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Source: *Biometrika*, Vol. 31, No. 1/2 (Jul., 1939), pp. 13-19
Published by: Oxford University Press on behalf of Biometrika Trust
Stable URL: https://www.jstor.org/stable/2334972

A TEST FOR THE SIGNIFICANCE OF THE DIFFERENCE BETWEEN THE TWO VARIANCES IN A SAMPLE FROM A NORMAL BIVARIATE POPULATION

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1. Derivation of likelihood ratio test

In a paper published in a recent issue of this Journal D. J. Finney (1938) considered the following questions. A sample of n pairs of variables (x, y) has been drawn from the bivariate normal distribution whose probability law is

$$p(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho_{12}^2)}}\exp\left[-\frac{1}{2(1-\rho_{12}^2)}\left\{\left(\frac{x-\xi_1}{\sigma_1}\right)^2 - 2\rho_{12}\frac{(x-\xi_1)(y-\xi_2)}{\sigma_1\sigma_2} + \left(\frac{y-\xi_2}{\sigma_2}\right)^2\right\}\right], \quad \ldots\ldots(1)$$

then

(i) What is the probability law of the ratio, ω, where

$$\omega = s_1'/s_2' \quad \ldots\ldots(2)$$

and

$$(s_1')^2 = \sum_i (x_i - \bar{x})^2/(n-1), \quad (s_2')^2 = \sum_i (y_i - \bar{y})^2/(n-1)? \quad \ldots\ldots(3)$$

(ii) Could this ratio be used as a criterion to test the hypothesis that $\sigma_1 = \sigma_2$?

Using a more direct method he was first able to confirm a previous result of Bose (1935), giving the probability distribution of ω in the case where $\sigma_1 = \sigma_2$, and hence to show that the chance that ω exceeds a given value, say Ω, could be obtained from the *Tables of the Incomplete Beta Function* (1934), using the relation

$$P\{\omega > \Omega\} = I_x\left(\frac{n-1}{2}, \frac{n-1}{2}\right), \quad \ldots\ldots(4)$$

where

$$x = \frac{1}{2}\left(1 - \frac{\Omega - \Omega^{-1}}{\sqrt{\{(\Omega + \Omega^{-})^2 - 4\rho_{12}^2\}}}\right). \quad \ldots\ldots(5)$$

Since the probability expression (4) is dependent on the population correlation ρ_{12}, which will be in general unknown, Finney pointed out that the ratio ω was not altogether a satisfactory criterion to use in testing the hypothesis $\sigma_1 = \sigma_2$, but he put forward a possible method of getting round this difficulty.

The question arises as to whether there is not some other more suitable criterion for testing whether the variances are equal, whose sampling distribution will be independent of ρ_{12} and of any other parameters whose values are not specified by the hypothesis itself. The likelihood ratio method of approach of Neyman and Pearson may be followed; this method has proved of service in a number of instances where the appropriate criterion was not immediately obvious. Summarized briefly, it involves the following steps:

(*a*) A specification of the set of admissible hypotheses. In the present case these will be defined by the joint probability law of the n pairs of observations,

$$p(x_1, \ldots, x_n; y_1, \ldots, y_n \mid \xi_1, \xi_2, \sigma_1, \sigma_2, \rho_{12})$$
$$= \{2\pi\sigma_1\sigma_2\sqrt{(1-\rho_{12}^2)}\}^{-n} \exp\left[-\frac{n}{2(1-\rho_{12}^2)}\left\{\left(\frac{\bar{x}-\xi_1}{\sigma_1}\right)^2 - 2\rho_{12}\frac{(\bar{x}-\xi_1)(\bar{y}-\xi_2)}{\sigma_1\sigma_2}\right.\right.$$
$$\left.\left. + \left(\frac{\bar{y}-\xi_2}{\sigma_2}\right)^2 + \frac{s_1^2}{\sigma_1^2} - \frac{2\rho_{12}r_{12}s_1s_2}{\sigma_1\sigma_2} + \frac{s_2^2}{\sigma_2^2}\right\}\right], \qquad \ldots\ldots(6)$$

where $\qquad -\infty \leqslant \xi_1, \xi_2 \leqslant \infty, \quad 0 \leqslant \sigma_1, \sigma_2 \leqslant \infty, \quad -1 \leqslant \rho_{12} \leqslant 1.$

In this expression $\bar{x}$ and $\bar{y}$ are the sample means, r_{12} the sample correlation coefficient, and

$$s_1^2 = \sum_i (x_i - \bar{x})^2/n, \quad s_2^2 = \sum_i (y_i - \bar{y})^2/n. \qquad \ldots\ldots(7)$$

(*b*) A determination of those values of the five unknown parameters, as functions of the observations, which jointly maximize the expression (6). The solution is known to be obtained when

$$\xi_1 = \bar{x}, \quad \xi_2 = \bar{y}, \quad \sigma_1 = s_1, \quad \sigma_2 = s_2, \quad \rho_{12} = r_{12}. \qquad \ldots\ldots(8)$$

The maximum value of (6) is thus

$$p_1(\max) = \{2e\pi s_1 s_2\sqrt{(1-r_{12}^2)}\}^{-n}. \qquad \ldots\ldots(9)$$

(*c*) A specification of the hypothesis tested. This hypothesis assumes that the probability law is of the form

$$p(x_1, \ldots, x_n; y_1, \ldots, y_n \mid \xi_1, \xi_2, \sigma, \rho_{12}),$$

where the function is obtained by putting $\sigma_1 = \sigma_2 = \sigma$ in (6).

(*d*) A determination of the values of the four unknown parameters ξ_1, ξ_2, σ, and ρ_{12} which maximize this expression. These values may be shown to be

$$\xi_1 = \bar{x}, \quad \xi_2 = \bar{y}, \quad \sigma = \sqrt{\{\tfrac{1}{2}(s_1^2+s_2^2)\}}, \quad \rho_{12} = 2r_{12}s_1s_2/(s_1^2+s_2^2). \quad \ldots\ldots(10)$$

The maximum value of the probability function defined in (*c*) then becomes

$$p_2(\max) = \{e\pi\sqrt{[(s_1^2+s_2^2)^2 - 4r_{12}^2 s_1^2 s_2^2]}\}^{-n}. \qquad \ldots\ldots(11)$$

(*e*) The likelihood ratio criterion is then

$$\lambda = \frac{p_2(\max)}{p_1(\max)} = \left\{\frac{4s_1^2s_2^2(1-r_{12}^2)}{(s_1^2+s_2^2)^2 - 4r_{12}^2s_1^2s_2^2}\right\}^{\frac{1}{2}n} \qquad \ldots\ldots(12a)$$
$$= \left\{1 - \frac{(s_1^2-s_2^2)^2}{(s_1^2+s_2^2)^2 - 4r_{12}^2s_1^2s_2^2}\right\}^{\frac{1}{2}n}. \qquad \ldots\ldots(12b)$$

(*f*) The hypothesis tested becomes less and less likely as λ moves from 1 to 0. To complete the test it is necessary to know the sampling distribution of λ, or of a single valued function of λ, when the hypothesis tested is true.

It will be seen at once that the test differs from Finney's because the criterion, unlike his ω, is a function of r_{12} as well as s_1^2 and s_2^2. The meaning of the criterion which has been picked out by the λ-method becomes clear if we make the following transformation of the original variables:

Write $$x = X+Y, \qquad y = X-Y, \qquad \text{......(13)}$$
so that $$X = \tfrac{1}{2}(x+y), \quad Y = \tfrac{1}{2}(x-y). \qquad \text{......(14)}$$

Then the population variances of x and y may be expressed as functions of the variances and correlation for X and Y, as follows:

$$\left.\begin{aligned}\sigma_1^2 &= \sigma_X^2+\sigma_Y^2+2\rho_{XY}\sigma_X\sigma_Y,\\ \sigma_2^2 &= \sigma_X^2+\sigma_Y^2-2\rho_{XY}\sigma_X\sigma_Y.\end{aligned}\right\} \qquad \text{......(15)}$$

The necessary and sufficient condition that the hypothesis tested is true, or that $\sigma_1 = \sigma_2$, is that

$$\rho_{XY} = 0. \qquad \text{......(16)}$$

Since X and Y are normally correlated variables, the appropriate criterion to test the hypothesis, $\rho_{XY} = 0$, is the sample correlation coefficient between the transformed variables, i.e. r_{XY}. If the hypothesis is true, this coefficient has the well-known probability law

$$p(r_{XY} \mid \rho_{XY} = 0) = \text{constant} \times (1-r_{XY}^2)^{\frac{1}{2}(n-4)}. \qquad \text{......(17)}$$

Making use of the transformations (14), it is found that

$$r_{XY} = \frac{s_1^2-s_2^2}{\{(s_1^2+s_2^2)^2-4r_{12}^2s_1^2s_2^2\}^{\frac{1}{2}}}. \qquad \text{......(18)}$$

Hence the likelihood criterion of (12*b*) is seen to be

$$\lambda = \{1-r_{XY}^2\}^{\frac{1}{2}n}, \qquad \text{......(19)}$$

and as the hypothesis tested becomes less and less likely, $\lambda \to 0$ or $r_{XY}^2 \to 1$. The test may therefore be carried out by (*a*) referring the r_{XY} of (18) to the probability distribution (17), or (*b*) alternatively referring

$$t = \frac{r_{XY}\sqrt{(n-2)}}{\sqrt{(1-r_{XY}^2)}} \qquad \text{......(20)}$$

to "Student's" distribution with degrees of freedom $f = n-2$, and (*c*) rejecting the hypothesis when $|r_{XY}|$ or $|t|$ fall beyond the desired probability level. The test, it will be seen, is independent of the unknown correlation ρ_{12} between x and y.

2. The power of the test

While the probability distribution of r_{XY} is independent of ρ_{12} if the hypothesis ($\sigma_1 = \sigma_2$) is true, the chance that in using the rejection rule of the test we shall detect real differences between σ_1 and σ_2 will depend on the value of ρ_{12}. Suppose that we fix a probability level $r(\alpha)$ such that

$$\alpha = 2\int_{r(\alpha)}^{1} c(1-r^2)^{\frac{1}{2}(n-4)}\,dr,^* \qquad \text{......(21)}$$

where α, for example, equals 0·05.

* c is the constant of the probability law (17).

Then the chance of rejecting the hypothesis that

$$\gamma = \sigma_1/\sigma_2 = 1, \qquad \text{......(22)}$$

when $\gamma = \gamma_1 \neq 1$, is given by the expression

$$P\{r^2_{XY} > r^2(\alpha) \mid \gamma = \gamma_1\} = P\{r^2 > r^2(\alpha) \mid \rho = \rho_1 \neq 0\}$$
$$= 1 - \int_{-r(\alpha)}^{r(\alpha)} p(r \mid \rho = \rho_1)\, dr. \qquad \text{......(23)}$$

This expression Neyman & Pearson (1936) have termed the power of the test of the hypothesis that $\gamma = \sigma_1/\sigma_2 = 1$ with regard to an alternative hypothesis $\gamma = \gamma_1$. In the expression (23), $p(r \mid \rho = \rho_1)$ denotes the general probability law for r in samples from a bivariate normal population, first obtained by R. A. Fisher (1915). It will be seen that a relation similar to (18) holds between ρ_{XY}, $\gamma = \sigma_1/\sigma_2$ and ρ_{12}, namely

$$\rho_{XY} = \frac{\gamma - \gamma^{-1}}{\{(\gamma - \gamma^{-1})^2 + 4(1-\rho_{12}^2)\}^{\frac{1}{2}}}. \qquad \text{......(24)}$$

Owing to the symmetry of the distribution of r when $\rho = 0$, the power of the test will be the same for alternatives ρ_{XY} and $-\rho_{XY}$; it will therefore be the same for alternatives γ and γ^{-1}. For example, the test is as likely to reject the hypothesis $(\sigma_1 = \sigma_2)$ when $\sigma_1 = 2\sigma_2$ as when $\sigma_1 = \frac{1}{2}\sigma_2$. This is clearly what we should expect, as σ_1 and σ_2 are in no way differentiated.

In Fig. 1 I have shown the power function of the test, taking $\alpha = 0{\cdot}10$* and sample size $n = 25$, for alternatives $\gamma > 1$, in the three cases $\rho_{12} = 0$, $\rho_{12} = 0{\cdot}5$, and $\rho_{12} = 0{\cdot}8$. It will be noticed that the test is more powerful when ρ_{12} is large. This of course follows from (24), since for a given value of γ, ρ_{XY} will be further from zero the nearer $|\rho_{12}|$ is to unity.

The computations were made with the help of F. N. David's recently published *Tables* (1938). The work was simplified by taking ρ_{XY} at convenient values $0{\cdot}1, 0{\cdot}2, \ldots, 0{\cdot}9$, and finding the corresponding values of γ by means of (24).

The table on p. 18 shows, in the columns headed Test (a), the values of the power function computed in this way for this case of $n = 25$ and also for $n = 12$ and $n = 100$.

3. Comparison with Finney's test in the case where ρ_{12} is known

In the case where ρ_{12} is not known, Finney has suggested that his test criterion ω might be used by making a double appeal to significance levels on the lines proposed in another case by Hirschfeld (1937). It does not, however, appear easy to determine numerically the power of the resulting test. In the case where ρ_{12} is known, it may be shown that the likelihood ratio criterion now becomes

$$\lambda = \left\{1 + \frac{(s_1 - s_2)^2}{2s_1 s_2(1 - r_{12}\rho_{12})}\right\}^{-n} = \left\{1 + \frac{(\omega - 1)^2}{2\omega(1 - r_{12}\rho_{12})}\right\}^{-n}. \qquad \text{......(25)}$$

* This means that the hypothesis is to be rejected when r_{XY} falls beyond the 5% level in either tail of the distribution (17), or for the case $n = 25$, when $|r_{XY}| > 0{\cdot}3365$.

This is not solely a function of Finney's criterion ω, since it depends again on the sample correlation r_{12}. The form of this expression is interesting, as it shows that if ρ_{12} is known and differs from zero, in view of the correlation which exists between s_1, s_2, and r_{12},* the sample values of all three are relevant in examining the hypothesis that $\sigma_1 = \sigma_2$. I have not succeeded in determining the sampling distribution of the λ of (25), or of any single-valued function of λ.

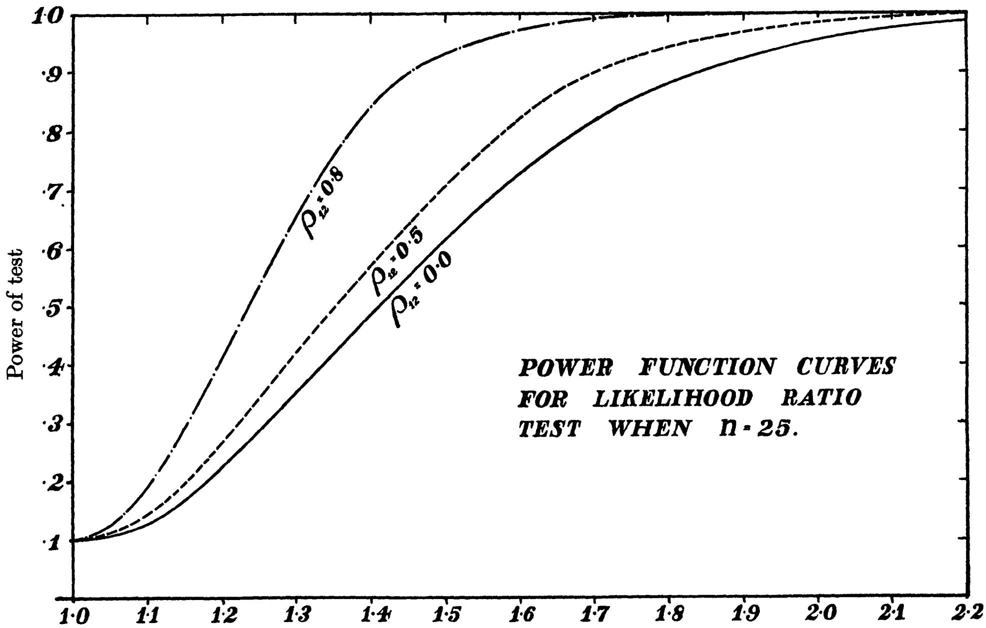

Fig. 1.

In the case of ρ_{12} known, it is, however, possible to compare Finney's test (involving ρ_{12}) with the likelihood test appropriate in the case where ρ_{12} is unknown, but still of course applicable when it is known. The power function for Finney's test may be computed as follows.

When using this test the hypothesis that $\gamma = 1$ is rejected if $\omega > \Omega$ or if $\omega < \Omega^{-1}$, where Ω is a constant chosen by using relations (4) and (5) so that

$$\int_{\Omega}^{\infty} p(\omega \mid \gamma = 1)\, d\omega = \tfrac{1}{2}\alpha. \qquad \text{......(26)}$$

The power of the test with regard to an alternative hypothesis, $\gamma \neq 1$, is therefore given by

$$F(\gamma) = 1 - P\{\Omega^{-1} < \omega < \Omega\} = \int_0^{\Omega^{-1}} p(\omega \mid \gamma)\, d\omega + \int_{\Omega}^{\infty} p(\omega \mid \gamma)\, d\omega, \qquad \text{......(27)}$$

where

$$p(\omega \mid \gamma) = \text{constant} \times (1-\rho_{12}^2)^{\frac{1}{2}(n-1)} \frac{\gamma^2+\omega^2}{\gamma\omega^2} \left\{ \left(\frac{\gamma^2+\omega^2}{\gamma\omega} \right)^2 - 4\rho_{12}^2 \right\}^{-\frac{1}{2}n}. \qquad \text{......(28)}$$

* See K. Pearson (1913).

The above integrals may be transformed to give

$$F(\gamma) = 1 - I_{x_1}\left(\frac{n-1}{2}, \frac{n-1}{2}\right) + I_{x_2}\left(\frac{n-1}{2}, \frac{n-1}{2}\right), \qquad \ldots\ldots(29)$$

where $x_1 = \tfrac{1}{2}[1 + (\gamma\Omega - \gamma^{-1}\Omega^{-1})\sqrt{\{(\gamma\Omega - \gamma^{-1}\Omega^{-1})^2 + 4(1-\rho_{12}^2)\}}]$

and $x_2 = \tfrac{1}{2}[1 - (\gamma^{-1}\Omega - \gamma\Omega^{-1})\sqrt{\{(\gamma^{-1}\Omega - \gamma\Omega^{-1})^2 + 4(1-\rho_{12}^2)\}}]$, $\ldots\ldots(30)$

so that values of the power function may be calculated using the *Tables of the*

Comparison of power functions of (a) Likelihood test (based on r_{XY}), (b) Finney's test (based on ω), when ρ_{12} is known

Case $\rho_{12} = 0{\cdot}0$		$n = 12$		$n = 25$		$n = 100$	
$\gamma = \sigma_1/\sigma_2$	ρ_{XY}	Test (*a*)	Test (*b*)	Test (*a*)	Test (*b*)	Test (*a*)	Test (*b*)
1·00	0	0·100	0·100	0·100	0·100	0·100	0·100
1·11	0·1	0·116	—	0·138	0·139	0·262	0·262
1·22	0·2	0·167	—	0·254	0·254	0·642	0·642
1·36	0·3	0·254	—	0·437	0·438	0·922	0·922
1·53	0·4	0·379	0·382	0·652	0·653	0·979	—
2·00	0·6	0·710	0·715	0·953	0·954	—	—
3·00	0·8	0·965	—	1·000	1·000	—	—
Case $\rho_{12} = 0{\cdot}5$		$n = 12$		$n = 25$		$n = 100$	
$\gamma = \sigma_1/\sigma_2$	ρ_{XY}	Test (*a*)	Test (*b*)	Test (*a*)	Test (*b*)	Test (*a*)	Test (*b*)
1·00	0	0·100	0·100	0·100	0·100	0·100	0·100
1·09	0·1	0·116	0·116	0·138	0·138	0·262	0·261
1·19	0·2	0·167	0·166	0·254	0·253	0·642	0·643
1·31	0·3	0·254	—	0·437	0·437	0·922	0·924
1·45	0·4	0·379	0·379	0·652	0·656	0·979	—
1·84	0·6	0·710	0·722	0·953	0·958	—	—
Case $\rho_{12} = 0{\cdot}8$		$n = 12$		$n = 25$		$n = 100$	
$\gamma = \sigma_1/\sigma_2$	ρ_{XY}	Test (*a*)	Test (*b*)	Test (*a*)	Test (*b*)	Test (*a*)	Test (*b*)
1·00	0	0·100	0·100	0·100	0·100	0·100	0·100
1·06	0·1	0·116	0·115	0·138	0·137	0·262	0·260
1·13	0·2	0·167	—	0·254	0·250	0·642	0·643
1·21	0·3	0·254	—	0·437	0·436	0·922	0·926
1·30	0·4	0·379	0·375	0·652	0·660	0·979	—
1·55	0·6	0·710	0·735	0·953	0·964	—	—
2·08	0·8	0·965	0·981	1·000	1·000	—	—

Incomplete Beta Function. These values are compared in the table on p. 18 above with those for the likelihood ratio test. It will be seen that:

(1) When $\rho_{12} = 0$, i.e. when the two variables are known to be independent, the test based on ω or the ratio of sample variances is the better. This is a result already known, but the table shows how small is the difference between the tests.

(2) When ρ_{12} is 0·5 or 0·8, however, the likelihood ratio test is somewhat more sensitive for the small departures in γ from unity, and less sensitive for large departures than the ω test. Practically, this means that when ρ_{12} is known, the ω test is slightly the better, since we are most interested in situations where the chance of detection of a difference becomes large, say greater than 0·5 at any rate. It is possible that a test based on the criterion given in equation (25), if it could be obtained, would be more powerful than either of the other two tests when ρ_{12} is known. In practical cases, however, it will nearly always happen that ρ_{12} is unknown, and in such cases the r_{XY} of (18) appears the appropriate test criterion to use.

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