

**Problem set # 6**

Due: Thursday, December 9th, by 4pm.

Ticker	Description
XLB	Materials
XLE	Energy
XLF	Financials
XLI	Industrials
XLK	Technology
XLP	Consumer Staples
XLU	Utilities
XLV	Healthcare
XLY	Consumer Discretionary

**1. Covariance Matrix Decomposition:**

- Download historical price data from January 1st 2010 until today for the sector ETFs in the table above. Clean the data for splits and any other anomalies.
- Calculate the covariance matrix of daily returns for the sector ETFs.
- Perform an eigenvalue decomposition of the covariance matrix. Plot the eigenvalues in order from largest to smallest. How many eigenvalues are positive? How many are negative? How many are zero? If any are negative, is this a problem? How many do you think are statistically significant?
- Generate a random matrix of the same size as your covariance matrix, where each element has a standard normal distribution.
- Perform an eigenvalue decomposition of this random matrix. Plot the eigenvalues in order from largest to smallest. Comment on the differences and similarities between the structure of the eigenvalues in this matrix and your historical covariance matrix.

**2. Portfolio Optimization:**

Recall that an unconstrained optimal portfolio can be found using the following formula:

$$\max_w w^T R - a w^T C w \quad (1)$$

The constant  $a$  indicates the amount of our risk aversion. Assume you are an investor with a risk aversion coefficient of 1.

- (a) Calculate the historical annualized returns for each sector ETFs.
- (b) Using the annualized returns as expected returns and the covariance matrix obtained above, calculate the weights of the unconstrained mean-variance optimal portfolio.
- (c) Create a new set of expected returns equal to the historical annualized return ( $\mu$ ) plus a random component, that is:

$$\mathbb{E}[r] = \mu + \sigma * Z \quad (2)$$

where  $\sigma$  is the volatility of the random component and  $Z$  is a standard normal random variable.

Using  $\sigma = 0.005$ ,  $\sigma = 0.01$ ,  $\sigma = 0.05$  and  $\sigma = 0.1$  calculate the unconstrained mean-variance optimal portfolio using the adjusted expected returns and the covariance matrix. Comment on the stability of your portfolio weights to changes in expected returns.

- (d) Consider a regularized covariance matrix that is a blend of the historical covariance matrix and a diagonal matrix with each asset's variance along the diagonals, that is:

$$\tilde{\Sigma} = \delta \Sigma_{\text{diag}} + (1 - \delta) \Sigma_{\text{full}} \quad (3)$$

where  $\delta$  is the weight of the diagonal matrix and  $(1 - \delta)$  is the weight of the historical covariance matrix.

- (e) Set  $\delta = 1$  and perform an eigenvalue decomposition of the regularized covariance matrix. What is the rank of the regularized covariance matrix now?
- (f) Try different values of  $\delta$  between 0 and 1 and perform eigenvalue decomposition on the regularized covariance matrix. How many eigenvalues are positive? How many are zero?
- (g) Repeat the exercise in (2.c) with the regularized covariance matrix for a few values of  $\delta$ . Compare the stability of your portfolio weights to what you got with the historical covariance matrix.

**NOTE: All code for completing these exercises should be completed either in Python or C++ and should be written generically. You may end up using this code on future assignments so I encourage you to code thoughtfully.**