Exam: Stochastic Processes

Date and Time: Monday January 6, 2014, 9:00–13:00.

This entire problem set contains **5 pages**. Please make sure that you have received all pages.

The exam is graded according to your answer as a whole: both quantity and quality count. We value concise arguments showing your command of the topics. Simply answering “Yes.” or “No.” will not do that!

We recommend that you read through each problem thoroughly before starting to solve it. Should you happen to get stuck at some point, we recommend that you continue and anyway try to solve the rest. You always have the opportunity to sketch or explain how you would have continued if you hadn’t got stuck.

It is allowed to use books, lecture notes, your own notes, calculators and computers during the exam. Communication to others during the exam is not allowed—therefore, the use of internet is strictly forbidden.

Problem 1:

The process $U(n)$, $n = 1, 2, \ldots$, is a strict-sense stationary (SSS) random process, of which we know the following information:

$$\begin{aligned} \mathbb{E}[U(10)] &= 2, \\ \mathbb{V}\mathrm{ar}[U(10)] &= 3, \\ \mathbb{E}[U(10)U(15)U(22)] &= 8. \end{aligned}$$

a) Using the above information and the properties of SSS processes, compute the following quantities. In your computations, indicate those steps in which you make use of the SSS property of $U(n)$:

- $\mathbb{E}[U(2)]$.
- $\mathbb{V}\mathrm{ar}[2U(37)]$.
- $\mathbb{E}[U^2(5)]$.
- $\mathbb{E}[U(2)U(7)U(14)]$.

b) Is $U(n)$ a wide-sense stationary (WSS) process?

Now, assume that $X(n)$ and $Y(n)$, $n = 1, 2, \ldots$, are two WSS processes with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2 and autocorrelation functions $R_X(k)$ and $R_Y(k)$, respectively. Furthermore, let $\{X(n)\}$ and $\{Y(n)\}$ be mutually independent. Based on $X(n)$ and $Y(n)$, we define the following random processes:

$$\begin{aligned} W(n) &= X(n)Y(n), \qquad n = 1, 2, \ldots, \\ Z(n) &= nX(n) - Y(n), \qquad n = 1, 2, \ldots. \end{aligned}$$

c) Compute the mean and variance functions of $W(n)$ and $Z(n)$ as a function the mean, variance and autocorrelation functions of $X(n)$ and $Y(n)$.

d) Compute the autocorrelation functions of $W(n)$ and $Z(n)$ as a function the mean, variance and autocorrelation functions of $X(n)$ and $Y(n)$.

e) Are the processes $W(n)$ and $Z(n)$ also WSS? Justify your answer.

The power spectral densities (PSD) $S_X(f)$ and $S_Y(f)$ of the processes $X(n)$ and $Y(n)$ are shown in Figure 1.

f) Write and/or sketch the autocorrelation function of $X(n)$. Random processes having a PSD and autocorrelation function such as that of $X(n)$ belong to a special type of processes. How are such processes usually called?

g) If we know that the area under the PSD $S_Y(f)$ is equal to that under $S_X(f)$, what can we say about the autocorrelation function of $Y(n)$?

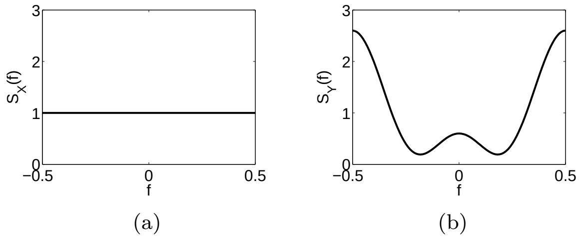


Figure 1: PSDs of (a) $X(n)$ and (b) $Y(n)$

Problem 2:

The random process $X(n)$, $n = 1, 2, \ldots$, is a white process with PSD

$$S_X(f) = 3, \qquad f \in (-0.5, 0.5]$$

which is input to a linear, time-invariant system with impulse response

$$h(n) = \delta(n) + a\delta(n-1) + b\delta(n-2).$$

The parameters $a, b \in \mathbb{R}$ can be tuned by an engineer to obtain an output process $Y(n)$ with the desired statistical properties.

a) Write the output random process $Y(n)$ in terms of the samples of the input process $X(n)$. What type of process is $Y(n)$ and what are its parameters?

b) Can you describe, at least, two different ways of calculating the autocorrelation function of $Y(n)$? Choose one of them and compute it.

c) Calculate the PSD of the output process $Y(n)$.

d) Assume that we want the output process $Y(n)$ to have an autocorrelation function fulfilling:

$$R_Y(1) = 2.55,$$
$$R_Y(2) = 2.1.$$

Find the values to which a and b should be set in order to obtain the desired outcome.

e) Write some pseudo-code (or MATLAB code) to generate multiple realizations of the process $Y(n)$. Use the values calculated in the previous item for a and b.

Next, assume that a different random process, $U(n)$, $n = 1, 2, \ldots$, is the input to the same system. The process $U(n)$ is a WSS process with mean $\mu_X = 2$.

f) Find new values for a and b so that the new output process is zero-mean and has no power at $f = 0.5$.

Problem 3:

A technician wants to measure the voltage level in an electric power line. Unfortunately, the measurement device he possesses is old and of bad quality, and it introduces a large amount of noise in the measurements taken. Desperate for a solution, the technician decides to ask his engineer colleague for help. The engineer proposes to digitize the signal at the output of the measurement device and apply digital signal processing techniques to mitigate the effects of the noise. He formulates the following model for the samples of the measured signal:

$$Y(n) = A\cos\left(\frac{\pi}{2}n\right) + W(n), \qquad n = 0, 1, 2, \ldots$$

where the noise samples $W(n)$ are modelled as a Gaussian i.i.d. process with zero mean and variance $\sigma_W^2 = 1$. The amplitude of the voltage signal is assumed to be uniformly distributed between $\pm 2\sqrt{3}$, i.e. $A \sim \mathcal{U}(-2\sqrt{3}, 2\sqrt{3})$. Furthermore, the voltage amplitude A and the noise samples $W(n)$ are assumed to be statistically independent.

Based on the above observation model, the engineer suggests to use a linear estimator based on the first 3 observed samples $Y(0), Y(1), Y(2)$, i.e. an estimator of the form

$$\hat{A} = b + \sum_{n=0}^{2} a_n Y(n). \tag{1}$$

Next, imagine that you are in the position of the engineer, and would like to use a linear minimum mean squared error estimator (LMMSEE).

a) What arguments would you use to convince the technician that this is a good estimator to use for this problem? Can you also think of any argument against it?

b) Assume that you managed to convince him. Now, find the values of b, a_0, a_1 and a_2 that need to be used for $\hat{A}$ to be an LMMSEE.[1]

c) Can you find an intuitive explanation for the value obtained for a_1?

Before implementing the estimator in a real-time digital signal processing device, the engineer decides to implement and test the performance of the estimator by computer simulations.

d) Write pseudocode (or MATLAB code) to generate one realization consisting of 100 samples of the process $Y(n)$.

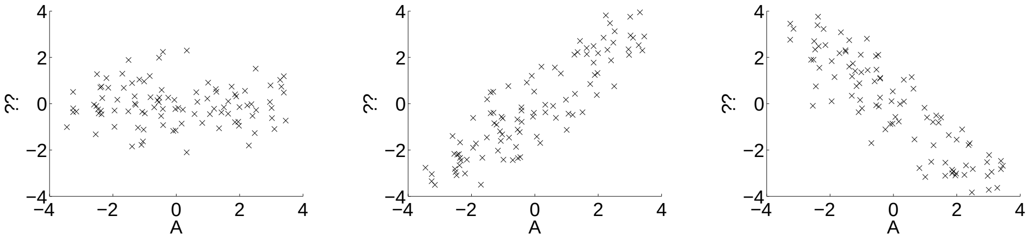
As a first step, the engineer decides to plot multiple realizations of the following pairs of random variables on scatter plots:

- A and $Y(0)$,
- A and $Y(1)$,
- A and $Y(2)$.

The scatter plots obtained are shown in Figure 2.

e) Unfortunately, the engineer forgot to include the label of the y-axis for these plots, and got them all mixed up. Can you identify to which pairs of random variables correspond each of the scatter plots shown in Figure 2? Explain the reasoning behind your choice.

[1]Hint: $\begin{bmatrix} 5 & 0 & -4 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix}^{-1} = \frac{1}{9}\begin{bmatrix} 5 & 0 & 4 \\ 0 & 9 & 0 \\ 4 & 0 & 5 \end{bmatrix}$



(a) (b) (c)

Figure 2: Scatter plots of A and the observations $Y(0)$, $Y(1)$, $Y(2)$.

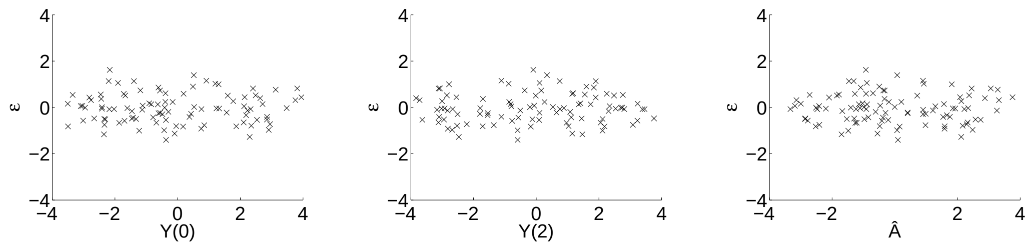
In addition, the engineer has also plotted multiple realizations of the estimation error $\varepsilon = \hat{A} - A$ versus realizations of the observations $Y(0)$, $Y(2)$ and of the estimate $\hat{A}$. The corresponding scatter plots are shown in Figure 3, and this time he did remember to include the labels for both axes.

f) In view of the properties of the LMMSEE, can you find an explanation of the results obtained for the scatter plots shown in Figure 3?

g) Compute the theoretical mean squared error (MSE) of $\hat{A}$.

h) Imagine that you have now derived a new LMMSEE of A using only the observations $Y(0)$ and $Y(2)$. Do you think the MSE of the new estimator will be larger or smaller than that of the estimator $\hat{A}$ in equation (1)? Justify your answer.

i) If you could pick any three samples of $Y(n)$ to compute your estimates of A, which samples would you choose? Why?



(a) (b) (c)

Figure 3: Scatter plots of the estimation error ε and (a) $Y(0)$, (b) $Y(2)$ and (c) $\hat{A}$.