

Theorem 1. Let $X = \varepsilon - \delta$, where $\varepsilon \sim N(0, \sigma_\varepsilon)$ and $\delta \sim N^+(0, \sigma_\delta)$. Then, X has density function

$$f(x) = \frac{2}{\sigma} \phi\left(\frac{x}{\sigma}\right) \left(1 - \Phi\left(\frac{x\lambda}{\sigma}\right)\right),$$

where $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, $\lambda = \sigma_\delta/\sigma_\varepsilon$, ϕ is the normal density function, and Φ is the normal distribution function.

Proof. First consider the distribution function of X , which is given by

$$\begin{aligned} F(x) &= \Pr(X \leq x) = \Pr(\varepsilon - \delta \leq x) \\ &= \int_{\varepsilon - \delta \leq x} f_\varepsilon(\varepsilon) f_\delta(\delta) d\delta d\varepsilon \\ &= \int_{\delta \in \mathbb{R}^+} f_\delta(\delta) \int_{\varepsilon \in (-\infty, x+\delta]} f_\varepsilon(\varepsilon) d\varepsilon d\delta. \end{aligned}$$

Substituting in known density functions yields

$$\begin{aligned} &\int_0^\infty 2\phi(\delta|0, \sigma_\delta) \int_{-\infty}^{x+\delta} \phi(\varepsilon|0, \sigma_\varepsilon) d\varepsilon d\delta \\ &= 2 \int_0^\infty \phi(\delta|0, \sigma_\delta) \Phi(x + \delta|0, \sigma_\varepsilon) d\delta. \end{aligned}$$

The density of X is then given by

$$f(x) = \frac{dF}{dx} = 2 \int_0^\infty \phi(\delta|0, \sigma_\delta) \phi(x + \delta|0, \sigma_\varepsilon) d\delta.$$

Using Sage to perform this integration, the result is given by

$$f(x) = -\frac{\left(\operatorname{erf}\left(\frac{\sigma_\delta x}{2\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}}\right) e^{\left(\frac{\frac{\sigma_\delta^2 x^2}{2(\sigma_\delta^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^4)}}\right)} - e^{\left(\frac{\frac{\sigma_\delta^2 x^2}{2(\sigma_\delta^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^4)}}\right)}\right) e^{\left(-\frac{x^2}{2\sigma_\varepsilon^2}\right)}}{2\sqrt{\pi}\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}}.$$

Defining $\lambda = \sigma_\delta/\sigma_\varepsilon$ and $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, the following can be simplified:

$$\begin{aligned} \frac{\sigma_\delta x}{2\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}} &= \frac{\lambda x}{\sigma\sqrt{2}} = \frac{x}{(\sigma/\lambda)\sqrt{2}}; \\ \frac{\sigma_\delta^2 x^2}{2(\sigma_\delta^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^4)} &= \frac{\lambda^2 x^2}{2\sigma^2} = \frac{x^2}{2(\sigma/\lambda)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= -\frac{\exp\left(\frac{x^2}{2(\sigma/\lambda)^2}\right) \left(\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right) - 1\right) \exp\left(-\frac{x^2}{2\sigma_\varepsilon^2}\right)}{\sigma\sqrt{2\pi}} \\ &= -\frac{\left(\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right) - 1\right) \exp\left(-x^2\left(\frac{1}{2\sigma_\varepsilon^2} - \frac{1}{2(\sigma/\lambda)^2}\right)\right)}{\sigma\sqrt{2\pi}}. \end{aligned}$$

Now,

$$\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right) - 1 = \left(1 + \operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right)\right) - 2 = 2\left(\frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right)\right) - 1\right)$$

$$= 2 \left(\Phi \left(\frac{x\lambda}{\sigma} \right) - 1 \right) = -2 \left(1 - \Phi \left(\frac{x\lambda}{\sigma} \right) \right).$$

Also,

$$\frac{1}{2\sigma_\varepsilon^2} - \frac{1}{2(\sigma/\lambda)^2} = \frac{1}{2\sigma_\varepsilon^2} - \frac{\sigma_\delta^2}{2\sigma_\varepsilon^2(\sigma_\delta^2 + \sigma_\varepsilon^2)} = \frac{\sigma_\delta^2 + \sigma_\varepsilon^2 - \sigma_\delta^2}{2\sigma_\varepsilon^2(\sigma_\delta^2 + \sigma_\varepsilon^2)} = \frac{\sigma_\varepsilon^2}{2\sigma_\varepsilon^2(\sigma_\delta^2 + \sigma_\varepsilon^2)} = \frac{1}{2\sigma^2}.$$

So,

$$f(x) = 2 \left(1 - \Phi \left(\frac{x\lambda}{\sigma} \right) \right) \frac{\exp \left(-\frac{x^2}{2\sigma^2} \right)}{\sigma\sqrt{2\pi}} = 2 \left(1 - \Phi \left(\frac{x\lambda}{\sigma} \right) \right) \phi(x|0, \sigma) = \frac{2}{\sigma} \phi \left(\frac{x}{\sigma} \right) \left(1 - \Phi \left(\frac{x\lambda}{\sigma} \right) \right).$$

□

Theorem 2. Let $X = \varepsilon + \delta$, where $\varepsilon \sim N(0, \sigma_\varepsilon)$ and $\delta \sim N_L^U(\mu_\delta, \sigma_\delta)$. Then, X has density function

$$f(x) = \frac{1}{\sigma} \phi \left(\frac{x - \mu_\delta}{\sigma} \right) \frac{\Phi \left(\frac{\eta(x, U)}{\sigma} \right) - \Phi \left(\frac{\eta(x, L)}{\sigma} \right)}{\Phi \left(\frac{U - \mu_\delta}{\sigma_\delta} \right) - \Phi \left(\frac{L - \mu_\delta}{\sigma_\delta} \right)},$$

where $\eta(x, B) = \lambda(B - x) + (B - \mu_\delta)(1/\lambda)$, $\lambda = \sigma_\delta/\sigma_\varepsilon$, $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, and ϕ and Φ are the standard normal density and distribution functions, respectively.

Proof. First consider the distribution function of X , which is given by

$$\begin{aligned} F(x) &= \Pr(X \leq x) = \Pr(\varepsilon - \delta \leq x) \\ &= \int_{\varepsilon + \delta \leq x} f_\varepsilon(\varepsilon) f_\delta(\delta) d\delta d\varepsilon \\ &= \int_{\delta \in [L, U]} f_\delta(\delta) \int_{\varepsilon \in (-\infty, x - \delta]} f_\varepsilon(\varepsilon) d\varepsilon d\delta. \end{aligned}$$

Substituting in known density functions yields

$$\begin{aligned} &\int_L^U \phi_L^U(\delta|\mu_\delta, \sigma_\delta) \int_{-\infty}^{x - \delta} \phi(\varepsilon|0, \sigma_\varepsilon) d\varepsilon d\delta \\ &= \int_L^U \phi_L^U(\delta|\mu_\delta, \sigma_\delta) \Phi(x - \delta|0, \sigma_\varepsilon) d\delta. \end{aligned}$$

The density of X is then given by

$$f(x) = \frac{dF}{dx} = \int_0^\infty \phi_L^U(\delta|\mu_\delta, \sigma_\delta) \phi(x - \delta|0, \sigma_\varepsilon) d\delta.$$

Using Sage to perform this integration, the result is given by

$$f(x) = \frac{\sqrt{\pi} e^{\left(\frac{\mu_\delta^2 \sigma_\varepsilon^2}{2(\sigma_\delta^4 + \sigma_\delta^2 \sigma_\varepsilon^2)} + \frac{\sigma_\delta^2 x^2}{2(\sigma_\delta^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^4)} + \frac{\mu_\delta x}{\sigma_\delta^2 + \sigma_\varepsilon^2} \right)} \left(\operatorname{erf} \left(\frac{L\sigma_\delta^2 + (L - \mu_\delta)\sigma_\varepsilon^2 - \sigma_\delta^2 x}{2\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}\sigma_\delta\sigma_\varepsilon} \right) - \operatorname{erf} \left(\frac{U\sigma_\delta^2 + (U - \mu_\delta)\sigma_\varepsilon^2 - \sigma_\delta^2 x}{2\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}\sigma_\delta\sigma_\varepsilon} \right) \right) e^{\left(-\frac{\mu_\delta^2}{2\sigma_\delta^2} - \frac{x^2}{2\sigma_\varepsilon^2} \right)}}{\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2} \left(2.0\pi \operatorname{erf} \left(\frac{\sqrt{2}(L - \mu_\delta)}{2\sigma_\delta} \right) - 2.0\pi \operatorname{erf} \left(\frac{\sqrt{2}(U - \mu_\delta)}{2\sigma_\delta} \right) \right)}.$$

Defining $\lambda = \sigma_\delta/\sigma_\varepsilon$ and $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, the following can be simplified:

$$\frac{\mu_\delta^2 \sigma_\varepsilon^2}{2(\sigma_\delta^4 + \sigma_\delta^2 \sigma_\varepsilon^2)} + \frac{\sigma_\delta^2 x^2}{2(\sigma_\delta^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^4)} + \frac{\mu_\delta x}{\sigma_\delta^2 + \sigma_\varepsilon^2} - \frac{\mu_\delta^2}{2\sigma_\delta^2} - \frac{x^2}{2\sigma_\varepsilon^2}$$

$$\begin{aligned}
&= \frac{\mu_\delta^2 \sigma_\varepsilon^2}{2\sigma_\delta^2(\sigma_\varepsilon^2 + \sigma_\delta^2)} + \frac{\sigma_\delta^2 x^2}{2\sigma_\varepsilon^2(\sigma_\varepsilon^2 + \sigma_\delta^2)} + \frac{\mu_\delta x}{\sigma_\varepsilon^2 + \sigma_\delta^2} - \frac{\mu_\delta^2}{2\sigma_\delta^2} - \frac{x^2}{2\sigma_\varepsilon^2} \\
&= \frac{\mu_\delta^2 \sigma_\varepsilon^4 + x^2 \sigma_\delta^4 + 2x\mu_\delta \sigma_\varepsilon^2 \sigma_\delta^2 - \mu_\delta^2 \sigma_\varepsilon^2(\sigma_\varepsilon^2 + \sigma_\delta^2) - x^2 \sigma_\delta^2(\sigma_\varepsilon^2 + \sigma_\delta^2)}{2\sigma_\varepsilon^2 \sigma_\delta^2(\sigma_\varepsilon^2 + \sigma_\delta^2)} \\
&= \frac{-\mu_\delta^2 \sigma_\delta^2 \sigma_\varepsilon^2 + 2\mu_\delta \sigma_\delta^2 \sigma_\varepsilon^2 x - \sigma_\delta^2 \sigma_\varepsilon^2 x^2}{2\sigma_\varepsilon^2 \sigma_\delta^2(\sigma_\varepsilon^2 + \sigma_\delta^2)} \\
&= -\frac{(x - \mu_\delta)^2 \sigma_\delta^2 \sigma_\varepsilon^2}{2\sigma_\varepsilon^2 \sigma_\delta^2(\sigma_\varepsilon^2 + \sigma_\delta^2)} = -\frac{(x - \mu_\delta)^2}{2\sigma^2}; \\
\frac{B\sigma_\delta^2 + (B - \mu_\delta)\sigma_\varepsilon^2 - \sigma_\delta^2 x}{2\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2 \sigma_\delta \sigma_\varepsilon}} &= \frac{B\lambda + (B - \mu_\delta)(1/\lambda) - \lambda x}{\sigma\sqrt{2}} = \frac{\lambda(B - x) + (B - \mu_\delta)(1/\lambda)}{\sigma\sqrt{2}};
\end{aligned}$$

Let

$$\eta(x, B) := \lambda(B - x) + (B - \mu_\delta)(1/\lambda).$$

So,

$$f(x) = \frac{\exp\left(-\frac{(x - \mu_\delta)^2}{2\sigma^2}\right) \left(\operatorname{erf}\left(\frac{\eta(x, L)}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\eta(x, U)}{\sigma\sqrt{2}}\right)\right)}{\sigma\sqrt{2\pi} \left(\operatorname{erf}\left(\frac{L - \mu_\delta}{\sigma_\delta\sqrt{2}}\right) - \operatorname{erf}\left(\frac{U - \mu_\delta}{\sigma_\delta\sqrt{2}}\right)\right)}.$$

Next,

$$\begin{aligned}
\operatorname{erf}\left(\frac{\eta(x, L)}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\eta(x, U)}{\sigma\sqrt{2}}\right) &= 2\left(\frac{1}{2}\operatorname{erf}\left(\frac{\eta(x, L)}{\sigma\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{\eta(x, U)}{\sigma\sqrt{2}}\right)\right) \\
&= 2\left(\frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\eta(x, L)}{\sigma\sqrt{2}}\right)\right) - \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{\eta(x, U)}{\sigma\sqrt{2}}\right)\right)\right) \\
&= 2\left(\Phi\left(\frac{\eta(x, L)}{\sigma}\right) - \Phi\left(\frac{\eta(x, U)}{\sigma}\right)\right).
\end{aligned}$$

Similarly,

$$\left(\operatorname{erf}\left(\frac{L - \mu_\delta}{\sigma_\delta\sqrt{2}}\right) - \operatorname{erf}\left(\frac{U - \mu_\delta}{\sigma_\delta\sqrt{2}}\right)\right) = 2\left(\Phi\left(\frac{L - \mu_\delta}{\sigma_\delta}\right) - \Phi\left(\frac{U - \mu_\delta}{\sigma_\delta}\right)\right).$$

Thus, altogether

$$\begin{aligned}
f(x) &= \frac{1}{\sigma}\phi\left(\frac{x - \mu_\delta}{\sigma}\right) \frac{\Phi\left(\frac{\eta(x, L)}{\sigma}\right) - \Phi\left(\frac{\eta(x, U)}{\sigma}\right)}{\Phi\left(\frac{L - \mu_\delta}{\sigma_\delta}\right) - \Phi\left(\frac{U - \mu_\delta}{\sigma_\delta}\right)} \\
&= \frac{1}{\sigma}\phi\left(\frac{x - \mu_\delta}{\sigma}\right) \frac{\Phi\left(\frac{\eta(x, U)}{\sigma}\right) - \Phi\left(\frac{\eta(x, L)}{\sigma}\right)}{\Phi\left(\frac{U - \mu_\delta}{\sigma_\delta}\right) - \Phi\left(\frac{L - \mu_\delta}{\sigma_\delta}\right)}.
\end{aligned}$$

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