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Binary complexes and algebraic K-theory

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ABSTRACT

FACULTY OF HUMAN AND SOCIAL SCIENCES
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Doctor of Philosophy

Binary complexes and algebraic K-theory

by Thomas K. Harris

Grayson has recently produced a purely algebraic description of the higher algebraic K-groups of an exact category. We take Grayson's presentation as the definition of the K-groups and examine the strength of the resulting theory. We provide the relevant background to state Grayson's theorem, and make a comparison between Grayson's presentation for K_1 and that of Nenashev. We give new algebraic proofs of the additivity, resolution, and cofinality theorems of algebraic K-theory. Finally we construct exterior power operations on the higher K-groups of schemes using Dold–Puppe complexes. We prove that these operations are homomorphisms on K_n for $n > 0$, and that they satisfy the second axiom of a λ-ring.

Contents

Declaration of authorship

I, Thomas Harris, declare that the thesis entitled *Binary complexes and algebraic K-theory* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
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- where I have consulted the published work of others, this is always clearly attributed;
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- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:

[HKT] Tom Harris, Bernhard Köck, and Lenny Taelman. *Exterior power operations on higher K-theory via binary complexes*, in preparation.

[Har15] Tom Harris. *Algebraic proofs of some fundamental theorems in algebraic K-theory*, Homology, Homotopy Appl. **17** (2015), no. 1, 267–280.

Signed:..

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Introduction

At the most basic level, K-theory consists of a series of invariants associated to various complicated mathematical objects X. The invariants $K_n(X)$ are abelian groups, and one hopes that from these K-groups one can recover information about the object X. The beginnings of K-theory can be found in the work of Grothendieck, who invented $K_0(X)$ for use in proving Riemann–Roch theorems (here X is a scheme), and the in work of Whitehead, whose obstruction to a homotopy equivalence being simple lives in the group $\mathrm{Wh}(\pi)$, which is now regarded as a quotient of $K_1(\mathbb{Z}\pi)$ (π is the fundamental group of the space in question). Shortly afterwards, Atiyah and Hirzebruch developed Grothendieck's ideas in the topological setting to give the first example of a generalised cohomology theory. Bass defined $K_1(R)$ for any ring R, and Milnor found the correct definition of $K_2(R)$. Higher algebraic K-groups were eventually defined by Quillen, and the theory flourished. Today various flavours of K-theory can be found at work in many parts of pure mathematics, from number theory and algebraic geometry, to algebraic topology and non-commutative geometry, and they have revealed hitherto unknown connections between these fields.

The history of algebraic K-theory can be roughly organised into two epochs, divided by Quillen's seminal articles ***On the Cohomology and K-Theory of the General Linear Groups Over a Finite Field*** [Qui72] and *Algebraic K-theory I* [Qui73]. One reason that Quillen's approach to K-theory was quickly popularly recognised to be the correct approach is that his methods allowed him to prove many deep and powerful theorems about higher K-groups that generalised known theorems for K_0, K_1, and K_2. Quillen achieved this by passing out of the realm of algebra and defining the higher K-groups as the homotopy groups of a K-theory space. While later constructions of K-groups work in more general settings (notably Waldhausen's K-theory for categories with cofibrations), the construction in [Qui73] defines higher algebraic K-groups of an ***exact category***. Exact categories are a weakening of abelian categories; an exact category comes with a class of short exact sequences, which behave like those in an abelian category even though the underlying category may not be abelian. A typical example is the category of vector bundles on some scheme X (in the affine case, just the projective modules over the coordinate ring). One may think of exact categories as the broadest 'algebraic' source for algebraic K-theory.

Assuming that one is interested in the K-theory groups rather than the space itself, we now have a rather unusual situation: to glean information about an algebraic object (the exact category) from simpler algebraic objects (the resulting abelian K-groups), one must make a detour through the realm of homotopy theory. We are reminded of the quote attributed to Jacques Hadamard: *"Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe." (The shortest path between two truths in the real domain passes through the complex domain.).* We would not like to deny the utility of the homotopy-theoretic methods, but it seems a natural question to ask whether such a detour is necessary. For a long time it seemed so. Following Quillen, other equivalent constructions of higher K-groups appeared, but all of them made use of homotopy theory in some way or other. In the 90s Alexander Nenashev published a series of papers ([Nen96], [Nen98b], [Nen98a]) describing K_1 purely algebraically in terms of *binary short exact sequences*: that is, pairs of short exact sequences with the same objects. In 2012, Dan Grayson published *Algebraic K-theory via binary complexes* [Gra12], which in spirit generalises Nenashev's result to all of the higher K-groups of an exact category $\mathcal{N}$. To do this, Grayson has to pass from binary short exact sequences to binary acyclic chain complexes, and from complexes to (many dimensional) multicomplexes, obtaining an exact category $(B^{q})^n\mathcal{N}$ of *n-dimensional bounded acyclic binary complexes* in $\mathcal{N}$. Grayson proves that $K_n\mathcal{N}$ is the quotient group of $K_0(B^{q})^n\mathcal{N}$ by the subgroup generated by all those binary multicomplexes that are not, loosely speaking, truly binary. Since K_0 has a simple presentation, this finally yields a description of $K_n\mathcal{N}$ that is purely algebraic.

This thesis is part of a project to assess the strengths of these algebraic foundations for K-theory, to test whether they can be used to recover known results, and to see if they can be used to obtain new results. After describing Grayson's presentation of $K_n\mathcal{N}$ in the first chapter, we thereafter take it to be the *definition* of the higher algebraic K-groups. In chapter 3 we obtain new proofs of some of the fundamental theorems of algebraic K-theory: the additivity, resolution, and cofinality theorems. In chapter 4 we use binary multicomplexes and the Dold–Kan correspondence to construct exterior power operations that satisfy the second λ-ring axiom on the higher K-groups of quasi-compact schemes. This widens the class of schemes for which such operations are known to exist. We also include in chapter 2 some material relating Grayson's presentation of K_1 to that of Nenashev.

Meanwhile, Taelman has outlined a proof that our exterior operations in fact make the K-theory of quasi-compact schemes schemes into a full λ-ring. Further to this and the results listed above, Grayson has used also his presentation to construct relative algebraic K-groups and produce the corresponding long exact sequence by elementary means [Gra13]. There are still many unanswered questions about the efficacy of this new construction, but we believe that it is a promising approach. We now describe the contents of each chapter of this thesis in more detail.

Summary of contents

Chapter 1 contains a detailed account of the necessary background for the results of this thesis. We give a self-contained introduction to exact categories and their Grothendieck groups in sections 1.1 and 1.2. After giving a short overview of the history of various constructions of the higher algebraic K-groups in section 1.3, we proceed with the background required to describe Grayson's new presentation of the higher K-groups of an exact category. Section 1.4 describes the theory of (acyclic) chain complexes in exact categories, and section 1.5 generalises this theory to binary complexes and multicomplexes. We finish that section, and the chapter, by stating Grayson's theorem on the presentation of of the higher K-groups (Theorem/Definition 1.40), which will be taken as a definition in chapters 2 – 4.

Chapter 2 contains some results concerning manipulations of binary (multi)complexes and the corresponding effects on K-theory, and their applications in simple cases. In section 2.1 we recall Nenashev's theorem about K_1 of an exact category (Theorem 2.1). We prove a generalisation of Nenashev's second relation for Grayson's presentation (Proposition 2.10), using a lemma that we have termed the 'shifting lemma' (Lemma 2.5), which turns out to be the most useful symmetry of binary complexes. In section 2.2 we make explicit the connections between the (algebraic) definitions of K_1 given by Bass, Nenashev, and Grayson. In particular when $\mathcal{P}$ is a split exact category, we exhibit naturally defined isomorphisms $K_1^{\mathrm{Bass}}\mathcal{P} \xrightarrow{\cong} K_1^{\mathrm{Nen}}\mathcal{P} \xrightarrow{\cong} K_1\mathcal{P}$. In the final part of the chapter, section 2.3, we make some remarks about a conjectural comparison homomorphism between the Milnor K-groups of a field F and Grayson's K-groups. Along the way we introduce a product structure on $K_*(F)$.

Chapter 3 is a slightly expanded version of the paper [Har15], which gives new proofs of some of the fundamental theorems of algebraic K-theory. Most of those theorems have simple proofs for K_0. Since Grayson's presentation describes $K_n\mathcal{N}$ (where $\mathcal{N}$ is an arbitrary exact category) as a certain quotient of the Grothendieck group $K_0(B^{\mathrm{q}})^n\mathcal{N}$ of a category of binary multicomplexes, we therefore prove the desired theorems for $K_n\mathcal{N}$ by first proving them for the exact category $(B^{\mathrm{q}})^n\mathcal{N}$ and then proving that the result passes to the relevant quotient. Using this approach, we prove the additivity theorem 3.2, the resolution theorem 3.7, and the cofinality theorem 3.18 in sections 3.1, 3.2, and 3.3 respectively.

Chapter 4 contains the most recent work. In it we use binary complexes to give a new construction of exterior power operations on the higher K-groups of a (quasi-compact) scheme. Let $\mathbf{P}(X)$ denote category of locally free $\mathcal{O}_X$-modules of finite rank over such a scheme X. We use the usual exterior powers Λ^r to induce functors $\Lambda^r_n\colon (B^{\mathrm{q}})^n\mathbf{P}(X) \to (B^{\mathrm{q}})^n\mathbf{P}(X)$ and show that these induce operations on $K_n(X)$ that have good properties. Since the original Λ^r is not exact (or even additive), we have to use the Dold–Kan correspondence to define these functors on categories of acyclic binary multicomplexes, in the same way that one defines derived functors of a general non-additive functor. In section 4.1 we give some motivation for the results of the chapter and outline our approach. Section 4.2 presents the necessary preliminaries from homological algebra, including the details of the Dold–Kan correspondence. Sections 4.3 and 4.4 do the main work of constructing the functors Λ_n^r in the affine case: the former section is rather more general. To show that our exterior powers define operations on K-theory, they must interact well with a tensor product. Since we define the exterior powers simplicially, the corresponding tensor product must also be constructed using the Dold–Kan correspondence: this is done in section 4.5, where we actually show that the resulting product vanishes (Proposition 4.38). The main theorem, that the functors $\Lambda_n^r \colon (B^{\mathrm{q}})^n\mathbf{P}(X) \to (B^{\mathrm{q}})^n\mathbf{P}(X)$ induce homomorphisms on $K_n(X)$ (Theorem 4.41), is proved in section 4.6. In section 4.7 we calculate some examples in simple cases. Finally, in section 4.8, we prove that our exterior powers on $K_*(X)$ satisfy the second λ-ring axiom (Theorem 4.45). We finish with a question that if answered affirmatively would imply the third λ-ring axiom and therefore complete the proof that our operations make $K_*(X)$ into a λ-ring. Since this section was written, Taelman has outlined an argument that appears to answer our question in the affirmative. These results will be included in the forthcoming paper [HKT].

In chapter 4 we use a slightly modified version of Grayson's construction of $K_n\mathcal{N}$. In the appendix to the chapter (4.A) we prove that these constructions are equivalent.

Notations and conventions

Keeping a consistent set of terminology and notations is difficult, even for short documents. I have tried to ensure there are no outright clashes in this thesis by setting out some intended conventions. However it is possible, perhaps likely, that some inconsistencies have slipped through. The following is a list of my good intentions.

1. All diagrams are commutative, unless stated otherwise.
2. Categories in the abstract are denoted with script letters, like these: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...
   More concrete categories are in bold: $\mathbf{Mod}(R)$, $\mathbf{Proj}(R)$, $\mathbf{Free}(R)$, $\mathbf{M}(X)$, $\mathbf{P}(X)$ ...
3. Objects of categories are denoted by upper-case letters: $A, B, C, \ldots$
   Elements of the objects are given lower-case letters (these are surprisingly rare).
4. Graded objects of a category (usually chain complexes or binary chain complexes) will have a dot-subscript, like these: $A_\bullet$, $B_\bullet$, $C_\bullet$.
5. Regarding graded objects, we use the phrase "degree-wise" to describe a property of a morphism that holds in every degree. For example: "$f\colon C_\bullet \to D_\bullet$ is degreewise an epimorphism" means that $f_i\colon C_i \to D_i$ is an epimorphism for every i.
6. We say that a graded object $N_\bullet$ is "supported on I" or "concentrated in degrees I" to mean that the object N_i is equal to 0 for i outside of I.
7. When a new term is being defined, it will be in ***bold italics***. If a definition cannot be found in the thesis, then a reference should be nearby.

Chapter 1

Exact categories, binary complexes, and K-theory

In this chapter we present some foundational material for the rest of the thesis. We are interested in the algebraic aspects of K-theory. In particular, we study the K-theory of those categories that can be considered in some was as being 'algebraic'—most commonly categories of modules over rings, or sheaves on schemes. The proper framework for this study is the notion of an *exact category*. We begin the chapter by introducing exact categories axiomatically and studying their basic properties. We define the Grothendieck group of an exact category and supply some basic examples, and also give a very brief history of the development of higher algebraic K-theory. In the final parts we define what it means for a chain complex in an exact category to be acyclic, and prove that the category of acyclic complexes is again an exact category. We introduce multicomplexes and binary multicomplexes, and finally end the chapter with the statement of Grayson's new presentation of the higher K-groups of an exact category. Nothing in this chapter is new: the main sources are Grayson's paper [Gra12], Weibel's K-book [Wei13], and Bühler's wonderful expository article on exact categories [Büh10].

1.1 Exact categories

In this section we define exact categories using the axioms given by Bühler [Büh10], and give some standard examples. We describe exact functors and exact subcategories, and state the Gabriel–Quillen embedding theorem. Finally we prove some elementary facts about exact categories that we shall need in the following sections. We assume familiarity with basic category theory, but do not assume any prior knowledge of exact categories. We briefly recall the definitions of additive and abelian categories, but the presumption is that the reader is already comfortable with these.

Recall that a category is ***pre-additive*** if its Hom-sets are abelian groups and composition of morphisms is bilinear; that is, a pre-additive category is a category enriched over **Ab**. An ***additive category*** is a pre-additive category which admits all finite biproducts. In particular, additive categories are pointed with a distinguished zero object, which we shall always denote by 0.

An ***abelian category*** is an additive category $\mathcal{A}$ such that:

1. Every morphism in $\mathcal{A}$ has a kernel and a cokernel.
2. Every monomorphism in $\mathcal{A}$ is the kernel of its cokernel.
3. Every epimorphism in $\mathcal{A}$ is the cokernel of its kernel.

If R is a ring, then the category of all R-modules is an abelian category. It is a standard fact that every small abelian category can be embedded as a full subcategory of the category of R-modules for some R, and furthermore that the embedding is exact in the sense of Definition 1.9 below. This is the Freyd–Mitchell embedding theorem.

Definition 1.1. A ***kernel-cokernel pair*** in an additive category is a composable pair of morphisms

$$A' \xrightarrow{i} A \xrightarrow{p} A''$$

such that (A', i) is the kernel of p and (A'', p) is the cokernel of i.

A sequence of morphisms:

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

is short exact in an abelian category $\mathcal{A}$ if and only if (f, g) is a kernel-cokernel pair. This condition is a little too rigid for several applications. An exact category is an additive category together with a distinguished subclass of its kernel/cokernel pairs that behaves like an abelian category in some important ways. Exact categories are a halfway house between additive and abelian categories. They are intended to describe various algebraic/geometric situations in which there is a good notion of 'short exact sequence', but without the strong regularity conditions linking monomorphisms/epimorphisms with kernels/cokernels that are present in an abelian category.

Definition 1.2. An ***exact structure*** on an additive category $\mathcal{N}$ is a class of kernel-cokernel pairs called ***admissible short exact sequences***, that is closed under isomorphisms of sequences and satisfies axioms (E1a) - (E3b) below. The kernel map is called an ***admissible monomorphism*** (or is said to be ***admissible monic***), and is denoted by an arrow with a tail ($\rightarrowtail$). The cokernel is called an ***admissible epimorphism*** (or is said to be ***admissible epic***), and is denoted by a two-headed arrow ($\twoheadrightarrow$). The axioms are:

(E1a) For every object N of $\mathcal{N}$, the identity 1_N is admissible monic.

(E1b) For every object N of $\mathcal{N}$, the identity 1_N is admissible epic.

(E2a) The composition of a pair of admissible monomorphisms is admissible monic.

(E2b) The composition of a pair of admissible epimorphisms is admissible epic.

(E3a) Push-outs of admissible monomorphisms along arbitrary morphisms exist and are themselves admissible monic.

(E3b) Pull-backs of admissible epimorphisms along arbitrary morphisms exist and are themselves admissible epic.

An ***exact category*** is an additive category with a given exact structure. When the exact structure is understood we shall simply say that a kernel-cokernel pair in the exact structure is a ***short exact sequence*** in $\mathcal{N}$. Caution is advised though: additive categories do not normally have a unique exact structure (see Lemma 1.12).

Remark 1.3. The axioms of Definition 1.2 are equivalent to those given by Quillen ([Qui73], §2). Neither system is minimal, but Bühler's are easy to use and pleasingly symmetric. A minimal system of axioms was given by Keller in [Kel90].

Our first example should come as no surprise.

Example 1.4. Every abelian category $\mathcal{A}$ is an exact category with the exact structure given by declaring the admissible short exact sequences to be the usual short exact sequences in $\mathcal{A}$. Axioms (E1) and (E2) hold trivially, and axioms (E3a) and (E3b) can be easily verified using the identifications $PO \cong \operatorname{coker}(\left[\begin{smallmatrix} f \\ -g \end{smallmatrix}\right] : A \to B \oplus C)$ and $PB \cong \ker([f\ g] : B \oplus C \to A)$ in the diagrams:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ C \arrow[r] & PO \end{tikzcd} \qquad\qquad \begin{tikzcd} PB \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D. \end{tikzcd}$$

In particular, the category of modules over a ring R is an exact category. In K-theory we are mostly interested in categories of finitely generated objects. We shall see that the category of finitely generated R-modules is an exact category for any R. In contrast, the category of finitely generated R-modules is not abelian unless R is Noetherian. If R is not Noetherian, then submodules of finitely-generated modules need not be finitely generated: the projection from a finitely generated module M to its quotient by a non-finitely generated submodule N is an epimorphism that does not have a kernel in the category of finitely generated R-modules, since any prospective kernel for $M \to M/N$ must contain all of the finitely generated submodules of N.

Similarly, the category of projective R-modules (resp. f.g. projective R-modules) is not abelian: the cokernel of the kernel of a map may not be equal to the kernel of its cokernel (consider $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$). These categories are exact because they embed 'nicely' in the larger abelian category of all R-modules.

Definition 1.5. Let $\mathcal{M}$ be a full subcategory of an exact category $\mathcal{N}$. We say that $\mathcal{M}$ is:

1. ***closed under extensions*** in $\mathcal{N}$ if, whenever there exists a short exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$ in $\mathcal{N}$ with M' and M'' in $\mathcal{M}$, then M is also an object of $\mathcal{M}$;

2. ***closed under kernels*** in $\mathcal{N}$ if, whenever there exists a short exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$ in $\mathcal{N}$ with M and M'' in $\mathcal{M}$, then M' is also an object of $\mathcal{M}$;

3. ***closed under cokernels*** in $\mathcal{N}$ if, whenever there exists a short exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$ in $\mathcal{N}$ with M' and M in $\mathcal{M}$, then M'' is also an object of $\mathcal{M}$.

Note that in each of these examples, $\mathcal{M}$ is closed under isomorphisms in $\mathcal{N}$.

Lemma 1.6. *Let $\mathcal{N}$ be an exact category and let $\mathcal{M} \subset \mathcal{N}$ be a full, additive subcategory that is closed under extensions in $\mathcal{N}$. Then $\mathcal{M}$ is an exact category with the short exact sequences inherited from $\mathcal{N}$.*

Proof. Axioms (E1) and (E2) are clear, even without the assumption that $\mathcal{M}$ is closed under extensions. To prove axiom (E3a) we first construct the diagram below by taking the pushout in the larger exact category $\mathcal{N}$ and adding in the cokernels (in $\mathcal{N}$) of the admissible epimorphisms:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[r, two heads] \arrow[d] & B/A \\ C \arrow[r, tail] & PO \arrow[r, two heads] & PO/C. \end{tikzcd}$$

By assumption, $A \rightarrowtail B$ is an admissible monomorphism of $\mathcal{M}$, so B/A is an object of $\mathcal{M}$. It is a standard fact in category theory that parallel morphisms in a pushout square have isomorphic cokernels, so PO/C is an object of $\mathcal{M}$. The bottom row of the diagram is short exact in $\mathcal{N}$ and its first and final objects are in $\mathcal{M}$, so PO is an object of $\mathcal{M}$. So the bottom row is also a short exact sequence in $\mathcal{M}$, and $C \rightarrowtail PO$ is admissible monic in $\mathcal{M}$, as was to be shown. The proof of (E3b) is analogous, using the dual fact that the kernels of parallel morphisms in a pullback square are isomorphic. □

Corollary 1.7. *Let R be a ring, and let $\mathfrak{P}$ be any property of modules that is preserved under taking extensions (and such that the zero module has property $\mathfrak{P}$). Then the full subcategory $\mathfrak{P}(R)$ of R-modules with the property $\mathfrak{P}$ is an exact category with exact sequences inherited from the category of all R-modules.*

Examples of such properties $\mathfrak{P}$ include: finitely generated, finitely presented (or, more generally, property $(FP)_n$ for any n), torsion-free, torsion, projective, injective, free, and flat. We will return to the categories of finitely generated/f.g. projective/f.g. free R-modules again and again, so it is convenient to fix some notation here: $\mathbf{Mod}_{\mathrm{f}}(R)$ is the exact category of f.g. R-modules, $\mathbf{Proj}_{\mathrm{f}}(R)$ is the exact category of f.g. projective R-modules, and $\mathbf{Free}_{\mathrm{f}}(R)$ is the exact category of f.g. free R-modules. The notations $\mathbf{Mod}(R)$, $\mathbf{Proj}(R)$, and $\mathbf{Free}(R)$ (no subscript f) are reserved for the categories of *all* R-modules (resp. projective/free R-modules). We will use these categories in chapter 4.

Example 1.8. The categories of finite rank real, complex or quaternionic vector bundles over a compact Hausdorff space X, with their usual classes of short exact sequences, are also exact categories. This follows from the Serre–Swan theorem (see, for example, §1.4 of [Ati67]), as the given categories are equivalent to the categories of f.g. projective modules over the rings of real, complex or quaternionic-valued continuous functions on X. This insight, following the definition of K_0 of a ring, led Atiyah and Hirzebruch to the definition of the functor K^0 on a topological space and the development of topological K-theory.

As usual after defining a structure, we describe the structure-preserving morphisms. Recall that an ***additive functor*** between additive categories is an **Ab**-enriched functor (a functor which is a homomorphism of abelian groups on the Hom-sets).

Definition 1.9. An additive functor $F\colon \mathcal{M} \to \mathcal{N}$ between exact categories is called an ***exact functor*** if $FM' \xrightarrow{Fi} FM \xrightarrow{Fp} FM''$ is short exact in $\mathcal{N}$ whenever $M' \xrightarrow{i} M \xrightarrow{p} M''$ is short exact in $\mathcal{M}$. Small exact categories and exact functors form a category, which we denote by **Ex**.

Examples 1.10.

1. Let R be a ring, and F a (finitely generated) flat R-module. Then the functor $-\otimes_R F\colon \mathbf{Mod}_{\mathrm{f}}(R) \to \mathbf{Ab}$ defined by $M \mapsto M \otimes F$ is exact.

2. If R is commutative, then the tensor product of R-modules is an R-module. Projectives are flat and the tensor product of a pair of projectives is projective, so $-\otimes_R P\colon \mathbf{Proj}_{\mathrm{f}}(R) \to \mathbf{Proj}_{\mathrm{f}}(R)$ is an exact functor, where P is any (f.g.) projective module R.

3. If $\mathcal{N}$ is an exact category and $\mathcal{M} \subset \mathcal{N}$ is a full, additive subcategory closed under extensions in $\mathcal{N}$ with the induced exact structure, then the inclusion $\mathcal{M} \hookrightarrow \mathcal{N}$ is an exact functor.

In the situation of the third part of preceding example, we say that $\mathcal{M}$ is an ***exact subcategory*** of $\mathcal{N}$. In particular each of the inclusions of the module categories above into the abelian category of all R-modules is exact. All of the exact categories we have considered so far are exact subcategories of an ambient abelian category, with the induced exact sequences; the following theorem asserts that this is always the case provided that our exact categories are small. Moreover, the embedding can be chosen functorially. We say that an exact functor $F\colon \mathcal{M} \to \mathcal{N}$ between exact categories ***reflects exactness*** if $FM' \xrightarrow{Fi} FM \xrightarrow{Fp} FM''$ is short exact in $\mathcal{N}$ implies that $M' \xrightarrow{i} M \xrightarrow{p} M''$ was short exact in $\mathcal{M}$.

Gabriel–Quillen embedding theorem 1.11 ([TT90], A.7.1, A.7.16). *Let $\mathcal{N}$ be a small exact category.*

1. *There exists an abelian category $\mathcal{A}$, and a fully faithful exact functor $i\colon \mathcal{N} \to \mathcal{A}$ that reflects exactness. Moreover, $\mathcal{N}$ is closed under extensions in $\mathcal{A}$.*

2. *The category $\mathcal{A}$ may be canonically chosen to be the category of left exact functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ and i chosen to be the Yoneda embedding $i(N) = \mathrm{Hom}_{\mathcal{N}}(-,N)$.*

We call an embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ of an exact category into an abelian category as in the first part of the theorem an ***admissible embedding***. Note that the choice of exact category $\mathcal{A}$ is not unique. For exact categories of modules obtained as in Corollary 1.7, their embedding into a category of all R-modules does the job: this functor is fully faithful, exact, and reflects exactness.

In light of the embedding theorem, we could define an exact category to be a full, additive, extensions-closed subcategory of some ambient abelian category. We mainly stick to the intrinsic point of view, but some times we will use the theorem to pass to a larger abelian category and carry out our arguments there, when it is more convenient to do so. This is the same tactic that one often uses for fiddly proofs in abelian categories: use the Freyd–Mitchell embedding theorem to pass to a category of modules and argue element-wise.

As in abelian categories, split monomorphisms/epimorphisms behave particularly well in exact categories. For objects N' and N'' of an additive category, we call a sequence isomorphic to the kernel-cokernel pair

$$N' \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} N' \oplus N'' \xrightarrow{[\,0\ 1\,]} N''$$

a ***direct sum sequence***, and call the pair itself the ***canonical*** one.

Lemma 1.12. *The class of direct sum sequences forms an exact structure on any additive category $\mathcal{N}$. Moreover, this class is minimal among all exact structures on $\mathcal{N}$.*

Proof. It suffices to show that the canonical direct sum sequences are short exact in any exact structure on $\mathcal{N}$, and that their isomorphism-closure is an exact structure. The first part can be proven immediately using the embedding theorem, but it also has a fairly simple proof directly from the axioms. We present it here to illustrate the approach. If $\mathcal{N}$ is an exact category, then the diagram

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & N'' \arrow[d, tail, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \\ N' \arrow[r, tail, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & N' \oplus N'' \end{tikzcd}$$

is a pushout square. The top arrow is an admissible monomorphism since it is the kernel of $N'' \xrightarrow{1} N''$, so the bottom arrow is admissible monic as well. The cokernel of the bottom arrow must therefore be admissible epic. Up to isomorphism this is the projection onto N'', so direct sum sequences are short exact in any exact structure. Showing that the class of all direct sum sequences forms an exact structure is straightforward: axioms (E1) and (E2) are trivial, and (E3a) and (E3b) hold as the pushout/pullback of a split monomorphism/epimorphism is a split monomorphism/epimorphism in any category. □

Given an exact category $\mathcal{N}$, we call this minimal exact structure the ***direct sum structure***, and use the notation $\mathcal{N}^{\oplus}$ to indicate that this is the structure we have in mind. Otherwise we are considering $\mathcal{N}$ with its natural structure (usually with exact sequences inherited from some category of modules). An exact category with the direct sum exact structure is called ***split exact***, so a split exact category $\mathcal{P}$ is one that satisfies $\mathcal{P}^{\oplus} = \mathcal{P}$. The category $\mathbf{Proj}_{\mathrm{f}}(R)$ is split exact for any ring R, as is $\mathbf{Free}_{\mathrm{f}}(R)$.

We shall see in §1.4 that chain complexes in an exact category are much like those in an abelian category, but not always quite so well behaved. In particular, some of the usual diagram lemmas for long exact sequences fail to hold in a general exact category. The well-known lemmas for short exact sequences do hold in general though. Most useful for us is the 3 by 3 lemma, which we shall often invoke.

3 **by** 3 **Lemma 1.13**. *Consider the following commutative diagram with short exact rows in an exact category:*

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & B' \arrow[r, two heads] \arrow[d] & C' \arrow[d] \\
A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\
A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C''.
\end{tikzcd}$$

1. *If the middle column and either of the outer two columns is short exact, then so is the third.*
2. *If the outer columns are short exact and the middle column composes to zero, then it is short exact as well.*

Proof. We may assume by the embedding theorem that there is an admissible embedding of the exact category into an abelian category. Admissible embeddings of exact categories reflect exactness, so the result follows immediately from the corresponding result for abelian categories. A somewhat more lengthy intrinsic proof from the axioms of an exact category can be found in §3 of [Büh10]. □

1.2 The Grothendieck group

In this section we introduce the Grothendieck group of an exact category. We give some examples and study some its basic properties.

Recall that there is a universal procedure called ***group completion*** that turns commutative monoids into abelian groups. If $\mathcal{N}$ is an additive category[1], the isomorphism classes of objects of $\mathcal{N}$ form a commutative monoid under direct sum so we may apply the group completion

[1]Actually one can form the Grothendieck group of any symmetric monoidal category in this way.

to this monoid to return an abelian group that is a simple invariant of the category. The Grothendieck group extends this procedure in a natural way to exact categories whose short exact sequences may not be split.

Definition 1.14. The ***Grothendieck group*** $K_0\mathcal{N}$ of an exact category is the abelian group having one generator $\langle N\rangle$ for each isomorphism class of objects in $\mathcal{N}$, and a relation $\langle N\rangle = \langle N'\rangle + \langle N''\rangle$ whenever there is a short exact sequence $N' \rightarrowtail N \twoheadrightarrow N''$ in $\mathcal{N}$. We denote the image of an object N of $\mathcal{N}$ in $K_0\mathcal{N}$ by $[N]$.

Remark 1.15. There are issues with foundations here. The above definition is only meaningful if the exact category $\mathcal{N}$ is skeletally small, which we implicitly require of all of our exact categories hereafter. All of the categories that we are actually interested in have objects that are 'finitely generated' in some sense, so they are small enough to avoid any problems. There are some categories that are not skeletally small that would at first seem to be of interest, such as the category of *all* modules over some ring, but it is in fact no great restriction to exclude them because of the ***Eilenberg swindle***.

Eilenberg swindle 1.16. Let R be a ring, and let κ be a sufficiently large[2] cardinal. Denote by $\mathbf{Mod}_\kappa(R)$ the category of all R-modules with cardinality less than κ. This is an exact category with the usual exact sequences, and it is skeletally small. For any R-module M, there is a (split) short exact sequence

$$0 \to M \to \bigoplus_{i=0}^{\infty} M \to \bigoplus_{i=1}^{\infty} M \to 0$$

where the first map is inclusion as the first coordinate, and the second map is the corresponding projection. Since $\bigoplus_{i=0}^{\infty} M \cong \bigoplus_{i=1}^{\infty} M$, we have $[M] = [\bigoplus_{i=0}^{\infty} M] - [\bigoplus_{i=0}^{\infty} M] = 0$ for any M, so $K_0\mathbf{Mod}_\kappa(R) = 0$. Similarly $K_0\mathbf{Proj}_\kappa(R) = 0$, where $\mathbf{Proj}_\kappa(R)$ is the exact category of projective modules with cardinality less than κ. So 'large' exact categories have uninteresting Grothendieck groups[3], which is why it is no restriction to exclude categories that are not skeletally small.

When we talk about the ***Grothendieck group of a ring*** R, we mean the Grothendieck group of the category of f.g. projective R-modules, $K_0(R) := K_0\mathbf{Proj}_{\mathrm{f}}(R)$. We also use the notation $K_0'(R)$ for the Grothendieck group $K_0\mathbf{Mod}_{\mathrm{f}}(R)$ of all f.g. R-modules. In the literature this group is often denoted $G_0(R)$ and is given the corresponding name of G-***theory***.

[2]It is enough that $\kappa > |R|$ and $\kappa \times \kappa = \kappa$.

[3]In fact it is not just their Grothendieck groups that are uninteresting: a standard consequence of the additivity theorem is that *all* of the K-groups of $\mathbf{Mod}_\kappa(R)$ and $\mathbf{Proj}_\kappa(R)$ vanish, not just K_0. This is proved by applying Proposition 3.6 to the short exact sequence of functors $1 \rightarrowtail 1 \oplus \infty \twoheadrightarrow \infty$, where $\infty(M) = \oplus_{i=0}^{\infty} M$

Examples 1.17.

1. Let F be a field. The only f.g. modules over F are the finite dimensional F-vector spaces, so $\mathbf{Mod}_{\mathrm{f}}(F) = \mathbf{Proj}_{\mathrm{f}}(F) = \mathbf{Free}_{\mathrm{f}}(F)$. Up to isomorphism there is one F-vector space for each natural number, so taking the Grothendieck group of this category amounts to adding formal inverses to $\mathbb{N}$. Hence $K_0(F) = K_0'(F) = \mathbb{Z}$. The same argument shows that $K_0\mathbf{Free}_{\mathrm{f}}(R) = \mathbb{Z}$ for any ring R with the ***invariant basis property***: $R^m \cong R^n \Rightarrow m = n$.

2. Let R be a Dedekind domain, and let $\mathrm{Cl}(R)$ be its ideal class group. Then $K_0(R) = \mathbb{Z}\oplus\mathrm{Cl}(R)$ (see, for example, §2 of chapter 1 of [Ros94]). In particular K_0 of any principle ideal domain is $\mathbb{Z}$.

3. $K_0(\mathbb{Z}) = K_0'(\mathbb{Z}) = \mathbb{Z}$. The previous example identifies $K_0(\mathbb{Z})$, as $\mathbb{Z}$ is a principle ideal domain. The second equality holds because the short exact sequences
$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$
force the classes of torsion modules to vanish in $K_0'(\mathbb{Z})$.

4. Let p be a prime, and let $\mathbf{Ab}_p$ and $\mathbf{Ab}_{\mathrm{fin}}$ denote the categories of all finite abelian p-groups and all finite abelian groups respectively, with the usual exact sequences. For a p-group A, Cauchy's theorem guarantees the existence of a short exact sequence
$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{m} A \longrightarrow A/(\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0.$$
Since $\mathbb{Z}/p\mathbb{Z}$ is the only group of order p, we use these sequences to calculate $K_0\mathbf{Ab}_p = \mathbb{Z}$, with generator $[\mathbb{Z}/p\mathbb{Z}]$. Together with the structure theorem for finite abelian groups, this implies that $K_0\mathbf{Ab}_{\mathrm{fin}} = \bigoplus_{p\text{ prime}}\mathbb{Z}$.

It is useful to be able to say when the classes of two objects of $\mathcal{N}$ become equal in the Grothendieck group. It turns out that $[N_1] = [N_2]$ in $K_0\mathcal{N}$ if and only if there are short exact sequences $Q' \rightarrowtail Q_1 \twoheadrightarrow Q''$ and $Q' \rightarrowtail Q_2 \twoheadrightarrow Q''$ of $\mathcal{N}$ such that $N_1 \oplus Q_1 \cong N_2 \oplus Q_2$. The proof is a standard exercise, which we omit. As a corollary we see that two objects P_1 and P_2 in a split exact category $\mathcal{P}$ satisfy $[P_1] = [P_2]$ if and only if they are ***stably isomorphic***: that is, there exists an object Q of $\mathcal{P}$ such that $P_1 \oplus Q \cong P_2 \oplus Q$. If $\mathcal{P}$ is $\mathbf{Proj}_{\mathrm{f}}(R)$ for some ring R then we can even choose Q to be a free module.

If R is a commutative ring, then $K_0(R)$ comes with extra structure: it is itself a commutative ring with product induced from $\langle P\rangle.\langle Q\rangle := \langle P\otimes Q\rangle$. Projective modules are flat, so this descends to a well-defined product on $K_0(R)$, with identity given by the class $[R]$. We'll have more to say about product structures on the K-theory of rings in chapter 4.

The Grothendieck group is a good invariant because it is a functor $K_0\colon \mathbf{Ex} \to \mathbf{Ab}$. If $F\colon \mathcal{M} \to \mathcal{N}$ is an exact functor, then the function $\langle M\rangle \mapsto \langle F(M)\rangle$ specifies a homomorphism

of free abelian groups $F_*\colon \bigoplus_{M\in\mathcal{M}}\mathbb{Z}.\langle M\rangle \to \bigoplus_{N\in\mathcal{N}}\mathbb{Z}.\langle N\rangle$. Being exact means precisely that F_* preserves the relations defining the Grothendieck group, so it induces a homomorphism $K_0F\colon K_0\mathcal{M}\to K_0\mathcal{N}$. The identities $K_0(FG)=(K_0F)(K_0G)$ and $K_0 1_{\mathcal{N}} = 1_{K_0\mathcal{N}}$ are immediate from this definition.
In particular, equivalent categories have isomorphic Grothendieck groups. Every category is equivalent to its skeleton, so we are justified in replacing a not necessarily small category with a small skeleton when defining K_0.

1.3 The higher K-theory of exact categories

Following Atiyah–Hirzebruch's development of topological K-theory, it was realised by Bass and others that the correct algebraic K_1 group of a ring R should be the quotient of the infinite general linear group $\mathrm{GL}(R)$ by the subgroup generated by elementary matrices (equivalently the abelianisation of $\mathrm{GL}(R)$). Similarly, Milnor gave a relatively simple algebraic definition of $K_2(R)$. No unifying principle could be identified to defined all higher $K_n(R)$ coherently though.

Quillen's first definition ([Qui72]) of higher K-groups only works for rings: he defines a space $B\mathrm{GL}(R)^+$ (a modification of the classifying space of $\mathrm{GL}(R)$) and defines $K_n(R)$ to be the homotopy group $\pi_n B\mathrm{GL}(R)^+$. This agrees with Bass and Milnor's K_1 and K_2 groups, and can also be made to agree with K_0: if we set the ***K-theory space*** $K(R)$ to be $K_0(R)\times B\mathrm{GL}(R)^+$, then $K_n(R)=\pi_n K(R)$[4]. This definition of higher K-groups is the most straightforward of the homotopy-theoretic approaches, and it has the advantage of being comparatively easy to compute (in the sense that at least some computations exist). The downside is that it is not broad enough to encompass other contexts in which we might want to define higher K-groups: of categories of all f.g R-modules, of schemes, or of more general exact categories.

Shortly after the first definition, Quillen proposed a new construction ([Qui73]) of higher algebraic K-groups, know as the ***Q–construction***, that is defined for all exact categories—in fact exact categories were introduced in the same article as a source for the K-theory functor. In this article Quillen produces from an exact category $\mathcal{N}$ a new category $Q\mathcal{N}$, in such a way that $\pi_n BQ\mathbf{Proj}_{\mathrm{f}}(R)\cong K_{n-1}(R)$ (here B refers to the classifying space of a category). Quillen therefore defines the K-theory space of the exact category $\mathcal{N}$ to be the loop space $K(\mathcal{N}) := \Omega BQ\mathcal{N}$, and again we have $K_n\mathcal{N}=\pi_n K(\mathcal{N})$. With this definition in hand, along with powerful new theorems in abstract homotopy theory, Quillen quickly proved a host of deep and important theorems. We will revisit some of these in chapter 3. Following Quillen's work, many other authors have extended and modified higher algebraic K-theory, most notably Waldhausen with his ***$S_\bullet$–construction*** [Wal85]. Like the Q–construction, the $S_\bullet$–construction defines the K-theory space as a certain loop space. Unlike the Q–construction,

[4]This construction is not functorial, however.

Waldhausen's construction of the higher K-groups is defined for categories that are not 'algebraic'.

We are interested in how much of algebraic K-theory can be done algebraically. That is, given that an exact category is an essentially algebraic object, and the higher K-groups are certainly algebraic, is there not some way to define $K_n\mathcal{N}$ from $\mathcal{N}$ without the use of homotopy theory? Grayson's answer is 'yes', but before we can understand his construction, we first need to review chain complexes in exact categories.

1.4 Chain complexes

In this section we study chain complexes in an exact category. In particular we examine the category of *acyclic* chain complexes in an exact category and show that it is an exact category in its own right. We emphasise the differences with the abelian case, and introduce the notion of an exact category that ***supports long exact sequences.*** This section and the next one are extended versions of the preliminary sections of [Gra12].

Abelian categories come with a good notion of an exact sequence, of which short exact sequences are a special case. We may recover the 'long' exact sequences of an abelian category from the short exact ones. This recovery suggests the correct definition of long exact sequences in a general exact category $\mathcal{N}$. We begin by considering the category of ***all*** chain complexes in $\mathcal{N}$.

Definition 1.18. By a ***chain complex*** in $\mathcal{N}$ we shall mean a $\mathbb{Z}$-graded collection of objects of $\mathcal{N}$, denoted $N_.$, together with a collection of morphisms $d_i\colon N_i \to N_{i-1}$ such that $d_{i-1}d_i = 0$, which we call the ***differential*** of the complex. We drop the subscripts and just write d for each d_i when there is no possibility of confusion[5].

Definition 1.19. We call a chain complex ***bounded*** if only finitely many of the objects N_i are non-zero. For $m \le n \in \mathbb{Z}$, we say that a chain complex $N_.$ is ***supported on*** $[m,n]$ if N_i is zero for $i < m$ and $i > n$.

A ***chain map*** $\phi\colon C_. \to D_.$ of chain complexes in $\mathcal{N}$ is a $\mathbb{Z}$-graded collection of morphisms $\phi_i\colon C_i \to D_i$ of $\mathcal{N}$ that commutes with the differentials of $C_.$ and $D_.$. The bounded[6] chain complexes and chain maps in $\mathcal{N}$ form a category, which we denote by $C\mathcal{N}$.

Since $\mathcal{N}$ is additive, the category of chain complexes in $\mathcal{N}$ is additive as well. All required properties are verified degree-wise. It is a standard fact of homological algebra that if $\mathcal{A}$ is abelian, then so is $C\mathcal{A}$; we will now prove that if $\mathcal{N}$ is an exact category, then so is $C\mathcal{N}$. We

[5] So the condition on the differential is simply $d^2 = 0$.

[6] The unbounded chain complexes also form a category (even an exact category), but we have no need of unbounded complexes until chapter 4 so we only consider bounded complexes here.

declare a composable pair of chain maps $N'_{\cdot} \xrightarrow{\phi} N_{\cdot} \xrightarrow{\psi} N''_{\cdot}$ to be short exact in $C\mathcal{N}$ if and only if the pair $N'_i \xrightarrow{\phi_i} N_i \xrightarrow{\psi_i} N''_i$ is short exact in $\mathcal{N}$ for every i.

Proposition 1.20. *These sequences define an exact structure on $C\mathcal{N}$.*

Proof. Two proofs are available for this proposition: a proof via the embedding theorem, and an intrinsic proof from the axioms. We present the embedding proof in detail, and then say something about the intrinsic proof at the end. Note first that for an abelian category $\mathcal{A}$, the short exact sequences in the abelian category $C\mathcal{A}$ agree with the short exact sequences defined above. By Theorem 1.11, there exists a full embedding of $\mathcal{N}$ into an abelian category such that the short exact sequences of $\mathcal{N}$ are precisely those inherited from $\mathcal{A}$. Then the full embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ induces a full embedding $C\mathcal{N} \hookrightarrow C\mathcal{A}$, and furthermore the short exact sequences specified for $C\mathcal{N}$ above are exactly those inherited from $C\mathcal{A}$. Therefore $C\mathcal{N} \hookrightarrow C\mathcal{A}$ is an admissible embedding and $C\mathcal{N}$ is an exact category with the given exact structure by Lemma 1.6.
If one wants to give an intrinsic proof, then axioms (E1) and (E2) of an exact category are straightforward to verify since the admissible monomorphisms/epimorphisms in $C\mathcal{N}$ are precisely the chain maps that are an admissible monomorphism/epimorphism in each degree. So as usual the only difficulty is in verifying axioms (E3a) and (E3b), but this is not much harder. The relevant pushout/pullback is constructed degree-wise in the underlying category of $\mathbb{Z}$-graded objects. That the resulting $\mathbb{Z}$-graded object of $\mathcal{N}$ comes equipped with a differential making the induced inclusion/projection maps into a chain map is now an elementary diagram chase. □

Obviously every admissible monomorphism/epimorphism in the exact category $C\mathcal{N}$ is an admissible monomorphism/epimorphism of $\mathcal{N}$ degree-wise. In fact this completely characterises the admissible monomorphisms/epimorphisms in $C\mathcal{N}$.

Proposition 1.21. *Let $f: C_{\cdot} \to D_{\cdot}$ be a morphism of chain complexes in $C\mathcal{N}$ that is an admissible monomorphism/epimorphism degree-wise. Then f is an admissible monomorphism/epimorphism in $C\mathcal{N}$.*

Proof. We prove the assertion when f is a degree-wise admissible monomorphism. The admissible epimorphism case is similar. Let d be the differential of $C_{\cdot}$ and d' be the differential of $D_{\cdot}$. Let $\operatorname{coker}(f)_{\cdot}$ denote the $\mathbb{Z}$-graded object of $\mathcal{N}$ that has $\operatorname{coker}(f)_i = \operatorname{coker}(f_i)$. By the universal property of cokernels, there exists a unique morphism $d''_i\colon \operatorname{coker}(f_i) \to \operatorname{coker}(f_{i-1})$ such that d' and d'' commute with the quotient map $D_{\cdot} \to \operatorname{coker}(f)_{\cdot}$. The composition $d'_{i-1} \circ d'_i$ must be zero as it is universal for the composition of the differential in (C,d), so $\operatorname{coker}(f)$ is a chain complex. Hence $C_{\cdot} \xrightarrow{f} D_{\cdot} \to \operatorname{coker}(f)_{\cdot}$ is a composable pair of morphisms in $C\mathcal{N}$ that is a short exact sequence degree-wise, so it is a short exact sequence in $C\mathcal{N}$ and f is an admissible monomorphism of $C\mathcal{N}$. □

So the category of chain complexes in an exact category is nice and well-behaved. It is an exact category with the expected exact structure, and the admissible monomorphisms and epimorphisms are exactly what we expect them to be. We now turn to the category of chain complexes that are 'long exact sequences', where the picture is a little less neat.

Recall that a chain complex with differential d in an abelian category $\mathcal{A}$ is called ***long exact***, or sometimes ***acyclic*** if $\operatorname{im}(d_i) = \ker(d_{i-1})$ for every i. Alternatively, the long exact sequences in $\mathcal{A}$ are those chain complexes with vanishing homology or, equivalently, those that are quasi-isomorphic to the zero complex. We denote the category of such complexes by $C^{\mathrm{q}}(\mathcal{A})$. Let $A_{\bullet}$ be a long exact sequence in $\mathcal{A}$, and let Z_i denote $\ker(d_i) = \operatorname{im}(d_{i+1})$. Then each sequence $0 \longrightarrow Z_i \longrightarrow A_i \xrightarrow{d_i} Z_{i-1} \longrightarrow 0$ is short exact in $\mathcal{A}$. This motivates the following definition.

Definition 1.22. A chain complex $(N_{\bullet}, d_{\bullet})$ in $\mathcal{N}$ is ***acyclic***[7] if there exist objects Z_i of $\mathcal{N}$ such that the differentials factor through the Z_i via short exact sequences of $\mathcal{N}$ as in the following diagram:

$$\begin{tikzcd}
 & & & Z_{i+1} \arrow[dr, tail] & & & & & \\
\cdots \arrow[rr, "d"] & & N_{i+1} \arrow[ur, two heads] \arrow[rr, "d"] & & N_i \arrow[dr, two heads] \arrow[rr, "d"] & & N_{i-1} \arrow[ur, two heads] \arrow[rr, "d"] & & \cdots \\
 & {} \arrow[ur, tail] & & & & Z_i \arrow[ur, tail] & & &
\end{tikzcd}$$

The acyclic bounded chain complexes form a full subcategory of the category of $C\mathcal{N}$. We denote it by $C^{\mathrm{q}}\mathcal{N}$.

The long exact sequences in an abelian category are the acyclic chain complexes as defined above. Therefore Definition 1.22 extends the definition of the category $C^{\mathrm{q}}(\mathcal{A})$, but the reader should take care not to think of $C^{\mathrm{q}}\mathcal{N}$ as the category of acyclic chain complexes in $\mathcal{N}$ that are quasi-isomorphic to zero, for this does not strictly make sense. The following remark is related to this.

Remark 1.23. A word of warning here. By the Gabriel–Quillen embedding theorem, every exact category $\mathcal{N}$ can be considered as a subcategory of an abelian category $\mathcal{A}$ with the inherited exact sequences. One could easily incorrectly assume that the acyclic sequences defined above are exactly the long exact sequences in $\mathcal{A}$ whose objects are in $\mathcal{N}$. This is not true in general. Let R be a ring and consider the exact subcategories $\mathbf{Proj}_{\mathrm{f}}(R)$ and $\mathbf{Free}_{\mathrm{f}}(R)$ of the abelian category of all R-modules. By definition, long exact sequences of modules factor through the kernels of the differentials of the sequences. For a bounded sequence of projective modules these kernels are again projective, but if the objects are free it does not

[7] To avoid overuse of the word 'exact', we prefer *acyclic* to *long exact sequence* for a chain complex in an exact category.

follow that the kernels are free. Hence in this case there are long exact sequences of free R-modules in the abelian category that are not considered acyclic in the sense of Definition 1.22. We now give an explicit example of such a complex.

Example 1.24. Let R be a ring such that there exists a finitely generated, stably free, non-free projective module P over R (so $P \oplus R^m \cong R^n$ for some m and n). There are short exact sequences of R-modules $0 \to P \xrightarrow{i} R^n \xrightarrow{p} R^m \to 0$ and $0 \to R^m \xrightarrow{j} R^n \xrightarrow{q} P \to 0$, where i, j, p and q are the obvious inclusions and projections. The sequence

$$0 \longrightarrow R^m \xrightarrow{\ j\ } R^n \xrightarrow{\ iq\ } R^n \xrightarrow{\ p\ } R^m \longrightarrow 0$$

is a chain complex in $\mathbf{Free}_{\mathrm{f}}(R)$, and it is exact as a complex of R modules, but it is not acyclic in $\mathbf{Free}_{\mathrm{f}}(R)$.

We make the following definition in light of this discrepancy between acyclic chain complexes in an exact category and the long exact sequences in its ambient abelian category.

Definition 1.25. An admissible embedding of an exact category $\mathcal{N}$ into an abelian category $\mathcal{A}$ ***supports long exact sequences*** if every bounded acyclic chain complex in $\mathcal{A}$ whose objects are in $\mathcal{N}$ has the images of its differentials in $\mathcal{N}$. That is to say that the bounded acyclic complexes in $\mathcal{N}$ are precisely the long exact sequences of $\mathcal{A}$ whose objects N_i are in $\mathcal{N}$. If such an embedding of $\mathcal{N}$ exists then we simply say that $\mathcal{N}$ supports long exact sequences.

Example 1.24 shows that the embedding of $\mathbf{Free}_{\mathrm{f}}(R)$ into the abelian category of *all* R-modules does not necessarily support long exact sequences. On the other hand, the embedding of $\mathbf{Proj}_{\mathrm{f}}(R)$ into the same abelian category does support long exact sequences. This follows from the next lemma.

Lemma 1.26. *Let $\mathcal{N} \hookrightarrow \mathcal{A}$ be an admissible embedding of an exact category into an abelian category. If $\mathcal{N}$ is closed under cokernels or closed under kernels in $\mathcal{A}$, in the sense of Definition 1.5, then the embedding supports long exact sequences.*

Proof. Suppose $\mathcal{N}$ is closed under kernels in $\mathcal{A}$. Let $N_.$ be a chain complex in $\mathcal{N}$ that is a long exact sequence in $\mathcal{A}$, and let d be its differential. Without loss of generality, we may suppose that $N_.$ is supported on $[0, n]$ for some $n > 0$. Let Z_{i+1} be the kernel of $d_i \colon N_i \to N_{i-1}$, which is equal to the the cokernel of $d_{i+1} \colon N_{i+1} \to N_i$. We claim that Z_i is in $\mathcal{N}$ for all i. Since $N_{-1} = 0$, it follows that $Z_1 \cong N_0$, so Z_1 is in $\mathcal{N}$. Assuming that Z_i is in $\mathcal{N}$, the short exact sequence $0 \to Z_{i+1} \to N_i \to Z_i \to 0$ in $\mathcal{A}$ shows that Z_{i+1} is in $\mathcal{N}$ as well, since N_{i+1} and Z_i are. The result then follows inductively. The corresponding proof for closure under cokernels is entirely analogous. □

Remark 1.27. The proof of Lemma 1.26 shows the lemma can be made slightly sharper: if $\mathcal{N}$ is closed under kernels in $\mathcal{A}$, then long exact sequences in $\mathcal{A}$ which are bounded below and have objects in $\mathcal{N}$ are acyclic in $\mathcal{N}$. Similarly, if $\mathcal{N}$ is closed under cokernels in $\mathcal{A}$, then long exact sequences in $\mathcal{A}$ that are bounded above and have objects in $\mathcal{N}$ are acyclic in $\mathcal{N}$. Some kind of boundedness condition is essential though: the category of projective $\mathbb{Z}/4\mathbb{Z}$-modules is closed under kernels in the category of all $\mathbb{Z}/4\mathbb{Z}$-modules, but the unbounded long exact sequence

$$\cdots \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \cdots$$

is not acyclic in $\mathbf{Proj}_{\mathrm{f}}(\mathbb{Z}/4\mathbb{Z})$ since the images of its differential are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, which is not a projective $\mathbb{Z}/4\mathbb{Z}$-module.

Remark 1.28. In some sense exact categories that support long exact sequences are sufficient for most of K-theory. For every exact category $\mathcal{N}$ there is an exact category $\overline{\mathcal{N}}$ called the ***idempotent completion*** of $\mathcal{N}$ such that $\overline{\mathcal{N}}$ supports long exact sequences, and such that $\mathcal{N}$ is an exact subcategory of $\overline{\mathcal{N}}$. Moreover, idempotent completion is functorial in $\mathcal{N}$. For details see Thomason [TT90], section 1.11 and appendix A. The embedding $\mathcal{N} \hookrightarrow \overline{\mathcal{N}}$ induces an isomorphism $K_n\mathcal{N} \cong K_n\overline{\mathcal{N}}$ for $n > 0$ and $K_0\mathcal{N} \hookrightarrow K_0\overline{\mathcal{N}}$. This follows from the ***cofinality*** of the embedding of $\mathcal{N}$ into $\mathcal{N}'$—see §3.3.

Much of the standard machinery of homological algebra holds in exact categories, but there are some notable exceptions. If an exact category $\mathcal{N}$ supports long exact sequences, then all of the usual diagram-chasing arguments in abelian categories can be performed in $\mathcal{N}$ by virtue of its well-behaved embedding. If the exact category does not support long exact sequences, then more care is needed, but most of the usual results hold in $\mathcal{N}$ as well. For the snake and five lemmas to hold it seems that an additional assumption of *weak idempotent completeness* (which implies that $\mathcal{N}$ supports long exact sequences) is necessary. Detailed proofs of various diagram lemmas can be found in Bühler's notes [Büh10]. Our main tool is the 3 by 3 lemma (Lemma 1.13), which holds regardless of whether $\mathcal{N}$ supports long exact sequences or not. We'll now examine the category of bounded acyclic complexes in $\mathcal{N}$, taking care not to assume that $\mathcal{N}$ supports long exact sequences.

Lemma 1.29. *The direct sum of acyclic chain complexes is acyclic.*

Proof. We first show that the direct sum of short exact sequences $A' \rightarrowtail A \twoheadrightarrow A''$ and $B' \rightarrowtail B \twoheadrightarrow B''$ in $\mathcal{N}$ is short exact. The map $A' \oplus B' \rightarrowtail A \oplus B'$ is admissible monic as it is the pushout of the diagram

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d] & A \\ A' \oplus B' & \end{tikzcd}$$

Similarly $A \oplus B' \rightarrowtail A \oplus B$ is admissible monic, so the composition $A' \oplus B' \rightarrowtail A \oplus B$ is admissible monic. Its cokernel is $A'' \oplus B''$ so $A' \oplus B' \rightarrowtail A \oplus B \twoheadrightarrow A'' \oplus B''$ is exact. Now let $C_.$ and $D_.$ be acyclic complexes with differentials factoring via short exact sequences through $W_.$ and $Z_.$ respectively. Then it is easy to check that the differentials of the direct sum $C_. \oplus D_.$ factor through the sequences $W_{i+1} \oplus Z_{i+1} \rightarrowtail C_i \oplus D_i \twoheadrightarrow W_i \oplus Z_i$, which are exact by the argument above. □

Lemma 1.29 shows that $C^{\mathrm{q}}\mathcal{N}$ is closed under direct sums in $C\mathcal{N}$. The zero complex is clearly acyclic, so C^{q} is an additive subcategory of $C\mathcal{N}$. In an exact category that has an embedding supporting long exact sequences, a sequence is acyclic if and only if it is long exact in the abelian category. This is easily seen to imply that the bounded acyclic chain complexes in such an exact category form an exact subcategory of the category of all chain complexes. This is actually true without the assumption that $\mathcal{N}$ supports long exact sequences.

Proposition 1.30. *For any exact category $\mathcal{N}$, the category $C^{\mathrm{q}}\mathcal{N}$ of bounded acyclic chain complexes in $\mathcal{N}$ is an exact category with the short exact sequences inherited from $C\mathcal{N}$.*

Proof. We have seen already that $C^{\mathrm{q}}\mathcal{N}$ is a full additive subcategory of $C\mathcal{N}$, so it suffices by Lemma 1.6 to show that $C^{\mathrm{q}}\mathcal{N}$ is closed under extensions in $C\mathcal{N}$. Our proof will use the embedding theorem 1.11, as a direct proof from the axioms is considerably more involved. Let $\mathcal{N} \hookrightarrow \mathcal{A}$ be an admissible embedding of $\mathcal{N}$ into an abelian category and let $C'_. \rightarrowtail C_. \twoheadrightarrow C''_.$ be a short exact sequence of chain complexes in $\mathcal{N}$. We may assume without loss of generality that $C'_., C_.$, and $C''_.$ are all supported on $[0, n]$ for some $n \geq 0$. Let d', d, and d'' be the differentials of $C'_., C_.$, and $C''_.$ respectively, and assume that $C'_.$ and $C''_.$ are acyclic. After embedding into $\mathcal{A}$ we have an exact sequence of bounded chain complexes in $\mathcal{A}$, with the outer two long exact in $\mathcal{A}$. By a diagram chase (or by appealing to the long exact sequence in homology induced by the short exact sequence of chain complexes) we conclude that $C_.$ is long exact in $\mathcal{A}$ as well. Since all of the complexes vanish in degree -1, the morphisms between degrees 1 and 0 must be epimorphisms. There is therefore a commuting diagram

$$\begin{tikzcd}
\ker(d'_1) \arrow[r, tail] \arrow[d] & C'_1 \arrow[r, two heads, "d'_1"] \arrow[d, tail] & C'_0 \arrow[d, tail] \\
\ker(d_1) \arrow[r, tail] \arrow[d] & C_1 \arrow[r, two heads, "d_1"] \arrow[d, two heads] & C_0 \arrow[d, two heads] \\
\ker(d''_1) \arrow[r, tail] & C''_1 \arrow[r, two heads, "d''_1"] & C''_0
\end{tikzcd}$$

whose arrows in the left column are induced by the universal property of kernels. This diagram satisfies the hypotheses of the 3 by 3 lemma, so the left column must also be short exact. Since the outer complexes are acyclic in $\mathcal{N}$, the kernels of d'_1 and d''_1 are objects of $\mathcal{N}$, so $\ker(d_1)$ must also be an object of $\mathcal{N}$ as $\mathcal{N}$ is closed under extensions in $\mathcal{A}$. Each of the complexes is exact in $\mathcal{A}$, so the image of d_i is equal to the kernel of d_{i-1}. The result now follows by an induction obtained by applying the same argument as above to the diagram

$$\begin{tikzcd}
\ker(d'_i) \arrow[r, tail] \arrow[d] & C'_i \arrow[r, two heads, "d'_i"] \arrow[d, tail] & \ker(d'_{i-1}) \arrow[d, tail] \\
\ker(d_i) \arrow[r, tail] \arrow[d] & C_i \arrow[r, two heads, "d_i"] \arrow[d, two heads] & \ker(d_{i-1}) \arrow[d, two heads] \\
\ker(d''_i) \arrow[r, tail] & C''_i \arrow[r, two heads, "d''_i"] & \ker(d''_{i-1})
\end{tikzcd}$$

to conclude that $\ker(d_i)$ is an object of $\mathcal{N}$ for all i. $\square$

In the proof of Proposition 1.30 we have only used the hypothesis that the chain complexes are bounded *below*. We could therefore also conclude that the category of acyclic chain complexes in $\mathcal{N}$ that are bounded below is also an exact category. Similarly, we could have started our induction from above and so conclude that the same is true for the category of acyclic complexes in $\mathcal{N}$ that are bounded above. As in Remark 1.27, some kind of boundedness assumption is essential: the proposition is not true for unbounded complexes.

Characterising admissible monomorphisms/epimorphisms in the exact category $C^{\mathrm{q}}\mathcal{N}$ is not as straightforward as it was for $C\mathcal{N}$ in Proposition 1.21. If the embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ does not support long exact sequences, then there may exist morphisms in $C^{\mathrm{q}}\mathcal{N}$ that are degree-wise admissible monic/epic in $\mathcal{N}$ but which are not themselves admissible monic/epic.

Example 1.31. Let R be a ring with a stably free, non-free f.g. projective module P, as in Example 1.24. Let i, j, p and q be the morphisms also in that example, and note that $iq + jp = 1$. Consider the following commuting diagram of free R-modules:

$$\begin{tikzcd}
0 \arrow[r] & R^m \arrow[r, "j"] \arrow[d, tail, "1"'] & R^n \arrow[r, "iq"] \arrow[d, tail, "\left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]"'] & R^n \arrow[r, "p"] \arrow[d, tail, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] & R^m \arrow[r, tail] \arrow[d, tail, "1"'] & 0 \\
0 \arrow[r] & R^m \arrow[r, tail, "\left[\begin{smallmatrix}j\\1\end{smallmatrix}\right]"] \arrow[d, two heads] & R^n \oplus R^m \arrow[r, "\left[\begin{smallmatrix}1 & -j\\-p & 1\end{smallmatrix}\right]"] \arrow[d, two heads, "{[-p\ 1]}"'] & R^n \oplus R^m \arrow[r, two heads, "{[p\ 1]}"] \arrow[d, two heads, "{[0\ 1]}"'] & R^m \arrow[r] \arrow[d, two heads] & 0 \\
 & 0 \arrow[r, tail] & R^m \arrow[r, "1"] & R^m \arrow[r, two heads] & 0 &
\end{tikzcd}$$

The third row is obviously acyclic in $\mathbf{Free}_{\mathrm{f}}(R)$, and so is the second row (the middle morphism of the middle row factors as

$$R^n \oplus R^m \xrightarrow{[1\ -j]} R^n \xrightarrow{\left[\begin{smallmatrix}1\\-p\end{smallmatrix}\right]} R^n \oplus R^m,$$

since $pj = 1$), so the morphism between the middle and bottom rows is a morphism of $C^{\mathrm{q}}\mathbf{Free}_{\mathrm{f}}(R)$. Each column of the diagram is a short exact sequence of $\mathbf{Free}_{\mathrm{f}}(R)$, so the morphism between the second and third rows is an admissible epimorphism degree-wise. However it is not an admissible epimorphism of $C^{\mathrm{q}}\mathbf{Free}_{\mathrm{f}}(R)$: its kernel—the top row—is the complex discussed in Example 1.24, which is exact as a sequence of R-modules, but is not acyclic in $\mathbf{Free}_{\mathrm{f}}(R)$.

Certainly having degree-wise maps that are admissible monomorphisms/epimorphisms is a necessary condition for a map in $C^{\mathrm{q}}\mathcal{N}$ to be admissible monic/epic in $C^{\mathrm{q}}\mathcal{N}$, but the above example shows that it is not sufficient. The trouble is that the cokernel/kernel of such a map may be a long exact sequence in $\mathcal{A}$ but not an acyclic complex in $\mathcal{N}$. A stronger assumption is needed. Let $C'_{\bullet} \to C_{\bullet}$ be a morphism in $C^{\mathrm{q}}\mathcal{N}$. Denote the image of the differential d_i of $C_{\bullet}$ by X_i, so that the differential factors as $C_i \twoheadrightarrow X_i \rightarrowtail C_{i-1}$. Similarly, denote the images of the differential of $C'_{\bullet}$ by X'_i. The chain map $C'_{\bullet} \to C_{\bullet}$ induces a morphism of images $X'_i \to X_i$ for every i.

Proposition 1.32. *Suppose the morphism $f\colon C'_{\bullet} \to C_{\bullet}$ in $C^{\mathrm{q}}\mathcal{N}$ is degree-wise an admissible monomorphism/epimorphism of $\mathcal{N}$. Then f is an admissible monomorphism/epimorphism of $C^{\mathrm{q}}\mathcal{N}$ if and only if the induced map $X'_i \to X_i$ is also an admissible monomorphism/epimorphism of $\mathcal{N}$ for every i.*

Proof. The proof is in the same vein as the proof of Proposition 1.30. We will prove the result for f a degree-wise admissible monomorphism (since the dual case has the dual proof). We assume without loss of generality that $C'_{\bullet}$ and $C_{\bullet}$ are supported on $[0, n]$. By Proposition 1.21, f is an admissible monomorphism of $C\mathcal{N}$, so it has a cokernel $C''_{\bullet}$. We shall show that this complex is acyclic, from which it follows that f is an admissible monomorphism. For each i, let X''_i be the cokernel of the induced map $X'_i \rightarrowtail X_i$ (note that $X'_1 = C'_0$, $X_1 = C_0$, and $X_0 = X'_0 = 0$). For each i, we have a commutative diagram

$$\begin{tikzcd}
X'_{i+1} \arrow[r, tail] \arrow[d, tail] & C'_i \arrow[r, two heads] \arrow[d, tail, "f_i"] & X'_i \arrow[d, tail] \\
X_{i+1} \arrow[r, tail] \arrow[d, two heads] & C_i \arrow[r, two heads] \arrow[d, two heads] & X_i \arrow[d, two heads] \\
X''_{i+1} \arrow[r] & C''_i \arrow[r] & X''_i
\end{tikzcd}$$

whose columns and upper two rows are short exact, and whose bottom row consists of the usual induced maps between cokernels. By the 3 by 3 lemma 1.13, this bottom row is also a short exact sequence of $\mathcal{N}$. By the uniqueness of induced maps, we see that the differential $C''_i \to C''_{i-1}$ is equal to the composition $C''_i \twoheadrightarrow X''_i \rightarrowtail C''_{i-1}$. Since its differential factors through short exact sequences of $\mathcal{N}$, we conclude that $C''_{\bullet}$ is acyclic.

For the converse we need to argue by induction as in the proof of Proposition 1.30. We now *define* the objects X''_i to be the images of the differential of the acyclic complex $C''_{\bullet}$. All complexes vanish in degree -1, so we have a commutative diagram

$$\begin{tikzcd}
X_2' \arrow[r, tail] \arrow[d] & C_1' \arrow[r, two heads] \arrow[d, tail, "f_1"] & C_0' \arrow[d, tail, "f_0"] \\
X_2 \arrow[r, tail] \arrow[d] & C_1 \arrow[r, two heads] \arrow[d, two heads] & C_0 \arrow[d, two heads] \\
X_2'' \arrow[r, tail] & C_1'' \arrow[r, two heads] & C_0''
\end{tikzcd}$$

whose leftmost column is proved to be short exact by the 3 by 3 lemma. Therefore $X_2' \to X_2$ is an admissible monomorphism in $\mathcal{N}$ (the corresponding statements for $i = 0, 1$ are obvious). The result now follows by induction using the 3 by 3 lemma again on the diagram:

$$\begin{tikzcd}
X_{i+1}' \arrow[r, tail] \arrow[d] & C_i' \arrow[r, two heads] \arrow[d, tail, "f_i"] & X_i' \arrow[d, tail] \\
X_{i+1} \arrow[r, tail] \arrow[d] & C_i \arrow[r, two heads] \arrow[d, two heads] & X_i \arrow[d, two heads] \\
X_{i+1}'' \arrow[r, tail] & C_i'' \arrow[r, two heads] & X_i''
\end{tikzcd}$$

to conclude that their leftmost arrow column is short exact. □

Just as we remarked following the proof of Proposition 1.30, the above result is not sharp in the sense that we can use its proof to characterise admissible monomorphisms/epimorphisms in the exact categories of acyclic chain complexes which are merely bounded below or bounded above.

Since the categories $C\mathcal{N}$ and $C^{\mathrm{q}}\mathcal{N}$ are themselves exact categories, we can also consider chain complexes of objects of $C\mathcal{N}$ and $C^{\mathrm{q}}\mathcal{N}$. Iterating this procedure, we arrive at the definition of a ***multicomplex*** in $\mathcal{N}$.

Definition 1.33. A ***bounded n-dimensional multicomplex*** in an exact category $\mathcal{N}$ is an object of the exact category $C^n\mathcal{N} := C(C(\cdots))\mathcal{N}$ of chain complexes of chain complexes of ... chain complexes of objects of $\mathcal{N}$. Equivalently, a bounded n-dimensional multicomplex in $\mathcal{N}$ is a $\mathbb{Z}^n$-graded collection of objects of $\mathcal{N}$ equipped with a differential in each of the n directions making the $\mathbb{Z}$-graded collection of objects in that direction into a chain complex in $\mathcal{N}$. We say that the multicomplex is ***acyclic*** if each of these complexes is acyclic in $\mathcal{N}$. Equivalently, an acyclic bounded n-dimensional multicomplex in $\mathcal{N}$ is an object of the subcategory $(C^{\mathrm{q}})^n\mathcal{N}$.

1.5 Binary complexes and Grayson's theorem

In the final part of this chapter we introduce binary (multi)complexes in an exact category, and state Grayson's presentation of the higher algebraic K-groups (Theorem/Definition 1.40).

Definition 1.34. A ***binary chain complex*** in $\mathcal{N}$ is a triple $(N_{\bullet}, d, d')$ such that each of $(N_{\bullet}, d)$ and $(N_{\bullet}, d')$ is a chain complex in $\mathcal{N}$. That is, a binary complex is a pair of chain complexes that happen to share the same underlying $\mathbb{Z}$-graded object. A morphism of binary complexes is a morphism of the underlying graded objects that commutes with each of the differentials. We denote the category of all bounded binary complexes in $\mathcal{N}$ by $B\mathcal{N}$.

Just as with $C\mathcal{N}$ in Proposition 1.20, $B\mathcal{N}$ is made into an exact category by declaring the short exact sequences of $B\mathcal{N}$ to be those composable pairs of morphisms that form a short exact sequence in each degree.

Lemma 1.35. *The following are exact functors with the obvious action on morphisms:*

1. *the diagonal functor* $\Delta\colon C\mathcal{N} \to B\mathcal{N}$ *given by* $(N_{\bullet}, d) \mapsto (N_{\bullet}, d, d)$*;*
2. *the top functor* $\top\colon B\mathcal{N} \to C\mathcal{N}$ *given by* $(N_{\bullet}, d, d') \to (N_{\bullet}, d)$*;*
3. *the bottom functor* $\perp\colon B\mathcal{N} \to C\mathcal{N}$ *given by* $(N_{\bullet}, d, d') \to (N_{\bullet}, d')$.

Furthermore Δ *is split by* $\top$ *and* $\perp$.

Proof. Clear. □

We call a binary complex ***acyclic*** if its image under both $\top$ and $\perp$ is an acyclic chain complex. We denote the category of all bounded acyclic binary chain complexes in $\mathcal{N}$ by $B^{\mathrm{q}}\mathcal{N}$. The functors Δ, $\top$, and $\perp$ restrict to functors between $C^{\mathrm{q}}\mathcal{N}$ and $B^{\mathrm{q}}\mathcal{N}$. In fact one can characterise objects of $B^{\mathrm{q}}\mathcal{N}$ as those binary complexes whose image under both $\top$ and $\perp$ is in $C^{\mathrm{q}}\mathcal{N}$.

Proposition 1.36. *The category* $B^{\mathrm{q}}\mathcal{N}$ *is exact with the short exact sequences inherited from* $B\mathcal{N}$.

Proof. We only need to show that $B^{\mathrm{q}}\mathcal{N}$ is closed under extensions in $B\mathcal{N}$. To do this one simply applies Proposition 1.30 separately to the top and bottom differential. □

Since $B^{\mathrm{q}}\mathcal{N}$ is itself an exact category, we can iterate the definition of binary complexes to define an exact category $(B^{\mathrm{q}})^n\mathcal{N}$ for each $n \geq 1$, just as we did in Definition 1.33. The following definition describes the objects of $(B^{\mathrm{q}})^n\mathcal{N}$ more concretely.

Definition 1.37. A n-***dimensional bounded acyclic binary multicomplex*** in $\mathcal{N}$ is a bounded $\mathbb{Z}^n$-graded collection of objects of $\mathcal{N}$ together with a pair of acyclic differentials d^i and $\tilde{d^i}$ in each direction i with $1 \le i \le n$, such that differentials in different directions commute. An illustration of part of such an object for $n = 2$ is given below.

$$\begin{tikzcd}
M_{1,1} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "\tilde{d^1}"'] \arrow[d, shift right, "\tilde{d^2}"'] \arrow[d, shift left, "d^2"] & M_{1,0} \arrow[d, shift right, "\tilde{d^2}"'] \arrow[d, shift left, "d_2"] \\
M_{0,1} \arrow[r, shift left, "d^1"] \arrow[r, shift right, "\tilde{d^1}"'] & M_{0,0}.
\end{tikzcd}$$

Remark 1.38. Another way to look at these commutativity restraints in Definition 1.37 is that the various subsets of the differentials form (non-binary) multicomplexes: for each $i = 1, \ldots, n$, choose d^i or $\tilde{d^i}$, and consider the object that has the same underlying $\mathbb{Z}^n$-graded object as $N_.$, but now has *one* bounded acyclic differential in each direction i, given by d^i or $\tilde{d^i}$, depending on our choice. For each of the 2^n choices of differentials, the resulting object is a bounded acyclic multicomplex in the sense of Definition 1.33 , i.e., it is an object of $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{N})$; conversely, given a pair of differentials $d^i, \tilde{d^i}$ in each direction, if the 2^n choices all form objects of $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{N})$, then the whole assembly is an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{N})$. This viewpoint will be useful in chapter 4.

The objects of $(B^{\mathrm{q}})^n\mathcal{N}$ are exactly the acyclic binary multicomplexes of dimension n in $\mathcal{N}$, and morphisms between these objects are morphisms of the underlying $\mathbb{Z}^n$-graded objects, that commute with all possible differentials. The differentials d^i and $\tilde{d^i}$ can be seen as the differentials in a binary complex of objects of $(B^{\mathrm{q}})^{n-1}\mathcal{N}$. In this way, starting from $n = 1$, it becomes clear that pairs of differentials of an object of $(B^{\mathrm{q}})^n\mathcal{N}$ that are in different directions must commute. In full, for any $i \neq j$, we have:

$$\begin{aligned}
d^i d^j &= d^j d^i; \\
d^i \tilde{d^j} &= \tilde{d^j} d^i; \\
\tilde{d^i} d^j &= d^j \tilde{d^i}; \\
\tilde{d^i} \tilde{d^j} &= \tilde{d^j} \tilde{d^i}.
\end{aligned}$$

A short exact sequence of acyclic binary multicomplexes is a composable pair of binary multicomplex morphisms that are degree-wise short exact (that is, it is a $\mathbb{Z}^n$-graded collection of short exact sequences commuting with all relevant differentials).

The functors $\top$ and $\perp$ of Lemma 1.35 extend to functors

$$\top^i, \perp^i : (B^{\mathrm{q}})^{n+1}\mathcal{N} \to C^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{N},$$

for $1 \le i \le n+1$ by forgetting the top/bottom differential in the i^{th} direction. Similarly, Δ extends to a functor

$$\Delta^i : C^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{N} \to (B^{\mathrm{q}})^{n+1}\mathcal{N}$$

by duplicating the differential in the i^{th} direction.

Definition 1.39. We say that an object of $(B^{\mathrm{q}})^n\mathcal{N}$ is a ***diagonal*** binary multicomplex if it is in the image of Δ^i for some i. That is, a binary multicomplex is diagonal if the differentials d^i and $\tilde{d}^i$ are identical for some i.

We can now state the main result (Corollary 7.4) of [Gra12]. This is the entirely algebraic description of $K_n\mathcal{N}$ on which this thesis is based.

Theorem / Definition 1.40. *Let $\mathcal{N}$ be an exact category. For all $n \geq 0$, the n^{th} Quillen K-theory group $K_n\mathcal{N}$ has the following presentation.*

1. *One generator for (the isomorphism class of) each object of $(B^{\mathrm{q}})^n\mathcal{N}$.*
2. *The relation $[B'] + [B''] = [B]$ whenever there is a short exact sequence of binary multicomplexes $B' \rightarrowtail B \twoheadrightarrow B''$ in $(B^{\mathrm{q}})^n\mathcal{N}$.*
3. *The relation $[B] = 0$ for any diagonal binary multicomplex B.*

A feature of Grayson's presentation is that it expresses $K_n\mathcal{N}$ as a quotient group of the Grothendieck group of the category $(B^{\mathrm{q}})^n\mathcal{N}$. Denote the subgroup of $K_0(B^{\mathrm{q}})^n\mathcal{N}$ generated by the K_0-classes of all diagonal multicomplexes in $(B^{\mathrm{q}})^n\mathcal{N}$ by $T^n_{\mathcal{N}}$. An equivalent statement of the theorem is that $K_n\mathcal{N}$ is the quotient group $K_0(B^{\mathrm{q}})^n\mathcal{N}/T^n_{\mathcal{N}}$. This viewpoint provides the main tool in chapter 3: we prove theorems about higher K-groups by adapting simple algebraic proofs for K_0 to the context of categories of binary multicomplexes.

The theorem is marked as a 'Theorem/Definition' for the following reason: from here on[8] we take this presentation as the *definition* of $K_n\mathcal{N}$. When we need to make clear that we are talking about higher K-groups constructed using the traditional methods of homotopy theory we will write $K_n^{\mathrm{Q}}\mathcal{N}$ (the Q, of course, is for Quillen).
Grayson's presentation of the higher K-groups did not arrive in a vacuum: his results build on similar work by Nenashev in the '90s. In [Nen98a], Nenashev describes $K_1\mathcal{N}$ as a groups generated by isomorphism classes of 'binary short exact sequences', modulo some relations. The two presentations are similar, but not immediately compatible. We will address this issue in the next chapter.

[8] With the exception of a minor modification in chapter 4.

Chapter 2

Manipulating binary complexes

In this chapter we describe the various ways acyclic binary multicomplexes can be manipulated, and the effects on their K-theory classes. We use these manipulations to examine Grayson's K-groups in simple low-dimensional cases: mainly K_1 and K_2 of split exact categories. Our main tools will be an analogue of the second relation in Nenashev's presentation of $K_1\mathcal{N}$ (Proposition 2.10), and the shifting lemma 2.5, which we introduce in the first section. In the second section we study maps between Grayson's $K_1\mathcal{N}$, the presentation $K_1^{\mathrm{Nen}}\mathcal{N}$ of Nenashev, and Bass' universal determinant group $K_1^{\mathrm{Bass}}\mathcal{N}$. In the case that $\mathcal{P}$ is a split exact category we give explicit isomorphisms between all of these groups. In the third and final section we make some remarks about K_2 of a field in Grayson's presentation, and give a conjectural isomorphism between that group and the group obtained by Matsumoto's theorem. We conjecture further that this isomorphism extends to describe a map between the higher K-theory of a field and its Milnor K-groups.

The material in this chapter is rather elementary, but has not, to our knowledge, appeared elsewhere in the literature in so explicit a form (Grayson makes implicit use of Proposition 2.10 and the shifting lemma in [Gra13]; Nenashev sketches the proof of Proposition 2.18 in [Nen98b]). Our proofs are rather diagram-heavy, but require no machinery beyond the results of the first chapter.

2.1 Nenashev's relation and the shifting lemma

Grayson's presentation of the higher algebraic K-theory groups of an exact category $\mathcal{N}$ is inspired by the presentation of $K_1\mathcal{N}$ given by Alexander Nenashev in [Nen98a]. Nenashev proves that $K_1\mathcal{N}$ is the group generated by binary short exact sequences of $\mathcal{N}$, modulo the diagonal binary short exact sequences and one further relation. The analogue of this further relation for acyclic binary complexes is a useful computational tool for working with Grayson's presentation of the K-groups. We first recall Nenashev's presentation of K_1.

Theorem 2.1 (Nenashev). *$K_1\mathcal{N}$ has the following presentation by generators and relations.*

1. *There is a generator for each binary short exact sequence*

$$0 \rightrightarrows A' \underset{d'}{\overset{d}{\rightrightarrows}} A \underset{d'}{\overset{d}{\rightrightarrows}} A'' \rightrightarrows 0.$$

2. *(i) The class of any diagonal binary short exact sequence vanishes.*

   (ii) For any pair of short exact sequences of short exact sequences with the same objects:

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[d] \\ B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[d] \\ C' \arrow[r] & C \arrow[r] & C'' \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[d] \\ B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[d] \\ C' \arrow[r] & C \arrow[r] & C'', \end{tikzcd}$$

   we have the relation

$$[r_0] - [r_1] + [r_2] = [c_0] - [c_1] + [c_2],$$

   where r_i and c_j are the binary short exact sequences obtained as i^{th} row and the j^{th} column of the double-arrowed diagram formed by overlaying the two diagrams above.

Remark 2.2. In the next section it will be useful to distinguish between the presentations of $K_1\mathcal{N}$ given by Nenashev and Grayson. We introduce the notation $K_1^{\text{Nen}}\mathcal{N}$ to denote the group defined in Proposition 2.1.

Remark 2.3. Nenashev's second relation is usually stated using a diagram of the form

$$\begin{tikzcd} A' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & A'' \arrow[d, shift left] \arrow[d, shift right] \\ B' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & B \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & B'' \arrow[d, shift left] \arrow[d, shift right] \\ C' \arrow[r, shift left] \arrow[r, shift right] & C \arrow[r, shift left] \arrow[r, shift right] & C'' \end{tikzcd}$$

to represent the pair of diagrams above. We have avoided this notation because of the obvious possibility of confusion with acyclic binary multicomplexes of dimension 2, which must obey more stringent commutativity hypotheses.

Before introducing the analogue of Nenashev's relation for acyclic binary complexes, we prove a simple lemma about the classes of 'shifted' acyclic binary complexes in Grayson's presentation. The lemma turns out to be remarkably useful, both in this chapter and in chapter 4.

Definition 2.4. Let $N_.$ be a binary complex with differentials d and $\tilde{d}$. The k^{th} ***shift*** of N, denoted $N[k]$, is the binary complex that has the same collection of objects as N, but 'shifted' k places, i.e., $(N[k])_i = N_{i-k}$, and differentials given by $(-1)^k d$ and $(-1)^k \tilde{d}$.

Shifting Lemma 2.5. *For any bounded acyclic binary complex* $N_.$ *the identity* $[N[k]] = (-1)^k [N]$ *holds in* $K_1\mathcal{N}$ *for any* $k \in \mathbb{Z}$.

Proof. Clearly it is enough to show that $[N[1]] = -[N]$. There is a short exact sequence

$$0 \to N_. \to \operatorname{cone}(N_.) \to N_.[1] \to 0,$$

where $\operatorname{cone}(N_.)$ denotes the mapping cone of the identity map $N_. \xrightarrow{1} N_.$ ($\operatorname{cone}(N_.)$ is a binary complex in the obvious way). So it suffices to show that $\operatorname{cone}(N_.)$ vanishes in $K_1\mathcal{N}$. Let N_n be the left-most non-zero object of $N_.$, and let $\operatorname{trun}(N_.)$ be the (not necessarily acyclic) binary complex formed by truncating $N_.$ to forget N_n: that is, $\operatorname{trun}(N_.)$ has a 0 in place of N_n. Then there is a short exact sequence

$$0 \to \operatorname{cone}(\operatorname{trun}(N_.)) \to \operatorname{cone}(N_.) \to \Delta(N_n \xrightarrow{1} N_n) \to 0,$$

where $\Delta(N_n \xrightarrow{1} N_n)$ is the diagonal binary complex

$$0 \rightrightarrows N_n \underset{1}{\overset{1}{\rightrightarrows}} N_n \rightrightarrows 0,$$

which is supported in degrees $n+1$ and n. Mapping cones of identities are always acyclic, so $\operatorname{cone}(\operatorname{trun}(N_.))$ is acyclic even when $\operatorname{trun}(N_.)$ is not. Since $\Delta(N_n \xrightarrow{1} N_n)$ is diagonal, its class vanishes in $K_1\mathcal{N}$, so the above short exact sequence yields the relation $[\operatorname{cone}(N_.)] = [\operatorname{cone}(\operatorname{trun}(N_.))]$. We iterate this procedure by repeatedly truncating $\operatorname{trun}(N_.)$ to show that $[\operatorname{cone}(N_.)]$ is zero. □

This lemma immediately generalises to binary multicomplexes.

Corollary 2.6. *Let* N *be an* n*-dimensional acyclic binary multicomplex,* $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$*, and define* $N[\underline{k}]$ *to be the binary multicomplex obtained from* N *by shifting by* k_1 *in the first direction, by* k_2 *in the second direction, and so on. Then* $[N[\underline{k}]] = (-1)^{k_1+\dots+k_n}[N]$ *in* $K_n\mathcal{N}$. □

To generalise Nenashev's second relation to the context of binary complexes of arbitrary (bounded) length, we introduce the notion of a ***binary double complex***.

Definition 2.7. A ***binary double complex*** in an exact category $\mathcal{N}$ is a pair of bounded double complexes in $\mathcal{N}$ that have the same objects. By a double complex we mean an object of the category of chain complexes of chain complexes in $\mathcal{N}$. We represent a binary double complex diagrammatically in the same way that we did for the second relation in Proposition

2.1—as a pair of diagrams with the same objects and with offset differentials:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet, \end{tikzcd}$$

Remark 2.8. The term ***double complex*** is usually used to refer to a diagram that anticommutes, whereas our double complexes are commutative. This is not problematic, since the two categories are equivalent via the ***sign trick*** ([Wei94], 1.2.5). By the ***total complex*** of a double complex N we mean the chain complex obtained by applying the sign trick to N and forming the usual total complex from the resulting anticommuting diagram.

The total complex of a double complex comes with two canonical filtrations: by rows and by columns. These filtrations pass also to binary double complexes. In detail, suppose without loss of generality that $N_{\cdot\cdot}$ is supported in degrees $[0,m]\times[0,n]$, so that $N_{i,j}=0$ for $(i,j)\notin[0,m]\times[0,n]$. For $1\le j\le n$, denote by $N_{\cdot\cdot}^{\mathrm{v},j}$ the double complex obtained by truncating $N_{\cdot\cdot}$ above the j^{th} row, so that $N_{\cdot\cdot}^{\mathrm{v},j}$ is supported in degrees $[0,m]\times[0,j]$. Observe that $N_{\cdot\cdot}^{\mathrm{v},0}$ is just the zeroth row $N_{\cdot,0}$ of $N_{\cdot\cdot}$, considered as a (binary) double complex supported on $[0,m]\times\{0\}$. The ***vertical filtration*** of $N_{\cdot\cdot}$ is the sequence of inclusions:

$$N_{\cdot,0}=N_{\cdot\cdot}^{\mathrm{v},0}\hookrightarrow N_{\cdot\cdot}^{\mathrm{v},1}\hookrightarrow\cdots\hookrightarrow N_{\cdot\cdot}^{\mathrm{v},m-1}\hookrightarrow N_{\cdot\cdot}^{\mathrm{v},m}=N_{\cdot\cdot}$$

The successive quotient $N_{\cdot\cdot}^{\mathrm{v},j}/N_{\cdot\cdot}^{\mathrm{v},j-1}$ of the vertical filtration is the binary complex $N_{\cdot,j}$, the j^{th} row of $N_{\cdot\cdot}$, considered as a binary double complex supported on $[0,m]\times\{j\}$. This filtration now induces the vertical filtration of $\mathrm{Tot}(N_{\cdot\cdot})$:

$$N_{\cdot,0}=\mathrm{Tot}(N_{\cdot\cdot}^{\mathrm{v},0})\hookrightarrow\mathrm{Tot}(N_{\cdot\cdot}^{\mathrm{v},1})\hookrightarrow\cdots\hookrightarrow\mathrm{Tot}(N_{\cdot\cdot}^{\mathrm{v},m-1})\hookrightarrow\mathrm{Tot}(N_{\cdot\cdot}^{\mathrm{v},m})=\mathrm{Tot}(N_{\cdot\cdot}),$$

and the successive quotient $\mathrm{Tot}(N_{\cdot\cdot}^{\mathrm{v},j})/\mathrm{Tot}(N_{\cdot\cdot}^{\mathrm{v},j-1})$ is the total complex of the j^{th} row of $N_{\cdot\cdot}$, concentrated in degrees $[0,m]\times\{j\}$, which is easily seen to be $N_{\cdot,j}[j]$, the j^{th} shift of the binary complex $N_{\cdot,j}$. We can play this whole game again in the horizontal direction, truncating by columns horizontally and establishing a ***horizontal filtration*** of $\mathrm{Tot}(N_{\cdot\cdot})$.

Lemma 2.9. *Let $N_{\cdot\cdot}$ be a bounded double complex in the exact category $\mathcal{N}$. If all of the rows or all of the columns of $N_{\cdot\cdot}$ are acyclic in $\mathcal{N}$, then the total complex $\mathrm{Tot}(N_{\cdot\cdot})$ is acyclic in $\mathcal{N}$.*

Proof. Without loss of generality, assume that $N_{\cdot\cdot}$ has acyclic rows in $\mathcal{N}$ (if instead the columns are acyclic, use the horizontal filtration instead of the vertical in what follows). Each of the terms in the vertical filtration of $\mathrm{Tot}(N_{\cdot\cdot})$ is a chain complex in $\mathcal{N}$, we will prove that all of these terms are in fact acyclic. The successive quotients of the vertical filtration are simply the (shifted) rows of $N_{\cdot\cdot}$, which are acyclic in $\mathcal{N}$. Since $\mathrm{Tot}(N_{\cdot\cdot}^{\mathrm{v},0})$ is equal to the zeroth row

$N_{.,0}$, this is also acyclic. We saw in the proof of Proposition 1.6 that $C^{\mathrm{q}}\mathcal{N}$ is closed under extensions in $C\mathcal{N}$, so we can iteratively conclude that each $\mathrm{Tot}(N_{..}^{\mathrm{v},j})$ is acyclic, since all of our complexes are bounded. The last object in the filtration is $\mathrm{Tot}(N_{..})$ itself, so the lemma is proved. □

The objects of the total complex of a double complex are independent of the differentials of the double complex. We can therefore form the total complex of a binary double complex, as this comprises a pair of chain complexes with the same objects, i.e., a binary complex. Given a bounded binary double complex $N_{..}$ whose rows or columns are acyclic binary complexes, the total complex $\mathrm{Tot}(N_{..})$ is a bounded acyclic binary complex.

The vertical and horizontal filtrations of the total complex a double complex extend with no problems to filtrations of the total complex of a binary double complex. Now if $N_{..}$ has acyclic rows, then each of the binary complexes $\mathrm{Tot}(N_{..}^{\mathrm{v},j})$ and $N_{.,j}[j]$ are acyclic, by Lemma 2.9. Short exact sequences of acyclic binary complexes give relations in $K_1\mathcal{N}$, so the vertical filtration yields the equation

$$[\mathrm{Tot}(N_{..})] = \sum_{j=0}^{n} [N_{.,j}[j]]$$

in $K_1\mathcal{N}$, which is equivalent by the shifting lemma 2.5 to

$$[\mathrm{Tot}(N_{..})] = \sum_{j=0}^{n} (-1)^j [N_{.,j}].$$

So the K_1-class of a bounded binary double complex in $\mathcal{N}$ with acyclic rows is equal to the alternating sum of the K_1-classes of its rows.

If the columns of $N_{..}$ are acyclic, this yields the equation

$$[\mathrm{Tot}(N_{..})] = \sum_{i=0}^{m} (-1)^i [N_{i,.}]$$

in $K_1\mathcal{N}$, so the class of the total complex of $N_{..}$ is equal to the alternating sum of the classes of its columns. Taken together, these two equations for $[\mathrm{Tot}(N_{..})]$ establish the following analogue of Nenashev's relation.

Proposition 2.10. *Let $N_{..}$ be an bounded binary double complex in $\mathcal{N}$ that is supported on $[0,m]\times[0,n]$, and whose rows and columns are acyclic. Let $N_{.,j}$ be its j^{th} row, and $N_{i,.}$ its i^{th} column. Then the equation*

$$\sum_{j=0}^{n} (-1)^j [N_{.,j}] = \sum_{i=0}^{m} (-1)^i [N_{i,.}]$$

holds in $K_1\mathcal{N}$. □

Corollary 2.11. *The map $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1\mathcal{N}$ given by considering a binary short exact sequence as an acyclic binary complex concentrated in degrees 0, 1, and 2 is well-defined.*

Of course, Proposition 2.10 does not rely in any significant way on the binary double complex $N_{\cdot\cdot}$ being supported on $[0,m]\times[0,n]$: for binary double complexes supported elsewhere one simply shuffles some indices in the proof. Neither is there anything special about K_1 in the proof, we simply chose it for simplicity of exposition. In fact we have the following more general version of Proposition 2.10, whose proof is almost exactly as above.

Proposition 2.12. *Let $N_{\cdot\cdot}$ be an bounded binary double complex of objects of $(B^{\mathrm{q}})^{n-1}\mathcal{N}$ that is supported on $[0,m]\times[0,n]$, and whose rows and columns are acyclic. Let $N_{\cdot,j}$ be its j^{th} row, and $N_{i,\cdot}$ its i^{th} column, and consider these as objects of $(B^{\mathrm{q}})^{n}\mathcal{N}$. Then the equation*

$$\sum_{j=0}^{n}(-1)^j[N_{\cdot,j}] = \sum_{i=0}^{m}(-1)^i[N_{i,\cdot}]$$

holds in $K_n\mathcal{N}$. □

We conclude this section with a simple application of Proposition 2.12.

Definition 2.13. The ***reflection*** of a binary complex $N_\cdot$, denoted $\tau(N_\cdot)$ is the binary complex with the same objects as $N_\cdot$, whose top differential is the bottom differential of $N_\cdot$ and vice versa.

Lemma 2.14. *Let $n > 1$, and let $N_\cdot$ be an object of $(B^{\mathrm{q}})^{n}\mathcal{N}$. Then $[\tau(N_\cdot)] = -[N_\cdot]$ in $K_n\mathcal{N}$.*

Proof. We regard $N_\cdot$ as a bounded acyclic binary complex of objects N_i of $(B^{\mathrm{q}})^{n-1}\mathcal{N}$. Let d and d' be its top and bottom differentials respectively. The binary double complex of objects of $(B^{\mathrm{q}})^{n-1}\mathcal{N}$ described by the pair of diagrams below has $N_\cdot \oplus \tau(N_\cdot)$ as its middle row, and diagonal complexes as its top and bottom rows.

$$\begin{tikzcd}
N_n \arrow[r, "d"] \arrow[d, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] & N_{n-1} \arrow[r, "d"] \arrow[d, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] & \cdots \\
N_n \oplus N_n \arrow[r, "\left[\begin{smallmatrix}d & 0\\0 & d'\end{smallmatrix}\right]"] \arrow[d, "{[0\ 1]}"'] & N_{n-1} \oplus N_{n-1} \arrow[r, "\left[\begin{smallmatrix}d & 0\\0 & d'\end{smallmatrix}\right]"] \arrow[d, "{[0\ 1]}"'] & \cdots \\
N_n \arrow[r, "d'"] & N_{n-1} \arrow[r, "d'"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
N_n \arrow[r, "d"'] \arrow[d, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] & N_{n-1} \arrow[r, "d"'] \arrow[d, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] & \cdots \\
N_n \oplus N_n \arrow[r, "\left[\begin{smallmatrix}d' & 0\\0 & d\end{smallmatrix}\right]"'] \arrow[d, "{[1\ 0]}"] & N_{n-1} \oplus N_{n-1} \arrow[r, "\left[\begin{smallmatrix}d' & 0\\0 & d\end{smallmatrix}\right]"'] \arrow[d, "{[1\ 0]}"] & \cdots \\
N_n \arrow[r, "d'"'] & N_{n-1} \arrow[r, "d'"'] & \cdots
\end{tikzcd}$$

So by Proposition 2.12, we have that $[N_. \oplus \tau(N_.)]$ is equal to the alternating sum of the classes of the columns, which are:

$$\begin{tikzcd} N_i \arrow[r, shift left, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[r, shift right, "\begin{bmatrix}0\\1\end{bmatrix}"'] & N_i \oplus N_i \arrow[r, shift left, "{[0\ 1]}"] \arrow[r, shift right, "{[1\ 0]}"'] & N_i. \end{tikzcd}$$

On the other hand, the alternating sum of these columns is zero in $K_n\mathcal{N}$, as we readily see by applying Proposition 2.12 to the binary double complex represented by the following pair of diagrams:

$$\begin{tikzcd} N_n \arrow[r, "d"] \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] & N_{n-1} \arrow[r, "d"] \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] & \cdots \\ N_n \oplus N_n \arrow[r, "\begin{bmatrix}d & 0\\0 & d\end{bmatrix}"] \arrow[d, "{[0\ 1]}"'] & N_{n-1} \oplus N_{n-1} \arrow[r, "\begin{bmatrix}d & 0\\0 & d\end{bmatrix}"] \arrow[d, "{[0\ 1]}"'] & \cdots \\ N_n \arrow[r, "d"] & N_{n-1} \arrow[r, "d"] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} N_n \arrow[r, "d"'] \arrow[d, "\begin{bmatrix}0\\1\end{bmatrix}"] & N_{n-1} \arrow[r, "d"'] \arrow[d, "\begin{bmatrix}0\\1\end{bmatrix}"] & \cdots \\ N_n \oplus N_n \arrow[r, "\begin{bmatrix}d & 0\\0 & d\end{bmatrix}"'] \arrow[d, "{[1\ 0]}"] & N_{n-1} \oplus N_{n-1} \arrow[r, "\begin{bmatrix}d & 0\\0 & d\end{bmatrix}"'] \arrow[d, "{[1\ 0]}"] & \cdots \\ N_n \arrow[r, "d"'] & N_{n-1} \arrow[r, "d"'] & \cdots \end{tikzcd}$$

This binary double complex has the same columns as the original binary double complex, but all of its rows are diagonal. Therefore the alternating sum of the rows vanishes in $K_n\mathcal{N}$, hence the alternating sum of the columns vanishes, so $[N_.] + [\tau(N_.)] = [N_. \oplus \tau(N_.)] = 0$. □

2.2 K_1 of a split exact category

Recall that a ***split exact category*** $\mathcal{P}$ is an exact category in which all of the short exact sequences are split. The usual example is to take $\mathcal{P} = \mathbf{Proj}_{\mathrm{f}}(R)$ for some ring R. Bass gave an early definition of the K_1 group of an exact category.

Definition 2.15. Let $\mathcal{N}$ be an exact category, we define the category $\mathrm{Aut}(\mathcal{N})$ as follows. Its objects are pairs (A, α), where A is an object of $\mathcal{N}$ and $\alpha\colon A \to A$ is an automorphism. A morphism $(A,\alpha) \to (B,\beta)$ is a morphism $A \to B$ of $\mathcal{N}$ that commutes with the automorphisms α and β. A sequence $(A,\alpha) \to (B,\beta) \to (C,\gamma)$ of $\mathrm{Aut}(\mathcal{N})$-morphisms is declared to be exact if $A \to B \to C$ is exact in $\mathcal{N}$. This class of sequences makes $\mathrm{Aut}(\mathcal{N})$ into an exact category. We can also think of $\mathrm{Aut}(\mathcal{N})$ as the category of representations on the infinite cyclic group $\mathbb{Z}$ in $\mathcal{N}$.

Definition 2.16. The ***Bass*** K_1 ***group*** of an exact category $\mathcal{N}$ is the quotient of $K_0 \operatorname{Aut}(\mathcal{N})$ by the relation

$$[(A,\alpha_1)] + [(A,\alpha_2)] = [(A,\alpha_1\alpha_2)]$$

for any $A \in \mathcal{N}$, and $\alpha_1, \alpha_2 \in \operatorname{Aut}(A)$. We denote this group by $K_1^{\mathrm{Bass}}\mathcal{N}$.

There is a homomorphism $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1^{\mathrm{Q}}\mathcal{N}$ (where the right hand side is Quillen's K-theory) that is known to be an isomorphism for split exact categories but neither injective or surjective in general [Ger73]. There is a natural candidate for the homomorphism $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1\mathcal{N}$ (or $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1^{\mathrm{Nen}}\mathcal{N}$) when we consider Grayson's (or Nenashev's) presentation of $K_1\mathcal{N}$: the class $[A,\alpha]$ is mapped to the $K_1\mathcal{N}$ class of the ***binary automorphism***

$$0 \rightrightarrows A \overset{\alpha}{\underset{1}{\rightrightarrows}} A \rightrightarrows 0$$

considered as an acyclic binary complex (resp. binary short exact sequence) concentrated in degrees 0 and 1. In the case of a split exact category we shall show that these are isomorphisms, as expected. We begin with a general exact category $\mathcal{N}$. By Corollary 2.11, there is a well-defined homomorphism $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1\mathcal{N}$ given by considering a binary short exact sequence as an acyclic binary complex concentrated in degrees 0, 1, and 2.

Lemma 2.17. *The map* $[(A,\alpha)] \mapsto [\, A \overset{\alpha}{\underset{1}{\rightrightarrows}} A \,]$ *induces a homomorphism* $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1^{\mathrm{Nen}}\mathcal{N}$.

Proof. We first show that

$$[\, A \overset{\alpha}{\underset{1}{\rightrightarrows}} A \,] + [\, A \overset{\beta}{\underset{1}{\rightrightarrows}} A \,] = [\, A \overset{\alpha\beta}{\underset{1}{\rightrightarrows}} A \,]$$

in $K_1^{\mathrm{Nen}}\mathcal{N}$. The application of Nenashev's relation (Proposition 2.1) to the binary double complex represented by the following pair of diagrams:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & A \arrow[r, "\beta"] \arrow[d, "1"] & A \arrow[d, "\alpha"] \\ 0 \arrow[r] & A \arrow[r, "\alpha\beta"] & A \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & A \arrow[r, "1"'] \arrow[d, "1"'] & A \arrow[d, "1"'] \\ 0 \arrow[r] & A \arrow[r, "1"'] & A \end{tikzcd}$$

yields the equation

$$[\, A \overset{\alpha}{\underset{1}{\rightrightarrows}} A \,] - [\, A \overset{1}{\underset{1}{\rightrightarrows}} A \,] = [\, A \overset{\alpha\beta}{\underset{1}{\rightrightarrows}} A \,] - [\, A \overset{\beta}{\underset{1}{\rightrightarrows}} A \,],$$

which is equivalent to the desired relation. We also need to show that a short exact sequence $(A,\alpha) \xrightarrow{f} (B,\beta) \xrightarrow{g} (C,\gamma)$ of $\operatorname{Aut}(\mathcal{N})$ gives a relation in $K_1^{\mathrm{Nen}}\mathcal{N}$. We view this sequence as a binary double complex

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "f"'] \arrow[d, "1"] & B \arrow[r, "g"'] \arrow[d, "1"] & C \arrow[d, "1"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C.
\end{tikzcd}$$

The desired relation now follows from Proposition 2.1, as the rows of this binary double complex are diagonal and therefore vanish in $K_1^{\mathrm{Nen}}\mathcal{N}$. $\square$

So we have the following commuting diagram of homomorphisms between presentations of K_1:

$$\begin{tikzcd}
K_1^{\mathrm{Bass}}\mathcal{N} \arrow[r] \arrow[dr] & K_1\mathcal{N} \\
 & K_1^{\mathrm{Nen}}\mathcal{N} \arrow[u]
\end{tikzcd}$$

We will return to the homomorphisms $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1\mathcal{N}$ and $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1\mathcal{N}$ shortly. First we review a result about $K_1^{\mathrm{Nen}}\mathcal{P}$.

Proposition 2.18. *Let $\mathcal{P}$ be a split exact category. Then the map $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1^{\mathrm{Nen}}\mathcal{P}$ in the diagram above is an isomorphism.*

Proof. We first prove that the binary automorphisms suffice to generate $K_1^{\mathrm{Nen}}\mathcal{P}$. Consider a binary short exact sequence in $\mathcal{P}$:

$$0 \rightrightarrows P' \underset{\tilde{j}}{\overset{j}{\rightrightarrows}} P \underset{\tilde{p}}{\overset{p}{\rightrightarrows}} P'' \rightrightarrows 0.$$

Since $\mathcal{P}$ is a split exact category, there exist splittings s and $\tilde{s}$ of j and $\tilde{j}$. Using these splittings we form the binary double complex represented by the following pair of diagrams:

$$\begin{tikzcd}
P' \arrow[r, "j"] \arrow[d, "1"] & P \arrow[r, "p"] \arrow[d, "{\left[\begin{smallmatrix} s \\ p \end{smallmatrix}\right]}"] & P'' \arrow[d, "1"] \\
P' \arrow[r, "{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}"] & P' \oplus P'' \arrow[r, "{[0\ 1]}"] & P''
\end{tikzcd}
\qquad
\begin{tikzcd}
P' \arrow[r, "\tilde{j}"'] \arrow[d, "1"'] & P \arrow[r, "\tilde{p}"'] \arrow[d, "{\left[\begin{smallmatrix} \tilde{s} \\ \tilde{p} \end{smallmatrix}\right]}"'] & P'' \arrow[d, "1"'] \\
P' \arrow[r, "{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}"'] & P' \oplus P'' \arrow[r, "{[0\ 1]}"'] & P''.
\end{tikzcd}$$

Applying Nenashev's relation to this binary double complex, and noting that the classes of the outer columns and the bottom row give diagonal binary complexes, we obtain the equation

$$[\ P' \overset{j}{\underset{\tilde{j}}{\rightrightarrows}} P \overset{p}{\underset{\tilde{p}}{\rightrightarrows}} P''\ ] = [\ 0 \rightrightarrows P \overset{\left[\begin{smallmatrix} s \\ p \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} \tilde{s} \\ \tilde{p} \end{smallmatrix}\right]}{\rightrightarrows}} P' \oplus P''\ ]$$

in $K_1^{\mathrm{Nen}}\mathcal{N}$. This second class is already a binary isomorphism, but we can easily go further and write it conveniently as a binary automorphism. Let $\tilde{r}\colon P'' \to P$ be a splitting of $\tilde{p}$ that is compatible with the splitting $\tilde{s}$ of $\tilde{j}$: that is, such that $P'' \xrightarrow{\tilde{r}} P \xrightarrow{\tilde{s}} P'$ is a short exact sequence. Then we have

$$[\ 0 \rightrightarrows P \overset{\left[\begin{smallmatrix} s \\ p \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} \tilde{s} \\ \tilde{p} \end{smallmatrix}\right]}{\rightrightarrows}} P' \oplus P''\ ] = [\ 0 \rightrightarrows P \overset{\tilde{j}s+\tilde{r}p}{\underset{1}{\rightrightarrows}} P\ ]$$

in $K_1^{\mathrm{Nen}}\mathcal{N}$, as is attested by Nenashev's relation applied to the binary double complex represented by the following pair of diagrams, whose commutativity follows from the definitions of s, $\tilde{s}$, and $\tilde{r}$:

$$\begin{tikzcd} P \arrow[r, "\left[\begin{smallmatrix} s \\ p \end{smallmatrix}\right]"] \arrow[d, "\tilde{j}s+\tilde{r}p"'] & P' \oplus P'' \arrow[d, "1"'] \\ P \arrow[r, "\left[\begin{smallmatrix} \tilde{s} \\ \tilde{p} \end{smallmatrix}\right]"] & P' \oplus P'' \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\left[\begin{smallmatrix} \tilde{s} \\ \tilde{p} \end{smallmatrix}\right]"'] \arrow[d, "1"] & P' \oplus P'' \arrow[d, "1"] \\ P \arrow[r, "\left[\begin{smallmatrix} \tilde{s} \\ \tilde{p} \end{smallmatrix}\right]"'] & P' \oplus P''. \end{tikzcd}$$

This proves that $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1^{\mathrm{Nen}}\mathcal{P}$ given by $[(P,\alpha)] \mapsto [\ 0 \rightrightarrows P \overset{\tilde{j}s+\tilde{r}p}{\underset{1}{\rightrightarrows}} P\ ]$ is surjective. It also suggests an inverse: for a class

$$x = [\ P' \overset{i}{\underset{j}{\rightrightarrows}} P \overset{p}{\underset{q}{\rightrightarrows}} P''\ ]$$

in $K_1^{\mathrm{Nen}}\mathcal{P}$, define $\phi(x)$ to be the element $[(P, js + rp)]$ in $K_1^{\mathrm{Bass}}\mathcal{P}$, where s and r are defined as above. In [Nen98b] Nenashev claims that this inverse is well-defined. $\square$

The second part of the map $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1\mathcal{N}$ is the map $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1\mathcal{N}$ given by considering a binary short exact sequence as an acyclic binary complex concentrated in degrees 0, 1, and 2. Since Nenashev's relation holds in $K_1\mathcal{N}$ (Proposition 2.10) as well as in $K_1^{\mathrm{Nen}}\mathcal{N}$, and since both presentations set diagonal binary complexes to vanish, this map is a well-defined homomorphism of groups for any exact category $\mathcal{N}$. We believe that this homomorphism is always an isomorphism, but we cannot prove it in general. If $\mathcal{P}$ is a split exact category we can show that the homomorphism is an isomorphism. Together with Proposition 2.18 this proves that $K_1\mathcal{P}$ is isomorphic to $K_1^{\mathrm{Bass}}\mathcal{P}$. Even the proof of $K_1^{\mathrm{Nen}}\mathcal{P} \xrightarrow{\cong} K_1\mathcal{P}$ is not straightforward though. A difficulty that arises in dealing with acyclic binary complexes of lengths greater than two (i.e., longer than binary short exact sequences), is that the images of the two differentials between a pair of objects may not be the same, or even be isomorphic. By way of example, let R be a ring such that there exists a finitely generated, stably free, non-free projective module P over R, as in Example 1.24 of chapter 1. Let i, p, j, and q be as in that example, and consider the acyclic binary complex:

$$0 \rightrightarrows R^m \underset{[1\ 0]}{\overset{j}{\rightrightarrows}} R^n \underset{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]}{\overset{iq}{\rightrightarrows}} R^n \underset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\overset{p}{\rightrightarrows}} R^m \rightrightarrows 0$$

(to make sense of the matrices we consider $R^n \cong R^m \oplus R^{n-m}$). Clearly the image of $\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$ is R^{n-m}, whereas the image of iq is P. These cannot be isomorphic, as P was assumed to be non-free.

If the images of the top and bottom differentials of an acyclic binary complex are isomorphic, then it is relatively easy to deal with—it may be thought of as being spliced together from binary short exact sequences in the same way that an acyclic chain complex is spliced together from short exact sequences. If the images are not equal then more work needs to be done. For this purpose we introduce an intermediate K_1 group.

Definition 2.19. Let $\mathcal{N}$ be an exact category. For an acyclic binary complex $N_.$ in $\mathcal{N}$, let Z_i and $\tilde{Z}_i$ respectively be the images of the i^{th} map in the top and bottom differential of $N_.$.

1. We say $N_.$ has ***isomorphic images*** if $Z_i \cong \tilde{Z}_i$ for every i (abstractly isomorphic is enough—the isomorphisms are not required to commute in any way with the differentials).
2. We define $K_1^{\text{iso}}\mathcal{N}$ to be the abelian group with one generator for each bounded acyclic binary complex in $\mathcal{N}$ *that has isomorphic images*, and with the relators coming from short exact sequences of such binary complexes, and from diagonal binary complexes, in the usual way.

We would hope that $K_1^{\text{iso}}\mathcal{N}$ is always a subgroup of $K_1\mathcal{N}$ (or even isomorphic to it). Its generators are a subset of the generators of $K_1\mathcal{N}$, and all relations in $K_1^{\text{iso}}\mathcal{N}$ also hold in $K_1\mathcal{N}$. It is conceivable though that there are further relations between the images of binary complexes with isomorphic images in $K_1\mathcal{N}$ that do not hold in $K_1^{\text{iso}}\mathcal{N}$. We will show in Lemma 2.23 that this is not possible for a split exact category. In any case, there is a natural candidate for a map $K_1^{\text{Nen}}\mathcal{N} \to K_1^{\text{iso}}\mathcal{N}$: consider a binary short exact sequence as an acyclic binary complex (which has isomorphic images—this is always true for binary short exact sequence) concentrated in degrees 0, 1, and 2. Unfortunately we can only show that this map is well-defined for split exact categories.

Lemma 2.20. *The shifting lemma 2.5 holds in* $K_1^{\text{iso}}\mathcal{N}$ *for any exact category* $\mathcal{N}$.

Proof. The proof goes as in the proof of Lemma 2.5. One only needs to check that all of the objects involved have isomorphic images. But the cone of *any* binary complex has isomorphic images, since the images of the cone of any chain complex are equal to the objects of the original complex. □

Lemma 2.21. *Nenashev's relation (Proposition 2.10) holds in $K_1^{\mathrm{iso}}\mathcal{P}$ for a split exact category $\mathcal{P}$.*

Proof. One only needs to check that all of the binary total complexes formed to prove Proposition 2.10 have isomorphic images if the rows and columns of the original binary double complex have isomorphic images. Clearly it is enough to consider only the vertical filtration: the horizontal filtration is entirely analogous. So we need to show that $\mathrm{Tot}(P_{..}^{\mathrm{v},j})$ has isomorphic images for each j if $P_{..}$ has acyclic rows with isomorphic images in $\mathcal{P}$. We examine the proof of Lemma 2.9. Assume inductively that $\mathrm{Tot}(P_{..}^{\mathrm{v},j})$ has isomorphic images (this is certainly true for $j = 0$). Since $\mathrm{Tot}(P_{..}^{\mathrm{v},j+1})$ is an extension of $\mathrm{Tot}(P_{..}^{\mathrm{v},i})$ and $N_{.,j+1}[j+1]$, the (shifted) $(j+1)^{\mathrm{th}}$ row of $P_{..}$, which has isomorphic images by assumption. So the images of $\mathrm{Tot}(P_{..}^{\mathrm{v},j+1})$ are extensions of those of $\mathrm{Tot}(P_{..}^{\mathrm{v},i})$ and $N_{.,j+1}[j+1]$. But in a split exact category the isomorphism class of an extension is determined entirely by the isomorphism classes of the kernel and quotient, so $\mathrm{Tot}(P_{..}^{\mathrm{v},j+1})$ must also have isomorphic images. □

Since it turns out that the shifting lemma holds in $K_1^{\mathrm{iso}}\mathcal{N}$ for any exact category, we might hope that the same is true of Nenashev's relation. We have neither a proof nor counterexample to this, but our feeling is that it is probably false in general. If it were true, then the following proposition would be true for a general exact category. We note only that part of the proof of the proposition implies the existence of a well-defined epimorphism $K_1^{\mathrm{iso}}\mathcal{N} \to K_1^{\mathrm{Nen}}\mathcal{N}$ for any exact category $\mathcal{N}$.

Proposition 2.22. *For a split exact category $\mathcal{P}$, the natural map $K_1^{\mathrm{Nen}}\mathcal{P} \to K_1^{\mathrm{iso}}\mathcal{P}$ is an isomorphism.*

Proof. We first note that the Lemma 2.21 implies that map is well-defined. We'll now show that it is an epimorphism. Let $P_.$ be a bounded acyclic binary complex that has isomorphic images. Without loss of generality, we may suppose that $P_.$ is supported on $[0, n]$. Let d and $\tilde{d}$ be the top and bottom differential respectively of $P_.$. Then obviously d and $\tilde{d}$ factor over not only non-isomorphic, but equal objects of $\mathcal{P}$: that is d_i factors as $P_i \overset{p_i}{\twoheadrightarrow} Z_i \overset{j_i}{\rightarrowtail} P_{i-1}$, and $\tilde{d}_i$ factors as $P_i \overset{\tilde{p}_i}{\twoheadrightarrow} Z_i \overset{\tilde{j}_i}{\rightarrowtail} P_{i-1}$. We now simplify $P_.$ by considering the binary double complex represented by the following pair of diagrams (here we are using $Z_1 = P_0$):

$$
\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r, "d_3"] \arrow[d, "1"] & P_2 \arrow[r, "p_2"] \arrow[d, "1"] & Z_2 \arrow[r] \arrow[d, "j_2"] & 0 \arrow[d] \\
\cdots \arrow[r] & P_3 \arrow[r, "d_3"] \arrow[d] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "p_1"] & P_0 \arrow[d, "1"] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z_1 \arrow[r, "1"] & P_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r, "\tilde{d}_3"'] \arrow[d, "1"'] & P_2 \arrow[r, "\tilde{p}_2"'] \arrow[d, "1"'] & Z_2 \arrow[r] \arrow[d, "\tilde{j}_2"'] & 0 \arrow[d] \\
\cdots \arrow[r] & P_3 \arrow[r, "\tilde{d}_3"'] \arrow[d] & P_2 \arrow[r, "\tilde{d}_2"'] \arrow[d] & P_1 \arrow[r, "\tilde{d}_1"'] \arrow[d, "\tilde{p}_1"'] & P_0 \arrow[d, "1"'] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z_1 \arrow[r, "1"'] & P_0.
\end{tikzcd}
$$

Applying Nenashev's relation (Lemma 2.21) to this binary double complex[1], we obtain the following equation in $K_1^{\mathrm{iso}}\mathcal{P}$

$$
\begin{aligned}
[P_.] &= [\ \cdots \rightrightarrows P_3 \overset{d_3}{\underset{\tilde{d}_3}{\rightrightarrows}} P_2 \overset{p_2}{\underset{\tilde{p}_2}{\rightrightarrows}} Z_2 \rightrightarrows 0\ ] + [\ Z_2 \overset{j_2}{\underset{\tilde{j}_2}{\rightrightarrows}} P_1 \overset{p_1}{\underset{\tilde{p}_1}{\rightrightarrows}} Z_1\ ] \\
&= [\ Z_2 \overset{j_2}{\underset{\tilde{j}_2}{\rightrightarrows}} P_1 \overset{p_1}{\underset{\tilde{p}_1}{\rightrightarrows}} Z_1\ ] - [\ \cdots \rightrightarrows P_3 \overset{d_3}{\underset{\tilde{d}_3}{\rightrightarrows}} P_2 \overset{p_2}{\underset{\tilde{p}_2}{\rightrightarrows}} Z_2\ ]
\end{aligned}
$$

(the second equality is by the shifting lemma 2.20). Iterating this argument, we obtain

$$
[P_.] = \sum_{i=1}^{n-1} (-1)^{i-1} [\ Z_{i+1} \overset{j_{i+1}}{\underset{\tilde{j}_{i+1}}{\rightrightarrows}} P_i \overset{p_i}{\underset{\tilde{p}_i}{\rightrightarrows}} Z_i\ ]
$$

in $K_1^{\mathrm{iso}}\mathcal{P}$, so the map $K_1^{\mathrm{Nen}}\mathcal{P} \to K_1^{\mathrm{iso}}\mathcal{P}$ is surjective.
We would like to show that the following obvious candidate for an inverse to the map $K_1^{\mathrm{Nen}}\mathcal{P} \to K_1^{\mathrm{iso}}\mathcal{P}$ suggested by the argument above is well-defined:

$$
[P_.] \longmapsto \sum_{i=1}^{n-1} (-1)^{i-1} [\ Z_{i+1} \overset{j_{i+1}}{\underset{\tilde{j}_{i+1}}{\rightrightarrows}} P_i \overset{p_i}{\underset{\tilde{p}_i}{\rightrightarrows}} Z_i\ ] \in K_1^{\mathrm{Nen}}\mathcal{P}.
$$

Clearly a diagonal acyclic binary complex in $\mathcal{P}$ is sent by this map to an alternating sum of diagonal binary short exact sequences, which vanishes in $K_1^{\mathrm{iso}}\mathcal{P}$, so there are no problems there. Suppose now that we have a short exact sequence of acyclic binary complexes $0 \to M_. \xrightarrow{f} N_. \xrightarrow{g} P_. \to 0$, all of which are assumed to have isomorphic images and to be supported

[1] Lemma 2.21 is not strictly necessary here: in this case one can directly compute that the total complexes have isomorphic images, even if $\mathcal{P}$ is not split exact.

on $[0,n]$. Let X_i, Y_i, and Z_i be the images of the differentials in $M_\bullet$, $N_\bullet$, and $P_\bullet$ respectively. Then for each i we have a binary double complex, whose outer vertical arrows are induced by f_i, represented by:

$$\begin{tikzcd}
X_{i+1} \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d, "f_i"] & X_i \arrow[d] \\
Y_{i+1} \arrow[r] \arrow[d] & N_i \arrow[r] \arrow[d, "g_i"] & Y_i \arrow[d] \\
Z_{i+1} \arrow[r] & P_i \arrow[r] & Z_i
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{i+1} \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d, "f_i"'] & X_i \arrow[d] \\
Y_{i+1} \arrow[r] \arrow[d] & N_i \arrow[r] \arrow[d, "g_i"'] & Y_i \arrow[d] \\
Z_{i+1} \arrow[r] & P_i \arrow[r] & Z_i.
\end{tikzcd}$$

The binary short exact sequence corresponding to the middle column of this binary double complex is diagonal, so by Nenashev's relation in $K_1^{\mathrm{Nen}}\mathcal{P}$ we have

$$\begin{aligned}
& [\, X_{i+1} \rightrightarrows M_i \rightrightarrows X_i \,] - [\, Y_{i+1} \rightrightarrows N_i \rightrightarrows Y_i \,] + [\, Z_{i+1} \rightrightarrows P_i \rightrightarrows Z_i \,] \\
= \; & [\, X_{i+1} \rightrightarrows Y_{i+1} \rightrightarrows Z_{i+1} \,] + [\, X_i \rightrightarrows Y_i \rightrightarrows Z_i \,].
\end{aligned}$$

Now the image of $[M_\bullet] - [N_\bullet] + [P_\bullet]$ under the proposed inverse map $K_1^{\mathrm{iso}}\mathcal{P} \to K_1^{\mathrm{Nen}}\mathcal{P}$ is the alternating sum over i of the elements:

$$[\, X_{i+1} \rightrightarrows M_i \rightrightarrows X_i \,] - [\, Y_{i+1} \rightrightarrows N_i \rightrightarrows Y_i \,] + [\, Z_{i+1} \rightrightarrows P_i \rightrightarrows Z_i \,],$$

which is equal to

$$\begin{aligned}
& \textstyle\sum_{i=1}^{n-1} (-1)^{i-1} ([\, X_{i+1} \rightrightarrows Y_{i+1} \rightrightarrows Z_{i+1} \,] + [\, X_i \rightrightarrows Y_i \rightrightarrows Z_i \,]) \\
= \; & (-1)^{n-1} [\, X_n \rightrightarrows Y_n \rightrightarrows Z_n \,] + [\, X_1 \rightrightarrows Y_1 \rightrightarrows Z_1 \,].
\end{aligned}$$

But we claim that both of the classes

$$[\, X_n \rightrightarrows Y_n \rightrightarrows Z_n \,] \quad \text{and} \quad [\, X_1 \rightrightarrows Y_1 \rightrightarrows Z_1 \,]$$

vanish in $K_1^{\mathrm{Nen}}\mathcal{P}$. To see this, note that $X_n = M_n$, $Y_n = N_n$, $Z_n = P_n$, and $X_1 = M_0$, $Y_1 = N_0$, $Z_1 = P_0$. Since the top *and* bottom arrows $X_n \to Y_n$ and $Y_n \to Z_n$ are induced from f and g, the top and bottom maps are in fact $M_n \xrightarrow{f_n} N_n$ and $N_n \xrightarrow{g_n} P_n$, so the relevant class $[\, X_n \rightrightarrows Y_n \rightrightarrows Z_n \,]$ is diagonal and therefore vanishes. Similarly for $[\, X_1 \rightrightarrows Y_1 \rightrightarrows Z_1 \,]$. Therefore the proposed inverse map $K_1^{\mathrm{iso}}\mathcal{P} \to K_1^{\mathrm{Nen}}\mathcal{P}$ sends $[M_\bullet] - [N_\bullet] + [N_\bullet]$ in $K_1^{\mathrm{iso}}\mathcal{P}$ to 0 in $K_1^{\mathrm{Nen}}\mathcal{P}$, so it is well defined. Hence the natural map $K_1^{\mathrm{Nen}}\mathcal{P} \to K_1^{\mathrm{iso}}\mathcal{P}$ is an isomorphism. □

Proposition 2.23. *The natural map $K_1^{\mathrm{iso}}\mathcal{P} \to K_1\mathcal{P}$ is an isomorphism.*

Proof. First we show that this map is surjective. Recall from §1.2 in the first chapter that objects P_1 and P_2 of a split exact category satisfy $[P_1] = [P_2]$ in $K_0\mathcal{P}$ if and only if P_1 and P_2 are stably isomorphic: that is, if and only if there exists an object Q of $\mathcal{P}$ such that

$P_1 \oplus Q \cong P_2 \oplus Q$.

Let $P_.$ be an acyclic binary complex, not necessarily with isomorphic images, and suppose without loss of generality that $P_.$ is supported on $[0,n]$. Let Z_i and $\tilde{Z}_i$ respectively be the images of the top and bottom differentials defined from P_i to P_{i-1}. Then it is not hard to see that $[Z_i] = [\tilde{Z}_i]$ in $K_0\mathcal{P}$ for every i: firstly $Z_1 \cong P_0 \cong \tilde{Z}_1$ so $i = 1$ is immediate; the short exact sequences $0 \to Z_2 \to P_1 \to Z_1 \to 0$ and $0 \to \tilde{Z}_2 \to P_1 \to \tilde{Z}_1 \to 0$ provided by the top and bottom differential of $P_.$ prove the claim for $i = 2$, and the general case follows similarly by induction. So there exist objects Q_i of $\mathcal{P}$, for $i = 1, \dots, n$, such that $Z_i \oplus Q_i \cong \tilde{Z}_i \oplus Q_i$. For each i, we can form the diagonal binary identity of Q_i, and consider it as an acyclic binary complex supported in degrees i and $i-1$. We let $D_.$ denote the direct sum of all of these diagonal complexes. That is:

$$D_. := \bigoplus_{i=1}^{n} \big(\, Q_i \overset{1}{\underset{1}{\rightrightarrows}} Q_i \, \big)[i-1],$$

where $(\, Q_i \overset{1}{\underset{1}{\rightrightarrows}} Q_i \,)$ represents a binary complex supported in degrees 0 and 1, and $[i-1]$ denotes a shift by $i-1$, as before. Since $D_.$ is diagonal, the classes $[P_.]$ and $[P_. \oplus D_.]$ are equal in $K_1\mathcal{P}$. The acyclic binary complex $P_. \oplus D_.$ has isomorphic images by construction: the image of the top differential defined from $P_i \oplus D_i$ to $P_{i-1} \oplus D_{i-1}$ is (isomorphic to) $Z_i \oplus Q_i$, while the image of the bottom differential at the same place is $\tilde{Z}_i \oplus Q_i$, and these are isomorphic. This proves that the class of every generator of $K_1\mathcal{P}$ is equivalent to the class of a binary complex with isomorphic images, therefore the map $K_1^{\mathrm{iso}}\mathcal{P} \to K_1\mathcal{P}$ is surjective.

This proof also supplies a candidate for the inverse map $K_1\mathcal{P} \to K_1^{\mathrm{iso}}\mathcal{P}$: send the class of $P_.$ in $K_1\mathcal{P}$ to the class of $P_. \oplus D_.$ in $K_1^{\mathrm{iso}}\mathcal{P}$. Diagonal binary complexes vanish in both groups, so it is clear that this is an inverse if it is well-defined. There are three things to show: that the choice of the Q_i does not affect the class of $P_. \oplus D_.$; that given a short exact sequence of acyclic binary complexes $0 \to P'_. \to P_. \to P''_. \to 0$, we can choose $D'_.$, $D_.$, and $D''_.$ so that $[P_. \oplus D_.] = [P_. \oplus D'_.] + [P_. \oplus D''_.]$ in $K_1^{\mathrm{iso}}\mathcal{P}$; and that $[P_. \oplus D_.]$ vanishes in $K_1^{\mathrm{iso}}\mathcal{P}$ if $P_.$ is diagonal. The last of these three is obvious, so it is only the first two that we need to address.

For the first claim, suppose for each i we have objects Q_i and R_i such that $Z_i \oplus Q_i \cong \tilde{Z}_i \oplus Q_i$ and $Z_i \oplus R_i \cong \tilde{Z}_i \oplus R_i$. Then we also have $Z_i \oplus (Q_i \oplus R_i) \cong \tilde{Z}_i \oplus (Q_i \oplus R_i)$. Let $D_.$ and $E_.$ be defined by

$$D_. := \bigoplus_{i=1}^{n} \big(\, Q_i \overset{1}{\underset{1}{\rightrightarrows}} Q_i \, \big)[i-1] \quad \text{and} \quad E_. := \bigoplus_{i=1}^{n} \big(\, R_i \overset{1}{\underset{1}{\rightrightarrows}} R_i \, \big)[i-1].$$

From the above we see that $P_. \oplus D_.$, $P_. \oplus E_.$, and $P_. \oplus (D_. \oplus E_.)$ all have isomorphic images. Furthermore we have short exact sequences $0 \to P_. \oplus D_. \to P_. \oplus (D_. \oplus E_.) \to E_. \to 0$ and $0 \to P_. \oplus E_. \to P_. \oplus (D_. \oplus E_.) \to D_. \to 0$, which give relations in $K_1^{\mathrm{iso}}\mathcal{P}$ because $D_.$ and $E_.$ have isomorphic images as they are diagonal. But this also means that the classes of $D_.$ and $E_.$ vanish in $K_1^{\mathrm{iso}}\mathcal{P}$, so $[P_. \oplus D_.] = [P_. \oplus (D_. \oplus E_.)] = [P_. \oplus E_.]$. This proves the first claim.

Now suppose $0 \to P'_. \to P_. \to P''_. \to 0$ is a short exact sequence of acyclic binary complexes.

There exist induced short exact sequences of the images of the top and bottom differentials, $0 \to Z'_i \to Z_i \to Z''_i \to 0$ and $0 \to \tilde{Z}'_i \to \tilde{Z}_i \to \tilde{Z}''_i \to 0$, for every i. Let Q'_i and Q''_i be such that $Z'_i \oplus Q'_i \cong \tilde{Z}'_i \oplus Q'_i$ and $Z''_i \oplus Q''_i \cong \tilde{Z}''_i \oplus Q''_i$. Then setting $Q_i = Q'_i \oplus Q''_i$, we have $Z_i \oplus Q_i \cong \tilde{Z}_i \oplus Q_i$. We therefore have a short exact sequence of the corresponding $D'_.$, $D_.$, and $D''_.$, and hence a short exact sequence $0 \to P'_. \oplus D'_. \to P_. \oplus D_. \to P''_. \oplus D''_. \to 0$, so the result follows. □

Theorem 2.24. *For $\mathcal{P}$ a split exact category, the map $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1\mathcal{P}$, $[(A, \alpha)] \mapsto [\ A \underset{1}{\overset{\alpha}{\rightrightarrows}} A\ ]$ (concentrated in degrees 0 and 1) is an isomorphism.*

Proof. The map factors as $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1^{\mathrm{Nen}}\mathcal{P} \to K_1^{\mathrm{iso}}\mathcal{P} \to K_1\mathcal{P}$. The first arrow in this composition is an isomorphism by Proposition 2.18, the second by Proposition 2.22, and the third by Proposition 2.23. □

We could of course have ignored Nenashev's $K_1^{\mathrm{Nen}}\mathcal{P}$ and the new group $K_1^{\mathrm{iso}}\mathcal{P}$ and studied the map $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1\mathcal{P}$ directly. We can prove that this map is surjective as follows: using part of Lemma 2.23 we see that every acyclic binary complex in $\mathcal{P}$ has the same $K_1\mathcal{P}$-class as one with isomorphic images, so it is enough to show that all of those are in the image of $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1\mathcal{P}$. Let $P_.$ be such an object, with top differential d and bottom differential $\tilde{d}$. Let d_i and $\tilde{d}_i$ factor as $P_i \overset{p_i}{\twoheadrightarrow} Z_i \overset{j_i}{\rightarrowtail} P_{i-1}$ and $P_i \overset{\tilde{p}_i}{\twoheadrightarrow} Z_i \overset{\tilde{j}_i}{\rightarrowtail} P_{i-1}$ as usual, and let s_i and $\tilde{s}_i$ be splittings for j_i and $\tilde{j}_i$. Then we have isomorphisms of acyclic complexes

$$\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r, "d_3"] \arrow[d, "\left[\begin{smallmatrix} s_4 \\ p_3 \end{smallmatrix}\right]"] & P_2 \arrow[r, "d_2"] \arrow[d, "\left[\begin{smallmatrix} s_3 \\ p_2 \end{smallmatrix}\right]"] & P_1 \arrow[r, "d_1"] \arrow[d, "\left[\begin{smallmatrix} s_2 \\ p_1 \end{smallmatrix}\right]"] & P_0 \arrow[d, "1"] \\
\cdots \arrow[r] & Z_4 \oplus Z_3 \arrow[r, "\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]"] & Z_3 \oplus Z_2 \arrow[r, "\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]"] & Z_2 \oplus Z_2 \arrow[r, "{[0\ 1]}"] & Z_1
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r, "\tilde{d}_3"'] \arrow[d, "\left[\begin{smallmatrix} \tilde{s}_4 \\ \tilde{p}_3 \end{smallmatrix}\right]"'] & P_2 \arrow[r, "\tilde{d}_2"'] \arrow[d, "\left[\begin{smallmatrix} \tilde{s}_3 \\ \tilde{p}_2 \end{smallmatrix}\right]"'] & P_1 \arrow[r, "\tilde{d}_1"'] \arrow[d, "\left[\begin{smallmatrix} \tilde{s}_2 \\ \tilde{p}_1 \end{smallmatrix}\right]"'] & P_0 \arrow[d, "1"'] \\
\cdots \arrow[r] & Z_4 \oplus Z_3 \arrow[r, "\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]"'] & Z_3 \oplus Z_2 \arrow[r, "\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]"'] & Z_2 \oplus Z_1 \arrow[r, "{[0\ 1]}"'] & Z_1
\end{tikzcd}$$

(since $Z_1 = P_0$ and $d_1 = p_1$, $\tilde{d}_1 = \tilde{p}_1$). These diagrams represent a binary double complex in $\mathcal{P}$, with diagonal bottom row. Nenashev's relation (Proposition 2.10) now yields

$$[P_.] = \sum_{i=0}^{n} (-1)^i [\ P_i \underset{\left[\begin{smallmatrix} \tilde{s}_{i+1} \\ \tilde{p}_i \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} s_{i+1} \\ p_i \end{smallmatrix}\right]}{\rightrightarrows}} Z_{i+1} \oplus Z_i\ ]$$

in $K_1\mathcal{P}$. We could now easily rewrite each of these classes as a binary automorphism, as was done at the end of the proof of Proposition 2.18. Therefore the map $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1\mathcal{P}$ is surjective. This also shows how to simplify classes in $K_1\mathcal{P}$ to alternating sums of classes of binary automorphisms, which are easier to handle. This simplification procedure also specifies the inverse to $K_1^{\mathrm{Bass}}\mathcal{P} \to K_1\mathcal{P}$, but showing that the map is well-defined is rather involved.

We conclude this section by saying something about the general case, in which the exact category $\mathcal{N}$ is not split exact. It is known that the maps $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1\mathcal{N}$ and $K_1^{\mathrm{Bass}}\mathcal{N} \to K_1^{\mathrm{Nen}}\mathcal{N}$ are not injective or surjective in general: see Section 5 in [Ger73]. We know that the groups $K_1^{\mathrm{Nen}}\mathcal{N}$ and $K_1\mathcal{N}$ are abstractly isomorphic; we would hope that the map $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1\mathcal{N}$ given by considering a binary short exact sequence as a binary complex concentrated in degrees 0, 1, and 2 is an isomorphism, but we have no proof of this. Grayson has shown that this map is surjective, but the proof is complicated [Gra14]. A reasonable approach might be to prove that there are intermediate isomorphisms $K_1^{\mathrm{Nen}}\mathcal{N} \xrightarrow{\cong} K_1^{\mathrm{iso}}\mathcal{N}$ and $K_1^{\mathrm{iso}}\mathcal{N} \xrightarrow{\cong} K_1\mathcal{N}$. However we cannot show this, and we suspect that $K_1^{\mathrm{iso}}\mathcal{N}$ is not a good object unless $\mathcal{N}$ is split exact.

2.3 Some remarks on K_2 and Milnor K-theory

In this short section we make some remarks concerning the relationship between the classical presentation of $K_2(F)$, where F is a field, and the presentation by binary multicomplexes. We do this by exploring a conjectural comparison map to the K-theory of F from its Milnor K-theory. In passing, we also introduce a product structure on the K-theory of rings.

Let F be a field. Following Matsumoto's identification (see [Mil71] for a self-contained proof) of $K_2(F)$ as the group:

$$(F^\times \otimes_{\mathbb{Z}} F^\times)/\langle x \otimes (1-x) \mid x \neq 0, 1\rangle,$$

Milnor gave a conjectural definition of the higher K-groups of a field. These are now called the ***Milnor*** K***-groups***, and are denoted $K_n^{\mathrm{M}}(F)$.

Definition 2.25. The Milnor K-groups of a field are defined as follows: consider $F^\times$ as an abelian group, and let $T(F^\times) = \bigoplus_{n=0}^{\infty}(F^\times)^{\otimes n}$ be its tensor algebra. Then $K_*^{\mathrm{M}}(F)$ is defined to be the quotient of $T(F^\times)$ by the two-sided ideal generated by elements of the form $x \otimes (1-x)$, where $x \neq 0, 1$. The group $K_n^{\mathrm{M}}(F)$ is the n^{th}-graded part of $K_*^{\mathrm{M}}(F)$.

From this definition it is immediate that $K_i^{\mathrm{M}}(F)$ is equal to $K_i(F)$ for $i = 0, 1, 2$ (since K_0 of any field is $\mathbb{Z}$). Unfortunately here the similarity ends, the Milnor K-groups do not in general agree with Quillen's groups for $i > 2$. However the two sets of groups are not unrelated. Furthermore, Milnor's K-groups are of much interest in their own right.

If R is a commutative ring, Loday proved that there is an associate, bilinear, graded-commutative product on Quillen K-groups: $K_p^{\mathrm{Q}}(R) \otimes K_q^{\mathrm{Q}}(R) \to K_{p+q}^{\mathrm{Q}}(R)$. Since $K_1^{\mathrm{Q}}(F) = F^\times$, this product induces a map $(F^\times)^{\otimes n} \to K_n^{\mathrm{Q}}(F)$ for every n, and every field F. The images of elements of $(F^\times)^{\otimes 2}$ under this map satisfy the Steinberg relations, so there is an induced homomorphism $K_n^{\mathrm{M}}(F) \to K_n^{\mathrm{Q}}(F)$ for each n. Unfortunately this comparison homomorphism is neither injective nor surjective in general (see [Wei13], V. Example 1.10.1), although useful things can be said about it in special cases. In the context of Grayson's presentation, Theorem 2.24 shows that the map $F^\times \to K_1(F)$, $x \mapsto [\ F \overset{x}{\underset{1}{\rightrightarrows}} F\ ]$ is an isomorphism (since it is well known that $x \mapsto [(F,x)]$, $F^\times \to K_1^{\mathrm{Bass}}(F)$ is an isomorphism). So it may be possible to give a new construction of the comparison homomorphism $K_n^{\mathrm{M}}(F) \to K_n(F)$ by defining a sensible product on the graded abelian group $K_*(F) := \oplus_{i=0}^{\infty} K_i(F)$. In fact such a product exists for any commutative ring R.

Let $P_{\cdot}$ and $Q_{\cdot}$ be bounded acyclic binary multicomplexes, of dimensions p and q respectively, of objects of $\mathbf{Proj}_{\mathrm{f}}(R)$. We define a new binary complex $P_{\cdot} \otimes Q_{\cdot}$ inductively as follows:

1. If $p = 0$ and $q = 1$, then $P_{\cdot}$ is just a projective module P. The binary complex $P \otimes Q_{\cdot}$ is formed by 'tensoring $Q_{\cdot}$ everywhere' by P.
2. If $p = 1$ and $q = 0$, then we do the same with the roles of $P_{\cdot}$ and $Q_{\cdot}$ reversed.
3. If p and q are both greater than 0, we consider $P_{\cdot}$ and $Q_{\cdot}$ as bounded acyclic binary complexes of objects P_i and Q_j of $(B^{\mathrm{q}})^{p-1}\mathbf{Proj}_{\mathrm{f}}(R)$ and $(B^{\mathrm{q}})^{q-1}\mathbf{Proj}_{\mathrm{f}}(R)$ respectively. Denote the differentials of $P_{\cdot}$ and $Q_{\cdot}$ by d^P, $\tilde{d^P}$, d^Q, and $\tilde{d^Q}$. The product $P_{\cdot} \otimes Q_{\cdot}$ is defined to be the binary multicomplex of dimension $p+q$:

$$
\begin{tikzcd}
& \vdots \arrow[d, shift left] \arrow[d, shift right] & & \vdots \arrow[d, shift left] \arrow[d, shift right] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & P_1 \otimes Q_1 \arrow[r, shift left, "d_1^P\otimes 1"] \arrow[r, shift right, "\tilde{d}_1^P\otimes 1"'] \arrow[d, shift right, "1\otimes\tilde{d}_1^Q"'] \arrow[d, shift left, "1\otimes d_1^Q"] & & P_0 \otimes Q_1 \arrow[d, shift right, "1\otimes\tilde{d}_1^Q"'] \arrow[d, shift left, "1\otimes d_1^Q"] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & P_1 \otimes Q_0 \arrow[r, shift left, "d_1^P\otimes 1"] \arrow[r, shift right, "\tilde{d}_1^P\otimes 1"'] & & P_0 \otimes Q_0
\end{tikzcd}
$$

Lemma 2.26. *The product $(P_{\cdot}, Q_{\cdot}) \mapsto P_{\cdot} \otimes Q_{\cdot}$ induces a well-defined, associative, bilinear multiplication $K_p(R) \times K_q(R) \to K_{p+q}(R)$.*

Proof. Clearly if $P_{\cdot}$ and $Q_{\cdot}$ are bounded, then so is $P_{\cdot} \otimes Q_{\cdot}$. Projective modules are flat, so 'tensoring an acyclic binary complex everywhere' by a projective module preserves acyclicity. Since the product of binary multicomplexes is built inductively from such products, it follows that if $P_{\cdot}$ and $Q_{\cdot}$ are acyclic, then so is $P_{\cdot} \otimes Q_{\cdot}$. For the same reason, if $0 \to P'_{\cdot} \to P_{\cdot} \to P''_{\cdot} \to 0$ is a short exact sequence in $(B^{\mathrm{q}})^p\mathbf{Proj}_{\mathrm{f}}(R)$, and $Q_{\cdot}$ is in $(B^{\mathrm{q}})^q\mathbf{Proj}_{\mathrm{f}}(R)$, then there is an induced short exact sequence $0 \to P'_{\cdot} \otimes Q_{\cdot} \to P_{\cdot} \otimes Q_{\cdot} \to P''_{\cdot} \otimes Q_{\cdot} \to 0$ in $(B^{\mathrm{q}})^{p+q}\mathbf{Proj}_{\mathrm{f}}(R)$, and similarly when the roles of $P_{\cdot}$ and $Q_{\cdot}$ are reversed. If either $P_{\cdot}$ or $Q_{\cdot}$ is diagonal, it is immediately clear from the diagram above that $P_{\cdot} \otimes Q_{\cdot}$ will be diagonal as well. Therefore the product induces a well-defined bilinear multiplication $K_p(R) \times K_q(R) \to K_{p+q}(R)$. That the multiplication is associative follows directly from the definition of the product $\otimes$, as it is clear that $P_\cdot \otimes (Q_\cdot \otimes R_\cdot) \cong (P_\cdot \otimes Q_\cdot) \otimes R_\cdot$. □

Corollary 2.27. *$K_*(R) := \bigoplus_{i=0}^{\infty} K_i(R)$ is a graded ring. In particular, each $K_i(R)$ is a $K_0(R)$-module.*

Remark 2.28. We suspect that this multiplication anti-commutes. Our reasoning is as follows: if $P_\cdot \otimes Q_\cdot$ is given by the diagram above, then $Q_\cdot \otimes P_\cdot$ is the diagram obtained by 'reflecting' it along the diagonal. In general one can reflect n dimensional binary multicomplexes along any one of their 'diagonals', and such reflections form exact endofunctors that square to the identity. These functors are are defined for the categories of binary multicomplexes in any exact category $\mathcal{N}$, and moreover commute with the functors on multicomplexes induced from exact functors $\mathcal{M} \to \mathcal{N}$. So the induced reflection maps should induce a multiplication by ± 1 on K-groups, and since the other symmetries such as shifting 2.5 and reflection 2.14 induce a -1, we suspect this one does too.

Remark 2.29. There is nothing particularly special about the category $\mathbf{Proj}_{\mathrm{f}}(R)$: the construction and lemma above hold in any exact category that has a good notion of tensor products (*cf.* Definition 4.27).

From the universal property of the tensor algebra, the map $F^\times \to K_1(F)$, $x \mapsto [\ F \overset{x}{\underset{1}{\rightrightarrows}} F\ ]$ induces a homomorphism of graded rings $T(F^\times) \to K_*(F)$. For x, $y \in F^\times$, let $\ell(x, y)$ denote the image of $x \otimes y$ under this map, that is $\ell(x, y)$ is the $K_2(F)$-class of the following acyclic binary multicomplex in $\mathbf{Proj}_{\mathrm{f}}(F)$:

$$\begin{array}{ccc} F & \overset{x}{\underset{1}{\rightrightarrows}} & F \\ {\scriptstyle 1}\Big\downarrow\Big\downarrow{\scriptstyle y} & & {\scriptstyle 1}\Big\downarrow\Big\downarrow{\scriptstyle y} \\ F & \overset{x}{\underset{1}{\rightrightarrows}} & F. \end{array}$$

We conjecture that $\ell(x, 1-x)$ vanishes in $K_2(F)$ for every $x \neq 0$, 1. If this is true, then the homomorphism $T(F^\times) \to K_*(F)$ induces a homomorphism of graded rings $K_*^{\mathrm{M}}(F) \to K_*(F)$, and furthermore the n^{th} graded piece $K_n^{\mathrm{M}}(F) \to K_n(F)$ is an isomorphism for $n = 0$, 1, 2. We would hope that this comparison homomorphism between Milnor and Quillen K-theories agrees with the one given by Loday.

Chapter 3

Fundamental theorems of K-theory via binary complexes

In this chapter we use Grayson's presentation to prove some of the fundamental theorems of algebraic K-theory. Specifically, we give new, entirely algebraic proofs of the additivity, resolution, and cofinality theorems. Each of the theorems above establishes isomorphisms between the K-groups of exact categories. These theorems have well-known algebraic folk proofs for the Grothendieck group K_0, so the general schema for our proofs is as follows. First we verify that the hypotheses on our exact categories of interest also hold for their associated categories of acyclic binary multicomplexes. Then we apply the algebraic proof for K_0 to obtain an isomorphism between their Grothendieck groups. Finally we verify that the required isomorphism still holds when we pass to the quotients defining the higher algebraic K-groups.

The remaining theorems regarded as fundamental in the algebraic K-theory of exact categories are the dévissage and localisation theorems. These theorems concern abelian categories, say $\mathcal{A}$ and $\mathcal{B}$. While the associated categories $(B^{\mathrm{q}})^n\mathcal{A}$ and $(B^{\mathrm{q}})^n\mathcal{B}$ are still exact, they will no longer be abelian, so a strategy more sophisticated than the approach of this chapter would be necessary to prove these theorems in the context of Grayson's definition of the higher algebraic K-groups.

This chapter has been published in a slightly more compact form as the paper [Har15]. I thank Dan Grayson for providing an essential idea behind the proof of the resolution theorem, as well as for many useful comments on a late draft of this material.

It is helpful to recall that we have defined $T^n_{\mathcal{N}}$ to be the subgroup of $K_0(B^{\mathrm{q}})^n\mathcal{N}$ generated by the classes of diagonal binary multicomplexes, so that $K_n\mathcal{N} = K_0(B^{\mathrm{q}})^n\mathcal{N}/T^n_{\mathcal{N}}$.

3.1 The additivity theorem

In this section we prove the additivity theorem for the K-theory for exact categories using the approach outlined above. The content of this theorem is that the K-theory of the category of short exact sequences in a given exact category is a product of two copies of the K-theory of that category (we'll actually prove a small generalisation of this). Quillen's original proof in [Qui73] involves homotopy theory in the form of his famous Theorem A. Many other proofs have been given (e.g., [McC93], [Gra11]), but all have some (possibly simplicial) homotopical input. Waldhausen's version of the additivity theorem for the $S_\bullet$-construction in [Wal85] revealed its true significance; it is now commonly regarded as *the* fundamental theorem of algebraic K-theory. Certainly this form of the theorem made possible the simpler proofs of the resolution, dévissage, localisation, and cofinality theorems given by Staffeldt [Sta89]. The proof we present in this section does not work in such wide generality, but it has the virtue of being entirely algebraic and, once the reader is happy with binary multicomplexes, almost completely elementary.

Let $\mathcal{A}$ and $\mathcal{C}$ be exact subcategories of an exact category $\mathcal{B}$.

Definition 3.1. The ***extension category*** $\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ is the category whose objects are short exact sequences $0 \to A \to B \to C \to 0$ of $\mathcal{B}$, with A in $\mathcal{A}$ and C in $\mathcal{C}$, and whose morphisms are commuting rectangles. We may view $\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ as a full subcategory of the exact category $C^{\mathrm{q}}_{[0,2]}\mathcal{B}$ of chain complexes in $\mathcal{B}$ that are supported on $[0,2]$[1]. It is clearly closed under extensions in $C^{\mathrm{q}}_{[0,2]}\mathcal{B}$, so $\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ is an exact category with the obvious short exact sequences.

Additivity Theorem 3.2. *$K_n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \cong K_n\mathcal{A} \times K_n\mathcal{C}$, for every $n \geq 0$.*

Proof for K_0. The exact functors

$$\begin{array}{rrcl} \phi\colon & \mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) & \to & \mathcal{A}\times\mathcal{C} \\ & (0\to A\to B\to C\to 0) & \mapsto & (A,C) \end{array}$$

and

$$\begin{array}{rrcl} \psi\colon & \mathcal{A}\times\mathcal{C} & \to & \mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \\ & (A,C) & \mapsto & (0\to A\to A\oplus C\to C\to 0) \end{array}$$

induce maps of Grothendieck groups $\phi_*\colon K_0\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \to K_0\mathcal{A}\times K_0\mathcal{C}$ and $\psi_*\colon K_0\mathcal{A}\times K_0\mathcal{C} \to K_0\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$. Since $\phi\psi = 1_{\mathcal{A}\times\mathcal{C}}$, so that ϕ_* is a left inverse of ψ_*, it suffices to show that $\psi_*\phi_* = 1_{K_0\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})}$. We have

$$\psi\phi((0\to A\to B\to C\to 0)) = (0\to A\to A\oplus B\to C\to 0),$$

[1] That this is an exact category is proved in exactly the same way as for $C^{\mathrm{q}}\mathcal{B}$—see Propositions 1.20 and 1.30.

so we only need to show the equality of the classes $[(0 \to A \to B \to C \to 0)]$ and $[(0 \to A \to A \oplus B \to C \to 0)]$ in $K_0\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$. The proof of this is provided by the following pair of short exact sequences of $\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$, which show that both classes are equal to $[(0 \to A \xrightarrow{1} A \to 0 \to 0)] + [(0 \to 0 \to C \xrightarrow{1} C \to 0)]$:

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, equal] \\
0 \arrow[r, tail] & C \arrow[r, equal] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, two heads] & A \oplus C \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, equal] \\
0 \arrow[r, tail] & C \arrow[r, equal] & C.
\end{tikzcd}
$$

□

To pass to higher K-theory groups, we need to consider the objects of $B^{q}\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$. A binary complex of objects of $\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ can represented as in the diagram below, reading horizontally.

$$
\begin{tikzcd}
\cdots \arrow[r, shift left] \arrow[r, shift right] & A_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & A_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & A_3 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & \cdots \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & B_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & B_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & B_3 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & \cdots \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & C_1 \arrow[r, shift left] \arrow[r, shift right] & C_2 \arrow[r, shift left] \arrow[r, shift right] & C_3 \arrow[r, shift left] \arrow[r, shift right] & \cdots
\end{tikzcd}
$$

But one may also read this diagram vertically as a short exact sequence of binary multicomplexes. The top row has objects in $\mathcal{A}$, the middle row in $\mathcal{B}$ and the bottom row in $\mathcal{C}$, so the diagram represents an object of $\mathcal{E}(B^{q}\mathcal{A}, B^{q}\mathcal{B}, B^{q}\mathcal{C})$ (it being clear that $B^{q}\mathcal{A}$ and $B^{q}\mathcal{C}$ are exact subcategories of $B^{q}\mathcal{B}$). This observation holds for binary multicomplexes of all dimensions and proves the following lemma. Let $\mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C})$ denote the extension category $\mathcal{E}((B^{q})^n\mathcal{A}, (B^{q})^n\mathcal{B}, (B^{q})^n\mathcal{C})$.

Lemma 3.3. *For each $n > 0$ there is an equivalence of exact categories*

$$(B^{q})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \cong \mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C}).$$

Together with the proof of the theorem for the Grothendieck group K_0, this simple observation is enough to prove the additivity theorem for higher K-groups.

Proof of Theorem 3.2. As in the proof for K_0 we define exact functors $\phi\colon \mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C}) \to (B^{q})^n\mathcal{A} \times (B^{q})^n\mathcal{C}$ and $\psi\colon (B^{q})^n\mathcal{A} \times (B^{q})^n\mathcal{C} \to \mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C})$, so that the induced map

$$\phi_*\colon K_0\mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C}) \to K_0(B^{q})^n\mathcal{A} \times K_0(B^{q})^n\mathcal{C}$$

is an isomorphism, with inverse ψ_*. By the preceding lemma we may identify the categories $\mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C})$ and $(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$, so we have an isomorphism

$$K_0(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \cong K_0(B^{\mathrm{q}})^n\mathcal{A} \times K_0(B^{\mathrm{q}})^n\mathcal{C}.$$

One sees by inspection that a binary multicomplex in $(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ is diagonal in direction i if and only if its constituent binary multicomplexes in $(B^{\mathrm{q}})^n\mathcal{A}, (B^{\mathrm{q}})^n\mathcal{B}$ and $(B^{\mathrm{q}})^n\mathcal{C}$ are also diagonal in direction i, so ϕ sends diagonal binary multicomplexes to diagonal binary multicomplexes. Similarly, if $A_.$ in $(B^{\mathrm{q}})^n\mathcal{A}$ and $C_.$ in $(B^{\mathrm{q}})^n\mathcal{C}$ are diagonal, then so are $(0 \to A_. \to A_. \to 0 \to 0)$ and $(0 \to 0 \to C_. \to C_. \to 0)$, so ψ also preserves diagonal binary multicomplexes. Therefore the isomorphism $K_0(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \cong K_0(B^{\mathrm{q}})^n\mathcal{A} \times K_0(B^{\mathrm{q}})^n\mathcal{C}$ restricts to an isomorphism $T^n_{\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})} \cong T^n_{\mathcal{A}} \times T^n_{\mathcal{B}}$. Passing to the quotient groups now yields the result. □

The additivity theorem is named for its application to exact sequences of functors.

Definition 3.4. Let F', F, and $F''\colon \mathcal{M} \to \mathcal{N}$ be exact functors between the exact categories $\mathcal{M}$ and $\mathcal{N}$, and let $F' \xrightarrow{\mu} F$ and $F \xrightarrow{\nu} F''$ be natural transformations. We say the sequence $F' \xrightarrow{\mu} F \xrightarrow{\nu} F''$ is ***short exact*** if the sequence

$$F'(M) \xrightarrow{\mu_M} F(M) \xrightarrow{\nu_M} F''(M)$$

is a short exact sequence of $\mathcal{N}$ for every object M of $\mathcal{M}$.

Example 3.5. For any exact category $\mathcal{N}$, we define exact functors s, t and $q\colon \mathcal{E}(\mathcal{N},\mathcal{N},\mathcal{N}) \to \mathcal{N}$ that take a short exact sequence to its sub, total, and quotient object respectively:

$$\begin{array}{rlcl} s\colon & (0 \to N' \to N \to N \to N'' \to 0) & \mapsto & N' \\ t\colon & (0 \to N' \to N \to N \to N'' \to 0) & \mapsto & N \\ q\colon & (0 \to N' \to N \to N \to N'' \to 0) & \mapsto & N''. \end{array}$$

These fit into an exact sequence $s \rightarrowtail t \twoheadrightarrow q$ with the obvious natural transformations between them.

Proposition 3.6. *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors from $\mathcal{M}$ to $\mathcal{N}$. Then the induced homomorphisms $K_n\mathcal{M} \to K_n\mathcal{N}$ satisfy $F'_* + F''_* = F_*$.*

Proof. We first consider the special case of the short exact sequence of functors $s \rightarrowtail t \twoheadrightarrow q$ introduced in the previous example. Note that the functor $(s,q)\colon (0 \to N' \to N \to N'' \to 0) \mapsto (N', N'')$ is exactly the functor ϕ that we saw induces the isomorphism $K_n\mathcal{E}(\mathcal{N},\mathcal{N},\mathcal{N}) \to K_n\mathcal{N} \times K_n\mathcal{N}$ in the proof of the additivity theorem. Therefore $(s,q)_*$ has ψ_* as its inverse, where ψ is defined by $(N', N'') \mapsto (0 \to N' \to N' \oplus N'' \to N'' \to 0)$. If $[N']$ and $[N'']$ are now the K_n-classes of acyclic binary multicomplexes, we see by inspection that

$$t_*\psi_*([N'],[N'']) = [N' \oplus N''] = [N'] + [N''] = (s_*\psi_* + q_*\psi_*)([M'],[M'']),$$

so $t_*\psi_* = (s_* + q_*)\psi_*$ on $K_n\mathcal{N}$, as they are equal on generators. But ψ_* is an isomorphism, so $t_* = (s_* + q_*)$.
Now assume the general situation. We define a functor

$$\begin{array}{rccl} G\colon & \mathcal{M} & \to & \mathcal{E}(\mathcal{N},\mathcal{N},\mathcal{N}) \\ & M & \mapsto & (F'(M) \rightarrowtail F(M) \twoheadrightarrow F''(M)) \end{array}$$

with the obvious action on morphisms. Then $sG = F'$, $tG = F$, and $qG = F''$, so we have $F'_* = F''_* = s_*G_* + q_*G_* = t_*G_* = F_*$. $\square$

3.2 The resolution theorem

The resolution theorem relates the K-theory of an exact category to that of a larger exact category whose objects all have a finite resolution by objects of the first category. Its most well-known application is that the K-theory of a regular ring is isomorphic to its G-theory (the K-theory of the exact category of *all* finitely generated R-modules). As in the proof of the additivity theorem, we adapt a simple proof for K_0 to work for all K_n. The main difficulty in this proof is verifying that the hypotheses of the theorem pass to exact categories of acyclic binary multicomplexes.

Resolution Theorem 3.7 ([Qui73], §4 Corollary 2). *Let $\mathcal{M}$ be an exact category and let $\mathcal{P}$ be a full, additive subcategory that is closed under extensions in $\mathcal{M}$. Suppose also that:*

1. *For any exact sequence $0 \to P' \to P \to P'' \to 0$ in $\mathcal{M}$, if P and P'' are in $\mathcal{P}$, then so is P'. That is, $\mathcal{P}$ is closed under kernels in $\mathcal{M}$ (see Definition 1.5).*
2. *Every object of $\mathcal{M}$ has a finite resolution by objects of $\mathcal{P}$.*

Then the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}$ induces an isomorphism $K_n(\mathcal{P}) \cong K_n(\mathcal{M})$ for all $n \geq 0$.

Quillen proves the resolution theorem as a purely formal consequence of the following result, which is Theorem 3 of [Qui73]. It is this theorem that we shall re-prove with binary multicomplexes.

Theorem 3.8. *Let $\mathcal{P}$ be a full, additive subcategory of an exact category $\mathcal{M}$ that is closed under extensions and satisfies:*

1. *For any exact sequence $0 \to P' \to P \to M'' \to 0$ in $\mathcal{M}$, if P is in $\mathcal{P}$, then so is P'*[2].
2. *For any M in $\mathcal{M}$ there exists a P in $\mathcal{P}$ and an admissible epimorphism $P \twoheadrightarrow M$.*

Then the inclusion functor $\iota\colon \mathcal{P} \hookrightarrow \mathcal{M}$ induces an isomorphism $K_n\mathcal{P} \cong K_n\mathcal{M}$ for all $n \geq 0$.

[2]Note that this is stronger than saying that $\mathcal{P}$ is closed under kernels in $\mathcal{M}$.

Proof of Theorem 3.8 for K_0. We claim that the induced homomorphism $\iota_*\colon K_0\mathcal{P} \to K_0\mathcal{M}$ has an inverse given by the map

$$\begin{array}{rccl} \phi\colon & K_0\mathcal{M} & \to & K_0\mathcal{P} \\ & [M] & \mapsto & [P]-[P'], \end{array}$$

where $0 \to P' \to P \to M \to 0$ is a short exact sequence of $\mathcal{M}$ with P and P' in $\mathcal{P}$ (which exists by the hypotheses on $\mathcal{P}$ and $\mathcal{M}$). If ϕ is well-defined, then it is clear that it is an inverse for ι_*, since $[P]-[P'] = [M]$ in $K_0\mathcal{M}$. It suffices therefore to show that ϕ is well-defined, i.e., that $\phi([M])$ is independent of the choice of resolution $0 \to P' \to P \to M \to 0$. This may be viewed as the analogue of Schanuel's lemma (see Proposition 3.12 of [Rot09], for example) in the situation of Theorem 3.8.

Let $0 \to P' \to P \xrightarrow{\epsilon_P} M \to 0$ and $0 \to Q' \to Q \xrightarrow{\epsilon_Q} M \to 0$ be resolutions of M. By the definition of an exact category, the pullback $P \times_M Q$ of ϵ_P and ϵ_Q exists, and the canonical projections $P \times_M Q \to P$ and $P \times_M Q \to Q$ are admissible epimorphisms. The object $P \times_M Q$ may not be an object of $\mathcal{P}$ but, by hypothesis (2) above, there exists an admissible epimorphism $R \twoheadrightarrow P \times_M Q$ for some object R of $\mathcal{P}$. Composing this with the pullback square we obtain an admissible epimorphism $R \twoheadrightarrow M$ that factors through ϵ_P and ϵ_Q separately. We therefore have the commuting diagram on the left below whose bottom row and right column are short exact sequences.

$$\begin{tikzcd}
 & & 0 \arrow[d, tail] \\
 & R \arrow[r, two heads] \arrow[d, two heads] & M \arrow[d, equal] \\
P' \arrow[r, tail] & P \arrow[r, two heads, "\epsilon_P"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
S_P \arrow[r, equal] \arrow[d] & S_P \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
R' \arrow[r, tail] \arrow[d] & R \arrow[r, two heads] \arrow[d, two heads] & M \arrow[d, equal] \\
P' \arrow[r, tail] & P \arrow[r, two heads, "\epsilon_P"'] & M
\end{tikzcd}$$

Denoting the kernel of $R \twoheadrightarrow P$ by S_P and the kernel of $R \twoheadrightarrow M$ by R', we add all induced maps between kernels to complete the diagram to obtain the 3 by 3 square on the right above. The top right square commutes since the middle column composes to 0, the bottom right square commutes by definition of R and the two left squares commute automatically from the definition of a kernel. All rows are exact and the rightmost two columns are exact, so the left column is also exact by the 3 by 3 lemma 1.13. Since $\mathcal{P}$ is closed under kernels in $\mathcal{M}$, all the objects in the left column are in $\mathcal{P}$, as P and R are in $\mathcal{P}$. The leftmost columns are short exact sequences of $\mathcal{P}$, so they give the relations $[R] = [S_P] + [P]$ and $[R'] = [S_P] + [P']$ in $K_0\mathcal{P}$, yielding the equation $[P]-[P'] = [R]-[R']$. But the construction of R and R' was symmetric in P and Q so we may repeat the above argument to obtain a diagram:

$$\begin{tikzcd}
S_Q \arrow[r, equal] \arrow[d, tail] & S_Q \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
R' \arrow[r, tail] \arrow[d, two heads] & R \arrow[r, two heads] \arrow[d, two heads] & M \arrow[d, equal] \\
Q' \arrow[r, tail] & Q \arrow[r, two heads, "\epsilon_Q"'] & M
\end{tikzcd}$$

and the equation $[Q]-[Q'] = [R]-[R']$. So for any pair $0 \to P' \to P \overset{\epsilon_P}{\to} M \to 0$ and $0 \to Q' \to Q \overset{\epsilon_Q}{\to} M \to 0$ of length 1 resolutions of M by objects of $\mathcal{P}$, we have $[P]-[P'] = [Q]-[Q']$ in $K_0\mathcal{P}$, hence $\phi\colon [M] \mapsto [P]-[P']$ is a well-defined inverse for ι_*. □

Proof of Theorem 3.8 for higher K-groups. A short exact sequence in $(B^{\mathrm{q}})^n\mathcal{M}$ is short exact in each $\mathbb{Z}^n$-graded degree, so if $\mathcal{P}$ is closed under extensions in $\mathcal{M}$, then $(B^{\mathrm{q}})^n\mathcal{P}$ is closed under extensions in $(B^{\mathrm{q}})^n\mathcal{M}$ for each n. By similar reasoning, one sees that if the inclusion $\mathcal{P} \hookrightarrow \mathcal{M}$ satisfies hypothesis (1) of the theorem, then so does the inclusion $(B^{\mathrm{q}})^n\mathcal{P} \hookrightarrow (B^{\mathrm{q}})^n\mathcal{M}$. The following proposition is about hypothesis (2).

Proposition 3.9. *Let $\mathcal{P}$ and $\mathcal{M}$ satisfy the hypotheses of Theorem 3.8. For every object $M_\cdot$ of $(B^{\mathrm{q}})^n\mathcal{M}$ there exists a short exact sequence $0 \to P'_\cdot \to P_\cdot \to M_\cdot \to 0$ of $(B^{\mathrm{q}})^n\mathcal{M}$ with $P'_\cdot$ and $P_\cdot$ in $(B^{\mathrm{q}})^n\mathcal{P}$. Furthermore, if $M_\cdot$ is a diagonal binary multicomplex, then we may choose $P_\cdot$ and $P'_\cdot$ to be diagonal as well.*

We shall prove Proposition 3.9 shortly. We now continue with the proof of Theorem3.8. Together with the known isomorphism for K_0, the first part of the proposition implies that the induced map $K_0(B^{\mathrm{q}})^n\mathcal{P} \to K_0(B^{\mathrm{q}})^n\mathcal{M}$ is an isomorphism for each n. Clearly this isomorphism sends elements of $T^n_{\mathcal{P}}$ to elements of $T^n_{\mathcal{M}}$. Since the value of ϕ is independent of the choice of resolution, the second part of the proposition implies that ϕ maps elements of $T^n_{\mathcal{M}}$ to elements of $T^n_{\mathcal{P}}$. The isomorphism therefore descends to an isomorphism $K_n\mathcal{P} \to K_n\mathcal{M}$. □

The rest of this section is concerned with the proof of Proposition 3.9, so we assume the hypotheses of Theorem 3.8 for $\mathcal{P} \hookrightarrow \mathcal{M}$ from here on. The idea of the proof is to construct, for each $M_\cdot$ in $(B^{\mathrm{q}})^n\mathcal{M}$, a morphism of acyclic binary chain complexes $P_\cdot \to M_\cdot$ that is an admissible epimorphism in every degree, i.e., $P_j \twoheadrightarrow M_j$ is admissible for each j. By the assumption on $\mathcal{P}$ and $\mathcal{M}$ each of these admissible epimorphisms is part of a short exact sequence $0 \to P'_j \to P_j \to M_j \to 0$ with the P_j in $\mathcal{P}$. The P'_j form a binary complex with the induced maps, and we show that this binary complex is in fact acyclic. The result will then follow from an induction on the dimension. We shall rely on the following lemma, and Proposition 1.32 of our first chapter, which states that a morphism $B_\cdot \to C_\cdot$ in $C^{\mathrm{q}}\mathcal{M}$ that is an admissible epimorphism of $\mathcal{M}$ in each degree has an acyclic kernel if and only if the induced maps between the images of the differentials of B and C are also admissible epimorphisms of $\mathcal{M}$.

Lemma 3.10. *Let $f_i\colon Q_i \to N$, $i = 1,\ldots,m$ be a family of morphisms in an exact category, at least one of which is an admissible epimorphism. Then the induced morphism*

$$[\, f_1 \ \cdots \ f_m \,]\colon \bigoplus_{i=1}^{m} Q_i \to N$$

is an admissible epimorphism as well.

Proof. The general case follows from the case $m = 2$. We may assume without loss of generality that f_1 is an admissible epimorphism. The morphism $[\, f_1 \ f_2 \,]$ factors as the composition

$$Q_1 \oplus Q_2 \xrightarrow{\begin{bmatrix} f_1 & 0 \\ 0 & 1 \end{bmatrix}} N \oplus Q_2 \xrightarrow{\begin{bmatrix} 1 & f_2 \\ 0 & 1 \end{bmatrix}} N \oplus Q_2 \xrightarrow{[\,1 \ 0\,]} N,$$

all of which are admissible epimorphisms. □

We begin resolving binary complexes in the less involved case, in which we assume $M_.$ to be diagonal.

Lemma 3.11. *Given a diagonal bounded acyclic binary complex $M_.$ in $B^{\mathrm{q}}\mathcal{M}$ there exists a short exact sequence $0 \to P'_. \to P_. \to M_. \to 0$ where $P'_.$ and $P_.$ are diagonal objects of $B^{\mathrm{q}}\mathcal{P}$.*

Proof. We may consider $M_.$ as an object of $C^{\mathrm{q}}\mathcal{N}$, as $\Delta\colon C^{\mathrm{q}}\mathcal{N} \to B^{\mathrm{q}}\mathcal{N}$ is a full embedding for any exact category $\mathcal{N}$. Represent $M_.$ in $C^{\mathrm{q}}\mathcal{N}$ as below. Without loss of generality we assume that $M_.$ ends at place 0.

$$0 \longrightarrow M_n \xrightarrow{d} \cdots \xrightarrow{d} M_k \xrightarrow{d} M_{k-1} \xrightarrow{d} \cdots \xrightarrow{d} M_0 \longrightarrow 0,$$

Since $\mathcal{P}$ and $\mathcal{M}$ satisfy the hypotheses of Theorem 3.8, there exists an object Q_k of $\mathcal{P}$ and an admissible epimorphism $\epsilon_k\colon Q_k \twoheadrightarrow M_k$ in $\mathcal{M}$ for each $0 \le k \le n$. The diagram below is a morphism in $B^{\mathrm{q}}\mathcal{M}$ with target $M_.$, and its upper row, the source of the morphism, is an object of $C^{\mathrm{q}}\mathcal{P}$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_k \arrow[r, equal] \arrow[d, two heads, "\epsilon_k"] & Q_k \arrow[r] \arrow[d, "d\epsilon_k"] & \cdots \arrow[r] & Q \arrow[r] \arrow[d] & Q \arrow[d] \\
0 \arrow[r] & M_n \arrow[r, "d"] & \cdots \arrow[r, "d"] & M_k \arrow[r, "d"] & M_{k-1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & M_0 \arrow[r] & 0
\end{tikzcd}$$

We denote the top row by $P^k_.$ and the morphism of complexes by $\zeta^k\colon P^k_. \to M_.$. We do this for each $k \in \{0,\ldots,n\}$ and form the sum

$$\zeta := [\, \zeta^n \ \ldots \ \zeta^0 \,]\colon \bigoplus P^k_. \to M_..$$

Call the direct sum $P_.$. Each of the complexes $P^k_.$ is acyclic, so $P_.$ is as well. Consider the morphism $\zeta_j \colon P_j \to M_j$, the part of ζ from the j^{th} term of P_j to the j^{th} term of M. By construction and Lemma 3.10, it is an admissible epimorphism of $\mathcal{M}$. We now have a morphism $\zeta \colon P_. \to M_.$ that is an admissible epimorphism in every degree. The kernels of these admissible epimorphisms are all in $\mathcal{P}$ by the hypotheses on $\mathcal{P}$ and $\mathcal{M}$. These kernels form a (as we have seen in Example 1.31, not *a priori* acyclic) bounded chain complex $P'_.$ in $\mathcal{P}$ under the induced maps between them.

It remains to show that this chain complex is acyclic (i.e., that $P'_.$ is in $C^q\mathcal{P}$), for then $0 \to P'_. \to P_. \to M_. \to 0$ will be a short exact sequence of acyclic complexes. Suppose that the differential on $M_.$ factors through objects Z_j of $\mathcal{M}$, and that $P_.$ factors through objects Y_j of $\mathcal{P}$. By Proposition 1.32, to show that $P'_.$ is acyclic in $\mathcal{M}$ it is enough to show that each $Y_j \to Z_j$ is an admissible epimorphism of $\mathcal{M}$. Since each $P^k_.$ is concentrated in degrees k and $k-1$, we have $Y_j = Q_j$ and the induced map $Q_j \to Z_j$ is the composition $Q_j \twoheadrightarrow M_j \twoheadrightarrow Z_j$, which is an admissible epimorphism of $\mathcal{M}$, so $P'_.$ is acyclic in $\mathcal{M}$ and its differentials factor through the kernels of the admissible epimorphisms $Q_j \twoheadrightarrow Z_j$ (call them X_j). But each Q_j is an object of $\mathcal{P}$, and so each X_j is an object of $\mathcal{P}$ as well, by the hypotheses on $\mathcal{P}$ and $\mathcal{M}$, so $P'_.$ is acyclic in $\mathcal{P}$.

Finally we consider $P'_.$, $P_.$ and $M_.$ as diagonal binary complexes (by applying Δ). Then $0 \to P'_. \to P_. \to M_. \to 0$ is the required short exact sequence of acyclic diagonal binary complexes. □

A little more work is required if the binary complex $M_.$ is not diagonal. The idea in this case is due to Grayson, and relies on the acyclicity of the chain complexes

$$0 \longrightarrow Q \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} Q\oplus Q \xrightarrow{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]} Q\oplus Q \xrightarrow{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]} Q\oplus Q \xrightarrow{[0\ 1]} Q \longrightarrow 0$$

and

$$0 \longrightarrow Q \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} Q\oplus Q \xrightarrow{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]} Q\oplus Q \xrightarrow{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]} Q\oplus Q \xrightarrow{[1\ 0]} Q \longrightarrow 0$$

of arbitrary length, where Q is an object of any exact category.

Lemma 3.12. *Given an arbitrary bounded acyclic binary complex $M_.$ in $B^q\mathcal{M}$ there exists a short exact sequence $0 \to P'_. \to P_. \to M_. \to 0$, where $P'_.$ and $P_.$ are objects of $B^q\mathcal{P}$.*

Proof. Let $M_.$ denote the element of $B^q\mathcal{M}$ given by the binary complex below

$$0 \rightrightarrows M_n \underset{d'}{\overset{d}{\rightrightarrows}} \cdots \underset{d'}{\overset{d}{\rightrightarrows}} M_k \underset{d'}{\overset{d}{\rightrightarrows}} M_{k-1} \underset{d'}{\overset{d}{\rightrightarrows}} \cdots \underset{d'}{\overset{d}{\rightrightarrows}} M_0 \rightrightarrows 0$$

and as before, for $0 \le k \le n$, let $\epsilon_k \colon Q_k \twoheadrightarrow M_k$ be admissible epimorphisms in $\mathcal{M}$ with Q_k in $\mathcal{P}$. For $0 \le k \le n$ and $1 \le l \le k$, inductively define two collections of morphisms

$\delta_{k,l}, \delta'_{k,l} \colon Q_k \to M_{k-l}$ by

$$\begin{cases} \delta_{k,1} = d \circ \epsilon_k \\ \delta'_{k,1} = d' \circ \epsilon_k \end{cases}$$

and

$$\begin{cases} \delta_{k,l+1} = d \circ \delta'_{k,l} \\ \delta'_{k,l+1} = d' \circ \delta_{k,l}. \end{cases}$$

Since each of its differentials is acyclic, the top row of the diagram below is an object of $B^{\mathrm{q}}\mathcal{P}$ for each $k \in \{0, \dots, n\}$.

$$\begin{tikzcd}[column sep=large]
{} \arrow[r, shift left] \arrow[r, shift right] & Q_k \arrow[r, shift left, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[r, shift right, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \arrow[d, two heads, "\epsilon_k"] & Q_k \oplus Q_k \arrow[r, shift left, "{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]}"] \arrow[r, shift right, "{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}"'] \arrow[d, "{[\,\delta_{k,1}\ \delta'_{k,1}\,]}"] & Q_k \oplus Q_k \arrow[r, shift left, "{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]}"] \arrow[r, shift right, "{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}"'] \arrow[d, "{[\,\delta_{k,2}\ \delta'_{k,2}\,]}"] & \cdots \arrow[r, shift left, "{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]}"] \arrow[r, shift right, "{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}"'] & Q_k \oplus Q_k \arrow[r, shift left, "{[\,1\ 0\,]}"] \arrow[r, shift right, "{[\,0\ 1\,]}"'] \arrow[d, "{[\,\delta_{k,k}\ \delta'_{k,k}\,]}"] & Q_k \arrow[d] \\
{} \arrow[r, shift left] \arrow[r, shift right] & M_k \arrow[r, shift left, "d"] \arrow[r, shift right, "d'"'] & M_{k-1} \arrow[r, shift left, "d"] \arrow[r, shift right, "d'"'] & M_{k-2} \arrow[r, shift left, "d"] \arrow[r, shift right, "d'"'] & \cdots \arrow[r, shift left, "d"] \arrow[r, shift right, "d'"'] & M_0 \arrow[r, shift left] \arrow[r, shift right] & 0.
\end{tikzcd}$$

The morphisms $\delta_{k,l}$ and $\delta'_{k,l}$ have been constructed so that the vertical morphisms commute with the top and bottom differentials, so the diagram represents a morphism in $B^{\mathrm{q}}\mathcal{M}$, which we shall again denote by $\zeta^k \colon P^k_{\bullet} \to M_{\bullet}$. Following the same method of proof as of the previous lemma, each $P^k_{\bullet}$ is acyclic so their direct sum is acyclic as well and so

$$\zeta := [\,\zeta^n \ \dots\ \zeta^0\,] \colon \bigoplus P^k_{\bullet} \to M_{\bullet}.$$

is a morphism in $B^{\mathrm{q}}\mathcal{M}$. By construction and Lemma 3.10 again, each morphism $\zeta^j \colon P_j \to M_j$ is an admissible epimorphism in $\mathcal{M}$. Each of these morphisms therefore has a kernel in $\mathcal{P}$ and these kernels form a binary complex with the induced maps.

We wish to show that both differentials of this binary complex are acyclic in $\mathcal{P}$. Consider the top differential first. Denote the objects that the top differentials of $M_{\bullet}$, $P_{\bullet}$ and each $P^k_{\bullet}$ factor through by Z_j, Y_j and Y^k_j (so Y_j is the sum of all the factors Y^k_j) As in the proof of Lemma 3.11, it is enough to show that each induced morphism $Y_j \to Z_j$ is an admissible epimorphism of $\mathcal{M}$. But since $Y_j \to Z_j$ is equal to the product of induced maps $\bigoplus Y^k_j \to Z_j$, it suffices, by Lemma 3.10, to show that, for each j, one of the morphisms $Y^k_j \to Z_j$ is an admissible epimorphism of $\mathcal{M}$. Since $Y^j_j = Q_j$, taking $k = j$ yields the result, as shown by the diagram below.

$$\begin{tikzcd}
Q_j \arrow[r, equal] \arrow[d, two heads] & Q_j \arrow[r, tail, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[d] & Q_j \oplus Q_j \arrow[d] \\
M_j \arrow[r, two heads] & Z_j \arrow[r, tail] & M_{j-1}.
\end{tikzcd}$$

The bottom differential is dealt with entirely analogously. □

Proof of Proposition 3.9. We proceed by induction on n. In the base case $n = 0$, there is nothing to show. For the inductive step, we view an acyclic binary multicomplex $M_.$ in $(B^{\mathrm{q}})^{n+1}\mathcal{M}$, as an acyclic binary multicomplex of objects of $(B^{\mathrm{q}})^n\mathcal{M}$, i.e., as an object of $B^q(B^{\mathrm{q}})^n\mathcal{M}$. By the inductive hypothesis, the inclusion of $(B^{\mathrm{q}})^n\mathcal{P}$ into $(B^{\mathrm{q}})^n\mathcal{M}$ satisfies the hypotheses of Theorem 3.8, so by Lemma 3.12 there exists a short exact sequence $0 \to P'_. \to P_. \to M_. \to 0$ in $B^q(B^{\mathrm{q}})^n\mathcal{M}$ with $P'_.$ and $P_.$ in $B^q(B^{\mathrm{q}})^n\mathcal{P} = (B^{\mathrm{q}})^{n+1}\mathcal{P}$, and so the first part follows. For the second part, suppose that $M_.$ is diagonal in some direction i. We consider $M_.$ as a diagonal acyclic binary complex of (not necessarily diagonal) objects of $(B^{\mathrm{q}})^n\mathcal{M}$, that is, we "expand" $M_.$ along the i direction. Then by Lemma 3.11 there exist diagonal acyclic binary complexes $P'_.$ and $P_.$ in $(B^{\mathrm{q}})^{n+1}\mathcal{P}$ that are diagonal in direction i, and an exact sequence $0 \to P'_. \to P_. \to M_. \to 0$, so the proof is complete. □

3.3 The cofinality theorem

While $K_0(R)$ is defined in terms of the category of (finitely generated) projective R-modules, the higher algebraic K-groups of R seem to depend only on the category of (f.g.) free R-modules, in that they are defined using the group $GL(R)$ which encodes all the morphisms of that category (see [Mil71] for $K_1(R)$ and $K_2(R)$, and [Qui72] for the higher K-groups). The cofinality theorem makes this heuristic precise: the inclusion $\mathbf{Free}_{\mathrm{f}}(R) \hookrightarrow \mathbf{Proj}_{\mathrm{f}}R$ induces an isomorphism $K_n\mathbf{Free}_{\mathrm{f}}(R) \xrightarrow{\cong} K_n\mathbf{Proj}_{\mathrm{f}}(R)$ for all $n > 0$.
Unlike the additivity and resolution theorems, the cofinality theorem was not proved by Quillen in [Qui73]. A proof for exact categories based on work by Gersten was given in [Gra79]. More general versions can be found in [Sta89] and [TT90]. It is proven in [Gra12] that the hypotheses of the cofinality theorem are satisfied by the appropriate exact categories of acyclic binary complexes; the main work in our proof is in ensuring that the results pass to the quotients defining K_n. The proof we give here is our original proof the cofinality theorem; at the end of the section we will describe a conceptually cleaner proof that was pointed out to us by the referee for the paper [Har15] (and is the proof given in the final version of that paper).

Definition 3.13. An exact subcategory (see Example 1.10 (3)) $\mathcal{M}$ of an exact category $\mathcal{N}$ is said to be ***cofinal*** in $\mathcal{N}$ if for every object N_1 of $\mathcal{N}$ there exists another object N_2 of $\mathcal{N}$ such that $N_1 \oplus N_2$ is isomorphic to an object of $\mathcal{M}$.

An obvious example of a cofinal exact subcategory is the category of free R-modules inside the category of projective R-modules, for any ring R. More generally, every exact category is cofinal in its ***idempotent completion*** ([TT90], Appendix A). The cofinality theorem relates the K-theory of the cofinal subcategory to the K-theory of the exact category containing it. Throughout this section $\mathcal{M} \hookrightarrow \mathcal{N}$ is the inclusion of a cofinal exact subcategory of an exact category $\mathcal{N}$.

Define an equivalence relation on the objects of $\mathcal{N}$ by declaring $N_1 \sim N_2$ if there exist objects M_1 and M_2 of $\mathcal{M}$ such that
$$N_1 \oplus M_1 \cong N_2 \oplus M_2.$$

Since $\langle M \rangle = 0$ for every M in $\mathcal{M}$, the cofinality of $\mathcal{M}$ in $\mathcal{N}$ ensures that equivalence classes of $\sim$ form a group under the natural operation $\langle N_1 \rangle + \langle N_2 \rangle = \langle N_1 \oplus N_2 \rangle$; we denote this group by $K_0(\mathcal{N} \text{ rel. } \mathcal{M})$. The following lemma and its corollary were first observed in the proof of Theorem 1.1 in [Gra79].

Lemma 3.14. *The sequence:*

$$\begin{array}{ccccccccc} 0 \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N} & \longrightarrow & K_0(\mathcal{N} \text{ rel. } \mathcal{M}) & \longrightarrow & 0 \\ & & & [N] & \longmapsto & \langle N \rangle & & \end{array}$$

is well-defined and exact.

Proof. Recall that $\mathcal{N}^{\oplus}$ denotes the category $\mathcal{N}$ together with the exact structure given by declaring only the short exact sequences to be those that are split only. So $\mathcal{N}^{\oplus}$ is a split exact category by definition. As we saw at the end of section 1.2, for objects N_1 and N_2 in $\mathcal{N}$, the equality $[N_1] = [N_2]$ holds in $K_0\mathcal{N}^{\oplus}$ if and only if there exists N in $\mathcal{N}$ such that $N_1 \oplus N \cong N_2 \oplus N$. Furthermore, since $\mathcal{M}$ is cofinal in $\mathcal{N}$, we can in fact choose N to be an object of $\mathcal{M}$. Now let M_1 and M_2 be in $\mathcal{M}$ such that $[M_1] - [M_2] = 0$ in $K_0\mathcal{N}^{\oplus}$, that is, $[M_1] - [M_2]$ is in the kernel of the induced map $K_0\mathcal{M}^{\oplus} \to K_0\mathcal{N}^{\oplus}$. Then there exists an M in $\mathcal{M}$ such that $M_1 \oplus M \cong M_2 \oplus M$. Therefore $[M_1] = [M_2]$ in $K_0\mathcal{M}^{\oplus}$, so the map is injective. Now consider an exact sequence $0 \to N_0 \to N_1 \to N_2 \to 0$ in $\mathcal{N}$ (with its usual exact sequences), and choose N_0' and N_2' in $\mathcal{N}$ such that $M_i = N_i \oplus N_i'$ is in $\mathcal{M}$ for $i = 0, 2$. Set $M_1 = N_0' \oplus N_1 \oplus N_2'$. By taking the direct sum of the exact sequences $0 \to N_0 \to N_1 \to N_2 \to 0$ and $0 \to N_0' \to N_0' \oplus N_2' \to N_2' \to 0$, we obtain an exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of $\mathcal{N}$. But $\mathcal{M}$ is closed under extensions in $\mathcal{N}$, so M_1 is an object of $\mathcal{M}$. Furthermore we have the following equality in $K_0\mathcal{N}^{\oplus}$

$$[M_1] - [M_0] - [M_2] = [N_1] - [N_0] - [N_2].$$

The element $[N_1] - [N_0] - [N_2]$ is in the kernel of the projection map $K_0\mathcal{N}^{\oplus} \to K_0\mathcal{N}$, so we have shown that
$$\ker(K_0\mathcal{N}^{\oplus} \to K_0\mathcal{N}) \subseteq \ker(K_0\mathcal{M}^{\oplus} \to K_0\mathcal{M}).$$

The reverse inclusion is trivial, so

$$\ker(K_0\mathcal{N}^{\oplus} \to K_0\mathcal{N}) = \ker(K_0\mathcal{M}^{\oplus} \to K_0\mathcal{M})$$

when $K_0\mathcal{M}^{\oplus}$ is considered as a subgroup of $K_0\mathcal{N}^{\oplus}$. Using the snake lemma in the category of abelian groups, we see that this implies that the induced map $K_0\mathcal{M} \to K_0\mathcal{N}$ is injective.

The snake lemma also shows that the induced map

$$\operatorname{coker}(K_0\mathcal{M}^{\oplus} \hookrightarrow K_0\mathcal{N}^{\oplus}) \to \operatorname{coker}(K_0\mathcal{M} \hookrightarrow K_0\mathcal{N})$$

is an isomorphism. On the other hand, one easily sees that the map

$$\begin{array}{rcl} K_0\mathcal{N}^{\oplus} & \longrightarrow & K_0(\mathcal{N} \text{ rel. } \mathcal{M}) \\ [N] & \longmapsto & \langle N \rangle. \end{array}$$

is a well-defined epimorphism, and that its kernel is $K_0\mathcal{M}^{\oplus}$. Therefore we may identify $K_0(\mathcal{N}$ rel. $\mathcal{M})$ with $\operatorname{coker}(K_0\mathcal{M}^{\oplus} \hookrightarrow K_0\mathcal{N}^{\oplus}) \cong \operatorname{coker}(K_0\mathcal{M} \hookrightarrow K_0\mathcal{N})$ and the proof is complete. □

Corollary 3.15. *For any pair of objects N_1, N_2 of $\mathcal{N}$ with the same class in $\operatorname{coker}(K_0\mathcal{M} \to K_0\mathcal{N})$ there exists an object N' in $\mathcal{N}$ such that each $N_i \oplus N'$ is in $\mathcal{M}$.*

Proof. By the lemma, if N_1 and N_2 have the same class in $\operatorname{coker}(K_0\mathcal{M} \to K_0\mathcal{N})$, then $\langle N_1 \rangle = \langle N_2 \rangle$. From cofinality there exists a P in $\mathcal{N}$ such that $N_1 \oplus P$ is in $\mathcal{M}$, so $\langle 0 \rangle = \langle N_1 \oplus P \rangle = \langle N_2 \oplus P \rangle$. Hence there exist objects P_1 and P_2 of $\mathcal{M}$ such that each $(N_i \oplus P) \oplus P_i$ is an object of $\mathcal{M}$. Setting $N' = P \oplus P_1 \oplus P_2$, each $N_i \oplus N'$ is an object of $\mathcal{M}$. □

We show now that cofinality of $\mathcal{M}$ in $\mathcal{N}$ passes to the associated categories of acyclic binary multicomplexes. We can say much more in general however.

Lemma 3.16. *For all $n \geq 0$, if $N_.$ is in $(B^{q})^n\mathcal{N}$ and $i \in \{1, \ldots, n\}$ is any direction, then there exists an object $T_.$ in $(B^{q})^n\mathcal{N}$ that is diagonal in direction i such that $N_. \oplus T_.$ is in $(B^{q})^n\mathcal{M}$. In particular, the exact subcategory $(B^{q})^n\mathcal{M}$ is cofinal in $(B^{q})^n\mathcal{N}$. Moreover, if N is diagonal in direction $j \in \{1, \ldots, n\}$, $j \neq i$, then T may be chosen to be diagonal in directions i and j.*

Proof. (Part of the following proof is adapted from the proof of Lemma 6.2 in [Gra12].) We proceed by induction on n. The statements for the base case $n = 0$ mean only that $\mathcal{M}$ is cofinal in $\mathcal{N}$, which is assumed throughout this section.
For the inductive step we fix $i \in \{1, \ldots, n+1\}$ and N in $(B^{q})^{n+1}\mathcal{N}$. We first assume that $N_.$ is diagonal in direction $j \neq i$, and "expand along j" to consider $N_.$ as a diagonal acyclic binary complex of objects of $(B^{q})^n\mathcal{N}$. Let C_k in $(B^{q})^n\mathcal{N}$ be the image of $d^j_k = \tilde{d}^j_k$ (between the terms k and $k-1$). By the inductive hypothesis, for each k there exists an object T_k in $(B^{q})^n\mathcal{N}$ that is diagonal in direction i such that $C_k \oplus T_k$ is an object of $(B^{q})^n\mathcal{M}$. Let (T', e) be the acyclic chain complex of objects of $(B^{q})^n\mathcal{N}$ given by taking the direct sum of the identity maps $T_k \to T_k$ concentrated in degrees k and $k-1$. That is, (T', e) is the complex

$$0 \to \cdots \to T_{k+2} \oplus T_{k+1} \to T_{k+1} \oplus T_k \to T_k \oplus T_{k-1} \to \cdots \to 0$$

with differential $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since each T_k is diagonal in direction i, the complex $T'_.$ is diagonal in direction i, when regarded as an object of $C^{q}(B^{q})^n\mathcal{N}$. The image of the differential

$d^j_k \oplus e_k = \tilde{d}^j_k \oplus e_k$ on the acyclic complex $\perp_j(N_.) \oplus T'_.$ is equal to $C_k \oplus T_k$, and hence belongs to $(B^{\mathrm{q}})^n\mathcal{M}$. Since $(B^{\mathrm{q}})^n\mathcal{M}$ is closed under extensions in $(B^{\mathrm{q}})^{n+1}\mathcal{M}$, we obtain that the complex $\perp_j(N_.) \oplus T'_.$ belongs to $C^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{M}$. We define a binary complex $T_. := \Delta_j(T'_.)$, which is an object of $B^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{N} = (B^{\mathrm{q}})^{n+1}\mathcal{N}$ and is diagonal in directions i *and* j. Then $N_. \oplus T_. = \Delta_j(\perp_j(N_.) \oplus T'_.)$ belongs to $B^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{M} = (B^{\mathrm{q}})^{n+1}\mathcal{M}$.

If $N_.$ is not diagonal in any direction different from i, we again consider $N_.$ as an acyclic binary complex

$$\cdots \underset{d'}{\overset{d}{\rightrightarrows}} N_{k+1} \underset{d'}{\overset{d}{\rightrightarrows}} N_k \underset{d'}{\overset{d}{\rightrightarrows}} N_{k-1} \underset{d'}{\overset{d}{\rightrightarrows}} \cdots,$$

of objects N_k in $(B^{\mathrm{q}})^n\mathcal{N}$, but now "expanded along" direction i. Let C_k and $\widetilde{C}_k$ in $(B^{\mathrm{q}})^n\mathcal{N}$ denote the images of the (normally different) differentials d^i_k and $\tilde{d}^i_k$ (between degrees k and $k-1$). The classes of both C_k and $\widetilde{C}_k$ are equal to the finite sum $\sum_{i=-\infty}^{k}(-1)^{k-i}[N_{k-i}]$ in $K_0(B^{\mathrm{q}})^n\mathcal{N}$, by a standard argument, so they have the same class in

$$\operatorname{coker}(K_0(B^{\mathrm{q}})^n\mathcal{M} \to K_0(B^{\mathrm{q}})^n\mathcal{N}).$$

By the inductive hypothesis and Corollary 3.15, there exists a single object T_k in $(B^{\mathrm{q}})^n\mathcal{N}$ such that $C_k \oplus T_k$ and $\tilde{C}_k \oplus T_k$ are both objects of $(B^{\mathrm{q}})^n\mathcal{N}$. As above, from the objects T_k we form the acyclic binary complex T in $B^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{N} = (B^{\mathrm{q}})^{n+1}\mathcal{N}$, which is diagonal in direction i. Then $N \oplus T$ is an object of $(B^{\mathrm{q}})^{n+1}\mathcal{M}$, as was to be shown. □

Following Lemmas 3.14 and 3.16, we now regard $K_0(B^{\mathrm{q}})^n\mathcal{M}$ as a subgroup of $K_0(B^{\mathrm{q}})^n\mathcal{N}$ for each n, and write $K_0(B^{\mathrm{q}})^n\mathcal{N}/K_0(B^{\mathrm{q}})^n\mathcal{M}$ for the cokernel of the inclusion. It is clear moreover that this inclusion respects the subgroups generated by the classes of diagonal binary multicomplexes, i.e., $T^n_{\mathcal{M}} \hookrightarrow T^n_{\mathcal{N}}$. The following proposition concerning representations of elements of $T^n_{\mathcal{N}}/T^n_{\mathcal{M}}$ is key to our proof of the cofinality theorem.

Proposition 3.17. *Let $x + T^n_{\mathcal{M}}$ be a class in $T^n_{\mathcal{N}}/T^n_{\mathcal{M}}$, for $n \geq 1$. Then $x + T^n_{\mathcal{M}} = [t] + T^n_{\mathcal{M}}$, where $[t]$ is the class in $K_0(B^{\mathrm{q}})^n\mathcal{N}$ of a diagonal acyclic binary multicomplex t in $(B^{\mathrm{q}})^n\mathcal{N}$.*

Proof. The idea is to take the class of a general element $x \in T^n_{\mathcal{N}}$ and transform it into a direct sum of diagonal complexes that are all diagonal in the same direction, without altering the class of x modulo $T^n_{\mathcal{M}}$. To begin, we write

$$x + T^n_{\mathcal{M}} = \sum_{j=1}^{n}([t'_j] - [t''_j]) + T^n_{\mathcal{M}},$$

where each t'_j and t''_j is an actual acyclic binary multicomplex diagonal in direction j, and pick a distinguished direction i. By Lemma 3.16, for each $j \neq i$ there exist acyclic binary multicomplexes s'_j and s''_j that are both diagonal in directions i *and* j such that $t'_j \oplus s'_j$ and $t''_j \oplus s''_j$ are objects of $(B^{\mathrm{q}})^n\mathcal{M}$. The binary complexes t'_j, s'_j, t''_j and s''_j are all diagonal in direction j, so their direct sums are also diagonal in direction j, and we have $[t'_j \oplus s'_j] \in T^n_{\mathcal{M}}$

and $[t''_j \oplus s''_j] \in T^n_{\mathcal{M}}$. Therefore $[s'_j] = -[t'_j]$ and $[s''_j] = -[t''_j]$ in $T^n_{\mathcal{N}}/T^n_{\mathcal{M}}$. But the s'_j and s''_j, along with t'_i and t''_i are all diagonal in direction i, so taking u_1 to be the sum of the positive classes in our new expansion of $x+T^n_{\mathcal{M}}$, and u_2 to be the sum of the negative classes, we have

$$x + T^n_{\mathcal{M}} = [u_1] - [u_2] + T^n_{\mathcal{M}},$$

where u_1 and u_2 are acyclic binary multicomplexes that are diagonal in direction i. Finally, we use Lemma 3.16 again to find an acyclic binary multicomplex u'_2 that is also diagonal in direction i, such that $[u_2 \oplus u'_2] \in T^n_{\mathcal{M}}$. Then

$$x + T^n_{\mathcal{M}} = [u_1] + [u'_2] + T^n_{\mathcal{M}},$$

and setting $t = u_1 \oplus u'_2$ yields the result. □

Cofinality Theorem 3.18. *The inclusion $\mathcal{M} \hookrightarrow \mathcal{N}$ induces an injection $K_0\mathcal{M} \hookrightarrow K_0\mathcal{N}$ and isomorphisms $K_n\mathcal{M} \cong K_n\mathcal{N}$ for $n > 0$.*

Proof. The case $n = 0$ is part of Lemma 3.14, so we proceed directly to $n > 0$. We have the following diagram of abelian groups, whose rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & T^n_{\mathcal{M}} \arrow[r] \arrow[d, hook] & K_0(B^{\mathrm{q}})^n\mathcal{M} \arrow[r] \arrow[d, hook] & K_n\mathcal{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T^n_{\mathcal{N}} \arrow[r] \arrow[d, two heads] & K_0(B^{\mathrm{q}})^n\mathcal{N} \arrow[r] \arrow[d, two heads] & K_n\mathcal{N} \arrow[r] & 0 \\
& T^n_{\mathcal{N}}/T^n_{\mathcal{M}} \arrow[r] & K_0(B^{\mathrm{q}})^n\mathcal{N}/K_0(B^{\mathrm{q}})^n\mathcal{M} & &
\end{tikzcd}$$

The snake lemma implies that the homomorphism $K_n\mathcal{M} \to K_n\mathcal{N}$ is an isomorphism if and only if the induced homomorphism

$$\begin{array}{rcl} T^n_{\mathcal{N}}/T^n_{\mathcal{M}} & \to & K_0(B^{\mathrm{q}})^n\mathcal{N}/K_0(B^{\mathrm{q}})^n\mathcal{M} \\ x + T^n_{\mathcal{M}} & \mapsto & x + K_0(B^{\mathrm{q}})^n\mathcal{M} \end{array}$$

is an isomorphism. Denote this homomorphism by ψ. We first show that ψ is surjective. Let b be a generic element of $K_0(B^{\mathrm{q}})^n\mathcal{N}$, so $b = [b_1] - [b_2]$, where b_1 and b_2 are acyclic binary multicomplexes of dimension n. By Lemma 3.16 there exist diagonal acyclic binary multicomplexes s_1 and s_2 such that $b_i \oplus s_i$ is an object of $(B^{\mathrm{q}})^n\mathcal{M}$ for $i = 1, 2$. Then $[b_1 \oplus s_1] - [b_2 \oplus s_2] \in K_0(B^{\mathrm{q}})^n\mathcal{M}$, and is therefore zero in $K_0(B^{\mathrm{q}})^n\mathcal{N}/K_0(B^{\mathrm{q}})^n\mathcal{M}$. Set $s = [s_1] - [s_2] \in T^n_{\mathcal{N}}$. Then $b + K_0(B^{\mathrm{q}})^n\mathcal{N}$ is the image of $-s + T^n_{\mathcal{M}}$ under ψ, so ψ is surjective.

For the injectivity of ψ, suppose that $x \in T^n_{\mathcal{N}}$ such that $x + T^n_{\mathcal{M}}$ is in $\ker(\psi)$. By Proposition 3.17 we may write $x + T^n_{\mathcal{M}} = [t] + T^n_{\mathcal{M}}$ for an actual acyclic binary multicomplex t diagonal in some direction i. Since $[t] + T^n_{\mathcal{M}}$ is in the kernel of ψ, we must have $[t] \in K_0(B^{\mathrm{q}})^n\mathcal{M}$ (considered as a subgroup of $K_0(B^{\mathrm{q}})^n\mathcal{N}$). In the notation of Lemma 3.14, we have $\langle t\rangle = 0$ in

$$K_0(B^{\mathrm{q}})^n\mathcal{N}/K_0(B^{\mathrm{q}})^n\mathcal{M} \cong K_0((B^{\mathrm{q}})^n\mathcal{N} \text{ rel. } (B^{\mathrm{q}})^n\mathcal{M}),$$

so there exist acyclic binary multicomplexes a_1 and a_2 in $(B^{\mathrm{q}})^n\mathcal{M}$ such that $t \oplus a_1 \cong a_2$. Consider the composite exact functor $\Delta_i\top_i\colon (B^{\mathrm{q}})^n\mathcal{N} \to (B^{\mathrm{q}})^n\mathcal{N}$, that replaces the bottom differential in direction i of an acyclic binary multicomplex with a second copy of the top differential. The binary multicomplexes $\Delta_i\top_i(a_1)$ and $\Delta_i\top_i(a_2)$ are diagonal in direction i, and $\Delta_i\top_i(t) = t$, since t is already diagonal in direction i. Applying the induced homomorphism on K_0 we have

$$[t] = K_0(\Delta_i\top_i)([t]) = K_0(\Delta_i\top_i)([a_1]-[a_2]) = [\Delta_i\top_i(a_1)] - [\Delta_i\top_i(a_2)] \in T^n_{\mathcal{M}}.$$

Hence $x \in T^n_{\mathcal{M}}$ for any x such that $x + T^n_{\mathcal{M}}$ is in the kernel of ψ, so the kernel of ψ is trivial and ψ is injective. □

The cleaner proof of the cofinality theorem that we mentioned at the beginning of this section uses the following lemma.

Lemma 3.19. *For each $n \geq 1$ we have a split short exact sequence:*

$$0 \longrightarrow K_{n-1}C^{\mathrm{q}}\mathcal{N} \xrightarrow{\Delta} K_{n-1}B^{\mathrm{q}}\mathcal{N} \longrightarrow K_n\mathcal{N} \longrightarrow 0,$$

which is functorial in $\mathcal{N}$.

Proof. This is already implicitly proven in [Gra12]. For completeness, we give a proof directly from Theorem/Definition 1.40. We have defined $K_n\mathcal{N}$ to be $K_0(B^{\mathrm{q}})^n\mathcal{N}$ modulo the sum of the images of the various diagonal maps

$$\Delta_i\colon K_0(B^{\mathrm{q}})^{i-1}C^{\mathrm{q}}(B^{\mathrm{q}})^{n-i}\mathcal{N} \longrightarrow K_0(B^{\mathrm{q}})^n\mathcal{N} \quad (i = 1,\ldots,n).$$

Similarly, $K_{n-1}B^{\mathrm{q}}\mathcal{N}$ is defined as the factor group of $K_0(B^{\mathrm{q}})^{n-1}B^{\mathrm{q}} = K_0(B^{\mathrm{q}})^n\mathcal{N}$ modulo the sum of the images of Δ_i for $i = 1,\ldots,n-1$. Therefore, the canonical epimorphism $K_0(B^{\mathrm{q}})^n\mathcal{N}$ naturally factorises as

$$K_0(B^{\mathrm{q}})^n\mathcal{N} \longrightarrow K_{n-1}B^{\mathrm{q}}\mathcal{N} \longrightarrow K_n\mathcal{N},$$

and $K_n\mathcal{N}$ is the factor group of $K_{n-1}B^{\mathrm{q}}\mathcal{N}$ modulo the image of the composition

$$K_0(B^{\mathrm{q}})^{n-1}C^{\mathrm{q}}\mathcal{N} \xrightarrow{\Delta_n} K_0(B^{\mathrm{q}})^n\mathcal{N} \longrightarrow K_{n-1}B^{\mathrm{q}}\mathcal{N}.$$

Furthermore, the epimorphism $K_0(B^{\mathrm{q}})^{n-1} \to K_{n-1}$ is a natural transformation of functors from the category of exact categories to abelian groups. Summarising, we obtain the following commutative diagram, whose bottom row is exact:

$$\begin{tikzcd}
K_0(B^{\mathrm{q}})^{n-1}C^{\mathrm{q}}\mathcal{N} \arrow[r, "\Delta_n"] \arrow[d, two heads] & K_0(B^{\mathrm{q}})^{n}\mathcal{N} \arrow[d, two heads] & & \\
K_{n-1}C^{\mathrm{q}}\mathcal{N} \arrow[r, "K_{n-1}(\Delta)"] & K_{n-1}B^{\mathrm{q}}\mathcal{N} \arrow[r, two heads] & K_n\mathcal{N} \arrow[r] & 0.
\end{tikzcd}$$

The top and bottom functors ($\top$ and $\perp$) each split Δ, so $K_{n-1}(\Delta)$ is split by either of $K_{n-1}(\top)$ or $K_{n-1}(\perp)$. Therefore the bottom row of the diagram is our required split short exact sequence. □

We can now prove the cofinality theorem without using second part of Lemma 3.16 or Proposition 3.17.

Alternative proof of the cofinality theorem. We consider $n = 1$ first. The inclusion $\mathcal{M} \hookrightarrow \mathcal{N}$ induces a morphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K_0C^{\mathrm{q}}\mathcal{M} \arrow[r, "\Delta"] \arrow[d] & K_0B^{\mathrm{q}}\mathcal{M} \arrow[r] \arrow[d] & K_1\mathcal{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0C^{\mathrm{q}}\mathcal{N} \arrow[r, "\Delta"] & K_0B^{\mathrm{q}}\mathcal{N} \arrow[r] & K_1\mathcal{N} \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 3.19, these short exact sequences are split. Furthermore, the downwards maps commute with the splittings $\top$ and $\perp$ of Δ, so the morphism is a morphism of split exact sequences. The induced map $K_0B^{\mathrm{q}}\mathcal{M} \to K_0B^{\mathrm{q}}\mathcal{N}$ is a monomorphism, as $B^{\mathrm{q}}\mathcal{M}$ is cofinal in $B^{\mathrm{q}}\mathcal{N}$, by Lemma 3.16. The map $K_1\mathcal{M} \to K_1\mathcal{N}$ is therefore a direct summand of a monomorphism and must be a monomorphism itself.

The induced map $K_1\mathcal{M} \to K_1\mathcal{N}$ sends the class $x+K_0C^{\mathrm{q}}\mathcal{M} \in K_1\mathcal{M}$ to the class $x+K_0C^{\mathrm{q}}\mathcal{N} \in K_1\mathcal{N}$, where $x \in K_0B^{\mathrm{q}}\mathcal{M}$. Let $x = [N_1] - [N_2]$ be a generic element of $K_0B^{\mathrm{q}}\mathcal{N}$, where N_1 and N_2 are acyclic binary complexes with objects in $\mathcal{N}$. By the first part of Lemma 3.16, there exist diagonal acyclic binary complexes T_1 and T_2 in $B^{\mathrm{q}}\mathcal{N}$ such that $N_i \oplus T_i \in B^{\mathrm{q}}\mathcal{M}$ for $i = 1, 2$. Then

$$[N_i] + K_0C^{\mathrm{q}}\mathcal{N} = [N_i \oplus T_i] + K_0C^{\mathrm{q}}\mathcal{N}$$

and therefore

$$x + K_0C^{\mathrm{q}}\mathcal{N} = [N_1] - [N_2] + K_0C^{\mathrm{q}}\mathcal{N} = [N_1 \oplus T_1] - [N_2 \oplus T_2] + K_0C^{\mathrm{q}}\mathcal{N}$$

is in the image of $K_1\mathcal{M} \to K_1\mathcal{N}$. Thus $K_1\mathcal{M} \to K_1\mathcal{N}$ is surjective and the case $n = 1$ of the theorem is proven.

The remaining cases now follow by an induction on n. By Lemma 3.19 again, the inclusion

$\mathcal{M} \hookrightarrow \mathcal{N}$ induces a morphism of split short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K_n C^{\mathsf{q}}\mathcal{M} \arrow[r, "\Delta"] \arrow[d] & K_n B^{\mathsf{q}}\mathcal{M} \arrow[r] \arrow[d] & K_{n+1}\mathcal{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_n C^{\mathsf{q}}\mathcal{N} \arrow[r, "\Delta"] & K_n B^{\mathsf{q}}\mathcal{N} \arrow[r] & K_{n+1}\mathcal{N} \arrow[r] & 0.
\end{tikzcd}$$

If $n \geq 1$, the induced map $K_n B^{\mathsf{q}}\mathcal{M} \to K_n B^{\mathsf{q}}\mathcal{N}$ is an isomorphism by the inductive hypothesis, since $B^{\mathsf{q}}\mathcal{M}$ is cofinal in $B^{\mathsf{q}}\mathcal{N}$, by Lemma 3.16 again. Therefore $K_{n+1}\mathcal{M} \to K_{n+1}\mathcal{N}$ is a direct summand of an isomorphism, so it is itself an isomorphism. $\square$

Chapter 4

Exterior power operations on higher K-groups of schemes

In this chapter we use Grayson's presentation to construct exterior power operations on the higher algebraic K-groups of an arbitrary quasi-compact scheme X, and study their properties. The approach is to use the classical Dold–Kan correspondence to define exterior powers of complexes, and to inductively extend this to produce exterior powers of binary multicomplexes. In Theorem 4.41 we prove that these operations induce well-defined homomorphisms on $K_n(X)$. We also prove that these operations satisfy the second axiom of a λ-ring, which concerns exterior powers of products (Theorem 4.45). We are unable to say whether our operations satisfy the third λ-axiom and therefore make $K_*(X)$ into a full-fledged λ-ring, but we make some remarks in this direction at the end of section 4.8. Similarly we have no comparison theorem relating our exterior power operations to those given for Quillen K-theory by other authors, but we have a partial result: in section 4.7 we show that our operations on $K_1(R)$ agree with those originally given by Quillen and Hiller in [Hil81] (here R is a commutative ring).

This chapter is essentially a preliminary version of the paper [HKT], written with the author's PhD supervisor Bernhard Köck. In particular, he provided detailed input for the scheme-theoretic parts in the final pages of this chapter. We have added a little more background than is given in that paper, but the results are the same. We also provide a proof in Appendix 4.A of a technical assumption that is merely stated in the paper.

4.1 Motivation: exterior powers and λ-rings

Let R be a commutative ring, and let M be an R-module. Recall that the ***exterior algebra*** of M, denoted $\Lambda(M)$, is the quotient of the tensor algebra $T(R) = \bigoplus_{i=0}^{\infty} M^{\otimes i}$ by the two-sided ideal generated by elements of the form $x \otimes x$. Since $T(M)$ is graded as an R-module, $\Lambda(M)$ inherits an R-module grading as well. The r^{th} ***exterior power*** of M, denoted $\Lambda^r(M)$ is the

r^{th} piece of $\Lambda(M)$ with respect to this grading.

If M is finitely generated then so is $\Lambda^r(M)$ for every r, and the exterior powers are functors $\Lambda^r\colon \mathbf{Mod}_{\mathrm{f}}(R) \to \mathbf{Mod}_{\mathrm{f}}(R)$, $r \geq 0$. Furthermore, exterior powers of projective modules are projective, so they restrict to $\Lambda^r\colon \mathbf{Proj}_{\mathrm{f}}(R) \to \mathbf{Proj}_{\mathrm{f}}(R)$.

In greater generality, let X be a quasi-compact scheme, and let $\mathbf{M}(X)$ be the category of quasi-coherent $\mathcal{O}_X$-modules. By patching together exterior powers of modules over affine subschemes of X, we can define global exterior powers $\Lambda^r\colon \mathbf{M}(X) \to \mathbf{M}(X)$. Furthermore, these restrict to $\Lambda^r\colon \mathbf{P}(X) \to \mathbf{P}(X)$, where $\mathbf{P}(X)$ is the category of locally free $\mathcal{O}_X$-modules of finite rank (vector bundles).

Since Λ^r is not exact for $r > 1$, the association $[V] \mapsto [\Lambda^r(V)]$ does not induce a self-homomorphism of $K_0(X)$. However, if $0 \to V' \to V \to V'' \to 0$ is a short exact sequence in $\mathbf{P}(X)$, then we can filter $\Lambda^r(V)$ to prove the formula $[\Lambda^r(V)] = \sum_{i=0}^{r}[\Lambda^i(V')][\Lambda^{r-i}(V'')]$ in $K_0(X)$ (the multiplication is induced by the tensor product). This formula is enough to show that the Λ^r induce well-defined operations on $K_0(X)$, and that these operations satisfy $\lambda^r(x+y) = \sum_{i=0}^{r} \lambda^i(x)\lambda^{r-i}(y)$ for all $x, y \in K_0(X)$.

With rather more work, one can show that these exterior power operations behave reasonably well with respect to products and composition: there exist universal integral polynomials P_r and $P_{r,s}$, in $2r$ and rs variables respectively, such that:

1. $[\Lambda^r(V \otimes W)] = P_r([\Lambda^1(V)], \ldots, [\Lambda^r(V)], [\Lambda^1(W)], \ldots, [\Lambda^r(W)])$
2. $[\Lambda^r(\Lambda^s(V))] = P_{r,s}([\Lambda^1(V)], \ldots, [\Lambda^{rs}(V)])$

in $K_0(X)$ for all vector bundles V and W. The definition of the polynomials P_r and $P_{r,s}$ can be found, for instance, in [AT69]. The proof that these formulas hold in $K_0(X)$ goes via the ***splitting principle*** and the ***projective bundle theorem*** (see chapters I. and V. of [FL85]). We will return to discuss these in more detail in §4.8.

To capture this general situation, Grothendieck introduced the formalism of λ-rings in an unpublished work of 1957, which he later included in [Gro58].

Definition 4.1. A ***λ-ring*** is a commutative ring K together with maps $\lambda^r\colon K \to K$, $r \geq 0$, such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$ for all $x \in K$, and such that the following axioms are satisfied for all $x, y \in K$ and all $r, s \geq 1$:

1. $\lambda^r(x+y) = \sum_{i=0}^{r} \lambda^i(x)\lambda^{r-i}(y)$,
2. $\lambda^r(xy) = P_r(\lambda^1(x), \ldots, \lambda^r(x), \lambda^1(y), \ldots, \lambda^r(y))$,
3. $\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \ldots, \lambda^{rs}(x))$,

where P_r ans $P_{r,s}$ are the universal polynomials described above.

Remark 4.2. We are using the modern terminology; in the old terminology the structure we have described is called a ***special λ-ring***, while the term λ-ring applies to a commutative ring with maps λ^r satisfying the conditions on λ^0, λ^1 and the first axiom above. In the modern terminology such a structure is called a ***pre-λ-ring***.

For X a scheme, the Grothendieck group $K_0(X)$ is a λ-ring with the λ-operations coming from the exterior powers as described above. It is natural to ask whether exterior powers exist on higher algebraic K-groups in a way that extends these operations. Several authors have studied this problem, beginning with Hiller and Quillen in [Hil81], in which it is shown that $K_*(R)$ is a λ-ring for any commutative ring R. Grayson [Gra89] has shown that the higher K-groups of any exact category with a suitable notion of exterior powers can be given λ-operations satisfying the first axiom of a λ-ring, but does not show that these satisfy the remaining axioms. He does show that his operations agree with those of Hiller–Quillen on the K-theory of a ring.

We define new exterior power operations on each the higher K-group $K_n(X)$ by producing endofunctors $\Lambda^r_n \colon (B^{\mathrm{q}})\mathbf{P}(X) \to (B^{\mathrm{q}})\mathbf{P}(X)$ that behave in the correct way with respect to short exact sequences. The construction of these endofunctors is quite involved and occupies much of sections 4.3 and 4.4, so it is worth taking a minute to outline the main idea.

Given an acyclic binary complex (of projective modules, say) $P_{\bullet}$, we wish to use the usual exterior power functor Λ^r in some way to produce a new acyclic binary complex from $P_{\bullet}$. Moreover, the way we do this should be the same across all the Λ^r, so that we have a chance at proving the usual formulas. A naive approach might be to simply apply the functor Λ^r everywhere to the binary complex $P_{\bullet}$, but this is hopeless: the result will be a binary complex, but it will not be acyclic as Λ^r is not an exact functor for $n > 1$. In fact Λ^r is not even additive, so even regular derived functors cannot help us.

Instead we define exterior powers of an acyclic chain complex in the same way that one does when describing the derived functors of Λ^r. Such complexes were introduced in [DP61], and are called ***Dold–Puppe complexes***. Dold and Puppe solved the problem of how to define the derived functors of a non-additive functor by passing to the simplicial world and back again using the procedure now known as the ***Dold–Kan correspondence***.

Let $\mathcal{A}$ be an abelian category. The Dold–Kan theorem states that the categories $C_{\geq 0}(\mathcal{A})$ and $A^{\Delta^{\mathrm{op}}}$ of non-negative chain complexes and simplicial objects in $\mathcal{A}$ are equivalent, and that the functors $N \colon \mathcal{A}^{\Delta^{\mathrm{op}}} \to C_{\geq 0}\mathcal{A}$ and $\Gamma \colon C_{\geq 0}\mathcal{A} \to \mathcal{A}^{\Delta^{\mathrm{op}}}$ realising this equivalence preserve homotopies[1].

The idea is to define Λ^r on acyclic complexes by pushing it around the Dold–Kan correspondence. Post-composition with Λ^r defines a functor $\mathcal{A}^{\Delta^{\mathrm{op}}} \to \mathcal{A}^{\Delta^{\mathrm{op}}}$, which we also denote by Λ^r, and this functor preserves homotopies (unlike the naive functor $C_{\geq 0}\mathcal{A} \to C_{\geq 0}\mathcal{A}$). The composite functor $N\Lambda^r\Gamma$ preserves homotopies, which in nice cases means that it sends

[1]Other authors (e.g., [Wei94]) use K in place of Γ; we avoid this notation for obvious reasons.

acyclic complexes to acyclic complexes. Schematically:

$$\begin{tikzcd} \text{acyclic complexes} \arrow[r, dashed] \arrow[d, "\Gamma"'] & \text{acyclic complexes} \\ \text{simplicial objects} \arrow[r, "\Lambda^r"'] & \text{simplicial objects.} \arrow[u, "N"'] \end{tikzcd}$$

The use of these Dold–Puppe complexes is the basic idea of this chapter, and it is a simple one; the implementation is considerably more complicated. To pass to categories of multicomplexes in many dimensions—and from there to categories of binary multicomplexes—requires quite a bit of work. Once that is done, it is still not straightforward to prove that our functors $\Lambda_n^r\colon (B^{\mathrm{q}})\mathbf{P}(X) \to (B^{\mathrm{q}})\mathbf{P}(X)$ induce operations on $K_n(X)$, as we then need to construct a new 'simplicial tensor product' that is compatible with these operations. This is done in section 4.5). Finally we produce results in sections 4.6, 4.7, and 4.8.

Warning 4.3. The Dold–Kan correspondence is phrased entirely in terms of chain complexes that are non-negative (those that vanish in degrees below 0). Since we use the correspondence heavily in this chapter, it will be useful to assume that all (binary) chain complexes and multicomplexes are non-negative. In particular, throughout this chapter we are working with a slightly different definition of the higher algebraic K-groups of an exact category than that of Theorem/Definition 1.40: we define $K_n\mathcal{N}$ to be the abelian group that has one generator for each bounded acyclic binary multicomplex in $\mathcal{N}$ *that is supported in non-negative degrees*, with the usual relations coming from short exact sequences and diagonal binary complexes. Keeping in mind the shifting lemma 2.5, it is not hard to believe that this is harmless. We give a proof in Appendix 4.A.
With the desire to keep burdensome notation to a minimum, in this chapter we use $C(\mathcal{N})$ to mean the category of ***non-negative*** chain complexes in $\mathcal{N}$, and similarly for $B(\mathcal{N})$. We will not always be working with complexes that are bounded above, so we use $C_{\mathrm{b}}(\mathcal{N})$ and $B_{\mathrm{b}}(\mathcal{N})$ when we are: these are the categories of bounded (binary) chain complexes in $\mathcal{N}$ that are supported in non-negative degrees. Various composites like $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{N}$ have the obvious meanings.

4.2 The Dold–Kan correspondence and other preliminaries from homological algebra

In this section we briefly recall, with few proofs, some preliminaries from homological algebra. For details of the proofs see §8 of [Wei94], for example. We give the construction of the Dold–Kan correspondence, and show how to use it to lift a non-additive functor to a functor on chain complexes that preserves contractibility. We also present some standard material linking contractibility to projective objects in the category of (bounded) chain complexes.

Recall that Δ denotes the ***simplex category***: the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are order-preserving maps. A ***simplicial object*** in a category $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$, and the natural transformations between such functors make $\mathcal{C}^{\Delta^{\mathrm{op}}}$ into a category. Equivalently, a simplicial object C in $\mathcal{C}$ can be specified to be a collection of objects C_n, $n \in \mathbb{N}$, of $\mathcal{C}$ together with face maps $\delta_i \colon C_n \to C_{n-1}$ and degeneracy maps $\sigma_j \colon C_n \to C_{n+1}$, $i, j = 0, \dots, n$, satisfying various combinatorial identities. A morphism between simplicial objects C and D is a collection of morphisms $C_n \to D_n$ that commutes with the faces and degeneracies. A homotopy $h \colon f \simeq g$ between simplicial maps $f, g \colon C \to D$ is a collection of morphisms $h_i \colon C_n \to D_{n+1}$, $i = 0, \dots, n$, which satisfies further combinatorial identities determined by compositions relating f, g, the h_i, and the faces and degeneracies of C and D.

If $F \colon \mathcal{C} \to \mathcal{D}$ is a covariant functor, then post-composition with F induces a functor between categories of simplicial objects $\mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{D}^{\Delta^{\mathrm{op}}}$. Abusing notation, we shall also call this functor F. Importantly, if $h \colon f \simeq g$ is a simplicial homotopy between $f, g \colon C \to D$, then $F(h) \colon F(f) \simeq F(g)$ is a simplicial homotopy between $F(f), F(g) \colon F(C) \to F(D)$. The Dold–Kan correspondence was introduced essentially because the analogous statement for chain homotopies between chain maps is not true if F is not additive. Let us now recall the functors realising the Dold–Kan correspondence.

Definition 4.4. Let A be a simplicial object in $\mathcal{A}$.

1. The ***associated chain complex*** $C(A)$ has objects $C(A)_n = A_n$ and differential

$$d_n = \sum_{i=0}^{n} (-1)^i \delta_i \colon C(A)_n \to C(A)_{n-1}.$$

2. The subcomplex

$$D(A)_n = \sum_{i=0}^{n} \mathrm{Im}(\sigma_i \colon A_{n-1} \to A_n)$$

is called the ***degenerate subcomplex*** of $C(A)$.

3. The ***normalized Moore complex*** $N(A)$ has objects

$$N_n(A) = A_n / D(A)_n$$

with the induced differential $\bar{d}_n$.

The associated chain complex splits globally as $C(A) = N(A) \oplus D(A)$.

The normalized Moore complex defines a functor $N \colon \mathcal{A}^{\Delta^{\mathrm{op}}} \to C(\mathcal{A})$. Writing down the inverse functor is a little more involved.

Definition 4.5. Given a chain complex $C. \in C(\mathcal{A})$, we define a simplicial object $\Gamma(C.) \in \mathcal{A}^{\Delta^{\mathrm{op}}}$ as follows.

1. **Objects**: Given $p \leq n$, let η range over all surjections $[n] \twoheadrightarrow [p]$ in Δ, and let $C_p\langle\eta\rangle$ denote a copy of C_p that is labelled by η. For each n, set

$$\Gamma(C)_n := \bigoplus_{p\leq n} \bigoplus_{\eta} C_p\langle\eta\rangle.$$

2. **Maps**: If $\alpha\colon [m] \to [n]$ is a morphism in Δ, we describe $\Gamma(\alpha)$ by describing each $\Gamma(\alpha, \eta)$, the restriction of $\Gamma(\alpha)$ to the summand $C_p\langle\eta\rangle$ of $\Gamma(C)_n$. Let

$$[m] \overset{\eta'}{\twoheadrightarrow} [q] \overset{\varepsilon}{\hookrightarrow} [p]$$

be the unique epi-monic factorisation of $\eta\alpha$. Then

$$\Gamma(\alpha,\eta) := \begin{cases} 1\colon C_p[\eta] \to C_p[\eta'] & \text{if } q = p, \\ d_p\colon C_p[\eta] \to C_{p-1}[\eta'] & \text{if } q = p-1 \text{ and } \varepsilon = \varepsilon_p, \\ 0 & \text{otherwise.} \end{cases}$$

This construction extends to a functor $\Gamma\colon C(\mathcal{A}) \to \mathcal{A}^{\Delta^{\mathrm{op}}}$.

Dold–Kan correspondence 4.6. *The functors $\Gamma\colon C(\mathcal{A}) \to \mathcal{A}^{\Delta^{\mathrm{op}}}$ and $N\colon \mathcal{A}^{\Delta^{\mathrm{op}}} \to C(\mathcal{A})$ are mutually inverse (up to natural isomorphism). Furthermore, Γ and N are exact and preserve homotopies.* □

This theorem implies that if $F\colon \mathcal{A} \to \mathcal{A}$ is a covariant functor satisfying $F(0) = 0$, then $NF\Gamma$ sends contractible chain complexes to contractible chain complexes. We are interested in acyclic complexes however, so we need to consider the relationship between acyclicity and contractibility in abelian and exact categories.

Contractible chain complexes in abelian categories are always acyclic. Let $\mathcal{N}$ be an exact subcategory of an ambient abelian category $\mathcal{A}$. The corresponding statement is not true for $\mathcal{N}$ in general without an additional assumption on the inclusion $\mathcal{N} \hookrightarrow \mathcal{A}$, for there could exist chain complexes in $C(\mathcal{N})$ that are acyclic in $C(\mathcal{A})$ but not in $C(\mathcal{N})$ (see Example 1.24). This is unnecessary for our purposes: we will only consider exact categories that are *idempotent complete*, and contractible complexes in idempotent complete exact categories are acyclic.

An acyclic chain complex in an abelian category is not usually contractible, but there is a useful criterion for contractibility. Recall that a chain complex $(C., d)$ is called ***split*** if there exist maps $s\colon C_n \to C_{n+1}$ such that $dsd = d$.

Lemma 4.7. *A chain complex in an abelian category is contractible if and only if it is acyclic and split.* □

For an abelian category $\mathcal{A}$, let $\mathbf{Proj}(\mathcal{A})$ denote the full subcategory of projective objects of $\mathcal{A}$. All short exact sequences of projective objects split, so a chain complex of objects of $\mathbf{Proj}(\mathcal{A})$ is contractible if and only if it is also in $C^{\mathrm{q}}(\mathcal{A})$. These turn out to be exactly the projective objects of $C(\mathcal{A})$, see the next lemma. This observation is crucial for our inductive proof of the fact that exterior powers of multi-dimensional acyclic complexes of projective objects are again acyclic, see Proposition 4.13 and Corollary 4.16.

Lemma 4.8. *For a chain complex $P_{\cdot}$ in $C(\mathcal{A})$, the following are equivalent.*

1. *$P_{\cdot}$ is in $C^{\mathrm{q}}(\mathbf{Proj}(\mathcal{A}))$.*
2. *$P_{\cdot}$ is a contractible complex of projectives.*
3. *$P_{\cdot}$ is a projective object of $C(\mathcal{A})$.* □

The category $(C^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A}))$ is the full subcategory of the abelian category $C^n(\mathcal{A})$ whose objects are the n-dimensional multicomplexes of projectives in $\mathcal{A}$, and whose differential in each direction is acyclic.

Lemma 4.9. *The category $\mathbf{Proj}(C^n(\mathcal{A}))$ is exactly $(C^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A}))$.*

Proof. The proof is by induction on n. The case $n = 1$ is the equivalence of statements (1) and (3) in Lemma 4.8. The inductive step is the chain of equalities

$$\begin{array}{rclcl} \mathbf{Proj}(C^{n+1}(\mathcal{A})) & = & \mathbf{Proj}(C(C^n(\mathcal{A}))) & & \\ & = & C^{\mathrm{q}}(\mathbf{Proj}(C^n(\mathcal{A}))) & = & C^{\mathrm{q}}((C^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A}))) \\ & & & = & (C^{\mathrm{q}})^{n+1}(\mathbf{Proj}(\mathcal{A})). \end{array}$$ □

We conclude our preliminaries by discussing functors of finite degree.

Definition 4.10. For any functor between abelian categories $F\colon \mathcal{A} \to \mathcal{B}$ that satisfies $F(0) = 0$, there is a functorial decomposition

$$F(X \oplus Y) = F(X) \oplus \mathrm{cr}_2(F)(X, Y) \oplus F(Y),$$

where $\mathrm{cr}_2(F)\colon \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is the second ***cross-effect functor*** (see [EML54]), which is defined as the kernel of the obvious projection $F(X \oplus Y) \to F(X) \oplus F(Y)$. The functor F is said to have ***degree*** ≤ 1 if it is additive (i.e., if $\mathrm{cr}_2(F)$ vanishes), and we say that F has degree $\leq d$ if $\mathrm{cr}_2(F)(X, Y)$ is of degree $\leq d - 1$ in each argument. If F is of degree $\leq d$, then F is of degree $\leq d'$ for all $d' \geq d$. We say that F ***has degree*** d if it has degree $\leq d$ but does not have degree $\leq d - 1$.

Example 4.11. For R a commutative ring, the exterior power $\Lambda^r\colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ has degree r for each $r > 0$. This follows from the canonical decomposition

$$\Lambda^r(X \oplus Y) \cong \Lambda^r(X) \oplus \left(\bigoplus_{i=1}^{r-1} \Lambda^{r-i}(X) \otimes \Lambda^i(Y) \right) \oplus \Lambda^r(Y).$$

If $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories, and if $P_\cdot$ is a bounded complex, then $NF\Gamma(P_\cdot)$ is certainly bounded again. This also holds true for functors of finite degree, as the following lemma shows.

Lemma 4.12 ([SK10], Corollary 4.6). *Let $P_\cdot$ be a chain complex in $C(\mathcal{A})$ of length ℓ, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of degree d between abelian categories. Then $NF\Gamma(P_\cdot)$ has length less than or equal to $d\ell$.* □

4.3 Operations on acyclic complexes

In this rather abstract section we describe how to use the Dold–Kan correspondence to lift a functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories to functors between categories of multicomplexes $F_n\colon C^n(\mathcal{A}) \to C^n(\mathcal{B})$, $n \geq 1$. By imposing suitable restrictions on F, we show that the lifted functors F_n send acyclic multicomplexes to acyclic multicomplexes and restrict to functors between the categories of projective objects in $C^n(\mathcal{A})$ and $C^n(\mathcal{B})$. We also show that if F is of finite degree, then each F_n preserves bounded multicomplexes and is also of finite degree.

Proposition 4.13. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a covariant functor between abelian categories with $F(0) = 0$. Let $F_1 := NF\Gamma\colon C(\mathcal{A}) \to C(\mathcal{B})$ denote the induced functor. Then the following statements hold true:*

1. *$F_1(0) = 0$.*
2. *F_1 sends contractible complexes to contractible complexes.*
3. *If F sends projectives to projectives, then so does F_1.*
4. *If F is of degree at most d, then F_1 sends bounded complexes to bounded complexes and F_1 is again of degree at most d.*

Proof. Part (1) is obvious.

For part (2), the functors $\Gamma\colon C(\mathcal{A}) \to \mathcal{A}^{\Delta^{\mathrm{op}}}$ and $N\colon \mathcal{B}^{\Delta^{\mathrm{op}}} \to C(\mathcal{B})$ preserve homotopies and send 0 to 0, so they both send contractible objects to contractible objects. Furthermore, F sends homotopies in $\mathcal{A}^{\Delta^{\mathrm{op}}}$ to homotopies in $\mathcal{B}^{\Delta^{\mathrm{op}}}$—if $h\colon f \sim g$ is a homotopy, then $F(h)\colon F(f) \sim F(g)$ is a homotopy. Since F also has the property that $F(0) = 0$, we see that if $A \simeq 0$ in $\mathcal{A}^{\Delta^{\mathrm{op}}}$, then $F(A) \simeq F(0) = 0$. Therefore $NF\Gamma(P_\cdot)$ is contractible in $C(\mathcal{A})$.

For part (3), let $P_\cdot$ be a projective object of $C(\mathcal{A})$. By Lemma 4.8, $P_\cdot$ is an acyclic complex of projectives, that is $P_\cdot$ is an object of $C^{\mathrm{q}}(\mathbf{Proj}(\mathcal{A}))$. Furthermore $P_\cdot$ is contractible. Therefore $F_1(P_\cdot)$ is contractible, by part (2). Contractible complexes are acyclic, so to show that $F_1(P_\cdot)$ is projective in $C(\mathcal{B})$ it suffices by Lemma 4.8 again to show that each $F_1(P)_n = NF\Gamma(P_\cdot)_n$ is projective. The objects of $\Gamma(P_\cdot)$ are projective as they are direct sums of the objects of $P_\cdot$, which are projective. Applying F, we have $(F\Gamma(P_\cdot))_n := F(\Gamma(P_\cdot)_n)$ for each $n \geq 0$, and F sends projectives to projectives, so the objects of $F\Gamma(P_\cdot)$ are projective as well. The complex

$NF\Gamma(P_\cdot)$ is a direct summand of the unnormalised Moore complex $CF\Gamma(P_\cdot)$, so $NF\Gamma(P_\cdot)_n$ is a direct summand of $CF\Gamma(P_\cdot)_n$. But $CF\Gamma(P_\cdot)_n$ is equal to $F\Gamma(P_\cdot)_n$, which, as we have just seen, is projective. Direct summands of projectives are projective, so the result follows.
Finally we consider part (4). The first part of this statement is Lemma 4.12. For the second part we note that, since N and Γ are additive, it is enough to show that $F\colon \mathcal{A}^{\Delta^{\mathrm{op}}} \to \mathcal{B}^{\Delta^{\mathrm{op}}}$ is of degree $\leq d$. This can be easily proved by induction on d. $\square$

The following example shows that we cannot expect the complex $NF\Gamma(C_\cdot)$ to be exact in general if we assume only that $C_\cdot$ is exact, i.e., that $C_\cdot$ is quasi-isomorphic rather than homotopy equivalent to the zero complex.

Example 4.14. Let F be the degree 2 endofunctor $A \mapsto A^{\otimes 2}$ on the abelian category of abelian groups, and let $C_\cdot$ be the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

considered as an acyclic complex concentrated in degrees 0, 1 and 2. Then $NF\Gamma(C_\cdot) = N\,\mathrm{diag}(\Gamma(C_\cdot)\otimes\Gamma(C_\cdot))$ is homotopy equivalent to $\mathrm{Tot}(C_\cdot \otimes C_\cdot)$ by the Eilenberg–Zilber theorem ([May92], §29)[2]. But one can check that the homology group $H_2(\mathrm{Tot}(C_\cdot \otimes C_\cdot))$ is $\mathbb{Z}/2\mathbb{Z}$, so $NF\Gamma(C_\cdot)$ is not exact. Furthermore, the short exact sequence

$$0 \longrightarrow N\Lambda^2\Gamma(C_\cdot) \longrightarrow NF\Gamma(C_\cdot) \longrightarrow N\,\mathrm{Sym}^2\,\Gamma(C_\cdot) \longrightarrow 0$$

shows that at least one of $N\Lambda^2\Gamma(C_\cdot)$ or $N\,\mathrm{Sym}^2\,\Gamma(C_\cdot)$ is not exact either.

We now describe induced functors on categories of acyclic multicomplexes.

Definition 4.15. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a covariant functor between abelian categories. We define functors

$$F_n\colon C^n(\mathcal{A}) \to C^n(\mathcal{B})$$

for all $n \geq 0$ recursively as follows:

1. $F_0 := F\colon \mathcal{A} \to \mathcal{B}$.

2. By regarding an object of $C^{n+1}(\mathcal{A})$ as a chain complex in the abelian category $C^n(\mathcal{A})$, we define $F_{n+1} := NF_n\Gamma$.

Corollary 4.16. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a covariant functor, which sends projectives of $\mathcal{A}$ to projectives of $\mathcal{B}$. Suppose further that $F(0) = 0$. Then for $n \geq 0$ the functors of Definition 4.15 restrict to functors*

$$F_n\colon (C^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A})) \to (C^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{B})).$$

[2] See also Definition 4.30, Lemma 4.31.

Furthermore, if F is of finite degree, then F_n sends bounded multicomplexes to bounded multicomplexes. That is, each F_n restricts to a functor

$$F_n\colon (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{A})) \to (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{B})).$$

Proof. We consider the unbounded case first. We use the equality

$$(C^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A})) = \mathbf{Proj}(C^n(\mathcal{A})),$$

which was shown in Lemma 4.9. We need to show that for each n, the functor F_n sends projectives of $C^n(\mathcal{A})$ to projectives of $C^n(\mathcal{B})$. As $F_n(0) = 0$ for all n by Proposition 4.13 (1), this immediately follows from Proposition 4.13 (3) by induction on n.
If $F_0 = F$ is of finite degree, then the same induction over n shows that F_n is of finite degree for every n, by Proposition 4.13 (4). In particular, for each $n \geq 1$, the functor $F_n = (F_{n-1})_1$ sends bounded complexes to bounded complexes, that is, it restricts to a functor

$$F_n\colon C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}})^{n-1}(\mathbf{Proj}(\mathcal{A}))) \to C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}})^{n-1}(\mathbf{Proj}(\mathcal{B}))).$$

But we can say more: considering $P_.$ in $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{A}))$ as a chain complex, each of its objects is in $(C^{\mathrm{q}}_{\mathrm{b}})^{n-1}(\mathbf{Proj}(\mathcal{A}))$, i.e., they are bounded. We claim that the objects of $F_n(P_.) = NF_{n-1}\Gamma(P_.)$ are also bounded. The objects of $\Gamma(P_.)$ are finite direct sums of the objects of $P_.$. Finite sums of bounded objects are bounded, so the objects of $\Gamma(P_.)$ are bounded. Therefore, by the inductive hypothesis, the objects of $F_{n-1}\Gamma(P_.)$ are also bounded. Finally, the objects of $NF_{n-1}\Gamma(P_.)$ are direct summands of the objects of $F_{n-1}\Gamma(P_.)$ (Definition 4.4), so they are bounded as well. Therefore F_n sends bounded chain complexes of bounded objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{A}))$ to bounded chain complexes of bounded objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{B}))$. This is exactly the statement that F_n restricts to a functor

$$F_n\colon (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{A})) \to (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{B})),$$

which was to be proved. □

Remark 4.17. Throughout this chapter we work with the inductive definition of $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{N})$, that is $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{N}) := C^{\mathrm{q}}_{\mathrm{b}}(C^{\mathrm{q}}_{\mathrm{b}})^{n-1}(\mathcal{N})$ for $n > 1$. As explained in Remark 1.38, one can think of objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{N})$ as $\mathbb{Z}^n_{\geq 0}$-graded objects of $\mathcal{N}$ (together with certain differentials) without specifying the order of directions in which the objects have been obtained in the inductive definition. The purpose of this remark is to convince the reader that the our construction of the functors F_n given in this section (and hence our construction of exterior powers in the sequel) does not depend on the order of directions either. Rather than including a complete proof, we sketch the idea in the case $n = 2$. Let $F = F_0$ be as before. The functor F_2 is defined as

$$N_h F_1 \Gamma_h = N_h N_v F_1 \Gamma_v \Gamma_h,$$

where the indices h and v indicated the horizontal and vertical directions respectively. It is easy to see that the composition $N_h N_v$ sends a bisimplicial object C to the double complex whose objects are obtained from the corresponding objects of C by dividing out the images of all of the horizontal and vertical degeneracy maps. This latter description of course does not depend on the order of N_h and N_v. One can show that the same holds for Γ_h and Γ_v by a similar argument, or just by recalling that Γ_h and Γ_v are adjoint to N_h and N_v respectively.

We can now describe the higher exterior power functors that we will use to induce operations on higher K-groups. The following example is the motivation for our work so far.

Main Example 4.18. Recall that $\mathbf{Mod}(R)$ is the category of modules over a commutative ring R, and that $\mathbf{Proj}(R)$ is the full subcategory of projective modules. For each $r > 0$, the usual exterior power functor $\Lambda^r\colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ satisfies the hypotheses of Corollary 4.16 (Λ^r has degree r). We therefore have induced functors

$$\Lambda^r_n\colon (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(R)) \to (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(R))$$

for all $n \geq 0$.

In general, the complex $N\Lambda^r\Gamma(P_.)$ is difficult to write down explicitly. The paper [SK10] gives an algorithm that addresses this problem. We conclude this section by computing $N\Lambda^r\Gamma(P_.)$ for a very simple choice of $P_.$.

Example 4.19. Let $\varphi\colon P \to Q$ be an isomorphism of invertible modules over some commutative ring R, considered as an acyclic complex concentrated in degrees 0 and 1:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{\varphi} & Q & \longrightarrow & 0, \\ {\scriptstyle 2} & & {\scriptstyle 1} & & {\scriptstyle 0} & & {\scriptstyle -1} \end{array}$$

or $P \xrightarrow{\varphi} Q$ for short. In Lemma 2.2 of [Köc01], the author's PhD supervisor gives an explicit calculation of $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ in terms of higher cross-effect functors (in fact, he does this for more general P, Q and φ). Specifically, in degree n we have:

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \mathrm{cr}_n(\Lambda^r)(P,\dots,P) \oplus \mathrm{cr}_{n+1}(\Lambda^r)(Q,P,\dots,P).$$

We do not wish to expound on the theory of cross-effect functors here: the interested reader can see [EML54] or section 1 of [Köc01]. Instead we merely quote the properties of $\mathrm{cr}_n(\Lambda^r)$ that we need. Firstly, $\mathrm{cr}_n(\Lambda^r) = 0$ for $n > r$, as Λ^r is of degree r; secondly, $\mathrm{cr}_r(\Lambda^r)(P_1,\dots,P_r) = P_1 \otimes \cdots \otimes P_r$; thirdly, if $n < r$ and if $P_1,\dots,P_n$ are all invertible, then $\mathrm{cr}_n(\Lambda^r)(P_1,\dots,P_n) = 0$. From these we see that:

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \begin{cases} P^{\otimes r} & \text{if } n = r, \\ Q \otimes P^{\otimes(r-1)} & \text{if } n = r-1, \\ 0 & \text{otherwise.} \end{cases}$$

We can also read off the differential $P^{\otimes r} \to Q \otimes P^{\otimes(r-1)}$ from Lemma 2.2 of *loc. cit.*: it is $\mathrm{cr}_r(\Lambda^r)(\varphi, 1, \dots, 1) = \varphi \otimes 1 \otimes \cdots \otimes 1$. So $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ is the acyclic complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & P \otimes P^{\otimes(r-1)} & \xrightarrow{\varphi\otimes 1} & Q \otimes P^{\otimes(r-1)} & \longrightarrow & 0. \\ {\scriptstyle r+1} & & {\scriptstyle r} & & {\scriptstyle r-1} & & {\scriptstyle r-2} \end{array}$$

Of particular note is the special case in which P and Q are equal to R considered as a module over itself, and φ is given by multiplication by some $x \in R^\times$. Then $N\Lambda^r\Gamma(R \xrightarrow{x} R)$ is equal to the complex $(R \xrightarrow{x} R)$ itself, shifted so that it is concentrated in degrees r and $r-1$.

4.4 Operations on binary multicomplexes

The goal of the first part this section is to extend the functors F_n between multicomplexes of the previous section to functors of *binary* multicomplexes. In the second part we specialise to the exterior powers $\Lambda^r \colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ and show that the higher exterior powers Λ^r_n send multicomplexes with finitely generated objects to multicomplexes with finitely generated objects. Together with the results of the previous section, this shows that if $P_.$ is a generator of $K_n(R)$ (in Grayson's presentation), then so is $\Lambda^r_n(P_.)$.

Categories of binary complexes are not so well behaved as categories of complexes. In particular, a bounded acyclic binary complex whose objects are projective is not necessarily a projective object in the category of all binary complexes, as shown in the following example.

Example 4.20. Let P be a projective object in an abelian category $\mathcal{A}$. The following diagram is an epimorphism in the category of bounded acyclic binary complexes:

$$\begin{tikzcd} P \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] \arrow[d, "1"] & P \oplus P \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\Sigma"] & P \arrow[d] \\ P \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & P \arrow[r, shift left] \arrow[r, shift right] & 0 \end{tikzcd}$$

(where i_1 and i_2 are the inclusions into the first and second summands, p_1 and p_2 are the corresponding projections, and $\Sigma = p_1 + p_2$). But there is no splitting $P \to P \oplus P$ that commutes with the both the top and bottom differentials, so the bottom row cannot be a projective object of $B^{\mathrm{q}}_{\mathrm{b}}(\mathcal{A})$.

This difficulty means that we cannot *a priori* recursively define exterior powers of binary multicomplexes in exactly the way we have for multicomplexes. This problem is resolvable: we shall show that if $P_.$ is an object of $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(R))$, then the objects of $\Lambda^r_n(P_.)$ are independent of the differentials of $P_.$. Therefore it will make sense to define the exterior power of a binary complex by applying the exterior powers we developed above individually to the two differentials of the binary complex. The resulting pair of complexes will have the same objects, so we consider them as a binary complex.

Lemma 4.21. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a covariant functor between abelian categories. If $P_.$ and $Q_.$ are chain complexes with the same underlying graded object, then $NF\Gamma(P_.)$ and $NF\Gamma(Q_.)$ have the same underlying graded object.*

Proof. Let $B \in \mathcal{B}^{\Delta^{\mathrm{op}}}$ be a simplicial object. The objects of the complex $N(B)$ are given by

$$N(B)_n := B_n \Big/ \left(\sum_{i=0}^{n} \mathrm{Im}(\sigma_i\colon B_{n-1} \to B_n) \right),$$

where the σ_i are the degeneracies of B. It is enough therefore to show that the objects and degeneracy maps of $F\Gamma(P_.)$ do not depend upon the differential of $P_.$. The objects of $\Gamma(P_.)$ are direct sums of the objects of $P_.$, indexed by the surjections out of $[n]$ in Δ, and do not depend on the differential. The degeneracy operator $\sigma_i\colon \Gamma(P_.)_{n-1} \to \Gamma(P_.)_n$ is the image of the degeneracy map $\eta_i\colon [n] \to [n-1]$ in Δ. For any surjection $\eta\colon [n-1] \twoheadrightarrow [p]$, the composition $\eta\eta_i$ is also a surjection, so the monomorphism in the epi-monic factorisation of $\eta\eta_i$ is just the identity on $[p]$. Therefore, the degeneracy operator σ_i acts on $\Gamma(P_.)_{n-1}$ by sending the summand corresponding to the surjection η by the identity to the summand of $\Gamma(P_.)_n$ corresponding to the surjection $\eta_i\eta$. Thus σ_i obviously does not depend on the differential of $P_.$. Since the objects and degeneracies of $\Gamma(P_.)$ only depend on the underlying graded object of $P_.$, the same is true of $F\Gamma(P_.)$. Therefore the objects of $NF\Gamma(P_.)$ only depend on the underlying graded object as well. □

Corollary 4.22. *Let $n \geq 1$, and let $P_.$, $Q_.$ be objects of $(C_{\mathrm{b}}^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A}))$. If $P_.$ and $Q_.$ have the same underlying $\mathbb{Z}^n$-graded object, then $F_n(P_.)$ and $F_n(Q_.)$ have the same underlying $\mathbb{Z}^n$-graded object.*

Proof. This is a straightforward induction on n. □

We are now ready at last to define exterior powers of acyclic binary multicomplexes. Let $P_.$ be an n-dimensional, bounded, acyclic binary multicomplex of projective modules, i.e., an object of $(B_{\mathrm{b}}^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A}))$. We view the commutativity constraints on the differentials of $P_.$ in the same way as described in Remark 1.38: as a collection of 2^n objects of $(C_{\mathrm{b}}^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A}))$.

Definition 4.23. For a functor F that satisfies the hypotheses of Corollary 4.16, we define induced functors

$$F_n\colon (B_{\mathrm{b}}^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A})) \longrightarrow (B_{\mathrm{b}}^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{B}))$$

by the following procedure. Let $P_.$ be an object of $(B_{\mathrm{b}}^{\mathrm{q}})^n(\mathbf{Proj}(\mathcal{A}))$, viewed as a collection of 2^n (non-binary) multicomplexes in the manner described above. Since these multicomplexes all have the same underlying $\mathbb{Z}^n$-graded object, the same is true of the 2^n multicomplexes obtained by applying F_n (the functor defined on $(C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(\mathcal{A}))$ in Corollary 4.16) to the multicomplexes describing $P_.$ by Corollary 4.22. We define $F_n(P_.)$ to be the binary multicomplex described by the resulting collection of multicomplexes.

We now turn to our main example of interest: the exterior power functors. For the rest of this section, let R be a commutative ring. Recall that $\mathbf{Mod}(R)$ is the abelian category of all R-modules, and $\mathbf{Proj}(R) = \mathbf{Proj}(\mathbf{Mod}(R))$ is the category of projective R-modules.

Example 4.24. We have seen in 4.18 that the usual exterior power operations Λ^r satisfy the hypotheses of 4.16, so the higher exterior powers

$$\Lambda^r_n\colon (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(R)) \to (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(R))$$

lift to exterior powers of binary multicomplexes

$$\Lambda^r_n\colon (B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(R)) \to (B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}(R))$$

for all $n \geq 0$.

Recall that the K-groups of R are defined to be the K-groups of the exact category $\mathbf{Proj}_{\mathrm{f}}(R)$ of ***finitely generated*** projective R-modules. Our goal is to use a functor $(B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}_{\mathrm{f}}(R)) \to (B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}_{\mathrm{f}}(R))$ to induce exterior power maps $\lambda^r\colon K_n(R) \to K_n(R)$. We would like to show that the exterior power endofunctors defined in Example 4.24 restrict to endofunctors:

$$\Lambda^r_n\colon (B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}_{\mathrm{f}}(R)) \to (B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}_{\mathrm{f}}(R)).$$

If the ring R is Noetherian this is not a problem. The category $\mathbf{Mod}_{\mathrm{f}}(R)$ of finitely generated R-modules is abelian, and its subcategory of projective objects is $\mathbf{Proj}_{\mathrm{f}}(R)$, so since the usual exterior powers send finitely generated modules to finitely generated modules, we obtain the desired operations by applying Definition 4.23 to the endofunctors $\Lambda^r\colon \mathbf{Mod}_{\mathrm{f}}(R) \to \mathbf{Mod}_{\mathrm{f}}(R)$. If R is not Noetherian, then the category $\mathbf{Mod}_{\mathrm{f}}(R)$ is not abelian, and we need an additional lemma. We call an object of $C^n(\mathbf{Mod}(R))$ (or of $B^n(\mathbf{Mod}(R))$) ***finitely generated*** if each of the objects of its underlying $\mathbb{Z}^n$-graded R-module is finitely generated.

Lemma 4.25. *Let $F\colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ be a functor that sends finitely generated modules to finitely generated modules. Then for every n, the functor $F_n\colon C^n(\mathbf{Mod}(R)) \to C^n(\mathbf{Mod}(R))$ (resp. $F_n\colon B^n(\mathbf{Mod}(R)) \to B^n(\mathbf{Mod}(R))$) sends finitely generated multicomplexes to finitely generated multicomplexes (resp. binary multicomplexes).*

Proof. The proof is by induction on n. The base case is tautological, as $F_0 = F$, so suppose that $F_n\colon C^n(\mathbf{Mod}(R)) \to C^n(\mathbf{Mod}(R))$ preserves finitely generated multicomplexes. Viewing an object $M_.$ of $C^{n+1}(\mathbf{Mod}(R))$ as a chain complex of objects of $C^n(\mathbf{Mod}(R))$, one sees that $M_.$ is finitely generated if and only if each of its constituent objects M_i is finitely generated. Now the functors N and Γ send chain complexes with finitely generated objects to simplicial objects with finitely generated objects, and *vice versa*, since direct sums and direct summands of finitely generated modules are finitely generated, and the induced functor $F_n\colon C^n(\mathbf{Mod}(R))^{\Delta^{\mathrm{op}}} \to C^n(\mathbf{Mod}(R))^{\Delta^{\mathrm{op}}}$ preserves simplicial objects with finitely generated objects by the hypothesis on F_n. So $F_{n+1} := NF_n\Gamma$ preserves finitely generated multicomplexes. □

As noted above, the exterior power functors $\Lambda^r\colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ preserve finitely generated objects, so the following corollary is immediate.

Corollary 4.26. *For $r > 0$ and $n \geq 0$, the functor Λ^r_n of Example 4.24 restricts to a functor*

$$\Lambda^r_n\colon (B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}_{\mathrm{f}}(R)) \to (B^{\mathrm{q}}_{\mathrm{b}})^n(\mathbf{Proj}_{\mathrm{f}}(R)).$$

4.5 Simplicial tensor products

In this long section we develop a tensor product for multicomplexes that is compatible with the exterior powers we have defined using the Dold–Kan correspondence. This 'simplicial tensor product' is proved to have all the desired properties: it preserves acyclicity, boundedness, and projective objects. We show that the class of this product vanishes in the appropriate K-group when lifted to binary complexes in a suitable exact category $\mathcal{P}$, which will eventually be the key to showing that higher exterior power operations are homomorphisms on higher K-groups. Although we are ultimately interested in the products induced from the usual tensor products of modules (or sheaves), it is convenient in this section to work in the rather more abstract setting of a generic abelian category with some form of well-behaved tensor product.

Definition 4.27. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{P} \subset \mathcal{A}$ be a full subcategory such that $\mathcal{P}$ is closed under extensions (so $\mathcal{P}$ is an exact category), and closed under kernels of epimorphisms in $\mathcal{A}$ between objects of $\mathcal{P}$ (i.e., in the sense of Definition 1.5). In particular, $\mathcal{P}$ is closed under direct summands. We say that a bi-additive bi-functor $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a ***tensor product*** on the pair $(\mathcal{A}, \mathcal{P})$ if the following hold:

1. if P is in $\mathcal{P}$, then $P \otimes -$ and $- \otimes P$ are exact functors on $\mathcal{A}$
2. if P and Q are in $\mathcal{P}$, then so is $P \otimes Q$.

For the rest of this section, we fix a pair $(\mathcal{A}, \mathcal{P})$ and a tensor product $\otimes$ on that pair. The reader may wish to keep in mind the example $(\mathcal{A}, \mathcal{P}) = (\mathbf{Mod}(R), \mathbf{Proj}_{\mathrm{f}}(R))$, with the usual tensor product of R-modules.

Definition 4.28. Let A be an object of $\mathcal{A}$, and let $(C_., d_C), (D_., d_D)$ be chain complexes in $\mathcal{A}$.

1. By $A \otimes C_\cdot$ we mean the chain complex whose i^{th} object is $A \otimes C_i$, with differential $1 \otimes d_C$. The complex $C_\cdot \otimes A$ is defined analogously.

2. By $\operatorname{Tot}(C_\cdot \otimes D_\cdot)$ we mean the chain complex formed by taking the total complex of the bicomplex whose $(i,j)^{\text{th}}$ object is $C_i \otimes D_j$, and whose differentials are $d^{\text{ver}} = d_C \otimes 1$ and $d^{\text{hor}} = 1 \otimes d_D$. This bicomplex's i^{th} row is $C_i \otimes D_\cdot$ and its j^{th} column is $C_\cdot \otimes D_j$

It is clear that if $C_\cdot$ and $D_\cdot$ are bounded complexes, then the products $A \otimes C_\cdot$ and $\operatorname{Tot}(C_\cdot \otimes D_\cdot)$ are bounded as well. We'll need a couple of properties of these products.

Lemma 4.29. *Let $P_\cdot$ be a chain complex in $\mathcal{A}$ whose objects are in $\mathcal{P}$ (i.e., $P_\cdot$ is in $C(\mathcal{P})$).*

1. *The functor*
$$\begin{array}{rccc} P_\cdot \otimes -: & \mathcal{A} & \to & C(\mathcal{A}) \\ & A & \mapsto & P_\cdot \otimes A \end{array}$$
is exact.

2. *If $C_\cdot$ is an acyclic complex in $\mathcal{A}$, then the complex* $\operatorname{Tot}(P_\cdot \otimes C_\cdot)$ *is acyclic.*

Proof. The first part is straightforward. If $C_\cdot$ is acyclic, then each of the complexes $P_n \otimes C_\cdot$ is acyclic, since acyclic complexes are spliced together from short exact sequences. Therefore the rows of of the bicomplex $P_\cdot \otimes C_\cdot$ are acyclic. Since our complexes are non-negative, the second statement now follows by the acyclic assembly lemma ([Wei94], 2.7.3). □

To define the simplicial tensor product of complexes we need to go beyond regular simplicial objects. A ***bisimplicial object*** B in $\mathcal{A}$ is a functor $B\colon \Delta^{\text{op}} \times \Delta^{\text{op}} \to \mathcal{A}$. The ***diagonal*** of B is the simplicial object defined by pre-composition with the usual diagonal functor $\operatorname{diag}\colon \Delta^{\text{op}} \to \Delta^{\text{op}} \times \Delta^{\text{op}}$:
$$\operatorname{diag}(B) := B \circ \operatorname{diag}\colon \Delta^{\text{op}} \to \Delta^{\text{op}} \times \Delta^{\text{op}} \to \mathcal{A}.$$

If C and D are simplicial objects in $\mathcal{A}$, then we define $C \otimes D$ to be the bisimplicial object given by $(C \otimes D)([m],[n]) = C_m \otimes D_n$ and $(C \otimes D)(\alpha, \beta) = C(\alpha) \otimes D(\beta)$ for $\alpha\colon [m] \to [m']$, $\beta\colon [n] \to [n']$. We can now push the tensor product around the Dold–Kan correspondence.

Definition 4.30. The ***simplicial tensor product*** of chain complexes $C_\cdot$ and $D_\cdot$ in $\mathcal{A}$ is defined to be
$$C_\cdot \otimes_\Delta D_\cdot := N(\operatorname{diag}(\Gamma(C_\cdot) \otimes \Gamma(D_\cdot))).$$

A word of warning here: although the tensor product is an additive functor in each variable, the complex $P_\cdot \otimes_\Delta Q_\cdot$ is not equal to the product complex $\operatorname{Tot}(P_\cdot \otimes Q_\cdot)$ discussed above. They are related by the Eilenberg–Zilber theorem, which we shall use in the proof of the following lemma.

Lemma 4.31. *Let $P_\cdot$, $Q_\cdot$ be chain complexes in $\mathcal{A}$. Suppose that $P_\cdot$ and $Q_\cdot$ have objects in $\mathcal{P}$, and that at least one of them is acyclic. Then $P_\cdot \otimes_\Delta Q_\cdot$ is acyclic and has objects in $\mathcal{P}$.*

Proof. We suppose, without loss of generality, that $Q_.$ is acyclic. By the Eilenberg–Zilber theorem ([May92] §29), the simplicial tensor product $P_. \otimes_\Delta Q_. = N\operatorname{diag}(\Gamma(P_.)\otimes\Gamma(Q_.))$ is homotopy equivalent to $\operatorname{Tot}(P_. \otimes Q_.)$, and is therefore acyclic by Lemma 4.29 (2). It is straightforward to see that $P_. \otimes_\Delta Q_.$ has objects in $\mathcal{P}$: the objects of $\Gamma(P_.)$ and $\Gamma(Q_.)$ are direct sums of objects of $\mathcal{P}$, so they are in $\mathcal{P}$; the subcategory $\mathcal{P}$ is closed under tensor products, as assumed in Definition 4.27 (2), so the objects of $\Gamma(P_.)\otimes\Gamma(Q_.)$ are in $\mathcal{P}$; finally, $\mathcal{P}$ is closed under taking direct summands, so $N\operatorname{diag}(\Gamma(P_.)\otimes\Gamma(Q_.))$ has objects in $\mathcal{P}$. □

Unlike for the total-complex tensor product, it is not immediately obvious that the simplicial tensor product of a pair of bounded multicomplexes is bounded. One cannot apply Lemma 4.12 without modifications, as $\operatorname{diag}(-\otimes-)$ is a bi-functor. Fortunately, the result is fairly straightforward from the definition.

Lemma 4.32. *If $P_.$ and $Q_.$ are both bounded chain complexes in $\mathcal{A}$, then $P_. \otimes_\Delta Q_.$ is bounded as well.*

Proof. Examining the Dold–Kan functors applied to a tensor product, one sees that the object $(P_. \otimes_\Delta Q_.)_n$ is equal to

$$N(\operatorname{diag}(\Gamma(P_.)\otimes\Gamma(Q_.)))_n = \bigoplus_{\varphi} P_i \otimes Q_j,$$

where φ runs over all injections $[n] \hookrightarrow [i]\times[j]$ whose composition with the projections onto $[i]$ and $[j]$ gives surjections $[n] \twoheadrightarrow [i]$ and $[n] \twoheadrightarrow [j]$ (this is derived in [Law12]). The complexes $P_.$ and $Q_.$ are bounded, so there exist k and l such that $P_i = 0$ and $Q_j = 0$ for all $i > k$ and $j > l$. But for sufficiently large n, there are no injections $[n] \hookrightarrow [i]\times[j]$ with $i \le k$ and $j \le l$, so $(P_. \otimes_\Delta Q_.)_n = 0$ for all n large enough. □

We now verify that $\otimes_\Delta$ is a tensor product in the sense of Definition 4.27.

Proposition 4.33. *The simplicial tensor product $\otimes_\Delta$ is a tensor product on the pair $(C(\mathcal{A}), C^{\mathrm{q}}_{\mathrm{b}}(\mathcal{P}))$.*

Proof. Extensions and kernels of epimorphisms between bounded acyclic chain complexes are again bounded and acyclic, so the inclusion $C^{\mathrm{q}}_{\mathrm{b}}(\mathcal{P}) \subset C(\mathcal{A})$ satisfies the necessary hypotheses. If $P_.$ and $Q_.$ are in $C^{\mathrm{q}}_{\mathrm{b}}(\mathcal{P})$, then so is $P_. \otimes_{\Delta,n} Q_.$, by Lemmas 4.31 and 4.32.
So it remains to show that $-\otimes_\Delta-$ is bi-additive, and that $P_.\otimes_\Delta-$ and $-\otimes_\Delta P_.$ are exact functors when $P_.$ is in $C^{\mathrm{q}}_{\mathrm{b}}(\mathcal{P})$. Since N and Γ are both additive and exact, we inspect $\operatorname{diag}(-\otimes-)$. Clearly for any simplicial objects A_1, A_2, B_1 and B_2 we have

$$\operatorname{diag}(A_1 \otimes (B_1 \oplus B_2)) \cong \operatorname{diag}(A_1 \otimes B_1) \oplus \operatorname{diag}(A_1 \otimes B_2)$$

and similarly for $\operatorname{diag}((A_1 \oplus A_2)\otimes B_2)$, so $\otimes_\Delta$ is bi-additive.
Let P be a simplicial object such that P_n is in $\mathcal{P}$ for every n. For a short exact sequence of

simplicial objects $0 \to A' \to A \to A'' \to 0$, the sequence

$$0 \to \mathrm{diag}(P \otimes A')_n \to \mathrm{diag}(P \otimes A)_n \to \mathrm{diag}(P \otimes A'')_n \to 0$$

is equal to

$$0 \to P_n \otimes A'_n \to P_n \otimes A_n \to P_n \otimes A''_n \to 0,$$

which is short exact since each $0 \to A'_n \to A_n \to A''_n \to 0$ is short exact and P_n is in $\mathcal{P}$. So the sequence

$$0 \to \mathrm{diag}(P \otimes A') \to \mathrm{diag}(P \otimes A) \to \mathrm{diag}(P \otimes A'') \to 0,$$

is short exact, which shows that the functor $\mathrm{diag}(P \otimes -)\colon \mathcal{A}^{\Delta^{\mathrm{op}}} \to \mathcal{A}^{\Delta^{\mathrm{op}}}$ is exact. The same is true for $\mathrm{diag}(- \otimes P)$. As already noted, if $P_\cdot$ has objects in $\mathcal{P}$, then so does $\Gamma(P_\cdot)$, so the functors $\mathrm{diag}(\Gamma(P_\cdot) \otimes -)$ and $\mathrm{diag}(- \otimes \Gamma(P_\cdot))$ are exact, and hence so are $P_\cdot \otimes_\Delta -$ and $- \otimes_\Delta P_\cdot$. □

Remark 4.34. In proving that $P_\cdot \otimes_\Delta -$ is an exact functor, we have not made use of the assumptions that $P_\cdot$ is acyclic and bounded. A more natural statement of Proposition 4.33 would be that $\otimes_\Delta$ is a tensor product on the pair $(C(\mathcal{A}), C(\mathcal{P}))$. Our choice of the subcategory $C^{\mathrm{q}}_{\mathrm{b}}(\mathcal{P})$ is motivated by our intended application.

We are now ready to iteratively define simplicial tensor products on categories of multicomplexes.

Definition 4.35. We define ***higher simplicial tensor products***

$$\otimes_{\Delta,n}\colon C^n(\mathcal{A}) \times C^n(\mathcal{A}) \to C^n(\mathcal{A})$$

for all $n \geq 0$ recursively:

1. $\otimes_{\Delta,0}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the usual tensor product $\otimes$, and
2. by regarding objects $C_\cdot$ and $D_\cdot$ of $C^{n+1}(\mathcal{A})$ as chain complexes in the abelian category with tensor products $C^n(\mathcal{A})$, we define $C_\cdot \otimes_{\Delta,n+1} D_\cdot := N(\mathrm{diag}(\Gamma(C_\cdot) \otimes_{\Delta,n} \Gamma(D_\cdot)))$.

We can now iterate Proposition 4.33.

Corollary 4.36. *For all $n \geq 0$, the higher simplicial tensor product $\otimes_{\Delta,n}$ is a tensor product in the sense of Definition 4.27 on the pair $(C^n(\mathcal{A}), (C^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{P}))$.*

Proof. The case $n = 0$ is an assumption of this section. The result is now straightforward by induction using Proposition 4.33, since $C^{n+1}(\mathcal{A}) = C(C^n(\mathcal{A}))$ and $(C^{\mathrm{q}}_{\mathrm{b}})^{n+1}(\mathcal{P}) = C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}}_{\mathrm{b}})^n(\mathcal{P}))$. □

In fact we can say a little more than this. The following lemma is crucial to the proof of the main result of this section.

Lemma 4.37. *Let $P_.$ be an object of $C_b(C_b^q)^n(\mathcal{P})$, and let $Q_.$ be an object of $(C_b^q)^{n+1}(\mathcal{P})$. Then $P_. \otimes_{\Delta,n+1} Q_.$ is an object of $(C_b^q)^{n+1}(\mathcal{P})$.*

Proof. Noting that $P_.$ and $Q_.$ both have their objects in $(C_b^q)^n(\mathcal{P})$, and that $Q_.$ is an acyclic complex of objects in that category, the lemma follows immediately from Lemmas 4.31 and 4.32 applied to the tensor product $\otimes_{\Delta,n}$ on the pair $(C^n(\mathcal{A}), (C_b^q)^n(\mathcal{P}))$. □

We can extend the higher simplicial tensor products to categories of binary complexes in the same way that we did for higher exterior powers in section 4.4. The simplicial tensor product of a pair of binary complexes $(C_., d_C, \tilde{d}_C)$ and $(D_., d_D, \tilde{d}_D)$ is obtained by considering the pair of chain complexes $(C, d_C) \otimes_\Delta (D, d_D)$ and $(C, \tilde{d}_C) \otimes_\Delta (D, \tilde{d}_D)$ as a binary complex (it is straightforward to prove that they have the same underlying graded object, in the same manner as Lemma 4.21). The analogue of Corollary 4.22 then follows, and we define the simplicial tensor product of binary multicomplexes just as we did for a functor of one variable in Definition 4.23.

We are almost ready to prove the main result of this section. Let $n > 0$, and let $P_.$ and $Q_.$ be n-dimensional bounded acyclic binary complexes of objects of $\mathcal{P}$. That is, $P_.$ and $Q_.$ are objects of $(B_b^q)^n(\mathcal{P})$. Then the simplicial tensor product $P_. \otimes_{\Delta,n} Q_.$ is in $(B_b^q)^n(\mathcal{P})$ as well by Corollary 4.36. Since the objects of $(B_b^q)^n(\mathcal{P})$ are the generators of $K_n\mathcal{P}$, one would like to use $\otimes_{\Delta,n}$ to induce a product $K_n\mathcal{P} \times K_n\mathcal{P} \to K_n\mathcal{P}$. Unfortunately this does not work, at least on first inspection, because the product $P_. \otimes_{\Delta,n} Q_.$ is not diagonal if only one of $P_.$ or $Q_.$ is diagonal. This is not a problem in the end though, since the whole product vanishes on $K_n\mathcal{P}$.

Proposition 4.38. *Let $n > 0$. For any pair of n-dimensional bounded acyclic multicomplexes $P_.$, $Q_.$ in $(B_b^q)^n(\mathcal{P})$, the class $[P_. \otimes_{\Delta,n} Q_.]$ vanishes in $K_n\mathcal{P}$.*

Proof. First we filter $P_.$ by degree. Regard $P_.$ as an acyclic binary complex of objects of $(B_b^q)^{n-1}(\mathcal{P})$. For $i \geq 0$, let $P|_{[0,i]}$ be the binary complex obtained by 'restricting' $P_.$ to be supported on $[0, i]$. That is, $(P|_{[0,i]})_j$ is equal to P_j if $0 \leq j \leq i$, and $(P|_{[0,i]})_j = 0$ otherwise. The differentials on $P|_{[0,i]}$ are inherited from $P_.$. We write $P_j[0]$ for P_j considered as a chain complex concentrated in degree 0. Then $P_j[j]$, which denotes P_j considered as a chain complex concentrated in degree j, is the quotient of the inclusion $P|_{[0,j-1]} \hookrightarrow P|_{[0,j]}$ (if $j \geq 1$). If $P_.$ is supported on $[0, n]$, so that $P_j = 0$ for $j > n$, we therefore have an n-stage filtration

$$P_0[0] = P|_{[0,0]} \hookrightarrow P|_{[0,1]} \hookrightarrow \cdots \hookrightarrow P|_{[0,n-1]} \hookrightarrow P|_{[0,n]} = P_.$$

whose successive quotients determine short exact sequences

$$0 \to P|_{[0,j-1]} \to P|_{[0,j]} \to P_j[j] \to 0.$$

We take the simplicial tensor product with $Q_.$ of this whole filtration, obtaining sequences

$$0 \to P|_{[0,j-1]} \otimes_{\Delta,n} Q_. \to P|_{[0,j]} \otimes_{\Delta,n} Q_. \to P_j[j] \otimes_{\Delta,n} Q_. \to 0 \qquad (*)$$

for $j = 1, \dots, n$, which are short exact by Corollary 4.36.
By Lemma 4.37, all of the objects are in the right category, so each of the short exact sequences of $(*)$ yields an equation

$$[P|_{[0,j]} \otimes_{\Delta,n} Q_.] = [P|_{[0,j-1]} \otimes_{\Delta,n} Q_.] + [P_j[j] \otimes_{\Delta,n} Q_.]$$

in $K_n\mathcal{P}$. Putting these together gives

$$[P_. \otimes_{\Delta,n} Q_.] = \sum_{j=0}^{n} [P_j[j] \otimes_{\Delta,n} Q_.].$$

To proceed we need to assume a small lemma, for which the second type of relation in $K_n\mathcal{P}$ (diagonal binary multicomplexes vanish) is crucial.

Lemma 4.39. *We have the following equality in $K_n\mathcal{P}$:*

$$[P_j[j] \otimes_{\Delta,n} Q_.] = (-1)^j [P_j[0] \otimes_{\Delta,n} Q_.].$$

Continuing with the main proof, our equation now reads

$$[P_. \otimes_{\Delta,n} Q_.] = \sum_{j=0}^{n} (-1)^j [P_j[0] \otimes_{\Delta,n} Q_.].$$

By inspection we see that $\Gamma(P_j[0])$ is the constant simplicial object which has P_j in each degree. The functor

$$\mathrm{diag}(\Gamma(P_j[0]) \otimes_{\Delta,n-1} -) \colon (C_\mathrm{b})^{n-1}(\mathcal{A})^{\Delta^{\mathrm{op}}} \to (C_\mathrm{b})^{n-1}(\mathcal{A})^{\Delta^{\mathrm{op}}}$$

is therefore isomorphic to the functor

$$P_j \otimes_{\Delta,n-1} - \colon (C_\mathrm{b})^{n-1}(\mathcal{A})^{\Delta^{\mathrm{op}}} \to (C_\mathrm{b})^{n-1}(\mathcal{A})^{\Delta^{\mathrm{op}}},$$

since they both have the same effect of 'tensoring everywhere by P_j'. This functor is additive, so we have an isomorphism of functors

$$N(P_j \otimes_{\Delta,n-1} \Gamma(-)) \cong P_j \otimes_{\Delta,n-1} -.$$

Hence

$$P_j[0] \otimes_{\Delta,n} Q_. = N\,\mathrm{diag}(\Gamma(P_j[0]) \otimes_{\Delta,n-1} \Gamma(Q_.)) \cong P_j \otimes_{\Delta,n-1} Q_.,$$

so we have

$$[P_. \otimes_{\Delta,n} Q_.] = \sum_{j=0}^{n} (-1)^j [P_j \otimes_{\Delta,n-1} Q_.].$$

There is an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0,$$

since $P_.$ is acyclic. The objects of $Q_.$ are in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}(\mathcal{P})$, so $- \otimes_{\Delta,n-1} Q_.$ is an exact functor by Lemma 4.29 (1), and so the sequence

$$0 \to P_n \otimes_{\Delta,n-1} Q_. \to P_{n-1} \otimes_{\Delta,n-1} Q_. \to \cdots \to P_1 \otimes_{\Delta,n-1} Q_. \to P_0 \otimes_{\Delta,n-1} Q_. \to 0$$

is exact. Exact sequences translate into alternating sums in the Grothendieck group, so this exact sequence gives exactly the identity

$$\sum_{j=0}^{n} (-1)^j [P_j \otimes_{\Delta,n-1} Q_.] = 0$$

in $K_0(B_{\mathrm{b}}^{\mathrm{q}})^n(\mathcal{P})$, and hence the same relation holds in $K_n\mathcal{P}$. Therefore $[P_. \otimes_{\Delta,n} Q_.] = 0$, as required. □

It remains to prove Lemma 4.39.

Proof of Lemma 4.39. Consider the following diagram as a short exact sequence of binary complexes concentrated in degrees j and $j-1$:

$$\begin{tikzcd}
0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & P_j \arrow[d, tail] \\
P_j \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] \arrow[d, two heads] & P_j \arrow[d, two heads] \\
P_j \arrow[r, shift left] \arrow[r, shift right] & 0.
\end{tikzcd}$$

We will use this diagram to show that $[P_j[j] \otimes_{\Delta,n} Q_.] = -[P_j[j-1] \otimes_{\Delta,n} Q_.]$. The argument can be iterated $j-1$ times to yield $[P_j[j] \otimes_{\Delta,n} Q_.] = (-1)^j [P_j[0] \otimes_{\Delta,n} Q_.]$ in $K_n\mathcal{P}$, as required. For lack of a better notation, we will denote the middle row of the diagram by $(P_j = P_j)$. Then the diagram represents a short exact sequence of binary complexes

$$0 \to P_j[j-1] \to (P_j = P_j) \to P_j[j] \to 0,$$

which upon tensoring with $Q_.$ becomes

$$0 \to P_j[j-1] \otimes_{\Delta,n} Q_. \to (P_j = P_j) \otimes_{\Delta,n} Q_. \to P_j[j] \otimes_{\Delta,n} Q_. \to 0.$$

Since $Q_.$ is acyclic and has objects in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}(\mathcal{P})$, each of the terms of this short exact sequence is an object of $(B_{\mathrm{b}}^{\mathrm{q}})^{n}(\mathcal{P})$ by Lemma 4.37, so we have a relation

$$[(P_j = P_j) \otimes_{\Delta,n} Q_.] = [P_j[j-1] \otimes_{\Delta,n} Q_.] + [P_j[j] \otimes_{\Delta,n} Q_.],$$

in $K_0(B_{\mathrm{b}}^{\mathrm{q}})^{n}(\mathcal{P})$, and hence in $K_n\mathcal{P}$. We claim that

$$[(P_j = P_j) \otimes_{\Delta,n} Q_.] = 0$$

in $K_n\mathcal{P}$, so that $[P_j[j] \otimes_{\Delta,n} Q_.] = -[P_j[j-1] \otimes_{\Delta,n} Q_.]$. We can filter $Q_.$ in the same manner that we have filtered $P_.$ in the main proof above,

$$Q_0[0] = Q|_{[0,0]} \hookrightarrow Q|_{[0,1]} \hookrightarrow \cdots \hookrightarrow Q|_{[0,n-1]} \hookrightarrow Q|_{[0,n]} = Q_.$$

giving short exact sequences

$$0 \to Q|_{[0,i-1]} \to Q|_{[0,i]} \to Q_i[i] \to 0.$$

Upon tensoring with $(P_j = P_j)$, we have short exact sequences:

$$0 \to (P_j = P_j) \otimes_{\Delta,n} Q|_{[0,i-1]} \to (P_j = P_j) \otimes_{\Delta,n} Q|_{[0,i]} \to (P_j = P_j) \otimes_{\Delta,n} Q_i[i] \to 0.$$

Furthermore, since $(P_j = P_j)$ is an acyclic binary complex of objects of $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}(\mathcal{P})$, each of the terms of these short exact sequences is an object of $(B_{\mathrm{b}}^{\mathrm{q}})^{n}(\mathcal{P})$, by Lemma 4.37. We therefore have the equation

$$[(P_j = P_j) \otimes_{\Delta,n} Q_.] = \sum_i [(P_j = P_j) \otimes_{\Delta,n} Q_i[i]]$$

in $K_0(B_{\mathrm{b}}^{\mathrm{q}})^{n}(\mathcal{P})$, and hence in $K_n\mathcal{P}$. But $(P_j = P_j)$ is a diagonal binary complex, as is each $Q_i[i]$ (trivially). The simplicial tensor product of a pair of diagonal complexes is again diagonal, so each of the acyclic binary complexes $(P_j = P_j) \otimes_{\Delta,n} Q_i[i]$ is diagonal and hence vanishes in $K_n\mathcal{P}$. Therefore $[(P_j = P_j) \otimes_{\Delta,n} Q_.] = 0$, so the desired relation holds. □

This finally completes the proof of Proposition 4.38. Having taken the trouble to set up an alternative product of bounded acyclic binary multicomplexes, one that is compatible with the higher exterior powers, we've now shown that (like the usual tensor product) it is always zero! It was not all for naught though: we know now that the induced operation $\otimes_{\Delta,n} \colon K_n\mathcal{P} \otimes K_n\mathcal{P} \to K_n\mathcal{P}$ is well-defined. Furthermore, the vanishing of this product proves that the higher exterior power operations induce homomorphisms on $K_n(R)$ (and more generally, on the higher K-groups of schemes). This is shown in the next section.

4.6 Exterior power operations on K-groups of schemes

The goal of this section is to extend the endofunctor Λ_n^r defined in section 4.4 to bounded acyclic multicomplexes of locally free modules of finite rank on a scheme X, and to prove that it induces a well-defined operation λ^r on the higher K-group $K_n(X)$. We will see that, for $n > 0$, this operation λ^r is not just a map but in fact a homomorphism.

Let X be a quasi-compact scheme, and let $\mathbf{M}(X)$ denote the category of quasi-coherent $\mathcal{O}_X$-modules. Furthermore, let $\mathbf{P}(X)$ denote the full subcategory of locally free $\mathcal{O}_X$-modules of finite rank. Then $\mathbf{P}(X)$ is an exact category in the usual sense. We write $K_n(X)$ for the K-group $K_n\mathbf{P}(X)$.
As in section 4.3, we inductively define an endofunctor Λ_n^r on $C^n\mathbf{M}(X)$ for $r \geq 1$ and $n \geq 0$ as follows: the functor Λ_0^r is the usual r^{th} exterior power functor on $C^0\mathbf{M}(X) = \mathbf{M}(X)$, and Λ_n^r is defined as $N\Lambda_{n-1}^r\Gamma$, with N and Γ as introduced in section 4.2.

Proposition 4.40. *For all $r, n > 0$, the functor Λ_n^r restricts to an endofunctor of the subcategory $(C_{\mathrm{b}}^{\mathrm{q}})^n\boldsymbol{P}(X)$ of $C^n\boldsymbol{M}(X)$.*

Proof. Given any open affine subscheme $U = \operatorname{Spec}(R)$ of X, a straightforward inductive argument shows that the following diagram commutes:

$$\begin{tikzcd} C^n\mathbf{M}(X) \arrow[r, "\Lambda_n^r"] \arrow[d] & C^n\mathbf{M}(X) \arrow[d] \\ C^n\mathbf{M}(U) \arrow[r, "\Lambda_n^r"] \arrow[d, equal] & C^n\mathbf{M}(U) \arrow[d, equal] \\ C^n\mathbf{Mod}(R) \arrow[r, "\Lambda_n^r"] & C^n\mathbf{Mod}(R); \end{tikzcd}$$

where the vertical functors are induced by the restriction functor $\mathbf{M}(X) \to \mathbf{M}(U)$, $M \mapsto M|_U$, and the lower horizontal functor is the functor Λ_n^r introduced in section 4.3. A complex in $C^n\mathbf{M}(X)$ is acyclic, or bounded, or in $C^n\mathbf{P}(X)$ if and only if its restriction to every open affine subscheme has the respective property, so Proposition 4.40 follows from the results of section 4.3 and Lemma 4.25. □

As in section 4.4, one easily deduces that, for any complex $M_.$ in $C^n\mathbf{M}(X)$, the objects in $\Lambda_n^r(M_.)$ do not depend on the differentials in $M_.$. We can therefore extend the endofunctor Λ_n^r to an endofunctor of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathbf{P}(X)$, which we denote by Λ_n^r again. The goal of the rest of this section is to prove the following theorem.

Theorem 4.41. *Let $n > 0$ and $r > 0$. The endofunctor Λ_n^r of $(B_{\mathrm{b}}^{\mathrm{q}})^n\boldsymbol{P}(X)$ induces a well-defined homomorphism $\lambda^r\colon K_n(X) \to K_n(X)$.*

Definition 4.42. The homomorphism λ^r in the previous theorem is called the r^{th} ***exterior power operation*** on $K_n(X)$.

Proof of Theorem 4.41. If $P_.$ is a diagonal multicomplex in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$, then the multicomplex $\Lambda^r_n(P_.)$ is diagonal as well, by definition of Λ^r_n. It therefore suffices to show that the association $[P_.] \mapsto [\Lambda^r_n(P_.)]$ induces a well-defined homomorphism of groups

$$\lambda^r\colon K_0(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X) \to K_n(X).$$

Thus we need to show that the equality

$$[\Lambda^r_n(P_.)] = [\Lambda^r_n(P'_.)] + [\Lambda^r_n(P''_.)]$$

holds in $K_n(X)$ for every short exact sequence $0 \to P'_. \to P_. \to P''_. \to 0$ in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$. The classes $[\Lambda^{r-i}_n(P'_.)\otimes_{\Delta,n}\Lambda^i_n(P''_.)]$, $i = 1,\dots,r-1$, vanish in $K_n(X)$ by Proposition 4.38 applied to the category $\mathcal{P} = \mathbf{P}(X)$, where the simplicial tensor product has been constructed inductively from the usual tensor product of quasi-coherent $\mathcal{O}_X$-modules. So the desired equality is equivalent in $K_n(X)$ to the more familiar-looking identity

$$[\Lambda^r_n(P_.)] = [\Lambda^r_n(P'_.)] + \sum_{i=1}^{r-1}[\Lambda^{r-i}_n(P'_.)\otimes_{\Delta,n}\Lambda^i_n(P''_.)] + [\Lambda^r_n(P''_.)].$$

In order to prove this latter formula, we cannot just apply the usual formula for the r^{th} exterior power of a direct sum because the given short exact sequence of binary complexes, $0 \to P'_. \to P_. \to P''_. \to 0$, does not split in general, even if X is affine (see Example 4.20). Instead, by induction on n, we construct for every sequence $0 \to P'_. \to P_. \to P''_. \to 0$ in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$ an induced filtration

$$\Lambda^r_n(P'_.) \hookrightarrow \Lambda^{r-1}_n(P'_.)\wedge_n\Lambda^1_n(P_.) \hookrightarrow \cdots \hookrightarrow \Lambda^1_n(P'_.)\wedge_n\Lambda^{r-1}_n(P_.) \hookrightarrow \Lambda^r_n(P_.)$$

of $\Lambda^r_n(P_.)$ by certain sub-objects $\Lambda^{r-i}_n(P'_.)\wedge_n\Lambda^i_n(P_.)$, $i = 0,\dots,r$ of $\Lambda^r_n(P_.)$, also belonging to $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$, together with short exact sequences

$$0 \longrightarrow \Lambda^{r-i+1}_n(P'_.)\wedge_n\Lambda^{i-1}_n(P_.) \longrightarrow \Lambda^{r-i}_n(P'_.)\wedge_n\Lambda^i_n(P_.) \longrightarrow \Lambda^{r-i}_n(P'_.)\otimes_{\Delta,n}\Lambda^i_n(P''_.) \longrightarrow 0, \quad (*)$$

$i = 1,\dots,n$.
For $n = 0$ and $i \in \{0,\dots,r\}$, the object $\Lambda^{r-i}_0(P')\wedge_0\Lambda^i_0(P)$ is defined to be what is usually meant by $\Lambda^{r-i}(P')\wedge\Lambda^i(P)$: the image of the canonical homomorphism $\Lambda^{r-i}(P')\otimes\Lambda^i(P) \to \Lambda^r(P)$. It is well-known that these objects come with the required short exact sequences $(*)$.

If $n > 0$ and if, for a moment, the sequence $0 \to P'_. \to P_. \to P''_. \to 0$ is given in $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$ rather than in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$, we first note that applying the exact functor Γ to the sequence (whose objects are considered as objects of $C^{\mathrm{q}}_{\mathrm{b}}(C^{\mathrm{q}}_{\mathrm{b}})^{n-1}\mathbf{P}(X)$), we get the short exact sequence $0 \to \Gamma(P'_.) \to \Gamma(P_.) \to \Gamma(P''_.) \to 0$ of simplicial objects in $(C^{\mathrm{q}}_{\mathrm{b}})^{n-1}\mathbf{P}(X)$. By the inductive hypothesis, the complexes $\Lambda^{r-i}_{n-1}(\Gamma(P'_.)_m)\wedge_{n-1}\Lambda^i_{n-1}(\Gamma(P)_m)$, $i = 0,\dots,n$, $m \geq 0$, belong to

$(C_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathbf{P}(X)$ and we have short exact sequences

$$\begin{aligned}0 \longrightarrow \Lambda_{n-1}^{r-i+1}(\Gamma(P')_m) \wedge_{n-1} \Lambda_{n-1}^{i-1}(\Gamma(P)_m)&\\ \longrightarrow \Lambda_{n-1}^{r-i}(\Gamma(P')_m) \wedge_{n-1} \Lambda_{n-1}^{i}(\Gamma(P)_m)&\\ \longrightarrow \Lambda_{n-1}^{r-i}(\Gamma(P')_m) \otimes_{\Delta,n-1} \Lambda_{n-1}^{i}(\Gamma(P'')_m) \longrightarrow 0,&\end{aligned}$$

$i = 1,\dots,r$, $m \geq 0$. These short exact sequences assemble to short exact sequences of simplicial objects of $(C_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathbf{P}(X)$. By applying the exact functor N we finally obtain the required objects

$$\Lambda_n^{r-i}(P') \wedge_n \Lambda_n^i(P) := N(\Lambda_{n-1}^{r-i}(\Gamma(P')) \wedge_{n-1} \Lambda_{n-1}^{i}(\Gamma(P))),$$

$i = 0,\dots,r$, and the required short exact sequences $(*)$. As the objects of the multicomplex $\Lambda_n^{r-i}(P') \wedge_n \Lambda_n^i(P)$ are independent of the differentials in the multicomplexes $P'_.$ and $P_.$, this construction of $\wedge_n$ passes to the category $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathbf{P}(X)$ as in section 4.4.
From Proposition 4.40 and section 4.5 we know that the complex $\Lambda_n^r(P_.)$ and the complexes $\Lambda_n^{r-i}(P'_.) \otimes_{\Delta,n} \Lambda_n^i(P''_.)$, $i = 0,\dots,r$, belong to $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathbf{P}(X)$. Now a straightforward downwards induction on i based on the short exact sequences $(*)$ shows that the complexes $\Lambda_n^{r-i}(P'_.) \wedge_n \Lambda_n^i(P_.)$, $i = 0,\dots,r$ are bounded and acyclic and are complexes with objects in $B^{n-1}\mathbf{P}(X)$, i.e., that they belong to $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathbf{P}(X)$, as was to be shown. □

4.7 Making things explicit

The following example describes the action of the exterior powers on certain simple elements of $K_1(X)$. For a nice ring $\Bbbk$ this already completely determines the operations $\lambda^r\colon K_1(\Bbbk) \to K_1(\Bbbk)$.

Example 4.43. Let $\phi, \psi\colon \mathcal{L}_1 \to \mathcal{L}_2$ be isomorphisms between line bundles in $\mathbf{P}(X)$, and let

$$\mathcal{L} = [\ \mathcal{L}_1 \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathcal{L}_2\ ]$$

be the element of $K_1(X)$ obtained by regarding the binary automorphism as an acyclic binary complex supported in degrees 0 and 1. Then

$$\lambda^r([(\mathcal{L})]) = (-1)^{r-1}[\mathcal{L}_1^{\otimes(r-1)} \otimes \mathcal{L}],$$

by Example 4.19 and the Shifting Lemma 2.5. In particular, if x is a unit of the commutative ring R, then

$$\lambda^r([\ R \overset{x}{\underset{1}{\rightrightarrows}} R\ ]) = (-1)^{r-1}[\ R \overset{x}{\underset{1}{\rightrightarrows}} R\ ].$$

If $\Bbbk$ is a commutative ring such that the map

$$\begin{array}{ccc} \Bbbk^\times & \longrightarrow & K_1\Bbbk \\ x & \longmapsto & [\ \Bbbk \overset{x}{\underset{1}{\rightrightarrows}} \Bbbk\ ] \end{array}$$

is an isomorphism (this class of rings includes all fields, commutative local rings, and Euclidean domains), then the above shows that the exterior power $\lambda^r\colon K_1\Bbbk \to K_1\Bbbk$ is multiplication by $(-1)^{r-1}$.

This computation of λ^r on K_1 of nice rings shows that our exterior powers agree with those given by Hiller [Hil81] in this case. In fact we can show that this is true in general.

Let R be a commutative ring. In addition to $K_0(R)$, we consider the Grothendieck group $K_0(\mathbb{Z},R)$ of representations of the infinite cyclic group $\mathbb{Z}$ on finitely generated projective R-modules. Equivalently, $K_0(\mathbb{Z},R)$ is the Grothendieck group of the category $\mathrm{Aut}(\mathbf{Proj}_{\mathrm{f}}(R))$ of Definition 2.15. The reduced Grothendieck group $\widetilde{K}_0(\mathbb{Z},R)$ is then defined to be the kernel of the epimorphism

$$K_0(\mathbb{Z},R) \to K_0(R), \quad [(P,\alpha)] \mapsto [P].$$

It is generated by differences of the form $[(P,\alpha)] - [(P,1)]$, with P, α as above.

Recall that Bass defines $K_1(R)$ as the quotient of $K_0(\mathbb{Z},R)$ by the relation $[(P,\alpha\beta)] = [(P,\alpha)] + [(P,\beta)]$, for any f.g. projective R-module P and $\alpha,\beta \in \mathrm{Aut}_R(P)$. In particular, there is a canonical homomorphism

$$q\colon \widetilde{K}_0(\mathbb{Z},R) \hookrightarrow K_0(\mathbb{Z},R) \twoheadrightarrow K_1(R)$$

which is obviously surjective.
The tensor product defines a multiplication on $K_0(\mathbb{Z},R)$. Furthermore, for any $r \geq 0$, the association $[(P,\alpha)] \mapsto [(\Lambda^r(P),\Lambda^r(\alpha))]$ can be extended to a well-defined map $\lambda^r\colon K_0(\mathbb{Z},R) \to K_0(\mathbb{Z},R)$, which we call the r^{th} exterior power operation on $K_0(\mathbb{Z},R)$, see [Köc91], (2.5). In fact $K_0(\mathbb{Z},R)$ is a λ-ring with these structures, see *loc. cit.* As the canonical map $K_0(\mathbb{Z},R) \to K_0(R)$ is a λ-ring homomorphism, the maps λ^r, $r \geq 1$, restrict to self-maps of $\widetilde{K}_0(\mathbb{Z},R)$.

Proposition 4.44. *For each $r \geq 1$, the following diagram commutes:*

$$\begin{tikzcd} \widetilde{K}_0(\mathbb{Z},R) \arrow[r, two heads, "q"] \arrow[d, "\lambda^r"] & K_1(R) \arrow[d, "\lambda^r"] \\ \widetilde{K}_0(\mathbb{Z},R) \arrow[r, two heads, "q"] & K_1(R). \end{tikzcd}$$

Proof. Let P be a finitely generated projective R-module, and let $\alpha \in \mathrm{Aut}_R(P)$. By Theorem 2.24, the element $[(P,\alpha)]$ in Bass' description of $K_1(R)$ corresponds to the class of the binary complex $P \underset{1}{\overset{\alpha}{\rightrightarrows}} P$ (placed in degrees 0 and 1) in Grayson's definition of $K_1(R)$. We therefore need to show that

$$q(\lambda^r([(P,\alpha)] - [(P,1)])) = \lambda^r([\, P \underset{1}{\overset{\alpha}{\rightrightarrows}} P \,]) \qquad (*)$$

in $K_1(R)$. By a formula on page 2 of [Gra89], the left-hand side of $(*)$ is equal to

$$\sum_{u=0}^{r}(-1)^u \sum_{a+b_1+\cdots+b_u=r} [(\Lambda^a(P)\otimes\Lambda^{b_1}(P)\otimes\cdots\otimes\Lambda^{b_u}(P), \Lambda^a(\alpha)\otimes 1\otimes\cdots\otimes 1)],$$

where the second sum is taken over a, b_i satisfying $a \geq 0$ and $b_i \geq 1$. The summands corresponding to $a = 0$ obviously vanish in $K_1(R)$, so after re-indexing we may rewrite the left-hand side of $(*)$ in the following way:

$$\sum_{u=1}^{r}(-1)^{u-1} \sum_{b_1+\cdots+b_u=r} [(\Lambda^{b_1}(P)\otimes\cdots\otimes\Lambda^{b_u}(P), \Lambda^{b_1}(\alpha)\otimes 1\otimes\cdots\otimes 1)],$$

where now each b_i is ≥ 1. The right-hand side of $(*)$ is equal, by definition, to the class of the binary complex $N\Lambda^r\Gamma(\, P \underset{1}{\overset{\alpha}{\rightrightarrows}} P \,)$ in $K_1(R)$. By Lemma 2.2 of [Köc01], this complex is the total complex of an acyclic binary double complex represented by:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{cr}_3(\Lambda^r)(P,P,P) \arrow[r] \arrow[d] & \mathrm{cr}_2(\Lambda^r)(P,P) \arrow[r] \arrow[d] & \Lambda^r(P) \arrow[d] \\
\cdots \arrow[r] & \mathrm{cr}_3(\Lambda^r)(P,P,P) \arrow[r] & \mathrm{cr}_2(\Lambda^r)(P,P) \arrow[r] & \Lambda^r(P)
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{cr}_3(\Lambda^r)(P,P,P) \arrow[r] \arrow[d] & \mathrm{cr}_2(\Lambda^r)(P,P) \arrow[r] \arrow[d] & \Lambda^r(P) \arrow[d] \\
\cdots \arrow[r] & \mathrm{cr}_3(\Lambda^r)(P,P,P) \arrow[r] & \mathrm{cr}_2(\Lambda^r)(P,P) \arrow[r] & \Lambda^r(P)
\end{tikzcd}$$

where the vertical maps are the isomorphisms $(\mathrm{cr}_u(\Lambda^r)(\alpha,1,\ldots,1),1)$, for $u = 1,\ldots,r$ (the definition of the horizontal maps is irrelevant for our considerations). It is easy to see that this binary complex can be filtered by acyclic binary subcomplexes such that the successive quotients are the acyclic binary complexes given by the vertical pairs of isomorphisms. Using the shifting lemma 2.5, we therefore obtain that the right-hand side of $(*)$ is equal to

$$\begin{aligned}
& \textstyle\sum_{u=1}^{r}(-1)^{u-1}[(\mathrm{cr}_u(\Lambda^r)(P,\ldots,P), \mathrm{cr}_u(\Lambda^r)(\alpha,1,\ldots,1))] \\
= \; & \textstyle\sum_{u=1}^{r}(-1)^{u-1}\sum_{b_1+\cdots+b_u=r}[(\Lambda^{b_1}(P)\otimes\cdots\otimes\Lambda^{b_u}(P), \Lambda^{b_1}(\alpha)\otimes 1\otimes\cdots\otimes 1)],
\end{aligned}$$

(since the cross effect $\mathrm{cr}_u(\Lambda^r)(P,\ldots,P)$ is $\bigoplus_{b_1+\cdots+b_u=r}\Lambda^{b_1}(P)\otimes\cdots\otimes\Lambda^{b_u}(P)$) and is therefore equal to the left-hand side. □

4.8 The λ-ring axioms for the higher K-groups of a scheme

Given a scheme X, there is an obvious 'trivial' way to equip the graded abelian group $K_*(X) := \bigoplus_{n\geq 0} K_n(X)$ with a multiplication[3], and to extend the exterior power operations defined in the previous section to $K_*(X)$ so that they are compatible with addition in $K_*(X)$ in the usual sense. The main result of this section is that they are also compatible with multiplication in the expected way—that is to say, they satisfy the second axiom of a λ-ring. Finally, we speculate on how to prove the third and final λ-ring axiom, relating to the composition of exterior power operations.

Let X be a quasi-compact scheme. We recall that $K_0(X)$ together with the usual exterior power operations $\lambda^r \colon K_0(X) \to K_0(X)$, $r \geq 0$, is a λ-ring as defined in section 4.1 (see chapter V of [FL85]). Furthermore, $K_n(X)$ is a $K_0(X)$-module via $[P].[Q_.] := [P \otimes Q_.]$, for P in $\mathbf{P}(X)$ and $Q_.$ in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$, *cf.* Definition 4.28 (1).
We define a multiplication on $K_*(X) := \bigoplus_{n\geq 0} K_n(X)$ by

$$(a_0, a_1, a_2, \dots) \,.\, (b_0, b_1, b_2, \dots) := (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_2b_0, \dots);$$

in particular, the product of any two elements in $\bigoplus_{n\geq 1} K_n(X)$ is defined to be zero. With this multiplication, $K_*(X)$ is a commutative ring. Furthermore, we define exterior power operations $\lambda^r \colon K_*(X) \to K_*(X)$, $r \geq 0$, by the formula

$$\lambda^r((a_0, a_1, a_2, \dots)) := (\lambda^r(a_0), \sum_{i=0}^{r-1} \lambda^i(a_0)\lambda^{r-i}(a_1), \sum_{i=0}^{r-1} \lambda^i(a_0)\lambda^{r-i}(a_2), \dots).$$

By definition, we then have $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K_*(X)$. A straightforward calculation using Theorem 4.41 and the fact that $K_0(X)$ satisfies axiom (1) of a λ-ring shows that $K_*(X)$ also satisfies axiom (1). The next theorem addresses axiom (2).

Theorem 4.45. *The ring $K_*(X)$ equipped with the exterior power operations λ^r, $r \geq 0$, defined above satisfies axiom (2) of a λ-ring.*

Proof. Axiom (2) holds for elements of the form $x = (a_0, 0, 0, \dots)$, $y = (b_0, 0, 0, \dots)$ in $K_*(X)$ because it holds for $K_0(X)$. It also holds for elements of the form $x = (0, a_1, a_2, \dots)$ and $y = (0, b_1, b_2, \dots)$ because $\lambda^r(0) = 0$ for all $r \geq 1$ and because every monomial in the ring $\mathbb{Z}[X_1, \dots, X_r, Y_1, \dots, Y_r]$ whose coefficient in $P_r(X_1, \dots, X_r, Y_1, \dots, Y_r)$ is non-zero is divisible by some product X_iY_j. Furthermore, it suffices to check axiom (2) for x, y belonging to a set of additive generators of $K_*(X)$ because $K_*(X)$ satisfies axiom (1) and because axiom (2) is equivalent (see [AT69]) to the multiplicativity of the homomorphism

$$\begin{array}{rccl} \lambda^t \colon & K_*(X) & \to & 1 + t.K_*(X)[[t]] \\ & x & \mapsto & \sum_{r\geq 0} \lambda^r(x)t^r. \end{array}$$

[3] This is not the multiplication on $K_*(X)$ defined in Lemma 2.26.

We are therefore reduced to showing that the equality

$$\lambda^r(xy) = P_r(\lambda^1(x),\dots,\lambda^r(x),\lambda^1(y),\dots,\lambda^r(y)) \tag{$*$}$$

holds in $K_n(X)$ for elements $y \in K_n(X)$ and $x \in K_0(X)$ of the form $x = [\mathcal{E}]$ for some locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank.

We now invoke the projective bundle theorem ([Qui73], §8, Theorem 2.1). We remark that its proof in *loc. cit.* only relies on the additivity and resolution theorems, and not, for instance, on the dévissage theorem or localisation sequence. We proved the additivity and resolution theorems within the context of Grayson's definition of higher K-groups in chapter 3, so the projective bundle theorem also has a proof within that context, without resorting to topological methods.

It is well-known that an iterated application of the projective bundle theorem yields the following splitting principle: there exists a projective morphism $f\colon Y \to X$ such that $f^*[\mathcal{E}]$ is the sum of invertible $\mathcal{O}_Y$-modules in $K_0(Y)$ and such that $f^*\colon K_n(X) \to K_n(Y)$ is injective. It is straightforward to check that $f^*\colon K_0(X) \to K_0(Y)$ is a homomorphism of (pre-)λ-rings. Using the above argument about additive generators again, we are therefore reduced to showing the equality ($*$) only when x is the class $[\mathcal{L}]$ of an invertible $\mathcal{O}_X$-module $\mathcal{L}$. In that case, ($*$) becomes the much simpler formula

$$\lambda^r([\mathcal{L}]\cdot y) = [\mathcal{L}^{\otimes r}]\cdot \lambda^r(y),$$

because $\lambda^2[\mathcal{L}] = \cdots = \lambda^r[\mathcal{L}] = 0$ and because P_r satisfies $P_r(1,0,\dots,0,Y_1,\dots,Y_r) = Y_r$ and has X-degree r (where X_i is defined to be of degree i for $i = 1,\dots,r$). Using the argument about additive generators again, it suffices to show that for any object $P_.$ of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$, the object $\Lambda^r_n(\mathcal{L}\otimes P_.)$ is isomorphic to $\mathcal{L}^{\otimes r}\otimes\Lambda^r_n(P_.)$. This is well-known if $n = 0$, and follows by induction on n from the following chain of isomorphisms applied to each of the 2^n multicomplexes associated with the binary multicomplex $P_.$ (which we again denote by $P_.$):

$$\begin{aligned}
\Lambda^r_n(\mathcal{L}\otimes P_.) &= N\Lambda^r_{n-1}\Gamma(\mathcal{L}\otimes P_.)\\
&\cong N(\mathcal{L}^{\otimes r}\otimes\Lambda^r_{n-1}\Gamma(P_.))\\
&\cong \mathcal{L}^{\otimes r}\otimes N\Lambda^r_{n-1}\Gamma(P_.)\\
&\cong \mathcal{L}^{\otimes r}\otimes\Lambda^r_n(P_.).
\end{aligned}$$

□

We end by making some comments about the third λ-ring axiom for the ring $K_*(X)$. It is well-known that the splitting principle used in the previous proof also implies axiom (3) for the Grothendieck group $K_0(X)$. This means that, for every $\mathcal{E}\in\mathbf{P}(X)$ and for any pair $(r,s)\in\mathbb{N}\times\mathbb{N}$, there exists: a number $k(\mathcal{E})\in\mathbb{N}$, short exact sequences

$$0 \to E'_j(\mathcal{E}) \to E_j(\mathcal{E}) \to E''_j(\mathcal{E}) \to 0$$

$j = 1,\dots,k(\mathcal{E})$, in $\mathbf{P}(X)$ and integers $m_j(\mathcal{E}) \in \mathbb{Z}$, $j = 1,\dots,k(\mathcal{E})$, such that the equality

$$[\Lambda^r(\Lambda^s(\mathcal{E}))] - P_{r,s}([\Lambda^1(\mathcal{E})],\dots,[\Lambda^{rs}(\mathcal{E})]) = \sum_{j=1}^{k(\mathcal{E})} m_j(\mathcal{E})([E_j(\mathcal{E})] - [E'_j(\mathcal{E})] - [E''_j(\mathcal{E})]) \qquad (**)$$

holds in the ring $\mathbb{Z}[\mathbf{P}(X)/\cong]$ associated with the monoid $(\mathbf{P}(X)/\cong,\otimes)$ of isomorphism classes of objects in $\mathbf{P}(X)$. While the splitting principle implies the existence of the short exact sequences above, it seems to be a hopeless problem to actually write them down explicitly. This problem may be viewed as a generalisation of the famous ***plethysm problem*** in classical representation theory, which is as yet (to the authors' knowledge) unsolved in general. Rather than trying to write these sequences down explicitly, we ask whether they exist in a functorial form in the following sense.

Question 4.46. Given $r,s \in \mathbb{N}$ and X as above, does there exist a constant $k \in \mathbb{N}$, integers $m_j \in \mathbb{Z}$, $j = 1,\dots,k$, and for every open subset U of X, endofunctors E'_j, E_j and E''_j, $j = 1,\dots,k$, of $\mathbf{M}(U)$ of finite degree, which are compatible with restriction with respect to any inclusion $V \subseteq U$ of open subsets U,V of X, and each of which induces an endofunctor of $\mathbf{P}(U)$, together with short exact sequences

$$0 \to E'_j \to E_j \to E''_j \to 0$$

$j = 1,\dots,k$, of endofunctors of $\mathbf{P}(X)$ such that the equality $(**)$ holds for all $\mathcal{E} \in \mathbf{P}(X)$?

Remark 4.47. If the answer to Question 4.46 is affirmative for X and for all $r,s \in \mathbb{N}$, then the ring $K_*(X)$ equipped with the exterior power operations λ^r, $r \geq 0$, defined above satisfies axiom (3) of a λ-ring as well, and is hence a λ-ring. Indeed, mimicking the earlier constructions, all of the endofunctors E'_j, E_j and E''_j and their accompanying short exact sequences can be extended to $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$ by induction on n and satisfy the obvious analogues of the equation $(**)$ for all $\mathcal{E}$ in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$. Note that in this analogue $\Lambda^r \circ \Lambda^s$ and every monomial in $P_{r,s}$ are turned into endofunctors of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathbf{P}(X)$ in the obvious way as well.

Example 4.48. If $r = s = 2$ and 2 is invertible on X, then the answer to Question 4.46 is affirmative.

Proof. We define the following endofunctors of $\mathbf{M}(U)$ for every open subset U of X: $E(\mathcal{E}) := \mathcal{E} \otimes \Lambda^3(\mathcal{E})$, $E'(\mathcal{E}) := \Lambda^4(\mathcal{E})$, and $E''(\mathcal{E}) := \Lambda^2(\Lambda^2(\mathcal{E}))$. These endofunctors obviously satisfy the conditions stated in Question 4.46. Furthermore, for each $\mathcal{E}$ in $\mathbf{P}(X)$, let the sequence

$$0 \longrightarrow \Lambda^4(\mathcal{E}) \xrightarrow{\alpha} \mathcal{E} \otimes \Lambda^3(\mathcal{E}) \xrightarrow{\beta} \Lambda^2(\Lambda^2(\mathcal{E})) \longrightarrow 0 \qquad (***)$$

be defined as follows: the map α is the usual Koszul differential, i.e.,

$$\alpha(a \wedge b \wedge c \wedge d) = a \otimes b \wedge c \wedge d - b \otimes a \wedge c \wedge d + c \otimes a \wedge b \wedge d - d \otimes a \wedge b \wedge c$$

for local sections a, b, c, d of $\mathcal{E}$; the map β is given by

$$\beta(a \otimes b \wedge c \wedge d) = (a \wedge b) \wedge (c \wedge d) - (a \wedge c) \wedge (b \wedge d) + (a \wedge d) \wedge (b \wedge c).$$

It is straightforward to check that α and β are well-defined, that $\beta \circ \alpha = 0$ and that $\mathrm{rk}(\mathcal{E} \otimes \Lambda^3(\mathcal{E})) = \mathrm{rk}(\Lambda^4(\mathcal{E})) + \mathrm{rk}(\Lambda^2(\Lambda^2(\mathcal{E})))$. Furthermore, it is well-known that α is an admissible monomorphism. Finally, the map β is surjective because

$$\beta(d \otimes a \wedge b \wedge c - c \otimes a \wedge b \wedge d) = 2(a \wedge b) \wedge (c \wedge d)$$

for all local sections a, b, c, d of $\mathcal{E}$. Hence the sequence (***) is exact and implies the desired equality

$$[\Lambda^2(\Lambda^2(\mathcal{E}))] - P_{2,2}([\Lambda^1(\mathcal{E})], \ldots, [\Lambda^4(\mathcal{E})]) = -([E(\mathcal{E})] - [E'(\mathcal{E})] - [E''(\mathcal{E})])$$

in $\mathbb{Z}[\mathbf{P}(X)/\cong]$ because $P_{2,2}(X_1, X_2, X_3, X_4) = X_1X_3 - X_4$ (this formula can be verified by direct computation, or found in [Hop06], Proposition 2.1). □

Remark 4.49. Without providing details, we remark that an analogue of Question 4.46 for axiom (2) rather than axiom (3) can be formulated so that the obvious analogue of Remark 4.47 holds true. Moreover, the paper [ABW82] by Akin, Buchsbaum, and Weyman provides an affirmative answer to the new question in full generality. In this way we obtain a second proof of Theorem 4.45, which does not rely on the projective bundle theorem (or the splitting principle), but which is more complicated from the combinatorial point of view.

Appendix

4.A Defining $K_n\mathcal{N}$ with positive multicomplexes is harmless

Because the Dold–Kan correspondence concerns the category of chain complexes that are *supported in positive degrees*, it is convenient in chapter 4 to suppose that all of our binary (multi)complexes satisfy the same assumption. A binary multicomplex $N_.$ is supported in positive degrees if the underlying $\mathbb{Z}^n$-graded object of $\mathcal{N}$ is. That is, objects of $N_.$ outside of $\mathbb{N}^n_{\geq 0}$ vanish. We denote the full subcategory of $(B^{\mathrm{q}})^n\mathcal{N}$ consisting of those objects that are supported in positive degrees by $(B^{\mathrm{q}})^n_{\geq 0}\mathcal{N}$. It is clear that this is an exact subcategory of $(B^{\mathrm{q}})^n\mathcal{N}$. In chapter 4 we use the following modified definition of higher algebraic K-groups.

Definition 4.50. Let $\mathcal{N}$ be an exact category. For all $n \geq 0$, we define $K_n^{\mathrm{pos}}\mathcal{N}$ to be the abelian group with the following presentation.

1. One generator for (the isomorphism class of) each object of $(B^{\mathrm{q}})^n_{\geq 0}\mathcal{N}$.
2. The relation $[B'] + [B''] = [B]$ whenever there is a short exact sequence of binary multicomplexes $B' \rightarrowtail B \twoheadrightarrow B''$ in $(B^{\mathrm{q}})^n_{\geq 0}\mathcal{N}$.
3. The relation $[B] = 0$ for any diagonal binary multicomplex B in $(B^{\mathrm{q}})^n_{\geq 0}\mathcal{N}$.

This is different to the presentation given in Theorem/Definition 1.40, but we claim that this additional constraint on the generators of $K_n\mathcal{N}$ is harmless. We suspect that all of the arguments of [Gra12] go through if we change Grayson's category of binary complexes to the one above which assumes an absolute lower bound, though we have not checked this in detail. Instead, we give an explicit isomorphism between the groups given by the two presentations.

For $i \geq 0$, let $(B^{\mathrm{q}})^n_{\geq -i}\mathcal{N}$ denote the full subcategory of $(B^{\mathrm{q}})^n\mathcal{N}$ consisting of n-dimensional bounded acyclic binary multicomplexes that are supported on $[-i, \infty)^n$. Then $K_n^{\mathrm{pos}}\mathcal{N}$ is the quotient of $K_0(B^{\mathrm{q}})^n_{\geq 0}\mathcal{N}$ by the subgroup generated by diagonal binary complexes. Furthermore, $\bigcup_i (B^{\mathrm{q}})^n_{\geq -i}\mathcal{N} = (B^{\mathrm{q}})^n\mathcal{N}$, so the quotient of $K_0(\bigcup_i (B^{\mathrm{q}})^n_{\geq -i}\mathcal{N})$ by the diagonal binary complexes is the usual $K_n\mathcal{N}$ given by Theorem/Definition 1.40.

Proposition 4.51. *The inclusion* $(B^{\mathrm{q}})^n_{\geq 0}\mathcal{N} \hookrightarrow (B^{\mathrm{q}})^n\mathcal{N}$ *induces an isomorphism* $K_n^{\mathrm{pos}}\mathcal{N} \xrightarrow{\cong} K_n\mathcal{N}$.

Proof. For ease of presentation we shall prove this for $n = 1$ only: there is no additional difficulty for $n > 1$. The obvious inclusions $B^{\mathrm{q}}_{\geq -i}\mathcal{N} \hookrightarrow B^{\mathrm{q}}_{\geq -(i+1)}\mathcal{N}$ are clearly exact functors, and they form a filtered system of exact categories whose limit is $\bigcup_i B^{\mathrm{q}}_{\geq -i}\mathcal{N} = B^{\mathrm{q}}\mathcal{N}$. It is clear that the compositions of all of these inclusions is equal to the inclusion $B^{\mathrm{q}}_{\geq 0}\mathcal{N} \hookrightarrow B^{\mathrm{q}}\mathcal{N}$, which is again an exact functor. The Grothendieck group functor commutes with filtered colimits (this is Exercise II. 7.9 in [Wei13]), so we obtain an isomorphism $\lim_i K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N} \cong K_0 B^{\mathrm{q}}\mathcal{N}$. We would like to show that this remains an isomorphism upon dividing out diagonal binary complexes.

Let T_i be the subgroup of $K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}$ generated by diagonal elements, and let T be the similarly defined subgroup of $K_0 B^{\mathrm{q}}\mathcal{N}$. The inclusions $B^{\mathrm{q}}_{\geq -i}\mathcal{N} \hookrightarrow B^{\mathrm{q}}_{\geq -(i+1)}\mathcal{N}$ induce maps $T_i \to T_{i+1}$, and these are compatible with the inclusions $T_i \hookrightarrow K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}$. Since the map $\lim_i K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N} \to K_0 B^{\mathrm{q}}\mathcal{N}$ is an isomorphism, it follows from this compatibility that $\lim_i T_i \to T$ is injective. Let $[D_.]$ be the class in $K_0 B^{\mathrm{q}}\mathcal{N}$ of a diagonal complex. Since all binary complexes are assumed to be bounded, $D_.$ is in the image of some $B^{\mathrm{q}}_{\geq -i}\mathcal{N} \hookrightarrow B^{\mathrm{q}}\mathcal{N}$, so $\lim_i T_i \to T$ is surjective as well. We therefore obtain an isomorphism

$$\lim_i (K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}/T_i) \cong \lim_i K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}/\lim_i T_i \xrightarrow{\cong} K_0 B^{\mathrm{q}}\mathcal{N}/T = K_1\mathcal{N}.$$

Now considering each inclusion $B^{\mathrm{q}}_{\geq -i}\mathcal{N} \hookrightarrow B^{\mathrm{q}}_{\geq -(i+1)}\mathcal{N}$ separately, we claim that in fact each $K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}/T_i \to K_0 B^{\mathrm{q}}_{\geq -(i+1)}\mathcal{N}/T_{i+1}$ is an isomorphism: an inverse is given by the map $K_0 B^{\mathrm{q}}_{\geq -(i+1)}\mathcal{N}/T_{i+1} \to K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}/T_i$ that is defined on generators by $[N_.] \mapsto -[N_.[1]]$. That is, we shift a binary multicomplex supported on $[-(i+1), \infty)$ by one place, so that it is supported on $[-i, \infty)$, and since shifting induces a -1 multiplication on these groups, essentially by Lemma 2.5[4], we introduce a -1 to correct for this (for $n > 1$ one needs to use Corollary 2.6 instead). This map is obviously well-defined, so each $K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}/T_i \to K_0 B^{\mathrm{q}}_{\geq -(i+1)}\mathcal{N}/T_{i+1}$ is an isomorphism. Therefore the composite $K_0 B^{\mathrm{q}}_{\geq 0}\mathcal{N}/T_0 \to \lim_i (K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}/T_i)$ is an isomorphism. But $K_0 B^{\mathrm{q}}_{\geq 0}\mathcal{N}/T_0$ is $K_1^{\mathrm{pos}}\mathcal{N}$ by definition, and we have shown that $\lim_i (K_0 B^{\mathrm{q}}_{\geq -i}\mathcal{N}/T_i) \cong K_n\mathcal{N}$ above, so the proof is complete. □

Since $K_n^{\mathrm{pos}}\mathcal{N}$ and $K_n\mathcal{N}$ are (naturally) isomorphic, we conclude that it is safe to use the presentation given in Definition 4.50 as the definition of $K_n\mathcal{N}$, as we have done in chapter 4.

[4]One needs to note that the cone of a complex in $B^{\mathrm{q}}_{\geq -i}\mathcal{N}$ is again in $B^{\mathrm{q}}_{\geq -i}\mathcal{N}$.

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