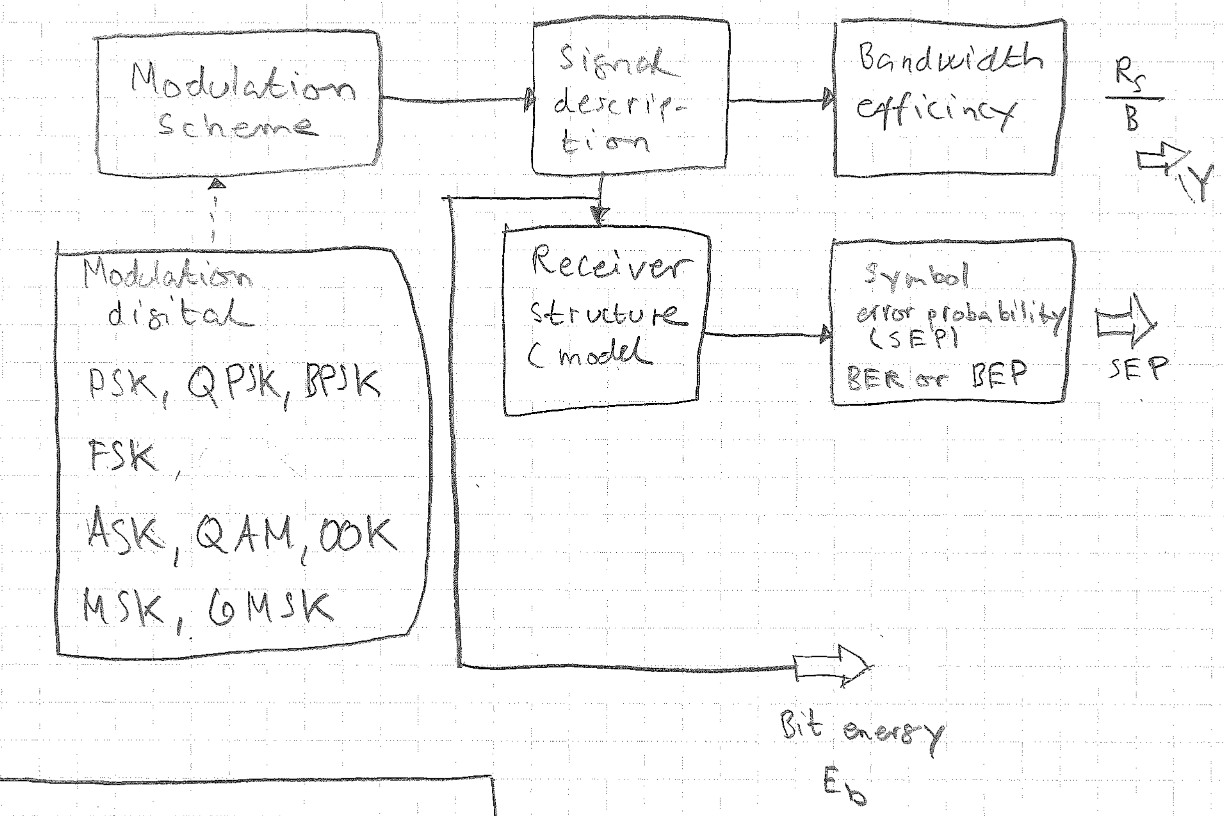
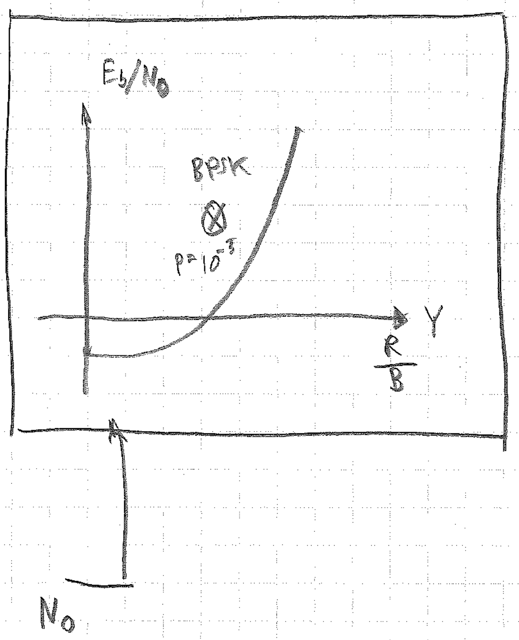
MM 6 - Channel Capacity

Channel capacity





Signal description
(random)

Thermal noise:

$$f_N(x) = \frac{1}{\sqrt{2\pi\sigma_N^2}} \cdot e^{-\frac{1}{2}\frac{(x-\mu_N)^2}{\sigma_N^2}}$$

For thermal noise $\mu_N = 0$

Abbreviation for normal distribution:

$$f_N(x) = N[x; \mu_N, \sigma_N^2]$$

The »standard normal« distribution:

$$N[x; 0, 1] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Auto covariance

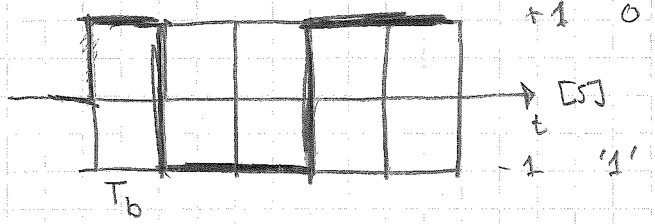
$$R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)] \quad [V^2]$$

The Power Spectral Density (PSD) is found by the Fourier transform of $R_{XX}(\tau)$:

$$G_X(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-2\pi f\tau} d\tau \quad [V^2/Hz]$$

$$[V^2 \cdot 1 \cdot s = \tfrac{1}{Hz}]$$

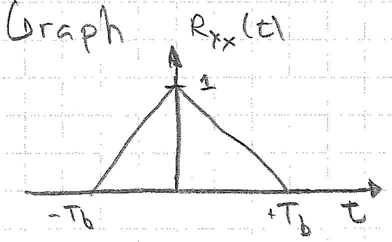
Spectrum for a random binary wave

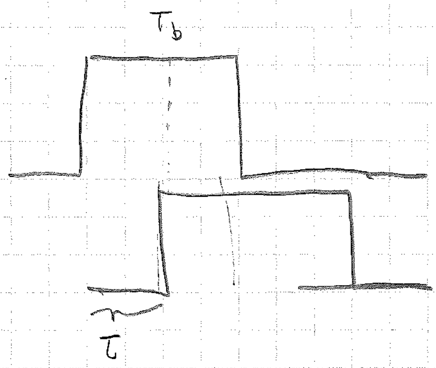


The autocovariance is found as:

$$R_{XX}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_b}, & \text{for } |\tau| \le T_b \\ 0, & \text{for } |\tau| \ge T_b \end{cases}$$

Graph



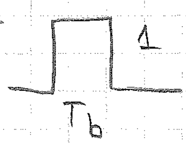


The PSD is calculated as:

$$G_X(f) = T_b \cdot \left(\frac{\sin(\pi f \tau)}{\pi f \tau} \right)^2$$

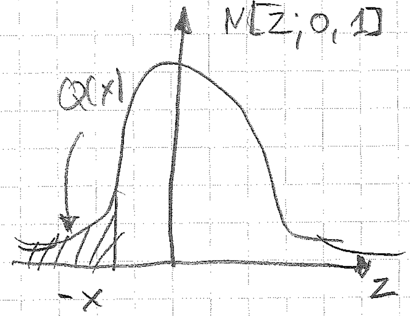
$$= T_b \cdot \left(\text{sinc}(f\tau) \right)^2$$

$$= \frac{1}{T_b} \cdot \left(T_b \cdot \text{sinc}(f\tau) \right)^2$$ Fourier of



95% of the power is found thus:

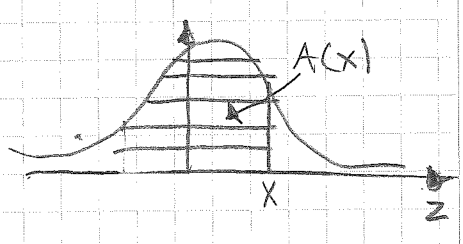
$$B_{95\%} = 1.5 \cdot \frac{1}{T_b} \, [Hz]$$



Useful functions

The »Gaussian Tail« function:

$$Q_{(x)} \triangleq \int_{-\infty}^{x} N[z; 0, 1] \, dz$$



The »Guassian Area« function:

$$A(x) \triangleq \int_{-\infty}^{x} N[z; 0, 1]$$

Formulas:

(The cdf)

$$Q(-x) = A(x)$$

$$Q(-x) = 1 - Q(x)$$

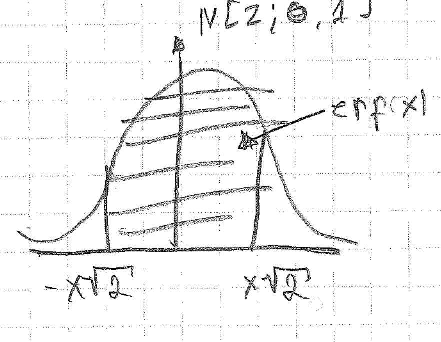
$$A(-x) = 1 - A(x)$$

$$A\left(\frac{x-\mu}{\sigma_x}\right) = F(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}}$$

Error Function

$$\mathrm{erf}(x) \triangleq 2 \cdot \int_{0}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2} dz = \int_{-x}^{x} N(z; 0, 1/2) \, dz$$

$$= \int_{-x\sqrt{2}}^{x\sqrt{2}} N[z; 0, 1] \, dz$$



Complementary error function:

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

Formulas

$$\mathrm{erf}(-x) = -\mathrm{erf}(x)$$

$$\mathrm{erf}(x) = 2A(x\sqrt{2}) - 1$$

$$A(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$

$$= 1 - \frac{1}{2}\mathrm{erfc}(x/\sqrt{2})$$

BPSK modulation

Signals: (coherent)

$$s_1 = \sqrt{\frac{2E_b}{T_b}} \cdot \cos(w_c t) = A_c \cdot \cos(wt)$$

$$s_2 = \sqrt{\frac{2E_b}{T_b}} \cdot \cos(w_c t + \pi) = -A_c \cdot \cos(wt)$$

Calculating the bit energy:

Power: $P = 1/2\, A_c^2$

Energy: $U = P \cdot T_b$

$$= 1/2\, A_c^2 \cdot T_b = 1/2\, \frac{2E_b}{T_b} \cdot T_b = E_b$$

New description:

$$S_1 = \sqrt{E_b} \cdot \phi_1(t)$$

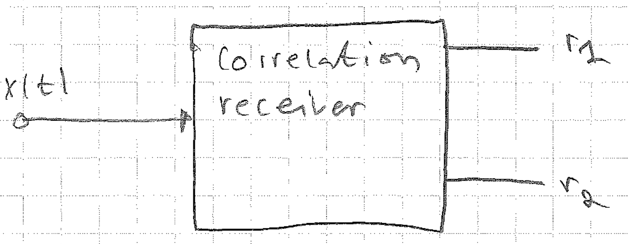
$$S_2 = -\sqrt{E_b} \cdot \phi_1(t)$$

Where:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cdot \cos(w_c t)$$

Base function

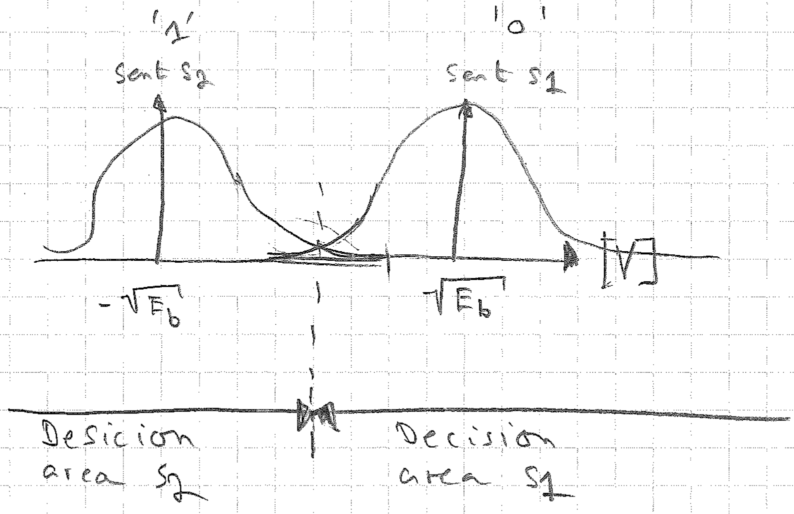
Correlation receiver:

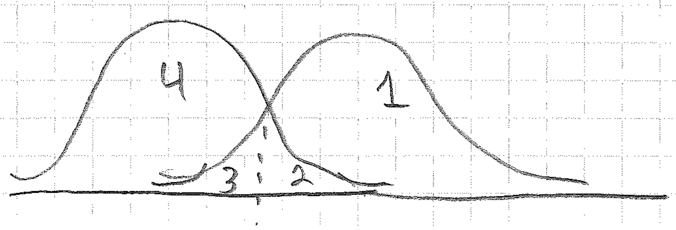


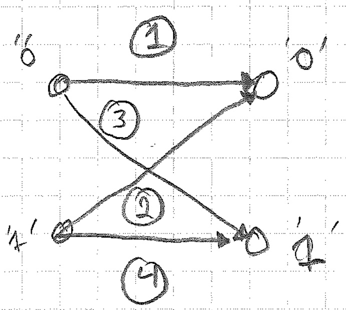
$$r_1: \int_0^{T_b} s_1(t)\,\phi_1(t)\,dt = \int_0^{T_b} \sqrt{E_b}\,\phi_1^2(t)\,dt = \sqrt{E_b}$$

$$r_2: \ldots\ldots = -\sqrt{E_b}$$

Detection







$$SEP = A\left(\frac{x-\mu_N}{\sigma_N}\right)$$

$$= A\left(\frac{-\sqrt{E_b}}{\sqrt{\frac{N_0}{2}}}\right) = 1 - A\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$\Rightarrow$

$$SEP = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$