Theorem 1. *Let $X = \varepsilon - \delta$, where $\varepsilon \sim N(0, \sigma_\varepsilon)$ and $\delta \sim N^+(0, \sigma_\delta)$. Then, X has density function*

$$f(x) = \frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right)\left(1 - \Phi\left(\frac{x\lambda}{\sigma}\right)\right),$$

where $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, $\lambda = \sigma_\delta/\sigma_\varepsilon$, ϕ is the normal density function, and Φ is the normal distribution function.

Proof. First consider the distribution function of X, which is given by

$$\begin{aligned} F(X) &= \Pr(X \leq x) = \Pr(\varepsilon - \delta \leq x) \\ &= \int_{\varepsilon - \delta \leq x} f_\varepsilon(\varepsilon) f_\delta(\delta) d\delta d\varepsilon \\ &= \int_{\delta \in \mathbb{R}^+} f_\delta(\delta) \int_{\varepsilon \in (-\infty, x+\delta]} f_\varepsilon(\varepsilon) d\varepsilon d\delta. \end{aligned}$$

Substituting in known density functions yields

$$\begin{aligned} &\int_0^\infty 2\phi(\delta|0, \sigma_\delta) \int_{-\infty}^{x+\delta} \phi(\varepsilon|0, \sigma_\varepsilon) d\varepsilon d\delta \\ &= 2\int_0^\infty \frac{\phi\left(\frac{\delta}{\sigma_\delta}\right)\Phi\left(\frac{x+\delta}{\sigma_\varepsilon}\right)}{\sigma_\delta} d\delta. \end{aligned}$$

Now, we can proceed with integration by parts. Let $u = \Phi\left(\frac{x+\delta}{\sigma_\varepsilon}\right)$ and $dv = \left(\phi\left(\frac{\delta}{\sigma_\delta}\right)/\sigma_\delta\right) d\delta$. Then, $du = \left(\phi\left(\frac{x+\delta}{\sigma_\varepsilon}\right)/\sigma_\varepsilon\right) d\delta$ and $v = \Phi\left(\frac{\delta}{\sigma_\delta}\right)$. So, the integral becomes

$$\begin{aligned} &2\left(\left[\Phi\left(\frac{x+\delta}{\sigma_\varepsilon}\right)\Phi\left(\frac{\delta}{\sigma_\delta}\right)\right]_0^\infty - \int_0^\infty \Phi\left(\frac{\delta}{\sigma_\delta}\right)\left(\frac{\phi\left(\frac{x+\delta}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon}\right) d\delta\right) \\ &= 2\left(\left(1 - \frac{\Phi\left(\frac{x}{\sigma_\varepsilon}\right)}{2}\right)\right) \end{aligned}$$

$\square$