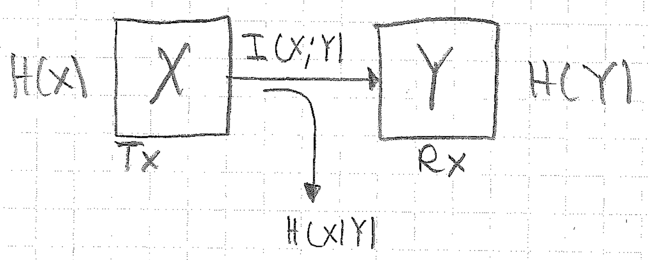
Channel Capacity

Information transfer system



$$I(X;Y) = \underbrace{H(X)}_{\text{sent info}} - \underbrace{H(X|Y)}_{\text{lost info}} \quad [\text{bit/symbol}]$$

Correlation

Full: $H(X|Y) = 0$

zero: $H(X|Y) = H(X)$

It can be shown that:

$$I(X;Y) = I(Y;X) = H(X) - H(X|Y)$$
$$= H(Y) - H(Y|X) \quad [\text{bit/symbol}]$$

The joint entropy:

$$H(X,Y) = -\sum_n \sum_k P(X_k, Y_n) \cdot \mathrm{ld}\, P(Y_k, Y_k) \quad [\text{bit/symbol}]$$

Theorem:

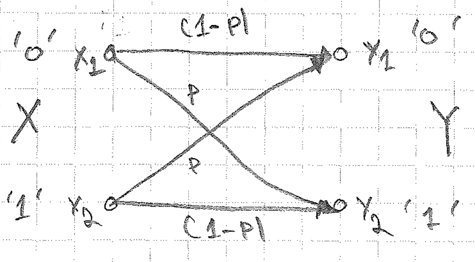
$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$
$$= H(Y|X) + H(X|Y) + I(X;Y) \quad [\text{bit/symbol}]$$

Alternative expression for the mutual information:

$$I(X;Y) = +\sum_n \sum_k P(X_k, Y_k) \cdot \mathrm{ld} \frac{P(X_k, Y_n)}{P(X_k)\, P(Y_n)} \quad [\text{bit/sym}]$$

Channel model

BSC
(Binary symmetric channel)



Symbols are binits or databits

p is BER rather:

BEP or SEP (Probability of error)

Channel matrix

$$\bar{\bar{P}}(Y|X) = \begin{array}{c} \\ X_1 \\ \\ X_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} Y_1 & \qquad\qquad Y_2 \end{array} \\ \begin{bmatrix} P(Y_1|X_1) & P(Y_2|X_1) \\ & \\ \cdot & \cdot \end{bmatrix} \end{array} = \begin{bmatrix} 1-p & p \\ & \\ p & 1-p \end{bmatrix}$$

A sthocastic matrix

The joint matrix:

$P(X,Y) = P(Y|X)\,P(X)$

$$\bar{\bar{P}}(X,Y) = \begin{array}{c} \\ X_1 \\ \\ X_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} Y_1 & \qquad\qquad Y_2 \end{array} \\ \begin{bmatrix} P(Y_1|X_1)\,P(X_1) & P(Y_2|X_1)\,P(X_1) \\ & \\ \cdot & \cdot \end{bmatrix} \end{array} \begin{array}{l} \rightarrow \Sigma = P(X_1) \\ \\ \\ \rightarrow \Sigma = P(X_2) \end{array}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ \Sigma = P(Y_1) & \Sigma = P(Y_2) \end{array}$$

PCC, cont

Now we have $P(X)$, $P(Y|X)$, $P(X,Y)$, $P(Y)$

We can calculate

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} = P(Y|X) \cdot \frac{P(X)}{P(Y)}$$

$$\bar{\bar{P}}(X|Y) = \begin{array}{c|cc} & Y_1 & Y_2 \\ \hline X_1 & P(Y_1|X_1)\,\frac{P(X_1)}{P(Y_1)} & \cdot \\ X_2 & \cdot & \cdot \end{array}$$

Finally we have

$$I(X;Y) = H(X) - H(X|Y) \quad [\text{bit/sym}]$$

or

$$I(X;Y) = H(Y) - H(Y|X) \quad [\text{— '' —}]$$

Information rates

$$\left.\begin{aligned} R_X &= H(X) \cdot r_s \\ R_Y &= H(Y) \cdot r_s \\ R_{XY} &= I(X;Y) \cdot r_s \end{aligned}\right\} [\text{bit/s}]$$

r_s = symbol rate [symb/s]

Channel capacity C

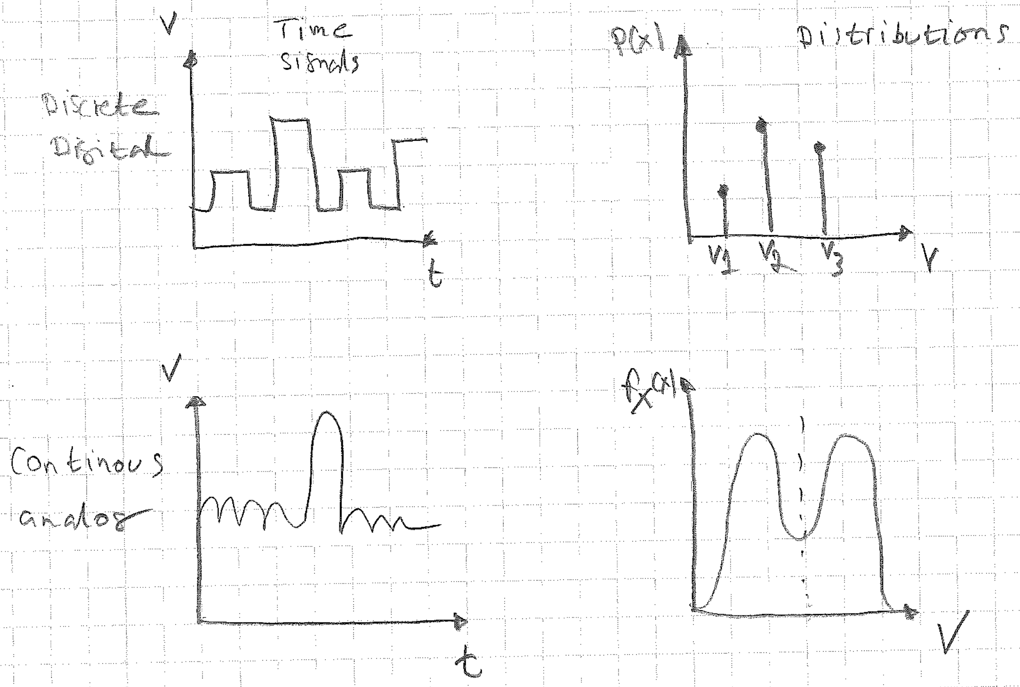
$$C \triangleq \max_{P(x)} [R_{XY}] = \max [I(X;Y) \cdot r_s]$$

$$= \max [I(X;Y)] \cdot \max [r_s] \; [\text{bit/s}]$$

Theorem

If we signal at a rate $\overset{\text{rate}}{R} < \overset{\text{capacity}}{C}$ then an aribitrily small BEP can be achireed.

Entropy of analog systems



Entropy:

$$H(X) = E\left[-\operatorname{ld} f_X(x)\right] = -\int_{-\infty}^{\infty} f_X(x) \cdot \operatorname{ld} f_X(x)\, dx \quad [\text{bit/symb}]$$

Equivacation:

$$H(X|Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X,Y) \cdot ld \frac{f(x,y)}{f_Y(Y)} \text{ [bit/symb]}$$

Mutual information:

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(X,Y)\ ld \cdot \frac{f_{XY}(x,y)}{f_X(x)\, f_Y(Y)} \text{ [bit/symb]}$$

The capacity of an AWGN channel

conditions:

1) The noise additive. (noise added to signal)

2) The noise is gaussian with mean 0. (thermal noise)

3) The signal power is known.

4) The noise power —"— (known how many watts is in the noise)

AWGN = Additive White Gaussian Noise

The capacity is:

$$C = B \cdot \overset{\to \log_2}{ld}\left(1 + \frac{S}{N}\right) \quad [bit/s]$$

where

B [Hz] is the bandwidth

S [W] is the signal power

N [W] —"— noise power

Ex.1

The capacity of a POTS line (Plane Old Telephone Service)

$B \approx 3000 Hz$ (300 - 3400 Hz)

$SNR \approx 1000 Hz$ (30dB)

$$C = B \cdot ld\left(1 + \frac{S}{N}\right) = 3000 \cdot ld(1001) \approx 3000 \cdot 10$$
$$= \underline{\underline{30\ kbit/s}}$$

Normalized capacity

$R_{XY} = I(X:Y)$

We define:

$$\left.\begin{array}{l} \text{Normalized } \overset{=r}{\text{rate}} : \dfrac{R \to \text{Entropy Rate}}{B \to \text{Bandwidth}} \\ \\ \text{Normalized capacity} : \dfrac{C}{B} \end{array}\right\} \left[\frac{bits/s}{Hz}\right]$$

Normalized signal/noise ratio (→ bitwise SNR)

$$\frac{E_b}{N_0} = \frac{S \cdot T_b^{\to \text{bit time}}}{N \cdot \frac{1}{B}} = \frac{S}{N} \cdot \frac{T_b}{\frac{1}{B}} = \frac{S}{N} \cdot \frac{B}{R} = \frac{S}{N} \cdot \frac{1}{r}$$

where:

$N_0 \cdot B = N$ [W]

$T_b = 1/R$ [s]

$r = \frac{R}{B}$ $\left[\frac{bit/s}{Hz}\right]$

We must signal slower than C:

$$R < C$$

$$R < B \cdot ld\left(1 + \frac{S}{N}\right)$$

$$\frac{R}{B} < ld\left(1 + \frac{E_b}{N_0} \cdot r\right)$$

$$r < ld\left(1 + \frac{E_b}{N_0} \cdot r\right)$$

$$2^r < 1 + \frac{E_b}{N_0} \cdot r$$

$$\frac{E_b}{N_0} > \frac{2^r - 1}{r}$$

For $r \to 0$ $\frac{E_b}{N_0} \to \ln 2 = 0.693 \Rightarrow -1.59\,dB$ Less then this no signal is possible

» The Shannon limit «

Normalized Capacity