1 Density function forms

Theorem 1. *Let $X = \varepsilon - \delta$, where $\varepsilon \sim N(0, \sigma_\varepsilon)$ and $\delta \sim N^+(0, \sigma_\delta)$. Then, X has density function*

$$f(x) = \frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right)\left(1 - \Phi\left(\frac{x\lambda}{\sigma}\right)\right),$$

where $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, $\lambda = \sigma_\delta/\sigma_\varepsilon$, ϕ is the normal density function, and Φ is the normal distribution function.

Proof. First consider the distribution function of X, which is given by

$$\begin{aligned}
F(x) &= \Pr(X \leq x) = \Pr(\varepsilon - \delta \leq x) \\
&= \int_{\varepsilon - \delta \leq x} f_\varepsilon(\varepsilon) f_\delta(\delta) d\delta d\varepsilon \\
&= \int_{\delta \in \mathbb{R}^+} f_\delta(\delta) \int_{\varepsilon \in (-\infty, x+\delta]} f_\varepsilon(\varepsilon) d\varepsilon d\delta.
\end{aligned}$$

Substituting in known density functions yields

$$\begin{aligned}
&\int_0^\infty 2\phi(\delta|0, \sigma_\delta) \int_{-\infty}^{x+\delta} \phi(\varepsilon|0, \sigma_\varepsilon) d\varepsilon d\delta \\
&= 2\int_0^\infty \phi(\delta|0, \sigma_\delta)\Phi(x + \delta|0, \sigma_\varepsilon) d\delta.
\end{aligned}$$

The density of X is then given by

$$f(x) = \frac{dF}{dx} = 2\int_0^\infty \phi(\delta|0, \sigma_\delta)\phi(x + \delta|0, \sigma_\varepsilon) d\delta.$$

Using Sage to perform this integration, the result is given by

$$f(x) = -\frac{\left(\operatorname{erf}\left(\frac{\sigma_\delta x}{2\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}\sigma_\varepsilon}\right) e^{\left(\frac{\sigma_\delta^2 x^2}{2\left(\sigma_\delta^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\right)}\right)} - e^{\left(\frac{\sigma_\delta^2 x^2}{2\left(\sigma_\delta^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\right)}\right)}\right) e^{\left(-\frac{x^2}{2\sigma_\varepsilon^2}\right)}}{2\sqrt{\pi}\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}}.$$

Defining $\lambda = \sigma_\delta/\sigma_\varepsilon$ and $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, the following can be simplified:

$$\frac{\sigma_\delta x}{2\sqrt{\frac{1}{2}\sigma_\delta^2 + \frac{1}{2}\sigma_\varepsilon^2}\sigma_\varepsilon} = \frac{\lambda x}{\sigma\sqrt{2}} = \frac{x}{(\sigma/\lambda)\sqrt{2}};$$

$$\frac{\sigma_\delta^2 x^2}{2\left(\sigma_\delta^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\right)} = \frac{\lambda^2 x^2}{2\sigma^2} = \frac{x^2}{2(\sigma/\lambda)^2}.$$

Thus,

$$\begin{aligned}
f(x) &= -\frac{\exp\left(\frac{x^2}{2(\sigma/\lambda)^2}\right)\left(\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right) - 1\right)\exp\left(-\frac{x^2}{2\sigma_\varepsilon^2}\right)}{\sigma\sqrt{2\pi}} \\
&= -\frac{\left(\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right) - 1\right)\exp\left(-x^2\left(\frac{1}{2\sigma_\varepsilon^2} - \frac{1}{2(\sigma/\lambda)^2}\right)\right)}{\sigma\sqrt{2\pi}}.
\end{aligned}$$

Now,

$$\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right)-1=\left(1+\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right)\right)-2=2\left(\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x}{(\sigma/\lambda)\sqrt{2}}\right)\right)-1\right)$$
$$=2\left(\Phi\left(\frac{x\lambda}{\sigma}\right)-1\right)=-2\left(1-\Phi\left(\frac{x\lambda}{\sigma}\right)\right).$$

Also,

$$\frac{1}{2\sigma_\varepsilon^2}-\frac{1}{2(\sigma/\lambda)^2}=\frac{1}{2\sigma_\varepsilon^2}-\frac{\sigma_\delta^2}{2\sigma_\varepsilon^2(\sigma_\delta^2+\sigma_\varepsilon^2)}=\frac{\sigma_\delta^2+\sigma_\varepsilon^2-\sigma_\delta^2}{2\sigma_\varepsilon^2(\sigma_\delta^2+\sigma_\varepsilon^2)}=\frac{\sigma_\varepsilon^2}{2\sigma_\varepsilon^2(\sigma_\delta^2+\sigma_\varepsilon^2)}=\frac{1}{2\sigma^2}.$$

So,

$$f(x)=2\left(1-\Phi\left(\frac{x\lambda}{\sigma}\right)\right)\frac{\exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}=2\left(1-\Phi\left(\frac{x\lambda}{\sigma}\right)\right)\phi(x|0,\sigma)=\frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right)\left(1-\Phi\left(\frac{x\lambda}{\sigma}\right)\right).$$

□

Theorem 2. *Let $X=\varepsilon+\delta$, where $\varepsilon\sim N(0,\sigma_\varepsilon)$ and $\delta\sim N_L^U(\mu_\delta,\sigma_\delta)$. Then, X has density function*

$$f(x)=\frac{1}{\sigma}\phi\left(\frac{x-\mu_\delta}{\sigma}\right)\frac{\Phi\left(\frac{\eta(x,U)}{\sigma}\right)-\Phi\left(\frac{\eta(x,L)}{\sigma}\right)}{\Phi\left(\frac{U-\mu_\delta}{\sigma_\delta}\right)-\Phi\left(\frac{L-\mu_\delta}{\sigma_\delta}\right)},$$

where $\eta(x,B)=\lambda(B-x)+(B-\mu_\delta)(1/\lambda)$, $\lambda=\sigma_\delta/\sigma_\varepsilon$, $\sigma^2=\sigma_\varepsilon^2+\sigma_\delta^2$, and ϕ and Φ are the standard normal density and distribution functions, respectively.

Proof. First consider the distribution function of X, which is given by

$$F(x)=\Pr(X\le x)=\Pr(\varepsilon+\delta\le x)$$
$$=\int_{\varepsilon+\delta\le x}f_\varepsilon(\varepsilon)f_\delta(\delta)d\delta d\varepsilon$$
$$=\int_{\delta\in[L,U]}f_\delta(\delta)\int_{\varepsilon\in(-\infty,x-\delta]}f_\varepsilon(\varepsilon)d\varepsilon d\delta.$$

Substituting in known density functions yields

$$\int_L^U\phi_L^U(\delta|\mu_\delta,\sigma_\delta)\int_{-\infty}^{x-\delta}\phi(\varepsilon|0,\sigma_\varepsilon)d\varepsilon d\delta$$
$$=\int_L^U\phi_L^U(\delta|\mu_\delta,\sigma_\delta)\Phi(x-\delta|0,\sigma_\varepsilon)d\delta.$$

The density of X is then given by

$$f(x)=\frac{dF}{dx}=\int_0^\infty\phi_L^U(\delta|\mu_\delta,\sigma_\delta)\phi(x-\delta|0,\sigma_\varepsilon)d\delta.$$

Using Sage to perform this integration, the result is given by

$$f(x)=\frac{\sqrt{\pi}e^{\left(\frac{\mu_\delta^2\sigma_\varepsilon^2}{2\left(\sigma_\delta^4+\sigma_\delta^2\sigma_\varepsilon^2\right)}+\frac{\sigma_\delta^2x^2}{2\left(\sigma_\delta^2\sigma_\varepsilon^2+\sigma_\varepsilon^4\right)}+\frac{\mu_\delta x}{\sigma_\delta^2+\sigma_\varepsilon^2}\right)}\left(\operatorname{erf}\left(\frac{L\sigma_\delta^2+(L-\mu_\delta)\sigma_\varepsilon^2-\sigma_\delta^2x}{2\sqrt{\frac{1}{2}\,\sigma_\delta^2+\frac{1}{2}\,\sigma_\varepsilon^2}\sigma_\delta\sigma_\varepsilon}\right)-\operatorname{erf}\left(\frac{U\sigma_\delta^2+(U-\mu_\delta)\sigma_\varepsilon^2-\sigma_\delta^2x}{2\sqrt{\frac{1}{2}\,\sigma_\delta^2+\frac{1}{2}\,\sigma_\varepsilon^2}\sigma_\delta\sigma_\varepsilon}\right)\right)e^{\left(-\frac{\mu_\delta^2}{2\,\sigma_\delta^2}-\frac{x^2}{2\,\sigma_\varepsilon^2}\right)}}{\sqrt{\frac{1}{2}\,\sigma_\delta^2+\frac{1}{2}\,\sigma_\varepsilon^2}\left(2.0\,\pi\operatorname{erf}\left(\frac{\sqrt{2}(L-\mu_\delta)}{2\,\sigma_\delta}\right)-2.0\,\pi\operatorname{erf}\left(\frac{\sqrt{2}(U-\mu_\delta)}{2\,\sigma_\delta}\right)\right)}.$$

Defining $\lambda = \sigma_\delta/\sigma_\varepsilon$ and $\sigma^2 = \sigma_\varepsilon^2 + \sigma_\delta^2$, the following can be simplified:

$$\frac{\mu_\delta^2\sigma_\varepsilon^2}{2\left(\sigma_\delta^4+\sigma_\delta^2\sigma_\varepsilon^2\right)} + \frac{\sigma_\delta^2 x^2}{2\left(\sigma_\delta^2\sigma_\varepsilon^2+\sigma_\varepsilon^4\right)} + \frac{\mu_\delta x}{\sigma_\delta^2+\sigma_\varepsilon^2} - \frac{\mu_\delta^2}{2\sigma_\delta^2} - \frac{x^2}{2\sigma_\varepsilon^2}$$

$$= \frac{\mu_\delta^2\sigma_\varepsilon^2}{2\sigma_\delta^2(\sigma_\varepsilon^2+\sigma_\delta^2)} + \frac{\sigma_\delta^2 x^2}{2\sigma_\varepsilon^2(\sigma_\varepsilon^2+\sigma_\delta^2)} + \frac{\mu_\delta x}{\sigma_\varepsilon^2+\sigma_\delta^2} - \frac{\mu_\delta^2}{2\sigma_\delta^2} - \frac{x^2}{2\sigma_\varepsilon^2}$$

$$= \frac{\mu_\delta^2\sigma_\varepsilon^4 + x^2\sigma_\delta^4 + 2x\mu_\delta\sigma_\varepsilon^2\sigma_\delta^2 - \mu_\delta^2\sigma_\varepsilon^2(\sigma_\varepsilon^2+\sigma_\delta^2) - x^2\sigma_\delta^2(\sigma_\varepsilon^2+\sigma_\delta^2)}{2\sigma_\varepsilon^2\sigma_\delta^2(\sigma_\varepsilon^2+\sigma_\delta^2)}$$

$$= \frac{-\mu_\delta^2\sigma_\delta^2\sigma_\varepsilon^2 + 2\mu\delta\sigma_\delta^2\sigma_\varepsilon^2 x - \sigma_\delta^2\sigma_\varepsilon^2 x^2}{2\sigma_\varepsilon^2\sigma_\delta^2(\sigma_\varepsilon^2+\sigma_\delta^2)}$$

$$= -\frac{(x-\mu_\delta)^2\sigma_\delta^2\sigma_\varepsilon^2}{2\sigma_\varepsilon^2\sigma_\delta^2(\sigma_\varepsilon^2+\sigma_\delta^2)} = -\frac{(x-\mu_\delta)^2}{2\sigma^2};$$

$$\frac{B\sigma_\delta^2 + (B-\mu_\delta)\sigma_\varepsilon^2 - \sigma_\delta^2 x}{2\sqrt{\frac{1}{2}\,\sigma_\delta^2+\frac{1}{2}\,\sigma_\varepsilon^2}\sigma_\delta\sigma_\varepsilon} = \frac{B\lambda + (B-\mu_\delta)(1/\lambda) - \lambda x}{\sigma\sqrt{2}} = \frac{\lambda(B-x)+(B-\mu_\delta)(1/\lambda)}{\sigma\sqrt{2}};$$

Let

$$\eta(x,B) := \lambda(B-x) + (B-\mu_\delta)(1/\lambda).$$

So,

$$f(x) = \frac{\exp\left(-\frac{(x-\mu_\delta)^2}{2\sigma^2}\right)\left(\operatorname{erf}\left(\frac{\eta(x,L)}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\eta(x,U)}{\sigma\sqrt{2}}\right)\right)}{\sigma\sqrt{2\pi}\left(\operatorname{erf}\left(\frac{L-\mu_\delta}{\sigma_\delta\sqrt{2}}\right) - \operatorname{erf}\left(\frac{U-\mu_\delta}{\sigma_\delta\sqrt{2}}\right)\right)}.$$

Next,

$$\operatorname{erf}\left(\frac{\eta(x,L)}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\eta(x,U)}{\sigma\sqrt{2}}\right) = 2\left(\frac{1}{2}\operatorname{erf}\left(\frac{\eta(x,L)}{\sigma\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{\eta(x,U)}{\sigma\sqrt{2}}\right)\right)$$

$$= 2\left(\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{\eta(x,L)}{\sigma\sqrt{2}}\right)\right) - \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{\eta(x,U)}{\sigma\sqrt{2}}\right)\right)\right)$$

$$= 2\left(\Phi\left(\frac{\eta(x,L)}{\sigma}\right) - \Phi\left(\frac{\eta(x,U)}{\sigma}\right)\right).$$

Similarly,

$$\left(\operatorname{erf}\left(\frac{L-\mu_\delta}{\sigma_\delta\sqrt{2}}\right) - \operatorname{erf}\left(\frac{U-\mu_\delta}{\sigma_\delta\sqrt{2}}\right)\right) = 2\left(\Phi\left(\frac{L-\mu_\delta}{\sigma_\delta}\right) - \Phi\left(\frac{U-\mu_\delta}{\sigma_\delta}\right)\right).$$

Thus, altogether

$$f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu_\delta}{\sigma}\right)\frac{\Phi\left(\frac{\eta(x,L)}{\sigma}\right) - \Phi\left(\frac{\eta(x,U)}{\sigma}\right)}{\Phi\left(\frac{L-\mu_\delta}{\sigma_\delta}\right) - \Phi\left(\frac{U-\mu_\delta}{\sigma_\delta}\right)}$$

$$= \frac{1}{\sigma}\phi\left(\frac{x-\mu_\delta}{\sigma}\right)\frac{\Phi\left(\frac{\eta(x,U)}{\sigma}\right) - \Phi\left(\frac{\eta(x,L)}{\sigma}\right)}{\Phi\left(\frac{U-\mu_\delta}{\sigma_\delta}\right) - \Phi\left(\frac{L-\mu_\delta}{\sigma_\delta}\right)}.$$

$\square$

2 Characteristics of distributions

Theorem 3. *Let $\varepsilon > 0$, $0 < \rho \leq 1$, and $\nu > 0$. Then, there exists $\sigma > 0$ so that*

$$\int_{-\varepsilon}^{\varepsilon} f_t(x|0, \sigma, \nu)dx > 1 - \rho,$$

where $f_t(x|0, \sigma, \nu)$ is the density of a t-distributed random variable with location 0, scale σ, and ν degrees of freedom.

Proof. First, note that

$$\int_{-\varepsilon}^{\varepsilon} f_t(x|0, \sigma, \nu)dx$$

$$= -2(\nu + 1)\sqrt{\nu}\sigma \arctan\left(\frac{\varepsilon}{\sqrt{\nu}\sigma}\right)$$

$$+2\varepsilon\left(\nu - \frac{\ln \pi}{2} + 1 - \frac{\ln \nu}{2} - \ln \sigma - \left(\frac{\nu + 1}{2}\right)\ln\left(\frac{\nu\sigma^2 + \varepsilon^2}{\nu\sigma^2}\right) + \ln\left(\Gamma\left(\frac{\nu + 1}{2}\right)\right) - \ln\left(\Gamma\left(\frac{\nu}{2}\right)\right)\right)$$

$\square$