REVIEW EXERCISES FROM PROBABILITY THEORY

Below is listed a number of short rehearsal exercises. It is our hope that none of these exercises should cause much trouble for you to solve. The concept of a stochastic process build directly upon concepts from basic probability theory, in particular, upon the definition of random variables and correlation among these. Hence, it is of paramount importance to be familiar and comfortable with these ideas and concepts. **If you happen to find a majority of the below exercises rather difficult, we strongly recommend you to do a comprehensive review of basic probability theory.** We advice you to consult Kay's book (chapter 1-15). You are also more than welcome to contact the course teachers for help and further advice. Write us an email or come directly to our offices.

Exercise 1:

Draw a bit-string of length ten at random (i.e. all $2^{10} = 1024$ strings are equally likely).

- What is the probability that *at least one bit* in the string is a 0?
- What is the probability that *exactly two bits* in the string are 1's?

Exercise 2:

Let A and B be identically distributed random variables. They are not necessarily independent. For notational convenience we define $\mu := \mathbb{E}[A]$ and $\sigma^2 := \mathbb{V}\mathrm{ar}[A]$, but notice that the random variable A is not assumed to be Gaussian/normally distributed (it's just notation).

- Is $A + B$ a random variable?
- Express $\mathbb{E}[2B]$ as a function of μ.
- Express $\mathbb{V}\mathrm{ar}[-2B]$ as a function of σ^2.

Exercise 3:

Let $X \sim \mathcal{U}\big(-\sqrt{3}\pi, \sqrt{3}\pi\big)$, i.e. X is uniformly distributed on the interval $\big[-\sqrt{3}\pi, \sqrt{3}\pi\big]$, which is symmetric around zero.

- Is X a discrete or continuous random variable?
- Is it important whether the two endpoints $\pm\sqrt{3}\pi$ are included in the interval or not?
- Find $\mathbb{V}\mathrm{ar}[X]$.

Exercise 4:

Let X and Y be independent and both distributed as a $\mathcal{U}\big(0, 1\big)$ random variable. Then, define two new random variables $Z = X + Y$ and $W = 1 - X$.

- What is the range of Z? Is Z uniformly distributed across its range?
- What is the range of W?
- Are the random variables X and W identically distributed? Are they independent?

Exercise 5:

Let $X \sim \mathcal{U}(0,2)$ and let A be the area of a square with side length X.

- Find $\mathbb{E}[A]$.
- Do you expect X and A to be positively correlated, negatively correlated or uncorrelated?
- Calculate $\mathbb{C}\text{ov}[A,X]$. Its sign reveals if your intuition turned out to be correct or not.

Exercise 6:

Let I be an indicator random variable[1] with success probability p where $0 < p < 1$, i.e.

$$P(I=i) = \begin{cases} p & , \quad i=1 \\ 1-p & , \quad i=0 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

- Is I a discrete or continuous random variable?
- Find $\mathbb{E}[I]$ and $\mathbb{V}\text{ar}[I]$.

Exercise 7:

Let $I_1, I_2, \ldots, I_{10}$ be independent and identically distributed indicator random variables and define

$$X = \sum_{n=1}^{10} I_n = I_1 + I_2 + \ldots + I_{10}.$$

- What is the range of X?
- Is the random variable X binomial distributed, gamma distributed, Poisson distributed or exponential distributed?

Exercise 8:

Consider three random variables X_1, X_2 and X_3. Assume that these three random variables are mutually independent and assume that

$$X_m \sim \mathcal{N}\big(m, m^2\big), \qquad m = 1,2,3.$$

Finally, we define a new random variable as

$$Y = X_1 + X_2 + X_3 + 4.$$

- Is the random variable Y Gaussian/normal distributed?
- Find $\mathbb{E}[Y]$ and $\mathbb{V}\text{ar}[Y]$.

Exercise 9:

Let X and Y be independent and $\mathcal{U}(0,1)$ distributed. Define new random variables $A = \min\{X,Y\}$ and $B = \max\{X,Y\}$, which are both functions of X and Y.

- Do you expect A and B to be positively correlated, negatively correlated or uncorrelated?
- Calculate $\mathbb{C}\text{ov}[A,B]$. Its sign reveals if your intuition turned out to be correct or not.

  Hint: Divide the unit square into two appropriate triangles to facilitate easy evaluation of the min/max-functions.

[1] Sometimes also called a Bernoulli random variable.