Olympiad Math History: The iTest

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Introduction

At the turn of the century, dozens of national math competitions were founded to challenge those in the rapidly growing student body interested in competition mathematics. These include the discontinued Mandelbrot competition and the ever-popular Harvard-MIT Math Tournament (HMMT). One of these novel competitions was the iTest.

Originally the American High School Internet Mathematics Competitions (AHSIMC), the iTest was founded in 2004 by author and future CEO of Voice Tech AI, Bradley Metrock. At its peak, it attracted competitors from 44 states and invited Polish students to compete in 2005. Teams consisted of five American students along with an adult advisor, and the only technologies allowed were graphing calculators, surfing the internet, and standard reference materials. The competition was held in the second week in September and lasted for five days. To take the iTest, competitors downloaded specialized software made specifically for the test. This platform could be used generally as a test creator, compiler, and generator (5). While looked upon favorably, it is seldom used today.

Prizes were awarded every year. Notably, in 2006, winners received a Wii, Nintendo DS Lite, and a copy of Brain Age (a puzzle game), but Wiis were not awarded that year due to them selling out (2). In addition, there were secondary awards, such as State Champion, Sponsor's Award for Leadership, and Best Team Name.

The 2005 through 2008 iTest problems are documented on the AoPS wiki and community forums, along with the 2007 and 2008 "Tournament of Champions" questions, an additional event for the top 16 scorers (6). Unfortunately, very few records of the 2004 iTest exist, and the iTest as a whole is seldom documented across the internet.

In the first two years, though it had some of the most unique questions on any Olympiad competition ever, the iTest was often criticized for its "problem quality and general terribleness" (3). However, after the recruitment of 2005 IMO Gold Medalist Thomas Mildorf to help Metrock write the questions, their quality dramatically improved, and many cite the final three years' questions as excellent.

The format of the iTest varied wildly between years. In 2005, there were 40 short answer questions, 3 "chain reaction" (relay-like) problems, 5 long answer questions, and 5 tiebreaker questions. Starting in 2006, there was a multiple choice section where the number of answers corresponded to the number of the question, so the first problem had one possible answer (seriously), whereas the 25th problem had 25 possible answers. These were followed by an equal number of free response questions. All the questions were roughly in ascending difficulty, from introductory problems to Olympiad-level questions on the later short answer questions. Following them was the 10-question Ultimate round, a relay in which answers to the last problem were used in the next problem, and some competitions also featured tiebreaker questions after that. The 2006 and 2007 tests lacked a clear theme, but the 2005 paper featured a story about Joe and Kathryn, two high school seniors over-obsessing on how to date each other. Similarly, in 2008, the test writers created a 100 question short answer paper featuring the wacky mathematical misadventures of the Kubik family (4). Unfortunately, in 2009 Metrock announced that he wished to focus on his startup instead of the iTest, and that he would return the following year. Needless to say, he did not, and the iTest fell into obscurity[1] (1).

The remainder of this paper is dedicated to showcasing the more unique problems on the iTest.

[1] Some sources on foreign sites still cite the AHSIMC as an open ongoing competition, and one person on Linkedin even cited themselves as a 2018 winner, but no definitive evidence of any tests past 2008 exists.

What a Subjective Objective!

$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = 3$$
$$f(3) = 5$$
$$f(4) = 9$$
$$f(5) = 11$$
$$f(6) = 29$$
$$f(11) = 31$$
$$f(20) = ?$$

(Long Answer Problem 2, 2005 iTest)

Obviously this problem is poorly defined, as one can define a function $f(x)$ over the integers such that $f(20)$ equals any integer one likes, be it 2, -41, or 31,415,926,535. Before self-imposing any stricter conditions, let us consider alternate interpretations of the problem.

Suppose the problem is taken as the continuation of the sequence 0, 1, 3, 5, 9, 11, 29, and 31. The first finite differences are 1, 2, 2, 4, 2, 18, and 2. It is reasonable to assume that the next finite difference would be the continuation of the sequence 1, 2, 4, and 18, which according to the first result on the OEIS (A295370) is 80, so we have one possible answer of $\boxed{111}$. Of course, there are many more continuations of the above sequence not given (there are 44 unique sequences recorded in the OEIS). Note that a direct search of the original sequence does not net any results.

Now assume that this problem asks for the polynomial f of lowest degree satisfying the given constraints. By the Lagrange Polynomial Interpolation formula, a set of n points can be interpolated by a unique polynomial of maximum degree $n-1$, so we seek a degree 7 polynomial for our given points. We can construct such a polynomial by multiplying linear factors of each given x-value except for one, and scale the resulting term to satisfy the given constraints. For example, to satisfy $f(5) = 11$, we construct the polynomial $kx(x-1)(x-2)(x-3)(x-4)(x-6)(x-11)$, so upon substituting any given value apart from 5, the term multiplies to 0. Then we find $k = 11/720$. Repeating this process for all the other terms results in

$$\begin{aligned} f(x) = &\frac{31}{1663200}x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \\ &- \frac{29}{3600}x(x-1)(x-2)(x-3)(x-4)(x-5)(x-11) \\ &+ \frac{11}{720}x(x-1)(x-2)(x-3)(x-4)(x-6)(x-11) \\ &- \frac{3}{112}x(x-1)(x-2)(x-3)(x-5)(x-6)(x-11) \\ &+ \frac{5}{288}x(x-1)(x-2)(x-4)(x-5)(x-6)(x-11) \\ &- \frac{1}{144}x(x-1)(x-3)(x-4)(x-5)(x-6)(x-11) \\ &+ \frac{1}{1200}x(x-2)(x-3)(x-4)(x-5)(x-6)(x-11) \end{aligned}$$

Finally, $f(20) = \boxed{-21258785/11}$, which is probably the intended answer as the long answer questions are not constrained to integers.

Fermi and Feynman's Fabled Feud

> Fermi and Feynman play the game *Probabicloneme* in which Fermi wins with probability a/b, where a and b are relatively prime positive integers such that $a/b < 1/2$. The rest of the time Feynman wins (there are no ties or incomplete games). It takes a negligible amount of time for the two geniuses to play *Probabicloneme* so they play many many times. Assuming they can play infinitely many games (eh, they're in Physicist Heaven, we can bend the rules), the probability that they are ever tied in total wins after they start (they have the same positive win totals) is 17/77. Find the value of a. (Problem U9, 2007 iTest)

There is a bijection between an infinite string of games that the men play and an infinite path from $(0, 0)$ always going up or right, which we call a "game path". Let the ordered pair (x, y) represent the number of Fermi wins followed by the number of Feynman wins, and call the probability of Fermi winning p.

Note that every possible game can first tie at 2 games, 4 games, etc. corresponding to game paths intersecting $(1, 1)$, game paths intersecting $(2, 2)$ but not $(1, 1)$, etc. Because these points lie on the line $y = x$, and we wish to have paths that don't intersect this diagonal until (n, n) (aka they stay underneath/above the diagonal until (n, n)), we consider the Catalan numbers.

Assume Fermi wins the first game. Then in order for them to first tie at (n, n), the game path must run under the diagonal, which is equivalent to a game path under or on the diagonal from $(1, 0)$ to $(n, n-1)$. This corresponds to C_{n-1} paths, where C_n is the nth Catalan number. Then they tie by Feynman winning the last game (how the path runs afterwards is irrelevant). Any suitable game path from $(0, 0)$ to (n, n) has n Fermi and Feynman wins each, so it happens with probability $(p(1-p))^n$. By symmetry, we double this probability due to starting with a Feynman win. Because these events are mutually exclusive between any distinct n, the cumulative probability of tieing is

$$2\sum_{n=1}^{\infty} C_{n-1}(p(1-p))^n = 2\sum_{n=0}^{\infty} C_n(p(1-p))^{n+1} = \frac{17}{77}$$

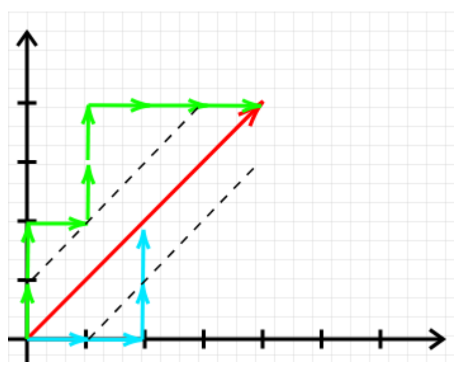

Figure 1: Sample paths corresponding to games that tie at 2 games apiece (cyan) and 4 games apiece (green)

Recall the generating function of the Catalan numbers and its closed form expression,

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

so

$$2\sum_{n=0}^{\infty} C_n(p(1-p))^{n+1} = 1 - \sqrt{1-4p(1-p)} = \frac{17}{77}$$

$$p = \frac{17}{154}, \frac{137}{154}$$

Because $p < 1/2$, the requested answer is $\boxed{17}$. Now why is the probability of Fermi winning half the probability that they tie...

Runaway Recurrence Relations

Let $\{X_n\}$ and $\{Y_n\}$ be sequences defined as follows: $X_0 = Y_0 = X_1 = Y_1 = 1$,

$$X_{n+1} = X_n + 2X_{n-1} \qquad (n = 1, 2, 3 \ldots),$$
$$Y_{n+1} = 3Y_n + 4Y_{n-1} \qquad (n = 1, 2, 3 \ldots).$$

Let k be the largest integer that satisfies all of the following conditions: $|X_i - k| \leq 2007$, for some positive integer i; $|Y_j - k| \leq 2007$, for some positive integer j; and $k < 10^{2007}$. Find the remainder when k is divided by 2007. (Problem 47, 2007 iTest)

First, we solve these linear recurrences. The characteristic polynomial of X_n is $x^2 = x + 2$ which has roots -1 and 2. Then using the initial values, $X_n = \lambda_1(-1)^n + \lambda_2(2)^n$ where λ_1 and λ_2 are the solutions to the system

$$\begin{cases} 1 = \lambda_1 + \lambda_2 \\ 1 = -\lambda_1 + 2\lambda_2 \end{cases}$$

Solving, we find $\lambda_1 = 1/3$ and $\lambda_2 = 2/3$, so $X_n = \frac{(-1)^n + 2^{n+1}}{3}$. Similarly, $Y_n = \frac{3(-1)^n + 2^{2n+1}}{5}$.

We ignore the $(-1)^n$ terms because they will be inconsequential compared to the 2^n terms, so define $x_n = \frac{2^{n+1}}{3}$ and $y_n = \frac{2^{2n+1}}{5}$. The problem conditions can be restated as $|X_a - Y_b| \leq 4014$ for some integers a and b so that we can set k to be the average of X_a and Y_b to guarantee a valid k. The inequality inspires us to analyze the equation $X_a \approx x_a = Y_b \approx y_b$, or $\frac{2^{a+1}}{3} = \frac{2^{2b+1}}{5}$. Solving for a results in $a = 2b - \log_2(5/3)$, and because a and b are integers, we assume $a = 2b - 1$.

Using our series approximations, $|X_a - Y_b| \approx |x_a - y_b|$, so it follows that $\left|\frac{2^{2b}}{3} - \frac{2^{2b+1}}{5}\right| \leq 4014$, which simplifies to $2^{2b} \leq 4014 \cdot 15$. Bashing or calculator use results in $b < 8$, so $b = 7$ and $a = 13$. Then, $X_{13} = 5461$ and $Y_7 = 6553$. Note that $X_{13} < Y_7$, so the largest k possible will be $X_{13} + 2007 = 7468$. The requested answer is 7468 mod 2007 = $\boxed{1447}$.

For completeness, we prove that $a = 2b - 1$ gives the maximum upper bound on b. Suppose $a = 2b - n$ for some integer n. Then $\left|\frac{2^{2b+n+1}}{3} - \frac{2^{2b+1}}{5}\right| \leq 4014$. We finish by casework.

1. If $n \geq 0$, then $\frac{2^{2b+n+1}}{3} - \frac{2^{2b+1}}{5} \leq 4014$ which implies $2^{2b} \leq \frac{4014(15)}{10(2^n)-6}$. The term on the RHS is maximized for small n, which implies $n = 0$.

2. If $n < 0$, then $\frac{2^{2b+1}}{5} - \frac{2^{2b+n+1}}{3} \leq 4014$ which implies $2^{2b} \leq \frac{4014(15)}{6-10(2^n)}$. Maximizing the term on the RHS is equivalent to minimizing $6 - 10(2^n)$ or maximizing n, which implies $n = -1$.

Finally, $n = 0$ gives $b \leq 4014(15)/4$ and $n = -1$ gives $b \leq 4014(15)$, so our proof is complete.

This answer is quite surprising especially given the huge upper bound on k in the problem statement. By intuition, one could assume that the sequences would eventually "overlap" by chance, but evidently not.

References

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