Stochastic Processes

Session 9 — Lecture

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Outline for Session 9 — Lecture

Linear time-invariant system with WSS input

1st and 2nd order characterization

Summary of results

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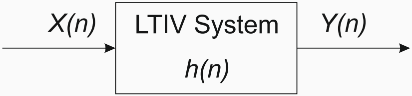
Linear time-invariant system with WSS input

We consider a *Linear time-invariant (LTIV) system* with input $\{X(n)\}$ and output $\{Y(n)\}$:



From linear system theory, we know that the input-output relation of a LTIV system is in the form of a *convolution* with an *impulse response* $h(n)$:

$$Y(n) = \sum_{k=-\infty}^{\infty} h(k)X(n-k) = h(n) \star X(n)$$

We consider for a weak sense stationary (WSS) $\{X(n)\}$ with mean μ_X and autocorrelation $R_X(k)$, the questions:

- How is the mean of $\{Y(n)\}$ related to properties of $\{X(n)\}$?
- What is the autocorrelation of $\{Y(n)\}$— how does it relate to properties of $\{X(n)\}$?
- Is $\{Y(n)\}$ WSS?
- If so, what can we say about the PSD?

Convolution and Fourier Relations

Knowledge of linear systems and Fourier transforms are prerequisites for the course. In this lecture, we make use of the following:

Convolution is a linear operator defined as:

$$g(n) \star h(n) := \sum_{k=-\infty}^{\infty} g(k)h(n-k) = \sum_{i=-\infty}^{\infty} g(n-i)h(i) = h(n) \star g(n)$$

Discrete Fourier transform is a linear operator defined as:

$$\mathcal{F}\{g(n)\}(f) := \sum_{n=-\infty}^{+\infty} g(n)e^{-j2\pi nf} = G(f).$$

Fourier transforms of convolutions simplify due to the *Convolution Theorem*:

$$\mathcal{F}\{g(n) \star h(n)\}(f) = G(f) \cdot H(f)$$

Fourier transforms of signals flipped in time *(Time Reversal Identity)*:

$$\mathcal{F}\{g(-n)\}(f) = G^*(f)$$

where * denotes complex conjugation.

In combination, the convolution theorem and the time reversal identity, give

$$\mathcal{F}\{g(n) \star g(-n)\}(f) = G(f) \cdot G^*(f) = |G(f)|^2$$

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Mean function of $\{Y(n)\}$

We compute the mean function of the output $\{Y(n)\}$ as[1]

$$\begin{aligned}\mu_Y := \mathbb{E}[Y(n)] &= \mathbb{E}[h(n) \star X(n)] \\ &= \mathbb{E}\left[\sum_{j=-\infty}^{\infty} h(j)X(n-j)\right] \\ &= \sum_j h(j)\mathbb{E}[X(n-j)] = \mu_X \sum_j h(j).\end{aligned}$$

The mean of $\{Y(n)\}$ is constant and can be related to the mean of $\{X(n)\}$ and the impulse response $h(n)$.

It is sometimes more convenient to write this relation in terms of the systems *transfer function* $H(f) = \mathcal{F}\{h(n)\}(f)$:

$$\mu_Y = H(0) \cdot \mu_X$$

Check this relation!

[1] If not otherwise mentioned, the summations are from $-\infty$ to $+\infty$.

Autocorrelation function for $\{Y(n)\}$ — Part 1 of 3

The derivation of ACF the output process is tedious but straightforward: Insert $\{Y(n)\}$ in the definition of autocorrelation and then simplify (carefully).

Since $\{Y(n)\}$ could be nonstationary, we use the general form of the ACF:

$$\begin{aligned}
R_Y(n, n+k) &= \mathbb{E}[Y(n)Y(n+k)] \\
&= \mathbb{E}[Y(n)\sum_j h(j)X(n+k-j)] \\
&= \sum_j h(j)\underbrace{\mathbb{E}[Y(n)X(n+k-j)]}_{=:R_{YX}(n,n+k-j)} \\
&= \sum_j h(j)R_{YX}(n, n+k-j) = h(k) \star R_{YX}(n, n+k)
\end{aligned}$$

where it still remains to derive the *cross-correlation function* $R_{YX}(n, n+k)$.

Autocorrelation function for $\{Y(n)\}$ — Part 2 of 3

We now derive an expression for the *cross-correlation function* $R_{YX}(n, n+k)$:

$$\begin{aligned} R_{YX}(n, n+k) &= \mathbb{E}[Y(n)X(n+k)] \\ &= \mathbb{E}\left[\sum_j h(j)X(n-j)X(n+k)\right] \\ &= \sum_j h(j)\underbrace{\mathbb{E}\left[X(n-j)X(n+k)\right]}_{=R_X(k+j)} \\ &= \sum_j h(j)R_X(k+j) \end{aligned}$$

This is a function of k only, and thus we write $R_{YX}(n, n+k) = R_{YX}(k)$. Then the cross-correlation can be written is the form

$$R_{YX}(k) = h(-k) \star R_X(k)$$

Autocorrelation function for $\{Y(n)\}$ — Part 3 of 3

Combining the pieces of the derivation of the ACF we finally achieve:

$$\begin{aligned} R_Y(n, n+k) &= h(k) \star R_{YX}(n, n+k) \\ &= h(k) \star R_{YX}(k) \\ &= h(k) \star h(-k) \star R_X(k). \end{aligned}$$

Since it has constant mean and its ACF is a function of k only, *the output process is WSS* with autocorrelation

$$R_Y(k) = h(k) \star h(-k) \star R_X(k).$$

Remark: The symmetric and positive semidefinite function $h(k) \star h(-k)$ is some-times called the "autocorrelation of the impulse response $h(k)$" and denoted by $R_h(k)$. This is because

$$h(k) \star h(-k) = \sum_j h(j)h(j+k).$$

The term is used for historical reasons, but is strictly speaking a misnomer, since it is no in accordance with the definition of ACF.

Power spectral density of the output process

Since the output process is WSS, its power spectral density is well defined. Starting from the autocorrelation of $\{Y(n)\}$

$$R_Y(k) = h(k) \star h(-k) \star R_X(k)$$

we achieve by Fourier transforming and using the convolution theorem and time reversal identity the relation:

$$S_Y(f) = H(f) \cdot H^*(f) \cdot S_X(f) = |H(f)|^2 S_X(f).$$

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Summary of results: 1st and 2nd order input-output relation

	Input	Output
Signal	$X(n)$	$Y(n) = h(n) \star X(n)$
Mean	μ_X	$\mu_Y = \mu_X \sum_n h(n) = H(0)\mu_X$
ACF	$R_X(k)$	$R_Y(k) = h(k) \star h(-k) \star R_X(k)$
WSS	Yes	Yes
PSD	$S_X(f)$	$S_Y(f) = \|H(f)\|^2 S_X(f)$

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Example: MA(1) process with zero mean

As a first example, we analyse the MA(1) process defined as

$$Y(n) = X(n) + X(n-1), \qquad \{X(n)\} \overset{iid.}{\sim} \mathcal{N}(0,3)$$

Thus, $X(n)$ is a white Gaussian process with ACF $R_X(k) = 3\delta(k)$.

We can write the output process as $Y(n) = h(n) \star X(n)$ with impulse response $h(n) = \delta(n) + \delta(n-1)$.

Mean: $\mathbb{E}[Y(n)] = \mathbb{E}[X(n)] \cdot (h(0) + h(1)) = 0 \cdot 2 = 0$
ACF: By noting that $h(k) \star h(-k) = \delta(k+1) + 2\delta(k) + 1\delta(k-1)$ we have the ACF:

$$R_Y(k) = \Big[\delta(k+1) + 2\delta(k) + \delta(k-1)\Big] \cdot 3 = 3\delta(k+1) + 6\delta(k) + 3\delta(k-1)$$

PSD : Note that $X(f)$ has PSD $S_X(f) = 3$. The transfer function reads $H(f) = 1 + \exp(-j2\pi f)$ and thus the PSD of $Y(n)$ is

$$S_Y(f) = |1 + \exp(-j2\pi f)|^2 3 = 6 \cdot (1 + \cos(2\pi f))$$

Example: MA(1) process with non-zero mean

As a second example, we analyse the MA(1) process defined as

$$Y(n) = X(n) + X(n-1), \qquad \{X(n)\} \overset{iid.}{\sim} \mathcal{N}(3, 15)$$

Here, the ACF of $X(n)$ is $R_X(k) = 9 + 15\delta(k)$ and we observe that $Y(n) = h(n) \star X(n)$ for $h(n) = \delta(n) + \delta(n-1)$.

Mean: $\mathbb{E}[Y(n)] = \mathbb{E}[X(n)] \cdot (h(0) + h(1)) = 3 \cdot 2 = 6$

ACF: Since $h(k) \star h(-k) = \delta(k+1) + 2\delta(k) + 1\delta(k-1)$, we have

$$\begin{aligned} R_Y(k) &= (9 + 15\delta(k+1)) + 2 \cdot (9 + 15\delta(k)) + (9 + 15\delta(k-1)) \\ &= 36 + 15\delta(k+1) + 30\delta(k) + 15\delta(k-1). \end{aligned}$$

PSD: The PSD for $X(f)$ is $S_X(f) = 15 + 9\delta(f)$ and the transfer function reads $H(f) = 1 + \exp(-j2\pi f)$, thus the PSD for $Y(n)$ is [2]

$$\begin{aligned} S_Y(f) &= |1 + \exp(-j\pi f)|^2 (15 + 9\delta(f)) \\ &= 2 \cdot (1 + \cos(2\pi f)) \cdot (15 + 9\delta(f)) \\ &= 30 \cdot (1 + \cos(2\pi f)) + 36\delta(f) \end{aligned}$$

[2] The Dirac impulse $36\delta(f)$ is due to the non-zero mean of $Y(n)$. The coefficient 36 equals the squared mean.