Frost Line Depth - SP 2022
Partial Differential Equations
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Problem Statement

Our contractor friend is in charge of a construction crew that builds houses in Preston, MN. When laying the foundation of a house, it is key that the foundation goes below the frost line, i.e. the depth at which the soil freezes, because otherwise the expansion and contraction of the soil that occurs with the freeze or thaw cycle can cause shifting and cracking in the foundation, which can lead to structural instability. The four main soil types our friend encounters are silt, clay, gravel, and loam. Here, we are going to explore whether the frost line is significantly different for different types of soil. We define frost line depth to be the lowest depth at which freezing temperature is reached at some point in the year.

It is appropriate to note that the units used throughout this paper will be: (°F) for temperature T, (ft) for depth x, (days) for time t, and $(\frac{ft^2}{day})$ for thermal diffusivity k.

1. Setting up Our IBVP

We start by establishing a PDE that models soil temperature (°F) as a function of depth (ft). For now, we denote the thermal diffusivity constant in our PDE as a general k. Suppose that the top boundary of the soil corresponds to $x = 0$ and that this boundary is exposed to an exterior air temperature that is sinusoidal with minimum and maximum temperatures of T_{min} and T_{max}, respectively. It also seems reasonable that the fluctuation in soil temperature decreases as the depth increases, and that for practical purposes, the soil temperature can be taken as a constant at depth c feet. The temperature at this depth is the average of T_{min} and T_{max}.

With these assumptions, we can set up the general IBVP:

$$
\begin{aligned}
u_t &= ku_{xx} && \text{(PDE)}\\
u(0,t) &= A + B \cdot \sin(Cx + D) && \text{(BC1)}\\
u(c,t) &= \frac{T_{max} + T_{min}}{2} && \text{(BC2)}\\
u(x,0) &= f(x) && \text{(IC)}
\end{aligned}
$$

where A, B, C, and D correspond to a vertical shift, the amplitude, the coefficient for period calculation, and a horizontal shift of a sinusoidal equation, respectively.

To determine T_{min} and T_{max}, we look at average temperature data for Preston, MN from January to December 2020 as noted from *wunderground.com*. Keep in mind that our assumptions consider air temperature to be sinusoidal. This is not the case in real life as there are large shifts in temperature day-to-day. However, this assumption is reasonable if we consider "temperature" as referring to "average monthly temperature", which then smooths out the large daily fluctuations by the average effect. Plots of monthly average temperatures are found in Figure 1. Thus, our values for T_{min} and T_{max} are not absolute temperatures, but rather "monthly average" temperatures encountered throughout the year. We found that $T_{min} = 17.88°F$ in February, and $T_{max} = 72.21°F$ in July.

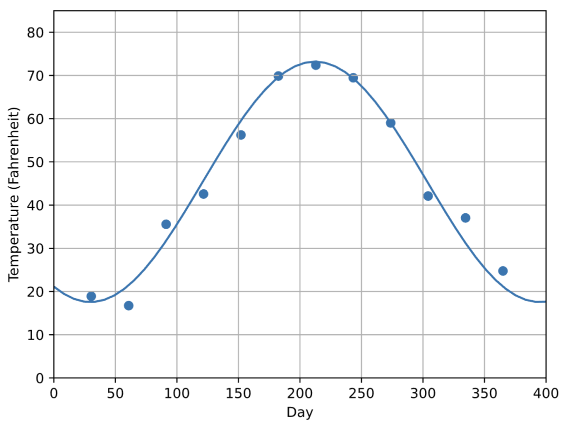


Figure 1: Monthly average temperature plotted with $u(0,t)$

We now calculate values of A, B, C, and D in our first boundary condition. The values of A and B were calculated from our collected data set and process through means of CoCalc. Values C and D were chosen to conform the boundary condition to our plotted data points—where the horizontal shift is chosen to be 4.2 as it coincides with the date between April and May where the mean average temperature occurs. Our first boundary condition was constructed to fit our data points and is plotted as such in figure 1. For $u(c,t)$, we chose the value to be the average of T_{min} and T_{max}, as mentioned earlier. For our initial condition, we choose a simple linear equation that matches our endpoints. Thus, we have the updated IBVP:

$$u_t = ku_{xx} \tag{PDE}$$

$$u(0,t) = 45.39 + 27.83 \cdot \sin(\frac{2\pi t}{365} + 4.2) \tag{BC1}$$

$$u(c,t) = 45.39 \tag{BC2}$$

$$u(x,0) = \frac{x}{40000} + 21.12 \tag{IC}$$

2. Solving PDE Using Numerical Methods in CoCalc

We use numerical methods in CoCalc to solve our IBVP, specifically using the implicit method as it is unconditionally stable, meaning it gives us more freedom to choose h and τ values compared to the explicit method. Note that h denotes the distance between neighboring spacial nodes, and τ denotes the distance between neighboring temporal nodes. In real life, the depth at which soil temperature becomes constant is typically below 20-30 ft. So, we've chosen a depth of 30 ft. for an approximation of our BC2, such that $u(30,t) = 45.39$. From this, we choose h to be 0.5 ft. as our step size. Note that since we are considering the temperature for 365 days, we decide to choose τ to be a 10 day step, for the efficiency of running CoCalc. Solutions of the IBVP are shown as a 3D plot in Figure 2.

The x-axis of this figure ranges from 0 to 30 ft. which is the soil depth we're observing, while the y-axis ranges from 0 to 365 days, and the z-axis represents the temperature from 0 to 150 °F. For the case of sand, this 3D plot represents the temperature profile of soil depth from

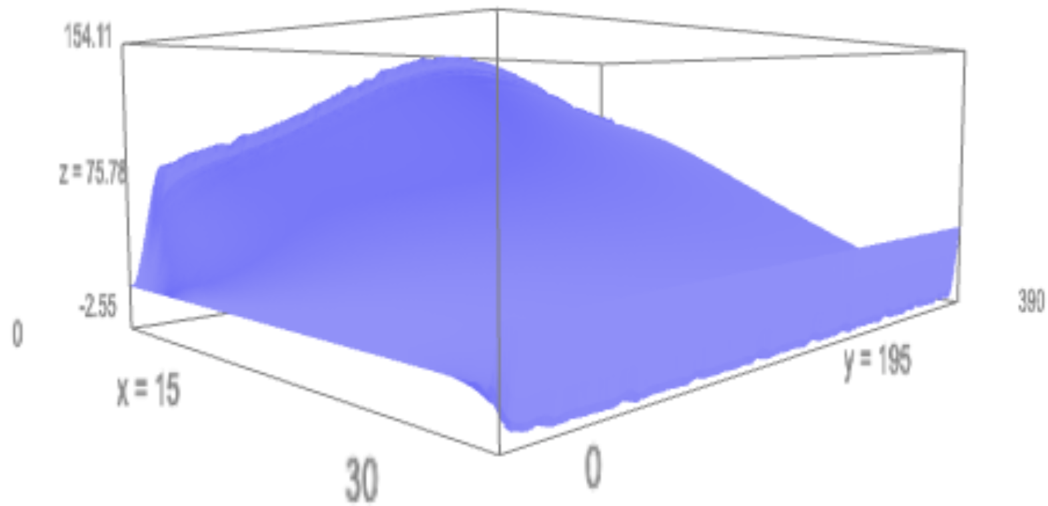


Figure 2: CoCalc solutions to IBVP for $k = 0.420\frac{ft^2}{day}$ (sand).

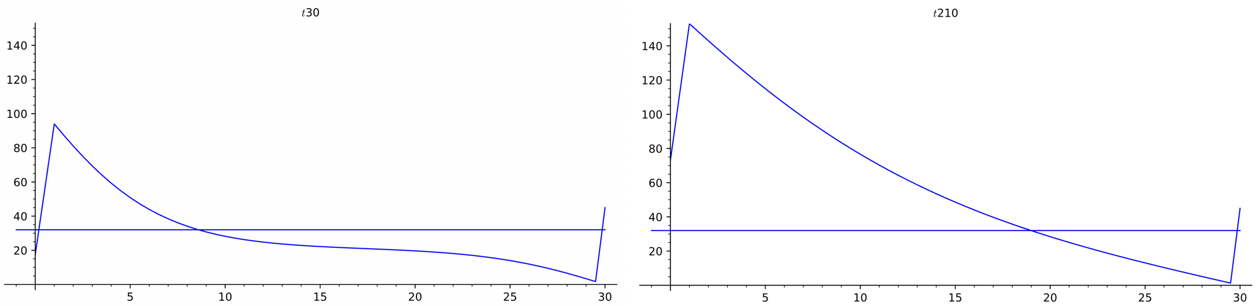
January through December. Notice that as temperature drops below 32 °F, our temperature stays consistent without much fluctuation. As depth approaches 30 ft., the temperature rises up to reach 45.39 °F. If we were to keep plotting x beyond 30ft., our graph would level at 45.39°F, becoming constant at this temperature as denoted by BC2.

3. Estimate Frost Line Depth

Next, we estimate the frost line depth for each type of soil. We choose two days of the year that correspond to the coldest and warmest days of the year, respectively. These days are $t = 30$ (between January and February) and $t = 210$ (between July and August), as calculated from our collected temperature data. We choose these points as they should correspond with our highest and lowest frost line.

The frost line depth is calculated from the cross section at $t = 30$ and $t = 210$ of the 3D plot above. From there, we plot a straight line at $y = 32$ as 32 °F is the freezing temperature of water, and hence, the freezing temperature for wet soil. The frost line depths are calculated from the intersection of these two plots. Figure 3 demonstrates this process.

Figure 3 is the cross section of the 3D plot at $t = 30$ and $t = 210$ for sand with $k = 0.420$. Figure 3a tells us the behavior of soil temperature from depth $x = 0$ to $x = 30$ at $t = 30$, the coldest day of the year. Notice that when x approaches 7.7 ft., it reaches 32° Fahrenheit, which is the freezing temperature, and the temperature decreases below the freezing point as we go deeper into the ground. However, as it approaches our chosen $c = 30$ ft., the temperature rises back up to 45.39 °F which is the temperature that will stay constant for any depth below $x = 30$. Similarly, Figure 3b tells us the behavior of soil temperature from depth $x = 0$ to $x = 30$ at $t = 210$, the hottest day. Notice that when x approaches 17.8 ft., it reaches the freezing temperature 32 °F. For the rest of the ground materials, we compile



(a) $t = 30$ (b) $t = 210$

Figure 3: Estimated frost line depth of the coldest and warmest days of the year.

their minimum frost line depths for the coldest and hottest days, as seen in Table 1.

Material	$\boldsymbol{k}$	Frost Line Depth, $t = 30$	Frost Line Depth, $t = 210$
Sand	0.420	7.7	17.8
Loam	0.452	7.8	18.1
Clay	0.495	8.1	18.6
Silt	0.538	8.5	18.9
Saturated Silt or Clay	0.635	9.2	19.8
Saturated Sand	0.850	10.6	21.0

Table 1: Frost Line Depths (ft.) for all materials, where k $(\frac{ft^2}{day})$ is the thermal diffusivity constant.

4. Concluding Remarks

Our goal for this project was to explore whether the frost line depth is significantly different for different types of soil. Our results show that for each type of material, as the thermal diffusivity increases, the frost line depth increases as well. For the coldest day, notice that the frost line depth ranges between 7.7 and 10.6 ft. Similarly, the hottest day's frost line depth ranges between 17.8 to 21.0 ft. So, in constructing a house that would last in all seasons, it would be wise to aim for the minimum frost line depth of the hottest day, that is between 17.8 to 21.0 ft. for Preston, MN depending on the type of soil.