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ASYMPTOTICALLY EFFICIENT ADAPTIVE RANK ESTIMATES IN LOCATION MODELS[1]

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This paper describes a new construction of uniformly asymptotically efficient rank estimates in the one and two-sample location models. The method adopted differs from van Eeden's (1970) earlier construction in three respects. First, the whole sample, rather than a vanishingly small fraction of the sample, is used in estimating the efficient score function. Secondly, a Fourier series estimator is used for the score function rather than a window estimator. Thirdly, the linearized rank estimates corresponding to the estimated score function provide the uniformly asymptotically efficient location estimates. These estimates are asymptotically efficient over a larger class of distributions than the van Eeden estimates and should approach their asymptotic behavior more rapidly.

1. Introduction. Suppose that $X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_n$ are random variables with joint density $\prod_{i=1}^m f(x_i - \mu_0) \prod_{j=1}^n f(y_j)$, where μ_0 is the difference in location of the two samples. Let F denote the distribution function corresponding to f and let $||\cdot||$ denote the norm in $L_2(0, 1)$. Under regularity conditions on F, there exists a rank estimate $\hat{\mu}(\phi_F)$ of μ_0, depending upon

$$\phi_F(t) = -f' \circ F^{-1}(t)/f \circ F^{-1}(t)\,, \tag{1.1}$$

such that the asymptotic distribution of $(mn/m + n)^{\frac{1}{2}}(\hat{\mu}(\phi_F) - \mu_0)$ is normal $(0, ||\phi_F||^{-2})$ (see Hodges and Lehmann (1963), Kraft and van Eeden (1970)). The estimate $\hat{\mu}(\phi_F)$ is asymptotically efficient in the sense that its asymptotic variance attains the Cramér–Rao lower bound.

Similarly, suppose that $X_1, X_2, \cdots, X_N$ are random variables with joint density $\prod_{i=1}^N f(x_i - \nu_0)$, where f is symmetric about the origin. Under regularity conditions on F, there exists a rank estimate $\hat{\nu}(\psi_F)$ of ν_0, depending upon

$$\psi_F(t) = \phi_F(\tfrac{1}{2} + t/2)\,, \tag{1.2}$$

such that the asymptotic distribution of $N^{\frac{1}{2}}(\hat{\nu}(\psi_F) - \nu_0)$ is normal $(0, ||\psi_F||^{-2})$ and the estimate is asymptotically efficient (see the references above).

In practice, a statistician analyzing data under a one or two-sample location model will have only approximate knowledge of ϕ_F. Even if ϕ_F is unknown, Stein (1956) noted the possibility of constructing nonparametric location estimates which are asymptotically efficient for all regular f. Uniformly

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asymptotically efficient rank estimates for location were first devised by van Eeden (1970). Her approach was to estimate ϕ_F from a vanishingly small fraction of the data, using a modified form of the estimator studied by Hájek (1962), and then find the Hodges–Lehmann rank estimate of location based upon the estimated score function and the remaining data.

This paper describes a new construction of uniformly asymptotically efficient rank estimates in the one and two-sample location models. The construction differs from van Eeden's in three respects. First, the whole sample is used in estimating ϕ_F. Secondly, a Fourier series estimator is used for the score function rather than a window estimator. Thirdly, linearized rank estimates corresponding to the estimate of ϕ_F provide the uniformly asymptotically efficient location estimates. These estimates are asymptotically efficient over a larger class of distributions F than the van Eeden estimates and should approach their asymptotic behavior more rapidly. However, the rate of convergence can still be very slow for particular F.

2. Estimation of ϕ_F. Suppose that the density f satisfies the following assumption:

A. f is absolutely continuous and $\phi_F \in L_2(0, 1)$.

Since $\phi_F \in L_2(0, 1)$, it has the Fourier expansion

$$\phi_F(t) \sim \textstyle\sum_{|k|=1}^{\infty} c_k \exp(2\pi i k t) , \tag{2.1}$$

where

$$c_k = \textstyle\int_0^1 \phi_F(t) \exp(-2\pi i k t)\, dt . \tag{2.2}$$

Let $Z_1, Z_2, \cdots$ be a sequence of independent identically distributed random variables, each of which has density f. In view of (2.1), a plausible estimate for ϕ_F based upon $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_N)$ is

$$\hat{\phi}_F(t) = \textstyle\sum_{|k|=1}^{M} \hat{c}_k \exp(2\pi i k t) \tag{2.3}$$

where $\hat{c}_k$ is an estimate of c_k based upon $\mathbf{Z}$ and $M \to \infty$ at a suitable rate relative to N. This section will develop a detailed version of this approach to estimating ϕ_F.

The first step is to estimate c_k, or more generally, a functional of the form

$$T(\phi) = \textstyle\int_0^1 \phi(t)\phi_F(t)\, dt = \int \left[\dfrac{d\phi \circ F(x)}{dx}\right] dF(x) \tag{2.4}$$

where ϕ is a real-valued function defined on $[0, 1]$ that possesses the following properties:

B. ϕ is twice differentiable and ϕ'' is continuous on $[0, 1]$.

The second expresssion for $T(\phi)$ suggests as an estimator

$$\begin{aligned} T_N(\mathbf{Z}, \phi) = \frac{1}{2N\theta_N} \textstyle\sum_{i=1}^{N} \Big[\phi\Big(\frac{1}{N-1} \sum_{j \neq i} v(Z_i - Z_j + \theta_N)\Big) \\ - \phi\Big(\frac{1}{N-1} \sum_{j \neq i} v(Z_i - Z_j - \theta_N)\Big)\Big] , \end{aligned} \tag{2.5}$$

where

$$v(x) = 1 \quad \text{if } x \geqq 0 \qquad = 0 \quad \text{if } x < 0, \tag{2.6}$$

and $\theta_N = N^{-\frac{1}{2}}\theta$ for some $\theta \neq 0$.

THEOREM 2.1. *Under assumptions* A *and* B, *the asymptotic distribution of* $N^{\frac{1}{2}}(T_N(\mathbf{Z}, \phi) - T(\phi))$ *as* $N \to \infty$ *is normal* $(0, \sigma^2(\phi))$, *where*

$$\sigma^2(\phi) = \int_0^1 \int_0^1 [\min(s, t) - st][2\phi'(s)\phi_F(s) - \phi''(s)f \circ F^{-1}(s)] \times [2\phi'(t)\phi_F(t) - \phi''(t)f \circ F^{-1}(t)]\, ds\, dt. \tag{2.7}$$

Moreover

$$\lim_{N\to\infty} NE[T_N(\mathbf{Z}, \phi) - T(\phi)]^2 = \sigma^2(\phi). \tag{2.8}$$

The proof of this theorem depends upon two lemmas and the following well-known result due to Skorokhod (1956). Let F_N denote the right continuous empirical distribution function based on $\mathbf{Z}$, let

$$W_N(t) = N^{\frac{1}{2}}[F_N \circ F^{-1}(t) - t], \tag{2.9}$$

and let $W(t)$ be the standard Brownian bridge. Then there exists a probability space $(\Omega, \mathscr{A}, \mathscr{P})$ and versions of W_N and W defined on that space such that

$$\lim_{N\to\infty} \sup_{0\leqq t\leqq 1} |W_N(t) - W(t)| = 0 \tag{2.10}$$

and the sample paths of W are all continuous. These uniformly convergent versions of W_N will be used as needed in the proofs.

Define G_{N1}, G_{N2} by

$$G_{N2}(x) = \frac{NF_N(x)}{N-1}, \qquad G_{N1}(x) = G_{N2}(x) - \frac{1}{N-1}. \tag{2.11}$$

Then

$$T_N(\mathbf{Z}, \phi) = \frac{1}{2\theta_N} \int [\phi \circ G_{N1}(x + \theta_N) - \phi \circ G_{N2}(x - \theta_N)]\, dF_N(x). \tag{2.12}$$

Let

$$\begin{aligned} T_{N1} = \frac{1}{2\theta_N} \int [\phi \circ F(x + \theta_N) &+ (G_{N1}(x + \theta_N) \\ &- F(x + \theta_N))\phi' \circ F(x + \theta_N) - \phi \circ F(x - \theta_N) \\ &- (G_{N2}(x - \theta_N) - F(x - \theta_N))\phi' \circ F(x - \theta_N)]\, dF_N(x), \end{aligned} \tag{2.13}$$

and without loss of generality, assume $\theta > 0$ throughout.

LEMMA 2.1. *If F is continuous and assumption* B *is satisfied, then*

$$\lim_{N\to\infty} NE[T_N(\mathbf{Z}, \phi) - T_{N1}]^2 = 0. \tag{2.14}$$

PROOF. By Taylor expansion of $\phi \circ G_{Ni}(x)$ about $\phi \circ F(x)$, we find that

$$N^{\frac{1}{2}}[T_N(\mathbf{Z}, \phi) - T_{N1}] = \frac{1}{4\theta}[I_{N1} - I_{N2}] \tag{2.15}$$

where

$$I_{N1} = N \int [G_{N1}(x) - F(x)]^2 \phi'' \circ \xi_{N1}(x)\, dF_N(x - \theta_N) \tag{2.16}$$
$$I_{N2} = N \int [G_{N2}(x) - F(x)]^2 \phi'' \circ \xi_{N2}(x)\, dF_N(x + \theta_N)$$

and $\xi_{Ni}(x)$ lies between $G_{Ni}(x)$ and $F(x)$ at the discontinuities of $F_N(x \pm \theta_N)$, as required. Since

$$N^{\frac{1}{2}}[G_{N2}(x) - F(x)] = W_N \circ F(x) + \frac{N^{\frac{1}{2}} F_N(x)}{N-1} \tag{2.17}$$
$$N^{\frac{1}{2}}[G_{N1}(x) - F(x)] = W_N \circ F(x) + \frac{N^{\frac{1}{2}}[F_N(x) - 1]}{N-1}$$

it follows under B that

$$\lim_{N\to\infty} I_{Ni} = \int [W \cdot F(x)]^2 \phi'' \circ F(x)\, dF(x) \tag{2.18}$$

for the Skorokhod versions of W_N and W. Hence $N^{\frac{1}{2}}[T_N(\mathbf{Z}, \phi) - T(\phi)] \to_p 0$. This may be strengthened to (2.14) because, by direct calculation, there exists a constant C such that $E\,|I_{Ni}|^3 < C$ for every N.

Let $\mathscr{H}_N$ denote the set of all statistics of the form $\sum_{i=1}^N h(Z_i)$ that have finite mean square. Hájek (1968) has shown that if $S_N = S_N(\mathbf{Z})$ is an arbitrary statistic with finite mean square, its projection $\hat{S}_N$ into $\mathscr{H}_N$ is given by

$$\hat{S}_N = \sum_{k=1}^N E(S_N \mid Z_k) - (N-1)E(S_N)\,. \tag{2.19}$$

In particular, suppose that

$$\begin{aligned} S_N = \frac{1}{2N\theta_N} \sum_{i=1}^N \Bigg\{\Bigg[&\frac{1}{N-1} \sum_{j\neq i} v(Z_i - Z_j + \theta_N) \\ &- F(Z_i + \theta_N)\Bigg] \phi' \circ F(Z_i + \theta_N) \\ &- \left[\frac{1}{N-1} \sum_{j\neq i} v(Z_i - Z_j - \theta_N) - F(Z_i - \theta_N)\right] \phi' \circ F(Z_i - \theta_N)\Bigg\}\,. \end{aligned} \tag{2.20}$$

Since

$$\begin{aligned} E\Bigg\{&\left[\frac{1}{N-1} \sum_{j\neq i} v(Z_i - Z_j \pm \theta_N) - F(Z_i \pm \theta_N)\right] \phi' \circ F(Z_i \pm \theta_N) \,\Big|\, Z_k\Bigg\} \\ &= 0 \qquad \text{if} \quad i = k \\ &= \frac{1}{N-1} \int [v(x - Z_k \pm \theta_N) \\ &\qquad - F(x \pm \theta_N)] \phi' \circ F(x \pm \theta_N)\, dF(x) \qquad \text{if} \quad i \neq k\,, \end{aligned} \tag{2.21}$$

the projection of S_N into $\mathscr{H}_N$ is, in this case,

$$\begin{aligned} \hat{S}_N &= \frac{1}{2N\theta_N} \sum_{k=1}^N \int \{[v(x - Z_k + \theta_N) - F(x + \theta_N)] \phi' \circ F(x + \theta_N) \\ &\qquad - [v(x - Z_k - \theta_N) - F(x - \theta_N)] \phi' \circ F(x - \theta_N)\}\, dF(x) \\ &= \frac{1}{2\theta_N} \int [F_N(x) - F(x)] \phi' \circ F(x) [f(x - \theta_N) - f(x + \theta_N)]\, dx\,. \end{aligned} \tag{2.22}$$

Since

$$T_{N1} = \frac{1}{2\theta_N} \int [\phi \circ F(x + \theta_N) - \phi \circ F(x - \theta_N)]\, dF_N(x) + S_N\,, \tag{2.23}$$

the projection of T_{N1} into $\mathscr{H}_N$ is

$$T_{N2} = \frac{1}{2\theta_N} \int [\phi \cdot F(x + \theta_N) - \phi \circ F(x - \theta_N)]\, dF_N(x) + \hat{S}_N\,. \tag{2.24}$$

LEMMA 2.2. *Under assumptions* A *and* B,

$$\lim_{N\to\infty} NE[T_{N1} - T_{N2}]^2 = 0\,. \tag{2.25}$$

PROOF. Since $\hat{S}_N$ is the projection of S,

$$NE[T_{N1} - T_{N2}]^2 = NE(S_N{}^2) - NE(\hat{S}_N{}^2)\,. \tag{2.26}$$

Defining $\{\Delta_{ij};\ 1 \leqq i, j \leqq N\}$ by

$$\begin{aligned}\Delta_{ij} = N^{\frac{1}{2}}\{&[v(Z_i - Z_j + \theta_N) - F(Z_i + \theta_N)]\phi' \circ F(Z_i + \theta_N) \\ &- [v(Z_i - Z_j - \theta_N) - F(Z_i - \theta_N)]\phi' \circ F(Z_i - \theta_N)\}\,,\end{aligned} \tag{2.27}$$

we have, from (2.20),

$$\begin{aligned}NE(S_N{}^2) = \frac{1}{4N^2(N-1)^2\theta_N{}^2} E[&\textstyle\sum\sum_{i\neq j} \Delta_{ij}^2 + \sum\sum_{i\neq j} \Delta_{ij}\Delta_{ji} \\ &+ \textstyle\sum\sum\sum_{i\neq j\neq k} \Delta_{ji}\Delta_{ki} + \sum\sum\sum_{i\neq j\neq k} \Delta_{ij}\Delta_{ik} \\ &+ \textstyle\sum\sum\sum_{i\neq j\neq k} \Delta_{ij}\Delta_{jk} + \sum\sum\sum_{i\neq j\neq k} \Delta_{ij}\Delta_{ki} \\ &+ \textstyle\sum\sum\sum\sum_{i\neq j\neq k\neq m} \Delta_{ij}\Delta_{km}]\,.\end{aligned} \tag{2.28}$$

By direct calculation, noting that f is of bounded variation under assumption A,

$$\begin{aligned}E(\Delta_{ij}^2) &= N \int [F(x + \theta_N) - F(x - \theta_N)][\phi' \circ F(x + \theta_N)]^2\, dF(x) \\ &\quad + N \int F(x - \theta_N)[\phi' \circ F(x + \theta_N) - \phi' \circ F(x - \theta_N)]^2\, dF(x) \\ &\quad - N \int [F(x + \theta_N)\phi' \circ F(x + \theta_N) \\ &\quad - F(x - \theta_N)\phi' \circ F(x - \theta_N)]^2\, dF(x) \\ &= O(N^{\frac{1}{2}}) \\ |E(\Delta_{ij}\Delta_{ji})| &\leqq E(\Delta_{ij}^2) = O(N^{\frac{1}{2}}) \\ E(\Delta_{ij}\Delta_{ik}) &= E(\Delta_{ij}\Delta_{jk}) = E(\Delta_{ij}\Delta_{ki}) = E(\Delta_{ij}\Delta_{km}) = 0 \\ E(\Delta_{ji}\Delta_{ki}) &= N \iint [\min(F(x), F(y)) \\ &\quad - F(x)F(y)]\phi' \circ F(x)\phi' \circ F(y)[f(x - \theta_N) \\ &\quad - f(x + \theta_N)][f(y - \theta_N) - f(y + \theta_N)]\, dx\, dy\,.\end{aligned} \tag{2.29}$$

By dominated convergence, since $|f'(x)|$ is integrable under assumption A,

$$\begin{aligned}&(4\theta^2)^{-1} \lim_{N\to\infty} E(\Delta_{ji}\Delta_{ki}) \\ &\quad = \iint [\min(F(x), F(y)) - F(x)F(y)]\phi' \circ F(x)\phi' \circ F(y) f'(x) f'(y)\, dx\, dy \\ &\quad = \lim_{N\to\infty} NE(\hat{S}_N{}^2)\,.\end{aligned} \tag{2.30}$$

The lemma follows.

PROOF OF THEOREM 2.1. (Using (2.22), 2.24) and integrating by parts, we obtain

$$N^{\frac{1}{2}}[T_{N2} - \frac{1}{2\theta_N} \int [\phi \circ F(x + \theta_N) - \phi \circ F(x - \theta_N)]\, dF(x)]$$

$$(2.31) \qquad = \frac{1}{2\theta_N} \int W_N \circ F(x)\{\phi' \circ F(x - \theta_N)f(x - \theta_N) - \phi' \circ F(x + \theta_N)f(x + \theta_N) + \phi' \circ F(x)[f(x - \theta_N) - f(x + \theta_N)]\}\, dx\,.$$

For Skorokhod versions of W_N and W, the right-hand side converges, by dominated convergence, to

$$(2.32) \qquad \int_0^1 W(t)[2\phi'(t)\phi_F(t) - \phi''(t) f \circ F^{-1}(t)]\, dt\,,$$

which has a normal $(0, \sigma^2(\phi))$ distribution.

Secondly, by Fubini's theorem,

$$N^{\frac{1}{2}} \left| \frac{1}{2\theta_N} \int [\phi \circ F(x + \theta_N) - \phi \circ F(x - \theta_N)]\, dF(x) - T(\phi) \right|$$

$$= \left| \frac{N^{\frac{1}{2}}}{2\theta_N} \int_0^{\theta_N} dt \int_0^t ds \int_{-\infty}^{\infty} f'(x)[\phi' \circ F(x - s)f(x - s) - \phi' \circ F(x + s)f(x + s)]\, dx \right|$$

$$(2.33) \qquad \leqq \frac{\theta}{2} \sup_{0 \leqq s \leqq \theta_N} \left| \int_{-\infty}^{\infty} f'(x)[\phi' \circ F(x - s)f(x - s) - \phi' \circ F(x + s)f(x + s)]\, dx \right|\,,$$

which tends to 0 as $N \to \infty$, by dominated convergence. The first part of Theorem 2.1 now follows from Lemmas 2.1, 2.2 and from the above.

Finally, to prove (2.8), use (2.31) and dominated convergence to show

$$(2.34) \qquad \lim_{N \to \infty} NE\left[T_{N2} - \frac{1}{2\theta_N} \int [\phi \circ F(x + \theta_N) - \phi \circ F(x - \theta_N)]\, dF(x) \right]^2 = \sigma^2(\phi)\,,$$

then consult Lemmas 2.1, 2.2 and (2.33).

We turn now to the random function $\hat{\phi}_F$ proposed earlier as an estimate for ϕ_F. Let $\{M_\alpha\}$, $\{N_\alpha\}$ be sequences of positive integers. Following (2.3), let $\hat{c}_{k,\alpha} = T_{N_\alpha}(\mathbf{Z}, \exp(-2\pi i k \cdot))$ and let

$$(2.35) \qquad \hat{\phi}_{F,\alpha}(t) = \sum_{|k|=1}^{M_\alpha} \hat{c}_{k,\alpha} \exp(2\pi i k t)\,.$$

THEOREM 2.2. *If assumption* A *is satisfied and if*

$$(2.36) \qquad \lim_{\alpha \to \infty} M_\alpha = \infty\,, \qquad \lim_{\alpha \to \infty} M_\alpha^{\frac{3}{2}}/N_\alpha = 0\,,$$

then

$$(2.37) \qquad \lim_{\alpha \to \infty} E\, ||\hat{\phi}_{F,\alpha} - \phi_F||^2 = 0\,.$$

PROOF. Note that (2.36) implies that $\lim_{\alpha \to \infty} N_\alpha = \infty$. For convenience, the subscript α will be dropped. If $\phi(t) = \cos(2\pi kt)$ or $\sin(2\pi kt)$, re-examination of the approximations used in establishing Theorem 2.1 shows that there exist constants $\{A_i\}$, independent of α, such that

$$E[T_N(\mathbf{Z}, \phi) - T_{N1}]^2 \leqq \frac{A_1 k^4}{N^{\frac{3}{2}}} + \frac{A_2 k^6}{N^2}$$

$$(2.38) \qquad E[T_{N1} - T_{N2}]^2 \leqq \frac{A_3 k^2}{N^{\frac{3}{2}}} + \frac{A_4 k^4}{N^2}$$

$$E\left[T_{N2} - \frac{1}{2\theta_N} \int [\phi \circ F(x + \theta_N) - \phi \circ F(x - \theta_N)]\, dF(x)\right]^2 \leqq \frac{A_5 k^2}{N}$$

$$\left[\frac{1}{2\theta_N} \int [\phi \circ F(x + \theta_N) - \phi \circ F(x - \theta_N)]\, dF(x) - T(\phi)\right]^2 \leqq \frac{A_6 k^2}{N}.$$

Hence, there exist constants $\{B_i\}$, independent of α, such that

$$(2.39) \qquad E[T_N(\mathbf{Z}, \phi) - T(\phi)]^2 \leqq \frac{B_1 k^2}{N} + \frac{B_2 k^4}{N^{\frac{3}{2}}} + \frac{B_3 k^6}{N^2}.$$

The first bound in (2.38) follows from the Taylor expansion

$$N^{\frac{1}{2}}[T_N(Z, \phi) - T_{N1}]$$
$$= \frac{1}{4\theta} \int [W_{N1}^2 \cdot F(x + \theta_N) - W_{N2}^2 \circ F(x - \theta_N)]\phi'' \circ F(x + \theta_N)\, dF_N(x)$$
$$(2.40) \qquad + \frac{1}{4\theta} \int [\phi'' \circ F(x + \theta_N) - \phi'' \circ F(x - \theta_N)] W_{N2}^2 \circ F(x - \theta_N)\, dF_N(x)$$
$$+ \frac{N^{-\frac{1}{2}}}{12\theta} \int [W_{N1}^3 \circ F(x + \theta_N)\phi''' \circ \zeta_{N1}(x)$$
$$- W_{N2}^3 \circ F(x - \theta_N)\phi''' \circ \zeta_{N2}(x)]\, dF_N(x)$$

where $\zeta_{Ni}(x)$ lies between $G_{Ni}(x \pm \theta_N)$ and $F(x \pm \theta_N)$ at the discontinuities of $F_N(x)$, as required, and

$$(2.41) \qquad W_{Ni} \circ F(x) = N^{\frac{1}{2}}[G_{Ni}(x) - F(x)].$$

For the second bound, observe from the proof of Lemma 2.2 that

$$(2.42) \qquad NE[T_{N1} - T_{N2}]^2 \leqq \frac{E(\Delta_{ij}^2)}{2\theta^2(N-1)} + \frac{NE(\hat{S}_N{}^2)}{N-1}$$

and consider the expressions for the two expectations on the right side. The remaining two bounds in (2.38) are straightforward.

In view of (2.39), there exist constants $\{C_i\}$ independent of α such that

$$(2.43) \qquad E \sum_{|k|=1}^{M} |\hat{c}_k - c_k|^2 \leqq \frac{C_1 M^3}{N} + \frac{C_2 M^5}{N^{\frac{3}{2}}} + \frac{C_3 M^7}{N^2}.$$

Since

$$(2.44) \qquad \|\hat{\phi}_F - \phi_F\|^2 = \sum_{|k|=1}^{M} |\hat{c}_k - c_k|^2 + \sum_{|k|=M+1}^{\infty} |c_k|^2,$$

the theorem follows.

3. Estimation in the two-sample model. Let $\{m_\alpha\}$, $\{n_\alpha\}$ be sequences of sample sizes and suppose that $X_1, X_2, \cdots, X_{m_\alpha}, Y_1, Y_2, \cdots, Y_{n_\alpha}$ have joint density $\prod_{i=1}^{m_\alpha} f(x_i - \mu_0) \prod_{j=1}^{n_\alpha} f(y_j)$, where the location difference μ_0 does not depend on α. For this two-sample model, let

$$\tilde{c}_{k,\alpha} = \frac{m_\alpha T_{m_\alpha}(\mathbf{X}, \exp(-2\pi i k\cdot)) + n_\alpha T_{n_\alpha}(\mathbf{Y}, \exp(-2\pi i k\cdot))}{m_\alpha + n_\alpha} \tag{3.1}$$

and for a sequence of integers $\{M_\alpha\}$, set

$$\tilde{\phi}_{F,\alpha}(t) = \sum_{k=1}^{M_\alpha} \tilde{c}_{k,\alpha} \exp(2\pi i k t) . \tag{3.2}$$

If $\lim_{\alpha\to\infty} M_\alpha = \infty$ and $\lim_{\alpha\to\infty} M_\alpha^{\frac{7}{2}}/\min(m_\alpha, n_\alpha) = 0$, it follows from Theorem 2.2 that

$$\lim_{\alpha\to\infty} E\,||\tilde{\phi}_{F,\alpha} - \phi_F||^2 = 0 . \tag{3.3}$$

Note that $\tilde{\phi}_{F,\alpha}$ is a location invariant estimate of ϕ_F.

Suppose that $\hat{\mu}_\alpha$ is an estimate of μ_0 that satisfies the following assumption:

C. $\hat{\mu}_\alpha$ is a location equivariant estimate of μ_0 and

$$\left(\frac{m_\alpha n_\alpha}{m_\alpha + n_\alpha}\right)^{\frac{1}{2}} (\hat{\mu}_\alpha - \mu_0) \text{ is bounded in probability as } \alpha \to \infty .$$

For every real number μ, let $(R_1(\mu), R_2(\mu), \cdots, R_{m_\alpha+n_\alpha}(\mu))$ denote the rank vector of $(X_1 - \mu, \cdots, X_{m_\alpha} - \mu, Y_1, \cdots, Y_{n_\alpha})$. When $\mu = 0$, we will write more simply $(R_1, R_2, \cdots, R_{m_\alpha+n_\alpha})$. As an adaptive estimate for μ_0, consider

$$\tilde{\mu}_\alpha = \hat{\mu}_\alpha + ||\tilde{\phi}_{F,\alpha}||^{-2} \left(\frac{m_\alpha + n_\alpha}{m_\alpha n_\alpha}\right) \sum_{j=1}^{m_\alpha} \tilde{\phi}_{F,\alpha}\left(\frac{R_j(\hat{\mu}_\alpha)}{m_\alpha + n_\alpha + 1}\right), \tag{3.4}$$

a form suggested by the linearized rank estimates studied by Kraft and van Eeden (1970).

THEOREM 3.1. *If assumptions* A *and* C *are satisfied and if*

$$\lim_{\alpha\to\infty} M_\alpha = \infty , \qquad \lim_{\alpha\to\infty} M_\alpha^{6}/\min(m_\alpha, n_\alpha) = 0 , \tag{3.5}$$

then the asymptotic distribution of $(m_\alpha n_\alpha/m_\alpha + n_\alpha)(\tilde{\mu}_\alpha - \mu_0)$ *is normal* $(0, ||\phi_F||^{-2})$.

The proof of this theorem is based upon two lemmas. For convenience, the subscript α will be dropped in the following calculations.

LEMMA 3.1. *If assumption* A *and* (3.5) *hold and if* $\mu_0 = 0$, *then*

$$\begin{aligned} \sup_{|\mu| \leqq C(m+n/mn)^{\frac{1}{2}}} \Bigg| &\left(\frac{m+n}{mn}\right)^{\frac{1}{2}} \sum_{j=1}^m \tilde{\phi}_F\left(\frac{R_j(\mu)}{m+n+1}\right) \\ &- \left(\frac{m+n}{mn}\right)^{\frac{1}{2}} \sum_{j=1}^m \tilde{\phi}_F\left(\frac{R_j}{m+n+1}\right) + \left(\frac{mn}{m+n}\right)^{\frac{1}{2}} \mu ||\phi_F||^2 \Bigg| \to_p 0 . \end{aligned} \tag{3.6}$$

for every $C > 0$.

PROOF. Let

$$H(x, \mu) = \frac{mF(x) + nF(x - \mu)}{m + n + 1} \tag{3.7}$$

$$H_{m,n}(x, \mu) = \frac{mF_m(x) + nG_n(x - \mu)}{m + n + 1},$$

where F_m and G_n are the right continuous empirical distribution functions based on X and Y respectively. Write $H(x)$, $H_{m,n}(x)$ for $H(x, 0)$, $H_{m,n}(x, 0)$ respectively. By Taylor expansion, setting $A_{m,n} = (m + n/mn)^{\frac{1}{2}}$,

$$\begin{aligned} A_{m,n} \textstyle\sum_{j=1}^m \tilde{\phi}_F\left(\frac{R_j(\mu)}{m+n+1}\right) - A_{m,n} \sum_{j=1}^m \tilde{\phi}_F\left(\frac{R_j}{m+n+1}\right) \\ = mA_{m,n} \textstyle\int [\tilde{\phi}_F \circ H_{m,n}(x, \mu) - \tilde{\phi}_F \circ H_{m,n}(x)]\, dF_m(x) \\ = mA_{m,n}\{\textstyle\int [\tilde{\phi}_F \circ H(x, \mu) - \tilde{\phi}_F \circ H(x)]\, dF_m(x) \\ + \textstyle\int \{[H_{m,n}(x, \mu) - H(x, \mu)] - [H_{m,n}(x) - H(x)]\}\tilde{\phi}_F' \circ H(x, \mu)\, dF_m(x) \\ + \textstyle\int [H_{m,n}(x) - H(x)][\tilde{\phi}_F' \circ H(x, \mu) - \tilde{\phi}_F' \circ H(x)]\, dF_m(x) \\ + \tfrac{1}{2} \textstyle\int [H_{m,n}(x, \mu) - H(x, \mu)]^2\tilde{\phi}_F'' \circ \delta_1(x)\, dF_m(x) \\ + \tfrac{1}{2} \textstyle\int [H_{m,n}(x) - H(x)]^2\tilde{\phi}_F'' \circ \delta_2(x)\, dF_m(x)\} = \sum_{i=1}^5 I_i \end{aligned} \tag{3.8}$$

where $\delta_1(x_j)$ lies between $H_{m,n}(X_j, \mu)$ and $H(X_j, \mu)$ while $\delta_2(X_j)$ lies between $H_{m,n}(X_j)$ and $H(X_j)$.

If $\tilde{\phi}_F^{(r)}$ denotes the rth derivative of $\tilde{\phi}_F$,

$$\begin{aligned} \sup_{0\leq t\leq 1} |\tilde{\phi}_F^{(r)}(t)| &\leqq [\textstyle\sum_{|k|=1}^M (2\pi k)^{2r}]^{\frac{1}{2}} \cdot [\sum_{|k|=1}^M |\tilde{c}_k|^2]^{\frac{1}{2}} \\ &= O(M^{\frac{1}{2}(2r+1)}) \cdot ||\tilde{\phi}_F|| . \end{aligned} \tag{3.9}$$

Hence, for $|\mu| \leqq A_{m,n} C$, $\sup |I_2|$ is $O_p(M^{\frac{3}{2}} \cdot \min (m^{\frac{1}{2}}, n^{\frac{1}{2}})) \cdot ||\tilde{\phi}_F||$ and $\sup |I_j|$ is $O_p(M^{\frac{5}{2}}/\min (m^{\frac{1}{2}}, n^{\frac{1}{2}})) \cdot ||\tilde{\phi}_F||$ for $3 \leqq j \leqq 5$. Since $||\tilde{\phi}_F|| \to_p ||\phi_F|| < \infty$ under the hypotheses of the lemma, all terms of the expansion (3.8) other than I_1 are asymptotically negligible.

Let

$$\begin{aligned} J_1 &= mA_{m,n} \textstyle\int [H(x, \mu) - H(x)]\tilde{\phi}' \circ H(x)\, dF_m(x) \\ &= A_{m,n} \frac{mn}{m+n+1} \textstyle\int [F(x - \mu) - F(x)]\tilde{\phi}_F' \circ H(x)\, dF_m(x) \\ J_2 &= -A_{m,n} \frac{mn}{m+n+1} \mu \textstyle\int f(x)\tilde{\phi}_F' \circ H(x)\, dF_m(x) \\ J_3 &= -A_{m,n}^{-1}\, \mu \textstyle\int f(x)\tilde{\phi}_F' \circ F(x)\, dF_m(x) \\ J_4 &= -A_{m,n}^{-1}\, \mu \textstyle\int f(x)\tilde{\phi}_F' \circ F(x)\, dF(x) \\ &= -A_{m,n}^{-1}\, \mu \textstyle\int_0^1 \tilde{\phi}_F(t)\phi_F(t)\, dt \\ J_5 &= -A_{m,n}^{-1}\, \mu\, ||\phi_F||^2 . \end{aligned} \tag{3.10}$$

Note that $\tilde{\phi}_F = (m\hat{\phi}_F{}^X + n\hat{\phi}_F{}^Y)/(m + n)$, where $\hat{\phi}_F{}^X$, $\hat{\phi}_F{}^Y$ are the estimates of ϕ_F based upon the first and second samples respectively. For $|\mu| \leqq A_{m,n} C$, the differences $\sup|I_1 - J_1|$, $\sup|J_2 - J_3|$, $\sup|J_3 - J_4|$ are all $O_p(M^{\frac{3}{2}}/\min(m^{\frac{1}{2}}, n^{\frac{1}{2}})) \cdot \|\tilde{\phi}_F\|$, while $\sup|J_1 - J_2|$ is $O_p(M^{\frac{3}{2}}/\min(m^{\frac{1}{2}}, n^{\frac{1}{2}})) \cdot \max(\|\hat{\phi}_F{}^X\|, \|\hat{\phi}_F{}^Y\|)$ and $\sup|J_4 - J_5|$ is $O_p(\|\tilde{\phi}_F - \phi_F\|)$. Under the assumptions, these bounds are all asymptotically negligible, so that the lemma follows.

LEMMA 3.2. *If assumption* A *is satisfied, if* $\mu_0 = 0$, *and if*

$$\lim_{\alpha\to\infty} M_\alpha = \infty\,, \qquad \lim_{\alpha\to\infty} M_\alpha{}^4/\min(m_\alpha, n_\alpha) = 0\,, \tag{3.11}$$

then the asymptotic distribution of $(m_\alpha + n_\alpha/m_\alpha n_\alpha)^{\frac{1}{2}} \sum_{j=1}^{m_\alpha} \tilde{\phi}_{F,\alpha}(R_j/m_\alpha + n_\alpha + 1)$ *is normal* $(0, \|\phi_F\|^2)$.

PROOF. Let $\phi_{F,M}(t) = \sum_{|k|=1}^{M} c_k \exp(2\pi ikt)$ and for $1 \leqq j \leqq m+n$, let $a_{m,n}(j) = E\phi_F(U^{(j)})$, where $U^{(1)} < \cdots < U^{(m+n)}$ form an ordered random sample from the uniform distribution on $(0, 1)$. Let $K_1 = (m + n/mn)^{\frac{1}{2}} \sum_{j=1}^{m} \tilde{\phi}_F(R_j/(m + n + 1))$ and define K_2, K_3 similarly by replacing $\tilde{\phi}_F(\cdot/(m+n+1))$ with $\phi_{F,M}(\cdot/(m+n+1))$; $a_{m,n}(\cdot)$ respectively. From the Cauchy–Schwarz inequality, some calculation, and (2.43),

$$\begin{aligned} E|K_1 - K_2| &\leqq \Big[\sum_{|k|=1}^{M} E|\tilde{c}_k - c_k|^2\Big]^{\frac{1}{2}} \\ &\qquad \times \left[\frac{m+n}{mn} \sum_{|k|=1}^{M} E\left|\sum_{j=1}^{m} \exp\left(\frac{2\pi ikR_j}{m+n+1}\right)\right|^2\right]^{\frac{1}{2}} \\ &= O(M^2/\min(m^{\frac{1}{2}}, n^{\frac{1}{2}})) + O(M^3/\min(m^{\frac{3}{4}}, n^{\frac{3}{4}})) \\ &\qquad + O(M^4/\min(m, n))\,, \end{aligned} \tag{3.12}$$

so that $K_1 - K_2 \to_p 0$ as $\alpha \to \infty$. Also, writing N for $m + n$,

$$\begin{aligned} E[K_2 - K_3]^2 &= \frac{1}{N-1} \sum_{j=1}^{N}\left[\phi_{F,M}\left(\frac{j}{N+1}\right) - a_{m,n}(j)\right]^2 \\ &\qquad + \frac{N}{mn}\left|E\sum_{j=1}^{m} \phi_{F,M}\left(\frac{R_j}{N+1}\right)\right|^2 \\ &= \frac{N}{N-1}\int_0^1\left[\phi_{F,M}\left(\frac{1+[tN]}{N+1}\right) - a_{m,n}(1+[tN])\right]^2 dt \\ &\qquad + O(M/\min(m, n))\,. \end{aligned} \tag{3.13}$$

Since $\int_0^1 [\phi_{F,M}((1 + [tN])/(N+1)) - \phi_{F,M}(t)]^2\,dt$ is $O(M^3/\min(m^2, n^2))$ and

$$\lim_{\alpha\to\infty} \int_0^1 [\phi_{F,M}(t) - \phi_F(t)]^2\,dt = 0\,, \quad \lim_{\alpha\to\infty} \int_0^1 [a_{m,n}(1 + [tN]) - \phi_F(t)]^2\,dt = 0\,,$$

the last limit being proved in Hájek and Šidák (1967), page 158, we conclude that $K_2 - K_3 \to_p 0$ as $\alpha \to \infty$. The lemma follows from the asymptotic normality of K_3 (see Hájek and Šidák (1967)).

PROOF OF THEOREM 3.1. Since $\hat{\mu}$ is location equivariant by assumption and $\tilde{\phi}_F$ is location invariant, we may assume $\mu_0 = 0$ without loss of generality. From (3.4), assumption C, and Lemma 3.1 follows

$$\left(\frac{mn}{m+n}\right)^{\frac{1}{2}}\hat{\mu} = \|\phi_F\|^{-2}\left(\frac{m+n}{mn}\right)^{\frac{1}{2}} \sum_{j=1}^{m} \tilde{\phi}_F\left(\frac{R_j}{m+n+1}\right) + o_p(1)\,. \tag{3.14}$$

Hence, by Lemma 3.2, $[mn/(m+n)]^{\frac{1}{2}}\tilde{\mu}$ is asymptotically normal $(0, ||\phi_F||^{-2})$.

4. Estimation in the one-sample model. Let $\{N_\alpha\}$ be a sequence of sample sizes and suppose that $X_1, X_2, \cdots, X_{N_\alpha}$ have joint density $\prod_{i=1}^{N_\alpha} f(x_i - \nu_0)$, where f is symmetric about the origin and ν_0 does not depend upon α. For this model, ϕ_F is skew-symmetric about $t = \frac{1}{2}$ and therefore has Fourier expansion

$$\phi_F(t) \sim \sum_{k=1}^{\infty} d_k \sin(2\pi kt) , \tag{4.1}$$

where $d_k = 2\int_0^1 \phi_F(t)\sin(2\pi kt)\,dt$. The estimate for ϕ_F becomes

$$\tilde{\tilde{\phi}}_{F,\alpha}(t) = \sum_{k=1}^{M_\alpha} \tilde{d}_{k,\alpha}\sin(2\pi kt) , \tag{4.2}$$

where $\tilde{d}_{k,\alpha} = 2T_{N_\alpha}(\mathbf{X}, \sin(2\pi k\,\cdot))$, and the corresponding estimate for ψ_F is $\tilde{\psi}_{F,\alpha}(t) = \tilde{\tilde{\phi}}_{F,\alpha}(\frac{1}{2} + t/2)$. If $\lim_{\alpha\to\infty} M_\alpha = \infty$ and $\lim_{\alpha\to\infty} M_\alpha^{\frac{3}{2}}/N_\alpha = 0$, it follows from the proof of Theorem 2.2 that $\lim_{\alpha\to\infty} E\,||\tilde{\psi}_{F,\alpha} - \psi_F||^2 = 0$. Note that $\tilde{\psi}_{F,\alpha}$ is a location invariant estimate of ψ_F.

Suppose that $\hat{\nu}_\alpha$ is an estimate of ν_0 which satisfies the following assumption:

D. $\hat{\nu}_\alpha$ is a location equivariant estimate of ν_0 and

$$N_\alpha^{\frac{1}{2}}(\hat{\nu}_\alpha - \nu_0) \text{ is bounded in probability as } \alpha \to \infty .$$

For every real number ν, let $(R_1^+(\nu), R_2^+(\nu), \cdots, R_{N_\alpha}^+(\nu))$ denote the rank vector of $(|X_1 - \nu|, \cdots, |X_{N_\alpha} - \nu|)$. When $\nu = 0$, we will write more simply $(R_1^+, R_2^+, \cdots, R_{N_\alpha}^+)$. As an adaptive estimate for ν_0, consider

$$\tilde{\nu}_\alpha = \hat{\nu}_\alpha + ||\tilde{\phi}_{F,\alpha}||^{-2} N_\alpha^{-1} \sum_{j=1}^{N_\alpha} \tilde{\psi}_{F,\alpha}\left(\frac{R_j^+(\hat{\nu}_\alpha)}{N_\alpha + 1}\right) \cdot \operatorname{sgn}(X_j - \hat{\nu}_\alpha) , \tag{4.3}$$

where $\operatorname{sgn}(x) = 1, -1$, or 0 according to whether x is positive, negative, or zero. As in the two-sample model, this estimate is suggested by the linearized rank estimates studied by Kraft and van Eeden (1970).

THEOREM 4.1. *If f is symmetric, if assumptions* A *and* D *are satisfied, and if*

$$\lim_{\alpha\to\infty} M_\alpha = \infty , \qquad \lim_{\alpha\to\infty} M_\alpha^{6}/N_\alpha = 0 \tag{4.4}$$

then the asymptotic distribution of $N_\alpha^{\frac{1}{2}}(\tilde{\nu}_\alpha - \nu_0)$ is normal $(0, ||\psi_F||^{-2})$.

The proof of this theorem rests on the following two lemmas. For convenience, the subscript α is dropped in their statements. Since the proofs of these lemmas are simply modifications of the proofs for Lemmas 3.1 and 3.2, we omit further details. Note that $||\psi_F|| = ||\phi_F||$.

LEMMA 4.1. *If f is symmetric, if assumption* A *and* (4.4) *hold, and if $\nu_0 = 0$, then*

$$\sup_{|\nu|\leq CN^{-\frac{1}{2}}} \Big| N^{-\frac{1}{2}} \sum_{j=1}^{N} \tilde{\psi}_F\left(\frac{R_j^+(\nu)}{N+1}\right) \operatorname{sgn}(X_j - \nu) - N^{-\frac{1}{2}} \sum_{j=1}^{N} \tilde{\psi}_F\left(\frac{R_j^+}{N+1}\right) \operatorname{sgn}(X_j) + N^{\frac{1}{2}}\nu\,||\psi_F||^2 \Big| \to_p 0 \tag{4.5}$$

for every $C > 0$.

LEMMA 4.2. *If f is symmetric, if assumption* A *is satisfied, if $\nu_0 = 0$, and if*

$$\lim_{\alpha\to\infty} M_\alpha = \infty\ , \qquad \lim_{\alpha\to\infty} M_\alpha^4/N_\alpha = 0 \tag{4.6}$$

then the asymptotic distribution of $N_\alpha^{-\frac{1}{2}} \sum_{j=1}^{N_\alpha} \tilde{\phi}_{F,\alpha}(R_j^+/N_\alpha + 1)\,\mathrm{sgn}\,(X_j)$ *is normal* $(0, ||\phi_F||^2)$.

5. Remarks. In proving Theorems 3.1 and 4.1, the only regularity assumptions made about the density f were absolute continuity and finite Fisher information. Consequently, the location estimates $\tilde{\mu}$, $\tilde{\nu}$ are asymptotically efficient over a larger class of distributions F than the corresponding van Eeden (1970) estimates.

For certain F, the estimates $\tilde{\mu}$, $\tilde{\nu}$ may approach their asymptotic behavior very slowly. For example, suppose that F is such that $\phi_F(t) = \sin(2\pi\lambda t)$, where λ is a large integer. In this case, $\tilde{\phi}_F$ or $\tilde{\tilde{\phi}}_F$ will be a poor estimate of ϕ_F until $N \gg M \geqq \lambda$. To avoid this difficulty, it would be necessary to choose the trigonometric basis for $\tilde{\phi}_F$ or $\tilde{\tilde{\phi}}_F$ in such a fashion as to omit all, or at least most, trigonometric functions of frequency less than λ. Thus, a selection problem arises which is similar to the classical problem of choosing useful regressors from the set of all possible regressors in a linear model.

In practice, the initial location estimates $\hat{\mu}$ or $\hat{\nu}$ must be chosen with care, because if these estimates are poor for a given sample, the modified estimates $\tilde{\mu}$ or $\tilde{\nu}$ may be even worse. Reasonable initial estimates when F is unimodal symmetric are the sample median in the one-sample model and the difference between sample medians in the two-sample model.

REFERENCES

[1] HÁJEK, J. (1962). Asymptotically most powerful rank order tests. *Ann. Math. Statist.* **33** 1124–1147.

[2] HÁJEK, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.* **39** 235–346.

[3] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests*. Academic Press.

[4] HODGES, J. L. and LEHMANN, E. L. (1963). Estimation of location based on rank test. *Ann. Math. Statist.* **34** 598–611.

[5] KRAFT, C. and VAN EEDEN, C. (1970). Efficient linearized estimates based on ranks. *Nonparametric Techniques in Statistical Inference*, (M. L. Puri, ed.). Cambridge Univ. Press. 267–273.

[6] SKOROKHOD, A. V. (1956). Limit theorems for stochastic processes. *Theor. Probability Appl.* **1** 261–290.

[7] STEIN, C. (1956). Efficient nonparametric testing and estimation. *Proc. Third Berkeley Symp. Math. Statist. Prob.* 187–195. Univ. of California Press.

[8] VAN EEDEN, C. (1970). Efficiency robust estimation of location. *Ann. Math. Statist.* **41** 172–181.

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