Financial Planning - FA 2019
Ordinary Differential Equations
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Introduction

Taking on the task of financial planning, we help advise one of our clients on how to save money to help pay for college education for their newborn baby in 18 years. In this report, we answer the following questions:

1. If the client saves $\$ 100$ per month, what can they expect the account value to be in 18 years?
2. How much would they need to save per month to be reasonably certain that they have at least $\$ 60,000$ in the account after 18 years?

Deterministic Case

First, we consider the simple deterministic model which only takes into account the relevant parameter values, in this case, the rate at which the investment grows, r. Using the $S\&P500$ data, we calculated the annual average returns rate (AAR) from 1980 through 2018 to be $r = 11.35\% = 0.1135$. Assuming the client saves $\$ 100$ per month, we have our yearly rate of contribution $M = 100 \cdot 12 = 1200$. With initial conditions: $A(0) = 0$ and $A(1) = 1200$.

$$\frac{dA}{dt} = rA(t) + M = 0.1135 \cdot A(t) + 1200$$

$A(t)$: amount in the investment at time t in years
M: rate of contributions made to the investment
r: rate of returns or the growth rate of investment

Solving for $A(18)$ where $t = 18$ would give us the value of the amount to expect in 18 years. By solving the differential equation using the integrating factor $e^{0.1135t}$, we get

$$A(t) = \frac{1200}{-0.1135} + \frac{1200}{0.1135} \cdot e^{0.1135t}$$

So then, $A(18) = \$ 70,961.45$

However, if we take away the assumption that the client must save $\$ 100$ per month and only take into account the fact that they want to reach at least $\$ 60,000$ after 18 years. Then we would solve for $A(18) \geq \$ 60,000$, which is:

$$\frac{1200}{-0.1135} + \frac{1200}{0.1135} \cdot e^{0.1135t} \geq \$ 60,000$$

We then get $M \geq 1015$. Thus, they would have to save at least $\frac{1015}{12} = \$ 85$ monthly to reach that goal.

Stochastic Case

The stock market is subjected to unpredictable events. Hence, we use the following discretized differential equation to take into account this randomness.

$$\Delta A = (rA + M)\Delta t + \sigma A \Delta B$$

σ: represents how "noisy" the investment is, where larger σ corresponds to higher volatility of the investment.
$B(t)$: a Brownian motion, it has normal distribution with mean 0 and standard deviation $\sqrt{\Delta t}$.

Assuming volatility of $\sigma = 0.15$ and average growth rate of $r = 0.1$, if the client saves \$ 100 per month, they can be reasonably confident to expect at least $A(18) = \$\,43,000$ after 18 years (see 3rd column in the table). Defining the term "reasonably confident" as the target amount $A(t)$ being at least in the 25^{th} percentile, in other words, the term "reasonably confidence" means having 75% confidence.

mean	median	25^{th} percentile	75^{th} percentile
67204.34	56488.90	43384.31	78628.28
68910.10	61294.95	44728.12	84593.03
68363.98	60863.72	43827.10	84574.16

Since we are taking into account the unpredictability of stock market, the amount they would have to save per month to reach \$ 60,000 can be acquired through solving for M in the inequality $A(18) \geq 60000$. Hence, the monthly contribution would have to be at least $\frac{M}{12} = \$\,140$ to be reasonably confident on reaching the 18 years goal. Again, that means the 25^{th} percentile $\geq \$\,60,000$.

Further Exploration

1. **The role of volatility**

   We look into when $r = 0.1$ and $\sigma = 0,\ 0.25,\ 0.5,\ 1$. When $\sigma = 0$, the mean and median both give \$ 60,056 (an approximation using Euler's method, but the result should be the same as the deterministic case, that is \$ 70,961.45). When $\sigma = 0.25,\ 0.5$ or 1, the median value starts diverging from the mean as volatility gets higher. To take a closer look, the 75^{th} percentile explodes to a significantly higher value as volatility is higher. Thus, with higher volatility, all possible values of $A(t)$ would scatter in a wider range. In comparison with lower volatility, $A(t)$ values are less spread out.

	$\sigma = 0$			$\sigma = 0.25$		
mean	60056.32	60056.32	60056.32	67685.12	66252.28	66656.62
median	60056.32	60056.32	60056.32	89569.82	90074.36	90657.28
25^{th} percentile	60056.32	60056.32	60056.32	39702.87	36530.73	37169.94
75^{th} percentile	60056.32	60056.32	60056.32	114449.2	118440.1	118785.7

	$\sigma = 0.5$			$\sigma = 1$		
mean	66630.35	62509.36	63379.66	134632.9	84952.31	75314.07
median	399970.4	351938.9	368072.0	1.2248e7	1.9954e7	1.7400e8
25^{th} percentile	22122.16	20231.95	20516.10	8782.257	8349.932	8871.297
75^{th} percentile	256615.5	194800.0	213815.4	1.4797e6	1.3394e6	1.0992e6

2. **Fund Options for Investment**

   Three different funds with different average returns and volatility are offered.

   Fund 1, with high r and high σ, gives high returns, but it is risky as the value of the 25^{th} percentile is roughly half that of the median. So, fund 1 could give the client a big win if they fall into the range from the median to the 99^{th} percentile, but not so much if they happen to be below the 25^{th} percentile.

   Fund 2, with moderate r and moderate σ, gives returns reasonably close to the \$ 60,000 that the client needs, it is less risky as the difference between the 25^{th} percentile and the median is not too far off from each other.

   Fund 3, with low r and low σ, would probably be the safest as the client could confidently reach at least \$ 32000 after 18 years. However, the return is only roughly half of the \$ 60,000 that the client needs.

	$r = 0.15$ and $\sigma = 0.35$			$r = 0.1$ and $\sigma = 0.15$			$r = 0.05$ and $\sigma = 0.03$		
mean	244132.7	288849.3	276000.2	74505.39	70197.71	70996.79	35195.93	35112.54	35154.66
median	109396.0	91316.35	99009.95	59205.60	57912.98	57569.36	35131.34	35050.67	35091.23
25^{th} pctl	44566.51	46115.35	46115.35	44191.89	41366.36	42925.68	33115.50	32672.25	32658.89
75^{th} pctl	280949.3	252455.5	257355.9	78451.88	81835.09	82344.81	37062.14	37274.90	37274.90

The figures calculated in the above table use the assumption that the client invests \$ 100 per month into strictly one fund. Now, if we consider a mix of investment across these 3 funds, it would be necessary to define how much risk the client is comfortable with taking, with the end goal of reaching \$ 60,000.

We do not want our 25^{th} percentile to fall below \$ 21,600 because that is what one gets if \$ 100 is saved over 18 without any external increment. But our 25^{th} percentile should still ideally be near \$ 60,000 give or take. So we consider the following three categories, not that the data points represent how \$ 100 is invested in funds 1, 2, and 3, respectively:

(a) High returns, high risk option, with average 25^{th} percentile $>$ \$ 60,000:
(90 10 0)

(b) Guaranteed-targeted returns, considerable risk, with \$ 50,000 $< 25^{th}$ percentile $<$ \$ 60,000:
(60 30 10), (80 10 10), (80 20 0)

(c) Near-targeted returns, moderate risk, with \$ 45,000 $< 25^{th}$ percentile $<$ \$ 50,000:
(30 70 0), (40 50 10), (50 50 0), (60 20 20), (60 40 0), (70 10 20)

Hence, we can at least say that considering a mix of investments across the 3 funds would be better than just going with only one.

3. $M(t)$ **is not constant, but increases over time.**

That means as time goes on, we take into consideration that the couple will advance in their careers and earn higher incomes. If $M(t)$ is not constant, then our stochastic differential equation $\frac{dA}{dt}$ would now take on two dependent variables, $A(t)$ and $M(t)$. Assuming that the rate at which the client's income increase will be the same rate of increment that applies to investment contributions $M(t)$, represented by w, we get the following system of differential equations:

$$\frac{dA}{dt} = r \cdot A + M(t) + \sigma \cdot A(t) \cdot \xi(t), \quad \frac{dM}{dt} = w \cdot M(t)$$

Assuming $r = 0.1$ and if their income grows by 50% from now until the year 18, the the annual rate of increment of the contribution would be $w = \frac{50}{18} = 27.78\% = 0.23$. Solving for $M(t)$ using the initial conditions $M(0) = 0$ and $M'(0) = 0$, we get $M(t) = -0.9 \cdot e^{0.3t} + 0.9 + 0.3t$. Plugging that back into our $\frac{dA}{dt}$ differential equation, we can then solve for $A(t)$.

Since we want our $A(t) \geq 60,000$, we want to find the minimum required $M(t)$ for that to hold. Applying the 75% confidence approach, we look at when the 25^{th} percentile of $A(t)$ where that value would be at least $60,000$, and by the solving the inequality, the amount the client would have to save per month would have to be at least $\frac{M}{12} = \$\,115$ to be reasonably confident to get to their goal.