Exam: Stochastic Processes

Date and Time: Friday January 4, 2013, 9:00–13:00.

This entire problem set contains **4 pages**. Please make sure that you have received all pages.

The exam is graded according to your answer as a whole: both quantity and quality count. We value concise arguments showing your command of the topics. Simply answering "Yes." or "No." will not do that!

We recommend that you read through each problem thoroughly before starting to solve it. Should you happen to get stuck at some point, we recommend that you continue and anyway try to solve the rest. You always have the opportunity to sketch or explain how you would have continued if you haven't got stuck.

It is allowed to use books, lecture notes, your own notes, calculators and computers during the exam. Communication to others during the exam is not allowed—therefore, the use of internet is strictly forbidden.

Problem 1:

Consider the following three random processes $\{X(n)\}$, $\{Y(n)\}$, and $\{Z(n)\}$:

$$X(n) = \frac{12}{|n|+2} + W(n), \tag{1}$$

$$Y(n) = 2 - W(n), \tag{2}$$

$$Z(n) = W(n) + 0.9W(n-1), \tag{3}$$

where $W(n) \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{2})$, $n = 0, \pm 1, \pm 2, \ldots$

1. Write pseudo-codes to simulate realizations of $\{X(n)\}$, $\{Y(n)\}$, and $\{Z(n)\}$.
2. Sketch qualitatively a "typical" realization of $\{X(n)\}$, $\{Y(n)\}$, and $\{Z(n)\}$ for $n = 0, 1, \ldots, 10$, respectively. Explain what properties of the random processes you took into consideration.
3. Investigate and comment on the stationarity properties of $\{X(n)\}$, $\{Y(n)\}$, and $\{Z(n)\}$.
4. Compute the mean and variance of the three random variables $X(2)$, $Y(2)$, and $Z(2)$.
5. Compute and sketch the autocorrelation function and the power spectral density of $\{Y(n)\}$ and $\{Z(n)\}$.

The random process $\{Z(n)\}$ can be thought of as the output of a linear time-invariant (LTIV) system with input random process $\{W(n)\}$.

6. Compute the impulse response $h(n)$ and the transfer function $H(f)$ of the LTIV system. Sketch both $h(n)$ and $|H(f)|^2$.
7. Compute the power spectrum, $S_{ZZ}(f)$, of the process $\{Z(n)\}$ by using the second order input-output relationship of the LTIV system.

Problem 2:

Consider the following random process with random and unknown coefficient θ:

$$Y(n) = X(n) + \theta X(n-1), \quad n = 1, 2, \ldots, \tag{4}$$

where $\theta \sim \mathcal{U}(0,1)$ and $\{X(n)\}$ is a wide sense stationary (WSS) Gaussian random process with mean $\mathbb{E}[X(n)] = 1$ and autocorrelation function $R_{XX}(k)$ which is given by

$$R_{XX}(k) = \begin{cases} 2, & k = 0 \\ 1, & |k| \geq 1 \end{cases}.$$

Furthermore, θ and $\{X(n)\}$ are independent. We wish to estimate the value of θ from an observation of $Y(1)$ and $Y(2)$.

1. Compute the mean and variance of θ. Compute the mean of $\{Y(n)\}$.
2. Verify that the autocorrelation function of $\{Y(n)\}$ is

$$R_{YY}(k) = \begin{cases} \frac{11}{3}, & k = 0 \\ \frac{17}{6}, & |k| = 1 \\ \frac{7}{3}, & |k| \geq 2 \end{cases}.$$

Now consider the linear minimum mean-squared error estimator (LMMSEE) of θ based on the observation of $\mathbf{Y} = [Y(1), Y(2)]^T$:

$$\hat{\theta} = h_0 + (\mathbf{h}^-)^T \mathbf{Y}, \tag{5}$$

where $\mathbf{h}^- = [h_1, h_2]^T$.

3. Find the values of $\mathbb{E}[\hat{\theta} - \theta]$ and $\mathbb{E}[(\hat{\theta} - \theta)\mathbf{Y}^T]$. *Hint: $\hat{\theta}$ is an LMMSEE.*
4. Compute the cross-covariance matrix $\Sigma_{\mathbf{Y}\theta}$ of $\mathbf{Y}$ and θ. Compute the autocovariance matrix $\Sigma_{\mathbf{YY}}$.
5. Compute h_0, h_1, and h_2. *Hint:* $\left(\frac{1}{12}\begin{bmatrix} 17 & 7 \\ 7 & 17 \end{bmatrix}\right)^{-1} = \frac{1}{20}\begin{bmatrix} 17 & -7 \\ -7 & 17 \end{bmatrix}$.
6. Compute the mean-squared error $\mathbb{E}[(\hat{\theta} - \theta)^2]$.
7. Suppose that a colleague proposes to use another linear estimator, $\tilde{\theta} = g_0 + \sum_{m=1}^{2} g_m Y(m)$ which he/she thinks can produce a smaller mean-squared error than $\hat{\theta}$. What would your comments to this colleague be? (Is he/she right or wrong and why?)

Problem 3:

A pathologist wants to detect whether a person suffers from a certain disease which strikes 0.2 % of the population. To do so, he needs a blood sample of the person to measure the amount of "antibody" particles in the blood. This is done by first smearing out the blood sample on a cover glass and subsequently counting the number of antibody particles using an electron microscope. The microscope has a circular field of view with a diameter of 0.2 µm on the cover glass. One blood sample with the antibody particles is illustrated in Fig. 1 (not to scale). The task for the pathologist is to count the number of antibody particles and make a diagnosis, i.e. make a decision on whether the person is infected by the disease or not.

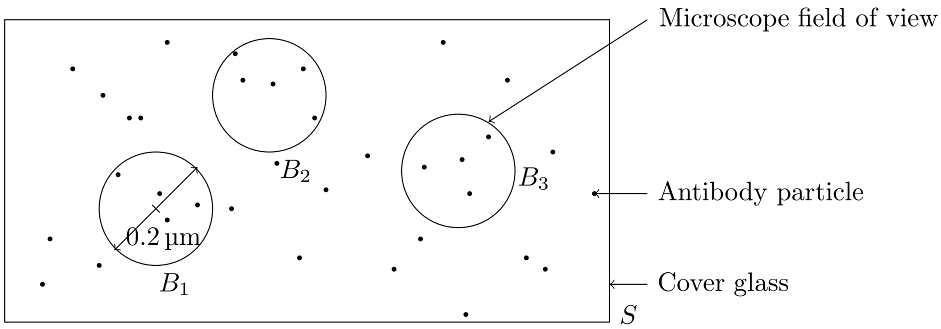

Figure 1: The particles on the cover glass, S, taken from one blood sample, where each dot denotes one antibody particle. Here, there are $K = 3$ circular disjoint regions B_1, B_2 and B_3.

The pathologist counts the number of antibody particles for each of the K disjoint regions $B_1, \ldots, B_K$. We model the positions of the antibody particles on the cover glass as a realization of a homogeneous 2D Poisson point process X. We assume that the intensity function under hypothesis H_0 (a given person is healthy) is $\varrho_X(s) = 200\,\mu\text{m}^{-2}$ and under hypothesis H_1 (a given person is ill) is $\varrho_X(s) = 800\,\mu\text{m}^{-2}$, respectively.

1. What is the probability mass function (PMF) for the counts in B_1 and in $B_1 \cup B_2 \cup \cdots \cup B_K$ under each of the two hypotheses?
2. The pathologist wishes to minimize the probability of making an incorrect decision. Which decision rule would you suggest the pathologist to use and why?
3. Derive the decision rule you chose and compute the threshold value.
4. Is it possible to reduce the probability of making an incorrect decision by taking more regions into account? If your answer is yes, then what is the smallest value of K to keep the probability of making an incorrect decision less than 10^{-5}? If your answer is no, then give your reasons.
   Hint: you can use the matlab function `poisscdf` *to solve this problem. If you do not have your computer with you, then write down the necessary steps instead.*