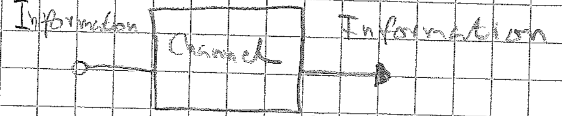
## Error control coding

A digital information transfer system




Capacity $C$ bit/s

Actual rate $R$ bit/s

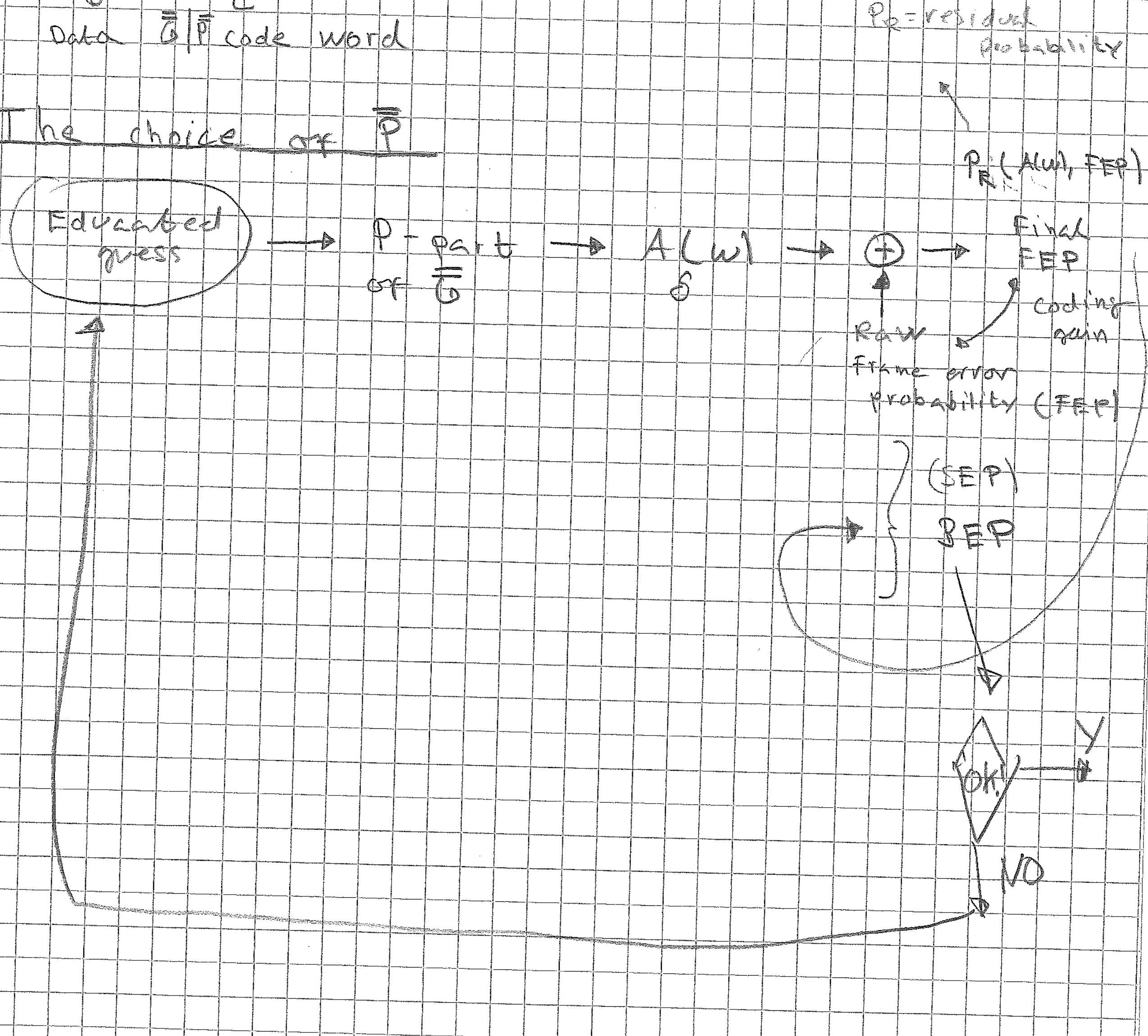
SEP

Error Control Coding ECC

$\overline{D} \longrightarrow \overline{C}$

Data $\overline{G}|\overline{P}$ code word

## The choice of $\overline{P}$



## SEC, Single Error Correcting

Hamming code

Correct 1 error: $\tau = 1$

Hence $\delta \geq 3$ as,

$$\tau = \left\lfloor \frac{\delta - 1}{2} \right\rfloor$$

$(\delta = d_{min})$

## Eks. 1

A (7, 4) code given by:

$$\underset{\substack{4\times 7 \\ k\times n}}{\bar{\bar{G}}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$\underbrace{\phantom{000}}_{P}$ 4x7

By decoding we will use:

$$\underset{n\times r}{\bar{\bar{H}}^T} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} P \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} I-\text{matrix} \end{array} \qquad \begin{array}{c} \text{Decimal} \\ 3 \\ 5 \\ 6 \\ 7 \\ 4 \\ 2 \\ 1 \end{array}$$

7x3

Receiving code words:

Calculation of the syndrome

$$\bar{S} = \bar{R} \cdot \bar{\bar{H}}^T = (\bar{C} + \bar{E}) \cdot \bar{\bar{H}}^T = \bar{E} \cdot \bar{\bar{H}}^T$$

If we assume an error in the $3^{rd}$ bit ( $\bar{E} = 0010.000$ )
We get:

$$\bar{S} = (110) \Rightarrow 3rd \text{ row}$$

Error correction

$$\tilde{C} = \bar{R} - \bar{E} = \bar{C} + \bar{E} - \bar{E} = \bar{C}$$

## DECODING

1. DETECTION
   (calculate syndrome)

2. CORRECTION
   1. Detection
   2. Location

## Requirements for SEC

$\#$ rows in $\bar{H}^T = n$

$\# \bar{S} = 2^r - 1$

We must have:

$$\# \bar{S} \geq \# \text{ rows}$$

$2^r - 1 \geq n$

$2^r \geq (n+1)$

$r \geq ld(n+1)$

$n - k \geq ld(n+1)$

$k \leq n - ld(n+1)$

EKS 2

$(k=8)$

We have 8 data bit. How many check bits are required to be able to correct 1 error

Trials:

$n = 11$ : $n - ld(n+1) = 11 - ld(12) = 11 - 3.6 = 7.4$

$n = 12$ : —— " —— $12 - ld(13) = 12 - 3.7 = 8.3$

So, $n=12$ is OK!, meaning that

$r = n - k = 12 - 8 = 4$

4 check bits to correct one error

Requirements for $\overline{\overline{P}}$

1. All rows must be different.
2. $k \leq n - ld(n+1)$.
3. Rows must hold at least 2 ones. (as $\delta \geq 3$)

Characterization of code words

$$\bar{C} = \bar{D} \cdot \overline{\overline{G}} \quad (1)$$

$$\bar{O} = \bar{C} \cdot \overline{\overline{H}}^T \quad (2)$$

It is seen that:

$\boxed{\text{The total set of legal code words is at the same time error patterns that cannot be detected}}$

$$\bar{S} = \bar{E} \cdot \bar{\bar{H}}^T$$

$$\Rightarrow \bar{S} = \bar{0}$$

$$\text{for } \bar{E} = \bar{C}$$

legal code words
Syndrome $\bar{S} = 0$
then errors will go undetected

## Weight spectrum

$$A(w) = \# \{ \bar{C} \mid w(\bar{C}) = w \}$$

The number of legal code words with weight $w$

## The Residual Error Rate

$P_R$ is a Frame Error Probability Rate (FEP)

$$P_R = P(\text{frame is declared »OK«, but has errors})$$

$$= \sum_{w=1}^{n} \underbrace{P\left(\begin{matrix}\text{frame has}\\ w \text{ errors}\end{matrix}\right)}_{P(w)} \cdot \underbrace{P\left(\begin{matrix}w \text{ errors will}\\ \text{go undetected}\end{matrix}\right)}_{r(w)}$$

## Distribution of errors (P(w))

$$P(w) = \binom{n}{w} \cdot P_e^{w} \, (1 - P_e)^{n-w}$$ the Binomial distribution

$$\binom{n}{w} = \frac{n!}{w!\,(n-w)!} = \frac{n}{w} \cdot \frac{n-1}{w-1} \cdot \frac{n-2}{w-2} \cdots$$ w factors

Error-less frame

$$P(0) = \binom{n}{0} P_e^{0} \, (1 - P_e)^{n} = (1 - P_e)^{n}$$

$$\approx 1 - n P_e$$

for $P_e \ll 1$

Probability of errors:

$$P(\text{errors}) = 1 - P(0) \approx n P_e \quad \text{for } P_e \ll 1$$

## Reduction function (r(w))

$$P\left(\begin{matrix} w \text{ errors} \\ \text{not caught} \end{matrix}\right) = r(w) = \frac{\#\{\bar{C} \mid w(\bar{C}) = w\}}{\#\{\bar{E} \mid w(\bar{E}) = w\}}$$

$$= \frac{A(w)}{\binom{n}{w}}$$

$$P_R = \sum_{w=1}^{n} P(w) \cdot r(w)$$