}$  can be cubulated, then  $(W, S)$  is of type  $\tilde{A}_n$  for some  $n \geq 1$ .*

Our final result provides one complete example of a Coxeter system where the converse to Theorem 1.1 holds. We use the explicit formulas in type  $\tilde{A}_2$  provided by Libedinsky–Patimo [LP23] and Burrull–Libedinsky–Plaza [BLP23] to give a constructive proof of the following statement, in Section 8.

**Theorem 1.5.** *Suppose  $(W, S)$  is of type  $\tilde{A}_2$ . Then for all (the infinitely many)  $y \in W$  such that  $P_{x,y} = 1$  whenever  $x \leq y$ , the Bruhat graph  $[1, y]_{\mathcal{B}}$  can be cubulated. That is, the converse to Theorem 1.1 holds in type  $\tilde{A}_2$ .*

**1.2. Organization of the paper.** This work involves ideas from several different areas of mathematics, so we have opted to provide the relevant background throughout the paper, interspersed with our contributions. In Section 2, we recall key concepts for Coxeter systems, including Bruhat order and the construction of the normal form forest. Section 3 gives some background on directed graphs, including the Cayley and Bruhat graphs for a Coxeter system, and then introduces our main new concept, the cubulation of a directed graph. We provide the recursive combinatorial definition of Kazhdan–Lusztig polynomials and review some relevant literature in Section 4.

The proof of Theorem 1.1 then follows in Section 5, after stating some required facts about palindromic polynomials. Section 6 investigates several special cases of the converse to Theorem 1.1, including standard parabolic Coxeter elements, dihedral groups, and the case

$y = w_0$  in Theorem 1.2, as well as proving Theorem 1.3. Section 7 reviews formal power series and volume growth and then proves Theorem 1.4. Finally, we provide a constructive proof of Theorem 1.5, which establishes the converse to Theorem 1.1 in type  $\tilde{A}_2$ , in Section 8.

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## 2. COXETER SYSTEMS AND NORMAL FORMS

Section 2.1 reviews basic terminology about Coxeter systems and their partial orderings and associated Poincaré series. In Section 2.2, we recall the construction of the lexicographically first normal form for a Coxeter system, as well as the normal form forest for finite Coxeter groups.

**2.1. Coxeter systems, partial orders, and Poincaré series.** In this section, we briefly review our notation for Coxeter systems and two natural partial orders they admit, and define the associated Poincaré series. Throughout this work,  $(W, S)$  is a Coxeter system with finite generating set  $S = \{s_1, \dots, s_n\}$ . We say that  $(W, S)$  is a *finite* Coxeter system if the group  $W$  is finite. Denote by  $[n] = \{1, \dots, n\}$  and by  $\mathbb{N} = \{0, 1, 2, \dots\}$ . The *support* of an element  $x \in W$  is the set of elements of  $S$  which appear in some (hence any) reduced expression for  $x$ .

We write  $\ell$  for the word length on  $W$  with respect to  $S$ . Given  $x \in W$  such that  $\ell(x) = k$ , any product of the form  $x = s_{i_1} s_{i_2} \cdots s_{i_k}$  where  $s_{i_j} \in S$  is a *reduced expression*. The empty word is the identity in  $W$ , which we denote by 1. Denote by  $D_S(x)$  the *right descent-set* of  $x \in W$ ; that is,  $D_S(x) = \{s \in S \mid \ell(xs) < \ell(x)\}$ .

We write  $\leq_S$  for the (right) *weak order* on  $(W, S)$ . This partial order is generated by the relations  $x \leq_S xs$  for all  $x \in W$  and  $s \in S$  such that  $\ell(xs) = \ell(x) + 1$ ; see, for example, [BB05, Def. 3.1.1]. Note that since  $s \in S$ , the condition  $\ell(xs) = \ell(x) + 1$  is equivalent to  $\ell(xs) > \ell(x)$ , or  $s \notin D_S(x)$ .

We write  $\mathcal{T} = \mathcal{T}(W, S)$  for the set of *reflections* in  $(W, S)$ ; that is

$$\mathcal{T} = \{xsx^{-1} \mid x \in W, s \in S\}.$$

We write  $\leq$  for the (right) *Bruhat order* on  $(W, S)$ ; this is sometimes referred to as *strong order*, to contrast with the weak order on  $(W, S)$ . Recall that the Bruhat order is generated by  $x \leq xt$  for all  $x \in W$  and  $t \in \mathcal{T}$  such that  $\ell(xt) > \ell(x)$ ; see, for example, [BB05, Def. 2.1.1]. We will also sometimes use the following characterization of Bruhat order via subwords: if  $x, y \in W$ , then  $x \leq y$  if and only if for every reduced expression  $s_{i_1} \cdots s_{i_k}$  for  $y$ , there exists a reduced expression for  $x$  which is a (possibly non-consecutive) subword of  $s_{i_1} \cdots s_{i_k}$ ; see, for example, [BB05, Theorem 2.2.2]. We write  $[x, y]$  for intervals in Bruhat order. Note that  $[x, y]$  is a graded poset, with rank function given by the word length  $\ell$ .

Given any subset  $A$  of  $W$ , the formal power series

$$A(z) = \sum_{a \in A} z^{\ell(a)}$$

is called the *Poincaré series* of  $A$ , or, if the set  $A$  is finite, the *Poincaré polynomial* of  $A$ . In this work, we consider Poincaré polynomials with  $A = [1, y]$ , and Poincaré series with  $A = W$ .

To avoid confusion, we reserve the variable  $q$  for Kazhdan–Lusztig polynomials (see Section 4), and we use the variable  $z$  for all other polynomials or power series.

**2.2. Normal forms and forests.** In this section, we follow [BB05, Sec. 3.4], based upon the work of du Cloux [dC99], to review the construction of the normal form forest associated to a Coxeter system.

A *normal form* for the Coxeter system  $(W, S)$  is a specific choice of reduced expression for every element of  $W$ , of which there are typically several systematic choices. Recall that we have fixed an indexing of the generating set  $S = \{s_1, \dots, s_n\}$ , as the normal form depends greatly on this choice of total ordering.

The normal form we consider is the *lexicographically first normal form*; that is, for each  $x \in W$ , we choose the reduced expression for  $x$  which is first in the chosen lexicographic order on  $S$ . Denote this reduced expression by  $\text{NF}(x)$ . Note that if  $J \subseteq S$  and  $x \in W_J$ , where  $W_J$  denotes the standard parabolic subgroup generated by  $J$ , then  $\text{NF}(x)$  is the same as the lexicographically first normal form of  $x$  regarded as an element of the subgroup  $W_J$ . Proposition 2.1 below describes a factorization for this lexicographically first normal form.

For any  $J \subseteq S$ , write  ${}^J W$  for the set of minimal-length representatives of the right cosets  $W_J \backslash W$ . An element  $x \in W$  is in  ${}^J W$  if and only if no reduced expression for  $x$  begins with a letter from  $J$ . In the special case where  $J = \{s_1, \dots, s_j\} = \{s_i \mid i \in [j]\}$  for some  $j \in [n]$ , we will write  $W_{[j]}$  for the subgroup  $W_J$  and  ${}^{[j]} W$  for the set of minimal-length representatives of the cosets  $W_{[j]} \backslash W$ . An element  $x \in W$  is in  ${}^{[j]} W$  if and only if no reduced expression for  $x$  begins with a letter  $s_i$  where  $i \in [j]$ . It will be convenient to define  $W_\emptyset = W_{[0]}$  to be the trivial subgroup of  $W$ , in which case  ${}^{[0]} W = W$ . Since  $W_{[j-1]} \leq W_{[j]}$  for all  $j \in [n]$ , we denote by  ${}^{[j-1]}(W_{[j]})$  the set of minimal-length representatives of the right cosets  $W_{[j-1]} \backslash W_{[j]}$ . An element  $x \in W_{[j]}$  is in  ${}^{[j-1]}(W_{[j]})$  if and only if every reduced expression for  $x$  begins with  $s_j$ .

**Proposition 2.1** (Proposition 3.4.2 of [BB05]). *Let  $(W, S)$  be any Coxeter system. Any  $x \in W$  can be written uniquely as  $x = x_1 \cdots x_n$  where  $x_j \in {}^{[j-1]}(W_{[j]})$  for all  $j \in [n]$ . Moreover,  $\text{NF}(x) = \text{NF}(x_1) \cdots \text{NF}(x_n)$ .*

Now suppose that  $(W, S)$  is a finite Coxeter system. The *normal form forest* of  $(W, S)$  consists of edge-labeled rooted trees  $\tau_1, \dots, \tau_n$ , with vertices and edges defined as follows. For a fixed  $j \in [n]$ , the vertices of  $\tau_j$  correspond to the elements of the set  ${}^{[j-1]}(W_{[j]})$ . The edges of  $\tau_j$  are labeled by elements of  $S$  such that the (unique) path from the root of  $\tau_j$  to the vertex  $x_j \in {}^{[j-1]}(W_{[j]})$  is labeled by the reduced expression  $\text{NF}(x_j)$ . Then by Proposition 2.1, the set of all normal forms  $\{\text{NF}(x) \mid x \in W\}$  is obtained by concatenating all (possibly empty) rooted paths in  $\tau_1, \dots, \tau_n$ , in this order.

**Example 2.2.** Let  $(W, S)$  have type  $A_3$ , in which case  ${}^{[0]} W = W \cong S_4$ . Label the nodes of the Dynkin diagram from left to right. We then have that  ${}^{[0]}(W_{[1]})$  is the set of minimal-length representatives of the right cosets  $W_{[0]} \backslash W_{[1]} = W_{[1]}$ , where  $W_{[1]} = \langle s_1 \rangle$ . The tree  $\tau_1$  is thus labeled by 1, and the vertices of  $\tau_1$  correspond to the elements  $\{1, s_1\}$ .

Now consider the set of minimal-length representatives of the right cosets  $W_{[1]} \backslash W_{[2]}$ , where  $W_{[2]} = \langle s_1, s_2 \rangle$ . As every reduced expression in  ${}^{[1]}(W_{[2]})$  begins with  $s_2$ , the edge label

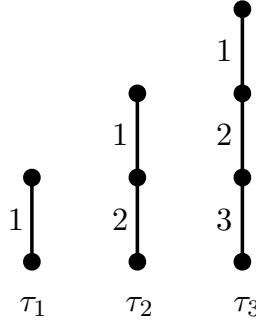


FIGURE 2. The normal form forest for type  $A_3$  obtained by labeling nodes of the Dynkin diagram from left to right.

immediately above the root vertex of  $\tau_2$  is 2. The number of minimal length elements in  $W_{[1]} \setminus W_{[2]}$  equals  $|W_{[2]}|/|W_{[1]}| = 6/2 = 3$ . The root and the adjacent vertex correspond to  $\{1, s_2\}$ , so there is one remaining vertex, which must be labeled by 1 in order that the elements of  $W_{[1]} \setminus W_{[2]}$  all begin with  $s_2$ . The tree  $\tau_2$  is thus labeled by 2, then 1, as shown in Figure 2, with the vertices corresponding to the elements  $\{1, s_2, s_2s_1\}$ .

This pattern continues, such that the label above the root vertex of  $\tau_3$  is 3. The additional edge-labels are given by the remaining indices listed in decreasing order, so that the non-identity elements of  ${}^{[2]}(W_{[3]}) = \{1, s_3, s_3s_2, s_3s_2s_1\}$  necessarily begin with  $s_3$ ; see Figure 2.

The normal form for  $w_0$ , for example, given by applying Proposition 2.1 to this normal form forest, is then  $w_0 = s_1 \cdot s_2s_1 \cdot s_3s_2s_1$ , obtained by concatenating the expressions corresponding to each of the paths  $\tau_1, \tau_2, \tau_3$ , where we read upward from the root of each tree.

### 3. DIRECTED GRAPHS AND CUBICAL LATTICES

This section begins with a brief review of several key concepts about directed graphs, in Section 3.1, where we also recall the Cayley and Bruhat graphs for a Coxeter system. We introduce our main new concept in Section 3.2: this is the notion of a cubulation, which forms a poset graded by the  $L^1$ -norm on  $\mathbb{Z}^N$ .

**3.1. Directed graphs.** In this section, we review basic terminology about directed graphs and recall two important examples: the Cayley graph and the Bruhat graph for a Coxeter system  $(W, S)$ .

Let  $\mathcal{D}$  be a directed graph. We write  $V(\mathcal{D})$  for the vertex set of  $\mathcal{D}$  and  $E(\mathcal{D})$  for the edge set of  $\mathcal{D}$ . All of the directed graphs  $\mathcal{D}$  appearing in this work will be *simple*, meaning that all edges have distinct start- and end-vertices, and that for any two distinct vertices  $u, v \in V(\mathcal{D})$ , there is at most one edge whose endpoints are  $\{u, v\}$ . Hence, we regard  $E(\mathcal{D})$  as a set of ordered pairs  $(u, v) \in V(\mathcal{D}) \times V(\mathcal{D})$  of distinct elements  $u \neq v$  of  $V(\mathcal{D})$ .

We will often consider the following two kinds of subgraphs of directed graphs.

**Definition 3.1** (Induced and spanning subgraphs). Let  $\mathcal{D}$  be a directed graph.

- (1) Let  $V'$  be a subset of  $V(\mathcal{D})$ . The *subgraph induced by  $V'$*  is the subgraph of  $\mathcal{D}$  with vertex set  $V'$  and edge set  $\{(u, v) \in E(\mathcal{D}) \mid u, v \in V'\}$ .
- (2) Let  $\mathcal{D}'$  be a subgraph of  $\mathcal{D}$ . We say that  $\mathcal{D}'$  is a *spanning subgraph* of  $\mathcal{D}$ , or that  $\mathcal{D}'$  *spans*  $\mathcal{D}$ , if  $V(\mathcal{D}') = V(\mathcal{D})$ .

Note that if  $\mathcal{D}'$  spans  $\mathcal{D}$ , then the edge set  $E(\mathcal{D}')$  will, in general, be a proper subset of  $E(\mathcal{D})$ .

We will consider several different directed graphs in this work. The following two graphs correspond to the (right) weak order  $\leq_S$  and the (right) Bruhat order  $\leq$ , respectively, which were defined in Section 2.1.

The (*directed right*) *Cayley graph*  $\mathcal{G} = \mathcal{G}(W, S)$  of  $W$  is the directed graph with vertex set  $V(\mathcal{G}) = W$  and an edge from  $x \in W$  to  $xs \in W$ , where  $s \in S$ , if and only if  $\ell(xs) = \ell(x) + 1$ . Equivalently, the edges of  $\mathcal{G}$  correspond to the covering relations in the (right) weak order  $\leq_S$  on  $W$ . For any  $x, y \in W$  with  $x \leq_S y$ , we write  $[x, y]_S$  for the subgraph of  $\mathcal{G}$  induced by the vertex set  $\{z \in W \mid x \leq_S z \leq_S y\}$ . In other words,  $[x, y]_S$  is the Hasse diagram for the partial order  $\leq_S$  on the interval between  $x$  and  $y$ . The Cayley graph for type  $A_2$  is seen in Figure 3 by taking only the black edges, oriented upwards.

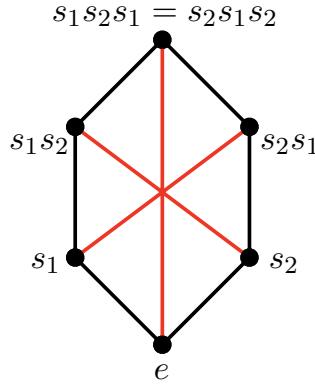


FIGURE 3. The Bruhat graph for type  $A_2$ , with all edges oriented upwards. The Hasse diagram for the Bruhat order consists of all edges except for the vertical red one, and the subgraph consisting of the black edges is the Cayley graph in type  $A_2$ .

The *Bruhat graph*  $\mathcal{B} = \mathcal{B}(W, S)$  is the directed graph with vertex set  $V(\mathcal{B}) = W$  and an edge from  $x \in W$  to  $xt \in W$ , for  $t \in T$ , if and only if  $\ell(xt) > \ell(x)$ . Equivalently, the edges of  $\mathcal{B}$  correspond to the relations which generate the (right) Bruhat order  $\leq$  on  $W$ . The Bruhat graph for type  $A_2$  is shown in Figure 3. For any  $x, y \in W$  with  $x \leq y$ , we write  $[x, y]_{\mathcal{B}}$  for the subgraph of  $\mathcal{B}$  induced by the vertex set  $[x, y]$ . By definition, the Hasse diagram for (the restriction of) the Bruhat order on  $[x, y]$  spans the subgraph  $[x, y]_{\mathcal{B}}$ . Since  $S \subseteq T$ , the subgraph  $[x, y]_S$  of the directed Cayley graph  $\mathcal{G}$  spans this Hasse diagram, and hence also spans  $[x, y]_{\mathcal{B}}$ .

**3.2. Cubical lattices.** We now introduce a special family of directed graphs which we call *cubical lattices*, and discuss their structure as a graded poset. Recall that by  $\mathbb{N}$  we mean the set of non-negative integers.

**Definition 3.2** (Cubical lattice). Fix a positive integer  $N$  and  $k_1, \dots, k_N \in \mathbb{N}$ . We define  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  to be the directed graph with vertex and edge sets

$$V(\mathcal{C}) = \{(m_1, m_2, \dots, m_N) \in \mathbb{Z}^N \mid 0 \leq m_i \leq k_i \text{ for each } i \in [N]\}$$

and

$$E(\mathcal{C}) = \{(u, v) \in V(\mathcal{C}) \times V(\mathcal{C}) \mid v - u = \vec{e}_i \text{ for some } i \in [N]\},$$

where  $\vec{e}_i$  denotes the  $i^{\text{th}}$  standard basis vector in  $\mathbb{Z}^N$ . A directed graph  $\mathcal{D}$  is a *cubical lattice* if  $\mathcal{D}$  is isomorphic to  $\mathcal{C}(k_1, \dots, k_N)$  for some  $N \geq 1$  and  $k_1, \dots, k_N \in \mathbb{N}$ .

If  $k_1 = \dots = k_N = 0$ , so that  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  is a single vertex (with no edges), we will sometimes write  $\mathcal{C}(0)$  instead of  $\mathcal{C}(0, \dots, 0)$ . We call  $\mathcal{C}(0)$  the *trivial cubical lattice*. A cubical lattice  $\mathcal{C}(k_1, \dots, k_N)$  with at least one  $k_i > 0$  is a *nontrivial cubical lattice*. We note that we do not require any ordering on the parameters  $k_1, \dots, k_N$ , since it will sometimes be convenient to consider  $k_1, \dots, k_N$  which are not (for example) weakly increasing. However, we will also use the following “canonical form” for cubical lattices.

**Lemma 3.3.** *Any nontrivial cubical lattice  $\mathcal{C}(k_1, \dots, k_N)$  is naturally isomorphic to a cubical lattice  $\mathcal{C}(k'_1, \dots, k'_{N'})$  where  $k'_j > 0$  for  $1 \leq j \leq N'$ , and in addition, if desired,  $k'_1 \leq \dots \leq k'_{N'}$ .*

*Proof.* By permuting coordinates in  $\mathbb{Z}^N$ , we see that for any permutation  $\sigma$  of  $[N]$ , the directed graph  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  is naturally isomorphic to  $\mathcal{C}(k_{\sigma(1)}, \dots, k_{\sigma(N)})$ . Hence in particular, for some  $0 \leq i < N$ , the graph  $\mathcal{C}$  is isomorphic to  $\mathcal{C}' = \mathcal{C}(k'_1, \dots, k'_{N'})$  where  $k'_1 = \dots = k'_i = 0$  and  $k'_j > 0$  for all  $i+1 \leq j \leq N$ . Moreover, since  $\mathcal{C}$  and hence  $\mathcal{C}'$  is nontrivial, by dropping the first  $i$  coordinates we see that  $\mathcal{C}'$  is naturally isomorphic to  $\mathcal{C}(k'_{i+1}, \dots, k'_{N'})$ . The result then follows by relabeling the parameters, and, if desired, permuting them so that they are weakly increasing.  $\square$

An equivalent formulation of Definition 3.2 is that the cubical lattice  $\mathcal{C}(k_1, \dots, k_N)$  is the Hasse diagram for the product of the subintervals  $[0, k_i]$  of  $\mathbb{Z}$ , with each such subinterval a poset under the usual ordering. This leads to the following result.

**Lemma 3.4.** *Let  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  be a cubical lattice. Then the  $L^1$ -norm on  $\mathbb{Z}^N$  given by*

$$\|(m_1, \dots, m_N)\|_1 = \sum_{i=1}^N m_i$$

*induces the structure of a graded poset on the vertex set  $V(\mathcal{C})$ . Moreover, this is the only possible rank function on  $V(\mathcal{C})$ .*

*Proof.* The vertex set of any product of subintervals of  $\mathbb{N}$ , with the usual ordering on each subinterval, is a graded poset with rank function induced by the  $L^1$ -norm on  $\mathbb{Z}^N$ . Hence, viewing the cubical lattice  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  as the product of the subintervals  $[0, k_i]$ , the  $L^1$ -norm on  $\mathbb{Z}^N$  induces the structure of a graded poset on  $V(\mathcal{C})$ .

For uniqueness, we induct on  $\sum_{i=1}^N k_i$ . Observe that the result holds trivially when this sum equals 0, equivalently  $\mathcal{C} = \mathcal{C}(0)$  is trivial. Now suppose  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  is nontrivial. By Lemma 3.3, we may assume, up to isomorphism of directed graphs (which will preserve any grading), that  $1 \leq k_1 \leq k_2 \leq \dots \leq k_N$ .

Define  $V' \subseteq V(\mathcal{C})$  to be the set of vertices  $(m_1, \dots, m_N) \in V(\mathcal{C})$  such that  $0 \leq m_1 \leq k_1 - 1$  and  $0 \leq m_i \leq k_i$  for  $i \in \{2, \dots, N\}$ , and let  $\mathcal{C}'$  be the subgraph of  $\mathcal{C}$  induced by  $V'$ . Then  $\mathcal{C}'$  is naturally isomorphic to the cubical lattice  $\mathcal{C}(k_1 - 1, k_2, \dots, k_N)$ . So by induction, there is a unique rank function on  $V(\mathcal{C}')$ , namely that induced by the  $L^1$ -norm on  $\mathbb{Z}^N$ .

Now the vertices of  $\mathcal{C}$  which are not in  $\mathcal{C}'$  are given by

$$V(\mathcal{C}) \setminus V' = \{(k_1, m_2, \dots, m_N) \mid 0 \leq m_i \leq k_i \text{ for } i \in \{2, \dots, N\}\}.$$

Let  $v = (k_1, m_2, \dots, m_N)$  be in  $V(\mathcal{C}) \setminus V'$ , and define  $v' = (k_1 - 1, m_2, \dots, m_N)$ . Notice that  $v' \in V'$ , and that there is an edge in  $\mathcal{C}$  from  $v'$  to  $v$ . Hence the only possible rank of  $v$  which is compatible with the unique rank function on  $V(\mathcal{C}')$  is

$$\|v'\|_1 + 1 = \left( (k_1 - 1) + \sum_{i=2}^N m_i \right) + 1 = k_1 + \sum_{i=2}^N m_i.$$

But this sum equals  $\|v\|_1$ , which completes the proof.  $\square$

In order to further investigate the cubical lattice  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  as a graded poset, define

$$d_{\mathcal{C}} = \sum_{i=0}^N k_i = \|(k_1, \dots, k_N)\|_1.$$

Then by Lemma 3.4, the maximum rank of any vertex of  $\mathcal{C}$  is  $d_{\mathcal{C}}$  (and the unique vertex of rank equal to  $d_{\mathcal{C}}$  is  $(k_1, \dots, k_N)$ ). For  $0 \leq j \leq d_{\mathcal{C}}$ , we now define natural numbers

$$b_j(\mathcal{C}) = \#\{v \in V(\mathcal{C}) \mid \|v\|_1 = j\}$$

and the polynomial

$$g_{\mathcal{C}}(z) = \sum_{j=0}^{d_{\mathcal{C}}} b_j(\mathcal{C}) z^j.$$

That is,  $b_j(\mathcal{C})$  is the number of vertices of the cubical lattice  $\mathcal{C}$  of rank exactly  $j$ , and  $g_{\mathcal{C}}(z)$  is the corresponding generating function.

The next result completely determines the polynomial  $g_{\mathcal{C}}(z)$ . Recall that a *quantum polynomial* is a polynomial of the form

$$\frac{z^{m+1} - 1}{z - 1} = 1 + z + z^2 + \dots + z^m$$

for some  $m \in \mathbb{N}$ .

**Lemma 3.5.** *Let  $\mathcal{C} = \mathcal{C}(k_1, \dots, k_N)$  be a cubical lattice. Then the generating function  $g_{\mathcal{C}}(z)$  is given by the following product of quantum polynomials:*

$$g_{\mathcal{C}}(z) = \frac{z^{k_1+1} - 1}{z - 1} \cdot \frac{z^{k_2+1} - 1}{z - 1} \cdots \frac{z^{k_N+1} - 1}{z - 1}.$$

*Proof.* For  $0 \leq j \leq d_{\mathcal{C}} = \sum_{i=1}^N k_i$ , the coefficient of  $z^j$  in the polynomial  $g_{\mathcal{C}}(z)$  is given by

$$\begin{aligned} b_j(\mathcal{C}) &= \#\{v \in V(\mathcal{C}) \mid \|v\|_1 = j\} \\ &= \# \left\{ (m_1, m_2, \dots, m_N) \in \mathbb{Z}^N \mid 0 \leq m_i \leq k_i \text{ and } \sum_{i=1}^N m_i = j \right\}. \end{aligned}$$

This final expression for  $b_j(\mathcal{C})$  is also clearly equal to the coefficient of  $z^j$  in the product

$$(1 + z + z^2 + \cdots + z^{k_1})(1 + z + z^2 + \cdots + z^{k_2}) \cdots (1 + z + z^2 + \cdots + z^{k_N}).$$

Thus

$$g_{\mathcal{C}}(z) = \frac{z^{k_1+1} - 1}{z - 1} \cdot \frac{z^{k_2+1} - 1}{z - 1} \cdots \frac{z^{k_N+1} - 1}{z - 1},$$

as required.  $\square$

Motivated by an application to Kazhdan–Lusztig polynomials, one of the main questions we consider in this paper is whether subgraphs  $[1, y]_{\mathcal{B}}$  of the Bruhat graph  $\mathcal{B}$  have spanning subgraphs which are cubical lattices. In order to discuss this phenomenon succinctly, we introduce the following terminology.

**Definition 3.6** (Cubulation). A directed graph  $\mathcal{D}$  can be *cubulated* if there is a cubical lattice  $\mathcal{C}$  which is isomorphic to a spanning subgraph of  $\mathcal{D}$ . In this case, we may say that  $\mathcal{C}$  *cubulates*  $\mathcal{D}$ , that  $\mathcal{D}$  is *cubulated* by  $\mathcal{C}$ , or that  $\mathcal{C}$  is a *cubulation* of  $\mathcal{D}$ .

See Figure 1 in the introduction for an example. We note that there may be more than one way of cubulating a given directed graph  $\mathcal{D}$ .

#### 4. KAZHDAN–LUSZTIG POLYNOMIALS

In Section 4.1, we provide the recursive combinatorial definition of  $R$ -polynomials and Kazhdan–Lusztig polynomials, and recall a fundamental result of Elias and Williamson [EW14]. In Section 4.2, we recall a result of Carrell and Peterson [Car94] characterizing trivial Kazhdan–Lusztig polynomials.

**4.1. Kazhdan–Lusztig polynomials.** We now define  $R$ -polynomials and Kazhdan–Lusztig polynomials, following [BB05, Sec. 5.1], which also provides detailed references to the primary literature; see also [Hum90, Ch. 7].

The *R-polynomial* is a polynomial in  $q$  with integer coefficients, depending on a pair of elements  $x, y \in W$ , denoted  $R_{x,y}(q)$  or often simply  $R_{x,y}$ . These  $R$ -polynomials can be defined recursively as follows.

**Theorem 4.1** (Theorem 5.1.1 of [BB05]). *Let  $(W, S)$  be any Coxeter system. There exists a unique family of polynomials*

$$\{R_{x,y}(q) \mid x, y \in W\} \subseteq \mathbb{Z}[q]$$

*such that all of the following hold:*

- (1) *if  $x \not\leq y$ , then  $R_{x,y}(q) = 0$ ;*
- (2) *if  $x = y$ , then  $R_{x,y}(q) = 1$ ; and*
- (3) *if  $s \in D_S(y)$ , then*

$$R_{x,y}(q) = \begin{cases} R_{xs,ys} & \text{if } s \in D_S(x) \\ qR_{xs,ys} + (q-1)R_{x,ys} & \text{if } s \notin D_S(x). \end{cases}$$

We now provide the recursive definition of the *Kazhdan–Lusztig polynomial*  $P_{x,y}(q) = P_{x,y}$  for  $x, y \in W$ . This definition also crucially involves the  $R$ -polynomials from Theorem 4.1.

**Theorem 4.2** (Theorem 5.1.4 of [BB05]). *Let  $(W, S)$  be any Coxeter system. There is a unique family of polynomials*

$$\{P_{x,y}(q) \mid x, y \in W\} \subseteq \mathbb{Z}[q]$$

such that all of the following hold:

- (1) if  $x \not\leq y$ , then  $P_{x,y}(q) = 0$ ;
- (2) if  $x = y$ , then  $P_{x,y}(q) = 1$ ;
- (3) if  $x < y$ , then  $\deg(P_{x,y}(q)) \leq \frac{1}{2}(\ell(y) - \ell(x) - 1)$ ; and
- (4) if  $x \leq y$ , then

$$q^{\ell(y)-\ell(x)} P_{x,y}\left(\frac{1}{q}\right) = \sum_{w \in [x,y]} R_{x,w}(q) P_{w,y}(q).$$

A fundamental result of Elias and Williamson [EW14] proves that in fact the Kazhdan–Lusztig polynomial  $P_{x,y}(q)$  has non-negative coefficients for all  $x, y \in W$ .

**Theorem 4.3** (Corollary 1.2 of [EW14]). *Let  $(W, S)$  be any Coxeter system. For any  $x, y \in W$ , the Kazhdan–Lusztig polynomial satisfies  $P_{x,y}(q) \in \mathbb{Z}_{\geq 0}[q]$ .*

**4.2. Trivial Kazhdan–Lusztig polynomials.** We say that  $P_{x,y}$  is *trivial* if  $P_{x,y}(q) = 1$  is constant. A result of Carrell and Peterson [Car94] provides several equivalent criteria for when Kazhdan–Lusztig polynomials are trivial. (Note that in [Car94] the left Bruhat order is used, but all results hold equally well for the right Bruhat order.) This result first requires some additional notation.

For any  $y \in W$ , denote by  $a(y)$  the *average length* of the elements of the interval  $[1, y]$ , that is,

$$a(y) = \frac{1}{\#[1, y]} \sum_{x \leq y} \ell(x).$$

For any  $y \in W$  and any  $0 \leq j \leq \ell(y)$ , define the nonnegative integer  $c_j(y)$  by

$$c_i(y) = \#\{x \leq y \mid \ell(x) = i\}.$$

We then write  $p_y(z)$  or sometimes just  $p_y$  for the Poincaré polynomial of the (finite) set  $[1, y] \subseteq W$ . That is,

$$p_y(z) = p_y = \sum_{j=0}^{\ell(y)} c_j(y) z^j.$$

Let  $f(z)$  be any polynomial of degree  $m \geq 0$ , given by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m.$$

We say that  $f(z)$  is *palindromic* if  $a_j = a_{m-j}$  for all  $0 \leq j \leq m$ .

**Theorem 4.4** (Theorem B of [Car94]). *Let  $(W, S)$  be any Coxeter system. Suppose  $y \in W$  is such that for each  $x \leq y$ , the polynomial  $P_{x,y}(q)$  has non-negative coefficients. Then the following are equivalent:*

- (1)  $P_{x,y}(q) = 1$  for each  $x \leq y$ ;

- (2)  $\#\{t \in \mathcal{T} \mid x < xt \leq y\} = \ell(y) - \ell(x)$  for all  $x \leq y$ ;
- (3)  $a(y) = \frac{\ell(y)}{2}$ ; and
- (4)  $p_y(z)$  is palindromic.

By combining Theorems 4.3 and 4.4, we obtain the following corollary. Note that we only record the implications that we actually use later, in Sections 5.2 and 8.1.

**Corollary 4.5.** *Let  $(W, S)$  be any Coxeter system, and let  $y \in W$ . Then  $p_y$  is palindromic if and only if  $P_{x,y} = 1$  for all  $x \leq y$ .*

## 5. CUBICAL LATTICES AND PALINDROMIC POINCARÉ POLYNOMIALS

The goal of this section is to prove Theorem 1.1, stating that if the Bruhat graph  $[1, y]_{\mathcal{B}}$  can be cubulated, then the Kazhdan–Lusztig polynomials  $P_{x,y}$  are trivial for all  $x \leq y$ . First, we gather some necessary facts about the family of palindromic polynomials in Section 5.1. The proof of Theorem 1.1 then follows in Section 5.2.

**5.1. Palindromic polynomials.** With the goal of applying Corollary 4.5, we now gather several additional facts and a definition concerning palindromic polynomials. We first record the elementary fact that the set of palindromic polynomials is closed under multiplication.

**Lemma 5.1.** *If  $f(z)$  and  $g(z)$  are palindromic polynomials, then  $f(z)g(z)$  is palindromic.*

One special family of palindromic polynomials are the *quantum polynomials*, which we defined in Section 3.2 as polynomials of the form

$$\frac{z^{m+1} - 1}{z - 1} = 1 + z + z^2 + \cdots + z^m$$

for some  $m \geq 0$ . Applying Lemma 5.1 gives us the following.

**Corollary 5.2.** *Any product of quantum polynomials is palindromic.*

**5.2. Cubulation implies trivial Kazhdan–Lusztig polynomials.** We now complete the proof of Theorem 1.1, using the following result, which recognizes the Poincaré polynomial for an element  $y \in W$  such that  $[1, y]_{\mathcal{B}}$  can be cubulated as a product of quantum polynomials. Recall from Section 3.2 that for any cubical lattice  $\mathcal{C}$ , we denote by  $g_{\mathcal{C}}(z)$  the polynomial in which the coefficient of  $z^j$  is the number of vertices of  $\mathcal{C}$  of rank exactly  $j$ .

**Proposition 5.3.** *Let  $(W, S)$  be any Coxeter system, and let  $y \in W$ . Suppose that the subgraph  $[1, y]_{\mathcal{B}}$  of the Bruhat graph  $\mathcal{B}$  is cubulated by  $\mathcal{C} = \mathcal{C}(k_1, k_2, \dots, k_N)$ , where  $N \in \mathbb{N}$  and  $k_1, \dots, k_N \in \mathbb{N}$ . Then the Poincaré polynomial of  $[1, y]$  is given by the following product of quantum polynomials:*

$$p_y(z) = \left( \frac{z^{k_1+1} - 1}{z - 1} \right) \left( \frac{z^{k_2+1} - 1}{z - 1} \right) \cdots \left( \frac{z^{k_N+1} - 1}{z - 1} \right).$$

Moreover, if  $k_j \geq 1$  for all  $1 \leq j \leq N$ , then  $N$  is the cardinality of the support of  $y$ .

*Proof.* Let  $\varphi$  be an isomorphism from  $\mathcal{C}$  to a spanning subgraph of  $[1, y]_{\mathcal{B}}$ . Then since  $\varphi(\mathcal{C})$  spans  $[1, y]_{\mathcal{B}}$ , we have  $V(\varphi(\mathcal{C})) = V([1, y]_{\mathcal{B}}) = [1, y]$ . Now the graded poset  $[1, y]$  and hence its spanning subgraph  $\varphi(\mathcal{C})$  has rank function  $\ell$ , while by Lemma 3.4, there is a unique rank function  $\|\cdot\|_1$  on  $V(\mathcal{C})$ . Thus for any  $v \in V(\mathcal{C})$  we have  $\ell(\varphi(v)) = \|v\|_1$ , and for all  $0 \leq j \leq \ell(y)$ , we have

$$\#\{x \leq y \mid \ell(x) = j\} = \#\{v \in V(\mathcal{C}) \mid \|v\|_1 = j\}.$$

This is exactly saying that the coefficient of  $z^j$  in the Poincaré polynomial  $p_y(z)$  is equal to the coefficient of  $z^j$  in the polynomial  $g_{\mathcal{C}}(z)$ . In other words,  $p_y(z) = g_{\mathcal{C}}(z)$ . The form of  $p_y(z)$  now follows from Lemma 3.5.

For the final claim, observe that  $s_i \in S$  is in the support of  $y$  if and only if  $s_i \in [1, y]$ , and that the set of elements of  $[1, y]$  of word length 1 is exactly the set  $S \cap [1, y]$ . Thus, by Lemma 3.4 again, we have that  $s_i \in S$  is in the support of  $y$  if and only if the corresponding vertex  $v$  of  $\mathcal{C}(k_1, \dots, k_N)$  satisfies  $\|v\|_1 = 1$ . If each  $k_j \geq 1$ , then there are  $N$  distinct vertices of  $\mathcal{C}(k_1, \dots, k_N)$  which have rank 1, namely the standard basis vectors of  $\mathbb{Z}^N$ , and so the support of  $y$  contains  $N$  elements.  $\square$

From Corollary 5.2 and Proposition 5.3, we immediately obtain the following.

**Corollary 5.4.** *Let  $(W, S)$  be any Coxeter system, and let  $y \in W$ . If  $[1, y]_{\mathcal{B}}$  can be cubulated, then  $p_y(z)$  is palindromic.*

We are now prepared to prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $(W, S)$  be any Coxeter system with  $S$  finite. Let  $y \in W$ , and suppose that the Bruhat graph  $[1, y]_{\mathcal{B}}$  can be cubulated. We now apply Corollary 5.4, which says that the Poincaré polynomial  $p_y$  is palindromic. Corollary 4.5 then says that  $P_{x,y} = 1$  for all  $x \leq y$ , which proves Theorem 1.1.  $\square$

In the remainder of this work, we consider the converse to Theorem 1.1.

## 6. INVESTIGATING THE CONVERSE TO THEOREM 1.1

In this section, we investigate the converse to Theorem 1.1. In Section 6.1, we consider two cases where it is straightforward to see that  $P_{x,y} = 1$  for all  $x \leq y$ , and also easy to see that  $[1, y]_{\mathcal{B}}$  is spanned by a cubical lattice. In Section 6.2, we consider the case of the longest element in a finite Coxeter system. We prove that  $\mathcal{B} = [1, w_0]_{\mathcal{B}}$  is spanned by a cubical lattice whenever  $W$  is of type  $A$  or  $B/C$ , verifying the converse to Theorem 1.1 for  $y = w_0$  in these cases. Then in Section 6.3 we show that if the converse to Theorem 1.1 fails for some Coxeter system  $(W, S)$ , then it fails for every Coxeter system which contains  $(W, S)$  as a subsystem.

**6.1. Several special cases.** This section considers the cases when no simple reflection appears more than once in any reduced expression, and when  $(W, S)$  is a dihedral group.

**6.1.1. Standard parabolic Coxeter elements.** An element  $y \in W$  is called *standard parabolic Coxeter* if each simple reflection in  $S$  is used at most once in any (equivalently every) reduced expression for  $y$ . As the terminology suggests, standard parabolic Coxeter elements are those that are Coxeter in some standard parabolic subgroup of  $W$ . (Note that standard parabolic Coxeter elements also appear by other names; for example, they are called *boolean* in some parts of the literature.)

Suppose  $y \in W$  is a standard parabolic Coxeter element. Then for any  $x \leq y$ , the interval  $[x, y]$  is isomorphic as a poset to the Boolean lattice  $B_{\ell(y)-\ell(x)}$ . Therefore, [Bre94, Cor. 6.8] says that  $P_{x,y} = 1$ . For any  $k \in \mathbb{N}$ , we write  $\mathcal{C}(1^k)$  for the cubical lattice  $\mathcal{C}(\underbrace{1, 1, \dots, 1}_{k \text{ times}})$ .

**Lemma 6.1.** *Let  $(W, S)$  be any Coxeter system. Suppose  $y \in W$  is a standard parabolic Coxeter element with a reduced expression of the form  $y = s_{i_1} \cdots s_{i_k}$ . Then  $[1, y]_{\mathcal{B}}$  is isomorphic to the cubical lattice  $\mathcal{C}(1^k)$ , hence the converse to Theorem 1.1 holds for this  $y \in W$ .*

*Proof.* For  $y = s_{i_1} \cdots s_{i_k}$  standard parabolic Coxeter, the poset  $[1, y]$  is isomorphic to the Boolean lattice  $B_{\ell(y)} = B_k$ . Now clearly, the Hasse diagram for  $B_k$  is isomorphic to the cubical lattice  $\mathcal{C}(1^k)$ . Thus the Hasse diagram for  $[1, y]$  is isomorphic to  $\mathcal{C}(1^k)$ .

By the Strong Exchange Property, for any element  $w = s_{j_1} \cdots s_{j_\ell}$  of the Bruhat interval  $[1, y]$  and any reflection  $t \in \mathcal{T}$ , we have  $wt = s_{j_1} \cdots \hat{s}_{j_m} \cdots s_{j_\ell}$  for some  $m \in [\ell]$ . Since  $y$  is standard parabolic Coxeter, so is  $w$ , and hence  $\ell(wt) = \ell(w) - 1$ . Therefore, in the case that  $y$  is standard parabolic Coxeter, the Bruhat graph  $[1, y]_{\mathcal{B}}$  is equal to the Hasse diagram for  $[1, y]$ . We conclude that  $[1, y]_{\mathcal{B}}$  is isomorphic to  $\mathcal{C}(1^k)$ , as required.  $\square$

For example, the left and middle of Figure 4 depict the graphs  $[1, s_i s_j]_{\mathcal{B}} \cong \mathcal{C}(1, 1)$  and  $[1, s_i s_j s_k]_{\mathcal{B}} \cong \mathcal{C}(1, 1, 1)$ , respectively, where  $s_i$ ,  $s_j$ , and  $s_k$  are three distinct simple generators.

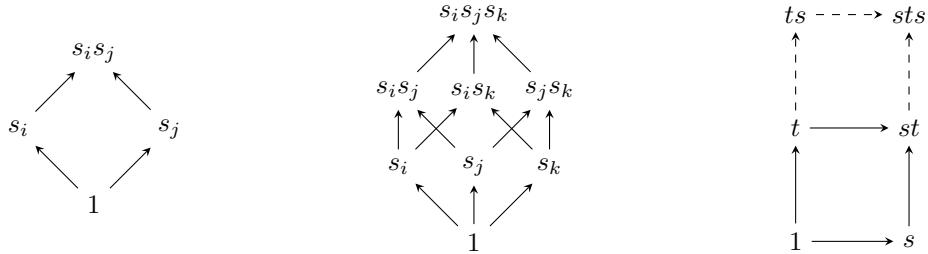


FIGURE 4. From left to right, we depict a cubulation for standard parabolic Coxeter elements of lengths 2 and 3, and for dihedral elements of length 3.

Note the following consequence for elements of short word-length, since every element  $y \in W$  such that  $\ell(y) \leq 2$  is necessarily standard parabolic Coxeter. By Exercise 7(a) in Chapter 5 of [BB05], we have  $P_{x,y} = 1$  for all  $x \leq y$  for all  $\ell(y) \leq 2$ . Therefore, short word-length provides another context where the converse to Theorem 1.1 holds.

**Corollary 6.2.** *Let  $(W, S)$  be any Coxeter system, and suppose  $y \in W$ . If  $\ell(y) \leq 2$ , then  $[1, y]_{\mathcal{B}}$  is isomorphic to a cubical lattice. In particular, the converse to Theorem 1.1 holds when  $\ell(y) \leq 2$ .*

**6.1.2. Dihedral groups.** If  $(W, S)$  is a dihedral group (finite or infinite) and  $y \in W$ , then  $P_{x,y} = 1$  for all  $x \leq y$ ; see [Hum90, Section 7.12(a)]. We show in the next result that the graph  $[1, y]_{\mathcal{B}}$  can be cubulated. We restrict to the case  $\ell(y) \geq 3$  since  $\ell(y) \leq 2$  is treated by Corollary 6.2 above.

**Proposition 6.3.** *Let  $(W, S)$  be a Coxeter system of type  $I_2(m)$  for  $m \geq 3$  or type  $\tilde{A}_1$ , and let  $y$  be an element of  $W$  such that  $\ell(y) \geq 3$ . Then  $[1, y]_{\mathcal{B}}$  is cubulated by  $\mathcal{C}(1, \ell(y) - 1)$ . In particular, the converse to Theorem 1.1 holds for dihedral groups.*

*Proof.* Let  $k = \ell(y)$ . Since  $(W, S)$  is dihedral, the generating set  $S$  has exactly two elements. To simplify notation we put  $S = \{s, t\}$ . Then, without loss of generality,  $y$  has reduced expression the alternating word  $sts \cdots$  containing  $k$  letters. We induct on  $k \geq 3$ .

Suppose first that  $k = 3$ ; see the right of Figure 4. By Lemma 6.1, the graph  $[1, st]_{\mathcal{B}}$  is isomorphic to the square  $\mathcal{C}(1, 1)$ ; the edges of this graph are the solid arrows on the right of Figure 4. Now the edge set of the graph  $[1, sts]_{\mathcal{B}}$  includes also the edges  $(t, ts)$ ,  $(st, sts)$ , and  $(ts, ts(sts)) = (ts, sts)$ ; these three edges are shown dashed on the right of Figure 4. Adding these three additional edges to the square  $[1, st]_{\mathcal{B}}$  results in a spanning subgraph of  $[1, sts]_{\mathcal{B}}$  which is isomorphic to  $\mathcal{C}(1, 2)$ . The proof of the inductive step is similar.  $\square$

**6.2. The longest element in a finite Coxeter group.** Let  $(W, S)$  be a finite Coxeter system, and let  $w_0$  be the longest element of  $W$ . Then  $P_{x,w_0} = 1$  for all  $x \in W$ ; see Exercise 7.14 in [Hum90]. In this section, we prove Theorem 1.2, which states that the Bruhat graph  $\mathcal{B} = [1, w_0]_{\mathcal{B}}$  is spanned by a cubical lattice whenever  $W$  is of type  $A$  or  $B/C$ .

**6.2.1. Normal form forests and cubulation.** In this section, we prove that if each tree in a normal form forest from Section 2.2 is a path, then the Bruhat graph  $\mathcal{B} = [1, w_0]_{\mathcal{B}}$  can be cubulated.

**Proposition 6.4.** *Let  $(W, S)$  be a finite Coxeter system with longest element  $w_0 \in W$ . Suppose  $(W, S)$  has a normal form forest  $\tau_1, \dots, \tau_n$  in which every rooted tree  $\tau_i$  is the path consisting of  $\ell_i \geq 1$  edges. Then  $\mathcal{B} = [1, w_0]_{\mathcal{B}}$  is cubulated by  $\mathcal{C}(\ell_1, \dots, \ell_n)$ .*

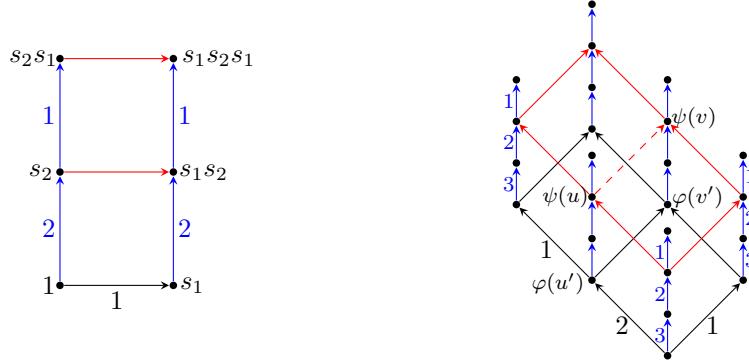


FIGURE 5. On the left (respectively, right), we depict the inductive construction in the proof of Proposition 6.4 in type  $A_n$  for  $n = 2$  (respectively,  $n = 3$ ).

We will illustrate the proof of Proposition 6.4 by following examples in types  $A_2$  and  $A_3$ , as depicted in Figure 5. Recall from Example 2.2 that  $(W, S)$  of type  $A_3$  has a normal form

forest of trees  $\tau_1, \tau_2, \tau_3$ , such that for  $i = 1, 2, 3$  the tree  $\tau_i$  is the path consisting of  $i$  edges. The trees  $\tau_1$  and  $\tau_2$  give a normal form forest of paths in type  $A_2$ . We will see that in type  $A_2$  (respectively,  $A_3$ ), the Bruhat graph  $\mathcal{B}$  is cubulated by  $\mathcal{C}(1, 2)$  (respectively,  $\mathcal{C}(1, 2, 3)$ ).

*Proof of Proposition 6.4.* We proceed by induction on  $n$ . If  $n = 1$ , then  $(W, S)$  is of type  $A_1$  and the graph  $[1, w_0]_{\mathcal{B}}$  is isomorphic to  $\mathcal{C}(1)$ .

For the inductive step, write  $\mathcal{B}_{n-1}$  for the Bruhat graph of the subsystem of  $(W, S)$  generated by  $\{s_1, \dots, s_{n-1}\}$ , and assume that there is an isomorphism  $\varphi$  from  $\mathcal{C}_{n-1} = \mathcal{C}(\ell_1, \dots, \ell_{n-1})$  to a subgraph of  $\mathcal{B}$  which spans  $\mathcal{B}_{n-1}$ . For example, in type  $A_n$  with  $n = 2$  (respectively,  $n = 3$ ), the graph  $\varphi(\mathcal{C}_{n-1}) \cong \mathcal{C}_{n-1}$  is shown in black on the left (respectively, right) of Figure 5.

Now, using Proposition 2.1, we have that every  $x \in W$  factors uniquely as  $x = x'x_n$ , where  $x' \in V(\varphi(\mathcal{C}_{n-1}))$  and  $x_n$  is the label of a vertex in the path  $\tau_n$ . For every fixed  $x' \in V(\varphi(\mathcal{C}_{n-1}))$ , we define  $\tau_{x'}$  to be the subgraph of  $\mathcal{B}$  induced by the vertex set  $\{x'x_n \mid x_n \text{ a vertex label in } \tau_n\}$ . Then each such  $\tau_{x'}$  is naturally isomorphic to the path  $\tau_n$ , with this isomorphism preserving all edge-labels, and the paths  $\{\tau_{x'} \mid x' \in V(\varphi(\mathcal{C}_{n-1}))\}$  are pairwise disjoint subgraphs of  $\mathcal{B}$ . Moreover, the Bruhat graph  $\mathcal{B}$  is spanned by the union of  $\varphi(\mathcal{C}_{n-1})$  and the collection of paths  $\{\tau_{x'} \mid x' \in V(\varphi(\mathcal{C}_{n-1}))\}$ . In type  $A_2$  (respectively,  $A_3$ ), the paths  $\{\tau_{x'}\}$  are depicted vertically in blue on the left (respectively, right) of Figure 5.

Observe that the cubical lattice  $\mathcal{C}_{n-1}$  naturally embeds as the subgraph  $\mathcal{C}(\ell_1, \dots, \ell_{n-1}, 0)$  of  $\mathcal{C}_n = \mathcal{C}(\ell_1, \dots, \ell_n)$ . For each vertex  $v' = (m_1, \dots, m_{n-1})$  of  $\mathcal{C}_{n-1}$ , we now define  $\tau_{v'}$  to be the subgraph of  $\mathcal{C}_n$  induced by the vertex set  $\{(m_1, \dots, m_{n-1}, m_n) \mid 0 \leq m_n \leq \ell_n\}$ . Then each such  $\tau_{v'}$  is naturally isomorphic to the path  $\tau_n$ , and the paths  $\{\tau_{v'} \mid v' \in V(\mathcal{C}_{n-1})\}$  are pairwise disjoint in  $\mathcal{C}_n$ . Write  $\mathcal{D}_n$  for the subgraph of  $\mathcal{C}_n$  which is the union of its naturally embedded copy of  $\mathcal{C}_{n-1}$ , together with all of the paths  $\{\tau_{v'} \mid v' \in V(\mathcal{C}_{n-1})\}$ . Then  $\mathcal{D}_n$  is a spanning subgraph of  $\mathcal{C}_n$ , and we can construct an isomorphism  $\psi$  from  $\mathcal{D}_n$  onto a spanning subgraph of  $\mathcal{B}$  by sending the copy of  $\mathcal{C}_{n-1}$  in  $\mathcal{D}_n$  onto  $\varphi(\mathcal{C}_{n-1})$  and, for any  $v' \in V(\mathcal{C}_{n-1})$ , mapping the subgraph  $\tau_{v'}$  of  $\mathcal{C}_n$  onto the subgraph  $\tau_{\varphi(v')}$  of  $\mathcal{B}$ . That is, in type  $A_n$  for  $n = 2$  (respectively,  $n = 3$ ), the graph  $\psi(\mathcal{D}_n) \cong \mathcal{D}_n$  is the union of the black and blue edges on the left (respectively, right) of Figure 5.

It now suffices to show that this map  $\psi : \mathcal{D}_n \rightarrow \mathcal{B}$  can be extended to an embedding of the entire cubical lattice  $\mathcal{C}_n$  into  $\mathcal{B}$ . This amounts to showing that for any edge  $(u, v)$  of  $\mathcal{C}_n$  which is not already in  $\mathcal{D}_n$ , there is an edge of  $\mathcal{B}$  from  $\psi(u)$  to  $\psi(v)$ . On both sides of Figure 5, we thus need to show that we can connect all pairs of vertices in  $\mathcal{B}$  which are at the same height and are such that the vertices directly underneath them in  $\varphi(\mathcal{C}_{n-1})$  are connected by a black edge. We depict the two such “missing” edges in red on the left of Figure 5, and (to avoid cluttering the image) only some of the “missing” edges in red on the right of Figure 5.

Let  $u = (p_1, \dots, p_{n-1}, p_n)$  and  $v = (q_1, \dots, q_{n-1}, q_n)$  be vertices of  $\mathcal{C}_n$  such that  $(u, v) \in E(\mathcal{C}_n) \setminus E(\mathcal{D}_n)$ . Then  $u$  and  $v$  must have the same final component  $p_n = q_n \geq 1$ , and there must be an edge from  $u' = (p_1, \dots, p_{n-1})$  to  $v' = (q_1, \dots, q_{n-1})$  in  $\mathcal{C}_{n-1}$ . Now since  $(u', v')$  is an edge of  $\mathcal{C}_{n-1}$ , by induction  $(\varphi(u'), \varphi(v'))$  is an edge of  $\mathcal{B}_{n-1} \subset \mathcal{B}$ . This means exactly that  $\ell(\varphi(v')) > \ell(\varphi(u'))$  and there is a reflection  $t' \in \mathcal{T}$  so that  $\varphi(v') = \varphi(u')t'$ . For example, in type  $A_3$  consider the vertices  $u = (0, 1, 2)$  and  $v = (1, 1, 2)$  of  $\mathcal{C}_3 = \mathcal{C}(1, 2, 3)$ . The dashed red edge from  $\psi(u) = s_2(s_3s_2)$  to  $\psi(v) = s_1s_2(s_3s_2)$ , on the right of Figure 5, corresponds to the black edge from  $\varphi(u') = s_2$  to  $\varphi(v') = s_1s_2$ , and we have  $t' = s_2s_1s_2$ .

To see that there is an edge of  $\mathcal{B}$  from  $\psi(u)$  to  $\psi(v)$ , let  $x_n$  be the label of the vertex at the end of the initial subpath of  $\tau_n$  which consists of  $p_n = q_n \geq 1$  edges. (For example, on the right of Figure 5 we have  $x_3 = s_3s_2$ .) Then by definition of  $\psi$ , we have  $\psi(u) = \psi(p_1, \dots, p_{n-1}, p_n) = \varphi(u')x_n$ , and similarly  $\psi(v) = \varphi(v')x_n$ . Since  $\varphi(u') \in V(\mathcal{B}_{n-1})$  and  $x_n$  is the label of a vertex of  $\tau_n$ , by Proposition 2.1 we obtain

$$\ell(\psi(u)) = \ell(\varphi(u')x_n) = \ell(\varphi(u')) + \ell(x_n) = \ell(\varphi(u')) + p_n,$$

and similarly  $\ell(\psi(v)) = \ell(\varphi(v')) + q_n$ . From the previous paragraph, we have  $\ell(\varphi(v')) > \ell(\varphi(u'))$  and  $p_n = q_n$ . Thus we obtain  $\ell(\psi(v)) > \ell(\psi(u))$ , as desired. Now let  $t \in \mathcal{T}$  be the reflection given by  $t = x_n^{-1}t'x_n$ . (For example, on the right of Figure 5 we have  $t = s_2s_3(s_2s_1s_2)s_3s_2$ .) Then

$$\psi(v) = \varphi(v')x_n = (\varphi(u')t')x_n = \varphi(u_0)x_n(x_n^{-1}t'x_n) = \psi(u)t.$$

Therefore  $(\psi(u), \psi(v))$  is an edge of  $\mathcal{B}$ , which completes the proof. Geometrically, the reflection  $t$  which labels the edge  $(\psi(u), \psi(v))$  is obtained by going down the path  $\tau_{\varphi(u')}$ , then across the edge  $(\varphi(u'), \varphi(v'))$ , then up the path  $\tau_{\varphi(v')}$ .  $\square$

**6.2.2. Normal form forests in the classical types.** Motivated by Proposition 6.4, we now, for types  $A$  and  $B/C$ , exhibit normal form forests in which every rooted tree is a path, thus providing a case-by-case proof of Theorem 1.2.

*Type  $A_n$ .* Suppose that  $(W, S)$  is of finite type  $A_n$  for  $n \geq 1$ . We shall construct a normal form forest  $\tau_1, \dots, \tau_n$  in which every rooted tree  $\tau_i$  is a path consisting of  $i$  edges, generalizing Example 2.2 illustrated by Figure 2.

Label the nodes of the Dynkin diagram  $1, \dots, n$  from left to right. Fix any  $j \in [n]$ , and consider the set  ${}^{[j-1]}(W_{[j]})$  of minimal-length representatives of the right cosets  $W_{[j-1]} \backslash W_{[j]}$ , where  $W_{[j]} = \langle s_1, s_2, \dots, s_j \rangle$ . In type  $A_j$ , the number of elements in  $W_{[j-1]} \backslash W_{[j]}$  equals  $|W_{[j]}|/|W_{[j-1]}| = \frac{j!}{(j-1)!} = j$ . As every reduced expression in  ${}^{[j-1]}(W_{[j]})$  begins with  $s_j$ , the edge label immediately above the root vertex of  $\tau_j$  is  $j$ .

In order that all other reduced expressions read from  $\tau_j$  also necessarily begin uniquely with  $s_j$ , we must label the remaining  $j - 1$  edges in turn by  $j - 1, j - 2, \dots, 1$ , due to the commuting relations encoded by the Dynkin diagram. Moreover, the expression  $s_js_{j-1} \cdots s_2s_1$  is clearly reduced, as a Coxeter element in the parabolic subgroup  $W_{[j]}$ . Each of the  $j$  initial subexpressions  $s_j \cdots s_k$  for  $k \in [j]$  is thus reduced and represents a distinct element of  ${}^{[j-1]}(W_{[j]})$ . Therefore by Proposition 2.1, the rooted tree  $\tau_j$  is the path consisting of  $j$  edges which are labeled successively by  $j, j - 1, \dots, 1$ .

By Proposition 6.4, this construction of a normal form forest consisting of paths of length  $1, 2, \dots, n$  proves that  $[1, w_0]_{\mathcal{B}}$  is spanned by the cubical lattice  $\mathcal{C}(1, 2, \dots, n)$  in type  $A_n$ .

*Types  $B_n$  and  $C_n$ .* Suppose that  $(W, S)$  is of finite type  $B_n$  for  $n \geq 2$ . We shall construct a normal form forest  $\tau_1, \dots, \tau_n$  in which every rooted tree  $\tau_i$  is a path consisting of  $2i - 1$  edges.

Label the nodes of the Dynkin diagram  $1, \dots, n$  from right to left, so that the last  $n - 1$  nodes form a type  $A_{n-1}$  subsystem, and the special node is indexed by 1; note that this is the reverse of the ordering from [Bou02]. Using this labeling,  $W_{[1]} = \langle s_1 \rangle$  has type  $A_1$ , and  $W_{[j]} = \langle s_1, s_2, \dots, s_j \rangle$  has type  $B_j$  for all  $j \geq 2$ .

We proceed by induction on  $n \geq 2$ . First note that  ${}^{[0]}(W_{[1]}) = \langle s_1 \rangle$  has type  $A_1$ , and so  $\tau_1$  is the path consisting of one edge labeled by 1, as seen in Figure 2. The group  $W_{[2]} = \langle s_1, s_2 \rangle$  has type  $B_2$ , and so there are exactly 4 distinct minimal-length coset representatives in  ${}^{[1]}(W_{[2]}) = \{1, s_2, s_2s_1, s_2s_1s_2\}$ . Since  $s_2s_1s_2$  is reduced and all other elements of  ${}^{[1]}(W_{[2]})$  are initial subexpressions of  $s_2s_1s_2$ , then  $\tau_2$  is a tree with 3 edges, labeled successively by 2, 1, 2, establishing the base case for  $n = 2$ .

Now for any  $j \geq 3$ , suppose that the  $j - 1$  rooted trees for the normal form forest of type  $B_{j-1}$  are paths consisting of  $2(j - 1) - 1$  edges, labeled successively by  $1, 3, \dots, 2(j - 1) - 1$ . The group  $W_{[j]}$  has type  $B_j$ , so that  ${}^{[j-1]}(W_{[j]})$  has  $\frac{2^j j!}{2^{j-1}(j-1)!} = 2j$  minimal-length coset representatives. By Theorem 1.1 of [Mil24] with the ordering of the nodes reversed, the expression  $t = s_js_{j-1} \cdots s_2s_1s_2 \cdots s_{j-1}s_j$  is reduced, and therefore so are all  $2j - 1$  of its initial subexpressions. Note by construction that both  $t$  and all initial subexpressions must begin with  $s_j$ . Together with the identity, we have thus identified the  $2j$  elements of  ${}^{[j-1]}(W_{[j]})$  as those words formed by reading the labels of the tree  $\tau_j$  with  $2j - 1$  edges, labeled successively by  $j, j - 1, \dots, 2, 1, 2, \dots, j - 1, j$ .

By induction and Proposition 6.4, this construction of a normal form forest consisting of paths of lengths  $1, 3, 5, \dots, 2n - 1$  proves that  $[1, w_0]_{\mathcal{B}}$  is spanned by the cubical lattice  $\mathcal{C}(1, 3, 5, \dots, 2n - 1)$ . Since the group  $W$  is identical, the result follows for type  $C_n$  as well.

**6.2.3. Proof of Theorem 1.2.** Altogether, we have shown in Section 6.2.2 that if  $(W, S)$  is of type  $A_n$  for  $n \geq 1$  or type  $B_n$  or  $C_n$  for  $n \geq 2$ , then  $[1, w_0]_{\mathcal{B}}$  is spanned by a cubical lattice. Since  $P_{x, w_0} = 1$  for all  $w \in W$  in  $(W, S)$  finite, then we have proved the converse to Theorem 1.1 for  $y = w_0$  in these types.

**6.2.4. Type  $D_n$ .** In type  $D_n$  for all  $n \geq 4$ , we leave it as an exercise for the reader to verify that there is no normal form forest consisting of path graphs. Therefore, the technique used in types  $A$  and  $B/C$  fails already in type  $D_4$ . We have verified the converse to Theorem 1.1 for all elements in type  $D_4$ , and constructed an explicit cubulation of the graph  $[1, w_0]_{\mathcal{B}}$  in types  $D_4$  and  $D_5$ , assisted by computer experimentation [Bis24]. However, we could not spot a pattern that could be continued inductively from these constructions in types  $D_4$  and  $D_5$ , and our code did not return any results in type  $D_6$  after running for more than four months on a cluster at the University of Sydney. Thus we pose the following question.

**Question 1.** Does the converse to Theorem 1.2 hold in type  $D_n$  for  $n \geq 5$ ? In particular, can the graph  $[1, w_0]_{\mathcal{B}}$  be cubulated in type  $D_n$  for  $n \geq 6$ ?

**6.2.5. Exceptional types.** Our exploration of the exceptional types has largely been computational, and the results concerning the converse to Theorem 1.1 negative. Using SageMath [Sag24], the first author has programmed a construction of Hasse diagrams for Bruhat intervals in any irreducible finite or affine Coxeter group, which also checks for spanning cubical lattices; the source code is available on Zenodo [Bis24].

In particular, we have found that the graph  $[1, w_0]_{\mathcal{B}}$  cannot be cubulated in types  $E_6, F_4$ , or  $H_4$ . Note also that if this code finds a spanning cubical lattice with  $N = 2$  or  $3$ , then it will further use this embedding to draw the Hasse diagram nicely, as seen in Figure 1.

**6.3. An embedding result.** In this section, we show that if the converse to Theorem 1.1 fails for some Coxeter system  $(W, S)$ , then it fails for every Coxeter system which contains  $(W, S)$  as a subsystem.

**Proposition 6.5.** *Suppose  $(W, S)$  is a Coxeter system so that, for some  $y \in W$ , we have  $P_{x,y} = 1$  for all  $x \leq y$ , but  $[1, y]_{\mathcal{B}}$  cannot be cubulated. Let  $(W', S')$  be any Coxeter system which contains  $(W, S)$  as a subsystem. Denote by  $\leq'$  the Bruhat order on  $(W', S')$ , and let  $\mathcal{B}'$  be the Bruhat graph of  $(W', S')$ .*

Write  $P'_{x',y'}$  for the Kazhdan–Lusztig polynomial for  $x', y' \in W'$ . Then, regarding  $y \in W$  as an element of  $W'$ , we have  $P'_{x',y} = 1$  for all  $x' \in W'$  with  $x' \leq' y$ , but  $[1, y]_{\mathcal{B}'}$  cannot be cubulated. In particular, the converse to Theorem 1.1 also fails for any Coxeter system which has  $(W, S)$  as a subsystem.

*Proof.* This follows from the observation that for  $x, y \in W$ , both the Kazhdan–Lusztig polynomial  $P_{x,y}$  and the graph  $[1, y]_{\mathcal{B}}$  are obtained using only elements of  $W$  which have reduced expressions which are subwords of some reduced expression for  $y$ . But any reduced expression in  $S'$  for  $y$ , regarding  $y$  now as an element of  $W'$ , involves only letters in  $S$ .  $\square$

*Proof of Theorem 1.3.* Apply Proposition 6.5 to the fact from Section 6.2.5 that  $[1, w_0]_{\mathcal{B}}$  cannot be cubulated in types  $E_6, F_4$ , or  $H_4$ .  $\square$

## 7. CUBULATIONS AND GROWTH IN COXETER GROUPS

The goal of this section is to prove Theorem 1.4 of the introduction. To prepare for this, in Section 7.1 we give some general background and a technical lemma concerning formal power series. We then recall some key definitions and results on growth in finitely generated groups in Section 7.2, and discuss growth in Coxeter groups in 7.3. The proof of Theorem 1.4 is then carried out in Section 7.4.

**7.1. Power series.** We begin by recalling some definitions concerning (formal) power series, and proving a key technical lemma. Let

$$F(z) = \sum_{j=0}^{\infty} a_j z^j \quad \text{and} \quad G(z) = \sum_{j=0}^{\infty} b_j z^j$$

be formal power series, with  $a_j, b_j \in \mathbb{C}$ . For any  $k \in \mathbb{N}$ , we denote by  $F(z)[k]$  the Taylor polynomial

$$F(z)[k] = \sum_{j=0}^k a_j z^j$$

consisting of all terms of  $F(z)$  of degree  $\leq k$ . Recall that the (Cauchy) product of  $F(z)$  and  $G(z)$  is the power series given by

$$F(z)G(z) = \sum_{j=0}^{\infty} c_j z^j \quad \text{where } c_j = \sum_{k=0}^j a_k b_{j-k}.$$

The next result is a straightforward consequence of these definitions. It says that if  $F(z)$  and  $G(z)$  agree on all terms of degree  $\leq k$ , then so do the products  $F(z)H(z)$  and  $G(z)H(z)$  for any power series  $H(z)$ .

**Lemma 7.1.** *Let  $F(z)$ ,  $G(z)$ , and  $H(z)$  be power series, and define*

$$\Phi(z) = F(z)H(z) \quad \text{and} \quad \Gamma(z) = G(z)H(z).$$

*Then for all  $k \in \mathbb{N}$ , if  $F(z)[k] = G(z)[k]$ , we have  $\Phi(z)[k] = \Gamma(z)[k]$ .*

The proof of Theorem 1.4 will rely on the following technical lemma. It says that if the truncations of a power series are all polynomials of a very particular form, which arises in our argument, then the power series itself is either constant or a polynomial of a similar form.

**Lemma 7.2.** *Let  $n \geq 1$ , and let  $(g_j(z))_{j=0}^{\infty}$  be a sequence of polynomials of the form*

$$g_j(z) = (1 - z^{a_{1,j}})(1 - z^{a_{2,j}}) \cdots (1 - z^{a_{n,j}}),$$

*where for all  $1 \leq i \leq n$  and all  $j \geq 0$ , the  $a_{i,j}$  are integers satisfying*

$$1 \leq a_{1,j} \leq a_{2,j} \leq \cdots \leq a_{n,j}.$$

*Suppose that  $F(z)$  is a power series such that for each  $j \geq 0$ ,*

$$F(z)[j] = g_j(z)[j].$$

*Then either  $F(z) = 1$ , or there exists a positive integer  $M$  and an integer  $N$  with  $1 \leq N \leq n$ , such that*

$$F(z) = (1 - z^{a_{1,M}})(1 - z^{a_{2,M}}) \cdots (1 - z^{a_{N,M}}),$$

*and for all  $j \geq M$ , we have  $g_j(z) = g_M(z)$ .*

*Proof.* We will prove the statement by induction on  $n$ . Suppose  $n = 1$ . Then for each  $j \geq 1$ , we have  $g_j(z) = 1 - z^{a_{1,j}}$ . If  $F(z) \neq 1$ , then there must be some  $m \in \mathbb{N}$  so that  $g_m(z)[m] \neq 1$ . Then

$$g_m(z)[m] = 1 - z^{a_{1,m}},$$

and so in particular,  $a_{1,m} \leq m$ . Now for all  $j \geq m$ , we have  $j \geq a_{1,m}$ , so

$$g_j(z)[j] = 1 - z^{a_{1,m}}$$

as well. Hence  $g_j(z) = g_m(z)$  for every  $j \geq m$ . But then for every  $j \geq m$ , we also have

$$F(z)[j] = 1 - z^{a_{1,m}},$$

and hence  $F(z) = 1 - z^{a_{1,m}}$ . Put  $M = m$  and  $N = n = 1$ , and we have established the result for  $n = 1$ .

For the inductive step, we again have that if  $F(z) \neq 1$  then  $g_m(z)[m] \neq 1$  for some  $m \in \mathbb{N}$ . Now  $a_{1,m} \leq \cdots \leq a_{n,m}$ , so if we expand out the product  $g_m(z)$  we obtain

$$g_m(z) = 1 - c_m z^{a_{1,m}} + \text{higher degree terms},$$

where the coefficient  $c_m \neq 0$  is the number of exponents  $a_{1,m}, \dots, a_{n,m}$  to be equal to  $a_{1,m}$ . We note that  $a_{1,m} \leq m$ . Hence for all  $j \geq m$ , since  $j \geq a_{1,m}$  we have

$$g_j(z)[j] = 1 - c_m z^{a_{1,m}} + \text{higher degree terms}.$$

Thus in particular, since  $a_{1,j} \leq \cdots \leq a_{n,j}$ , we have  $a_{1,j} = a_{1,m}$  for all  $j \geq m$ . So for all  $j \geq m$  we have

$$\frac{g_j(z)}{1 - z^{a_{1,m}}} = (1 - z^{a_{2,j}}) \cdots (1 - z^{a_{n,j}}).$$

We can thus define a sequence of polynomials  $(h_j(z))_{j=0}^\infty$  by

$$h_j(z) = \frac{g_{m+j}(z)}{1 - z^{a_{1,m}}} = (1 - z^{a_{2,m+j}}) \cdots (1 - z^{a_{n,m+j}}).$$

Now as the quotient  $1/(1 - z^{a_{1,m}})$  is itself a power series, we can view each  $h_j(z)$  as the product of the polynomial  $g_{m+j}(z)$  with this power series, and we can also define the product

$$G(z) = F(z) \left( \frac{1}{1 - z^{a_{1,m}}} \right) = \frac{F(z)}{1 - z^{a_{1,m}}}.$$

Since  $F(z)[j] = g_j(z)[j]$  for all  $j \geq 0$ , we can thus apply Lemma 7.1 to see that for all  $j \geq 0$ ,

$$\begin{aligned} G(z)[m+j] &= \left( \frac{F(z)}{1 - z^{a_{1,m}}} \right) [m+j] \\ &= \left( \frac{g_{m+j}(z)}{1 - z^{a_{1,m}}} \right) [m+j] \\ &= h_j(z)[m+j]. \end{aligned}$$

Therefore  $G(z)[j] = h_j(z)[j]$  for all  $j \geq 0$ .

By inductive assumption, since each  $h_j(z)$  is a product of  $(n-1)$  factors, either  $G(z) = 1$ , or there is an  $M' \geq 1$  and an integer  $N'$  with  $1 \leq N' - 1 \leq n-1$ , such that  $G(z)$  is the product of  $(N'-1)$  factors as follows:

$$G(z) = (1 - z^{a_{2,M'}}) \cdots (1 - z^{a_{N',M'}}),$$

and for all  $j \geq M'$ , we have  $h_j(z) = h_{M'}(z)$ . Recall also from above that  $a_{1,j} = a_{1,m}$  for each  $j \geq m$ .

If  $G(z) = 1$ , then  $F(z) = 1 - z^{a_{1,m}}$  and  $g_j(z) = g_m(z)$  for all  $j \geq m$ , and we are done with  $M = m$  and  $N = 1 \leq n$ . Now assume  $G(z) \neq 1$ . Then we have in particular that  $a_{i,m+j} = a_{i,m+M'}$  for all  $2 \leq i \leq N'$  and all  $j \geq M'$ . Put  $M = m + M'$ . Then  $a_{1,m} = a_{1,M}$  and  $a_{i,M'} = a_{i,M}$  for all  $2 \leq i \leq n$ , and so  $F(z) = (1 - z^{a_{1,m}})G(z)$  is the product of  $1 + (N' - 1) = N'$  factors as follows:

$$F(z) = (1 - z^{a_{1,M}})(1 - z^{a_{2,M}}) \cdots (1 - z^{a_{N',M}}),$$

and for each  $j \geq M$ , we have  $g_j(z) = g_M(z)$ . Letting  $N = N'$ , this completes the proof of the inductive step.  $\square$

**7.2. Growth in finitely generated groups.** In this section, we give background on volume growth for finitely generated groups, mostly following the exposition in [Löh17, Chapter 6]. We then establish a specialized lemma, for use in our proof of Theorem 1.4. In Section 7.3, we will consider volume growth in the setting of Coxeter groups.

Throughout this section,  $G$  is an arbitrary finitely generated group, with finite generating set  $S \subseteq G$ . We define  $S^{-1} = \{s^{-1} \mid s \in S\}$ . Let  $\ell_S : G \rightarrow \mathbb{N}$  be the *word length function* on  $G$  with respect to  $S$ . That is, for any  $g \in G$ ,

$$\ell_S(g) = \min\{k \mid g = s_{i_1} \cdots s_{i_k} \text{ where } s_{i_j} \in S \cup S^{-1} \text{ for } 1 \leq j \leq k\}.$$

The corresponding *word metric*  $d_S : G \times G \rightarrow \mathbb{N}$  is given by  $d_S(g, h) = \ell_S(g^{-1}h)$  for any  $g, h \in G$ , and it is straightforward to verify that  $d_S$  is indeed a metric on  $G$ . For any  $k \in \mathbb{N}$ , define

$$B_{G,S}(k) = \{g \in G \mid d_S(1, g) \leq k\} = \{g \in G \mid \ell_S(g) \leq k\}$$

to be the (closed) ball in  $G$  of radius  $k$  around the identity element 1, with respect to the word metric  $d_S$ . We note that, since  $S$  is finite, the ball  $B_{G,S}(k)$  has finitely many elements for every  $k \geq 0$ . We can thus make the following definitions.

**Definition 7.3.** The *volume growth function of  $G$  with respect to  $S$*  is the function  $\beta_{G,S} : \mathbb{N} \rightarrow \mathbb{N}$  given by

$$\beta_{G,S}(k) = \#B_{G,S}(k).$$

The *volume growth series of  $G$  with respect to  $S$*  is the power series

$$\Gamma_{G,S}(z) = \sum_{k=0}^{\infty} \beta_{G,S}(k) z^k.$$

That is, the coefficient  $\beta_{G,S}(k)$  of  $z^k$  in  $\Gamma_{G,S}(z)$  counts the number of elements of  $G$  of word length at most  $k$ , with respect to the generating set  $S$ . Note that if the group  $G$  is infinite, the function  $\beta_{G,S} : \mathbb{N} \rightarrow \mathbb{N}$  is strictly increasing, meaning that for all  $k \in \mathbb{N}$ , we have  $\beta_{G,S}(k) < \beta_{G,S}(k+1)$ .

Obviously,  $\beta_{G,S}(k)$ , and hence  $\Gamma_{G,S}(z)$ , depend upon the choice of generating set  $S$ . Although for Coxeter groups we will work with a fixed (finite) generating set, in order to apply the more general theory of volume growth we will need to consider arbitrary finite generating sets. The idea is that, as illustrated by the following example, changing the generating set does not substantially affect the volume growth function.

**Example 7.4.** Let  $G = \mathbb{Z}$  with its standard generating set  $S = \{1\}$ . Then for all  $k \in \mathbb{N}$ , we have  $\beta_{G,S}(k) = 2k + 1$ . If instead we take the generating set  $S' = \{2, 3\}$ , then we obtain that for all integers  $k \geq 2$ , we have  $\beta_{G,S'}(k) = 6k + 1$ . Notice that both of these volume growth functions are (for large enough  $k$ ) polynomials of degree 1.

To take this further, we recall a partial order and equivalence relation on *generalized growth functions*, which are just strictly increasing functions  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ . By [Löh17, Example 6.2.3], if  $G$  is infinite then for any finite generating set  $S$  for  $G$ , the volume growth function  $\beta_{G,S}$  induces a generalized growth function, given by

$$r \mapsto \beta_{G,S}(\lceil r \rceil) \quad \text{for all } r \geq 0.$$

Let  $\beta_1$  and  $\beta_2$  be generalized growth functions. We say that  $\beta_1$  is *quasi-dominated* by  $\beta_2$ , denoted  $\beta_1 \prec \beta_2$ , if there exists  $c \in \mathbb{N}$  so that for all  $r \geq 0$ ,

$$\beta_1(r) \leq c\beta_2(cr + c) + c.$$

We then define  $\beta_1$  and  $\beta_2$  to be *quasi-equivalent*, denoted  $\beta_1 \sim \beta_2$ , if  $\beta_1 \prec \beta_2$  and  $\beta_2 \prec \beta_1$ .

Now let  $G_1$  and  $G_2$  be infinite, finitely generated groups, with finite generating sets  $S_1$  and  $S_2$ , respectively. We extend the definitions in the previous paragraph to the volume growth functions  $\beta_{G_i, S_i} : \mathbb{N} \rightarrow \mathbb{N}$  for  $i = 1, 2$ , by considering the associated generalized growth functions  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ . Explicitly, we have  $\beta_{G_1, S_1} \prec \beta_{G_2, S_2}$  if and only if there exists  $c \in \mathbb{N}$  so that for all  $k \in \mathbb{N}$ ,

$$\beta_{G_1, S_1}(k) \leq c\beta_{G_2, S_2}(ck + c) + c.$$

The next result gathers some useful statements concerning these definitions.

**Lemma 7.5** (Section 6.2.1 of [Löh17]). *Let  $\beta_1$  and  $\beta_2$  be generalized growth functions.*

- (1) If  $\beta_1(r) = r^{d_1}$  and  $\beta_2(r) = r^{d_2}$ , where  $d_1$  and  $d_2$  are positive integers, then  $\beta_1 \prec \beta_2$  if and only if  $d_1 \leq d_2$ . That is, the relation of quasi-equivalence detects the degree of monomials.
- (2) If  $\beta_1(r) = a_1^r$  and  $\beta_2(r) = a_2^r$ , where  $a_1, a_2 > 1$ , then  $\beta_1 \sim \beta_2$ . That is, all exponential functions with base  $> 1$  are quasi-equivalent.
- (3) If  $\beta_1(r) = r^d$  with  $d$  a positive integer, and  $\beta_2 = a^r$  with  $a > 1$ , then  $\beta_1 \not\sim \beta_2$ . That is, monomial and exponential functions are never quasi-equivalent.
- (4) The relation  $\sim$  is an equivalence relation on generalized growth functions and hence on volume growth functions of infinite, finitely generated groups. Moreover, the relation of quasi-domination induces a partial order on the corresponding equivalence classes.

We now give a special case of Proposition 6.2.4 of [Löh17]. We note that if  $G$  is finitely generated and  $H$  is a finite-index subgroup of  $G$ , then  $H$  is also finitely generated, and  $H$  is quasi-isometric to  $G$  (see, for example, [Löh17, Definition 5.1.6 and Corollary 5.4.5]).

**Proposition 7.6.** *Let  $G$  be an infinite, finitely generated group, and let  $H$  be a finite-index subgroup of  $G$ . Then for any finite generating set  $S$  for  $G$  and any finite generating set  $T$  for  $H$ , the volume growth functions  $\beta_{G,S}$  and  $\beta_{H,T}$  are quasi-equivalent. In particular, for any two finite generating sets  $S_1$  and  $S_2$  for  $G$ , we have  $\beta_{G,S_1} \sim \beta_{G,S_2}$ .*

We can hence make the following important definitions.

**Definition 7.7** (Polynomial and exponential growth). Let  $G$  be an infinite, finitely generated group, and let  $S$  be any finite generating set of  $G$ . The *growth type* of  $G$  is the quasi-equivalence class of any (hence all) volume growth functions of  $G$ , with respect to finite generating sets. We say that  $G$  has:

- *polynomial growth* if there is a positive integer  $d$  so that  $\beta_{G,S} \prec (r \mapsto r^d)$ ; and
- *exponential growth* if it has the growth type of  $r \mapsto e^r$ , meaning that  $\beta_{G,S} \sim (r \mapsto r^d)$ .

By Lemma 7.5(3), a group cannot be of both polynomial and exponential growth. Two key examples are as follows; see [Löh17, Section 6.1] for more details.

**Example 7.8.** For any  $n \in \mathbb{N}$ , the free abelian group  $\mathbb{Z}^n$  of rank  $n$  has polynomial growth.

**Example 7.9.** A finite rank nonabelian free group has exponential growth.

We will use the following standard result, which follows from Proposition 7.6 and Example 7.8.

**Corollary 7.10.** *Let  $G$  be an infinite, finitely generated group. If  $G$  has a finite-index subgroup  $H \cong \mathbb{Z}^n$ , for some integer  $n \in \mathbb{N}$ , then  $G$  has polynomial growth.*

Finally, we establish a specialized statement which we will need for our proof of Theorem 1.4. This says that if the volume growth series is a rational function of a very particular form, then the group  $G$  has polynomial growth.

**Lemma 7.11.** *Let  $G$  be an infinite, finitely generated group with finite generating set  $S$ . Suppose that for some polynomial  $f(z)$  with non-negative integer coefficients and some positive integer  $m$ , the volume growth series  $\Gamma_{G,S}(z)$  is given by the rational function*

$$\Gamma_{G,S}(z) = \frac{f(z)}{(1-z)^m}.$$

*Then  $G$  has polynomial growth.*

*Proof.* We first note that  $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$ . Now, treating  $z$  as a real variable and carrying out formal differentiation, a straightforward induction shows that for all  $m \geq 2$ ,

$$\frac{1}{(1-z)^m} = \frac{1}{(m-1)} \sum_{j=m-1}^{\infty} \frac{d^{m-1}}{dz^{m-1}}(z^j) = \frac{1}{(m-1)} \sum_{k=0}^{\infty} (k+m-1)(k+m-2)\cdots(k+1)z^k.$$

Put  $b_{1,k} = 1$  for all  $k \in \mathbb{N}$ , and for all  $m \geq 2$  and  $k \in \mathbb{N}$ , define

$$b_{m,k} = \frac{(k+m-1)(k+m-2)\cdots(k+1)}{(m-1)}.$$

Note that, since the numerator here contains  $m-1$  consecutive positive integers, each  $b_{m,k}$  is a positive integer. Also observe that for all  $m \geq 1$  and all  $k \in \mathbb{N}$ , we have  $b_{m,k} \leq b_{m,k+1}$ . That is, for all  $m \geq 1$ , we have

$$\frac{1}{(1-z)^m} = \sum_{k=0}^{\infty} b_{m,k} z^k,$$

where the coefficients  $b_{m,k}$  form a weakly increasing sequence of positive integers.

Now write  $f(z) = a_0 + a_1 z + \cdots + a_M z^M$ , where the  $a_j$  are non-negative integers (and  $a_M \neq 0$ ). Then the coefficient of  $z^k$  in the volume growth series  $\Gamma_{G,S}(z) = f(z)/(1-z)^m$  is given by

$$\beta_{G,S}(k) = \sum_{j=0}^k a_j b_{m,k-j}.$$

(We note that, since the volume growth function  $\beta_{G,S}(k)$  is strictly increasing, if  $m = 1$  then the polynomial  $f(z)$  must have degree  $M \geq 1$ .) Let  $A$  be the positive integer  $A = \max_{0 \leq j \leq M} a_j$ . Then since the sequence  $(b_{m,k})_{k=0}^{\infty}$  is weakly increasing, we have that for all  $m \geq 1$  and all  $k \in \mathbb{N}$ ,

$$\beta_{G,S}(k) \leq A \sum_{j=0}^k b_{m,k-j} \leq A(k+1)b_{m,k}.$$

If  $m = 1$  then since  $b_{1,k} = 1$  this implies

$$\beta_{G,S}(k) \leq A(k+1),$$

while for  $m \geq 2$  we get

$$\beta_{G,S}(k) \leq \frac{A(k+1)(k+m-1)(k+m-2)\cdots(k+1)}{(m-1)}.$$

Thus for all  $m \geq 1$ , we obtain that  $\beta_{G,S}(k)$  is bounded above by a polynomial in  $k$  (of degree  $m$ ). Therefore  $\beta_{G,S}$  is quasi-dominated by a monomial (of degree  $m$ ), and so  $G$  has polynomial growth, as required.  $\square$

**7.3. Growth in Coxeter groups.** In this section we record some useful results for volume growth in the setting of Coxeter groups. Let  $(W, S)$  be any Coxeter system (with  $S$  finite).

We first recall that the Poincaré series of the group  $W$  is given by

$$W(z) = \sum_{x \in W} z^{\ell(x)}.$$

(We caution that in some of the literature on growth in Coxeter groups, this series is referred to instead as the growth series, or the spherical growth series.) Now let  $\Gamma_{W,S}(z)$  be the volume growth series for  $W$  with respect to the generating set  $S$ , as defined above in Definition 7.3. The series  $W(z)$  and  $\Gamma_{W,S}(z)$  are related via the following easy observation (which, however, we could not find in the literature).

**Lemma 7.12.** *Let  $(W, S)$  be a Coxeter system. Then  $W(z) = (1 - z)\Gamma_{W,S}(z)$ .*

*Proof.* Both  $W(z)$  and  $\Gamma_{W,S}(z)$  have constant term 1. For all  $k \geq 1$ , the coefficient of  $z^k$  in  $\Gamma_{W,S}(k)$  is the number of elements of the ball  $B_{W,S}(k)$ , while the coefficient of  $z^k$  in  $W(z)$  is the number of elements of  $B_{W,S}(k) \setminus B_{W,S}(k-1)$ . The result follows.  $\square$

Next, we record the fact that affine Coxeter groups have polynomial growth. This is well-known, and is stated without proof in, for example, the introduction to [BY24].

**Lemma 7.13.** *If  $(W, S)$  is an irreducible affine Coxeter system,  $W$  has polynomial growth.*

*Proof.* If  $|S| = n + 1$  then the translation subgroup of  $W$  is free abelian of rank  $n$ , and has finite index in  $W$ . Hence by Corollary 7.10, the group  $W$  has polynomial growth.  $\square$

We now use a result of Terragni [Ter16] to give a short proof of the following converse to Lemma 7.13. We expect this statement is known to experts, and that it could be proved by other means, but could not find it written down explicitly. For  $k \in \mathbb{N}$ , denote by  $c_k$  the coefficient of  $z^k$  in the Poincaré series  $W(z)$ . The (*exponential*) *growth rate* of  $(W, S)$  is then defined by  $\omega(W, S) = \limsup_k \sqrt[k]{c_k}$ .

**Theorem 7.14** (Polynomial growth implies affine). *Let  $(W, S)$  be an irreducible Coxeter system. If  $W$  is infinite and has polynomial growth, then  $(W, S)$  is (irreducible) affine.*

*Proof.* First, as noted in the introduction to [Ter16], if  $(W, S)$  has polynomial growth then  $\omega(W, S) \leq 1$ . But by Theorem (B) of [Ter16], if  $(W, S)$  is infinite non-affine then  $\omega(W, S) \geq \tau$ , where  $\tau = 1.13\dots$  is a specified algebraic integer (which equals the growth rate for  $(W, S)$  of type  $E_{10}$ ). The result follows.  $\square$

In the remainder of this section, motivated by Theorem 7.14, we specialize further to the case of irreducible affine Coxeter systems. We will use the following theorem of Bott [Bot79], which gives a precise formula for the Poincaré series in this setting.

Suppose now that  $(W, S)$  is an irreducible affine Coxeter system of type  $\tilde{X}_n$ , where  $X \in \{A, B, C, D, E, F, G, H\}$  with appropriate restrictions on  $n$ . Then we will denote the  $(n+1)$  elements of the affine generating set  $S$  by  $s_0, \dots, s_n$ , and write  $(W_0, S_0)$  for the associated spherical Coxeter system of type  $X_n$ , with generating set  $S_0 = \{s_1, \dots, s_n\} \subset S$ . By [BB05,

Theorem 7.1.5], for example, there are positive integers  $e_1, \dots, e_n$  such that the order of the finite group  $W_0$  is given by the product  $\prod_{i=1}^n (e_i + 1)$ , and the number of reflections in  $W_0$  is given by the sum  $\sum_{i=1}^n e_i$ . These integers  $e_1, \dots, e_n$  are called the *exponents* of the irreducible finite Coxeter system  $(W_0, S_0)$ , and for all types can be found in [BB05, Table I, Appendix A1].

**Theorem 7.15** (Bott, Theorem 7.1.10 of [BB05]). *Let  $(W, S)$  be an irreducible affine Coxeter system with  $|S| = n+1 \geq 2$ , and let  $e_1, \dots, e_n$  be the exponents of the corresponding irreducible finite Coxeter system. Then the Poincaré series of  $W$  is given by the rational function*

$$W(z) = \prod_{i=1}^n \frac{(1 - z^{e_i+1})}{(1 - z)(1 - z^{e_i})}.$$

**Corollary 7.16.** *Let  $(W, S)$  be an irreducible affine Coxeter system with  $|S| = n+1 \geq 2$ . The following are equivalent:*

- (1) *all poles of the Poincaré series  $W(z)$  are at  $z = 1$ ;*
- (2)  *$(W, S)$  is of type  $\tilde{A}_n$ .*

*Proof.* The exponents in finite type  $A_n$  are

$$\{e_1, e_2, \dots, e_n\} = \{1, 2, \dots, n\} = [n]$$

and thus the Poincaré series in type  $\tilde{A}_n$  is  $(1 - z^{n+1})/(1 - z)^{n+1}$ . For all other types, we see from [BB05, Table I, Appendix A1] that  $W(z)$  has a pole at a  $k^{\text{th}}$  root of unity  $e^{2\pi i/k}$  with  $k \geq 3$ , and hence  $e^{2\pi i/k} \neq 1$ . Specifically, if  $(W, S)$  is of type:

- $\tilde{B}_n$  or  $\tilde{C}_n$  for  $n \geq 2$ , then  $W(z)$  has a pole at  $e^{2\pi i/k}$  with  $k = (2n - 1) \geq 3$ ;
- $\tilde{D}_n$  for  $n \geq 4$ , then  $W(z)$  has a pole at  $e^{2\pi i/k}$  with  $k = (2n - 3) \geq 5$ ;
- $\tilde{E}_6$  or  $\tilde{F}_4$ , then  $W(z)$  has a pole at  $e^{2\pi i/k}$  with  $k = 11$ ;
- $\tilde{E}_7$ , then  $W(z)$  has a pole at  $e^{2\pi i/k}$  with  $k = 17$ ;
- $\tilde{E}_8$  or  $\tilde{H}_4$ , then  $W(z)$  has a pole at  $e^{2\pi i/k}$  with  $k = 29$ ;
- $\tilde{G}_2$ , then  $W(z)$  has a pole at  $e^{2\pi i/k}$  with  $k = 5$ ; and
- $\tilde{H}_3$ , then  $W(z)$  has a pole at  $e^{2\pi i/k}$  with  $k = 19$ .

This completes the proof.  $\square$

Combining Lemma 7.12 and Corollary 7.16, we obtain the following.

**Corollary 7.17.** *Let  $(W, S)$  be an irreducible affine Coxeter system with  $|S| = n+1 \geq 2$ . The following are equivalent:*

- (1) *all poles of the volume growth series  $\Gamma_{W,S}(z)$  are at  $z = 1$ ;*
- (2)  *$(W, S)$  is of type  $\tilde{A}_n$ .*

**7.4. Cubulations, growth, and type  $\tilde{A}_n$ .** We now prove Theorem 1.4. For this, we will establish two key results, Propositions 7.20 and 7.21, and then combine Proposition 7.21 with statements from Sections 7.2 and 7.3.

Our results in this section concern the following class of Coxeter systems.

**Definition 7.18.** Let  $(W, S)$  be a Coxeter system with  $S = \{s_i \mid i \in [n]\}$ . We say that  $(W, S)$  is *minimal nonspherical* if  $W$  is an infinite group, but for each  $i \in [n]$ , the standard parabolic subgroup generated by  $S \setminus \{s_i\}$  is finite.

Equivalently,  $(W, S)$  is minimal nonspherical if every proper parabolic subgroup of  $W$  is finite. Note that if  $(W, S)$  is minimal nonspherical, then  $|S| = n \geq 2$ .

**Remark 7.19.** If  $(W, S)$  is minimal nonspherical, then as discussed in Section 6.9 and Example 14.2.3 of [Dav08], the generating set  $S$  is the set of reflections in the faces of a compact simplex in either Euclidean or hyperbolic space. Hence  $(W, S)$  is either irreducible affine, or irreducible hyperbolic as classified by Lannér; see [Dav08, Table 6.2].

We now establish Proposition 7.20. This says that in minimal nonspherical systems, the ball  $B_{W,S}(k)$  is contained in the Bruhat interval  $[1, y]$  for all long enough  $y$  (depending on  $k$ ).

**Proposition 7.20.** *Let  $(W, S)$  be a minimal nonspherical Coxeter system. Then there is an explicit constant  $L = L(W, S) \geq 2$  such that for all  $k \in \mathbb{N}$  and all  $y \in W$  with  $\ell(y) \geq kL$ , we have*

$$B_{W,S}(k) \subseteq [1, y].$$

*Proof.* Since  $(W, S)$  is minimal nonspherical, for each  $i \in [n]$  the standard parabolic subgroup  $W_{S \setminus \{s_i\}}$  of  $W$  is finite. Hence for each  $i \in [n]$ , we may define the positive integer  $\ell_i$  to be the length of the longest element of  $W_{S \setminus \{s_i\}}$ . We then define  $L = L(W, S) \in \mathbb{N}$  by

$$L = 1 + \max_{i \in [n]} \ell_i \geq 2.$$

We note that if  $y \in W$  is such that  $\ell(y) \geq L$ , then the support of  $y$  must equal  $S$ . Otherwise,  $y$  would be contained in some proper standard parabolic subgroup of  $W$ , which would in turn imply  $\ell(y) \leq L - 1$ .

We now fix  $k \in \mathbb{N}$ , and let  $y \in W$  be any element such that  $\ell(y) \geq kL$ . Then there is reduced expression for  $y$  given by  $y = y_1 \dots y_k$ , where for each  $1 \leq j \leq k$ , the subword  $y_j$  is reduced, and  $\ell(y_j) \geq L$ . Hence each  $y_j$  has support equal to  $S$ .

To complete the proof, let  $x \in B_{W,S}(k)$ . Then there is a reduced expression for  $x$  given by  $x = s_{i_1} \dots s_{i_m}$ , where  $s_{i_j} \in S$  and  $m \leq k$ . For all  $1 \leq j \leq m$ , since  $y_j$  has support the entire set  $S$ , the letter  $s_{i_j} \in S$  is a (proper) subword of the reduced expression  $y_j$ . Hence the reduced expression  $s_{i_1} \dots s_{i_m}$  for  $x$  is a subword of the reduced expression  $y_1 \dots y_k$  for  $y$ . Thus  $x \leq y$  in Bruhat order, and so  $B_{W,S}(k)$  is contained in the Bruhat interval  $[1, y]$  as desired.  $\square$

We will use Proposition 7.20 to prove our second key result, Proposition 7.21, which describes the volume growth series for minimal nonspherical Coxeter systems  $(W, S)$  in which  $[1, y]_{\mathcal{B}}$  can be cubulated for infinitely many distinct  $y \in W$ . The proof of Proposition 7.21 also makes essential use of Proposition 5.3, which describes the Poincaré polynomial  $p_y$  when  $[1, y]_{\mathcal{B}}$  can be cubulated, and the results of Section 7.1 concerning power series.

**Proposition 7.21.** *Let  $(W, S)$  be a minimal nonspherical Coxeter system with  $|S| = n \geq 2$ . If there are infinitely many distinct  $y \in W$  such that the Bruhat graph  $[1, y]_{\mathcal{B}}$  can be cubulated, then either*

$$\Gamma_{W,S}(z) = \frac{1}{(1-z)^{n+1}}$$

*or there is an integer  $N$  with  $1 \leq N \leq n$ , and integers  $2 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ , such that*

$$\Gamma_{W,S}(z) = \left(\frac{1-z^{a_1}}{1-z}\right) \left(\frac{1-z^{a_2}}{1-z}\right) \cdots \left(\frac{1-z^{a_N}}{1-z}\right) \frac{1}{(1-z)^{n+1-N}}.$$

*Proof.* Let  $L = L(W, S) \geq 2$  be the constant from Proposition 7.20. Set  $y_0 = 1$ . Then choose an infinite sequence  $(y_j)_{j=0}^{\infty}$  of elements of  $W$  such that:

- for each  $j \geq 0$ , the graph  $[1, y_j]_{\mathcal{B}}$  can be cubulated; and
- for each  $j \geq 1$ , we have  $\ell(y_j) \geq Lj$ .

Thus  $B_{W,S}(0) = \{1\}$  and  $[1, y_0] = \{1\}$ , and by Proposition 7.20, we have that for all  $j \geq 1$ , the ball  $B_{W,S}(j)$  is contained in  $[1, y_j]$ . Hence, in particular, for all  $j \geq 1$  the element  $y_j$  has support equal to  $S$ .

To simplify notation we now write  $p_j(z)$  for the Poincaré polynomial  $p_{y_j}(z)$ . We note that each  $p_j(z)$  has degree equal to  $\ell(y_j)$ , hence  $\deg p_j(z) \geq Lj \geq j + 1$  for all  $j \geq 1$ . As  $B_{W,S}(j)$  is a subset of  $[1, y_j]$ , we thus obtain that for all  $j \geq 0$ ,

$$W(z)[j] = p_j(z)[j],$$

where we recall by Lemma 7.12 that  $W(z) = (1-z)\Gamma_S(z)$  is the Poincaré series for  $W$ .

By assumption, for each  $j \geq 1$ , the graph  $[1, y_j]_{\mathcal{B}}$  can be cubulated by, say, the (nontrivial) cubical lattice  $\mathcal{C}(k_{1,j}, k_{2,j}, \dots, k_{N_j,j})$ . We may assume  $1 \leq k_{1,j} \leq k_{2,j} \leq \cdots \leq k_{N_j,j}$ . Then by Proposition 5.3, for all  $j \geq 1$  the Poincaré polynomial  $p_j(z)$  is the product of quantum polynomials

$$p_j(z) = \frac{z^{k_{1,j}+1}-1}{z-1} \cdot \frac{z^{k_{2,j}+1}-1}{z-1} \cdots \frac{z^{k_{N_j,j}+1}-1}{z-1},$$

where, since each  $k_{i,j} \geq 1$ , in fact each  $N_j = n = |S|$  is the cardinality of the support of  $y_j$ . We hence simplify notation by putting  $a_{i,j} = k_{i,j} + 1$  for all  $1 \leq i \leq n$  and  $j \geq 1$ . Then for all  $j \geq 1$ , we have  $2 \leq a_{1,j} \leq a_{2,j} \leq \cdots \leq a_{n,j}$  and

$$p_j(z) = \prod_{i=1}^n \left( \frac{z^{a_{i,j}} - 1}{z - 1} \right) = \prod_{i=1}^n \left( \frac{1 - z^{a_{i,j}}}{1 - z} \right).$$

We now define a sequence of polynomials  $(g_j(z))_{j=0}^{\infty}$  by

$$g_j(z) = (1-z)^n p_j(z),$$

and define a power series  $F(z)$  by

$$F(z) = (1-z)^n W(z) = (1-z)^{n+1} \Gamma_{W,S}(z).$$

By Lemma 7.1, since  $W(z)[j] = p_j(z)[j]$ , we have  $F(z)[j] = g_j(z)[j]$  for each  $j \geq 0$ .

If  $F(z) = 1$  then it is immediate that  $\Gamma_{W,S}(z) = \frac{1}{(1-z)^{n+1}}$ . Otherwise, by Lemma 7.2,

$$F(z) = (1 - z^{a_{1,M}})(1 - z^{a_{2,M}}) \cdots (1 - z^{a_{N,M}})$$

for some  $M \geq 1$  and some integer  $N$  with  $1 \leq N \leq n$ . Upon dividing this polynomial expression for  $F(z)$  through by  $(1 - z)^{n+1}$ , and putting  $a_i = a_{i,M} \geq 2$  for  $1 \leq i \leq N$ , we obtain the desired expression for the volume growth series  $\Gamma_{W,S}(z)$ .  $\square$

**Corollary 7.22.** *Let  $(W, S)$  be a minimal nonspherical Coxeter system. If there are infinitely many distinct  $y \in W$  such that  $[1, y]_{\mathcal{B}}$  can be cubulated, then  $W$  has polynomial growth.*

*Proof.* Apply Lemma 7.11 to the form of the volume growth series  $\Gamma_{W,S}(z)$  established in Proposition 7.21, with  $f(z)$  the product of quantum polynomials

$$f(z) = \left( \frac{1 - z^{a_1}}{1 - z} \right) \left( \frac{1 - z^{a_2}}{1 - z} \right) \cdots \left( \frac{1 - z^{a_N}}{1 - z} \right)$$

and  $m = n + 1 - N \geq 1$ .  $\square$

We now complete the proof of Theorem 1.4 by combining Proposition 7.21 and Corollary 7.22 with results from Sections 7.2 and 7.3.

*Proof of Theorem 1.4.* Let  $(W, S)$  be a minimal nonspherical Coxeter system, and suppose that there are infinitely many distinct  $y \in W$  such that  $[1, y]_{\mathcal{B}}$  can be cubulated. By Theorem 7.14 and Corollary 7.22, since  $W$  has polynomial growth, then the Coxeter system  $(W, S)$  is irreducible affine.

If  $|S| = n + 1 \geq 2$ , then from Proposition 7.21 with  $n$  replaced by  $n + 1$  in its statement, we see that all poles of the volume growth series  $\Gamma_{W,S}(z)$  are at  $z = 1$ . Hence, by Corollary 7.17, we have that  $(W, S)$  is of type  $\tilde{A}_n$  for some  $n \geq 1$ . This completes the proof of Theorem 1.4.  $\square$

**Remark 7.23.** We now sketch an alternative approach to part of the proof of Theorem 1.4. We will freely use some standard concepts and arguments from geometric group theory (see [Löh17], as well as the reference [BH99]). Suppose  $(W, S)$  is minimal nonspherical. Then by Remark 7.19, either  $(W, S)$  is irreducible affine, or the group  $W$  acts properly discontinuously and cocompactly by isometries on  $n$ -dimensional hyperbolic space  $\mathbb{H}^n$  (where  $|S| = n + 1$ ). In the latter case,  $W$  is quasi-isometric to  $\mathbb{H}^n$ , hence is a Gromov-hyperbolic group, and therefore has exponential growth. Thus once it is known that  $W$  has polynomial growth (or even just subexponential growth), we can deduce that  $(W, S)$  is irreducible affine without using Theorem 7.14.

## 8. CONSTRUCTION OF CUBULATIONS IN TYPE $\tilde{A}_2$

We conclude by giving a constructive proof of Theorem 1.5. In Section 8.1, we show that if  $(W, S)$  is of type  $\tilde{A}_2$ , it suffices to consider one infinite family  $\{y_m\}_{m \in \mathbb{N}}$  of elements of  $W$ , together with certain standard parabolic Coxeter elements of length  $\leq 3$ , which are already addressed by Lemma 6.1. We construct a cubulation of each graph  $[1, y_m]_{\mathcal{B}}$  in Section 8.2.

**8.1. Reduction.** In this section we carry out a reduction for the proof of Theorem 1.5, using results from Libedinsky–Patimo [LP23] and Burrull–Libedinsky–Plaza [BLP23].

In order to obtain this reduction, we first record another condition equivalent to  $P_{x,y} = 1$  for all  $x \leq y$ , complementing Theorem 4.4. Following the notation of [LP23], and using the same normalizations as in that work, the Hecke algebra of  $W$  is a  $\mathbb{Z}[v, v^{-1}]$  module with two

distinguished bases: the standard basis  $\{\mathbf{H}_y\}_{y \in W}$ , and the canonical (or Kazhdan–Lusztig) basis  $\{\underline{\mathbf{H}}_y\}_{y \in W}$ . These bases are related via the equation

$$(8.1.1) \quad \underline{\mathbf{H}}_y = \sum_{x \leq y} h_{x,y} \mathbf{H}_y$$

where the  $h_{x,y} = h_{x,y}(v)$  are the Kazhdan–Lusztig polynomials in Soergel’s normalization [Soe97]. As in [LP23], for  $y \in W$  put

$$\mathbf{N}_y = \sum_{x \leq y} v^{\ell(y)-\ell(x)} \mathbf{H}_y.$$

We now relate these notions to triviality of the Kazhdan–Lusztig polynomials  $P_{x,y} = P_{x,y}(q)$  as defined recursively in Section 4.1.

**Lemma 8.1.** *Let  $(W, S)$  be any Coxeter system, and let  $y \in W$ . The following are equivalent:*

- (1)  $\underline{\mathbf{H}}_y = \mathbf{N}_y$ ;
- (2) for all  $x \leq y$ , we have  $h_{x,y} = v^{\ell(y)-\ell(x)}$ ; and
- (3) for all  $x \leq y$ , we have  $P_{x,y} = 1$ .

*Proof.* The equivalence of (1) and (2) is due to  $\{\mathbf{H}_y\}_{y \in W}$  being a basis for the Hecke algebra, Equation (8.1.1), and the definition of  $\mathbf{N}_y$ . The equivalence of (2) and (3) is then obtained by changing the normalization of Kazhdan–Lusztig polynomials. Specifically, making the identification  $q = v^{-2}$  we get that  $P_{x,y} = v^{\ell(x)-\ell(y)} h_{x,y}$ , and hence  $P_{x,y} = 1$  exactly when  $h_{x,y} = v^{\ell(y)-\ell(x)}$ .  $\square$

For  $y, y' \in W$ , write  $y \sim y'$  if there is a diagram automorphism  $\phi$  of  $(W, S)$  such that  $\phi(y) = y'$ . The next statement, which applies Theorems 4.3 and 4.4, is similar to Proposition 2.14 of [BLP23].

**Corollary 8.2.** *Let  $(W, S)$  be any Coxeter system, and let  $y \in W$ . The following are equivalent:*

- (1)  $\underline{\mathbf{H}}_y = \mathbf{N}_y$ ;
- (2) for all  $x \leq y$ , we have  $P_{x,y} = 1$ ; and
- (3) for all  $y' \in W$  such that  $y \sim y'$ , and for all  $x \leq y'$ , we have  $P_{x,y'} = 1$ .

*Proof.* Observe that both word length and the set of reflections in  $W$  are invariant under diagram automorphisms. Hence if  $y \sim y'$ , the posets  $[1, y]$  and  $[1, y']$  are isomorphic, and so in particular, the Poincaré polynomials  $p_y(z)$  and  $p_{y'}(z)$  are identical. Therefore utilizing both implications in Corollary 4.5 (which follows from Theorems 4.3 and 4.4), we have  $P_{x,y} = 1$  for all  $x \leq y$  if and only if  $P_{x,y'} = 1$  for all  $x \leq y'$ . The result then follows from Lemma 8.1.  $\square$

In [BLP23, Section 2.1], it is established that every element of  $W$  is  $\sim$ -equivalent to an element in the disjoint union of four infinite families, denoted  $X = \{x_n\}_{n \in \mathbb{N}}$ ,  $\Theta = \{\theta(m, n)\}_{m, n \in \mathbb{N}}$ ,  $\Theta_1 = \{\theta(m, n)s_{m,n}\}_{m, n \in \mathbb{N}}$ , and  $\Theta_2 = \{s_0\theta(m, n)s_{m,n}\}_{m, n \in \mathbb{N}}$ . We refer the reader to [BLP23] for the full definitions of these families, since we will only be interested in certain instances, which we define after the next statement.

**Theorem 8.3.** Let  $(W, S)$  be of type  $\tilde{A}_2$ .

- (1)  $\underline{\mathbf{H}}_{x_n} = \mathbf{N}_{x_n}$  if and only if  $n \leq 3$ .
- (2)  $\underline{\mathbf{H}}_{\theta(m,n)} = \mathbf{N}_{\theta(m,n)}$  if and only if at least one of  $m$  and  $n$  is equal to 0.
- (3) If  $y \in \Theta_1 \sqcup \Theta_2$ , then  $\underline{\mathbf{H}}_y \neq \mathbf{N}_y$ .

*Proof.* Parts (1) and (2) are immediate from the formulas for  $\underline{\mathbf{H}}_y$  given in parts (i) and (ii) of Theorem 1 of [LP23], respectively. The last part is then immediate from the more explicit formulas for  $\underline{\mathbf{H}}_y$  given for  $y \in \Theta_1$  (respectively,  $y \in \Theta_2$ ) by Proposition 3.1 (respectively, Proposition 3.3) of [BLP23].  $\square$

We now define certain of the elements of  $W$  which appear in Theorem 8.3. Put  $S = \{s_0, s_1, s_2\}$ , where  $\{s_1, s_2\}$  is the set of generators for the corresponding spherical Coxeter system of type  $A_2$ . This notation is consistent with that of [BLP23]; whereas in [LP23], the generating set is instead denoted  $\{s_1, s_2, s_3\}$ . In both [LP23] and [BLP23], the generator  $s_i$  is sometimes denoted by  $i$ , and  $i$  is often taken modulo 3.

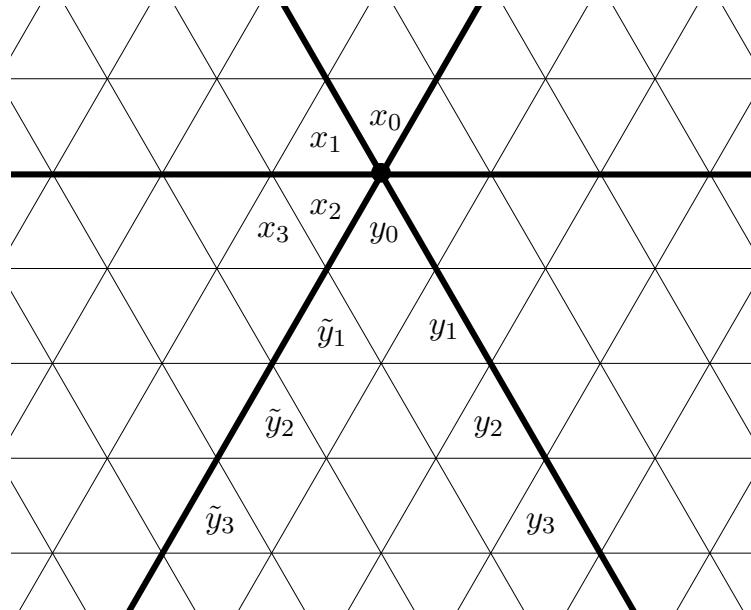


FIGURE 6. The Coxeter complex in type  $\tilde{A}_2$ , showing the elements  $x_0 = 1$ ,  $x_1 = s_1$ ,  $x_2 = s_1 s_2$ ,  $x_3 = s_1 s_2 s_0$ , together with  $y_m$  and  $\tilde{y}_m$  for small  $m$ . The heavy lines are the hyperplanes bounding the Weyl chambers.

Returning to our notation, we have  $x_0 = 1$ ,  $x_1 = s_1$ ,  $x_2 = s_1 s_2$ , and  $x_3 = s_1 s_2 s_0$ . Then for  $m \in \mathbb{N}$ , the element  $\theta(m, 0)$  of  $W$  is defined by the initial subword of length  $3 + 2m$  of the reduced word

$$(s_1 s_2 s_1)(s_0 s_2 s_1)^m.$$

In order to simplify notation we will write  $y_m = \theta(m, 0)$ . So  $y_0 = s_1 s_2 s_1$  is the longest element in type  $A_2$ , and we have

$$y_1 = (s_1 s_2 s_1)s_0 s_2, \quad y_2 = (s_1 s_2 s_1)s_0 s_2 s_1 s_0, \quad y_3 = (s_1 s_2 s_1)s_0 s_2 s_1 s_0 s_2 s_1,$$

and so forth. The family  $\{\theta(0, n)\}_{n \in \mathbb{N}}$  also appears in the statement of Theorem 8.3. However, we observe that the diagram automorphism which swaps 1 and 2 (in the notation of both [LP23] and [BLP23]) swaps the elements  $\theta(m, 0)$  and  $\theta(0, m)$ , for all  $m \in \mathbb{N}$ ; that is,  $y_m \sim \theta(0, m)$  for every  $m \in \mathbb{N}$ . We thus denote by  $\tilde{y}_m = \theta(0, m)$  for all  $m \in \mathbb{N}$ . See Figure 6 for the locations of these elements of  $W$  in the Coxeter complex.

**Corollary 8.4.** *Let  $(W, S)$  be of type  $\tilde{A}_2$ . Suppose that for all*

$$y \in \{1, s_1, s_1s_2, s_1s_2s_0\} \cup \{y_m\}_{m \in \mathbb{N}},$$

*the graph  $[1, y]_{\mathcal{B}}$  can be cubulated. Then for every  $y' \in W$  such that  $P_{x, y'} = 1$  for all  $x \leq y'$ , the graph  $[1, y']_{\mathcal{B}}$  can be cubulated.*

*Proof.* Let  $y' \in W$  be such that  $P_{x, y'} = 1$  for all  $x \leq y'$ . By Corollary 8.2, we have  $\underline{\mathbf{H}}_{y'} = \mathbf{N}_{y'}$ . Since  $(W, S)$  is of type  $\tilde{A}_2$ , by Theorem 8.3, the observations about the four families of elements from [BLP23, Section 2.1], and the fact that  $y_m \sim \tilde{y}_m$  for all  $m \in \mathbb{N}$ , we thus have  $y' \sim y$  for some  $y \in \{1, s_1, s_1s_2, s_1s_2s_0\} \cup \{y_m\}_{m \in \mathbb{N}}$ . Now by assumption,  $[1, y]_{\mathcal{B}}$  can be cubulated. Since diagram automorphisms induce isomorphisms of Bruhat graphs, the graph  $[1, y']_{\mathcal{B}}$  can thus be cubulated as well.  $\square$

We thus have proved that in order to establish Theorem 1.5, it suffices to cubulate the infinite family of graphs  $\{[1, y_m]_{\mathcal{B}}\}_{m \in \mathbb{N}}$ , since the handful of special cases  $[1, y]_{\mathcal{B}}$  for  $y \in \{1, s_1, s_1s_2, s_1s_2s_0\}$  are all treated by Lemma 6.1.

**8.2. Construction of cubulations.** We now complete the proof of Theorem 1.5, by cubulating  $[1, y_m]_{\mathcal{B}}$  for every  $m \in \mathbb{N}$ . First, the element  $y_0 = s_1s_2s_1$  is contained in a subsystem of type  $A_2$ , which is dihedral. Hence by Proposition 6.3, the graph  $[1, y_0]_{\mathcal{B}}$  is cubulated by  $\mathcal{C}(1, 2)$ . Our starting point for the cases  $m \geq 1$  is the following observation.

**Lemma 8.5.** *For any  $m \geq 1$ , the following sets are all of cardinality  $3(m+1)(m+2)$ :*

- (1) *the Bruhat interval  $[1, y_m] \subset W$ ;*
- (2) *the vertex set of the graph  $[1, y_m]_{\mathcal{B}}$ ; and*
- (3) *the vertex set of the cubical lattice  $\mathcal{C}(2, m, m+1)$ .*

*Proof.* Part (1) is a special case of Lemma 1.4 of [LP23], and (2) follows by definition. From Definition 3.2 it is clear that for  $m \geq 1$ , the cubical lattice  $\mathcal{C}(2, m, m+1)$  has

$$(2+1)(m+1)((m+1)+1) = 3(m+1)(m+2)$$

distinct vertices, which proves (3).  $\square$

The next statement, which completes the proof of Theorem 1.5, is motivated by the numerics in Lemma 8.5. We will illustrate its proof by several figures, in which we write  $s_{i_1 \dots i_k}$  for the product  $s_{i_1} \dots s_{i_k}$ , to save space.

**Proposition 8.6.** *For all integers  $m \geq 1$ , the graph  $[1, y_m]_{\mathcal{B}}$  can be cubulated by  $\mathcal{C}(2, m, m+1)$ .*

To prove this proposition, we first establish some useful terminology and notation. To simplify notation, we will sometimes put  $\mathcal{C}_m = \mathcal{C}(2, m, m + 1)$ . For  $k = 0, 1, 2$ , we define the *level k vertices* of  $\mathcal{C}_m$ , denoted  $V_k(\mathcal{C}_m)$ , to be all vertices of  $\mathcal{C}_m$  with first coordinate  $k$ . That is,

$$V_k(\mathcal{C}_m) = \{(k_1, k_2, k_3) \in V(\mathcal{C}_m) \mid k_1 = k\}.$$

Now for  $k = 0, 1, 2$ , we define the *level k edges* of  $\mathcal{C}_m$ , denoted  $E_k(\mathcal{C}_m)$ , to be all edges in  $\mathcal{C}_m$  which connect two elements of  $V_k(\mathcal{C}_m)$ . The *horizontal edges* of  $\mathcal{C}_m$  are then given by the (disjoint) union of the level 0, 1, and 2 edges. Finally, the *vertical edges* of  $\mathcal{C}_m$  are those from a level  $k$  to a level  $k + 1$  vertex, for  $k = 0, 1$ . Thus in particular, every edge of  $\mathcal{C}_m$  is either horizontal or vertical (not both).

We can now formulate a precise (but rather technical) statement, which will be key to the proof of Proposition 8.6, as follows.

**Lemma 8.7.** *For each  $m \geq 1$ , there is a bijection  $\phi = \phi_m : V(\mathcal{C}_m) \rightarrow [1, y_m]$  such that all of the following hold:*

- (1) *For every level 0 edge  $(v, v')$  in  $\mathcal{C}_m$ , the graph  $[1, y_m]_{\mathcal{B}}$  has an edge  $(\phi(v), \phi(v'))$ .*
- (2) *For  $k = 1, 2$ :*
  - (a)  $\phi(k, k_2, k_3) = s_k \phi(k - 1, k_2, k_3)$ ; and
  - (b)  $\ell(\phi(k, k_2, k_3)) = \ell(\phi(k - 1, k_2, k_3)) + 1$ .

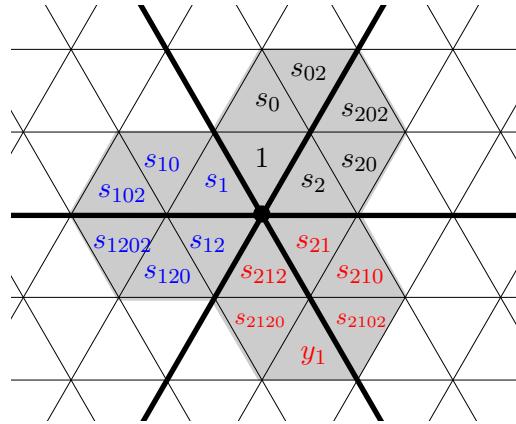


FIGURE 7. The Bruhat interval  $[1, y_1]$  for  $y_1 = (s_1 s_2 s_1) s_0 s_2$ , labeled to illustrate the proof of the case  $m = 1$  in Lemma 8.7.

*Proof.* We carry out induction on  $m \geq 1$ . For  $m = 1$ , we have  $y_1 = (s_1 s_2 s_1) s_0 s_2$ , and that the interval  $[1, y_1]$  contains 18 elements, by Lemma 8.5. These 18 elements are shaded gray in Figure 7. We define  $\phi$  on the level 0 vertices of  $\mathcal{C}_1 = \mathcal{C}(2, 1, 2)$  by:

$$\begin{aligned} \phi(0, 0, 0) &= 1, & \phi(0, 0, 1) &= s_2, & \phi(0, 0, 2) &= s_2 s_0, \\ \phi(0, 1, 0) &= s_0, & \phi(0, 1, 1) &= s_0 s_2, & \phi(0, 1, 2) &= s_2 s_0 s_2. \end{aligned}$$

Identifying  $\mathcal{C}(1, 2)$  with the subgraph  $\mathcal{C}(0, 1, 2)$  of  $\mathcal{C}_1$ , this labeling of the level 0 vertices is depicted on the left of Figure 8. It is then straightforward to verify that the restriction of  $\phi$  to  $V_0(\mathcal{C}_1)$  is a bijection onto a subset of  $[1, y_1]$ . This subset is labeled in black in Figures 7 and 8.

Since  $\phi(V_0(\mathcal{C}_1))$  is the set of elements of the parabolic subgroup of  $W$  of type  $A_2$  generated by  $s_0$  and  $s_2$ , it is easy to see that part (1) holds.

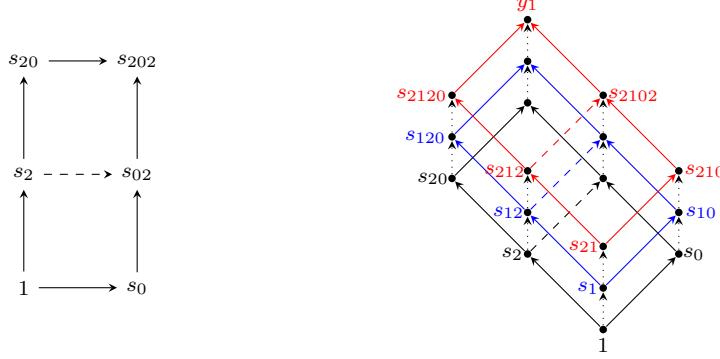


FIGURE 8. The labeling  $\phi$  of the vertices of  $\mathcal{C}(0, 1, 2) \cong \mathcal{C}(1, 2)$  on the left, and of all vertices of  $\mathcal{C}(2, 1, 2)$  on the right. On both the left and right, the solid edges correspond to right-multiplication by a simple generator (either  $s_0$  or  $s_2$ ) and the dashed edges correspond to right-multiplication by  $s_{202}$ . The dotted edges on the right correspond to left-multiplication by  $s_k$  (to go from level  $k$  to level  $k + 1$ ), for  $k = 1, 2$ .

On the remaining vertices of  $\mathcal{C}_1$ , we define  $\phi(1, k_2, k_3) = s_1\phi(0, k_2, k_3)$  and then  $\phi(2, k_2, k_3) = s_2\phi(1, k_2, k_3)$ . From the description of the Bruhat intervals  $[1, y_m] = [1, \theta(m, 0)]$  developed in the introduction to [LP23], one can verify that  $\phi$  is then a bijection  $V(\mathcal{C}_1) \rightarrow [1, y_1]$ . In particular, for the unique vertex of  $\mathcal{C}_1 = \mathcal{C}(2, 1, 2)$  of maximal rank we have

$$\phi(2, 1, 2) = s_2\phi(1, 1, 2) = s_2s_1\phi(0, 1, 2) = s_2s_1s_2s_0s_2 = s_1s_2s_1s_0s_2 = y_1.$$

In Figures 7 and 8, the elements  $\phi(1, k_2, k_3) = s_1\phi(0, k_2, k_3)$  are labeled in blue, and the elements  $\phi(2, k_2, k_3) = s_2\phi(1, k_2, k_3)$  are labeled in red. Now (2)(a) holds by construction, and (2)(b) can be checked quickly. That is, for  $m = 1$  we have constructed a bijection  $\phi : V(\mathcal{C}_1) \rightarrow [1, y_1]$  satisfying the statement.

Assume by induction that we have a bijection  $\phi_m : V(\mathcal{C}_m) \rightarrow [1, y_m]$  satisfying the statement, for  $m \geq 1$ . The Bruhat intervals  $[1, y_m]$  for  $m = 1, 2, 3$  are depicted in Figure 9. We identify  $\mathcal{C}_m = \mathcal{C}(2, m, m+1)$  with its natural image in  $\mathcal{C}_{m+1} = \mathcal{C}(2, m+1, m+2)$ , and define  $\phi_{m+1}(v) = \phi_m(v)$  for all  $v \in V(\mathcal{C}_m)$ . We now explain how to define  $\phi_{m+1}$  on the remaining vertices of  $\mathcal{C}_{m+1}$ . This construction will depend on the value of  $m$  modulo 3, and so we now work with simple generators  $s_i$  where the subscript  $i$  is taken modulo 3. For all  $0 \leq k_2 \leq m$ , we define

$$\phi_{m+1}(0, k_2, m+2) = \phi_m(0, k_2, m+1)s_m$$

and for all  $0 \leq k_3 \leq m$ , we define

$$\phi_{m+1}(0, m+1, k_3) = \phi_m(0, m, k_3)s_m.$$

We then define  $\phi_{m+1}$  on the remaining two level 0 vertices of  $\mathcal{C}_{m+1}$  by

$$\phi_{m+1}(0, m+1, m+1) = \phi_{m+1}(0, m+1, m)s_{m-1}$$

and then

$$\phi_{m+1}(0, m+1, m+2) = \phi_{m+1}(0, m+1, m+1)s_m.$$

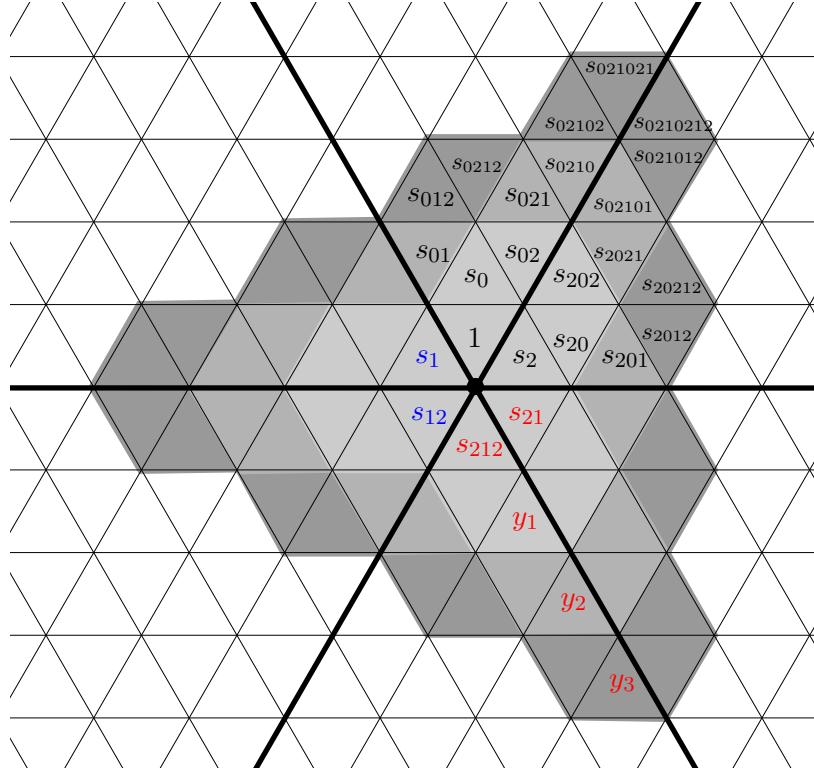


FIGURE 9. The Bruhat intervals  $[1, y_m]$  for  $m = 1, 2, 3$ , in successively darker shades of gray. The elements labeling the level 0 vertices of  $\mathcal{C}_m$  are indicated in black, and certain elements at level 1 (respectively, level 2) are indicated in blue (respectively, red).

Identifying  $\mathcal{C}(m, m+1)$  with the subgraph  $\mathcal{C}(0, m, m+1)$  of  $\mathcal{C}_m$ , this labeling of the level 0 vertices is depicted in Figure 10, for  $m = 3$ . Finally, as in the base case of the induction, we define  $\phi_{m+1}(1, k_2, k_3) = s_1 \phi_{m+1}(0, k_2, k_3)$  and  $\phi_{m+1}(2, k_2, k_3) = s_2 \phi_{m+1}(1, k_2, k_3)$ , so that part (2)(a) of the statement holds by construction.

Using the description of the Bruhat interval  $[1, y_m]$  from [LP23], it is straightforward to check that  $\phi_{m+1}$  is a bijection from  $V(\mathcal{C}_{m+1})$  to  $[1, y_{m+1}]$ . It can also be easily verified from this description and our construction that if  $(v, v')$  is a level 0 edge which is in  $\mathcal{C}_{m+1}$  but not in  $\mathcal{C}_m$ , then  $\ell(\phi(v')) = \ell(\phi(v)) + 1$ . Moreover, for all such edges  $(v, v')$ , the element  $\phi(v')$  is obtained from  $\phi(v)$  by either right-multiplication by a single generator, or right-multiplication by the longest element in a subsystem of type  $A_2$ . Therefore by induction, part (1) holds. Property (2)(b) can then be obtained without difficulty from the description in [LP23], by considering inversion sets for the relevant elements of  $W$ .  $\square$

We can now prove Proposition 8.6, from which Theorem 1.5 immediately follows.

*Proof of Proposition 8.6.* Fix  $m \geq 1$  and let  $\phi : V(\mathcal{C}_m) \rightarrow [1, y_m]$  be as in Lemma 8.7. Since  $\phi$  is a bijection from the vertex set of  $\mathcal{C}_m$  to the vertex set of  $[1, y_m]_{\mathcal{B}}$ , we just need to see that for every edge  $(v, v')$  in  $\mathcal{C}_m$ , there is an edge  $(\phi(v), \phi(v'))$  in  $[1, y_m]_{\mathcal{B}}$ . Part (1) of Lemma 8.7 gives this for the level 0 edges of  $\mathcal{C}_m$ .

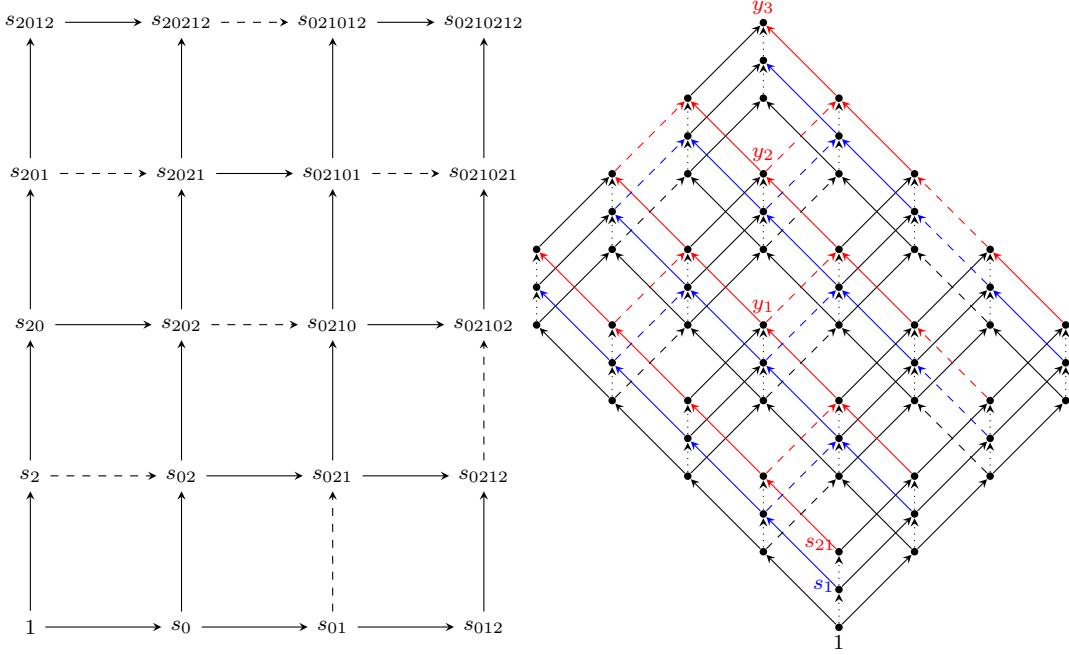


FIGURE 10. The labeling  $\phi$  of the vertices of  $\mathcal{C}(0, 3, 4) \cong \mathcal{C}(3, 4)$  on the left, and of certain vertices of  $\mathcal{C}(2, 3, 4)$  on the right. On both the left and right, the solid edges correspond to right-multiplication by a simple generator, and the dashed edges to right-multiplication by the longest element in a subsystem of type  $A_2$ . On the right, the dotted vertical edges correspond to left-multiplication by  $s_k$  (to go from level  $k$  to level  $k + 1$ ), for  $k = 1, 2$ .

Now suppose  $(v, v')$  is a level 1 edge of  $\mathcal{C}_m$ . Write  $v = (1, k_2, k_3)$  and  $v' = (1, k'_2, k'_3)$  and put  $v_0 = (0, k_2, k_3)$  and  $v'_0 = (0, k'_2, k'_3)$ . Then since  $(v, v')$  is a horizontal edge of  $\mathcal{C}_m$ , there must be a (horizontal) edge  $(v_0, v'_0)$ . Hence by (1) of Lemma 8.7, there is an edge  $(\phi(v_0), \phi(v'_0))$  in  $[1, y_m]_{\mathcal{B}}$ . This means exactly that there is a reflection  $r$  such that  $\phi(v_0)r = \phi(v'_0)$  and  $\ell(\phi(v_0)r) > \ell(\phi(v_0))$ . Now by (2)(a) of Lemma 8.7 with  $k = 1$ , we have  $\phi(v) = s_1\phi(v_0)$  and  $\phi(v') = s_1\phi(v'_0)$ . Thus

$$\phi(v)r = s_1\phi(v_0)r = s_1\phi(v'_0) = \phi(v').$$

Using (2)(b) of Lemma 8.7 with  $k = 1$ , we have  $\ell(\phi(v)) = \ell(\phi(v_0)) + 1$  and  $\ell(\phi(v')) = \ell(\phi(v'_0)) + 1$ . Hence

$$\ell(\phi(v)r) = \ell(\phi(v'_0)) + 1 = \ell(\phi(v_0)r) + 1 > \ell(\phi(v_0)) + 1 = \ell(\phi(v)).$$

Thus there is an edge in  $[1, y_m]_{\mathcal{B}}$  from  $\phi(v)$  to  $\phi(v')$ . The argument is similar for the level 2 edges of  $\mathcal{C}_m$ .

It remains to consider the vertical edges of  $\mathcal{C}_m$ . Suppose  $v_0 = (0, k_2, k_3)$  is a level 0 vertex of  $\mathcal{C}_m$  and put  $v_1 = (1, k_2, k_3)$ , so that there is a vertical edge  $(v_0, v_1)$ . Then by (2)(a) of Lemma 8.7, we have  $\phi(v_1) = s_1\phi(v_0)$ , and by (2)(b) of Lemma 8.7, we have  $\ell(\phi(v_1)) = \ell(\phi(v_0)) + 1$ . Hence in particular,  $\ell(\phi(v_1)) > \ell(\phi(v_0))$ . To see that  $\phi(v_1)$  is obtained from  $\phi(v_0)$  by right-multiplication by a reflection, let  $r = \phi(v_0)^{-1}s_1\phi(v_0)$ . Then  $r$  is a reflection and we have  $\phi(v_1) = \phi(v_0)r$ . Thus there is an edge  $(\phi(v_0), \phi(v_1))$  in  $[1, y_m]_{\mathcal{B}}$ . A similar argument holds for the vertical edges from level 1 to level 2. Thus  $[1, y_m]_{\mathcal{B}}$  is cubulated by  $\mathcal{C}_m$ , as required.  $\square$

*Proof of Theorem 1.5.* Let  $(W, S)$  be of type  $\tilde{A}_2$ . By Lemma 6.1, Proposition 6.3, and Proposition 8.6, the graphs  $[1, y]_{\mathcal{B}}$  for all  $y \in \{1, s_1, s_1s_2, s_1s_2s_0\} \cup \{y_m\}_{m \in \mathbb{N}}$  can be cubulated. Therefore, Corollary 8.4 says that whenever  $P_{x,y'} = 1$  for all  $x \leq y'$ , the Bruhat graph  $[1, y']_{\mathcal{B}}$  can be cubulated. In other words, the converse to Theorem 1.1 holds.  $\square$

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