Derivation of Angle Dependence in Boost and Rotation

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For this derivation, I am using the natural units system (c = 1). For this case we have 2 protons with 100 GeV colliding nearly head on.

$$p^{blue} = \left(\sqrt{100^2 + m_{proton}^2}, 100\sin(\theta^{blue}), 0, 100\cos(\theta^{blue})\right) \tag{1}$$

$$p^{yellow} = \left(\sqrt{100^2 + m_{proton}^2}, 100\sin(\theta^{yellow} + \pi), 0, 100\cos(\theta^{yellow} + \pi)\right) \tag{2}$$

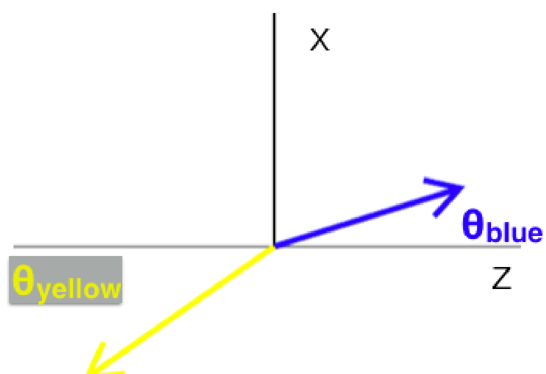

Figure 1: The vector diagram for the blue and yellow 4 momentum vectors.

Assuming θ^{blue} and θ^{yellow} are small (dropping $\theta^2 order$), we can reduce these components to

$$p^{blue} = \left(\sqrt{100^2 + m_{proton}^2}, 100\theta^{blue}, 0, 100\right) \tag{3}$$

$$p^{yellow} = \left(\sqrt{100^2 + m_{proton}^2}, -100\theta^{yellow}, 0, -100\right). \tag{4}$$

The total momentum vector is

$$p^{CMS} = (p^{blue} + p^{yellow}). \tag{5}$$

From this, we calculate the boost velocity vector to be

$$v^B = (p_x^{CMS}, 0, 0)/E^{CMS}. \tag{6}$$

Using the small angle approximation and small mass limit, we obtain

$$v_x^B = \frac{100(\theta^{blue} - \theta^{yellow})}{2\sqrt{100^2 + m_{proton}^2}} \approx 0.5(\theta^{blue} - \theta^{yellow}). \tag{7}$$

The resulting boost matrix (for the x-direction) is

$$B = \begin{bmatrix} \gamma & -\gamma v_x^B & 0 & 0 \\ -\gamma v_x^B & 1 + (\gamma - 1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For a given 4 momentum vector of a single particle, we have

$$p = (E, p_x, p_y, p_z). \tag{8}$$

The momentum in the boosted frame (in the limit that $\gamma \approx 1$) is

$$p_x\prime = -\gamma v_x^B E + p_x \approx -v_x^B E + p_x \tag{9}$$

$$p_z\prime \approx p_z. \tag{10}$$

The rotation angle after the boost (assuming $p_z\prime >> p_x\prime$) is

$$\theta_{xz} = \arctan\left(\frac{p_x^{blue}\prime}{p_z^{blue}\prime}\right) \approx \frac{p_x^{blue}\prime}{p_z^{blue}\prime}. \tag{11}$$

The resulting rotation matrix is

$$R = \begin{pmatrix} \cos(\theta_{xz}) & -\sin(\theta_{xz}) \\ \sin(\theta_{xz}) & \cos(\theta_{xz}) \end{pmatrix}.$$

The momentum in the rotated frame is

$$p_x\prime\prime = p_x\prime \cos(\theta_{xz}) - p_z\prime \sin(\theta_{xz}) \approx p_x\prime - p_z\prime\theta_{xz}. \tag{12}$$

Rewriting $p_x\prime$ in terms of the unprimed momentum we get

$$p_x\prime\prime = p_x - E v_x^B - p_z \frac{p_x^{blue} - E^{blue} v_x^B}{p_z^{blue}} \tag{13}$$

and then rewriting the boost velocity in terms of the beam angle, we obtain

$$p_x\prime\prime = p_x - 0.5E(\theta^{blue} - \theta^{yellow}) - \left(\frac{p_z p_x^{blue}}{p_z^{blue}}\right) + \frac{0.5 p_z E^{blue}(\theta^{blue} - \theta^{yellow})}{p_z^{blue}}. \tag{14}$$

Using the approximations $p_x^{blue}/p_z^{blue} \approx \theta^{blue}$ and $E^{blue}/p_z^{blue} \approx 1$, we obtain

$$p_x\prime\prime = p_x - 0.5E(\theta^{blue} - \theta^{yellow}) - (p_z\theta^{blue}) + 0.5p_z(\theta^{blue} - \theta^{yellow}). \tag{15}$$

If we define $F_{Blue} = 0.5(E + p_z)$ and $F_{Yellow} = 0.5(E - p_z)$, we obtain

$$p_x\prime\prime = p_x - \theta_{Blue}F_{Blue} - \theta_{Yellow}F_{Yellow}. \tag{16}$$

- For a charged pion at η = -3.5 (average BBCs detector location) with p_T = 250 MeV, E = 4.14 GeV and p_z = -4.13 GeV/c, we get the numerical results F_{Blue} = 0.00495 and $F_{Yellow} = -4.14$.

- For a charged pion at η = -1.5 (average FVTXs location) with p_T = 250 MeV, E = 0.604 GeV and p_z = -0.532 GeV/c, we get the numerical results F_{Blue} = 0.036 and $F_{Yellow} = -0.57$.

- For a charged pion at η = 0 (average central arm location) with p_T = 250 MeV, E = 0.286 GeV and p_z = 0.0 GeV/c, we get the numerical results F_{Blue} = 0.143 and $F_{Yellow} = -0.143$.

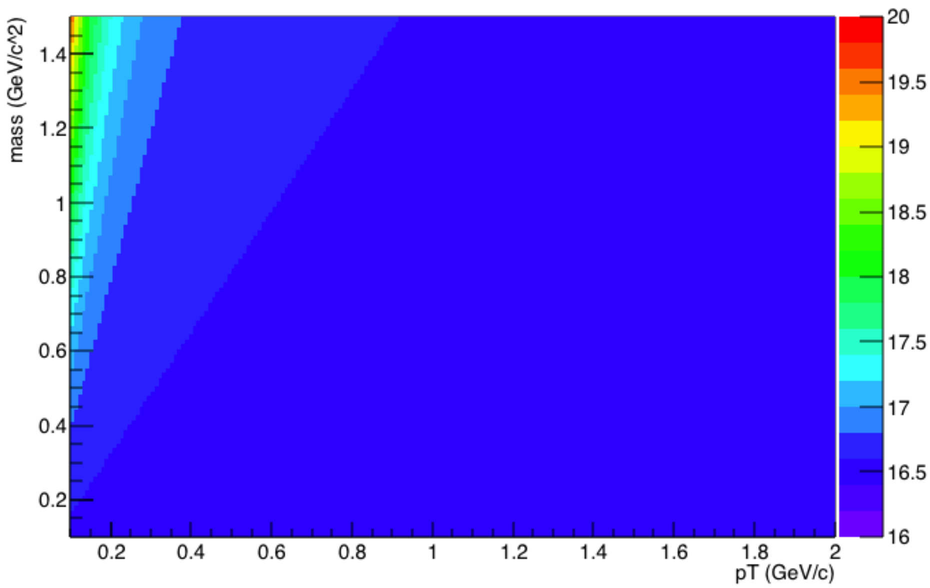

Figure 2: Diagram showing the relevant beam angles. The z-axis is F_{yellow}/p_T for a particle with η = -3.5