

HOMOLOGY GROUPS OF CUBICAL SETS WITH CONNECTIONS

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ABSTRACT. Toward defining commutative cubes in all dimensions, Brown and Spencer introduced the notion of “connection” as a new kind of degeneracy. In this paper, for a cubical set with connections, we show that the connections generate an acyclic subcomplex of the chain complex of the cubical set. In particular, our results show that the homology groups of a cubical set with connections are independent of whether we normalize by the connections or we do not, that is, connections do not contribute to any nontrivial cycle in the homology groups of the cubical set.

1. INTRODUCTION

Cubical sets stemmed naturally from the development of homology theory of various spaces. Instead of simplices, cubes were, for the first time, used by Serre to develop (co)homology theory for fiber spaces [18], and Eilenberg and MacLane [12] developed the singular, cubical homology theory of topological spaces. Massey’s classical book [17] presents a comprehensive treatment of singular homology using the cubical approach.

Kan introduced and studied abstract cubical sets for the purpose of developing a general homotopy theory, see [15]. Cubical sets come with a singular homology theory [10, Section 14.7] and a geometric realization [10, Definition 11.1.11]. Federer [13, Theorem 3.9.12] showed that the singular homology groups of a cubical set and that of its geometric realization are isomorphic.

Toward the development of a general abstract homotopy theory, Brown and Spencer [11] identified the need, in higher dimensions, for what they call “commutative” cubes, and introduced a new kind of degeneracy which they call “connections.” Cubical sets with connections were then introduced and studied by Brown and Higgins in [7]. The recent paper [6] explains the origin of the notion of connection as well as the need for it.

Not all cubical sets admit connections. However, cubical sets with connections have been shown to have many desirable properties [8], and have characteristics similar to that of simplicial sets [14]. For examples, cubical abelian groups with connections are equivalent to chain complexes [9], and cubical groups with connections are Kan fibrant [20], a property shared with simplicial sets. Recently, in [16], it was shown that cubical sets with connections form a strict test category. In particular, the geometric realization of the product of cubical sets with connections has the “right” homotopy type; a property that cubical set (without connections) do not have in general.

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In this note we study the singular homology groups of cubical sets with connections. We were originally motivated by computational considerations encountered in [5]. Since the chain groups are very large, we explored cutting down the size of the the chain complex by dropping connection cubes. For this purpose, we investigate the contribution of connections to the nontrivial cycles in the homology groups. We do so by studying the relations between the singular cubical differential, the face maps, the degeneracy maps and the connections maps. This study culminates in [Theorem 3.1](#) from which we then deduce in [Corollary 3.2](#) that connections generate a chain subcomplex of the singular chain complex of the cubical set. Furthermore, using a chain homotopy given in [Theorem 3.8](#) we deduce in [Corollary 3.10](#) that the homology groups of this subcomplex are trivial. In particular, the quotient of the singular chain complex of the cubical set by the subcomplex generated by the connection cubes computes the same homology as the singular chain complex itself.

In an appendix we provide the arguments showing that this quotient complex indeed is the cellular chain complex of the canonical CW-structure on the geometric realization of a cubical set with connections (see [Theorem 3.17](#)). In particular, for a cubical set with connections, we state in [Corollary 3.18](#) that the singular homology groups of the geometric realizations with and without connection identifications coincide.

The latter result is also a consequence of a result by Antolini [2], who states that the two realizations are homotopy equivalent. Since we consider Antolini's arguments hard to penetrate, we see some value of our down to earth derivation.

2. BACKGROUND AND NOTATIONS

In this section we recall the definition of a cubical set with connections and the homology theory of cubical sets. Then we give two examples of such sets to demonstrate the motivation for this study.

Throughout the paper, R denotes a commutative ring with unit which shall be the ring of coefficients. For any positive integer n , let $[n] := \{1, \dots, n\}$.

Definition 2.1 ([15]). A *cubical set* K is a collection of sets $\{K_n\}_{n \geq 0}$ together with, for each $n \geq 1$ and each $i \in [n]$,

- (1) two maps $f_i^+, f_i^- : K_n \longrightarrow K_{n-1}$, which are called *face maps*, and
- (2) a map $\varepsilon_i : K_{n-1} \longrightarrow K_n$, which is called a *degeneracy map*,

satisfying the following relations: For $\alpha, \beta \in \{+, -\}$,

- (i) $f_i^\alpha f_j^\beta = f_{j-1}^\beta f_i^\alpha \quad \text{if } i < j.$
- (ii) $\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \quad \text{if } i \leq j.$
- (iii) $f_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1} f_i^\alpha & \text{if } i < j; \\ \varepsilon_j f_{i-1}^\alpha & \text{if } i > j; \\ id & \text{if } i = j. \end{cases}$

In a cubical set K , an element $\sigma \in K_n$ is called a *singular n -cube*. A singular n -cube σ is said to be degenerate if $\sigma = \varepsilon_i f_i^+ \sigma$ for some $i \in [n]$. Otherwise, σ is called *non-degenerate*.

Definition 2.2 ([1]). A *cubical set with connections* is a cubical set K together with, for $n \geq 1$ and each $i \in [n]$, two additional maps (called *connections*)

$$\Gamma_i^+, \Gamma_i^- : K_n \longrightarrow K_{n+1}.$$

such that, for $\alpha, \beta \in \{+, -\}$ and $i, j \in [n]$, the following relations are satisfied:

- (i) $\Gamma_i^\alpha \Gamma_j^\beta = \Gamma_{j+1}^\beta \Gamma_i^\alpha \quad \text{if } i \leq j.$
- (ii) $\Gamma_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j+1} \Gamma_i^\alpha & \text{if } i < j; \\ \varepsilon_j \Gamma_{i-1}^\alpha & \text{if } i > j; \\ \varepsilon_i^2 = \varepsilon_{i+1} \varepsilon_i & \text{if } i = j. \end{cases}$
- (iii) $f_i^\alpha \Gamma_j^\beta = \begin{cases} \Gamma_{j-1}^\beta f_i^\alpha & \text{if } i < j; \\ \Gamma_j^\beta f_{i-1}^\alpha & \text{if } i > j+1; \\ id & \text{if } i = j, j+1, \alpha = \beta; \\ \varepsilon_i f_i^\alpha & \text{if } i = j, j+1, \alpha \neq \beta. \end{cases}$

Homology Groups of Cubical Sets. Let K be a cubical set and let R be the ring of coefficients. For each $n \geq 0$, let $\mathcal{L}_n(K)$ be the free R -module generated by the singular n -cubes with coefficients from R , that is,

$$\mathcal{L}_n(K) := \left\{ \sum_{\sigma \in S} r_\sigma \sigma : S \text{ finite subset of } K_n \text{ and } r_\sigma \in R \right\}.$$

For $n > 0$, define the map $\partial_n : \mathcal{L}_n(K) \longrightarrow \mathcal{L}_{n-1}(K)$ such that, for each singular n -cube σ ,

$$\partial_n(\sigma) = \sum_{i=1}^n (-1)^i (f_i^- \sigma - f_i^+ \sigma)$$

and extend linearly to all elements of $\mathcal{L}_n(K)$. Furthermore, define the map $\partial_0 : \mathcal{L}_0(K) \longrightarrow \mathcal{L}_{-1}(K)(= \{0\})$ to be the zero map, that is $\partial_0(\sigma) = 0$ for all $\sigma \in \mathcal{L}_0$.

For each $n \geq 1$, let $\mathcal{D}_n(K)$ be the R -submodule of $\mathcal{L}_n(K)$ that is generated by all degenerate singular n -cubes, and let $\mathcal{C}_n(K)$ be the free R -module $\mathcal{L}_n(K)/\mathcal{D}_n(K)$, whose elements are called n -chains. Clearly, the cosets of non-degenerate singular n -cubes freely generate $\mathcal{C}_n(K)$.

Using Definition 2.1(iii), it is easy to check that $\partial_n[\mathcal{D}_n(K)] \subseteq \mathcal{D}_{n-1}(K)$ and, for $n \geq 1$, $\partial_{n-1}\partial_n = 0$, see [4, 17]. Hence, $\partial_n : \mathcal{C}_n(K) \longrightarrow \mathcal{C}_{n-1}(K)$ is a boundary operator, and $\mathcal{C}(K) = (\mathcal{C}_\bullet(K), \partial_\bullet)$ is a chain complex of free R -modules. We call $\mathcal{C}(K)$ the *non-degenerate chain complex* of the cubical set K .

The homology groups of K are defined to be the homology groups of the chain complex $\mathcal{C}(K)$, that is, $\mathcal{H}_n(K) := \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$, see [15]. For more information about the homology and homotopy of cubical sets see [10, Sections 14.7 and 13.1].

Cubical Sets of Topological Spaces. Let X be a topological space, and, for $n \geq 0$, let I^n be the geometric n -dimensional cube, that is, $I^n := \{(x_1, \dots, x_n) : x_i \in [0, 1], i \in [n]\}$ with the standard topology. Define KX_n to be the set of all continuous maps $\sigma : I^n \longrightarrow X$.

For each $i \in [n]$ and $\sigma \in KX_n$, define face maps $f_i^+ \sigma, f_i^- \sigma \in KX_{n-1}$ such that, for $(a_1, \dots, a_{n-1}) \in I^{n-1}$,

$$\begin{aligned} (f_i^+ \sigma)(a_1, \dots, a_{n-1}) &:= \sigma(a_1, \dots, a_{i-1}, 1, a_i, \dots, a_{n-1}), \\ (f_i^- \sigma)(a_1, \dots, a_{n-1}) &:= \sigma(a_1, \dots, a_{i-1}, 0, a_i, \dots, a_{n-1}). \end{aligned}$$

Also, define $\varepsilon_i \sigma \in KX_{n+1}$ such that, for $(a_1, \dots, a_{n+1}) \in I^{n+1}$,

$$(\varepsilon_i \sigma)(a_1, \dots, a_{n+1}) := \sigma(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}).$$

It is easy to check that $KX := \{KX_n\}_{n \geq 0}$ along with the face maps f_i^\pm and degeneracy maps ε_i is a cubical set.

Furthermore, KX is a cubical set with connections defined as follows. For each $i \in [n]$, set

$$\Gamma_i^\varepsilon \sigma(a_1, \dots, a_{n+1}) := \sigma(a_1, \dots, a_{i-1}, m_\varepsilon(a_i, a_{i+1}), a_{i+2}, \dots, a_{n+1})$$

where

$$m_\varepsilon(x, y) = \begin{cases} \min(x, y) & \text{if } \varepsilon = +; \\ \max(x, y) & \text{if } \varepsilon = -. \end{cases}$$

The set KX was initially constructed by Eilenberg and Mac Lane [12] and was used to define the cubical singular homology groups of X , which turned out to be the same as the (classical) singular homology groups of X , that is, $H_n(X) = H_n(KX)$ for all n , see [17, Section 2, Chapter II]. Furthermore, the geometric realization $|KX|$ of KX and X are weakly homotopy equivalent [10, Proposition 11.1.16], in particular $H_n(|KX|)$ and $H_n(X)$ are isomorphic for all n , see [19, Theorem 7.6.25].

Discrete Cubical Sets of Graphs. Another cubical set with connections arises from the development of a discrete homology theory for metric spaces [3, 4]. For a given metric space X , the singular (n, r) -cubes are defined to be the r -Lipschitz maps from the n -dimensional Hamming cube to the metric space X , and the (discrete) homology groups of the metric space X are defined to be the singular homology groups of the resulting singular chain complex.

In a recent paper [5] we study the theory from [4] in the combinatorially interesting case where the singular n -cubes are the graph homomorphisms from the n -dimensional Hamming cube to a given undirected, simple graph G . This results in a cubical set KG which is used to define a (discrete) cubical homology of the graph G .

For $n \geq 0$, let Q_n be the Hamming n -dimensional cube, that is, $Q_n := \{(x_1, \dots, x_n) : x_i \in \{0, 1\}, i \in [n]\}$. Define KG_n to be the set of all graph homomorphisms $\sigma : Q_n \rightarrow G$. For each $i \in [n]$ and $\sigma \in KG_n$, define face maps $f_i^+ \sigma, f_i^- \sigma \in KG_{n-1}$ such that, for $(a_1, \dots, a_{n-1}) \in Q_{n-1}$,

$$\begin{aligned} (f_i^+ \sigma)(a_1, \dots, a_{n-1}) &:= \sigma(a_1, \dots, a_{i-1}, 1, a_i, \dots, a_{n-1}), \\ (f_i^- \sigma)(a_1, \dots, a_{n-1}) &:= \sigma(a_1, \dots, a_{i-1}, 0, a_i, \dots, a_{n-1}). \end{aligned}$$

Also, define $\varepsilon_i \sigma \in KG_{n+1}$ such that, for $(a_1, \dots, a_{n+1}) \in Q_{n+1}$,

$$(\varepsilon_i \sigma)(a_1, \dots, a_{n+1}) := \sigma(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}).$$

Furthermore, for each $i \in [n]$, define connection maps $\Gamma_i^+ \sigma, \Gamma_i^- \sigma \in KG_{n+1}$ such that

$$\Gamma_i^\varepsilon \sigma(a_1, \dots, a_{n+1}) := \sigma(a_1, \dots, a_{i-1}, m_\varepsilon(a_i, a_{i+1}), a_{i+2}, \dots, a_{n+1}),$$

where

$$m_\varepsilon(x, y) = \begin{cases} \min(x, y) & \text{if } \varepsilon = +; \\ \max(x, y) & \text{if } \varepsilon = -. \end{cases}$$

The proof of the following lemma is straightforward and is similar to that of KX being a cubical set with connections.

Lemma 2.3. *The collection $KG := \{KG_n\}_{n \geq 0}$ along with the face maps f_i^\pm , degeneracy maps ε_i and connections Γ_i^\pm is a cubical set with connections.*

Even though we were able to compute the homology groups of many classes of graphs [5, Sections 4 and 7], in general such computations are not feasible and, once again, the need for better understanding of the cubical set itself is evident. Investigating the role of the connections in the nontrivial cycles in the homology groups of KG seems a natural step.

3. HOMOLOGY OF THE CONNECTION CHAIN SUBCOMPLEX

Let K be a cubical set with connections and let $\mathcal{C}(K)$ be its non-degenerate chain complex. It is easy to see that the set of connections of K does not form a cubical subset of K as not all faces of a connection are necessarily connections. However, we will show in this section that the connections generate a chain subcomplex of $\mathcal{C}(K)$. Furthermore, the homology groups of this subcomplex are trivial.

Theorem 3.1. *Let K be a cubical set and $\mathcal{C}(K)$ be its chain complex as above. Let $\tau \in K_n$ be a singular n -cube and $\beta \in \{+, -\}$. Then,*

- (i) $\partial_{n+1} \Gamma_1^\beta(\tau) = -\Gamma_1^\beta \sum_{i=2}^n (-1)^i (f_i^- - f_i^+)(\tau)$.
- (ii) $\partial_{n+1} \Gamma_n^\beta(\tau) = \Gamma_{n-1}^\beta \sum_{i=1}^{n-1} (-1)^i (f_i^- - f_i^+)(\tau)$.
- (iii) For any $1 < t < n$,

$$\partial_{n+1} \Gamma_t^\beta(\tau) = \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) - \Gamma_t^\beta \sum_{i=t+1}^n (-1)^i (f_i^- - f_i^+)(\tau).$$

Proof. Let $\tau \in K_n$ be a singular n -cube and $\beta \in \{+, -\}$. Then, for $t \in [n]$,

$$\partial_{n+1} \Gamma_t^\beta(\tau) = \sum_{i=1}^{n+1} (-1)^i (f_i^- - f_i^+) (\Gamma_t^\beta(\tau)).$$

By Definition 2.2(iii), $f_t^\alpha \Gamma_t^\beta(\tau) = f_{t+1}^\alpha \Gamma_t^\beta(\tau)$ and $f_i^\alpha \Gamma_t^\beta = \begin{cases} \Gamma_{t-1}^\beta f_i^\alpha & \text{if } i < t; \\ \Gamma_t^\beta f_{i-1}^\alpha & \text{if } i > t+1. \end{cases}$

Now [Theorem 3.1\(i\)](#), i.e. when $t = 1$, and [Theorem 3.1\(ii\)](#), i.e. when $t = n$, follow immediately. For $1 < t < n$, the following computation implies [Theorem 3.1\(iii\)](#),

$$\begin{aligned} \partial_{n+1}\Gamma_t^\beta(\tau) &= \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) + \Gamma_t^\beta \sum_{i=t+2}^{n+1} (-1)^i (f_{i-1}^- - f_{i-1}^+)(\tau) \\ &= \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) - \Gamma_t^\beta \sum_{i=t+1}^n (-1)^i (f_i^- - f_i^+)(\tau). \end{aligned}$$

□

Let K be a cubical set with connections and let $\mathcal{C}(K)$ be its non-degenerate chain complex. For $n \geq 0$, let $\text{Con}_{n+1}(K)$ be the R -submodule of $\mathcal{C}_{n+1}(K)$ that is generated by the cosets of $\Gamma_i^\beta(\tau)$ where $\tau \in K_n$, $i \in [n]$, and $\beta \in \{+, -\}$.

The following is an immediate consequence of [Theorem 3.1](#).

Corollary 3.2. *Let $\theta \in \text{Con}_{n+1}(K)$ then $\partial_{n+1}(\theta) \in \text{Con}_n(K)$. In particular, $\text{Con}(K) = (\text{Con}_\bullet, \partial_\bullet)$ is a chain subcomplex of the chain complex $\mathcal{C}(K)$.*

We call $\text{Con}(K)$ the *connection chain complex* of K .

Clearly, $\text{Con}_{n+1}(K)$ is generated by the cosets of $\Gamma_i^\beta(\tau)$ where τ is a non-degenerate singular n -cubes. In particular, $\text{Con}_1(K) = (0)$.

Corollary 3.3. *Let $\tau \in \mathcal{C}_n(K)$ be a singular n -cube. Then, for $\beta \in \{+, -\}$ and $t \in [n]$, the following equations are true.*

(i)

$$\partial_{n+1}\Gamma_1^\beta(\tau) + \Gamma_1^\beta\partial_n(\tau) = \Gamma_1^\beta(f_i^+ - f_i^-)(\tau).$$

(ii) For $2 \leq t \leq n$,

$$\partial_{n+1}\Gamma_t^\beta(\tau) + \Gamma_t^\beta\partial_n(\tau) = \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) + \Gamma_t^\beta \sum_{i=1}^t (-1)^i (f_i^- - f_i^+)(\tau).$$

(iii) For $t \in [n]$,

$$\partial_{n+1} \sum_{j=1}^t (-1)^j \Gamma_j^\beta(\tau) + \sum_{j=1}^t (-1)^j \Gamma_j^\beta \partial_n(\tau) = (-1)^t \Gamma_t^\beta \sum_{j=1}^t (-1)^j (f_j^- - f_j^+)(\tau).$$

(iv) For $t = n \geq 2$,

$$\partial_{n+1} \sum_{j=1}^n (-1)^j \Gamma_j^\beta(\tau) + \sum_{j=1}^{n-1} (-1)^j \Gamma_j^\beta \partial_n(\tau) = 0.$$

Proof. [Corollary 3.3\(i\)](#) follows by adding the term $\Gamma_t^\beta\partial_n(\tau)$ to both sides of [Theorem 3.1\(iii\)](#). [Corollary 3.3\(i\)](#) is a special case of [Corollary 3.3\(ii\)](#) without the first sum on the right hand side. Using alternating summation, [Corollary 3.3\(iii\)](#) follows from [Corollary 3.3\(ii\)](#). Finally, [Corollary 3.3\(iv\)](#) is the case $t = n$ of [Corollary 3.3\(iii\)](#). □

Corollary 3.4. Let $\theta = \Gamma_t^\beta(\tau)$ where $\tau \in K_{n-1}$, $t \in [n-1]$ and $\beta \in \{+, -\}$. The following equations are true.

(i)

$$\partial_{n+1}\Gamma_t^\beta(\theta) + \Gamma_t^\beta\partial_n(\theta) = (-1)^{t+1}\beta\theta + 2\Gamma_t^\beta\Gamma_{t-1}^\beta \sum_{i=1}^{t-1}(-1)^i(f_i^- - f_i^*)(\tau).$$

(ii)

$$\partial_{n+1} \sum_{j=1}^t (-1)^j \Gamma_j^\beta(\theta) + \sum_{j=1}^t (-1)^j \Gamma_j^\beta \partial_n(\theta) = -\beta\theta + (-1)^t \Gamma_t^\beta \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^*)(\tau).$$

Proof. We know from Corollary 3.3(ii) that

$$\partial_{n+1}\Gamma_t^\beta(\theta) + \Gamma_t^\beta\partial_n(\theta) = \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^*)(\theta) + \Gamma_t^\beta \sum_{i=1}^t (-1)^i (f_i^- - f_i^*)(\theta).$$

By Definition 2.2(iii), the coset $\Gamma_t^\beta[(-1)^t(f_t^- - f_t^*)(\theta)] = (-1)^{t+1}\beta\theta$ and $(f_i^- - f_i^*)(\Gamma_t^\beta(\tau)) = \Gamma_{t-1}^\beta((f_i^- - f_i^*)(\tau))$. Thus

$$\begin{aligned} \partial_{n+1}\Gamma_t^\beta(\theta) + \Gamma_t^\beta\partial_n(\theta) &= (-1)^{t+1}\beta\theta + (\Gamma_{t-1}^\beta + \Gamma_t^\beta)\Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^*)(\tau) \\ &= (-1)^{t+1}\beta\theta + 2\Gamma_t^\beta\Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^*)(\tau), \end{aligned}$$

since $\Gamma_{t-1}^\beta\Gamma_{t-1}^\beta = \Gamma_t^\beta\Gamma_{t-1}^\beta$. This concludes the proof of Corollary 3.4(i). Now Corollary 3.4(ii) follows directly from Corollary 3.3(iii), namely,

$$\begin{aligned} \partial_{n+1} \sum_{j=1}^t (-1)^j \Gamma_j^\beta(\theta) + \sum_{j=1}^t (-1)^j \Gamma_j^\beta \partial_n(\theta) &= (-1)^t \Gamma_t^\beta \sum_{j=1}^t (-1)^j (f_j^- - f_j^*)(\Gamma_t^\beta(\tau)) \\ &= -\beta\theta + (-1)^t \Gamma_t^\beta \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^*)(\tau). \end{aligned}$$

□

Lemma 3.5. Let $\theta = \Gamma_t^\beta(\tau)$, for some $\tau \in K_{n-1}$, $t \in [n-1]$ and $\beta \in \{+, -\}$. Then

$$\partial_{n+1}[(-1)^{t+1}\Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta](\theta) + [(-1)^{t+1}\Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\alpha] \partial_n(\theta) = \beta\theta.$$

Proof. Follows directly from Corollary 3.4. By multiplying the equation from Corollary 3.4(i) by $(-1)^t$ and subtracting from that twice the equation from Corollary 3.4(ii),

we get

$$\partial_{n+1}[(-1)^t \Gamma_t^\beta - 2 \sum_{j=1}^t (-1)^j \Gamma_j^\beta](\theta) + [(-1)^t \Gamma_k^\beta - 2 \sum_{j=1}^t (-1)^j \Gamma_j^\alpha] \partial_n(\theta) = \beta \theta.$$

Hence

$$\partial_{n+1}[(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta](\theta) + [(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\alpha] \partial_n(\theta) = \beta \theta.$$

□

Remark 3.6. Notice that it is possible for a singular n -cube θ which is a connection to be written using different connection maps, say $\theta = \Gamma_t^\alpha(\sigma) = \Gamma_s^\beta(\tau)$ for some $t, s \in [n]$, $\alpha, \beta \in \{+, -\}$, and $\sigma, \tau \in K_{n-1}$.

If $s = t$ or $s = t + 1$, however, then either $\beta = \alpha$ (and hence $\sigma = \tau$) or θ is degenerate. Thus if θ is a non-degenerate singular n -cube that is a connection, then θ can be written uniquely as $\theta = \Gamma_t^\alpha(\sigma)$ where t is the smallest such index. The following lemma follows.

Lemma 3.7. *Let θ be a non-degenerate connection n -cube, and suppose that $\theta = \Gamma_t^\alpha(\sigma) = \Gamma_s^\beta(\tau)$ where $t \leq s$. Then either*

- (i) $s = t$ or $s = t + 1$, and hence $\sigma = \tau$ and $\alpha = \beta$, or
- (ii) $s > t + 1$, and in this case $\theta = \Gamma_t^\alpha(\Gamma_{s-1}^\beta(\delta)) = \Gamma_s^\beta(\Gamma_t^\alpha(\delta))$ where $\delta = f_t^\alpha(\tau) = f_{t+1}^\alpha(\tau) = f_s^\beta(\sigma) = f_{s-1}^\beta(\sigma)$.

For the rest of this section, whenever we write a non-degenerate connection n -cube θ as $\theta = \Gamma_t^\beta(\tau)$ we assume t is the smallest index for which such a representation exists.

Let $\theta = \Gamma_t^\beta(\tau)$ be a non-degenerate connection. Define

$$\phi_n(\theta) = \beta \left[(-1)^{t+1} \Gamma_t^\beta(\theta) - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta(\theta) \right].$$

The map ϕ_n extends linearly to a map $\phi_n : \text{Con}_n(K) \longrightarrow \text{Con}_{n+1}(K)$ such that

$$\phi_n \left(\sum_{j=1}^s r_{ij} \Gamma_{ij}^{\beta_j}(\sigma_j) \right) = \sum_{j=1}^s r_{ij} \phi_n(\Gamma_{ij}^{\beta_j}(\sigma_j)).$$

Theorem 3.8. *For any $\theta \in \text{Con}_n(K)$,*

$$\partial_{n+1} \phi_n(\theta) + \phi_{n-1} \partial_n(\theta) = \theta.$$

Proof. Recall that $\text{Con}_n(K)$ is freely generated by the cosets of $\theta = \Gamma_t^\beta(\tau)$ where τ is non-degenerate and $\beta = +, -$. Using [Lemma 3.5](#), to conclude the proof we just need to show that

$$\phi_{n-1} \partial_n(\theta) = \beta \left[(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\alpha \right] \partial_n(\theta).$$

Recall that

$$\partial_n(\theta) = \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) - \Gamma_t^\beta \sum_{i=k+1}^{n-1} (-1)^i (f_i^- - f_i^+)(\tau).$$

Now

$$\begin{aligned} \phi_{n-1} \partial_n(\theta) &= \phi_{n-1} \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) - \phi_{n-1} \Gamma_t^\beta \sum_{i=t+1}^{n-1} (-1)^i (f_i^- - f_i^+)(\tau) \\ &= \beta \left[(-1)^t \Gamma_{t-1}^\beta - 2 \sum_{j=1}^{t-2} (-1)^j \Gamma_j^\beta \right] \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) \\ &\quad - \beta \left[(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta \right] \Gamma_t^\beta \sum_{i=t+1}^{n-1} (-1)^i (f_i^- - f_i^+)(\tau) \\ &= \beta \left[(-1)^{t+1} \Gamma_{t-1}^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta \right] g \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) \\ &\quad - \beta \left[(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta \right] \Gamma_t^\beta \sum_{i=t+1}^{n-1} (-1)^i (f_i^- - f_i^+)(\tau) \\ &= \beta \left[(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta \right] \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+)(\tau) \\ &\quad - \beta \left[(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta \right] \Gamma_t^\beta \sum_{i=t+1}^{n-1} (-1)^i (f_i^- - f_i^+)(\tau) \\ &= \beta \left[(-1)^{t+1} \Gamma_t^\beta - 2 \sum_{j=1}^{t-1} (-1)^j \Gamma_j^\beta \right] \partial_n(\theta). \end{aligned}$$

□

Corollary 3.9. *The map ϕ is a chain homotopy between the identity and zero chain maps. In particular, we have $\mathcal{H}_n(\text{Con}(K)) = 0$ for all n .*

Corollary 3.10. *The short exact sequence of chain complexes*

$$0 \longrightarrow \text{Con}_n(K) \hookrightarrow \mathcal{C}_{n+1}(K) \twoheadrightarrow \mathcal{C}_{n+1}(K)/\text{Con}_n(K) \longrightarrow 0$$

induces a long exact sequence of homology groups, and since $\mathcal{H}_n(\text{Con}(K))$ is trivial, we have $\mathcal{H}_n(\mathcal{C}(K)) \cong \mathcal{H}_n(\mathcal{C}(K)/\text{Con}(K))$.

It is well-known that, over a suitable category, the category of chain complexes and the category of crossed complexes are equivalent [9]. It would be interesting to see whether the results in this paper can be properly stated and extended to the context of crossed complexes.

APPENDIX: HOMOLOGY OF CUBICAL SETS AND HOMOLOGY OF THEIR GEOMETRIC
REALIZATION

Recall that I^n is the geometric n -dimensional cube $[0, 1]^n$. Let $(f_i^\alpha)^* : I^{n-1} \rightarrow I^n$ be the map sending $(x_1, \dots, x_{n-1}) \in I^{n-1}$ to $(x_1, \dots, x_{i-1}, y, x_i, \dots, x_{n-1})$ where $y = 0$ if $\alpha = -$ and $y = 1$ if $\alpha = +$. Let further $(\varepsilon_i)^* : I^n \rightarrow I^{n-1}$ be the map sending (x_1, \dots, x_n) to $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. The geometric realization $|K|$ of a cubical set is the quotient space of the disjoint union $\coprod I^n \times K_n$ by the equivalence relation \sim , which is generated by the following elementary equivalences: For $(x_1, \dots, x_n) \in I^n$ and $\sigma \in K_{n-1}$ we set

$$(1) \quad ((x_1, \dots, x_n), \varepsilon_i(\sigma)) \sim ((\varepsilon_i)^*((x_1, \dots, x_n))), \sigma)$$

and, for $(x_1, \dots, x_{n-1}) \in I^{n-1}$ and $\sigma \in K_n$, we set

$$(2) \quad ((x_1, \dots, x_{n-1}), f_i^\alpha(\sigma)) \sim ((f_i^\alpha)^*((x_1, \dots, x_{n-1}))), \sigma).$$

Then $|K|$ can be given the structure of a CW-complex whose (open) n -cells are the images $e_\sigma^{(n)}$ of the cells $\overset{\circ}{I}{}^n \times \{\sigma\}$ in $|K|$ for $\sigma \in K_n^{nd}$. Here K_n^{nd} denotes the set of non-degenerate n -cubes in K , see [10, Remark 11.1.14]. Let $S(K)$ be the cellular chain complex of $|K|$. By the definition of $S(K)$ the cells $e_\sigma^{(n)}$ for $\sigma \in K_n^{nd}$ form a basis of its n^{th} chain group $S_n(K)$. It is well known (see [13, Corollary 3.9.11]) that identifying $\sigma \in K_n^{nd}$ with $e_\sigma^{(n)}$ yields the following isomorphism of chain complexes.

Lemma 3.11 (Corollary 3.9.11 [13]). $\mathcal{C}(K) \cong S(K)$.

If the cubical set K is a cubical set with connections then there is an associated geometric realization $|K|'$ which is the quotient of the disjoint union $\coprod I^n \times K_n$ by the equivalence relation \sim' , which is generated by (1), (2) and the relation

$$(3) \quad ((\Gamma_i^\alpha)^*((x_1, \dots, x_n))), \sigma) \sim' ((x_1, \dots, x_n), \Gamma_i^\alpha(\sigma))$$

for $\sigma \in K_{n-1}$ and $(x_1, \dots, x_n) \in I^n$. Here $(\Gamma_i^\alpha)^* : I^n \rightarrow I^{n-1}$ is defined by

$$(\Gamma_i^\alpha)^*((x_1, \dots, x_n)) = \begin{cases} (x_1, \dots, x_{i-1}, \max(x_i, x_{i+1}), x_{i+2}, \dots, x_n) & \text{if } \alpha = - \\ (x_1, \dots, x_{i-1}, \min(x_i, x_{i+1}), x_{i+2}, \dots, x_n) & \text{if } \alpha = + \end{cases}.$$

In particular, \sim' is coarser than \sim and hence $|K|'$ can be seen as a quotient of $|K|$ by the additional identifications implied by (3). Let K_n^{ndc} be the set of n -cubes in K that are neither degenerate nor connections.

In order to understand the relation between $|K|$ and $|K|'$ we need to understand the face structure of cubes in K_n^{ndc} . For that we consider for any cube $\sigma \in K_n$ the set of all of its faces τ ; i.e. all cubes τ such that $\tau = f_{i_1}^{\alpha_1}(\dots(f_{i_r}^{\alpha_r}(\sigma))\dots)$ for a choice of i_1, \dots, i_r and $\alpha_1, \dots, \alpha_r$. For $\sigma \in K$ we denote by F_σ the set of its faces. We order the cubes from K by saying that τ is smaller than σ if τ is a face of σ . With this notation we are in position to formulate the following structural result on the role of non-degenerate and non-connection cubes in the face structure.

Lemma 3.12. *For any $\tau \in K_n$ there is a unique face ρ of τ that is maximal with the property that it is neither degenerate nor a connection. Moreover, if $\tau = \varepsilon_i(\sigma)$ or $\tau = \Gamma_i^\alpha(\sigma)$ then ρ is a subspace of σ and $\tau = g_k \cdots g_1(\rho)$ for suitably chosen connection and degeneracy maps g_1, \dots, g_k for some $k \geq 0$.*

Proof. We prove the assertion by induction on the dimension n .

If $n = 0$ then τ is non-degenerate and non-connection. Hence τ itself is the maximal face we are looking for.

Let $n > 0$. If τ is neither degenerate nor a connection then again τ itself is the unique maximal face.

Let τ be degenerate, say $\tau = \varepsilon_i(\sigma)$ for some $i \in [n]$ and some $(n - 1)$ -cube σ . Then, by (iii) of [Definition 2.1](#), $f_j^\alpha(\tau) = \sigma$ if $i = j$, and $f_j^\alpha(\tau) = \varepsilon_{i-1}(f_j^\alpha(\sigma))$ if $j < i$ and $= \varepsilon_i(f_{j-1}^\alpha(\sigma))$ if $j > i$. By induction, we know that there is an unique maximal non-degenerate and non-connection face ρ of σ . We claim that ρ is the unique maximal non-degenerate and non-connection face of τ . By induction we know that each $\varepsilon_r(f_s^\beta(\sigma))$ has a unique maximal non-degenerate, non-connection face which is a subspace of $f_s^\beta(\sigma)$ and hence of σ . In particular, they must be subspaces of ρ . It follows by induction that $\sigma = g_k \cdots g_1 \rho$ for a sequence of degeneracy and connection maps g_1, \dots, g_k and $k \geq 0$. Then $\tau = \varepsilon_i g_k \cdots g_1 \rho$.

Finally, consider the case that τ is a connection. Say $\tau = \Gamma_i^\alpha(\sigma)$ for some $i \in [n]$ and some $(n - 1)$ -cube σ . Notice that, by (iii) of [Definition 2.2](#), every $(n - 1)$ -face of τ other than σ is either $\Gamma_j^\beta(f_t^\alpha(\sigma))$ or $\varepsilon_j(f_t^\alpha(\sigma))$ for some $j \in [n]$, $t \in [n]$, and $\alpha \in \{+, -\}$. By induction σ and any $\Gamma_j^\beta(f_t^\alpha(\sigma))$ have an unique maximal non-degenerate, non-connection face. Again by induction the latter are subspaces of σ . In particular, they must be subspaces of the unique maximal non-degenerate, non-connection face ρ of σ . From the induction hypothesis it follows $\sigma = g_k \cdots g_1 \rho$ for a sequence of degeneracy and connection maps g_1, \dots, g_k and $k \geq 0$. Then $\tau = \Gamma_i^\alpha g_k \cdots g_1 \rho$. \square

Note that along the same lines one can show that for any cube there is a unique maximal non-degenerate face.

The relations among the degeneracy and connection maps allow the following strengthening of [Lemma 3.12](#).

Lemma 3.13. *For any $\tau \in K_n$ there is a unique face ρ of τ that is maximal with the property that it is neither degenerate nor a connection. Moreover, if τ is non-degenerate then $\tau = g_k \cdots g_1(\rho)$ for suitably chosen connection maps g_1, \dots, g_k and some $k \geq 0$.*

Proof. From [Lemma 3.12](#) it follows that there is a unique maximal face ρ of τ that is neither degenerate nor a connection. It also follows from that lemma that $\tau = g_k \cdots g_1 \rho$, for degeneracy and connection maps g_1, \dots, g_k . If all g_i are connection maps we are done. Assume there is an i such that g_i is a degeneracy map. We claim that then τ is degenerate. We prove the claim by downward induction on the maximal i such that g_i is a degeneracy map. If $i = k$ then τ is degenerate, contradicting the assumptions. If $i < k$ then by [Definition 2.2](#)(iii) there is a connection or degeneracy map g'_i and a degeneracy map g'_{i+1} such that

$$\tau = g_k \cdots g_{i+2} g'_{i+1} g'_i g_{i-1} \cdots g_1 \rho.$$

By induction this implies that τ is degenerate. \square

Now we apply the results on the face structure in order to understand the attachment of cells in $|K|$ and $|K'|$. We assume without stating the proofs the following fact:

- Let $(x, \sigma), (y, \sigma) \in I^{\dim \sigma} \times \{\sigma\}$. Then $(x, \sigma), (y, \sigma)$ are identified through the equivalence relation generated by (1), (2) (resp. (1), (2) and (3)) on $\coprod_{\tau \in K} I^{\dim \tau} \times \{\tau\}$ if and only if they are identified by the equivalence relation generated by (1), (2) (resp. (1), (2) and (3)). on $\coprod_{\tau \in F_\sigma} I^{\dim \tau} \times \{\tau\}$.

This fact allows us to consider the identifications by the equivalence relations we consider as local identifications among points in the cells corresponding to the faces of a given cell.

Lemma 3.14. *Let $\tau \in K_n$ be such that $\tau = g_k \cdots g_1 \rho$ for some cube ρ and connection maps g_1, \dots, g_k . Let \sim_τ be the restriction of the equivalence relation generated by (1), (2), (3) to $M_\tau = \coprod_{\sigma \in F_\tau} I^{\dim \sigma} \times \{\sigma\}$ and define \sim_ρ analogously. Then there is a retraction $p_\tau : M_\tau / \sim_\tau \rightarrow M_\rho / \sim_\rho$.*

Proof. We construct the retraction by induction on k . For $k = 0$ the identity is the desired retraction.

Let $k \geq 1$ and assume that for $\tau' = g_{k-1} \cdots g_1 \rho$ there is such a retraction $p_{\tau'} : M_{\tau'} / \sim_{\tau'} \rightarrow M_\rho / \sim_\rho$. Then $\tau = g_k \tau'$. The equivalence relation on $I^{\dim \tau} \times \{\tau\}$ induced by the connection map $g_k = \Gamma_i^\beta$ has equivalence classes being sets with fixed maximum or minimum of the i^{th} and $(i+1)^{\text{st}}$ coordinate depending on β being + or -. Each equivalence class has exactly two points that via the face maps f_i^β and f_{i+1}^β are identified with points in $I^{\dim \tau'} \times \{\tau'\}$, indeed both points are identified with the same point. The map that sends each equivalence class to the image of this point in $M_{\tau'} / \sim_{\tau'}$ provides a retraction from M_τ / \sim_τ to $M_{\tau'} / \sim_{\tau'}$. Composing this retraction with the retraction from $p_{\tau'}$ provides the asserted retraction. This concludes the induction step. \square

We now introduce the concept of pushing cells for a general CW-complex which we will then match with the process of passing from $|K|$ to $|K'|$ in our case. Let X be a CW-complex where, for $n \geq 0$, $X_n = (e_\sigma^{(n)})_{\sigma \in J_n}$ is the set of open n -cells in X for some indexing set J_n . For each $\sigma \in J_n$ let $g_\sigma : \partial e_\sigma^{(n)} \rightarrow X^{(n-1)}$ be the attaching map. For some fixed $N \geq 0$, let $\bar{J}_N \subseteq J_N$ be a subset of the index set of the cells in dimension N such that, for each $\sigma \in \bar{J}_N$,

- there is a $\tau \in J_\ell$ for some $\ell < N$ such that $\overline{\text{Img}}_\sigma \subseteq \overline{e_\tau^{(\ell)}}$, and
- for this τ there is a retraction $p_\sigma : e_\sigma^{(N)} \rightarrow \overline{e_\tau^{(\ell)}}$.

Now let X^{push} be the CW-complex with $X_n^{\text{push}} = (\tilde{e}_\sigma^{(n)})_{\sigma \in J'_n}$ the open n -cells in X^{push} where $J'_n = J_n$ for $n \neq N$ and $J'_N = J_N \setminus \bar{J}_N$ and attaching maps $g'_\tau(x) = g_\tau(x)$ if $g_\tau(x) \notin \overline{e_\sigma^{(N)}}$ for some $\sigma \in \bar{J}_N$ and $g'_\sigma(x) = p_\tau(g_\sigma(x))$ otherwise. In this situation we say that X^{push} arises from X by pushing the cells $e_\sigma^{(N)}$ for $\sigma \in \bar{J}_N$.

Next we show that $|K|$ and $|K'|$ are examples of CW-complexes that arise from each other by pushing cells.

Lemma 3.15. *The geometric realization $|K|'$ is a CW-complex that arises from the CW-complex of the geometric realization $|K|$ by pushing the cells corresponding to connections successively by dimension in increasing order. In particular, $|K|'$ can be given the structure of a CW-complex with n -cells indexed by the K_n^{ndc} .*

Proof. Since the first connection cells (that are not already degenerate) arise in dimension 2, we can assume the following situation. For some $n \geq 2$ we have constructed a complex X such that

- (a) X arises from $|K|$ by pushing all cells that correspond to connections of dimensions $< n$ where $n \geq 2$.
- (b) $|K|/\sim_n \cong X$ where \sim_n is the equivalence relation which has singleton equivalence classes outside the closure of the cells of dimension $< n$ and equals (3) when applied to the union of the closures of all other cells.

Now let $\sigma \in K_n$ be a connection that is non-degenerate. Then by Lemma 3.12 there is a unique maximal face $\tau \in K_\ell$ of σ which is non-degenerate and non-connection. Since all proper connection faces of σ have been pushed the attaching map g_σ of the N -cell I^N corresponding to σ has as its image the ℓ -cell corresponding to τ . Furthermore, by Lemma 3.13 the conditions of Lemma 3.14 are satisfied and there is a retraction p_σ from then closure of the N -cell corresponding to σ to the closure of the ℓ -cell corresponding to σ . Moreover, by Lemma 3.14 the map σ identifies the exactly those elements which lie in the same equivalence class of \sim_n .

Hence the conditions for a pushing to the cells corresponding to non-degenerate connections σ are satisfied. It follows that (a) and (b) are satisfied for \sim_n . \square

Finally, we need to understand the impact of pushing cells on the cellular chain complex of a CW-complex.

Lemma 3.16. *Let X be a CW-complex with cells $X_n = (e_\sigma^{(n)})_{\sigma \in J_n}$, $n \geq 0$. Assume that there is a dimension N such that X^{push} arises from X by pushing the cells $e_\sigma^{(N)}$ for $\sigma \in \bar{J}_N \subseteq J_N$. Let*

$$\partial e_\sigma^{(n)} = \sum_{\sigma' \in J_{n-1}} d_{\sigma, \sigma'} e_{\sigma'}^{(n-1)}$$

be the differential of the cellular chain complex associated to X . Then for $\sigma \in J_n \setminus J_N$, $\sigma' \in J_{n-1} \setminus J_N$ the coefficient $d_{\sigma, \sigma'}^{\text{push}}$ in the differential of the cellular chain complex of X^{push} we have $d_{\sigma, \sigma'}^{\text{push}} = d_{\sigma, \sigma'}$.

Proof. The coefficient $d_{\sigma, \sigma'}$ is given as the degree of the composition

$$S^{n-1} \cong \partial \overline{e^{(n)}} \xrightarrow{g_\sigma} X^{(n-1)} \rightarrow X^{(n-1)} / (X^{(n-1)} \setminus e_{\sigma'}^{(n-1)}) \cong S^{n-1}.$$

The composition depends on the attaching maps g_σ of the cells corresponding to σ only. Now consider the same sequence in X^{push} , which in particular implies $\sigma, \sigma' \neq \tau$. Let g'_σ be the corresponding attaching maps. If $g_\sigma(x) \notin \overline{e_\tau^{(N)}}$ for some $\tau \in \bar{J}_N$ then $g_\sigma(x) = g'_\sigma(x)$. If $g_\sigma(x) \in \overline{e_\tau^{(N)}}$ for some $\tau \in \bar{J}_N$ then $g'_\sigma(x) = p_\sigma(g_\sigma(x))$ for a retraction p_σ . But in the latter

case $g_\sigma(x)$ and $g'_\sigma(x)$ lie in the complement of any $(n - 1)$ cell different from $e_\tau^{(N)}$. In that situation the composition is again determined by g_σ . It follows that $d_{\sigma,\sigma'} = d_{\sigma,\sigma'}^{\text{push}}$. \square

By definition $\mathcal{C}_n(K)/\text{Con}_n(K)$ has a basis indexed by K_n^{ndc} . The differential of the complex $\mathcal{C}_n(K)/\text{Con}_n(K)$ arises from the differential in $\mathcal{C}(K)$ in the following way. Let $\partial\alpha$ is the differential of $\alpha \in K_n^{\text{ndc}}$ in $\mathcal{C}_n(K)$ then we set all coefficients of element from $K_{n-1}^{\text{nd}} \setminus K_{n-1}^{\text{ndc}}$ to 0. Now the following theorem is an immediate consequence of [Lemma 3.16](#) and [Lemma 3.15](#).

Theorem 3.17. *The cellular chain complex $S'(K)$ of $|K|'$ is isomorphic to the quotient complex $\mathcal{C}(K)/\text{Con}(K)$. In particular,*

$$H_i(|K|') \cong H_i(S'(K)) \cong H_i(\mathcal{C}(K)/\text{Con}(K)).$$

Proof. The assertion follows immediately from [Lemma 3.15](#) and [Lemma 3.16](#). \square

The theorem together with [Corollary 3.10](#) implies the following.

Corollary 3.18. *Let K be a cubical set with connections. Then*

$$H_i(|K|') \cong H_i(S'(K)) \cong H_i(\mathcal{C}(K)/\text{Con}(K)) \cong H_i(\mathcal{C}(K)) \cong H_i(|K|).$$

This fact provides another motivation for the study of connections.

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REFERENCES

- [1] F. A. AL-AGL, R. BROWN, AND R. STEINER, *Multiple categories: The equivalence of a globular and a cubical approach*, Advances in Mathematics, 170 (2002), pp. 71 – 118. [3](#)
- [2] R. ANTOLINI, *Geometric realisations of cubical sets with connections, and classifying spaces of categories*, Applied Categorical Structures, 10 (2002), pp. 481–494. [2](#)
- [3] E. BABSON, H. BARCELO, M. DE LONGUEVILLE, AND R. LAUBENBACHER, *Homotopy theory of graphs*, Journal of Algebraic Combinatorics, 24 (2006), pp. 31–44. [4](#)
- [4] H. BARCELO, V. CAPRARO, AND J. A. WHITE, *Discrete homology theory for metric spaces*, Bull. London Math. Soc., 46 (2014), pp. 889–905. [3, 4](#)
- [5] H. BARCELO, C. GREENE, A. S. JARRAH, AND V. WALKER, *Discrete cubical and path homologies of graphs*, Algebraic Combinatorics, (2018). to appear, arXiv:1803.07497. [2, 4, 5](#)
- [6] R. BROWN, *Modelling and computing homotopy types: I*, Indagationes Mathematicae, 29 (2018), pp. 459 – 482. [1](#)
- [7] R. BROWN AND P. J. HIGGINS, *The equivalence of ω -groupoids and cubical T-complexes*, Cahiers de Topologie et Géométrie Différentielle Catégorique, 22 (1981), pp. 349–370. [1](#)
- [8] R. BROWN AND P. J. HIGGINS, *On the algebra of cubes*, Journal of Pure and Applied Algebra, 21 (1981), pp. 233 – 260. [1](#)
- [9] R. BROWN AND P. J. HIGGINS, *Cubical abelian groups with connections are equivalent to chain complexes*, Homology Homotopy Appl., 5 (2003), pp. 49–52. [1, 9](#)
- [10] R. BROWN, P. J. HIGGINS, AND R. SIVERA, *Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids*, EMS Tracts in Mathematics, European Mathematical Society, August 2011. [1, 3, 4, 10](#)

- [11] R. BROWN AND C. SPENCER, *Double groupoids and crossed modules*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 17 (1974), pp. 343–362. [1](#)
- [12] S. EILENBERG AND S. MACLANE, *Acyclic models*, Amer. J. Math, 75 (1953), pp. 189–199. [1](#), [4](#)
- [13] H. FEDERER, *Lectures in Algebraic Topology*, Brown University, Providence, R.I., 1962. [1](#), [10](#)
- [14] M. GRANDIS AND L. MAURI, *Cubical sets and their site*, Theory Appl. Categ., 11 (2003), pp. 185–211. [1](#)
- [15] D. M. KAN, *Abstract homotopy I*, Proceedings of the National Academy of Sciences, 41 (1955), pp. 1092–1096. [1](#), [2](#), [3](#)
- [16] G. MALTSINIOTIS, *La catégorie cubique avec connexions est une catégorie test stricte*, Homology Homotopy Appl., 11 (2009), pp. 309–326. [1](#)
- [17] W. MASSEY, *Singular Homology Theory*, vol. 70 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1980. [1](#), [3](#), [4](#)
- [18] J.-P. SERRE, *Homologie singulière des espaces fibres*, Annals of Mathematics, 54 (1951), pp. 425–505. [1](#)
- [19] E. H. SPANIER, *Algebraic Topology*, McGraw Hill, New York, N.Y., 1966. [4](#)
- [20] A. TONKS, *Cubical groups which are Kan*, Journal of Pure and Applied Algebra, 81 (1992), pp. 83 – 87. [1](#)

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