Data Structure and Algorithm HW1

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1. What if you became a DSA TA?

Proof of conjecture 1

Consider $f(n) \geq 0$ and $g(n) \geq 0$ that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, where $c \in R$
To prove $\lim_{n\to\infty} \frac{f(n)}{g(n)} \geq 0$ for all $n \in R$, we can consider h(n) = 0
Since $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \in R$, $\frac{f(n)}{g(n)} \geq h(n)$ for all n
therefore,

$$\lim_{n\to\infty} h(n) \geq \lim_{n\to\infty} \frac{f(n)}{g(n)}$$

$$\lim_{n\to\infty} h(n) = 0 \Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = c \geq 0$$

By difinition of limit, there exists $\epsilon > 0$, $n_0 > 0$ such that for all $n > n_0$,

$$|\frac{f(n)}{g(n)} - c| < \epsilon$$

$$-\epsilon < \frac{f(n)}{g(n)} - c < \epsilon$$

$$-\epsilon + c < \frac{f(n)}{g(n)} < \epsilon + c$$

In case that $0 < c < \epsilon$

$$-\epsilon + c < 0 < \frac{f(n)}{g(n)} < \epsilon + c$$

In case that $c \geq \epsilon$

$$0 \leq -\epsilon + c < \frac{f(n)}{g(n)} < \epsilon + c$$

$$0 \leq f(n) < (\epsilon + c)g(n)$$

Let $(\epsilon + c) = c_1$

$$0 \leq f(n) \leq c_1 g(n)$$

which shows $f(n) = O(g(n))$, QED

1.1 - all by myself

Consider $c_1, c_2 > 0$ for all $n > 0$
if

$$\frac{c_1}{c_2} n \geq 1$$

then

$$c_1 n^3 \geq c_2 n^2$$

let

$$n_0 = \frac{c_2}{c_1}$$

$$\frac{c_1}{c_2} n_0 = 1$$

consider $k > 0$

$$\frac{c_1}{c_2}(n_0 + k) = 1 + \frac{c_1}{c_2} k$$

where

$$\frac{c_1}{c_2} k > 0$$

that is, for all $n \geq n_0$,

$$0 \leq c_2 n^2 \leq c_1 n^3$$

therefore, if

$$0 \leq f(x) \leq c_3 n^2$$

for all $n > n_1$ where $n_1, c_3 > 0$
then

$$0 \leq f(n) \leq c_3 n^2 \leq \frac{c_3 c_1}{c_2} n^3$$

for all $n > max(n_0, n_1)$, QED

1-2 - all by myself

Let $t_m = the\ number\ of\ while\ checks$, $k = the\ index\ of\ key$, $d_n = timecostofeachlinetoexecuteonce$
Since the value of m is assigned to l in the beginning of each while loop, m will start from 1 and increase by 1 when while loop is executed once, moreover, $A[m] \leq key$ will always be satisfied.
therefore,

$$t_m = k$$

in worst case, key is not in the array, in this case

$$T(n) = d_1 n + d_2(n-1) + d_3(n-1) + d_4 * 0 + d_5(n-1) + d_6 * 0 + d_7(n-1) + d_8(n-1) + d_9$$

$$T(n) = c_1 n + c_2$$

where c_1, $c_2 > 0$

$$\lim_{n \to \infty} \frac{c_1 n + c_2}{n} = c_1$$

By conjecture 1, time complexity of the algorithm is $O(n)$

1-3

ref: `https://math.stackexchange.com/questions/925053/using-limits-to-determine-big-o-big-omega-and-big-theta#comment6149810_925053`

$$f(n) = \Theta(n^2) \iff \textit{there exists positive } (n_0, c_1, c_2) \textit{ such that } c_1 n^2 \leq f(n) \leq c_2 n^2 \textit{ for all } n \geq n_0$$

Let $c_1 = 1$, $c_2 = 10$ and $f(n) = n^2(sin(n) + 2) + n$
for all $n > 0$, $n^2 \leq f(n) \leq 10n^2$
Suppose

$$\lim_{n\to\infty} f(n) = L$$

where $L \in R$, then there exist $\epsilon > 0$, $n > 0$ such that for all $n > n_0$,

$$|f(n) - L| < \epsilon$$

Let $f(n_1) = \epsilon_1$ where $n_1 > n_0$ and $\epsilon_1 < \epsilon$

$$f(n_1 + 2\pi k) - f(n_1) = ((n_1 + 2\pi k)^2 - n_1^2)(sin(n_1) + 2) + 2\pi k$$

where $k \in n$

$$\lim_{k\to\infty} f(n_1 + 2\pi k) - f(n_1) \neq 0$$

$$\Rightarrow |f(n + 2\pi k) - L| > \epsilon$$

for some $k > k_0$ where $k_0 \in n$
That is, $\lim_{n\to\infty} \frac{f(n)}{n^2}$does not exists, therefore, the proposition is WRONG.