Balance control using CoM height variation: limitations imposed by unilateral contact

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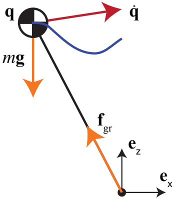

Motivation: balance mechanisms

- Stepping
- Changing center of pressure
- Using angular momentum

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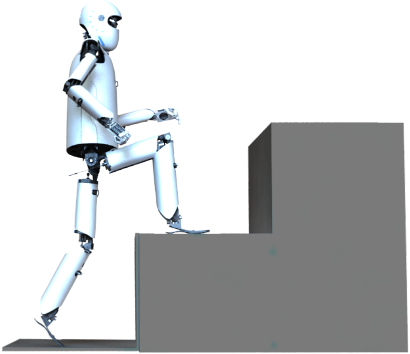
- ~~Changing center of pressure~~
- ~~Using angular momentum~~
- Changing CoM height
  - not studied as much
  - typically: CoM assumed to move on a plane (Linear Inverted Pendulum [Kajita and Tani, 1991])
  - often: horizontal plane

Step-up

Standard LIP assumption violated.



But, main focus of this talk: CoM height variation *as a balance mechanism*.

Problem statement

How to use CoM height variation to achieve balance,
in the sense of convergence to upright equilibrium?

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in the sense of convergence to upright equilibrium?

How much can CoM height variation help?

Variable height inverted pendulum

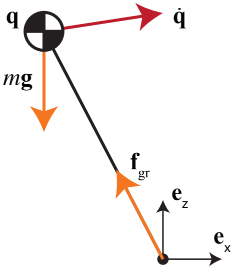
CoM dynamics:

$$m\ddot{\mathbf{q}} = m\mathbf{g} + \mathbf{f}_{\mathrm{gr}}$$

Constant centroidal angular momentum $\Rightarrow$ ground reaction force in direction of CoM:

$$\mathbf{f}_{\mathrm{gr}} = m\mathbf{q}u$$

with control input: $u \geq 0$
(no pulling on ground!)



$$\mathbf{q} = \begin{pmatrix} x \\ z \end{pmatrix}$$

Also studied in e.g. [Pratt and Drakunov, 2007], [Ramos and Hauser, 2015].

Goal: convergence to upright equilibrium

When is it *impossible* to achieve balance?

One extreme: choose $u = 0$ for all time $\Rightarrow$ ballistic trajectory

Lemma

'If z-intercept of ballistic trajectory is below CoP, then balance is not achievable without pulling on the ground.'

Argument: If z-intercept < 0, then $\frac{d}{dt}$ (z-intercept) ≤ 0 along trajectories, no matter what $u \geq 0$.

Control law

Design virtual constraint [Shiriaev et al., 2005, Pratt and Drakunov, 2007]:

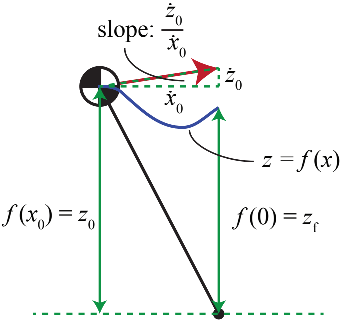
$$z = f(x)$$

Linear constraints:

1. Initial height
2. Initial slope
3. Final height
4. Come to a stop: zero orbital energy!
   [Pratt and Drakunov, 2007]

Cubic height trajectory:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$



Input trajectory and control law

Given height trajectory $f(x)$, can find input trajectory **in closed form**:

$$u = U(x, \text{initial state})$$

For a given initial state, a **rational** function **of** x**-coordinate only**.

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Feedback control law: plug in current state for initial state:

$$\begin{aligned} u &= U(x, \text{current state}) \\ &= \frac{\dot{x}\left(g^2 x^4 + 7 g x^3 \dot{x} \dot{z} - 7 g x^2 \dot{x}^2 z - 3 z_f g x^2 \dot{x}^2 + 10 x^2 \dot{x}^2 \dot{z}^2 - 20 x \dot{x}^3 z \dot{z} + 10 \dot{x}^4 z^2\right)}{g x^4 (\dot{x} z - x \dot{z})} \end{aligned}$$

Example: low initial horizontal velocity

Example: high initial horizontal velocity

Trouble: very low initial horizontal velocity

Region of attraction

For what set of initial conditions will

$$u \geq 0$$

be satisfied *for all* $t \geq 0$?

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be satisfied *for all* $t \geq 0$?
Same as requiring that:

$$U\,(x, \text{initial state}) \geq 0 \text{ for all } x \text{ between } x_0 \text{ and } 0$$

Quantifier elimination

Given quantified expression:

$$U\,(x, \text{initial state}) \geq 0 \text{ for all } x \text{ between } x_0 \text{ and } 0$$

Cylindrical **A**lgebraic **D**ecomposition algorithm [Collins, 1975] finds equivalent quantifier free expression:

$$a < 0, \quad \frac{7g}{a} + 20b \geq \sqrt{3g\left(\frac{3g}{a^2} + 40z_{\mathrm{f}}\right)}$$

$$\text{where } a = \frac{\dot{x}_0}{x_0}, \quad b = \dot{z}_0 - az_0$$

Explicit description of region of attraction!

Inside region of attraction

(note: phase plot is a 2D slice of a 4D space)

Outside region of attraction: pulling on the ground

(note: phase plot is a 2D slice of a 4D space)

A simple tweak

What if we just clip the orbital energy controller at 0?

$$u = \max\left(0, U\left(x, \text{current state}\right)\right)$$

Fact

If z-intercept of ballistic trajectory is above CoP, then state will eventuallly enter region of attraction of orbital energy controller.

$\Rightarrow$ controller achieves balance for all states from which balance is achievable!

A simple tweak

Contributions

For 2D **variable height inverted pendulum**:

- **Necessary condition** for balance, subject to unilateral contact
- Two **control laws** + their regions of attraction in closed form
- Region of attraction of one control law matches necessary condition

  $\Rightarrow$ **necessary condition is also sufficient.**

See `github.com/tkoolen/VariableHeightInvertedPendulum`.

References

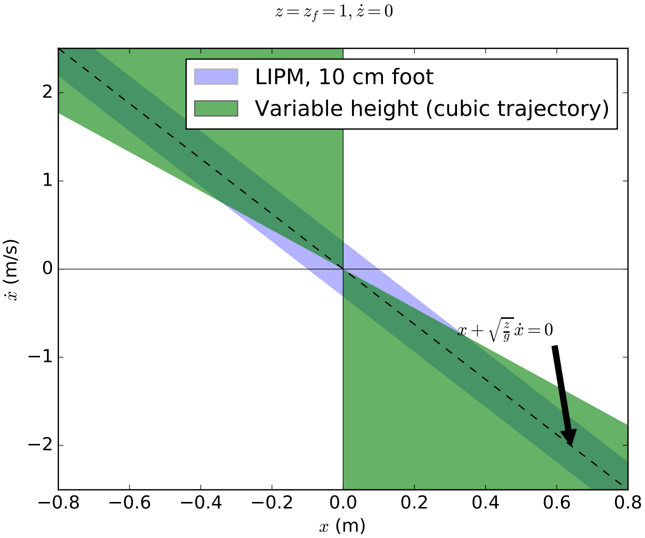
Collins, G. E. (1975).
Quantifier elimination for real closed fields by cylindrical algebraic decompostion.
In *Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern, May 20–23, 1975*, pages 134–183. Springer.

Kajita, S. and Tani, K. (1991).
Study of dynamic biped locomotion on rugged terrain-derivation and application of the linear inverted pendulum mode.
In *Robotics and Automation, 1991. Proceedings., 1991 IEEE International Conference on*, pages 1405–1411 vol.2.

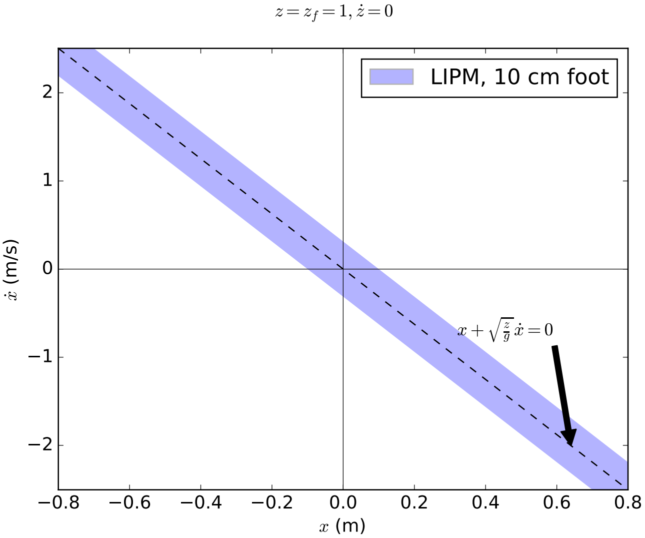
Pratt, J. E. and Drakunov, S. V. (2007).
Derivation and application of a conserved orbital energy for the inverted pendulum bipedal walking model.
In *Robotics and Automation, 2007 IEEE International Conference on*, pages 4653–4660.

Step up

Comparison to CoP control



Comparison to CoP control



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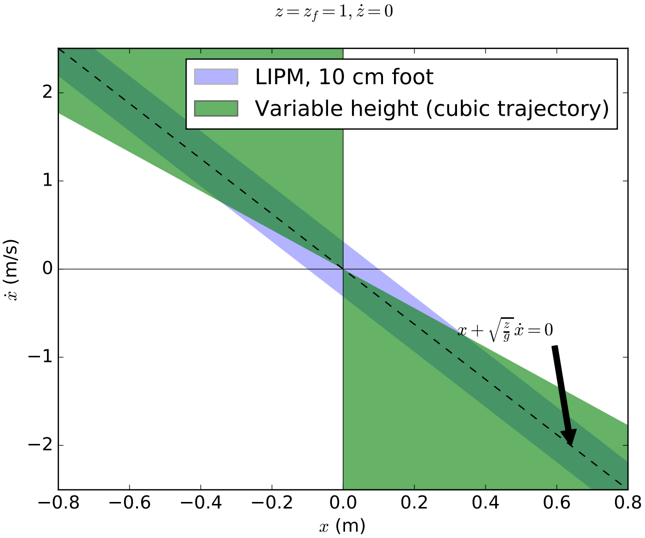

Solving for height trajectory

Height trajectory:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Solve for coefficients:

$$\begin{array}{r} \text{Initial height:} \\ \text{Initial slope:} \\ \text{Final height:} \\ \text{Zero orbital energy:} \end{array} \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & 2x_0 & 3x_0^2 \\ 1 & 0 & 0 & 0 \\ \frac{3}{2} g x_0^2 & g x_0^3 & \frac{3}{4} g x_0^4 & \frac{3}{5} g x_0^5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} z_0 \\ \dot{z}_0 / \dot{x}_0 \\ z_f \\ k \end{pmatrix}$$

$$\text{with } k = \frac{1}{2} \left(\dot{x}_0 z_0 - \dot{z}_0 x_0 \right)^2 + g x_0^2 z_0$$

Control law in terms of a and b

Controller:

$$\begin{aligned} u(\mathbf{x}) &= U(x, \mathbf{x}) \\ &= \frac{\dot{x}\left(g^2x^4 + 7gx^3\dot{x}\dot{z} - 7gx^2\dot{x}^2z - 3z_f gx^2\dot{x}^2 + 10x^2\dot{x}^2\dot{z}^2 - 20x\dot{x}^3z\dot{z} + 10\dot{x}^4z^2\right)}{gx^4(\dot{x}z - x\dot{z})} \\ &= \frac{3a^3z_{\mathrm{f}} - ga}{b} - \frac{10a^3b}{g} - 7a^2 \end{aligned}$$

with

$$a = \frac{\dot{x}_0}{x_0}, \quad b = \dot{z}_0 - az_0$$

Results

Slice of the 4D state space at $\dot{z} = 0$: