

REDUCED WORDS FOR REFLECTIONS IN WEYL GROUPS

ELIZABETH MILIĆEVIĆ

ABSTRACT. The reflections in a Coxeter group are defined as conjugates of a single generator, and thus admit palindromic expressions as products of generators. Our main result gives closed formulas providing a palindromic reduced expression for each reflection in any finite Weyl group. There exist algorithmic methods for determining such reduced expressions, but explicit formulas have not been recorded outside of well-known special cases.

1. INTRODUCTION

Given any Coxeter group W with set of generators S , every element $w \in W$ can be written as a product $w = s_1 s_2 \cdots s_k$ for some $s_i \in S$. If k is minimal, this expression for w is reduced. Every $w \in W$ admits at least one reduced expression, and all expressions can be transformed into a reduced expression by doing a sequence of nil, braid, and/or commuting relations on the generators. Identifying reduced expressions with desirable properties, as well as enumerating reduced expressions, have motivated substantial work on the combinatorics of Coxeter groups.

1.1. Reduced words for reflections. Every Coxeter group has a distinguished subset of reflections $T = \{ws w^{-1} \mid s \in S, w \in W\}$. Equivalently, every $t \in T$ has a palindromic expression in the generators, meaning that the product reads the same forwards or backwards. It is a straightforward exercise to prove that every reflection admits a palindromic expression which is also reduced; see Exercise 1.10 in [BB] and a proof in either Lemma 4.1 of [BFP] or the response to this question on MathOverflow [Sam]. Explicit formulas for reduced expressions for reflections do not appear in the literature on Coxeter groups, however, apart from well-known special cases such as $W = S_n$ the symmetric group.

We now restrict to the class of Coxeter groups which arise as Weyl groups of a crystallographic root system. This family plays a central role in the classification of semisimple Lie algebras, and almost all finite irreducible Coxeter groups are Weyl groups. If W is a Weyl group, each $t \in T$ acts geometrically on Euclidean space as a reflection in the hyperplane orthogonal the corresponding root vector. The purpose of this paper is to provide closed formulas for palindromic reduced expressions for all reflections in any Weyl group.

Theorem 1.1. *The following are palindromic reduced expressions, one for each distinct reflection in the Weyl group of the type specified.*

(A_n) *For $1 \leq i < j \leq n$, we have*

$$s_{e_i - e_j} = s_i s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1} s_i.$$

(B_n) *For $1 \leq i < j \leq n$, we have*

$$\begin{aligned} s_{e_i - e_j} &= s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i \\ s_{e_i + e_j} &= (s_j s_{j+1} \cdots s_{n-1} \cdot s_i s_{i+1} \cdots s_{n-2}) (s_n s_{n-1} s_n) (s_{n-2} \cdots s_{i+1} s_i \cdot s_{n-1} \cdots s_{j+1} s_j) \\ s_{e_i} &= s_i s_{i+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{i+1} s_i. \end{aligned}$$

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(C_n) For $1 \leq i < j \leq n$, we have

$$\begin{aligned} s_{e_i - e_j} &= s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i \\ s_{e_i + e_j} &= (s_j s_{j+1} \cdots s_{n-1} \cdot s_i s_{i+1} \cdots s_{n-2}) (s_n s_{n-1} s_n) (s_{n-2} \cdots s_{i+1} s_i \cdot s_{n-1} \cdots s_{j+1} s_j) \\ s_{2e_i} &= s_i s_{i+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{i+1} s_i. \end{aligned}$$

(D_n) We have

$$\begin{aligned} s_{e_i - e_j} &= s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i \quad \text{for } 1 \leq i < j \leq n, \\ s_{e_i + e_n} &= s_i s_{i+1} \cdots s_{n-2} s_n s_{n-2} \cdots s_{i+1} s_i \quad \text{for } 1 \leq i < n, \\ s_{e_i + e_j} &= (s_j s_{j+1} \cdots s_{n-2} \cdot s_i s_{i+1} \cdots s_{n-3}) (s_{n-1} s_{n-2} s_n s_{n-2} s_{n-1}) (s_{n-3} \cdots s_i \cdot s_{n-2} \cdots s_j) \\ &\quad \text{for } 1 \leq i < j < n. \end{aligned}$$

(E₆) See Table 4.

(E₇) See Tables 4 and 5.

(E₈) See Tables 4, 5, 6, and 7.

(F₄) See Table 3.

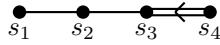
(G₂) See Table 2.

The expansion for each root vector in terms of a basis of simple roots is also provided in Section 3 for the classical types, and in each table in Section 4 for the exceptional types. It is the hope of the author that these formulas will be viewed as an additional reference expanding upon the helpful plates at the end of [Bou]. We remark that we also correct 5 distinct typos in item (II) of those plates within the body of this work, in each of types C_n, D_n, E₈, and G₂.

There exist algorithmic methods for determining reduced expressions for general $w \in W$, such as the numbers game; see Section 4.3 of [BB]. In response to a Stack Exchange question explicitly requesting the results of Theorem 1.1, Reading explains a method for starting with a word for wsw^{-1} and reducing it while maintaining its palindromic property. He also provides an alternate algorithm using the geometric realization of (W, S) ; see [Rea]. Lemma 1.7 in [BD] provides a closed formula for a palindromic reduced expression of $t \in T$, given another reduced expression for t as input. Several computational software packages have implemented similar algorithms, such as the `associated_reflection()` command in Sage, which returns a reduced word given a linear combination of simple roots [Sag]. The Maple packages `coxeter` and `weyl` by Stembridge support the `vec2fc` command, which performs the geometric algorithm in [Rea].

1.2. Discussion of the results. We continue by illustrating Theorem 1.1 in types C₄ and E₈.

Example 1.2. Let e_i denote the i^{th} standard basis vector in \mathbb{R}^4 , and choose the simple roots to be $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, 2e_4\}$. The set Δ corresponds to the simple reflections $\{s_1, s_2, s_3, s_4\}$, labeling the Dynkin diagram as follows:



The $4^2 = 16$ positive roots with respect to Δ are

$$\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq 4\} \cup \{e_i + e_j \mid 1 \leq i < j \leq 4\} \cup \{2e_i \mid i \in [4]\}.$$

The $\binom{4}{2} = 6$ positive roots in the first set are the roots for the underlying type A₃ root system generated by the first 3 nodes on the Dynkin diagram. Those non-simple type A₃ reflections thus have the following familiar palindromic reduced expressions

$$s_{e_1 - e_3} = s_1 s_2 s_1, \quad s_{e_2 - e_4} = s_2 s_3 s_2, \quad s_{e_1 - e_4} = s_1 s_2 s_3 s_2 s_1.$$

There are 3 additional non-simple reflections of type A₄, all conjugates of the generator s_4 to avoid duplication:

$$s_{2e_1} = s_1 s_2 s_3 s_4 s_3 s_2 s_1, \quad s_{2e_2} = s_2 s_3 s_4 s_3 s_2, \quad s_{2e_3} = s_3 s_4 s_3.$$

The remaining 6 reflections are more interesting, and also fully illustrate the key idea for recording palindromic reduced expressions for all the non-type A reflections. First note that there is one obvious unused palindromic reduced expression $s_4s_3s_4$, which is not equivalent to $s_3s_4s_3$ in type C_4 . The remaining reflections are all conjugates of $s_4s_3s_4$ by the minimal length coset representatives of the quotient of the subgroup $\langle s_1, s_2, s_3 \rangle$ by the maximal parabolic subgroup $\langle s_1, s_3 \rangle$. These representatives index the Schubert cells in the Grassmannian $Gr(2, 4)$ of 2-planes in \mathbb{C}^4 , and are in bijection with the Young diagrams contained in a 2×2 rectangle. A corresponding reduced word is read by overlaying the Young diagram on the grid labeled by

$$\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \longleftrightarrow s_2s_3s_1s_2.$$

We build a word by reading the labels up each column, starting with the rightmost column, as shown in the figure above. The root $e_i + e_j$ corresponds to the diagram whose first column extends down to label i and second column to j . For example, the rectangle above corresponds to the root $e_1 + e_2$, and the associated reflection is $s_{e_1+e_2} = s_2s_3s_1s_2 \cdot s_4s_3s_4 \cdot s_2s_1s_3s_2$. Likewise,

$$s_{e_3+e_4} = s_4s_3s_4, \quad s_{e_2+e_4} = s_2 \cdot s_4s_3s_4 \cdot s_2, \quad s_{e_1+e_4} = s_1s_2 \cdot s_4s_3s_4 \cdot s_2s_1,$$

$$s_{e_2+e_3} = s_3s_2 \cdot s_4s_3s_4 \cdot s_2s_3, \quad s_{e_1+e_3} = s_3s_1s_2 \cdot s_4s_3s_4 \cdot s_2s_1s_3,$$

which completes our list of palindromic reduced expressions for the 16 reflections in type C_4 .

In the exceptional types, there are always embedded copies of Dynkin diagrams for lower-rank classical types, permitting use of those results to collect palindromic reduced expressions for many reflections, perhaps after relabeling. Each exceptional type also has reflections distinct to that type, however, which must be identified systematically in some other way.

Example 1.3. Figure 1 on the next page illustrates our method for identifying those reflections which are distinct to type E . The vertices of the graph in Figure 1 correspond to the 67 reflections distinct to types E_6 , E_7 , and E_8 , colored blue, red, and green, respectively, with length increasing going upward. The type E_6 reflection of minimal length is $s_\theta = s_{16524342561}$, where we write $s_is_js_k = s_{ijk}$ for brevity.

The edge labels in Figure 1 indicate the simple reflection with which to conjugate to obtain the reflection indexing the adjacent vertex. For example, the lowest red vertex represents the element $s_7s_9s_7$. By traveling along the lefthand side of Figure 1, for example, we also see that

$$s_{\tilde{\alpha}} = s_{87654312435426543765487} \cdot s_{16524342561} \cdot s_{78456734562453421345678},$$

where $\tilde{\alpha}$ is the highest root in type E_8 . Not all possible conjugacy relations are depicted in Figure 1; however, a sufficient subset of relations is provided to demonstrate a clear path from s_θ to each of the other 66 reflections distinct to type E .

Acknowledgments. Theorem 1.1 was obtained in service of a preliminary case-by-case proof of Lemma 4.9 in [MV], and has thus benefitted from feedback by Eva Viehmann. The author is grateful to Anne Thomas, whose tikz code was adapted to create the Dynkin diagrams displayed in this paper. Part of this work was carried out while the author was a Director's Mathematician in Residence at the Budapest Semesters in Mathematics, jointly supported by the Rényi Alfréd Matematikai Kutatóintézet. Calculations in some exceptional types were assisted by Sage, and the author thanks the Sage developers for implementing related procedures [Sag].

2. BACKGROUND

In this section, we review all preliminary definitions, notation, and results we need concerning Coxeter groups and their associated root systems. For more detailed references, we refer the reader to textbooks such as [BB, Bou, Hum].

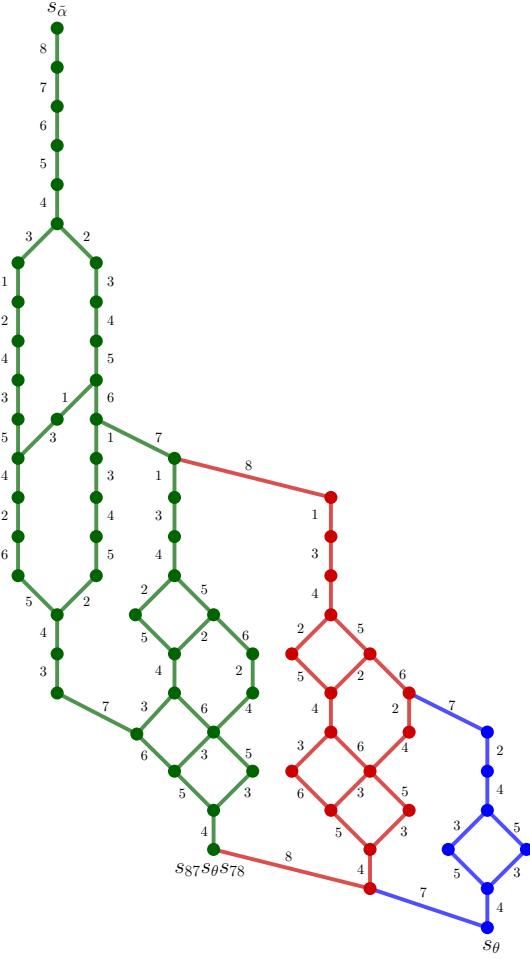


FIGURE 1. Some conjugacy relations among type E reflections, where $s_\theta = s_{16524342561}$.

2.1. Coxeter groups. Let (W, S) be a finite irreducible Coxeter system of rank n . The generating set $S = \{s_1, \dots, s_n\}$ are the simple reflections, and we denote the index set by $[n] = \{1, \dots, n\}$. Further suppose that W is the Weyl group of a crystallographic root system; equivalently, let W be a finite Weyl group. The Dynkin diagram for (W, S) can then be classified as one of the families displayed in Table 1 on the next page.

Every element $w \in W$ can be written as a product $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ for some $s_{i_j} \in S$. If k is minimal, then this product for w is called a reduced expression. In case of a reduced expression, we say that the length of w equals k , denoted $\ell(w) = k$.

Denote by $T = \{ws w^{-1} \mid w \in W, s \in S\}$ the set of reflections of (W, S) . By choosing any expression for $w = s_{i_1} \cdots s_{i_k}$, we can decompose $ws w^{-1} = (s_{i_1} \cdots s_{i_k})s(s_{i_k} \cdots s_{i_1})$ into a palindromic expression, meaning that the sequence of generators reads the same forwards and backwards. Note that such a palindromic expression is not guaranteed to be reduced.

2.2. Root systems. The group W acts by linear transformations on a real n -dimensional vector space V , which we may identify with \mathbb{R}^n . The vector space V admits an ordered basis $\Delta = (\alpha_i)_{i \in [n]}$ of simple roots, and a symmetric bilinear form $B(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m(i,j)}$, where $m(i,j)$ is the (i,j) -entry of the associated Coxeter matrix.

Type	Dynkin Diagram	Type	Dynkin Diagram
$A_n, n \geq 1$		E_6	
$B_n, n \geq 2$		E_7	
$C_n, n \geq 2$		E_8	
$D_n, n \geq 4$		F_4	
		G_2	

TABLE 1. Dynkin diagrams for all types; labels consistent with [Bou] and [Sag].

Given any $s_i \in S$ and $\alpha_j \in \Delta$, the action of W on V is defined by

$$(1) \quad s_i(\alpha_j) = \alpha_j - c_{ji}\alpha_i,$$

where c_{ji} is the (j, i) -entry of the associated Cartan matrix. The set

$$\Phi = \{w(\alpha_i) \mid w \in W, i \in [n]\} \subset V$$

forms the root system for (W, S) , and the elements are called roots. Given any roots $\alpha, \beta \in \Phi$, the action of W on V can also be expressed via

$$(2) \quad s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha,$$

where (\cdot, \cdot) denotes the Euclidean dot product. Let Φ^+ (resp. Φ^-) denote the positive (resp. negative) roots in Φ , meaning those which have non-negative (resp. non-positive) coefficients when expanded in terms of the basis Δ . Let ρ denote the half-sum of the positive roots in Φ^+ .

Given any $\alpha \in \Phi^+$, write $\alpha = w(\alpha_i)$ for some $w \in W$ and $\alpha_i \in \Delta$. The assignment

$$(3) \quad s_\alpha = ws_iw^{-1}$$

is a bijective correspondence between positive roots in Φ^+ and reflections in T ; see Proposition 4.5 in [BB]. In particular, we have $s_{\alpha_i} = s_i$. Moreover, the element $ws_iw^{-1} \in T$ acts on V as the reflection in the hyperplane orthogonal to the root vector $\alpha = w(\alpha_i) \in \Phi^+$. For any $w \in W$, denote by

$$N(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}.$$

Recall from [BB, Proposition 4.4.4] that $\ell(w) = |N(w)|$ for any $w \in W$. For any $\alpha \in \Phi^+$, we denote $N(\alpha) = N(s_\alpha)$. In particular, we then have $\ell(s_\alpha) = |N(\alpha)|$.

Given any $i \in [n]$, define a linear functional $\alpha_i^\vee \in V^*$ by $\langle \alpha_i^\vee, v \rangle := 2B(\alpha_i, v)$ for any $v \in V$, where $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{Z}$ denotes the evaluation pairing. The ordered set $\Delta^\vee = (\alpha_i^\vee)_{i \in [n]}$ is then a basis of simple coroots for the dual space V^* . For any root $\alpha = w(\alpha_i)$, the corresponding coroot α^\vee is defined by $\langle \alpha^\vee, v \rangle := 2B(\alpha, v)$ for $v \in V$. Identifying V and V^* , we may also write $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ for any root $\alpha \in \Phi$. The set $\Phi^\vee = \{w(\alpha_i)^\vee \mid w \in W, i \in [n]\} \subset V^*$ forms the

system of coroots for (W, S) . The height of the coroot $\alpha^\vee \in \Phi^\vee$ is the sum of the coefficients c_i in the expansion $\alpha^\vee = \sum c_i \alpha_i^\vee$ via Δ^\vee . Equivalently, the height of α^\vee equals $\text{ht}(\alpha^\vee) = \langle \alpha^\vee, \rho \rangle$.

3. CLASSICAL TYPES

This section is dedicated to the proof of Theorem 1.1 in the classical types. We refer the reader to the plates in [Bou], which provide explicit formulas for the roots and Cartan matrices. Note that we correct a typo in item (II) in each of the plates for types C_n and D_n below.

3.1. Type A_n . In type A_n , the roots are

$$\Phi = \{e_i - e_j \mid 1 \leq i, j \leq n+1\},$$

where here e_i denotes the i^{th} standard basis vector in \mathbb{R}^{n+1} . We choose the simple roots to be those of the form $\Delta = \{\alpha_i\} = \{e_i - e_{i+1} \mid i \in [n]\}$, so that the positive roots are

$$\Phi^+ = \{\alpha_{ij} := e_i - e_{j+1} \mid 1 \leq i < j \leq n\}.$$

In the geometric realization of (W, S) , the reflection s_{α_i} acts by interchanging coordinates i and $i+1$ and is thus identified with the simple transposition $s_i \in S_{n+1} \cong W$.

Since $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ for any $\alpha \in \Phi$, we observe that $\alpha^\vee = \alpha$ for all $\alpha \in \Phi$ in type A_n . The corresponding basis of simple coroots is thus $\Delta^\vee = \{\alpha_i^\vee = \alpha_i\}$, and we can write the positive coroot $\alpha_{ij}^\vee = (e_i - e_j)^\vee = \alpha_i^\vee + \cdots + \alpha_j^\vee$. In particular, note that $\text{ht}(\alpha_{ij}^\vee) = j - i + 1$. Since A_n is simply-laced, Lemma 4.3 of [BFP] says that $\ell(s_\alpha) = \langle 2\rho, \alpha^\vee \rangle - 1 = 2\text{ht}(\alpha^\vee) - 1$ for all $\alpha \in \Phi^+$. Therefore, $\ell(s_{\alpha_{ij}}) = 2(j - i + 1) - 1 = 2(j - i) + 1$.

For any fixed $\alpha_{ij} \in \Phi^+$, we claim that

$$(4) \quad s_{e_i - e_{j+1}} = s_{\alpha_{ij}} = s_i s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1} s_i$$

is a reduced expression for the reflection corresponding to $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$. In type A , this fact is well-known, but we include a proof for the sake of completeness, since there are large type A subsystems in the other types, and we thus rely inductively on this type A result.

Note that the expression in (4) uses exactly $\ell(s_{\alpha_{ij}}) = 2(j - i) + 1$ simple generators, and is thus reduced if it indeed represents $s_{\alpha_{ij}}$. Let $w = s_i s_{i+1} \cdots s_{j-1}$. Using the bijective correspondence between positive roots and reflections from (3), if we can prove that $w(\alpha_j) = \alpha_{ij}$, then $s_{\alpha_{ij}} = ws_j w^{-1}$ is indeed the reflection corresponding to α_{ij} . By (1), in type A_n we have

$$(5) \quad s_i(\alpha_j) = \begin{cases} -\alpha_j & \text{if } i = j \\ \alpha_i + \alpha_j & \text{if } |i - j| = 1 \\ \alpha_j & \text{otherwise.} \end{cases}$$

We can now directly compute via (5) that

$$w(\alpha_j) = s_i s_{i+1} \cdots s_{j-1}(\alpha_j) = \alpha_i + \cdots + \alpha_j = \alpha_{ij},$$

as required. Therefore, the expressions in (4) form a system of palindromic reduced expressions for the $\binom{n}{2} = \frac{n(n-1)}{2}$ distinct reflections in W of type A_n .

3.2. Type C_n . In type C_n , the roots are

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid i \in [n]\},$$

where here e_i denotes the i^{th} standard basis vector in \mathbb{R}^n . We choose the simple roots to be $\Delta = \{e_i - e_{i+1} \mid i \in [n-1]\} \cup \{2e_n\}$, where $\alpha_i = e_i - e_{i+1}$ as in type A_{n-1} , but $\alpha_n = 2e_n$ is distinct to type C_n . The n^2 positive roots then come in three natural families:

$$\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{e_i + e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid i \in [n]\}.$$

We study each of these families separately.

For the $\binom{n}{2} = \frac{n(n-1)}{2}$ positive roots in $\{e_i - e_j \mid 1 \leq i < j \leq n\}$, we can write

$$e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1},$$

which are the roots for the underlying type A_{n-1} root system generated by the first $n-1$ nodes of the Dynkin diagram. Therefore, for any fixed $e_i - e_j \in \Phi^+$, we have the following palindromic reduced expression

$$(6) \quad s_{e_i - e_j} = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i$$

by our proof in type A .

For any fixed $i \in [n]$, we claim that

$$(7) \quad s_{2e_i} = s_i s_{i+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{i+1} s_i$$

is a reduced expression for each of the reflections corresponding to the n positive roots $2e_i$. Let $w = s_i \cdots s_{n-1}$. We first show that $w(\alpha_n) = 2e_i$. Write

$$2e_i = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n,$$

correcting a typo in item (II) of the type C_n plate in [Bou], which omits the 2s on the righthand side of this expression. By (1), in type C_n we have

$$(8) \quad s_i(\alpha_j) = \begin{cases} -\alpha_j & \text{if } i = j \\ \alpha_i + \alpha_j & \text{if } |i - j| = 1 \text{ and } j \neq n \\ 2\alpha_i + \alpha_j & \text{if } i = n-1 \text{ and } j = n \\ \alpha_j & \text{otherwise.} \end{cases}$$

We can now directly compute via (8) that

$$w(\alpha_n) = s_i s_{i+1} \cdots s_{n-1}(\alpha_n) = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n = 2e_i,$$

as required to verify that $s_{2e_i} = ws_j w^{-1}$ via (3).

Note that the expression in (7) uses exactly $2(n-i) + 1$ simple reflections. To see that this expression for s_{2e_i} is reduced, we prove that $\ell(s_{2e_i}) = |N(2e_i)| = 2(n-i) + 1$. Recall that $i \in [n]$ is fixed here, and define the following subset of positive roots:

$$\Phi_{2e_i}^+ = \{e_i - e_j \mid i < j \leq n\} \cup \{e_i + e_j \mid i < j \leq n\} \cup \{2e_i\}.$$

We claim that $\Phi_{2e_i}^+ = N(2e_i)$. For each $\alpha \in \Phi_{2e_i}^+$, we use (2) to directly calculate $s_{2e_i}(\alpha) = \alpha - (2e_i, \alpha)e_i$. For any $\alpha = e_i - e_j$ with $i < j \leq n$, we have $(2e_i, e_i - e_j) = 2$ and so $s_{2e_i}(e_i - e_j) = (e_i - e_j) - 2e_i = -e_i - e_j \in \Phi^-$. Similarly, for any $\alpha = e_i + e_j$ with $i < j \leq n$, we have $s_{2e_i}(e_i + e_j) = (e_i + e_j) - 2e_i = -e_i + e_j \in \Phi^-$. Finally, $s_{2e_i}(2e_i) = -2e_i \in \Phi^-$, completing the proof that $\Phi_{2e_i}^+ \subseteq N(2e_i)$.

Note that there are exactly $n-i$ elements in each of the first and second subsets of $\Phi_{2e_i}^+$, so that altogether $|\Phi_{2e_i}^+| = 2(n-i) + 1 \leq |N(2e_i)| = \ell(s_{2e_i})$. Since the expression in (7) uses exactly $2(n-i) + 1$ simple reflections, then this inequality implies that the expression for the reflection s_{2e_i} in (7) is reduced for any $i \in [n]$. Consequently, we have $\ell(s_{2e_i}) = 2(n-i) + 1$, in which case $\Phi_{2e_i}^+ = N(2e_i)$.

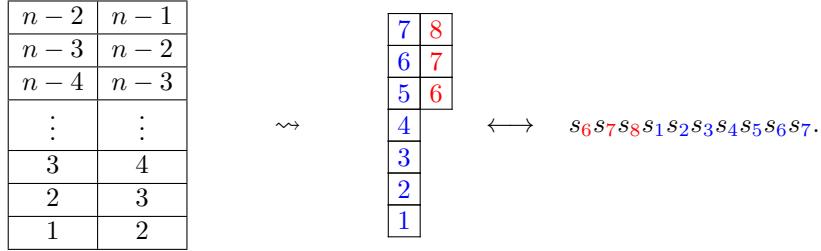
For the remaining $\binom{n}{2} = \frac{n(n-1)}{2}$ positive roots $\{e_i + e_j \mid 1 \leq i < j \leq n\}$, we claim that

$$(9) \quad s_{e_i + e_j} = (s_j s_{j+1} \cdots s_{n-1} \cdot s_i s_{i+1} \cdots s_{n-2}) (s_n s_{n-1} s_n) (s_{n-2} \cdots s_{i+1} s_i \cdot s_{n-1} \cdots s_{j+1} s_j)$$

is a reduced expression. Here, the product $s_j s_{j+1} \cdots s_{n-1}$ is trivial if $j = n$, and the product $s_i s_{i+1} \cdots s_{n-2}$ is trivial if $i = n-1$.

We first provide some context for the choice of conjugating element in (9), generalizing our discussion in Example 1.2. Consider the subgroup of W of type A_{n-1} given by $\langle s_1, \dots, s_{n-1} \rangle \cong S_n$, and the maximal parabolic subgroup P obtained by omitting the reflection s_{n-2} . The minimal length coset representatives in S_n/P are in bijection with the Young diagrams contained

in an $(n - 2) \times 2$ rectangle, and a corresponding reduced word can be read by overlaying the Young diagram in English notation on the $(n - 2) \times 2$ grid labeled by:



Our convention will be to build the word from left to right by reading the labels up each column of the diagram, moving from the right column to the left column, as shown in the $n = 9$ example in the figure above. Denote by w_{ij} the element corresponding to the Young diagram whose first column extends down to label i and second column extends down to label j . If the second column is empty, define $j = n$, and if the first column is empty, define $i = n - 1$ so that both products are trivial. (In the example above, we have $i = 1$ and $j = 6$, and $w_{16} = s_6 s_7 s_8 s_1 s_2 s_3 s_4 s_5 s_6 s_7$.) Explicitly, the conjugating element in (9) is then

$$w_{ij} = s_j s_{j+1} \cdots s_{n-1} \cdot s_i s_{i+1} \cdots s_{n-2}.$$

In this notation, we aim to prove that

$$s_{e_i + e_j} = w_{ij}(s_n s_{n-1} s_n) w_{ij}^{-1}$$

is a reduced expression for the reflection corresponding to the positive root $e_i + e_j$. Write

$$e_i + e_j = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } j \neq n, \\ \alpha_i + \cdots + \alpha_n & \text{if } j = n. \end{cases}$$

From (8), compute that $s_n(\alpha_{n-1}) = \alpha_{n-1} + \alpha_n = e_{n-1} + e_n$, and so $s_{e_{n-1} + e_n} = s_n s_{n-1} s_n$ by (3), which is clearly reduced. We show that the roots $e_i + e_j$ correspond bijectively to the conjugates of $s_{e_{n-1} + e_n}$ by the $\binom{n}{n-2} = \binom{n}{2}$ distinct minimal length coset representatives of S_n/P .

Consider the case where $j = n$. Compute using (8) that

$$w_{in} s_n(\alpha_{n-1}) = w_{in}(\alpha_{n-1} + \alpha_n) = s_i s_{i+1} \cdots s_{n-2}(\alpha_{n-1} + \alpha_n) = \alpha_i + \cdots + \alpha_n = e_i + e_n,$$

verifying (9) for $j = n$ via (3). Now consider the case where the second column of the Young diagram for w_{ij} is nonempty, equivalently $j \leq n - 1$. Compute using (8) and the $j = n$ case that

$$\begin{aligned} w_{ij} s_n(\alpha_{n-1}) &= w_{ij}(\alpha_{n-1} + \alpha_n) = s_j s_{j+1} \cdots s_{n-1} \cdot s_i s_{i+1} \cdots s_{n-2}(\alpha_{n-1} + \alpha_n) = \\ &= s_j s_{j+1} \cdots s_{n-1}(\alpha_i + \cdots + \alpha_n) = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n = e_i + e_j, \end{aligned}$$

verifying (9) for $j \neq n$ via (3). We have proved that $s_{e_i + e_j} = w_{ij}(s_n s_{n-1} s_n) w_{ij}^{-1}$ for all $1 \leq i < j \leq n$.

Note that the expression in (9) uses exactly $4n - 2(i + j) + 1$ simple reflections. To see that this expression for $s_{e_i + e_j}$ is reduced, we prove that $\ell(s_{e_i + e_j}) = |N(e_i + e_j)| = 4n - 2(i + j) + 1$. Define the following subset of positive roots:

$$\begin{aligned} \Phi_{e_i + e_j}^+ &= \{e_k - e_l \mid i = k < l \neq j, \text{ or } j = k < l\} \\ &\cup \{e_k + e_l \mid i = k \text{ and } j \leq l, \text{ or } j = l \text{ and } i < k \neq j\} \\ &\cup \{2e_i, 2e_j\}. \end{aligned}$$

We claim that $\Phi_{e_i + e_j}^+ = N(e_i + e_j)$. For each $\alpha \in \Phi_{e_i + e_j}^+$, we use (2) to directly calculate $s_{e_i + e_j}(\alpha) = \alpha - (e_i + e_j, \alpha)(e_i + e_j)$. For any $\alpha = e_k - e_l$ such that $i = k < l \neq j$, we have $(e_i + e_j, e_i - e_l) = 1$ and so $s_{e_i + e_j}(e_i - e_l) = (e_i - e_l) - (e_i + e_j) = -e_l - e_j \in \Phi^-$. Note that there are

exactly $n-i-1$ positive roots of the form $e_i - e_l$ with $l \neq j$. Similarly, for any $\alpha = e_k - e_l$ such that $j = k < l$, we have $(e_i + e_j, e_j - e_l) = 1$ and so $s_{e_i+e_j}(e_j - e_l) = (e_j - e_l) - (e_i + e_j) = -e_i - e_l \in \Phi^-$. Note that there are exactly $n-j$ positive roots of the form $e_j - e_l$. Altogether, we have identified $(n-i-1) + (n-j) = 2n - i - j - 1$ positive roots of the form $e_k - e_l$ in $N(e_i + e_j)$.

Now consider $\alpha = e_k + e_l$ such that $i = k$ and $j \leq l$. If $\alpha = e_i + e_j$, then clearly $s_{e_i+e_j}(e_i + e_j) = -e_i - e_j \in \Phi^-$. If $j < l$, then $(e_i + e_j, e_i + e_l) = 1$ and so $s_{e_i+e_j}(e_i + e_l) = -e_j + e_l \in \Phi^-$ since $j < l$. Note that there are exactly $n-j+1$ positive roots of the form $e_i + e_l$ such that $j \leq l$. Now consider $\alpha = e_k + e_l$ such that $j = l$ and $i < k \neq j$. Then $(e_i + e_j, e_k + e_j) = 1$ and so $s_{e_i+e_j}(e_k + e_j) = -e_i + e_k \in \Phi^-$ since $i < k$. Note that there are exactly $n-i-1$ positive roots of the form $e_k + e_j$ such that $i < k \neq j$. Altogether, we have identified $(n-j+1) + (n-i-1) = 2n - i - j$ positive roots of the form $e_k + e_l$ in $N(e_i + e_j)$.

Finally, $s_{e_i+e_j}(2e_i) = -2e_j \in \Phi^-$ and $s_{e_i+e_j}(2e_j) = -2e_i \in \Phi^-$, which concludes our proof that $\Phi_{e_i+e_j}^+ \subseteq N(e_i + e_j)$. We thus have $|\Phi_{e_i+e_j}^+| = (2n - i - j - 1) + (2n - i - j) + 2 = 4n - 2(i+j) + 1 \leq |N(e_i + e_j)| = \ell(s_{e_i+e_j})$. Since the expression in (9) uses exactly $4n - 2(i+j) + 1$ simple reflections, then this inequality implies that the expression for the reflection $s_{e_i+e_j}$ in (9) is reduced for any $1 \leq i < j \leq n$. Consequently, we have $\ell(s_{e_i+e_j}) = 4n - 2(i+j) + 1$, in which case $\Phi_{e_i+e_j}^+ = N(e_i + e_j)$.

3.3. Type B_n . The Weyl groups in types B_n and C_n are identical, and therefore so is the set of reflections $T = \{ws w^{-1} \mid w \in W, s \in S\}$. However, the roles of the roots and coroots are reversed in types B_n and C_n , and so the explicit correspondence between reduced words for reflections and positive roots requires minor relabeling in type B_n .

In type B_n , the roots are

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid i \in [n]\},$$

where e_i denotes the i^{th} standard basis vector in \mathbb{R}^n . We choose the simple roots to be $\Delta = \{e_i - e_{i+1} \mid i \in [n-1]\} \cup \{e_n\}$, where $\alpha_i = e_i - e_{i+1}$ as in type A_{n-1} , but here $\alpha_n = e_n$ is distinct to type B_n . The n^2 positive roots are then

$$\Phi^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid i \in [n]\}.$$

Compute directly that $(e_i \pm e_j)^\vee = e_i \pm e_j$, whereas $e_i^\vee = 2e_i$. Therefore, the reflections corresponding to the roots $e_i \pm e_j \in \Phi^+$ are labeled identically to those in type C_n , whereas the reflections s_{2e_i} in type C_n are relabeled by s_{e_i} in type B_n . We thus have the following palindromic reduced expressions

$$\begin{aligned} s_{e_i - e_j} &= s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i \\ s_{e_i} &= s_i s_{i+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{i+1} s_i \\ s_{e_i + e_j} &= (s_j s_{j+1} \cdots s_{n-1} \cdot s_i s_{i+1} \cdots s_{n-2}) (s_n s_{n-1} s_n) (s_{n-2} \cdots s_{i+1} s_i \cdot s_{n-1} \cdots s_{j+1} s_j) \end{aligned}$$

for the respective $\binom{n}{2} + n + \binom{n}{2} = n^2$ reflections in type B_n .

3.4. Type D_n . In type D_n , the roots are

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\},$$

where e_i denotes the i^{th} standard basis vector in \mathbb{R}^n . We choose the simple roots to be $\Delta = \{e_i - e_{i+1} \mid i \in [n-1]\} \cup \{e_{n-1} + e_n\}$, where $\alpha_i = e_i - e_{i+1}$ as in all other classical types, but here $\alpha_n = e_{n-1} + e_n$ is distinct to type D_n . The $n^2 - n$ positive roots are then

$$\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{e_i + e_j \mid 1 \leq i < j \leq n\}.$$

For the $\binom{n}{2} = \frac{n(n-1)}{2}$ positive roots in $\{e_i - e_j \mid 1 \leq i < j \leq n\}$, we can write

$$e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

which are the roots for the underlying type A_{n-1} root system generated by the first $n-1$ nodes of the Dynkin diagram. (Note that this corrects a typo in item (II) of the type D_n plate in [Bou], which omits α_i from this expansion.) Therefore, for any fixed $e_i - e_j \in \Phi^+$, we have the following palindromic reduced expression

$$(10) \quad s_{e_i - e_j} = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i$$

of length $2(j-i)-1$, by our proof in type A .

Next we consider the type A_{n-1} subsystem generated by $\{s_1, \dots, s_{n-2}, s_n\}$. The reflections which do not use s_n in any reduced expression already appear among the reflections $s_{e_i - e_j}$ expanded in (10). For the remaining $n-1$ reflections in this system, we claim that

$$(11) \quad s_{e_i + e_n} = s_i s_{i+1} \cdots s_{n-2} s_n s_{n-2} \cdots s_{i+1} s_i$$

is a reduced expression for all $1 \leq i < n$, where the product $s_i s_{i+1} \cdots s_{n-2}$ is trivial if $i = n-1$. In case $i = n-1$, recall that $\alpha_n = e_{n-1} + e_n$ is a simple reflection, and so $s_{\alpha_n} = s_n$ by definition. For $1 \leq i \leq n-2$, compute using (1) that in type D_n

$$s_i s_{i+1} \cdots s_{n-2}(\alpha_n) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2} + \alpha_n = e_i + e_n.$$

Therefore, (11) provides expressions for each of the $n-1$ reflections $s_{e_i + e_n}$ by (3).

To prove that the expressions in (11) are reduced, we observe that $\alpha^\vee = \alpha$ for all $\alpha \in \Phi$ since type D_n is simply-laced. The basis of simple coroots is thus $\Delta^\vee = \{\alpha_i^\vee = \alpha_i\}$. In particular,

$$(e_i + e_n)^\vee = (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2} + \alpha_n)^\vee = \alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_{n-2}^\vee + \alpha_n^\vee,$$

and so $\text{ht}(e_i + e_n)^\vee = n-i$. Since D_n is simply-laced, Lemma 4.3 of [BFP] says that $\ell(s_\alpha) = 2\text{ht}(\alpha^\vee) - 1$ for all $\alpha \in \Phi^+$. Therefore, $\ell(s_{e_i + e_n}) = 2\text{ht}(e_i + e_n)^\vee - 1 = 2(n-i) - 1$ for all $1 \leq i \leq n-1$. Since the expression for $s_{e_i + e_n}$ in (11) uses exactly $2(n-i) - 1$ simple reflections, it must be reduced.

For the remaining $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ positive roots $\{e_i + e_j \mid 1 \leq i < j < n\}$, we claim that

$$(12) \quad s_{e_i + e_j} = (s_j s_{j+1} \cdots s_{n-2} \cdot s_i s_{i+1} \cdots s_{n-3}) (s_{n-1} s_{n-2} s_n s_{n-2} s_{n-1}) (s_{n-3} \cdots s_i \cdot s_{n-2} \cdots s_j)$$

is a reduced expression. Here, the product $s_j s_{j+1} \cdots s_{n-2}$ is trivial if $j = n-1$, and the product $s_i s_{i+1} \cdots s_{n-3}$ is trivial if $i = n-2$. To contextualize this conjugating element, consider the subgroup of W of type A_{n-2} generated by $\langle s_1, \dots, s_{n-2} \rangle \cong S_{n-1}$, and the maximal parabolic subgroup P obtained by omitting s_{n-3} . The minimal length coset representatives in S_{n-1}/P are in bijection with the Young diagrams contained in the $(n-3) \times 2$ rectangle labeled by

$n-3$	$n-2$
$n-4$	$n-3$
\vdots	\vdots
2	3
1	2

and read in the same manner as in type C_{n-1} . We show that the remaining roots $e_i + e_j$ in type D_n correspond to the conjugates of the reflection $s_{n-1} s_{n-2} s_n s_{n-2} s_{n-1}$ by the elements of the quotient S_{n-1}/P .

Denote the conjugating element in (12) by

$$w_{ij} = s_j s_{j+1} \cdots s_{n-2} \cdot s_i s_{i+1} \cdots s_{n-3}.$$

We aim to prove that $s_{e_i + e_j} = w_{ij} (s_{n-1} s_{n-2} s_n s_{n-2} s_{n-1}) w_{ij}^{-1}$ is a reduced expression for the reflection corresponding to the positive root $e_i + e_j$. First consider the case $i = n-2$ and $j = n-1$, equivalently $w_{ij} = 1$. Since the root subsystem spanned by $\alpha_{n-1}, \alpha_{n-2}, \alpha_n$ is type A_3 , with an ordering on the roots which reverses the roles of α_{n-2} and α_{n-1} , we know from

our type A results that the root $e_{n-2} + e_{n-1} = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ corresponds to the reflection $s_{n-1}s_{n-2}s_n s_{n-2}s_{n-1}$. For $1 \leq i < j < n$, we have

$$e_i + e_j = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & \text{if } j < n-1, \\ \alpha_i + \cdots + \alpha_n & \text{if } j = n-1. \end{cases}$$

From the entries of the Cartan matrix in type D_n , we have from (1) that

$$(13) \quad s_{n-2}(\alpha_n) = \alpha_{n-2} + \alpha_n, \quad s_{n-1}(\alpha_n) = \alpha_n, \quad s_n(\alpha_{n-2}) = \alpha_{n-2} + \alpha_n, \quad s_n(\alpha_{n-1}) = \alpha_{n-1},$$

and otherwise, the action $s_i(\alpha_j)$ agrees with the type A_n formula from (5). First consider the case $j = n-1$. Compute using (5) and (13) that

$w_{i,n-1}(\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = s_i s_{i+1} \cdots s_{n-3}(\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = \alpha_i + \cdots + \alpha_n = e_i + e_{n-1}$, verifying (12) for $j = n-1$ by (3). Finally, consider the case where the second column of the Young diagram for w_{ij} is nonempty, equivalently $j < n-1$. Compute using (5) and (13), as well as the $j = n-1$ case, that

$$\begin{aligned} w_{ij}(\alpha_{n-2} + \alpha_{n-1} + \alpha_n) &= s_j s_{j+1} \cdots s_{n-2} \cdot s_i s_{i+1} \cdots s_{n-3}(\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = \\ &= s_j s_{j+1} \cdots s_{n-2}(\alpha_i + \cdots + \alpha_n) = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n = e_i + e_j, \end{aligned}$$

verifying (12) for $j < n-1$ by (3). We have thus proven that $s_{e_i + e_j} = w_{ij}(s_{n-1}s_{n-2}s_n s_{n-2}s_{n-1})w_{ij}^{-1}$ for all $1 \leq i < j < n$.

To prove that the expressions in (12) are reduced, we again use the fact that type D_n is simply-laced. Since $\alpha^\vee = \alpha$ for all $\alpha \in \Phi$, we have

$$(e_i + e_j)^\vee = \begin{cases} \alpha_i^\vee + \cdots + \alpha_{j-1}^\vee + 2\alpha_j^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee & \text{if } j < n-1, \\ \alpha_i^\vee + \cdots + \alpha_n^\vee & \text{if } j = n-1. \end{cases}$$

and in particular, $\text{ht}(e_i + e_j)^\vee = 2n - i - j$ for all $1 \leq i < j < n$. By Lemma 4.3 of [BFP], we have $\ell(s_{e_i + e_j}) = 2\text{ht}(e_i + e_j)^\vee - 1 = 4n - 2(i + j) - 1$ for all $1 \leq i < j < n$. Since the expression for $s_{e_i + e_j}$ in (12) uses exactly $2[(n-1-j) + (n-2-i)] + 5 = 4n - 2(i + j) - 1$ simple reflections for all $1 \leq i < j < n$, it must be reduced.

This concludes our proof of Theorem 1.1 in the classical types. \square

4. EXCEPTIONAL TYPES

This section is dedicated to the proof of Theorem 1.1 in the exceptional types. We discuss the exceptional types in order of increasing rank. We refer the reader to the plates in [Bou], which provide explicit formulas for the roots and Cartan matrices. Note that we correct a typo in item (II) for type G_2 , as well as two of the positive root expansions in item (II) of type E_8 .

Beyond type G_2 , we rely heavily upon our results in the classical types, as each exceptional group contains several parabolic subgroups of type A, B, C and/or D . Throughout our discussion of the exceptional types, we typically denote $s_i s_j s_k = s_{ijk}$ for brevity.

Root $\alpha \in \Phi^+$	Expand α in Δ	Reflection s_α
$\varepsilon_1 - \varepsilon_2$	α_1	s_1
$-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$	α_2	s_2
$-\varepsilon_1 + \varepsilon_3$	$\alpha_1 + \alpha_2$	s_{212}
$\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3$	$3\alpha_1 + \alpha_2$	s_{121}
$-\varepsilon_2 + \varepsilon_3$	$2\alpha_1 + \alpha_2$	s_{12121}
$-\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3$	$3\alpha_1 + 2\alpha_2$	s_{21212}

TABLE 2. Palindromic reduced expressions for the 6 reflections in type G_2 .

4.1. Type G_2 . In type G_2 , we choose the basis of simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$, where ε_i denotes the projection of the standard basis vector $e_i \in \mathbb{R}^3$ onto the hyperplane $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$. The positive roots are then

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},$$

where we point out that item (II) in the G_2 plate of [Bou] omits α_2 from Φ^+ . Table 2 provides the expansion for each positive root in terms of $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Using the Cartan matrix in type G_2 together with (1), we have

$$\begin{aligned} s_1(\alpha_2) &= 3\alpha_1 + \alpha_2 & s_2(\alpha_1) &= \alpha_1 + \alpha_2 \\ s_1s_2(\alpha_1) &= 2\alpha_1 + \alpha_2 & s_2s_1(\alpha_2) &= 3\alpha_1 + 2\alpha_2. \end{aligned}$$

We thus immediately obtain the palindromic expressions in Table 2 via (3), all of which are reduced by inspection.

Root $\alpha \in \Phi^+$	Expand α in Δ	Reflection s_α
$e_2 - e_3$	α_1	s_1
$e_3 - e_4$	α_2	s_2
$e_2 - e_4$	$\alpha_1 + \alpha_2$	s_{121}
e_2	$\alpha_1 + \alpha_2 + \alpha_3$	s_{12321}
e_3	$\alpha_2 + \alpha_3$	s_{232}
e_4	α_3	s_3
$e_3 + e_4$	$\alpha_2 + 2\alpha_3$	s_{323}
$e_2 + e_4$	$\alpha_1 + \alpha_2 + 2\alpha_3$	s_{13231}
$e_2 + e_3$	$\alpha_1 + 2\alpha_2 + 2\alpha_3$	$s_{2132312}$
$\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$	α_4	s_4
$\frac{1}{2}(e_1 - e_2 - e_3 + e_4)$	$\alpha_3 + \alpha_4$	s_{434}
$\frac{1}{2}(e_1 - e_2 + e_3 - e_4)$	$\alpha_2 + \alpha_3 + \alpha_4$	s_{42324}
$\frac{1}{2}(e_1 - e_2 + e_3 + e_4)$	$\alpha_2 + 2\alpha_3 + \alpha_4$	$s_{3423243}$
$e_1 - e_2$	$\alpha_2 + 2\alpha_3 + 2\alpha_4$	s_{43234}
$\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$s_{1423241}$
$\frac{1}{2}(e_1 + e_2 - e_3 + e_4)$	$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$	$s_{134232431}$
$\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$	$s_{21342324312}$
$\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$	$s_{3213423243123}$
$e_1 - e_3$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$	$s_{1432341}$
$e_1 - e_4$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$	$s_{214323412}$
$e_1 + e_4$	$\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$	$s_{32143234123}$
$e_1 + e_3$	$\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$s_{2321432341232}$
e_1	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	$s_{432134232431234}$
$e_1 + e_2$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$s_{123214323412321}$

TABLE 3. Palindromic reduced expressions for the 24 reflections in type F_4 .

4.2. Type F_4 . In type F_4 , we choose the simple roots $\Delta = \{\alpha_i\}$ to be $\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$. The positive roots are then

$$\Phi^+ = \{e_i \mid 1 \leq i \leq 4\} \cup \{e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \left\{ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

Table 3 provides the expansion for each positive root in terms of Δ .

It is clear from the Dynkin diagram and the choice of $\alpha_1, \alpha_2, \alpha_3$ that $W/\langle s_4 \rangle$ has type B_3 . We thus obtain the first 9 entries in Table 3, by simply reindexing the standard basis vectors $e_i \mapsto e_{i+1}$ in type B_3 . In addition, $W/\langle s_1 \rangle$ has type C_3 , though the first and last labels are swapped compared to our labeling in type C . Noting that the 4 reflections $s_2, s_3, s_{232}, s_{323}$ are common to $W/\langle s_1 \rangle$ and $W/\langle s_4 \rangle$, we obtain 5 distinct type C_3 reflections, which appear next in Table 3. The remaining 10 reflections are distinct to type F_4 , and are obtained by successively conjugating the 3 longest type C_3 reflections by simple reflections which increase the length.

Root $\alpha \in \Phi^+$	Expand α in Δ	Reflection s_α
01111	α_1	s_1
10111	$\alpha_1 + \alpha_3$	s_{131}
11011	$\alpha_1 + \alpha_3 + \alpha_4$	s_{13431}
00011	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$s_{1342431}$
$-\varepsilon_1 + \varepsilon_2$	α_3	s_3
$-\varepsilon_1 + \varepsilon_3$	$\alpha_3 + \alpha_4$	s_{343}
$-\varepsilon_2 + \varepsilon_3$	α_4	s_4
$\varepsilon_1 + \varepsilon_2$	α_2	s_2
$\varepsilon_1 + \varepsilon_3$	$\alpha_2 + \alpha_4$	s_{424}
$\varepsilon_2 + \varepsilon_3$	$\alpha_2 + \alpha_3 + \alpha_4$	s_{34243}
$-\varepsilon_3 + \varepsilon_4$	α_5	s_5
$-\varepsilon_2 + \varepsilon_4$	$\alpha_4 + \alpha_5$	s_{454}
$-\varepsilon_1 + \varepsilon_4$	$\alpha_3 + \alpha_4 + \alpha_5$	s_{34543}
11101	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$	$s_{1345431}$
00101	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$s_{132454231}$
01001	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$s_{41324542314}$
10001	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$	$s_{3413245423143}$
$\varepsilon_1 + \varepsilon_4$	$\alpha_2 + \alpha_4 + \alpha_5$	s_{24542}
$\varepsilon_2 + \varepsilon_4$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$s_{3245423}$
$\varepsilon_3 + \varepsilon_4$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	$s_{432454234}$
$-\varepsilon_4 + \varepsilon_5$	α_6	s_6
$-\varepsilon_3 + \varepsilon_5$	$\alpha_5 + \alpha_6$	s_{656}
$-\varepsilon_2 + \varepsilon_5$	$\alpha_4 + \alpha_5 + \alpha_6$	s_{65456}
$-\varepsilon_1 + \varepsilon_5$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$s_{6543456}$
$\varepsilon_1 + \varepsilon_5$	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$	$s_{6542456}$
$\varepsilon_2 + \varepsilon_5$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$s_{652434256}$
$\varepsilon_3 + \varepsilon_5$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$s_{46524342564}$
$\varepsilon_4 + \varepsilon_5$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$s_{5465243425645}$
11110	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$s_{134565431}$
00110	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$s_{16524342561}$
01010	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$s_{1465243425641}$
01100	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$s_{154652434256451}$
10010	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	$s_{314652434256413}$
10100	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$s_{31546524342564513}$
11000	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$s_{4315465243425645134}$
00000	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$s_{243154652434256451342}$

TABLE 4. Palindromic reduced expressions for the 36 reflections in type E_6 .

4.3. Type E_6 . In type E_6 , let ε_i denote the projection of the standard basis vector $e_i \in \mathbb{R}^8$ onto the subspace $\{(x_1, \dots, x_8) \mid x_6 = x_7 = -x_8\}$. We choose the basis of simple roots $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2$, $\alpha_3 = \varepsilon_2 - \varepsilon_1$, $\alpha_4 = \varepsilon_3 - \varepsilon_2$, $\alpha_5 = \varepsilon_4 - \varepsilon_3$, $\alpha_6 = \varepsilon_5 - \varepsilon_4$. The positive roots are then

$$\Phi^+ = \{\pm \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq 5\} \cup \left\{ \frac{1}{2} \left(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^5 (-1)^{\nu(i)} \varepsilon_i \right) \mid \sum_{i=1}^5 \nu(i) \in 2\mathbb{Z} \right\},$$

where $\nu(i) \in \{0, 1\}$. Table 4 provides the expansion for each positive root in terms of Δ . For roots in the second set, we record only the values $\nu(1), \dots, \nu(5)$ for brevity.

It is clear from the Dynkin diagram that $W/\langle s_6 \rangle$ has type D_5 . We thus obtain the first 20 entries in Table 4, by reindexing the simple reflections $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 2$ from type D_5 . Similarly, $W/\langle s_1 \rangle$ has type D_5 , and by reindexing the simple reflections $1 \mapsto 6$, $2 \mapsto 5$, $3 \mapsto 4$, $4 \mapsto 2$, $5 \mapsto 3$, ignoring all reflections in $W/\langle s_1, s_6 \rangle$ of type D_4 that already appear, we obtain the next 8 reflections in Table 4. There is 1 additional reflection from the type A_5 system $W/\langle s_2 \rangle$. The 7 remaining reflections are distinct to type E_6 , and are obtained by conjugating the 3 longest type D_5 reflections in $W/\langle s_1 \rangle$ by s_1 , and then again by those subwords of s_{243} which increase length. These reflections correspond to the 7 blue vertices in Figure 1.

Root $\alpha \in \Phi^+$	Expand α in Δ	Reflection s_α
$-\varepsilon_5 + \varepsilon_6$	α_7	s_7
$-\varepsilon_4 + \varepsilon_6$	$\alpha_6 + \alpha_7$	s_{767}
$-\varepsilon_3 + \varepsilon_6$	$\alpha_5 + \alpha_6 + \alpha_7$	s_{76567}
$-\varepsilon_2 + \varepsilon_6$	$\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{7654567}$
$-\varepsilon_1 + \varepsilon_6$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{765434567}$
$\varepsilon_1 + \varepsilon_6$	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{765424567}$
$\varepsilon_2 + \varepsilon_6$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{76524342567}$
$\varepsilon_3 + \varepsilon_6$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{4765243425674}$
$\varepsilon_4 + \varepsilon_6$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$	$s_{547652434256745}$
$\varepsilon_5 + \varepsilon_6$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{65476524342567456}$
111110	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{13456765431}$
001110	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{7165243425617}$
010110	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{714652434256417}$
011010	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$	$s_{71546524342564517}$
100110	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$s_{73146524342564137}$
101010	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$	$s_{7315465243425645137}$
110010	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$	$s_{743154652434256451347}$
000010	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$	$s_{72431546524342564513427}$
011100	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{6715465243425645176}$
101100	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{673154652434256451376}$
110100	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{67431546524342564513476}$
000100	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{6724315465243425645134276}$
111000	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{5674315465243425645134765}$
001000	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{256743154652434256451347652}$
010000	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{42567431546524342564513476524}$
100000	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{3425674315465243425645134765243}$
$-\varepsilon_7 + \varepsilon_8$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$s_{134256743154652434256451347652431}$

TABLE 5. Reduced expressions for the 27 type E_7 reflections which are not type E_6 .

4.4. Type E_7 . In type E_7 , let ε_i denote the projection of the standard basis vector $e_i \in \mathbb{R}^8$ onto onto the subspace $\{(x_1, \dots, x_8) \mid x_7 = -x_8\}$. We choose the basis of simple roots $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2$, $\alpha_3 = \varepsilon_2 - \varepsilon_1$, $\alpha_4 = \varepsilon_3 - \varepsilon_2$, $\alpha_5 = \varepsilon_4 - \varepsilon_3$, $\alpha_6 = \varepsilon_5 - \varepsilon_4$, $\alpha_7 = \varepsilon_6 - \varepsilon_5$. The positive roots are then

$$\Phi^+ = \{\pm \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq 6\} \cup \{\varepsilon_8 - \varepsilon_7\} \cup \left\{ \frac{1}{2} \left(\varepsilon_8 - \varepsilon_7 + \sum_{i=1}^6 (-1)^{\nu(i)} \varepsilon_i \right) \mid \sum_{i=1}^6 \nu(i) \notin 2\mathbb{Z} \right\},$$

where $\nu(i) \in \{0, 1\}$. Table 5 provides the expansion for each positive root in terms of Δ . For roots in the third set, we record only the values $\nu(1), \dots, \nu(6)$ for brevity.

It is clear from both the choice of simple roots and the Dynkin diagram that $W/\langle s_7 \rangle$ has type E_6 , without any necessary relabeling. We thus obtain the first 36 reflections in E_7 directly from Table 4 for type E_6 . Similarly, $W/\langle s_1 \rangle$ has type D_6 , and by reindexing the D_6 simple reflections $1 \mapsto 7$, $2 \mapsto 6$, $3 \mapsto 5$, $5 \mapsto 2$, $6 \mapsto 3$, ignoring all reflections in $W/\langle s_1, s_7 \rangle$ of type D_5 that have already been identified, we obtain the first 10 reflections in Table 5. There is 1 additional reflection not yet identified from the type A_6 system $W/\langle s_2 \rangle$. The 16 remaining reflections are distinct to type E_7 , and are obtained by conjugating the 7 type E_6 reflections in $W/\langle s_7 \rangle$ by s_7 , then by s_6 provided that length increases, and finally by conjugating the second longest of the resulting reflections by s_5, s_2, s_4, s_3, s_1 in turn; these reflections correspond to the 16 red vertices in Figure 1. The 27 type E_7 reflections additional to type E_6 appear in Table 5.

Root $\alpha \in \Phi^+$	Expand α in Δ	Reflection s_α
$-\varepsilon_6 + \varepsilon_7$	α_8	s_8
$-\varepsilon_5 + \varepsilon_7$	$\alpha_7 + \alpha_8$	s_{878}
$-\varepsilon_4 + \varepsilon_7$	$\alpha_6 + \alpha_7 + \alpha_8$	s_{87678}
$-\varepsilon_3 + \varepsilon_7$	$\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{8765678}$
$-\varepsilon_2 + \varepsilon_7$	$\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{876545678}$
$-\varepsilon_1 + \varepsilon_7$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{87654345678}$
$\varepsilon_1 + \varepsilon_7$	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{87654245678}$
$\varepsilon_2 + \varepsilon_7$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{8765243425678}$
$\varepsilon_3 + \varepsilon_7$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{487652434256784}$
$\varepsilon_4 + \varepsilon_7$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{54876524342567845}$
$\varepsilon_5 + \varepsilon_7$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{6548765243425678456}$
$\varepsilon_6 + \varepsilon_7$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{765487652434256784567}$
1111110	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{1345678765431}$

TABLE 6. Reduced expressions for the 13 type A_7 and D_7 reflections in type E_8 .

4.5. Type E_8 . In type E_8 , we let ε_i denote the standard basis vector in \mathbb{R}^8 , for ease of comparison with types E_6 and E_7 . We choose the basis of simple roots $\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$, $\alpha_2 = \varepsilon_1 + \varepsilon_2$, $\alpha_3 = \varepsilon_2 - \varepsilon_1$, $\alpha_4 = \varepsilon_3 - \varepsilon_2$, $\alpha_5 = \varepsilon_4 - \varepsilon_3$, $\alpha_6 = \varepsilon_5 - \varepsilon_4$, $\alpha_7 = \varepsilon_6 - \varepsilon_5$, $\alpha_8 = \varepsilon_7 - \varepsilon_6$. The positive roots are then

$$\Phi^+ = \{\pm \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq 8\} \cup \left\{ \frac{1}{2} \left(\varepsilon_8 + \sum_{i=1}^7 (-1)^{\nu(i)} \varepsilon_i \right) \mid \sum_{i=1}^7 \nu(i) \in 2\mathbb{Z} \right\},$$

where $\nu(i) \in \{0, 1\}$. For roots in the second set, we record only the values $\nu(1), \dots, \nu(7)$ for brevity. Tables 6 and 7 provide the expansion for each positive root in terms of Δ . Note that item (II) in the E_8 plate of [Bou] with these expansions contains 2 typos. Namely, writing $(c_1 \dots c_8)$ for $\sum c_i \alpha_i \in \Phi^+$, the root (12232211) is recorded twice, and an erroneous root (11233321) appears. These roots are replaced by (12232111) and (11233221) in Table 7 below.

It is clear from both the choice of simple roots and the Dynkin diagram that $W/\langle s_8 \rangle$ has type E_7 , without any necessary relabeling. We thus obtain the first 63 reflections in E_8 directly from Tables 4 and 5 displaying reduced words for the reflections in types E_6 and E_7 , respectively. In addition, $W/\langle s_1 \rangle$ has type D_7 , and by reindexing the D_7 simple reflections $1 \mapsto 8, 2 \mapsto 7, 3 \mapsto 6, 4 \mapsto 5, 5 \mapsto 4, 6 \mapsto 2, 7 \mapsto 3$, ignoring all reflections in $W/\langle s_1, s_8 \rangle$ of type D_6 that have already been identified, we obtain the first 12 reflections in Table 6. There is one additional reflection from the type A_7 system $W/\langle s_2 \rangle$, which has both s_1 and s_8 in its support, shown as the last entry in Table 6.

The 44 remaining reflections are distinct to type E_8 , and can be obtained by conjugating the 16 type E_7 reflections in $W/\langle s_8 \rangle$ by s_8 , and then performing additional conjugations as depicted by Figure 1. As explained in Example 1.3, the vertices of the graph in Figure 1 correspond to the 67 reflections of types E_6, E_7 , and E_8 , colored blue, red, and green respectively, with length increasing going upward. The type E_6 reflection of minimal length is $s_\theta = s_{16524342561}$, obtained from the bottom portion of Table 4. The edge labels in Figure 1 indicate the simple reflection with which to conjugate to obtain the reflection indexing the adjacent vertex. For example, conjugating s_θ by $w = s_{24354}$ reading up the blue edges along the righthand side gives one reduced expression for the longest reflection in type E_6 ; compare the final entry of Table 4. Similarly, the red vertices in Figure 1 correspond to the 17 reflections of type E_7 , and the green vertices correspond to the 44 remaining reflections which are distinct to type E_8 . Starting from $s_{87}s_\theta s_{78}$ at the bottom of the green vertices and conjugating by the labels on the green edges, we obtain the 44 reduced expressions recorded in Table 7 on the next page.

This concludes our proof of Theorem 1.1 in the exceptional types. \square

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ELIZABETH MILIĆEVIĆ, DEPARTMENT OF MATHEMATICS & STATISTICS, HAVERFORD COLLEGE, 370 LANCASTER AVENUE, HAVERFORD, PA, 19041, USA

Email address: emilicevic@haverford.edu

Roots $\alpha \in \Phi^+$	Expand α in Δ	w for $s_\alpha = ws_\theta w^{-1}$
0011110	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	s_{87}
0101110	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	s_{487}
1001110	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	s_{3487}
1010110	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	s_{53487}
1100110	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	s_{453487}
0000110	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$s_{2453487}$
0001010	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{62453487}$
0110110	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	s_{5487}
0111010	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	s_{65487}
1011010	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	s_{365487}
1101010	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{4365487}$
1110010	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{54365487}$
0010010	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{254365487}$
0100010	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{4254365487}$
1000010	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{34254365487}$
$-\varepsilon_6 + \varepsilon_8$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	$s_{134254365487}$
0111100	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	s_{765487}
1011100	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{3765487}$
1101100	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{43765487}$
0001100	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{243765487}$
0010100	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{5243765487}$
0100100	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{45243765487}$
1000100	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{345243765487}$
$-\varepsilon_5 + \varepsilon_8$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{1345243765487}$
1110100	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{543765487}$
1111000	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{6543765487}$
0011000	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{26543765487}$
0101000	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{426543765487}$
1001000	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{3426543765487}$
$-\varepsilon_4 + \varepsilon_8$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{13426543765487}$
$-\varepsilon_3 + \varepsilon_8$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{513426543765487}$
$-\varepsilon_2 + \varepsilon_8$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{4513426543765487}$
$-\varepsilon_1 + \varepsilon_8$	$2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{34513426543765487}$
0110000	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{5426543765487}$
1010000	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{35426543765487}$
1100000	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{435426543765487}$
0000000	$\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{2435426543765487}$
$\varepsilon_1 + \varepsilon_8$	$2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{12435426543765487}$
$\varepsilon_2 + \varepsilon_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{312435426543765487}$
$\varepsilon_3 + \varepsilon_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{4312435426543765487}$
$\varepsilon_4 + \varepsilon_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{54312435426543765487}$
$\varepsilon_5 + \varepsilon_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8$	$s_{654312435426543765487}$
$\varepsilon_6 + \varepsilon_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8$	$s_{7654312435426543765487}$
$\varepsilon_7 + \varepsilon_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	$s_{87654312435426543765487}$

TABLE 7. Reduced expressions for 44 reflections distinct to type E_8 , with $s_\theta = s_{16524342561}$.