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ESTIMATING TECHNICAL AND ALLOCATIVE INEFFICIENCY RELATIVE TO STOCHASTIC PRODUCTION AND COST FRONTIERS

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Earlier papers by Aigner, Lovell and Schmidt and by Meeusen and van den Broeck have considered stochastic frontier production functions. This paper extends that work by considering the duality between stochastic frontier production and cost functions, under the assumptions of exact cost minimization (technical inefficiency only) and of inexact cost minimization (technical and allocative inefficiency). We show how to measure both types of inefficiency, and the associated cost of inefficiency. The techniques are illustrated using data on steam-electric generating plants.

1. Introduction

Recent interest in the formulation and estimation of production frontiers has centered on three problems. First, since it is the production frontier rather than some fitted 'average' function that corresponds to the theoretical notion of a production function, interest naturally has centered on determining the shape and location of the frontier. Second, there has been some interest in examining the nature of the relationship between the production frontier and the fitted 'average' function. For if the former is a neutrally scaled transform of the latter, then the shape, but not the placement, of the frontier can be inferred from the fitted 'average' function. But if the former is not a neutrally scaled transform of the latter, then virtually nothing can be learned about the frontier from estimation of the 'average' function. Finally, and most importantly, production frontiers have implications for the technical efficiency of production. Although under certain conditions a fitted 'average' function permits a ranking of observations by technical efficiency, it provides no quantitative information on technical inefficiency in the sample. An estimated frontier carries the promise of shedding light on the actual

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magnitude of technical inefficiency; if not for each observation, then at least on average for the sample as a whole.

The way in which econometricians look at production frontiers has undergone substantial modification in recent years. The earliest work on frontiers assumed what we would call a *deterministic* frontier. This idea was developed by Farrell (1957), Farrell and Fieldhouse (1962) and Afriat (1972), and tested by Aigner and Chu (1968), Seitz (1971), Richmond (1974), Førsund and Jansen (1977), and others. Apart from all conceptual questions of whether the frontier really ought to be deterministic, there are severe statistical problems with deterministic frontiers. In some cases, although some disturbance must implicitly be assumed, no assumptions are made about its properties; as a result, the parameters are not estimated in any statistical sense, but are merely 'computed' via mathematical programming techniques. In other cases, a one-sided (non-positive) disturbance has explicitly been assumed, of some particular form (e.g., gamma). However since the regularity conditions for application of maximum likelihood are violated by this specification, estimation of deterministic frontiers is not completely straightforward. [This point is discussed in a recent paper by Greene (1978).] Finally, deterministic frontiers are extremely sensitive to outliers.

It is this second deficiency that led to the development of *probabilistic* production frontiers by Timmer (1971). In this approach a deterministic frontier is computed by mathematical programming techniques, after which supporting data points are discarded and a new deterministic frontier is computed. The process continues until the computed frontier stabilizes. The probabilistic frontier approach thus 'solves' the outlier problem by discarding outliers from the sample, and no attempt is made to reconcile their placement above the final frontier with the theoretical concept of a frontier as an upper bound on output. Moreover, since a probabilistic frontier is just a deterministic frontier computed from a subset of the original sample, it remains vulnerable to the first objection raised against deterministic frontiers: since it is computed rather than estimated, hypothesis testing is impossible.

Finally, recent papers by Aigner, Lovell and Schmidt [hereafter ALS] (1977) and Meeusen and van den Broeck (1977) have sought to ameliorate the problems associated with both deterministic and probabilistic production frontiers by specifying a *stochastic* production frontier. In such a specification the output of each firm is bounded above by a frontier that is stochastic in the sense that its placement is allowed to vary randomly across firms. From an economic standpoint this technique permits firms to be technically inefficient relative to their own frontier rather than to some sample norm. Interfirm variation of the frontier presumably captures the effects of exogenous shocks, favorable and unfavorable, beyond the control of the firms. Errors of observation and measurement on output constitute another source of variation in the frontier. The statistical specification is that the disturbance term is made up of two parts: a symmetric (normal) component capturing randomness outside the control of the firm; and a one-sided (non-positive) component capturing randomness under the control of the firm (i.e., inefficiency). Estimation of the frontier is then a statistical problem, and the usual types of statistical inference are possible.

Although a stochastic production frontier is a useful construct, there is one serious limitation in the information it contains. A production process can be inefficient in two ways, only one of which can be detected by an estimated production frontier. It can be technically inefficient, in the sense that it fails to produce maximum output from a given input bundle; technical inefficiency results in an equiproportionate overutilization of all inputs. It can also be allocatively inefficient in the sense that the marginal revenue product of an input might not be equal to the marginal cost of that input; allocative inefficiency results in utilization of inputs in the wrong proportions, given input prices. Since estimation of production frontiers is carried out with observations on output and inputs only, such an exercise cannot provide evidence bearing on the matter of allocative inefficiency, and hence cannot be used to draw inferences about total, or economic, inefficiency.

The primary purpose of this paper is to demonstrate that it is possible to obtain evidence bearing on total inefficiency and its technical and allocative components by means of a straightforward extension of the analysis of ALS (1977) and Meeusen and van den Broeck (1977). We make the behavioral assumption that the firm seeks to minimize the cost of producing its desired rate of output, subject to a stochastic production frontier constraint. If the firm is technically inefficient it operates beneath its stochastic production frontier, and if the firm is allocatively inefficient it operates off its least cost expansion path. Incorporating these features into the analysis leads to the derivation of a system of stochastic factor demand frontiers and, from them, a stochastic cost frontier. Since both contain factor prices as arguments, estimation or either one provides evidence bearing on the magnitude and the cost of total inefficiency and its technical and allocative components.

As we develop our techniques, we will also provide an empirical illustration of this application. Specifically, we will estimate stochastic frontiers for a sample of US steam-electric generating plants. Since the aim of this paper is primarily methodological, this illustration should be viewed as an indication of the potential usefulness of the technique rather than as a careful study of steam-electric generation.

The outline of the paper is as follows. In section 2 we derive and estimate a stochastic cost frontier under the assumption of exact cost minimization, where the only source of inefficiency is technical. In section 3 we derive and estimate a stochastic cost frontier under the assumption of inexact cost minimization, in which both technical and allocative inefficiency are permitted. In section 4 we modify the analysis of section 3 to permit allocative inefficiency to have a systematic component to it, as suggested, for example, by the Averch–Johnson hypothesis. Finally, section 5 contains our conclusions.

2. Technical inefficiency only

In this section we assume that the firm seeks to minimize the cost of producing its desired rate of output subject to a stochastic production frontier constraint. We permit the firm to be technically inefficient by allowing it to operate beneath its stochastic production frontier, but we require the firm to be allocatively efficient by requiring it to operate on its least cost expansion path. The firm's production technology is characterized by a production function of the form

$$y = a \prod_{i=1}^{n} x_i^{\alpha_i} e^{\varepsilon},$$

where y is the output of the firm, the x_i are the inputs to the production process, ε is a random disturbance, and a and the α_i are parameters to be estimated. Also, following ALS (1977) and Meeusen and van den Broeck (1977), the disturbance is assumed to be of the form

$$\varepsilon = v - u.$$

Here v is distributed as $N(0, \sigma_v^2)$, and captures random variation in output due to factors outside the control of the firm (weather, acts of God, etc.). On the other hand, u is a non-positive disturbance, reflecting technical inefficiency. We can then write the production function in log-linear form as

$$\ln y = A + \sum_{i=1}^{n} \alpha_i \ln x_i + (v - u), \tag{1}$$

where

$$A = \ln a,$$

and all other notation is as previously defined. Note that $\ln y$ is bounded from above by the stochastic production frontier

$$A + \sum_{i=1}^{n} \alpha_i \ln x_i + v,$$

with technical efficiency relative to the frontier given by u percent.

Since the firm is assumed to be allocatively efficient, it makes no mistakes in selecting the cost minimizing factor proportions, which are given by the solution to

$$\ln x_1 - \ln x_i = B_i, \qquad i = 2, \ldots, n, \tag{2}$$

where

$$B_i = \ln(p_i \alpha_1 / p_1 \alpha_i),$$

and $p_1, p_2, \ldots, p_n$ are prices of the n inputs. From these we can derive the *factor demand equations*

$$\ln x_i = \ln k_i + \frac{1}{r} \ln y + \ln \left[\prod_{j=1}^{n} p_j^{\alpha_j/r} / p_i \right] - \frac{1}{r}(v - u), \qquad i = 1, \ldots, n. \tag{3}$$

Here

$$r = \sum_{i=1}^{n} \alpha_i = \text{returns to scale},$$

and

$$k_i = \alpha_i \left[a \prod_{i=1}^{n} \alpha_i^{\alpha_i} \right]^{-1/r}$$

Note that $\ln x_i$ is bounded from below by the stochastic factor demand frontier

$$\ln k_i + \frac{1}{r} \ln y + \ln \left[\sum_{j=1}^{n} p_j^{\alpha_j/r} / p_i \right] - \frac{1}{r} v.$$

The excess of factor demand above this frontier is solely due to technical inefficiency, and equals $(1/r)u$ percent for each input.

Finally we find that the *cost function* is of the form

$$\ln C = K + \frac{1}{r} \ln y + \sum_{i=1}^{n} \frac{\alpha_i}{r} \ln p_i - \frac{1}{r}(v - u), \tag{4}$$

where

$$K = \ln \left[\sum_{i=1}^{n} k_i \right] = \ln r - \frac{1}{r} A - \frac{1}{r} \ln \left[\prod_{i=1}^{n} \alpha_i^{\alpha_i} \right].$$

Note that $\ln C$ is also bounded from below by the stochastic cost frontier

$$K + \frac{1}{r} \ln y + \sum_{i=1}^{n} \frac{\alpha_i}{r} \ln p_i - \frac{1}{r} v,$$

which represents, in logarithmic terms, the minimum possible cost of producing output y and prices p_i. The cost frontier is stochastic, just as the production and factor demand frontiers are, because of the randomness of v, which reflects shocks beyond the control of the firm. The term $(1/r)u$ in (4) then represents the percent by which actual cost exceeds the cost frontier. In other words, *it measures the extra cost of producing below the production frontier* – that is, of technical inefficiency.[1] Later we will allow for allocative inefficiency, which clearly will contribute another positive component to actual cost. For now, with allocative efficiency assumed, a firm can be above its cost frontier only by being below its production frontier. Clearly the cost of such technical inefficiency varies inversely with returns to scale.

The correspondence between production and cost function parameters and disturbances, which is clear from (1) and (4), is of course the same apart from all questions of which equation is more appropriate to estimate, from an econometric point of view. In ALS (1977), we relied on the Zellner, Kmenta and Drèze (1966) assumption of maximization of expected profit, and estimated the frontier production function. From such estimates, estimates of the frontier cost function can be derived. Furthermore, the asymptotic distribution of the estimates of the cost function parameters can be derived from the asymptotic distribution of the estimates of the production function parameters by the usual method (linearization by Taylor series).

However under other assumptions the reverse procedure may be more reasonable. For example, in our empirical illustration we will be dealing with steam-electric generating plants. There exists a considerable literature suggesting that, in this industry, output is exogenous to the plant; see, for example, Nerlove (1963).[2] The plant then maximizes profit simply by minimizing the cost of producing given output. Assuming that input prices are also fixed to the plant, the exogenous variables in this model are then the quantity of output, and input prices; the endogenous variables are quantities of inputs, and cost.

Under these circumstances direct estimation of the production function by the methods described in ALS (1977) would be inconsistent. On the other hand, since all the variables on the right-hand side of the cost function are

[1]This cost differential of $(1/r)u$, which we refer to as the cost of technical inefficiency, no doubt captures in part the cost of planned plant flexibility to react to varying factor supply situations. To quote Fuss and McFadden (1971, p. 1): 'The cost of adding flexibility is usually a loss in economic efficiency relative to a 'best practice' design for a specific static operating environment.' This efficiency–flexibility tradeoff is of particular relevance to steam-electric generation, which faces peak/off-peak demand cycles and highly variable coal, natural gas and oil prices.

[2]Since the Nerlove study, the estimation of production technology in steam electric generation has been the subject of a large number of studies. Most writers have followed Nerlove's cost minimization assumption on the grounds that output is exogenous to the plant. Some writers, primarily those interested in the Averch–Johnson hypothesis, have instead used the profit maximization assumption. For a survey of these studies, and a defense of the cost minimization assumption, see Cowing and Smith (1977).

exogenous, the cost function can be estimated without worrying about complications due to simultaneity. From these estimates we can derive estimates of the production function parameters. We therefore proceed to discuss estimation of the cost function parameters, under the assumption of exogenous output.

Since the cost function is linearly homogeneous in input prices we rewrite it as

$$\ln(C/p_n) = K + \frac{1}{r}\ln y + \sum_{i=1}^{n-1} \frac{\alpha_i}{r}\ln(p_i/p_n) - \frac{1}{r}(v-u). \tag{5}$$

It makes no difference, economically or statistically, which price is used to normalize the equation, of course. So we are now ready to concentrate on the estimation of (5).

At this point, we need to make some assumption about the distribution of the one-sided error term u. We will assume, as in ALS (1977), that u is half-normal; it is the absolute value of a variable distributed as $N(0,\sigma_u^2)$. Estimation under alternative distributions for u would of course also be possible, with the changes required in the subsequent discussion being fairly straightforward.[3]

Eq. (4) is a standard Cobb–Douglas cost function except for the form of the disturbance. However, the presence of the half-normal component $(1/r)u$ in the disturbance can be handled in ways completely analogous to those suggested for the case of frontier production functions; we need only to change a few signs. Specifically, eq. (4) can be estimated in the following two ways:

(i) Run OLS on the cost function as given in (5). This gives consistent estimates of the parameters $1/r, \alpha_1/r, \ldots, \alpha_{n-1}/r$. From these we can solve for consistent estimates of r, $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Next, calculate the second and third moments of the OLS residuals. These are consistent estimates of the second and third moments, say μ_2 and μ_3, of the disturbance of the cost function. Since

$$\mu_2 = \left[\sigma_v^2 + \frac{\pi-2}{\pi}\sigma_u^2\right] \bigg/ r^2$$

$$\mu_3 = \left\{\sqrt{\frac{2}{\pi}}\left[\frac{4-\pi}{\pi}\right]\sigma_u^3\right\} \bigg/ r^3,$$

[3]It is certainly true that there is no particularly good reason to assume that u is half-normal. The problem is basically that there is not much theoretical guidance in picking one-sided distributions. The usual types of arguments along the lines of the central limit theorem (that a disturbance is the sum of various independent effects) lead to a normal disturbance, which is not one-sided. It seems to us that the only real solution is to try various alternative distributions for u, and see which fits best. This we have not yet done, though we hope to in future work.

it is easily seen that

$$\sigma_u^2 = \left[\sqrt{\frac{\pi}{2}}\left(\frac{\pi}{4-\pi}\right)\mu_3\right]^{2/3} r^2,$$
$$\sigma_v^2 = r^2\mu_2 - \left(\frac{\pi-2}{\pi}\right)\sigma_u^2. \tag{6}$$

Using our consistent estimates $\hat{\mu}_2$ and $\hat{\mu}_3$ (from the OLS residuals) in place of μ_2 and μ_3 in (6), we get consistent estimates $\hat{\sigma}_u^2$ and $\hat{\sigma}_v^2$ of the parameters in the distributions of v and u.

Finally, we need to correct the estimated constant term. If $\hat{K}$ is the OLS estimate of the constant term K, then

$$E(\hat{K}) = \text{plim}\hat{K} = K + \frac{1}{r}E(u) = K + \frac{1}{r}\sqrt{\frac{2}{\pi}}\sigma_u.$$

So, given that we have already derived consistent estimates $\hat{r}$ and $\hat{\sigma}_u$ of r and σ_u, we can form a consistent estimate of K as

$$\tilde{K} = \hat{K} - \frac{1}{\hat{r}}\sqrt{\frac{2}{\pi}}\hat{\sigma}_u.$$

We have therefore derived consistent estimates of all of the parameters of the model – $\alpha_1, \alpha_2, \ldots, \alpha_n$, r, K, σ_u^2, σ_v^2. We will refer to these as the OLS/MOMENTS estimators since they are derived from the OLS estimates of the cost function and from the moments of the associated OLS residuals.

Since the estimates of $\alpha_1, \alpha_2, \ldots, \alpha_n$ and of r are functions of the OLS estimates of the cost function parameters, we can find their asymptotic distribution by linearization by Taylor series. The asymptotic distributions of our estimates of K, σ_u^2 and σ_v^2 are more complicated, though not intractable. We can derive the asymptotic distribution of the moment estimators $\hat{\mu}_2$ and $\hat{\mu}_3$ using the central limit theorem. (These expressions will be messy since they will involve the moments of the disturbance up to order six.) Since the estimates of K, σ_u^2 and σ_v^2 are then differentiable functions of these estimated moments, their asymptotic distribution is also calculable by linearization by Taylor series. For an explicit derivation, see Olson, Schmidt and Waldman (1978).

(ii) Our second possible estimation procedure is to estimate the parameters of the cost function by maximum likelihood. The likelihood function is very similar to that for the frontier production function case, as given in ALS (1977); we give the details in the appendix.

Maximization of the likelihood function, by numerical techniques, gives the maximum likelihood estimates of the cost function parameters – K, $1/r$, $\alpha_1/r,\ldots,\alpha_{n-1}/r$, λ, and σ_*^2 (where $\lambda=\sigma_u/\sigma_v$ and $\sigma_*^2=(1/r^2)[\sigma_v^2+\sigma_u^2]$). From these we can solve for the maximum likelihood estimates of the production function parameters – A, $\alpha_1,\ldots,\alpha_n$, λ, and σ^2 (where $\sigma^2=\sigma_v^2+\sigma_u^2$). Because of the one-to-one correspondence between cost function parameters and production function parameters, the invariance property of maximum likelihood estimates guarantees that these estimates of the production function parameters are the same estimates as those obtained by maximizing the likelihood function with respect to the production function parameters. As a result we are justified in calling these estimates likelihood estimates, and in claiming that they are consistent *and* asymptotically efficient.

Of course, it should be emphasized that the OLS/Moments or MLE estimates of the production function parameters which are generated in this way (from estimates of the cost function parameters, under the assumption of exogenous output) are not the same as the OLS/Moments or MLE estimates that we would get by directly estimating the production function (under the assumption of exogenous input quantities).

The main advantage of the OLS/Moment estimators is that they are easier to compute than the MLE estimators. Their main disadvantage is that they are less efficient than the MLE estimators. It should also be pointed out that there is no guarantee that the OLS/Moments estimates of σ_u^2 and σ_v^2 will be non-negative. If $\hat{\mu}_3$ (the third order moment of the OLS residuals) is negative, $\hat{\sigma}_u$ will be negative; even though $\hat{\sigma}_u^2$ is still positive, this is troubling. It is also possible to find $\hat{\mu}_2$ and $\hat{\mu}_3$ such that $\hat{\sigma}_v^2<0$. It is easy to prevent this in calculating MLE's, on the other hand. If one is sure of the specification, this last point is a further advantage of MLE. However it might alternatively be argued that it is an advantage of OLS/Moments that it is possible to get an easily understood warning (negative $\hat{\mu}_3$) to make one rethink the error specification.

We now turn to our empirical illustration. The data we use are drawn from a previously collected sample of 150 new privately-owned steam-electric generating plants constructed in the US between 1947 and 1965.[4] Our subsample consists of data on output, total cost, and prices and quantities of three inputs (capital, fuel and labor) for each of 111 plants. Output is electricity generated (10^6 KWH) in the first year of operation; capital is the actual cost of the plant; fuel is the actual consumption (10^6 BTU) of fuel (coal, oil or gas) in the first year of operation; labor is the design labor force measured in total employee manhours (total employees $\times 2000$); the price of

[4]We are indebted to Thomas G. Cowing for making these data available to us. The data are described in greater detail in Cowing (1970).

capital is given by the firm's bond rate prior to installation; the price of fuel is the actual price ($\$/10^6$ BTU) of the dominant fuel used by the plant, averaged over the first three years of operation; and the price of labor is the regional average industry wage rate ($/hour), averaged over the two years prior to plant installation.

One advantage of this data set is that it enables us to test for inefficiency using truly microeconomic data. Most previous tests have been conducted on data that have been aggregated to some degree. In ALS (1977) we estimated a stochastic production frontier and found very little evidence of technical inefficiency, using two aggregated samples. We suggested then that evidence on technical (or other) inefficiency may be lost in the aggregation process. If this conjecture is correct, then we should expect to find clearer evidence of inefficiency using microeconomic data.

We present in table 1 our estimates of the cost function, and the implied estimates of the production function, for the steam-electric data just described. Results are given for both methods of estimation. (Numbers in parentheses are asymptotic standard errors.)

Note that the OLS/moments method gives a negative estimate of σ_v^2. Other than that, the results are fairly reasonable. Returns to scale are about 1.25. The coefficient of the fuel input is very large (0.9649) by MLE) and highly significant. The coefficient of the capital input is only marginally significant, and the coefficient of the labor input is insignificant, at the usual confidence levels.

These results look all the more reasonable if we compare them to the 'direct' estimates of the production function. For example, if we estimate the stochastic frontier production function by MLE, assuming input quantities exogenous [as discussed in ALS (1977)], we find the following coefficients and standard errors:

Capital:	0.04770	(0.01289)
Fuel:	1.0796	(0.01475)
Labor:	−0.01119	(0.01998)

It is disturbing that the coefficient of labor is negative, that the coefficient of capital is so small, and that the coefficient of fuel is (significantly) larger than one. The OLS/Moments estimates are very similar. Of course, if we are correct in our assumption that output is exogenous, while inputs are endogenous, for these data, then direct estimation of the production function is not appropriate. It is reassuring that the estimates obtained through the cost function are more reasonable.

We can also note that the maximum likelihood value of λ is 4.39, reflecting $\sigma_u^2=0.03897$ and $\sigma_v^2=0.00202$. This is evidence of substantial technical inefficiency. One way to see this is to note that the variance of u, $(\pi-2)\sigma_u^2/\pi$,

Table 1

Estimates of cost function (technical inefficiency only).

Method	Constant	Output	P_K/P_L	P_F/P_L	σ_u^2	σ_v^2	λ
OLS/Moments	10.8551 (0.2743)	0.7956 (0.01131)	0.1370 (0.07172)	0.7291 (0.03125)	0.03731	−0.002646	undefined
MLE	10.9357 (0.2549)	0.7998 (0.00998)	0.1340 (0.06802)	0.7717 (0.02366)	0.02483 (0.00490)	0.001295 (0.000708)	4.388 (1.468)

Derived estimates of production function (technical inefficiency only).

Method	Constant	Capital	Fuel	Labor	σ_u^2	σ_v^2	λ	RTS
OLS/Moments	−12.8719	0.1722 (0.08932)	0.9164 (0.03920)	0.1683 (0.09445)	0.05894	−0.004180	undefined	1.2569 (0.01785)
MLE	−12.8077	0.1675 (0.07648)	0.9649 (0.02924)	0.1179 (0.08543)	0.03897 (0.00766)	0.002024 (0.00111)	4.3876 (1.468)	1.2503 (0.01560)

equals 0.01416, and is seven times as large as the variance of v, σ_v^2. This means that the variance of output below the frontier is seven times as large as the variance of the frontier itself. However, the implication of this inefficiency is perhaps clearer in considering that the mean of the one-sided production function disturbance $(-u)$ is -0.1575, which implies that output is, on average, almost 16% below the frontier. The mean of the one-sided disturbance in the cost function, $(1/r)u$, is 0.1260, which implies that cost is, on average, 12.6% above the frontier level. This is the cost of technical inefficiency.

Two related points are worth mentioning. First, although we can estimate the mean of u, we cannot estimate its values for each observation. In other words, we cannot compare the technical efficiency of the various plants. Second, under our assumption of allocative efficiency, the 12.6% increase in cost is caused by a 12.6% overutilization of all three inputs. This symmetry can be avoided by relaxing the assumption of allocative efficiency, which we do in the next section.[5]

3. Technical and allocative inefficiency

In this section we continue to assume that the firm seeks to minimize the cost of producing its desired rate of output subject to a stochastic production frontier constraint, and we continue to assume that the firm may be technically inefficient. However, we now allow for the possibility that the firm may also be allocatively inefficient by permitting it to operate off its least cost expansion path. Allocative inefficiency is modelled by permitting the cost minimizing conditions, which define the least cost expansion path in implicit form, to fail to hold exactly. Errors in choosing cost minimizing factor proportions then correspond to disturbances from the exact satisfaction of the first-order conditions for cost minimization.[6] This leads to the specification

$$\begin{aligned} \ln y &= A + \sum_{i=1}^{n} \alpha_i \ln x_i + (v-u) \\ \ln x_1 - \ln x_i &= B_i + \varepsilon_i, \qquad i = 2,\ldots,n, \end{aligned} \tag{7}$$

[5] It could also be avoided by replacing the Cobb–Douglas function with some non-homothetic functional form.

[6] Lau and Yotopoulos (1971) model allocative inefficiency by assuming that the first-order cost minimization conditions are deterministic, and that the marginal rate of technical substitution is *proportional* to the factor price ratio, a non-unitary factor of proportionality indicating allocative inefficiency. They do not, however, obtain estimates of the factors of proportionality by firm.

where as before $B_i = \ln(p_i\alpha_1/p_1\alpha_i)$, $v \sim N(0,\sigma_v^2)$, u is the absolute value of a $N(0,\sigma_u^2)$ variable and ε_i represents the amount by which the ith first-order condition for cost minimization fails to hold.

We now need to assume something about the ε_i. For the time being we will assume away the possibility of *systematic* errors in cost minimization, by assuming that the errors ε_i are random, with a mean of zero. (The zero mean of the ε_i implies that there is no systematic tendency to over- or under-utilize any input relative to any other input.) This assumption will be relaxed in section 4. For now, however, we assume that the vector

$$\varepsilon = (\varepsilon_2, \ldots, \varepsilon_n)$$

has a multivariate normal distribution with mean zero and covariance matrix Σ. We also assume that ε is independent of v and u. This last assumption will be relaxed in a later paper.

We should point out that, although our discussion will proceed in terms of exogenous output, a very similar discussion would hold for a profit maximizing firm for which output was endogenous. We would simply replace the $n-1$ cost minimization equations in (7) with n profit maximization equations, with random errors. This would convert (7) from a system of n equations in n endogenous variables (the inputs x_i) to a system of $n+1$ equations in $n+1$ endogenous variables (output y and the inputs x_i). Such systems have been widely discussed before – see, e.g., Zellner, Kmenta and Drèze (1966), and the references therein – but not with a one-sided disturbance included in the production function specification.

From the equations in (7) we can derive the *factor demand equations*

$$\begin{aligned} \ln x_1 &= \ln k_1 + \frac{1}{r}\ln y + \ln\left[\sum_{j=1}^{n} p_j^{\alpha_j/r}/p_1\right] - \frac{1}{r}(v-u) + \sum_{j=2}^{n}\frac{\alpha_j}{r}\varepsilon_j, \\ \ln x_i &= \ln k_i + \frac{1}{r}\ln y + \ln\left[\sum_{j=1}^{n} p_j^{\alpha_j/r}/p_i\right] - \frac{1}{r}(v-u) - \varepsilon_i + \sum_{j=2}^{n}\frac{\alpha_j}{r}\varepsilon_j, \\ & \qquad i = 2,\ldots,n. \end{aligned} \tag{8}$$

(The k_i are defined in section 2.) These are the same as the factor demand equations in eq. (3), except for the added terms involving the ε's. Given the assumptions on the ε's, it follows that particular factors may be over- or under-utilized, relative to the case of allocative efficiency – though not systematically so.

From the factor demand equations, we can proceed to the *cost function*

$$\ln C = K + \frac{1}{r}\ln y + \sum_{i=1}^{n}\frac{\alpha_i}{r}\ln p_i - \frac{1}{r}(v-u) + (E - \ln r), \tag{9}$$

where

$$E=\sum_{j=2}^{n}\frac{\alpha_j}{r}\varepsilon_j+\ln\left[\alpha_1+\sum_{j=2}^{n}\alpha_j e^{-\varepsilon_j}\right]. \tag{10}$$

It is easy to show that the term E is minimized by $\varepsilon_2=\varepsilon_3=\ldots\varepsilon_n=0$ (allocative efficiency), and then equals $\ln r$. In this case, the cost function (9) is the same as (4). Otherwise, the (non-negative) value of $E-\ln r$ is the addition to $\ln C$ attributable to allocative inefficiency.

To summarize, the (stochastic) cost frontier is the same as in section 2 – it is given by

$$K+\frac{1}{r}\ln y+\sum_{i=1}^{n}\frac{\alpha_i}{r}\ln p_i-\frac{1}{r}v.$$

Actual cost exceeds the frontier for two reasons: (i) *technical inefficiency*, reflected in the term $(1/r)u$; and (ii) *allocative inefficiency*, reflected in the term $(E-\ln r)$.

Before turning to the question of estimating the model, it is worth pointing out exactly what information this will reveal. One thing that will be obtained is a set of estimates of the parameters of the production function – A, $\alpha_1,\ldots,\alpha_n$, and r. These parameters are of interest in themselves, and they allow us to measure the non-stochastic part (the mean) of the production and cost frontiers. We will also obtain an estimate of σ_v^2, which allows us to specify the distribution of either frontier. However, as in section 2, we cannot obtain individual values (by observation) for v; we can calculate the distribution of either (stochastic) frontier, for each observation, but not its actual value in our sample. Similarly, we can obtain an estimate of σ_u^2. From this we can obtain an estimate of the mean of u, so that we can estimate the *average* deviation from either frontier that is due to technical inefficiency. However, as in section 2, we cannot assign individual values of u by observation.

Looking at allocative inefficiency, however, we can generate more information. We obtain estimates of the elements of Σ, which specifies the distribution of the cost minimizing disturbances. More interesting, however, is the fact that we can obtain, *for each observation*, individual estimates of the ε's, simply as the residuals of the fitted cost minimizing conditions. *This enables us to estimate the extra cost due to allocative inefficiency, for each observation*, as well as its average value. [Incidentally, the average extra cost of allocative inefficiency is most easily calculated simply by averaging the values of the individual observations. The reason is that the expected value of E in (10) is very complicated even for $n=2$, and is completely intractable for $n\geqq 3$.]

We now return to the question of how to estimate the parameters of the model. We could, of course, estimate the cost function (9) by OLS, if all we wanted was consistent estimates of the α_i. However since the moments of the disturbance in (9) are intractable, we cannot manipulate them to obtain consistent estimates of the other parameters of the model (as we did, in the last section, under the assumption of exact cost minimization). If we wish to obtain consistent estimates of all of the parameters, we need to look to either the original system (7) or the set of input demand equations (8).

Conceptually, the simpler choice is to apply MLE to the original system (7), bearing in mind that we are still assuming output exogenous but the inputs endogenous. By assumption, the density of ε is the density of $N(0,\Sigma)$,

$$(2\pi)^{-(n-1)/2}|\Sigma|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}\varepsilon'\Sigma^{-1}\varepsilon].$$

The density of $v-u$, from ALS (1977) is

$$\frac{2}{\sigma}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(v-u)^2\right]\{1-F[(v-u)\lambda/\sigma]\}.$$

(Here F is the cumulative distribution function of the standard normal distribution.) Since $(v-u)$ is assumed to be independent of ε, the joint density of ε and $(v-u)$ is just the product of these two terms. Finally, the Jacobian of the transformation from $[\varepsilon', v-u]'$ to $[\ln x_1,\ldots,\ln x_n]'$ is r. We therefore find that the likelihood function of the sample is

$$\begin{aligned} L = {} & 2^T(2\pi)^{-nT/2}(\sigma^2)^{-T/2}|\Sigma|^{-T/2}r^T \\ & \times\exp\left\{-\tfrac{1}{2}\sum_{t=1}^{T}\left[\varepsilon_t'\Sigma^{-1}\varepsilon_t+\frac{1}{\sigma^2}\xi_t^2\right]\right\} \\ & \times\prod_{t=1}^{T}[1-F(\xi_t\lambda/\sigma)], \end{aligned} \tag{11}$$

where

$$\varepsilon_t=\begin{bmatrix}\ln x_{1t}-\ln x_{2t}-B_{2t}\\ \vdots \\ \ln x_{1t}-\ln x_{nt}-B_{nt}\end{bmatrix},$$

and

$$\xi_t=\ln y_t-A-\sum_{i=1}^{n}\alpha_i\ln x_{it}.$$

This likelihood function (or, typically, its logarithm) can be maximized numerically with respect to the parameters (A, $\alpha_1,\ldots,\alpha_n$, σ^2, λ, and the elements of Σ) to obtain the maximum likelihood estimates. Their asymptotic variances can be gotten from the information matrix.

The computational burden of obtaining maximum likelihood estimates can be eased somewhat by concentrating the likelihood function with respect to Σ. At the maximum of L, the following is true:

$$\sigma_{ij}=\frac{1}{T}\sum_t \varepsilon_{ti}\varepsilon_{tj}$$

$$=\frac{1}{T}\sum_t (\ln x_{1t}-\ln x_{it}-B_{it})(\ln x_{1t}-\ln x_{jt}-B_{jt}), \qquad i,j=2,\dots,n. \quad (12)$$

This expresses the elements of Σ as a function of the α's (which make up the B's. We can substitute these functions of the α's for Σ in (11) to get the concentrated likelihood function, which depends only on A, $\alpha_1,\dots,\alpha_n$, σ^2 and λ. These parameters can be estimated by maximizing the concentrated likelihood function. The maximum likelihood estimate of Σ can then be formed according to (12).

Finally, we could also estimate the parameters of the model by applying OLS to the factor demand equations (8). These estimates would be consistent, but inefficient. They would also most likely be messy since we would get n estimates of each of the α_i, and since the estimates of σ_u^2, σ_v^2 and Σ would have to be solved for from the second and third moments of the residuals. As a result, we will not discuss these estimates further.

We now return to the empirical example discussed in section 2. In table 2 we give the maximum likelihood estimates of the production function parameters, and the implied estimates of the cost function parameters.

The results for both production and cost function parameters are broadly similar to those obtained under the assumption of allocative efficiency (and presented in table 1). The biggest differences are that the coefficient of the labor input has decreased markedly, and all of the standard errors have decreased. Also the values of σ_u^2 and λ have decreased considerably. This is a reflection of finding less technical inefficiency; we will return to this point shortly.

The mean of the one-sided disturbance in the production function $(-u)$ is -0.09889, indicating that output averages 9.9% below the frontier. The mean of the one-sided technical inefficiency disturbance in the cost function $(1/r)u$ is 0.08455, which indicates that technical inefficiency raises cost an average of 8.5% above the cost frontier. In addition, the average value of $E - \ln r$ is 0.08059, which indicates that allocative inefficiency raises cost an additional 8.1% above the cost frontier. In other words, we find that, on average, cost is 16.5% above its minimal (frontier) level. Roughly half of this is due to technical inefficiency (failure to produce maximal output from the inputs chosen), and half is due to allocative inefficiency (failure to choose inputs correctly).

Also, it is worth pointing out that the sample average values of ε_2 and ε_3 are 0.7291 and 0.3630, respectively. These are both considerably (and very

Table 2

Estimates of production function (technical and allocative inefficiency).

Constant	Capital	Fuel	Labor	σ_u^2	σ_v^2	λ	RTS
−11.7099	0.12945	0.97661	0.06349	0.015361	0.003584	2.0701	1.1696
(0.2041)	(0.02158)	(0.02634)	(0.00821)			(0.7962)	(0.01354)

Derived estimates of cost function (technical and allocative inefficiency).

Constant	Output	P_K/P_L	P_F/P_L	σ_u^2	σ_v^2	λ
10.4076	0.8550	0.1107	0.8350	0.01123	0.00262	2.0701
	(0.00990)	(0.01816)	(0.01733)			(0.7963)

Incidental parameters: $\sigma_{22}=0.9626$, $\sigma_{33}=0.7220$, $\sigma_{23}=0.4980$

significantly) different from zero, which ought to make one question the specification that the means of ε_2 and ε_3 are zero. In the next section this assumption is relaxed. For now we just note that the positive average values of ε_2 and ε_3 indicate overcapitalization. This is true because ε_2 and ε_3 are, respectively, the percentage deviations from the cost minimizing capital/fuel and capital/labor ratios. In other words, we find that the capital/fuel ratio and capital/labor ratio are, on average, 73% and 36% higher, respectively, than the cost minimizing ratios.

Some of the individual observations are also interesting in this regard. The most allocatively inefficient observation in the sample has a value of $E-\ln r$ of 0.5091, indicating extra cost of over 50% due to allocative inefficiency alone. This corresponds to $\varepsilon_2=2.0705$ and $\varepsilon_3=-0.4709$; that is, to a capital/fuel ratio that is 207% too high, and a capital/labor ratio that is 47% too low. Another observation has a value of $E-\ln r$ of 0.4953; the next highest value, after these two, is 0.2178. The most allocatively efficient observation has a value of $E-\ln r$ of 0.0007, corresponding to $\varepsilon_2=0.1137$ and $\varepsilon_3=0.0697$.

4. Technical and systematic allocative inefficiency

In the previous section we considered allocative inefficiency, with the disturbances in the cost minimizing equations assumed to have means of zero. This assumption implies that there is no systematic tendency to over- or under-utilize any input relative to any other.

There are at least two reasons for relaxing this assumption. One is the simple observation that, in our empirical example, the sample means of the (estimated) disturbances in the cost minimizing equations are very significantly different from zero. The other reason is that there is a well-known argument (the so-called Averch–Johnson hypothesis) that suggests that firms which are subject to rate of return regulation will tend to over-capitalize systematically; that is, they will employ higher ratios of capital to other inputs than cost minimization would dictate.[7] We can allow for this possibility by allowing a disturbance with a (possibility) non-zero mean in the cost minimizing condition. By testing whether these means are zero, we can test whether or not there are in fact systematic deviations from the cost minimizing input ratios. This could be considered, for example, as a test of the Averch–Johnson hypothesis.

We therefore consider the specification as given in eq. (7) of the previous section, which for convenience we recopy here:

$$\begin{aligned} \ln y &= A+\sum_{i=1}^{n}\alpha_i\ln x_i+(v-u), \\ \ln x_1-\ln x_i &= B_i+\varepsilon_i, \qquad i=2,\ldots,n. \end{aligned} \tag{13}$$

[7]With essentially the same data we are using, Cowing (1975) finds evidence supporting the Averch–Johnson hypothesis.

As in section 3, $B_i = \ln(p_i \alpha_1 / p_1 \alpha_i)$, $v \sim N(0, \sigma_v^2)$, and u is the absolute value of a variable distributed as $N(0, \sigma_u^2)$. However, we now assume that $\varepsilon = (\varepsilon_2, \ldots, \varepsilon_n)'$ has a multivariate normal distribution with mean μ and covariance matrix Σ. This reduces to the specification of the last section if $\mu = 0$.

The factor demand equations and cost function are of precisely the same form as in the last section, as given by eqs. (8) and (9), respectively. The only difference is of course that ε, which appears in the disturbances of the factor demand equations and cost function, no longer has a mean of zero.

Estimation of this model is very similar to the estimation of the model of the previous section. The likelihood function is

$$\begin{aligned} L = {} & 2^T (2\pi)^{-nT/2} (\sigma^2)^{-T/2} |\Sigma|^{-T/2} r^T \\ & \times \exp\left\{ -\tfrac{1}{2} \sum_{t=1}^{T} \left[(\varepsilon_t - \mu)' \Sigma^{-1} (\varepsilon_t - \mu) + \frac{1}{\sigma^2} \xi_t^2 \right] \right\} \\ & \times \prod_{t=1}^{T} [1 - F(\xi_t \lambda / \sigma)], \end{aligned} \tag{14}$$

which reduces to the previous expression (11) if $\mu = 0$. The likelihood function can be 'concentrated' with respect to Σ and μ, since at the maximum

$$\begin{aligned} \mu_i = \bar{\varepsilon}_i &= \frac{1}{T} \sum_t \varepsilon_{ti} \\ &= \frac{1}{T} \sum_t (\ln x_{1t} - \ln x_{it} - B_{it}), \end{aligned} \tag{15}$$

and

$$\sigma_{ij} = \frac{1}{T} \sum_t (\varepsilon_{ti} - \bar{\varepsilon}_i)(\varepsilon_{tj} - \bar{\varepsilon}_j). \tag{16}$$

Substituting these expressions for μ and Σ into (11) makes the likelihood function depend only on the reduced set of parameters A, $\alpha_1, \ldots, \alpha_n$, σ^2, λ. These parameters can be estimated by numerical maximization of the likelihood function, and then the maximum likelihood estimates of μ and Σ can be found according to (15) and (16).

We now return to the empirical example discussed in the previous two sections. In table 3 we give the maximum likelihood estimates of the production function parameters, and the implied estimates of the cost function parameters. The results for both the production function parameters and the cost function parameters are quite similar to those reported in table 2. Note that the coefficient of fuel in the production function is greater than one, but not significantly so (at reasonable confidence levels).

Our estimates of total inefficiency, and its breakdown, are also quite similar to those of the last section. The mean of the one-sided disturbance in the production function $(-u)$ is -0.10132. This indicates that output averages 10.1 % below the frontier (which is only very slightly more than the

Table 3

Estimates of production function (technical and systematic allocative inefficiency).

Constant	Capital	Fuel	Labor	σ_u^2	σ_v^2	λ	RTS
−12.0226	0.11685	1.0099	0.06271	0.01613	0.00205	2.8061	1.1895
(0.2942)	(0.02429)	(0.02186)	(0.02529)			(1.1595)	(0.02279)

Derived estimates of cost function (technical and systematic allocative inefficiency).

Constant	Output	P^K/P_L	P_F/P_L	σ_u^2	σ_v^2	λ
10.6293	0.84069	0.09823	0.84901	0.01140	0.00145	2.8061
	(0.01610)	(0.01867)	(0.02159)			(1.1595)

Incidental parameters: $\mu_2=0.9280$, $\mu_3=0.4530$, $\sigma_{22}=0.3352$, $\sigma_{33}=0.5902$, $\sigma_{23}=0.2104$

9.9 % found in the last section). The mean of $(1/r)u$ is 0.08518, which indicates that technical inefficiency raises cost an average of 8.5 % above the frontier (which is essentially identical to the 8.5 % found in the previous section). The average value of $E-\ln r$ is 0.09198, which indicates that allocative inefficiency raises cost an additional 9.2% (compared to the 8.1% found previously). As a result, cost is, on average, a total of 17.7% above the frontier level; slightly over half of this is due to allocative inefficiency, and slightly under half is due to technical inefficiency.

The estimates of μ_2 and μ_3 are 0.9280 and 0.4530, respectively. This indicates very considerable systematic over-capitalization; the capital/fuel and capital/labor ratios are respectively 92.8% and 45.3% higher (on average) than the cost minimizing ratios. This is certainly consistent with the Averch–Johnson hypothesis. Incidentally, the likelihood ratio test statistic for testing the hypothesis $\mu_2=\mu_3=0$ is 140.2. Since the relevant (asymptotic) distribution under the null hypothesis is χ_2^2, the hypothesis is strongly rejected.

The values of ε_2, ε_3 and $E-\ln r$ for individual observations are also surprisingly similar to those obtained from the previous model. For example, the most allocatively inefficient observation has cost 54% above the frontier (compared to 50.9% previously), with $\varepsilon_2=2.206$ (vs. 2.071) and $\varepsilon_3=-0.381$ (vs. -0.471).

5. Conclusions

In this paper we have investigated the relationship among stochastic production, factor demand and cost frontiers. We have demonstrated how a (technically and/or allocatively) inefficient production process can be modelled in an empirically useful way using these frontiers. We have also developed various techniques appropriate for the estimation of such stochastic frontiers under three different assumptions concerning the magnitude and the nature of allocative inefficiency. The output of the estimation procedures includes consistent estimates of the shape and placement of the frontiers, measures of the average extent and cost of both technical and allocative inefficiency over the sample, and a measure of the extent and cost of allocative inefficiency for each observation in the sample.

In our application of these techniques to U.S. steam-electric generating plants we found substantial evidence of both types of inefficiency. Technical inefficiency averaged 10.1% over the sample, pushing actual cost 8.5% above the cost frontier. Allocative inefficiency in the form of excessively high capital/fuel and capital/labor ratios led to an additional 9.2% increase in actual cost. Thus the total cost of inefficiency was about 17.7% of frontier cost, and was about evenly allocated between the technical and allocative components. The cost of allocative inefficiency alone at the plant level ranged from 0.1% to 54% of frontier cost.

Since our technique provides only sample mean estimates of the extent and cost of technical inefficiency, we are unable to conduct a search for its sources. But we do obtain estimates of allocative inefficiency by plant, and so a search for its sources is feasible. With the data at our disposal we are able to conduct only a few tests, most of which turn out negative. Allocative inefficiency is uncorrelated with various measures of allowed rate of return. It is also uncorrelated with the year of plant installation. It is, however, negatively and significantly correlated with size of plant, as measured by output.

Much work remains to be done on this topic. For example, in our most general statistical model set out in section 4, we assumed that the two types of inefficiency are uncorrelated. This assumption can be relaxed, though not in any simple fashion; what makes things difficult is that presumably we want u to be correlated not with ε_i but with $|\varepsilon_i|$, $i=2,\ldots,n$. This we hope to do in a future paper. A second obvious extension is to relax the homogeneous Cobb–Douglas assumption to see what effect functional form has on the measurement of inefficiency. In this regard we consider an extension from homogeneity to homotheticity to be of particular interest. Thirdly, we hope to consider alternatives to the cost minimization hypothesis. Finally, it would be desirable to obtain estimates of the extent and cost of technical inefficiency by plant. At present we do not see how to solve this problem.

Appendix A

Our previous paper [Aigner, Lovell and Schmidt (1977)] considered maximum likelihood estimation of an equation with a disturbance of the form $v-u$, where v is normal and u is half-normal. We now consider a disturbance of the form $v+u$.

Suppose that $V \sim N(0,\sigma_v^2)$, and that u is the absolute value of a variable which is distributed as $N(0,\sigma_u^2)$. Define $\lambda=\sigma_u/\sigma_v$ and $\sigma^2=\sigma_u^2+\sigma_v^2$. Then density of

$$\varepsilon=v+u$$

is

$$\frac{2}{\sigma} f\left(\frac{\varepsilon}{\sigma}\right)[1-F(-\varepsilon\lambda\sigma^{-1})],$$

where f and F represent the density and cumulative distribution function of the $N(0,1)$ distribution.

Suppose we write the model in the general form

$$y_i=\beta' x_i+\varepsilon_i, \qquad 1,\ldots,N,$$

where i is the observation index and N is the number of observations, y represents the dependent variable, x represents the vector of explanatory variables, with coefficients β, and ε is the disturbance. Then the log-likelihood function is of the form

$$L = N\ln\sqrt{\frac{2}{\pi}} - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta' x_i)^2 + \sum_{i=1}^{n}\ln(1-F_i),$$

where

$$F_i = F[-(y_i - \beta' x_i)\lambda\sigma^{-1}].$$

This is identical to the likelihood function of the previous paper, except for the minus sign inside the distribution function F.

To compute the asymptotic variances of the MLE's, we use the matrix of second-order derivatives of L. To define these, let F_i be as above and let f_i be the standard normal density evaluated at $-(y_i - \beta' x_i)\lambda\sigma^{-1}$. Then we have the following:

$$\frac{\partial^2 L}{\partial\lambda^2} = \frac{1}{\sigma^2}\sum_i \frac{f_i}{(1-F_i)^2}\left[-f_i - \frac{\lambda}{\sigma}(1-F_i)\varepsilon_i\right]\varepsilon_i^2,$$

$$\frac{\partial^2 L}{\partial\beta\,\partial\beta'} = -\frac{1}{\sigma^2}\sum_i x_i x_i' + \frac{\lambda^2}{\sigma^2}\sum_i \frac{f_i}{(1-F_i)^2}\left[-f_i - \frac{\lambda}{\sigma}\varepsilon_i(1-F_i)\right]x_i x_i',$$

$$\frac{\partial^2 L}{\partial(\sigma^2)^2} = \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}\sum_i \varepsilon_i^2 + \frac{\lambda}{4\sigma^5}\sum_i \frac{f_i}{(1-F_i)^2}\left[3(1-F_i)\varepsilon_i - \frac{\lambda^2}{\sigma^2}(1-F_i)\varepsilon_i^3 - \frac{\lambda}{\sigma}f_i\varepsilon_i^2\right],$$

$$\frac{\partial^2 L}{\partial\lambda\,\partial\beta} = -\frac{1}{\sigma}\sum_i \frac{f_i}{(1-F_i)^2}\left[(1-F_i) - \frac{\lambda}{\sigma}f_i\varepsilon_i - \frac{\lambda^2}{\sigma^2}(1-F_i)\varepsilon_i^2\right]x_i,$$

$$\frac{\partial^2 L}{\partial\sigma^2\,\partial\beta} = -\frac{1}{\sigma^4}\sum_i \varepsilon_i x_i + \frac{\lambda}{2\sigma^3}\sum_i \frac{f_i}{(1-F_i)^2}\left[(1-F_i) - \frac{\lambda}{\sigma}f_i\varepsilon_i - \frac{\lambda^2}{\sigma^2}\varepsilon_i^2(1-F_i)\right]x_i,$$

$$\frac{\partial^2 L}{\partial\sigma^2\,\partial\lambda} = \frac{1}{2\sigma^3}\sum_i \frac{f_i}{(1-F_i)^2}\left[-(1-F_i)\varepsilon_i + \frac{\lambda}{\sigma}f_i\varepsilon_i^2 + \frac{\lambda^2}{\sigma^2}(1-F_i)\varepsilon_i^3\right].$$

These results are identical to the results of the previous paper, except for a few changes of sign.

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