

# UPPER BOUNDS FOR THE LAGRANGIAN COBORDISM RELATION ON LEGENDRIAN LINKS

JOSHUA M. SABLOFF, DAVID SHEA VELA-VICK, AND C.-M. MICHAEL WONG

**ABSTRACT.** Lagrangian cobordism induces a preorder on the set of Legendrian links in any contact 3-manifold. We show that any finite collection of null-homologous Legendrian links in a tight contact 3-manifold with a common rotation number has an upper bound with respect to the preorder. In particular, we construct an exact Lagrangian cobordism from each element of the collection to a common Legendrian link. This construction allows us to define a notion of minimal Lagrangian genus between any two null-homologous Legendrian links with a common rotation number.

## 1. INTRODUCTION

The relation  $\preceq$  defined by (exact, orientable) Lagrangian cobordism between Legendrian submanifolds in the symplectization of the contact manifold raises a host of surprisingly subtle structural questions. While the Lagrangian cobordism relation is trivially a preorder (i.e. is reflexive and transitive), it is not symmetric [BS18, Cha10, CNS16]; it is unknown whether the relation is a partial order. Further, not every pair of Legendrians is related by Lagrangian cobordism, with the first obstructions coming from the classical invariants: For links  $\Lambda_{\pm}$  in  $\mathbb{R}^3$ , if  $\Lambda_- \preceq \Lambda_+$  via the Lagrangian  $L \subset \mathbb{R} \times \mathbb{R}^3$ , then  $r(\Lambda_+) = r(\Lambda_-)$  and  $tb(\Lambda_+) - tb(\Lambda_-) = -\chi(L)$  [Cha10]. A growing toolbox of non-classical obstructions has been developed to detect this phenomenon; see, just to begin, [BS18, BLW19, EHK16, GJ19, Pan17, ST13].

If two Legendrians are not related by a Lagrangian cobordism, one may still ask if they have a common upper or lower bound with respect to  $\preceq$ . Implicit in the work of Boranda, Traynor, and Yan [BTY13] is that any finite collection of Legendrian links in the standard contact  $\mathbb{R}^3$  with the same rotation number has a lower bound with respect to  $\preceq$ . In another direction, Lazarev [Laz20] has shown that any finite collection of formally isotopic Legendrians in a contact  $(2n+1)$ -manifold with  $n \geq 2$  has an upper bound with respect to a moderate generalization of  $\preceq$ .

The goal of this paper is to find both lower and upper bounds for finite collections of Legendrian links in any tight contact 3-manifold. On one hand, in contrast to the diagrammatic methods of [BTY13], our topological techniques allow us to find lower bounds in any tight contact 3-manifold, though we also present a refinement of the proof in [BTY13] that better suits our goal of constructing upper bounds. On the other hand, in contrast to Lazarev's use of an  $h$ -principle, which restricts his results to higher dimensions, our direct constructions of upper bounds work for Legendrian links in dimension 3.

**Theorem 1.1.** *Let  $\Lambda$  and  $\Lambda'$  be oriented Legendrian links in a tight contact 3-manifold  $(Y, \xi)$ , and suppose that there exist Seifert surfaces  $\Sigma$  and  $\Sigma'$  for which  $r_{[\Sigma]}(\Lambda) = r_{[\Sigma']}(\Lambda')$ . Then there exist oriented Legendrian links  $\Lambda_{\pm} \subset (Y, \xi)$  such that  $\Lambda_- \preceq \Lambda \preceq \Lambda_+$  and  $\Lambda_- \preceq \Lambda' \preceq \Lambda_+$ .*

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**Remark 1.2.** For Legendrian links in  $\mathbb{R}^3$ , the rotation number may be defined without reference to Seifert surfaces, and the hypotheses merely require  $r(\Lambda) = r(\Lambda')$ .

**Example 1.3.** In [Figure 1](#), we display an upper bound for the maximal Legendrian right-handed trefoil and a Legendrian  $m(5_2)$  knot. These two Legendrian knots are not related by Lagrangian cobordism. To see why, note that any Lagrangian cobordism between them must be a concordance since they have the same Thurston–Bennequin number, but no such concordance exists even topologically.

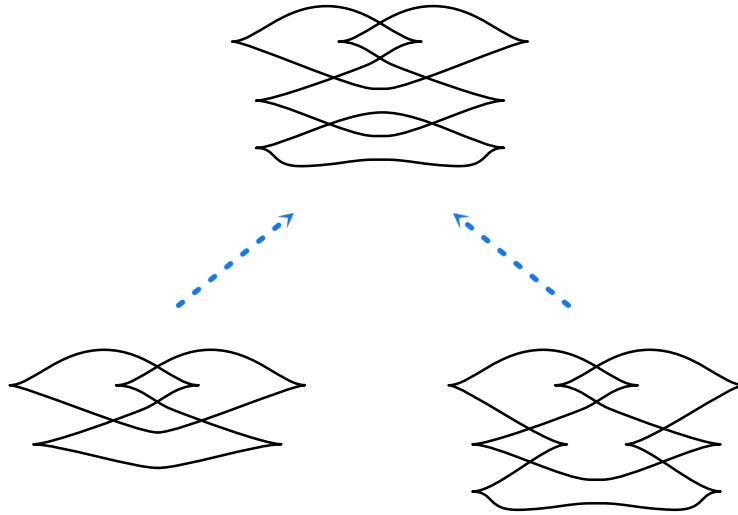


FIGURE 1. An upper bound for the maximal right-handed trefoil and an  $m(5_2)$  knot.

**Example 1.4.** In [Figure 2](#), we display an upper bound for the maximal Legendrian unknot and the maximal Legendrian figure-eight knot. Once again, these two Legendrian knots are not related by Lagrangian cobordism. The fact that the figure-eight has lower Thurston–Bennequin number shows that there cannot be a cobordism from the unknot to the figure-eight; the fact that the figure-eight has two normal rulings shows that there cannot be a cobordism from the figure-eight to the unknot [[CNS16](#), Theorem 2.7].

In fact, we prove the following strengthened version of [Theorem 1.1](#).

**Proposition 1.5.** *Under the same hypotheses of [Theorem 1.1](#), there exist oriented Legendrian links  $\Lambda_-, \Lambda_+ \subset Y$  and oriented exact decomposable Lagrangian cobordisms  $L$  and  $L'$  from  $\Lambda_-$  to  $\Lambda_+$ , such that*

- The Legendrian link  $\Lambda$  appears as a collared slice of  $L$ ;
- The Legendrian link  $\Lambda'$  appears as a collared slice of  $L'$ ; and
- $L$  and  $L'$  are exact-Lagrangian isotopic.

**Remark 1.6.** There are statements analogous to [Theorem 1.1](#) and [Proposition 1.5](#) that hold for unoriented Legendrian links and unoriented (and possibly unorientable) exact Lagrangian cobordisms, for which there are no requirements on the rotation number.

The main theorem has several interesting consequences. First, we recall that not every Legendrian knot has a Lagrangian filling. The figure-eight knot in [Figure 2](#) is one such example. By transitivity,

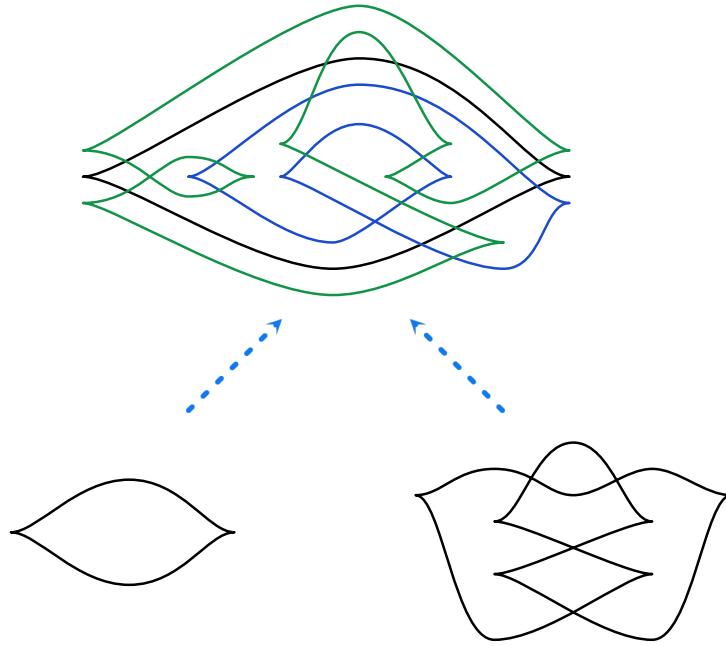


FIGURE 2. An upper bound for the maximal unknot and the maximal figure-eight knot. The colors in the diagram of the upper bound are only meant to distinguish components of the link to improve readability.

this implies that not every Legendrian knot lies at the top of a Lagrangian cobordism from a fillable Legendrian. On the other hand, we have the following corollary of the main theorem:

**Corollary 1.7.** *For any Legendrian link  $\Lambda$ , there exists a Legendrian link  $\Lambda_+$  with a Lagrangian filling and a Lagrangian cobordism from  $\Lambda$  to  $\Lambda_+$ .*

The proof simply requires us to apply [Theorem 1.1](#) with  $\Lambda$  being the given Legendrian and  $\Lambda'$  being the maximal Legendrian unknot. The upper bound  $\Lambda_+$  is Lagrangian fillable since there is a cobordism to it from the unknot.

A second consequence of the main theorem is that we are able to define a notion of the minimal genus of a Lagrangian cobordism between *any* two Legendrian links with the same rotation number. Roughly speaking, we define a Lagrangian quasi-cobordism between  $\Lambda$  and  $\Lambda'$  to be a sequence  $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_n = \Lambda'$  of Legendrian links together with upper (or lower) bounds between each of  $\Lambda_i$  and  $\Lambda_{i+1}$ . The genus of the quasi-cobordism is the genus of the (smooth) composition of the underlying Lagrangian cobordisms between the  $\Lambda_i$  and their bounds; we may then define  $g_L(\Lambda, \Lambda')$  to be the minimal genus of such a Lagrangian quasi-cobordism. When there is a Lagrangian cobordism from  $\Lambda$  to  $\Lambda'$  and  $\Lambda$  is fillable,  $g_L(\Lambda, \Lambda')$  agrees with the relative smooth genus  $g_s(\Lambda, \Lambda')$ ; see [Lemma 6.7](#).

The remainder of the paper is organized as follows: In [Section 2](#), we review key ideas in the definition and construction of Lagrangian cobordisms between Legendrian links. We also define the notion of a Legendrian handle graph, which will form the basis of our later constructions. In [Section 3](#) and [Section 4](#), we prove that any two Legendrians in a tight contact 3-manifold have a lower bound with respect to  $\preceq$ , and encode the Lagrangian cobordisms involved with Legendrian handle graphs. We present two approaches to this goal: In [Section 3](#), we prove the claim for general

tight contact 3-manifolds using convex surface theory, while in [Section 4](#), we provide a diagrammatic proof in  $\mathbb{R}^3$ , refining a proof of [\[BTY13\]](#). We then proceed in [Section 5](#) to prove [Proposition 1.5](#), and hence [Theorem 1.1](#). We end the paper in [Section 6](#) by beginning an exploration of Lagrangian quasi-cobordisms and their genera, finishing with some open questions.

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## 2. A DESCRIPTION OF LAGRANGIAN COBORDISMS

In this section, we describe Lagrangian cobordisms, how to construct them, and how to keep track of those constructions.

**2.1. Lagrangian cobordisms.** We begin with the formal definition of a Lagrangian cobordism between Legendrian links.

**Definition 2.1.** Let  $\Lambda_-$  and  $\Lambda_+$  be Legendrian links in the contact manifold  $(Y, \xi)$ , where  $\xi = \ker(\alpha)$  for a contact 1-form  $\alpha$ . An (exact, orientable) *Lagrangian cobordism*  $L$  from  $\Lambda_-$  to  $\Lambda_+$  is an exact, orientable, properly embedded Lagrangian submanifold  $L \subset (\mathbb{R} \times Y, d(e^t\alpha))$  that satisfies the following:

- There exists  $T_+ \in \mathbb{R}$  such that  $L \cap ([T_+, \infty) \times Y) = [T_+, \infty) \times \Lambda_+$ ;
- There exists  $T_- < T_+$  such that  $L \cap ((-\infty, T_-] \times Y) = (-\infty, T_-] \times \Lambda_-$ ; and
- The primitive of  $(e^t\alpha)|_L$  is constant (rather than locally constant) at each cylindrical end of  $L$ .

Note that the last condition enables us to concatenate Lagrangian cobordisms while preserving exactness.

We will use three constructions of Lagrangian cobordisms in this paper, which we will call the *elementary Lagrangian cobordisms*:

**0-handle:** Adding a disjoint, unlinked maximal Legendrian unknot  $\Upsilon$  to  $\Lambda$  induces an exact Lagrangian cobordism from  $\Lambda$  to  $\Lambda \sqcup \Upsilon$  [\[BST15, EHK16\]](#).

**Legendrian isotopy:** A Legendrian isotopy from  $\Lambda$  to  $\Lambda'$  induces an exact Lagrangian cobordism from  $\Lambda$  to  $\Lambda'$ , though the construction is more complicated than simply taking the trace of the isotopy [\[BST15, EHK16, EG98\]](#).

**Legendrian ambient surgery:** We describe this construction in more detail in [Section 2.2](#), and we will develop a method for keeping track of a set of ambient surgeries in [Section 2.3](#).

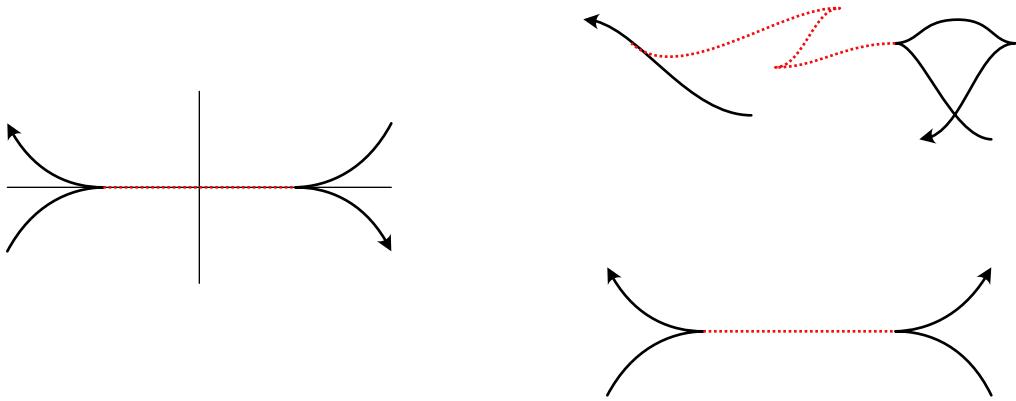
**2.2. Legendrian ambient surgery.** Our next step is to explain Dimitroglou Rizell's Legendrian ambient surgery construction in the 3-dimensional setting [\[Dim16\]](#). Similar constructions appear in [\[BST15\]](#) and [\[EHK16\]](#), though Dimitroglou Rizell's more flexible language is best suited for our purposes. In dimension 3, Legendrian ambient surgery begins with the data of an oriented Legendrian link  $\Lambda \subset (Y, \xi)$  and an embedded Legendrian curve  $D$  with endpoints on  $\Lambda$  that is, in a sense to be defined, compatible with the orientation of  $\Lambda$ . The construction then produces a Legendrian  $\Lambda_D$ , contained in an arbitrarily small neighborhood of  $\Lambda \cup D$ , that is obtained from  $\Lambda$  by ambient surgery along  $D$ . Further, the construction produces an exact Lagrangian cobordism from  $\Lambda$  to  $\Lambda_D$ .

More precisely, given  $\Lambda \subset (Y, \xi)$  with contact 1-form  $\alpha$ , a *surgery disk* is an embedded Legendrian arc  $D \subset Y$  such that

- (1)  $D \cap \Lambda = \partial D$ ,
- (2) The intersection  $D \cap \Lambda$  is transverse, and
- (3) The vector field  $H \subset T_p \Lambda$  defined for all  $p \in \partial D$  (up to scaling) by  $d\alpha(G, H(p)) > 0$  for all outward-pointing vectors  $G$  in  $T_p D$  either completely agrees with or completely disagrees with the framing on  $\partial D$  induced by the orientation of  $\Lambda$ .

For an unoriented surgery, we need not specify a framing for  $\partial D$ , and the last condition is no longer relevant.

The standard model for such a surgery disk appears in [Figure 3 \(a\)](#). In fact, up to an overall orientation reversal on  $\Lambda$ , there is a neighborhood  $U$  of  $D$  in  $Y$  that is contactomorphic to a neighborhood of the standard model for  $\Lambda_0$  and  $D_0$  [[Dim16](#), Section 4.4.1]. Working in the standard model, we may replace  $\Lambda_0$  by the Legendrian arcs  $\Lambda_1$  as in [Figure 4](#), a process that realizes the ambient surgery on  $\Lambda_0$  along  $D_0$ . Pulling this construction back to the neighborhood of  $D$  in  $Y$ , we call the resulting link *Legendrian ambient surgery* on  $\Lambda$  along  $D$ .



[FIGURE 3.](#) (a) The standard model in  $(\mathbb{R}^3, \alpha_0)$  of a surgery disk  $D_0$  with endpoints on a Legendrian  $\Lambda_0$ , with (b) another example of a surgery disk, and (c) a disk that fails condition (3).

**Theorem 2.2** (Dimitroglou Rizell [[Dim16](#)]). *Given an oriented Legendrian link  $\Lambda$  and a surgery disk  $D$ , let  $\Lambda_D$  be the Legendrian link obtained from  $\Lambda$  by Legendrian ambient surgery along  $D$ . Then there exists an exact Lagrangian cobordism from  $\Lambda$  to  $\Lambda_D$  arising from the attachment of a 1-handle to  $(-\infty, T] \times \Lambda$ .*

**Remark 2.3.** The construction of Legendrian ambient surgery and the associated Lagrangian cobordism is local. In particular, for a small neighborhood  $U$  of  $D$ , the surgery construction does not alter  $\Lambda \cap (Y \setminus U)$ , and the cobordism  $L$  outside of  $\mathbb{R} \times U$  is cylindrical over  $\Lambda \cap (Y \setminus U)$ .

**2.3. Legendrian handle graphs.** In this section, we introduce a structure for keeping track of independent ambient surgeries. We use the notion of a Legendrian graph, following the conventions in [[OP12](#)].

Before we begin, recall from e.g. [[OP12](#)] that two Legendrian graphs in  $(\mathbb{R}^3, \xi_0)$  are Legendrian isotopic if and only if their front diagrams are related by planar isotopy and six Reidemeister moves, as seen in [Figure 5](#).

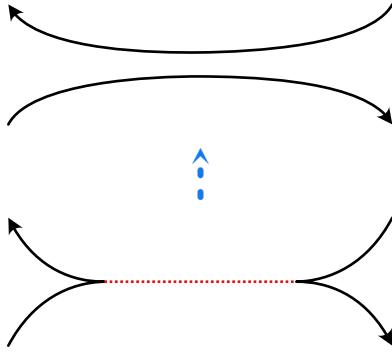


FIGURE 4. Surgery on the standard model  $\Lambda_0 \cup D$  yields a new Legendrian  $\Lambda_1$ .

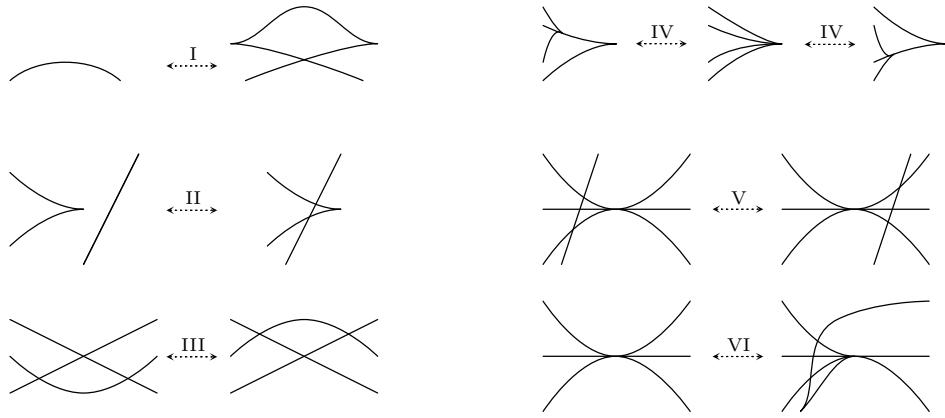


FIGURE 5. Reidemeister moves for Legendrian graphs in  $\mathbb{R}^3$ .

**Definition 2.4.** A *Legendrian handle graph* is a pair  $(G, \Lambda)$ , where  $G \subset (Y, \xi)$  is a trivalent Legendrian graph and  $\Lambda \subset (Y, \xi)$  is a Legendrian link (called the *underlying link*), such that

- $\Lambda \subset G$ ;
- The vertices of  $G$  lie on  $\Lambda$ ; and
- $G \setminus \Lambda$  is the union of a finite collection of pairwise disjoint Legendrian arcs  $\gamma_1, \dots, \gamma_m$  whose closures satisfy the conditions of surgery disks for  $\Lambda$ .

We also say that  $G$  is a *Legendrian handle graph on  $\Lambda$* . The set of closures of the components of  $G \setminus \Lambda$  is denoted by  $\mathcal{H}$ .

See the bottom of [Figure 6](#) for an example of a Legendrian handle graph whose underlying Legendrian link is a Legendrian Hopf link in  $(\mathbb{R}^3, \xi_0)$ .

**Definition 2.5.** Let  $(G, \Lambda)$  be a Legendrian handle graph and let  $\mathcal{H}_0$  be a subset of the arcs in  $\mathcal{H}$ . The *Legendrian ambient surgery*  $\text{Surg}(G, \Lambda, \mathcal{H}_0)$  is the Legendrian handle graph  $(G', \Lambda')$  resulting from performing Legendrian ambient surgery along each arc in  $\mathcal{H}_0$ , as described in [Section 2.2](#).

We will, at times, abuse notation and refer to the underlying Legendrian link  $\Lambda'$  by  $\text{Surg}(G, \Lambda, \mathcal{H}_0)$ ; we will also use  $\text{Surg}(G, \Lambda)$  when  $\mathcal{H}_0 = \mathcal{H}$ . For example, in [Figure 6](#), the Legendrian link at the top is  $\text{Surg}(G, \Lambda)$  for the Legendrian handle graph  $(G, \Lambda)$  at the bottom.

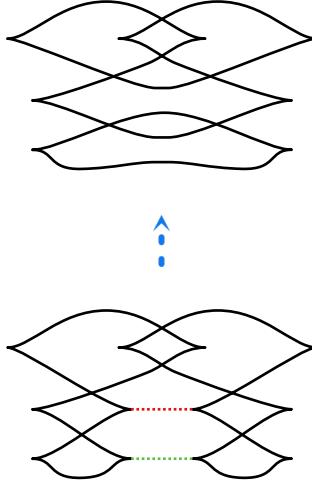


FIGURE 6. The Legendrian link at the top of the figure is the Legendrian ambient surgery on the Legendrian handle graph  $(G, \Lambda)$  at the bottom.

By the work of Dimitroglou Rizell [Dim16] as described in Section 2.2, Legendrian ambient surgery on any given Legendrian arc corresponds to an exact Lagrangian cobordism; this implies that, given an order  $\mathbf{o} = (\gamma_{j_1}, \dots, \gamma_{j_m})$  of the components of  $\mathcal{H}_0$ , one obtains an exact Lagrangian cobordism  $L(G, \Lambda, \mathcal{H}_0, \mathbf{o})$  from  $\Lambda$  to  $\text{Surg}(G, \Lambda, \mathcal{H}_0)$  by performing Legendrian ambient surgery in the order given by  $\mathbf{o}$ . The order, in fact, does not matter.

**Proposition 2.6.** *Suppose  $(G, \Lambda)$  is a Legendrian handle graph, and  $\mathbf{o}_1$  and  $\mathbf{o}_2$  are orders of the components of  $\mathcal{H}_0$ . The Lagrangian cobordisms  $L(G, \Lambda, \mathcal{H}_0, \mathbf{o}_1)$  and  $L(G, \Lambda, \mathcal{H}_0, \mathbf{o}_2)$  are exact-Lagrangian isotopic.*

*Proof.* It suffices to consider the case where  $\mathbf{o}_1$  and  $\mathbf{o}_2$  differ by an adjacent transposition  $(\gamma_{j_1}, \gamma_{j_2}) \rightarrow (\gamma_{j_2}, \gamma_{j_1})$ . The cobordism  $L(G, \Lambda, \mathcal{H}_0, \mathbf{o})$  is defined by composing the elementary Lagrangian cobordisms associated to the arcs  $\gamma_1, \dots, \gamma_m$ . Since there are finitely many of these, by shrinking the neighborhoods of  $\gamma_j$  as in Remark 2.3, we may assume the neighborhoods to be pairwise disjoint. This implies that the elementary Lagrangian cobordisms associated to  $\gamma_{j_1}$  and  $\gamma_{j_2}$  may be constructed simultaneously and shifted past each other along the cylindrical parts of the cobordism. Thus, the parameter given by the relative heights of these two cobordisms gives an exact Lagrangian isotopy between  $L(G, \Lambda, \mathcal{H}_0, \mathbf{o}_1)$  and  $L(G, \Lambda, \mathcal{H}_0, \mathbf{o}_2)$ .  $\square$

Proposition 2.6 allows us to associate an *isotopy class*  $L(G, \Lambda, \mathcal{H}_0)$  of exact Lagrangian cobordisms to a Legendrian handle graph  $(G, \Lambda)$  and  $\mathcal{H}_0 \subset \mathcal{H}$ .

**Remark 2.7.** It is not clear that every decomposable cobordism can be described using a Legendrian handle graph. As shown in Figure 7, one may need to perform one ambient surgery in order for another to be possible; this would violate Proposition 2.6.

### 3. LOWER BOUNDS VIA CONTACT TOPOLOGY

In this section, for a pair of Legendrian links with the same rotation number, we construct a pair of exact Lagrangian cobordisms from a common lower bound, encoded by Legendrian handle graphs with the same underlying link.

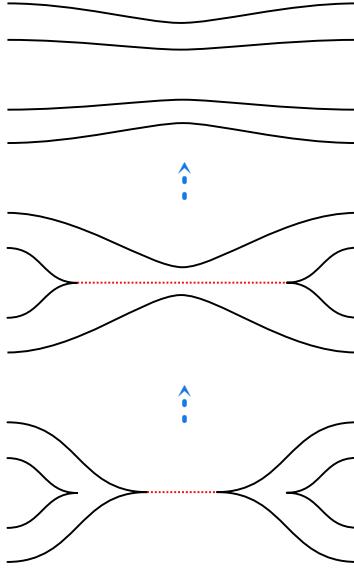


FIGURE 7. The surgery joining the two inner cusps cannot be performed until after the surgery joining the two outer cusps.

**Proposition 3.1.** *Let  $\Lambda$  and  $\Lambda'$  be oriented Legendrian links in a tight contact manifold  $(Y, \xi)$  and suppose that there exist Seifert surfaces  $\Sigma$  and  $\Sigma'$  for which  $r_{[\Sigma]}(\Lambda) = r_{[\Sigma']}(\Lambda')$ . Then there exists an oriented Legendrian link  $\Lambda_- \subset (Y, \xi)$  and handle graphs  $G$  and  $G'$  on  $\Lambda_-$  such that  $\text{Surg}(G, \Lambda_-)$  (resp.  $\text{Surg}(G', \Lambda_-)$ ) is Legendrian isotopic to  $\Lambda$  (resp.  $\Lambda'$ ).*

Our proof of [Proposition 3.1](#) relies on convex surface theory applied to the Seifert surfaces  $\Sigma$  and  $\Sigma'$ . In preparation for this, we begin by establishing the following lemma, which extends the work of Boranda, Traynor, and Yan [[BTY13](#)] by placing their result in the context of Legendrian handle graphs.

**Lemma 3.2** (cf. [[BTY13](#), Lemma 3.3]). *Let  $\Lambda$  be an oriented Legendrian link in a tight contact manifold  $(Y, \xi)$ , and  $S_+ \circ S_-(\Lambda)$  the result of successive negative and positive stabilization on a component of  $\Lambda$ . Then there is a handle graph  $G$  on  $S_+ \circ S_-(\Lambda)$  such that  $\text{Surg}(G, S_+ \circ S_-(\Lambda))$  is Legendrian isotopic to  $\Lambda$ .*

*Proof.* The proof is essentially contained in [Figure 8](#), which explicitly identifies a local model for the desired handle graph.  $\square$

**Lemma 3.3.** *Let  $\Lambda$  be oriented, null-homologous Legendrian link in a tight contact manifold  $(Y, \xi)$ . Then there exists an oriented Legendrian unknot  $\Lambda_U \subset (Y, \xi)$  and a handle graph  $G$  on  $\Lambda_U$  such that  $\text{Surg}(G, \Lambda_U)$  is Legendrian isotopic to  $\Lambda$ .*

*Proof.* Suppose that  $\Sigma$  is a Seifert surface for  $\Lambda$ . Applying [Lemma 3.2](#) to successively double-stabilize each component of  $\Lambda$  if necessary, we obtain a handle graph  $(G_1, \Lambda_1)$  and a Seifert surface  $\Sigma_1$  for  $\Lambda_1$  isotopic to  $\Sigma$ , such that the twisting of  $\xi$  relative to  $\Sigma_1$  along each component of  $\partial\Sigma_1 = \Lambda_1$  is negative, and  $\text{Surg}(G_1, \Lambda_1)$  is Legendrian isotopic to  $\Lambda$ . Below, we will denote this first condition by the shorthand notation  $\text{tw}(\xi, \Sigma_1) < 0$ , and similarly for other surfaces.

By work of Kanda [[Kan98](#)], since  $\text{tw}(\xi, \Sigma_1) < 0$ , there is an isotopy of  $\Sigma_1$  relative to  $\partial\Sigma_1 = \Lambda_1$  such that the resulting surface  $\Sigma_2$  is convex. (While we will not use this, we may assume that this isotopy

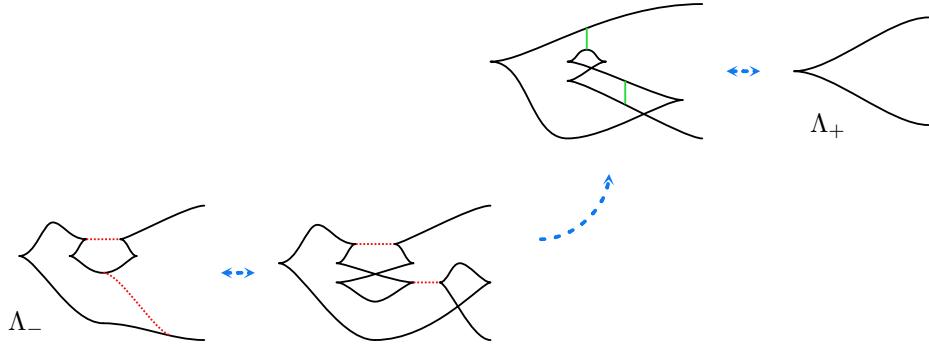


FIGURE 8. A handle graph giving rise to a Lagrangian cobordism from  $S_+ \circ S_-(\Lambda)$  to  $\Lambda$ .

is a  $C^0$  perturbation near the boundary, followed by a  $C^\infty$  perturbation of the interior.) Further, by possibly Legendrian-isotoping the handle arcs of  $G_1$ , we obtain a handle graph  $(G_2, \Lambda_2 = \Lambda_1)$  whose handle arcs  $G_2 \setminus \Lambda_2$  intersect  $\Sigma_2$  transversely in a finite number of points.

To aid the discussion to follow, we picture the convex Seifert surface  $\Sigma_2$  in disk-band form; see Figure 9. Below, we shall distinguish the bands from the disks, by fixing the disk-band decomposition. (Note that Figure 9 is an abstract diagram of  $\Sigma_2$ ; as  $\Sigma_2$  is embedded in  $Y$ , the bands may be “linked”.) Since  $\text{tw}(\xi, \Sigma_2) < 0$ , the dividing set must intersect each component of  $\Lambda$ .

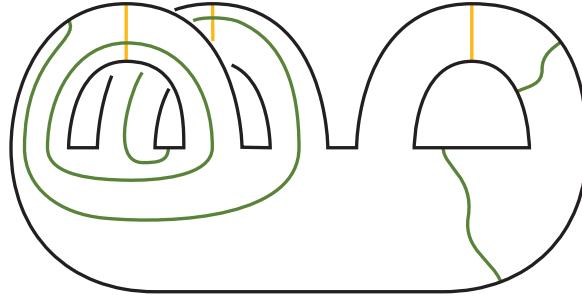


FIGURE 9. The convex Seifert surface  $\Sigma_2$ , dividing set  $\Gamma_\Sigma$ , and arc basis, viewed in disk-band form.

To obtain the desired handle graph  $(G, \Lambda_U)$ , our strategy is to cut the bands of  $\Sigma_2$ . Precisely, let  $\{a_1, \dots, a_g\}$  be an arc basis for  $\Sigma_2$  consisting of a collection of properly embedded arcs in  $\Sigma_2$ , such that the intersection of each  $a_i$  with  $G_2 \setminus \Lambda_2$  is empty. Figure 10 depicts a band of  $\Sigma_2$  and a corresponding basis arc  $a_i$ .

Now construct a (not necessarily Legendrian) link  $\Lambda_{2.5}$  as follows: Take a parallel push-off of  $\Lambda_2$  in  $\Sigma_2$  and, for each basis arc  $a_i$  that intersects the dividing set  $\Gamma_{\Sigma_2}$ , perform a finger move across each of the dividing curves involved and back, as shown in Figure 11. Since  $a_i \cap (G_2 \setminus \Lambda_2) = \emptyset$  for each  $i$ , we may assume that the finger moves avoid all intersection points between  $G_2 \setminus \Lambda_2$  and  $\Sigma_2$ .

Since  $\text{tw}(\xi, \Sigma_2) < 0$ , each component of  $\Sigma_2 \setminus \Lambda_{2.5}$  intersects  $\Gamma_{\Sigma_2}$ , and we can apply the Legendrian Realization Principle (LeRP) to  $\Lambda_{2.5} \subset \Sigma_2$  to obtain an isotopy of  $\Sigma_2$  to a convex surface  $\Sigma_{2.5}$  with  $\partial \Sigma_{2.5} = \Lambda_2$ , such that the image  $\Lambda_3$  of  $\Lambda_{2.5}$  under the isotopy is Legendrian, and  $\Gamma_{\Sigma_{2.5}}$  is the image of  $\Gamma_{\Sigma_2}$ ; see [Kan98] and [Hon00, Section 3].

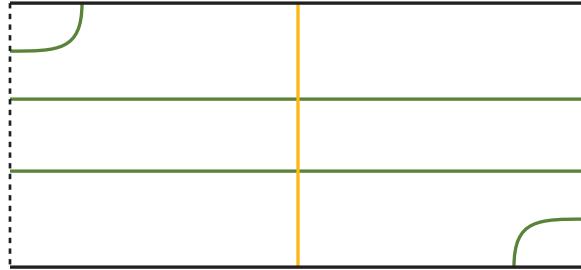


FIGURE 10. A band of the convex Seifert surface  $\Sigma_2$  and a corresponding basis arc  $a_i$ .

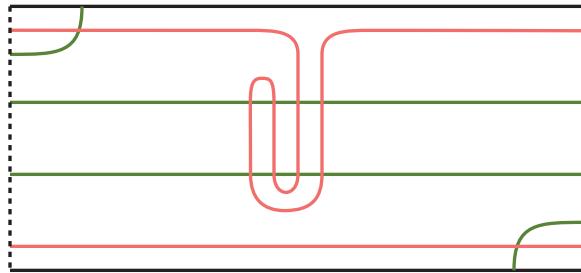


FIGURE 11. A double-stabilization  $\Lambda_3$  of  $\Lambda_2$  whose new Seifert surface  $\Sigma_3$  contains an arc basis disjoint from its dividing set.

Since the finger moves giving rise to  $\Lambda_{2.5}$ —and hence  $\Lambda_3$ —each involved isotoping across elements of the dividing set an even number of times, we have that the Legendrian link  $\Lambda_3$  is necessarily an iterated double-stabilization of  $\Lambda_2$ . In fact,  $\Lambda_3$  is Legendrian isotopic to  $\Lambda_2$  outside of a tubular neighborhood  $U_a$  of the  $a_i$ 's. We now construct a handle graph on  $\Lambda_3$  as follows: First, extend the Legendrian isotopy between  $\Lambda_2 \setminus U_a$  and  $\Lambda_3 \setminus U_a$  to a local contact isotopy, and apply the local contact isotopy to the handle arcs in  $G_2 \setminus \Lambda_2$ , obtaining handle arcs that are attached to  $\Lambda_3$ . Second, by [Lemma 3.2](#), we may add a collection  $\mathcal{H}_3$  of handle arcs to this collection to obtain a handle graph  $(G_3, \Lambda_3)$  such that  $\text{Surg}(G_3, \Lambda_3, \mathcal{H}_3)$  is Legendrian isotopic to  $(G_2, \Lambda_2)$ . In particular, this means that  $\text{Surg}(G_3, \Lambda_3)$  is Legendrian isotopic to  $\Lambda$ . As before, by possibly Legendrian-isotoping the handle arcs of  $G_3$ , we may assume that the handle arcs in  $(G_3, \Lambda_3)$  intersect  $\Sigma_{2.5}$  transversely in a finite number of points. Note that, by [Figure 8](#) in the proof of [Lemma 3.2](#), the handle arcs of  $\mathcal{H}_3$  can be taken to be contained in an arbitrarily small tubular neighborhood of the  $a_i$ 's, implying that the complication in [Remark 2.7](#) does not arise, since the handle arcs in  $\mathcal{H}_3$  are contained in a neighborhood disjoint from  $G_{2.5} \setminus \Lambda_3$ .)

Let  $\Sigma_3$  be the closure of the component of  $\Sigma_{2.5} \setminus \Lambda_3$  that does not intersect  $\partial\Sigma_{2.5}$ . Then we have obtained a Legendrian link  $\Lambda_3$  bounding a convex Seifert surface  $\Sigma_3$  that contains an arc basis  $\{a'_1, \dots, a'_g\}$  that does not intersect the dividing set  $\Gamma_{\Sigma_3}$ , and a handle graph  $G_3$  on  $\Lambda_3$  such that  $\text{Surg}(G_3, \Lambda_3)$  is Legendrian isotopic to  $\Lambda$ .

We are now ready to construct the unknot  $\Lambda_U$  and desired Legendrian handle graph  $G$  on  $\Lambda_U$ . We begin as above by taking a parallel push-off of  $\Lambda_3$  in  $\Sigma_3$  and topologically surgering it along the basis arcs  $a'_i$ , to obtain a (not necessarily Legendrian) topological unknot  $\Lambda_{3.5}$ , together with a collection  $\{b_1, \dots, b_g\}$  of dual surgery arcs, as depicted in [Figure 12](#). Again, we may assume that  $b_i \cap (G_3 \setminus \Lambda_3) = \emptyset$  for each  $i$ .

As above, since  $\text{tw}(\xi, \Sigma_3) < 0$  (as  $\partial\Sigma_3 = \Lambda_3$  is a double-stabilization of  $\Lambda_2 = \partial\Sigma_2$ , which has the same property), each component of  $\Sigma_3 \setminus (\Lambda_{3.5} \cup \bigcup_i b_i)$  intersects  $\Gamma_{\Sigma_3}$ . We may therefore again apply

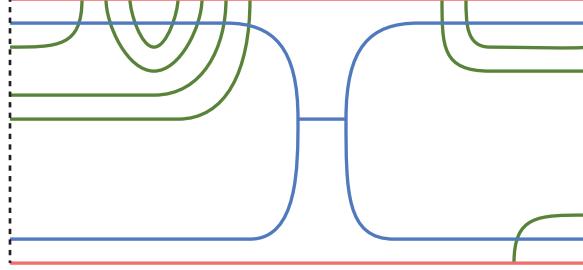


FIGURE 12. A Legendrian unknot  $\Lambda_U = \Lambda_4$  contained in  $\Sigma_3$  together with dual surgery arcs  $\{b'_1, \dots, b'_g\}$ .

the LeRP to  $\Lambda_{3.5} \cup \bigcup_i b_i \subset \Sigma_3$  to obtain an isotopy of  $\Sigma_3$  to a convex surface  $\Sigma_4$  with  $\partial\Sigma_4 = \Lambda_3$ , such that the image  $\Lambda_4$  of  $\Lambda_{3.5}$  and the image  $b'_i$  of each  $b_i$  under the isotopy is Legendrian, and  $\Gamma_{\Sigma_4}$  is the image of  $\Gamma_{\Sigma_3}$ .

Again,  $\Lambda_4$  is Legendrian isotopic to  $\Lambda_3$  outside of a tubular neighborhood  $U_{a'}$  of the  $(a'_i)$ 's; see [Hon00, Section 3]. We now construct a handle graph on  $\Lambda_4$  as follows: First, extend the Legendrian isotopy between  $\Lambda_3 \setminus U_{a'}$  and  $\Lambda_4 \setminus U_{a'}$  to a local contact isotopy, and apply the local contact isotopy to the handle arcs in  $G_3 \setminus \Lambda_3$ , obtaining handle arcs that are attached to  $\Lambda_4$ . Second, we add to this collection the collection  $\mathcal{H}_4 = \{b'_i\}$ , to obtain a handle graph  $(G_4, \Lambda_4)$  such that  $\text{Surg}(G_4, \Lambda_4, \mathcal{H}_4)$  is Legendrian isotopic to  $(G_3, \Lambda_3)$ . In particular, this means that  $\text{Surg}(G_4, \Lambda_4)$  is Legendrian isotopic to  $\Lambda$ .

Let  $\Lambda_U = \Lambda_4$  and  $G = G_4$ , and our proof is complete.  $\square$

We are now ready to prove the main result of this section.

*Proof of Proposition 3.1.* According to Lemma 3.3, there are oriented Legendrian unknots  $\Lambda_U$  and  $\Lambda'_U$  and handle graphs  $G$  and  $G'$ , such that  $\text{Surg}(G, \Lambda_U)$  (resp.  $\text{Surg}(G', \Lambda'_U)$ ) is Legendrian isotopic to  $\Lambda$  (resp.  $\Lambda'$ ).

Since  $r_{[\Sigma]}(\Lambda) = r_{[\Sigma']}(\Lambda')$ , it follows that  $r(\Lambda_U) = r(\Lambda'_U)$ . This also implies that  $\text{tb}(\Lambda_U)$  and  $\text{tb}(\Lambda'_U)$  differ by a multiple of 2. Without loss of generality, assume that  $\text{tb}(\Lambda_U) \geq \text{tb}(\Lambda'_U)$ ; then by successively applying Lemma 3.2 to  $\Lambda_U$  if necessary, we obtain a handle graph  $(\bar{G}, \bar{\Lambda}_U)$  such that  $\text{tb}(\bar{\Lambda}_U) = \text{tb}(\Lambda'_U)$  and  $\text{Surg}(\bar{G}, \bar{\Lambda}_U)$  is Legendrian isotopic to  $G$ . Again, by Figure 8 in the proof of Lemma 3.2, the handle arcs of  $\bar{G}$  can be taken to be contained in an arbitrarily small neighborhood of a point; thus, we may combine these handle arcs with those of  $G$ , as in the proof of Lemma 3.3 to obtain a handle graph  $(\tilde{G}, \bar{\Lambda}_U)$  such that  $\text{Surg}(\tilde{G}, \bar{\Lambda}_U)$  is Legendrian isotopic to  $\Lambda$ .

Now  $\bar{\Lambda}_U$  and  $\Lambda'_U$  are unknots in the tight contact 3-manifold  $(Y, \xi)$  with the same Thurston–Bennequin and rotation numbers. By the classification of Legendrian unknots in tight contact manifolds by Eliashberg and Fraser [EF98], we have that there is a contact isotopy  $\phi_t$  of  $(Y, \xi)$  taking  $\bar{\Lambda}_U$  to  $\Lambda'_U$ .

We now apply the isotopy  $\phi_t$  to the Legendrian handle graph  $\tilde{G}$  and perturb the result so that the attached handles are disjoint from those of  $G'$ . We then obtain a pair of Legendrian handle graphs for the unknot  $\Lambda'_U$ , surgery along which yields Legendrian links isotopic to  $\Lambda$  and  $\Lambda'$  respectively.  $\square$

#### 4. LOWER BOUNDS VIA DIAGRAMS

In this section, we re-prove Lemma 3.3 for Legendrian links in the standard contact  $\mathbb{R}^3$  using diagrammatic techniques rather than convex surface theory. This proof refines that of [BTY13] to produce a handle graph as well as a Lagrangian cobordism from an unknot. We begin with a

sequence of lemmas that reduce the number of crossings of the front diagram of the Legendrian link in a Legendrian handle graph at the expense of increasing the number of handles. But first, we state a technical general position result.

**Lemma 4.1.** *For any Legendrian handle graph  $(G, \Lambda)$ , there exists a  $C^0$ -close, isotopic Legendrian handle graph  $(G', \Lambda')$  such that all singular points of the front diagram of  $G$  have distinct  $x$  coordinates.*

*Proof.* While this lemma simply expresses general position for the graph  $G$ , we note in Figure 13 (a) that moving a triple point off of a cusp of  $\Lambda$  is tantamount to using a Reidemeister VI move.  $\square$

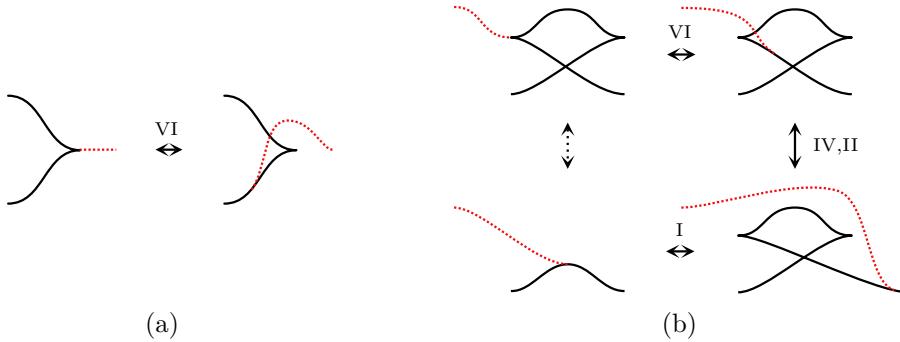


FIGURE 13. (a) Moving a triple point off of a cusp using a Reidemeister VI move from [OP12] and (b) Clearing a cusp of  $\Lambda$  for a Reidemeister I move.

First, we remove negative crossings.

**Lemma 4.2.** *Given a Legendrian link  $\Lambda_+$ , whose front diagram has a negative crossing, and a Legendrian handle graph  $G_+$  on  $\Lambda_+$ , there exists a Legendrian handle graph  $(G_-, \Lambda_-)$  and a subset  $\mathcal{H}_0$  of handles of  $G_-$ , such that  $\text{Surg}(G_-, \Lambda_-, \mathcal{H}_0)$  is Legendrian isotopic to  $(G_+, \Lambda_+)$ , and the front diagram of  $\Lambda_-$  has one fewer negative crossing than that of  $\Lambda_+$ .*

*Proof.* After applying Lemma 4.1 to isolate negative crossings, the proof of Lemma 4.2 is contained in Figure 14.  $\square$

Next, we remove positive crossings.

**Lemma 4.3.** *Given a Legendrian link  $\Lambda_+$ , the leftmost crossing of whose front diagram is positive, and a Legendrian handle graph  $G_+$  on  $\Lambda_+$ , there exists a Legendrian handle graph  $(G_-, \Lambda_-)$  and a subset  $\mathcal{H}_0$  of handles of  $G_-$ , such that  $\text{Surg}(G_-, \Lambda_-, \mathcal{H}_0)$  is Legendrian isotopic to  $(G_+, \Lambda_+)$  and the front diagram of  $\Lambda_-$  has one fewer positive crossing than that of  $\Lambda_+$ .*

*Proof.* Apply Lemma 4.1 to isolate crossings and cusps of  $\Lambda_+$  from handles of  $G_+$ . Consider the leftmost crossing  $X_0$  of  $\Lambda_+$ . Without loss of generality, we may assume that  $\Lambda_+$  is oriented from right to left on both strands of  $X_0$ . The upper-left strand incident to  $X_0$  must thus next return to  $X_0$ ; the same is true for the bottom-left strand. Since there are no crossings of  $\Lambda_+$  to the left of  $X_0$ , either the upper left strand must next cross the  $x$ -coordinate of  $X_0$  above  $X_0$  or the lower left strand must next cross the  $x$ -coordinate of  $X_0$  below  $X_0$ . Without loss of generality, assume that this holds for the upper left strand as in the upper-right portion of Figure 15. Let  $-\eta \subset \Lambda_+$  be the compact 1-manifold that starts at  $X_0$  and traverses along the upper-left strand of  $X_0$  until

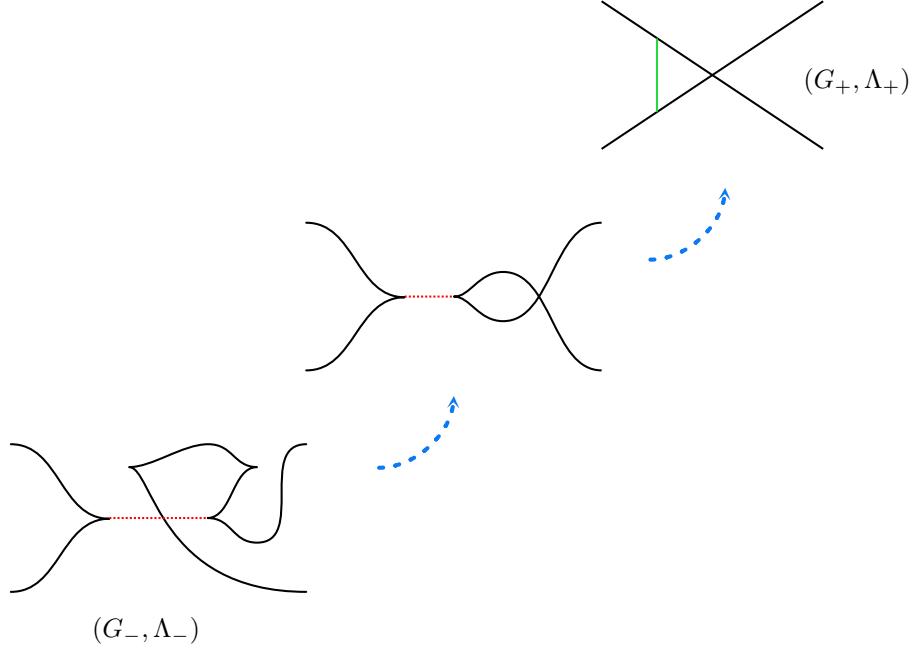


FIGURE 14. The Legendrian handle graph  $(G_-, \Lambda_-)$  has one fewer negative crossing than  $(G_+, \Lambda_+)$ . Red curves represent surgery disks, i.e. cores of handles, while green curves represent co-cores.

returning to the same  $x$ -coordinate, and let  $\eta$  be  $-\eta$  with the orientation reversed, so that  $X_0$  is at the end of  $\eta$ .

As in the second diagram down in Figure 15, create a finger of  $\Lambda_+$  parallel to  $\eta$  using a Reidemeister I move at the initial point of  $\eta$  and next to every cusp of  $\eta$  along with Reidemeister II moves to pass the lead cusp of the finger through handles of  $G_+$  that are incident to  $\eta$ . Move the end of the finger just to the right of  $X_0$ . Place a co-core of a handle inside the finger just below each crossing created by a Reidemeister I move. Place two additional co-cores from the finger to the original link on either side of the crossing  $X_0$ .

Finally, replace the co-cores by surgery disks to create a new Legendrian handle graph as in the third row of Figure 15. Isotope the new Legendrian handle graph as at the bottom of Figure 15, using a combination of the move in Figure 13 (b) to move the handles away and Reidemeister I moves to remove the crossings. The result is a Legendrian handle graph that has many more surgery disks, but whose underlying Legendrian link has one fewer crossing than before.  $\square$

The procedure above may produce a disconnected Legendrian link. We next see how to join these components.

**Lemma 4.4.** *Let  $(G_+, \Lambda_+)$  be a Legendrian handle graph, where  $\Lambda_+$  has  $n \geq 2$  components, which are mutually disjoint in the front diagram. Suppose that there exists a path  $\gamma$  in the front diagram of  $G_+$  that starts on component  $\Lambda'_+ \subset \Lambda_+$ , ends on  $\Lambda''_+ \subset \Lambda_+$ , and does not intersect  $\Lambda_+$  otherwise. Then there exists a Legendrian handle graph  $(G_-, \Lambda_-)$  and a subset  $\mathcal{H}_0$  of handles of  $G_-$ , such that  $\text{Surg}(G_-, \Lambda_-, \mathcal{H}_0)$  is Legendrian isotopic to  $(G_+, \Lambda_+)$ , one component of  $\Lambda_-$  is topologically the*

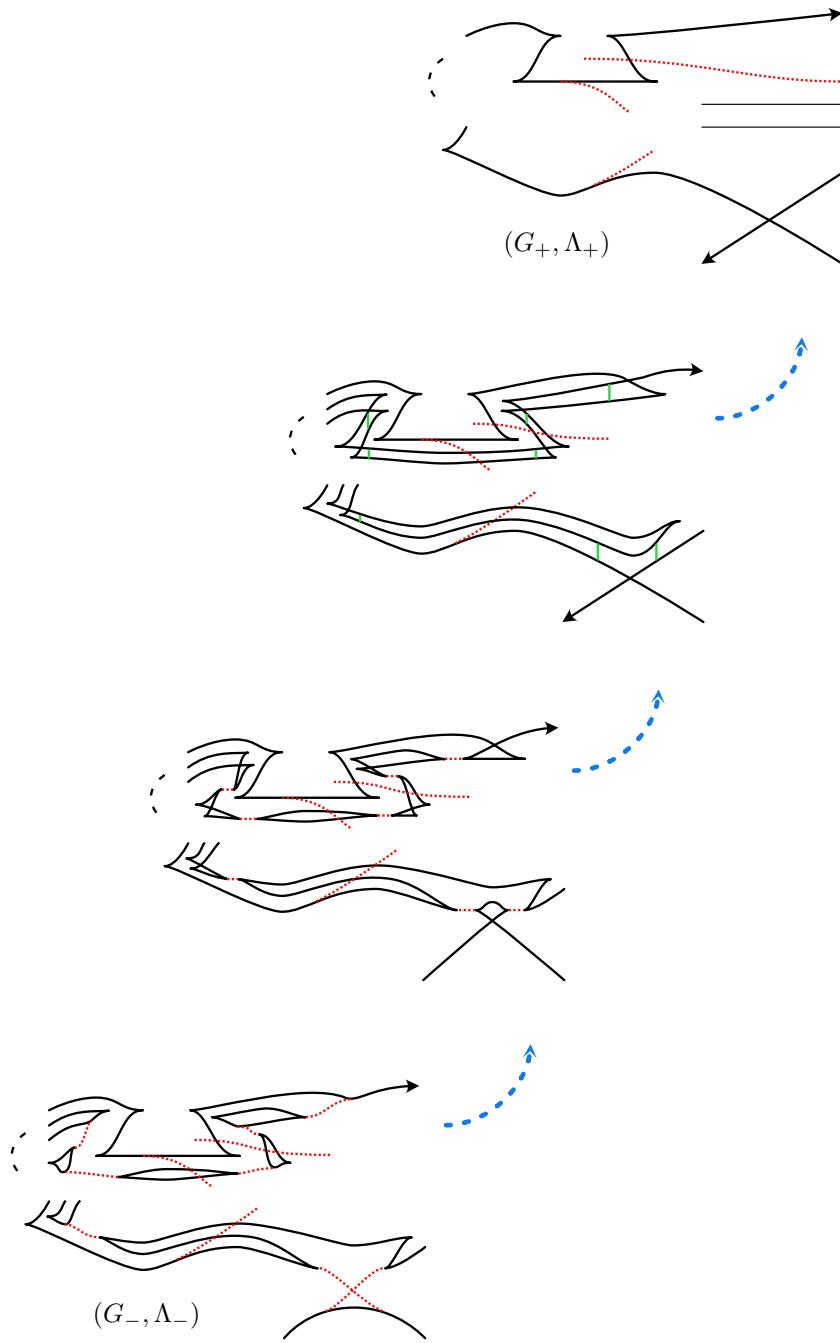


FIGURE 15. The Legendrian handle graph  $(G_-, \Lambda_-)$  has one fewer positive crossing than  $(G_+, \Lambda_+)$ . Red curves represent cores of handles, while green curves represent co-cores.

*connected sum of  $\Lambda'_+$  and  $\Lambda''_+$ , the other components of  $\Lambda_-$  match the remaining components of  $\Lambda_+$ , and none of the components of  $\Lambda_-$  intersect in the front diagram.*

*Proof.* We may assume that  $\gamma$  intersects  $\Lambda'_+$  and  $\Lambda''_+$  away from triple points, crossings, and cusps. Create a finger of  $\Lambda'_+$  that follows  $\gamma$ , starting with a Reidemeister I move and using Reidemeister II moves to cross handles of  $G_+$  and additional Reidemeister I moves when  $\gamma$  has a vertical tangent; see the middle diagram of Figure 16. Stop the finger just before  $\gamma$  intersects  $\Lambda''_+$ , performing an additional Reidemeister I move if necessary to ensure that the orientations of parallel strands of the finger and  $\Lambda''_+$  are opposite. Place a co-core of a handle between those two parallel strands. Finally, replace the co-core by a core of a handle to create a new Legendrian handle graph  $(G_-, \Lambda_-)$  as in the bottom-left portion of Figure 16.

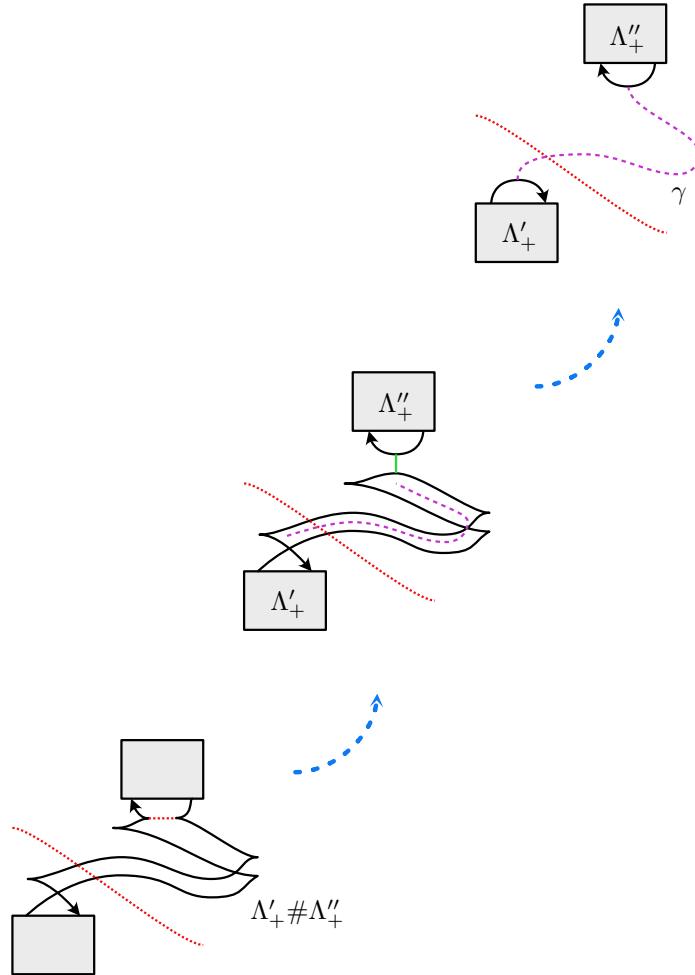


FIGURE 16. The Legendrian link  $\Lambda_-$  has a component that is topologically the connected sum of two components of  $\Lambda_+$ .

That the new component of  $\Lambda_-$  is the connect sum of  $\Lambda'_+$  and  $\Lambda''_+$  comes from the facts that the diagrams of  $\Lambda'_+$  and  $\Lambda''_+$  are disjoint and that  $\gamma$  is disjoint from the diagram of  $\Lambda_+$  on its interior. The final two conclusions of the lemma follow immediately from the construction.  $\square$

With the tools above in place, we are ready to re-prove [Lemma 3.3](#) using the diagrammatic techniques of this section.

*Diagrammatic proof of Lemma 3.3 in  $(\mathbb{R}^3, \xi_0)$ .* Given a Legendrian  $\Lambda$ , use [Lemma 4.2](#) repeatedly, and then [Lemma 4.3](#) repeatedly, to obtain a Legendrian handle graph  $(G_1, \Lambda_1)$  such that the front diagram of  $\Lambda_1$  has no crossings, and  $\text{Surg}(G_1, \Lambda_1)$  is Legendrian isotopic to  $\Lambda$ . Use [Lemma 4.4](#) to find a Legendrian handle graph  $(G_2, \Lambda_2)$  with a subset  $\mathcal{H}_0$  of handles, such that  $\Lambda_2$  is connected, and  $\text{Surg}(G_2, \Lambda_2, \mathcal{H}_0)$  is Legendrian isotopic to  $\text{Surg}(G_1, \Lambda_1)$ , which implies that  $\text{Surg}(G_2, \Lambda_2)$  is Legendrian isotopic to  $\Lambda$ . Finally, note that  $\Lambda_2$  is a smooth unknot since it is the connected sum of smooth unknots.  $\square$

## 5. UPPER BOUNDS

With constructions of a common lower bound and corresponding handle graphs for  $\Lambda$  and  $\Lambda'$  in hand, we are ready to find an upper bound. The structure of the following proof parallels that of Lazarev [[Laz20](#)] in higher dimensions.

*Proof of Proposition 1.5.* Given oriented Legendrian links in  $\Lambda$  and  $\Lambda'$  in  $(Y, \xi)$ , [Proposition 3.1](#) implies that there exist an oriented Legendrian link  $\Lambda_-$  and Legendrian handle graphs  $(G, \Lambda_-)$  and  $(G', \Lambda_-)$  such that  $\text{Surg}(G, \Lambda_-)$  (resp.  $\text{Surg}(G', \Lambda_-)$ ) is Legendrian isotopic to  $\Lambda$  (resp.  $\Lambda'$ ).

We Legendrian isotope the handles  $\mathcal{H}'$  of  $G'$  to be in general position with respect to the handles  $\mathcal{H}$  of  $G$ . In particular, we may assume that the Legendrian handle graph  $(G', \Lambda_-)$  has  $\mathcal{H}'$  disjoint from  $\mathcal{H}$ , with  $\text{Surg}(G', \Lambda_-)$  still Legendrian isotopic to  $\Lambda'$ .

Define the Legendrian graph  $G_+ = G \cup G'$ ; it is clear that  $(G_+, \Lambda_-)$  is a Legendrian handle graph. Note that  $\text{Surg}(G_+, \Lambda_-, \mathcal{H})$  is Legendrian isotopic to a Legendrian handle graph  $(G_{+,1}, \Lambda)$ ; similarly,  $\text{Surg}(G_+, \Lambda_-, \mathcal{H}')$  is Legendrian isotopic to a Legendrian handle graph  $(G_{+,2}, \Lambda')$ .

Observe that both  $\text{Surg}(G_{+,1}, \Lambda)$  and  $\text{Surg}(G_{+,2}, \Lambda')$  are Legendrian isotopic to  $\text{Surg}(G_+, \Lambda_-)$ , which we denote by  $\Lambda_+$ . Let  $L: \Lambda_- \rightarrow \Lambda_+$  be the concatenation of  $L(G, \Lambda_-)$  with  $L(G_{+,1}, \Lambda)$ ; similarly, let  $L': \Lambda_- \rightarrow \Lambda_+$  be the concatenation of  $L(G', \Lambda_-)$  with  $L(G_{+,2}, \Lambda')$ . Then it is clear that  $\Lambda$  (resp.  $\Lambda'$ ) appears as a collared slice of  $L$  (resp.  $L'$ ). At the same time, [Proposition 2.6](#) implies that  $L$  and  $L'$  are exact-Lagrangian isotopic, since they are both obtained from the same Legendrian handle graph  $(G_+, \Lambda_-)$  by Legendrian ambient surgery, only in a different order—in other words, they both belong to the isotopy class  $L(G_+, \Lambda_-)$ .  $\square$

**Example 5.1.** [Figure 17](#) and [Figure 18](#) display the full process of creating the upper bounds in [Figure 1](#) and [Figure 2](#), respectively.

## 6. THE LAGRANGIAN COBORDISM GENUS

In this section, we use the construction of upper and lower bounds for a pair of Legendrian knots to define a new quantity, the relative Lagrangian genus, and a new relation, Lagrangian quasi-concordance. We explore foundational properties and immediate examples, leaving deeper explorations, as embodied in the list of open questions at the end, for future work. For ease of notation, we work with Legendrian links in the standard contact  $\mathbb{R}^3$ , though our definitions may easily be adapted to Legendrians in any tight contact 3-manifold.

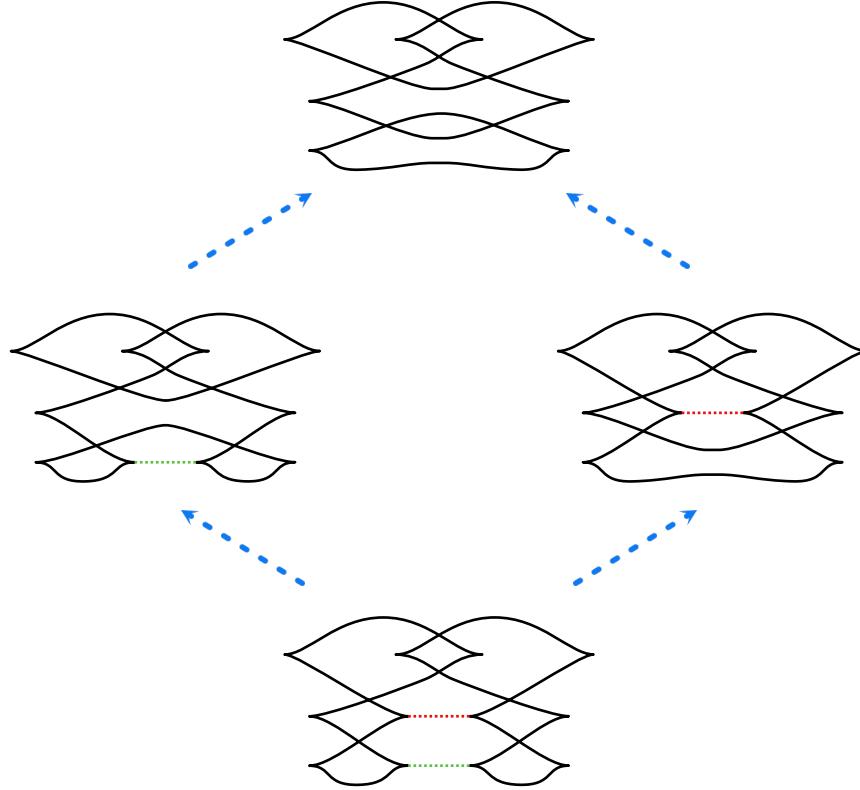


FIGURE 17. The handle graph at the bottom of the figure is used to create the upper bound of the trefoil and an  $m(5_2)$  knot that appeared in [Figure 1](#).

**6.1. Lagrangian Quasi-Cobordism.** We begin with a definition that undergirds the two concepts referred to above.

**Definition 6.1.** A *Lagrangian quasi-cobordism* between Legendrian links  $\Lambda$  and  $\Lambda'$  consists of an ordered set of  $n + 1$  Legendrian links

$$\boldsymbol{\Lambda} = (\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_n = \Lambda'),$$

another ordered set of  $n$  nonempty Legendrian links

$$\boldsymbol{\Lambda}^* = (\Lambda_1^*, \dots, \Lambda_n^*),$$

such that  $\Lambda_i^*$  is an upper or lower bound for the pair  $(\Lambda_{i-1}, \Lambda_i)$ , and the Lagrangian cobordisms

$$\mathbf{L} = (L_1^<, L_1^>, L_2^<, L_2^>, \dots, L_n^<, L_n^>)$$

that realize the upper or lower bound constructions.

There are several quantities associated to a Lagrangian quasi-cobordism.

**Definition 6.2.** Given a Lagrangian quasi-cobordism  $(\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^*, \mathbf{L})$ , its *length* is one less than the number of elements in  $\boldsymbol{\Lambda}$ , while its *Euler characteristic*  $\chi(\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^*, \mathbf{L})$  is the sum of the Euler characteristics of the Lagrangians in  $\mathbf{L}$  and its *genus*  $g(\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^*, \mathbf{L})$  defined, as usual, in terms of the Euler characteristic.

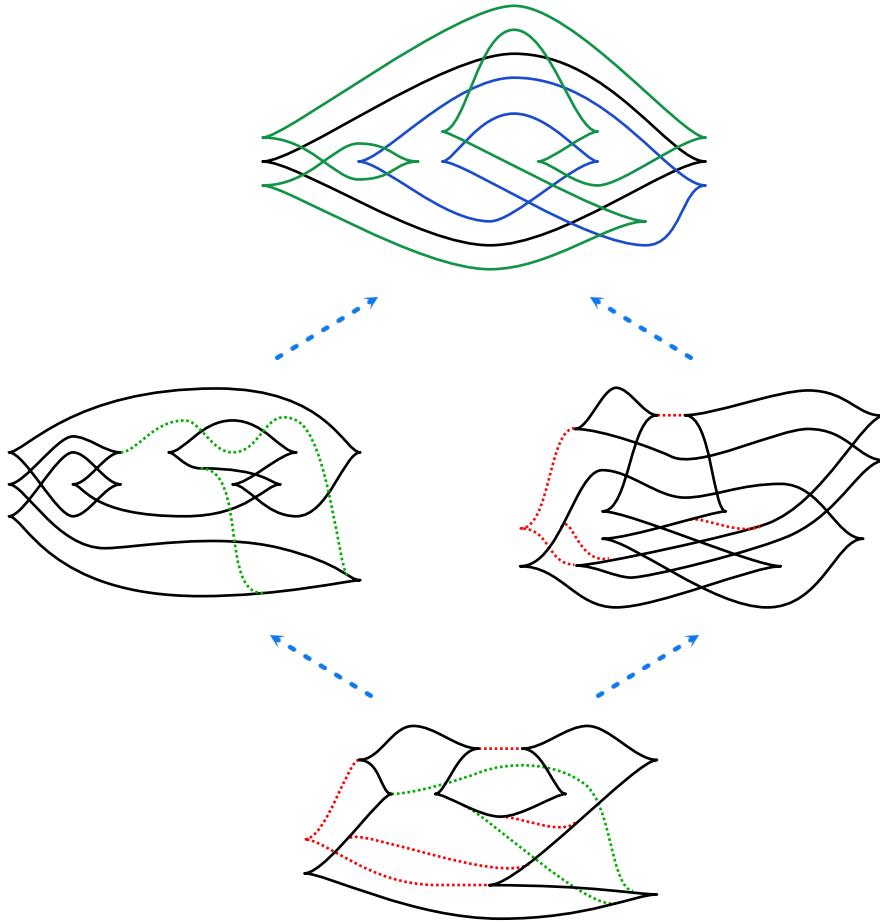


FIGURE 18. The handle graph at the bottom of the figure is used to create the upper bound of the figure eight knot and the unknot that appeared in [Figure 2](#). Note that the Legendrian knots in the handle graphs in the middle level are isotopic to the unknot (left) and the figure eight (right).

Further, we define the *relative Lagrangian genus*  $g_L(\Lambda, \Lambda')$  between the Legendrian links  $\Lambda$  and  $\Lambda'$  as the minimum genus of any Lagrangian quasi-cobordism between them. Two Legendrian links  $\Lambda$  and  $\Lambda'$  are *Lagrangian quasi-concordant* if  $g_L(\Lambda, \Lambda') = 0$ .

**Example 6.3.** Let  $\Upsilon$  be the maximal Legendrian unknot, and let  $\Lambda$  be a maximal Legendrian representative of  $m(6_2)$ . Note that both  $\Upsilon$  and  $\Lambda$  have Thurston–Bennequin number  $-1$  and that the smooth 4-genus of  $6_2$  is equal to  $1$  [LM]. It follows from the behavior of the Thurston–Bennequin invariant under Lagrangian cobordism that there cannot be a Lagrangian cobordism joining  $\Upsilon$  and  $\Lambda$  in either direction. Nevertheless, there is a genus-1 Lagrangian quasi-cobordism between the two; see [Figure 19](#).

Lagrangian quasi-cobordism induces an equivalence relation on the set of isotopy classes of Legendrian links. As in the smooth case, this equivalence relation is uninteresting, as shown by the following immediate corollary of [Theorem 1.1](#) or [Proposition 3.1](#):

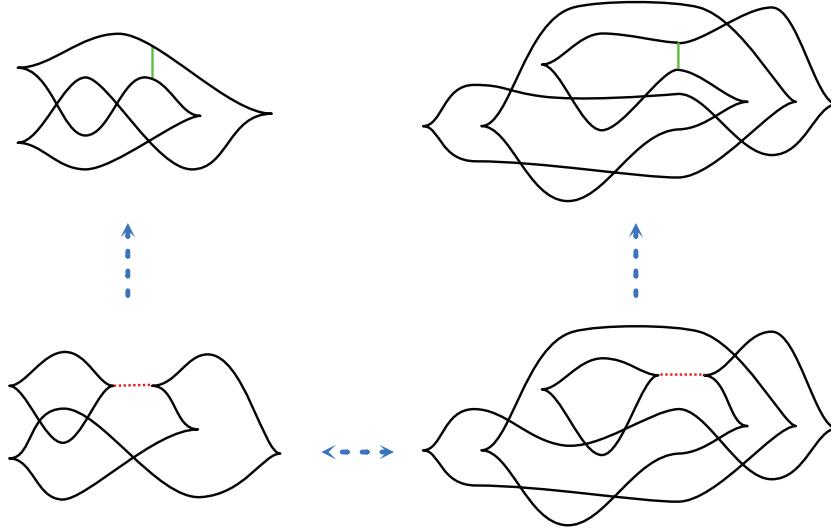


FIGURE 19. A genus 1 Lagrangian quasicobordism between the maximal unknot  $\Upsilon$  and a maximal representative  $\Lambda$  of the mirror of the  $6_2$  knot. The quasi-cobordism was produced using the ideas in [BTY13, Section 5], especially Figure 25 and Figure 27.

**Corollary 6.4.** *Any two Legendrian links with the same rotation number are Lagrangian quasi-cobordant. In fact, the quasi-cobordism may be chosen to have length 1.*

The corollary shows that the relative Lagrangian genus is defined for any two Legendrian links of the same rotation number.

On the other hand, Lagrangian quasi-concordance also clearly induces an equivalence relation on the set of isotopy classes of Legendrian links. The relative Lagrangian genus descends to Lagrangian quasi-concordance classes. Both the rotation number and the Thurston–Bennequin number [Cha10] are invariants of Lagrangian quasi-concordance, though non-classical invariants coming from Legendrian Contact Homology or Heegaard Floer theory will have a more complicated relationship with quasi-concordance.

**6.2. Relation to Smooth Genus.** To connect the relative Lagrangian genus to smooth constructions, note that we may define the smooth cobordism genus between two smooth knots  $K_1$  and  $K_2$  to be the minimum genus of all cobordisms between them; we denote this by  $g_s(K_1, K_2)$ . Chantraine proved that Lagrangian *fillings* minimize the smooth 4-ball genus of a Legendrian knot [Cha10], and so one might ask if this minimization property extends to  $g_L$ . We begin with a simple lemma.

**Lemma 6.5.** *Given Legendrian knots  $\Lambda$  and  $\Lambda'$ , we have  $g_s(\Lambda, \Lambda') \leq g_L(\Lambda, \Lambda')$ .*

*Proof.* Let  $(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$  be a Lagrangian quasi-cobordism between  $\Lambda$  and  $\Lambda'$ . Assume for ease of notation that each  $\Lambda^* \in \mathbf{\Lambda}^*$  is an upper bound. Let  $\bar{L}_i^>$  be the smooth cobordism from  $\Lambda_i^*$  to  $\Lambda_i$  obtained from reversing  $L_i^>$ ; note that  $\bar{L}_i^>$  is not, in general, a Lagrangian cobordism. Since Euler characteristic is additive under gluing, the smooth cobordism  $L_1^< \circ \bar{L}_1^> \circ L_2^< \circ \cdots \circ \bar{L}_n^>$  has genus  $g(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$ , and hence  $g_s(\Lambda, \Lambda') \leq g_L(\Lambda, \Lambda')$ .  $\square$

It is natural to ask under what conditions on  $\Lambda_1$  and  $\Lambda_2$ —as Legendrian or as smooth knots—is the inequality in [Lemma 6.5](#) an equality? On one hand, we cannot expect to achieve equality in all cases.

**Example 6.6.** Let  $\Lambda$  be any Legendrian knot, and let  $\Lambda'$  be a double stabilization of  $\Lambda$  with the same rotation number as  $\Lambda$ . Since  $\Lambda$  and  $\Lambda'$  have the same underlying smooth knot type, we have  $g_s(\Lambda, \Lambda') = 0$ . On the other hand, let  $(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L})$  be a Lagrangian quasi-cobordism between  $\Lambda$  and  $\Lambda'$ . Note that  $\chi(L_i^<), \chi(L_i^>) \leq 0$  for all  $i$ , since each of  $L_i^<$  and  $L_i^>$  has at least two boundary components. Since  $\text{tb}(\Lambda) > \text{tb}(\Lambda')$ , some pair  $\Lambda_i, \Lambda_{i+1}$  in  $\mathbf{\Lambda}$  must have different Thurston–Bennequin numbers. In particular, the bound  $\Lambda_i^*$  must have a different Thurston–Bennequin number than at least one of  $\Lambda_i$  or  $\Lambda_{i+1}$ . It follows that  $\chi(L_i^<) + \chi(L_i^>) < 0$ , and hence that  $\chi(\mathbf{\Lambda}, \mathbf{\Lambda}^*, \mathbf{L}) < 0$ . Since  $\Lambda$  and  $\Lambda'$  are knots, this implies that  $g(\Lambda, \Lambda', \mathbf{L}) > 0$ . In particular, we have  $g_L(\Lambda, \Lambda') > 0$  even though  $g_s(\Lambda, \Lambda') = 0$ .

On the other hand, there is a simple sufficient condition for equality in the lemma above.

**Lemma 6.7.** *If the Legendrian knot  $\Lambda$  has a Lagrangian filling, and there exists a Lagrangian cobordism from  $\Lambda$  to  $\Lambda'$ , then  $g_s(\Lambda, \Lambda') = g_L(\Lambda, \Lambda')$ .*

*Proof.* We begin by setting notation. Let  $L_0$  be the Lagrangian filling of  $\Lambda$  and let  $L_1^<$  be the Lagrangian cobordism from  $\Lambda$  to  $\Lambda'$ . Taking  $L_1^>$  to be the trivial cylindrical Lagrangian cobordism from  $\Lambda'$  to itself, and taking  $\Lambda_1^* = \Lambda'$ , we see that

$$(6.8) \quad g_L(\Lambda, \Lambda') \leq g(L_1^<).$$

Let  $\Sigma$  be the smooth cobordism from  $\Lambda$  to  $\Lambda'$  that minimizes the smooth cobordism genus. We know that  $L_0 \circ L_1^<$  is a Lagrangian filling of  $\Lambda'$ , and hence that  $g(L_0 \circ L_1^<) \leq g(L_0 \circ \Sigma)$ . Since  $\Lambda$  is a knot, the genus is additive under composition of cobordisms, and we obtain

$$(6.9) \quad g(L_1^<) \leq g(\Sigma).$$

Combining (6.8) and (6.9), we obtain

$$g_L(\Lambda, \Lambda') \leq g(\Sigma) = g_s(\Lambda, \Lambda').$$

The lemma now follows from [Lemma 6.5](#). □

**6.3. Open Questions.** We end with a list of questions about Lagrangian quasi-cobordism and quasi-concordance beyond the motivating question above about the relationship between the relative Lagrangian genus and the relative smooth genus.

- (1) Building off of [Example 6.3](#), is there an example of a pair  $\Lambda$  and  $\Lambda'$  that are Lagrangian quasi-concordant but not Lagrangian concordant?
- (2) Taking the previous question further, for two Lagrangian quasi-concordant Legendrians  $\Lambda$  and  $\Lambda'$ , what is the minimal length of any Lagrangian quasi-concordance between them? Are there examples for which this minimal length is arbitrarily high?
- (3) Even more generally, define  $g_L(\Lambda, \Lambda', n)$  to be the minimal genus of any Lagrangian quasi-cobordism between  $\Lambda$  and  $\Lambda'$  of length at most  $n$ . The sequence  $(g_L(\Lambda, \Lambda', n))_{n=1}^\infty$  decreases to and stabilizes at  $g_L(\Lambda, \Lambda')$ . Are there examples for which the number of steps it takes the sequence to stabilize is arbitrarily long?
- (4) Can  $g_L(\Lambda, \Lambda') - g_s(\Lambda, \Lambda')$  be arbitrarily large when  $\Lambda$  and  $\Lambda'$  both have maximal Thurston–Bennequin invariant?
- (5) Can the hypotheses of [Lemma 6.7](#) be weakened to  $\Lambda$  having only an augmentation instead of a filling?

- (6) Is there a version of this theory for Maslov 0 Lagrangians, which would better allow the use of Legendrian Contact Homology, especially the tools in [Pan17]?

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