Exam: Stochastic Processes

Date and Time: Thursday January 5, 2012, 9:00–13:00.

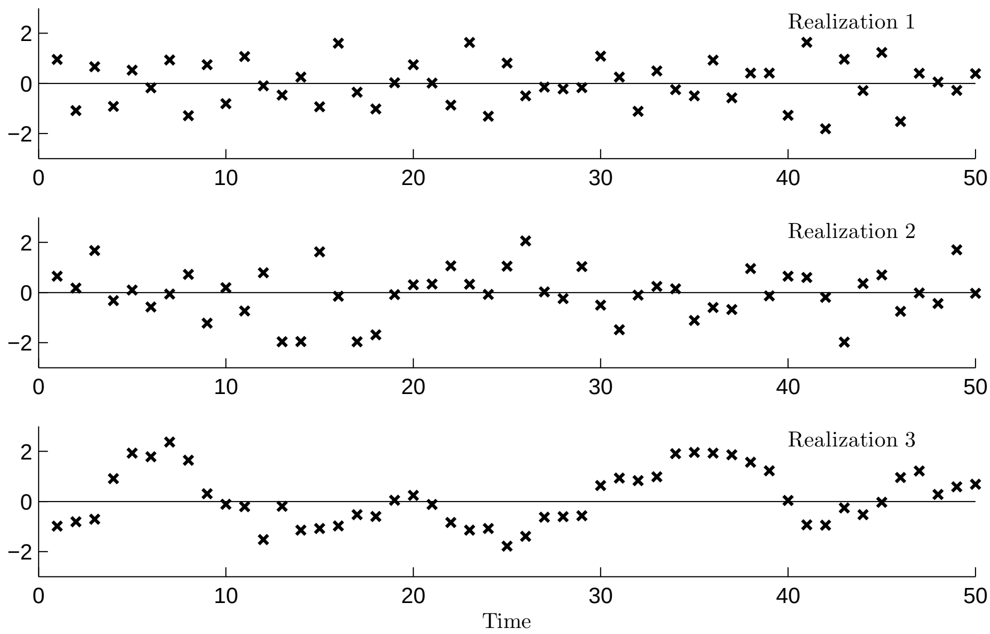
This entire problem set contains **6 pages** (including this cover page). Please make sure that you have received all pages.

The overall problem set is divided into 4 main problems each subdivided into 5-6 smaller problems. All subproblems are assigned the same number of points.

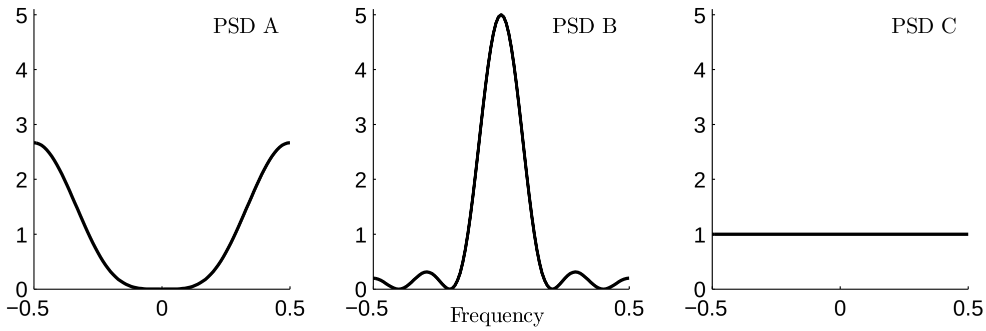
For each problem, we strongly advise to spend the time necessary to thoroughly read *and* understand the details before starting to solve it. Should you happen to get stuck at some point, we recommend that you continue and anyway try to solve the rest. You always have the opportunity to sketch or explain how you would have continued if you haven't got stuck.

Problem 1:
Random processes, stationarity and power spectral densities

Realizations of three different discrete-time random processes are displayed in the figure below. All three processes are wide-sense stationary (WSS) with zero-mean and unit variance.



Power spectral densities (PSD's) corresponding to three different discrete-time WSS processes are shown in the figure below.

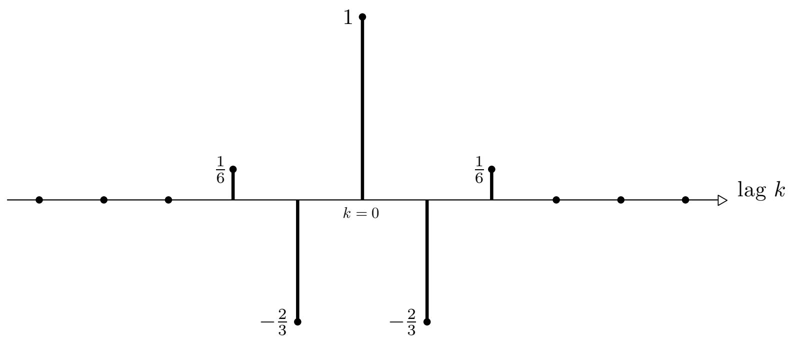


a) Match each of the three realizations to one of the three PSD's. For each pair, explain why you have grouped together that particular realization with that particular PSD.

b) It can be identified directly from PSD A, PSD B and PSD C, that all three of them correspond to processes with zero mean. Explain how.

This problem continues on the following page.

c) Which of the power spectral densities A, B or C corresponds to the autocorrelation function shown in the figure below? Justify your answer.



Now, let $\{X_n\}$ be a discrete-time WSS random process. Then, define a new random process $\{Y_n\}$ with individual samples generated via some transformation of the samples of $\{X_n\}$, i.e.

$$Y_n = \mathcal{T}\big(\ldots, X_{n-1}, X_n, X_{n+1}, \ldots\big), \qquad n \in \mathbb{Z},$$

where $\mathcal{T}$ denotes the transformation.

d) If $\mathcal{T}$ is such that $Y_n = cX_n$ for some fixed constant c, show that $\{Y_n\}$ is WSS.

e) Find a transformation $\mathcal{T}$ such that the process $\{Y_n\}$ is *not* WSS. Show or argue why $\{Y_n\}$ is not WSS with your particular choice of transformation.

f) Give a specific example of a random process which is strict-sense stationary (SSS). Justify that it is impossible for a random process to be SSS without being WSS.

Problem 2:

Poisson random process, compound Poisson process, and thinning

Every day customers arrive at a certain petrol station to fuel up their vehicles. To model this scenario we make the following assumptions which are motivated by feedback from the manager of the petrol station:

- customers arrive according to a homogeneous Poisson process
- on average, 8 customers arrive per hour
- any given customer needs a random amount of fuel which is uniformly distributed between 30 and 50 liters.

a) What is the probability that no customers show up for refueling during the first half of any one-hour period?

b) What is the probability that no customers show up for refueling during the first *and* the last quarter of any one-hour period? Comment on the relationship with part a).

c) How many liters of fuel are expected to be sold in total during the first *and* the last quarter of any one-hour period? If we know that no fuel were sold at all during the first quarter, should we then expect twice the amount of fuel to be sold during the last quarter? Why or why not?

The manager of the petrol station is eager to verify the above calculations by computer simulation.

d) Write a pseudo-code for simulating a 24-hour realization of the arrival times of refueling customers. Describe how this code can be used to verify the calculations in part c).

As a consequence of the financial crisis, the petrol station's manager suddenly decides to double the fuel prices. On average, every fourth arriving customer now finds the fuel prices too high and drives on without refueling (hoping to find cheaper fuel somewhere else).

e) Taking the new behavior of the customers into account, state the probability mass function (pmf) for the number of customers who still refuels during a one-hour period.

Problem 3:

The bivariate Gaussian distribution and LMMSE estimation

Consider the two random variables

$$X_1 = Y + N_1 \qquad \text{and} \qquad X_2 = Y + N_2,$$

where N_1 and N_2 are jointly Gaussian distributed with zero mean and associated covariance matrix given by

$$\Sigma_N = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Furthermore, N_1 and N_2 are both independent of the random variable $Y \sim \mathcal{N}(0,1)$.

a) State the joint probability density function (pdf) of N_1 and N_2 and determine the correlation coefficient between N_1 and N_2.

b) State the joint pdf of X_1 and X_2 and calculate the variance of the difference $X_2 - X_1$.

c) Find the linear minimum mean-squared error (LMMSE) estimator for Y based on a joint observation of X_1 and X_2:

$$\hat{Y} = h_0 + \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

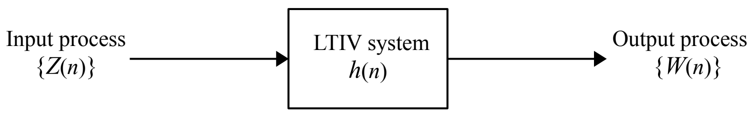
d) Given that $X_1 = -2$ and $X_2 = 1$, compute the associated LMMSE estimate $\hat{Y}$. Explain whether you find it reasonable that the coefficient h_1 has a smaller magnitude than the coefficient h_2. Also, is it reasonable that h_0 is zero? Why or why not?

e) Calculate the mean-squared error $\mathbb{E}\big[(Y - \hat{Y})^2\big]$.

Problem 4:

Fitting MA and AR models to a WSS random process

Consider the linear time-invariant (LTIV) system



where $\{Z(n)\}$ is a discrete-time white Gaussian process with unit variance and the system's impulse response reads

$$h(n) = -\frac{1}{2}\delta(n+1) + \delta(n) - \frac{1}{2}\delta(n-1),$$

with δ denoting the Kronecker delta. Notice that the above LTIV system is non-causal.

a) Show that the output process $\{W(n)\}$ is a Gaussian process, that it has zero mean, and that its autocorrelation function reads

$$R_{WW}(k) = \begin{cases} \frac{3}{2} & k = 0 \\ -1 & |k| = 1 \\ \frac{1}{4} & |k| = 2 \\ 0 & |k| \geq 3. \end{cases}$$

Hint: Determine $R_{hh}(k)$ from a convolution involving the impulse response $h(n)$. Sketch $h(n)$ and do the convolution graphically to avoid too many calculations.

We now wish to find a simple causal model having an autocorrelation function as similar to $R_{WW}(k)$ as possible. We consider the following two candidate models:

Model 1:	$X(n) = \theta Z(n-1) + Z(n)$
Model 2:	$Y(n) = \phi Y(n-1) + Z(n),$

where θ and ϕ are fixed parameters to be determined and $\{Z(n)\}$ is the same process as defined in the figure above.

b) What types of discrete linear stochastic models are the processes $\{X(n)\}$ and $\{Y(n)\}$, respectively? Describe the main structural difference between these two models.

c) For Model 1, determine the two possible values of θ such that $R_{XX}(0) = R_{WW}(0)$. Sketch for both of these values the resulting $R_{XX}(k)$ for $|k| \leq 3$. Which of the two values of θ do you find the most suitable for matching the autocorrelation properties of $\{W(n)\}$? Explain why.

d) For Model 2, find the two values of ϕ satisfying the equation $R_{YY}(0) = R_{WW}(0)$, and sketch for both values the resulting $R_{YY}(k)$ for $|k| \leq 3$. Argue and explain which value of ϕ you find the most suitable for matching the autocorrelation properties of $\{W(n)\}$.

e) What is the common interpretation of the areas under the graphs of the power spectral densities $S_{WW}(f)$, $S_{XX}(f)$ and $S_{YY}(f)$? How are these three areas related and why?