MF796-Project (FX Markets and Their Option Implied Risk-Neutral Densities)

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March 6, 2022

1 Introduction

Foreign exchange rates are very liquid markets, and thus there are quite a lot of data sources to pull from. There are different baskets of exchange rates that classify them into broad categories, the ones we look at are the G10, EMEA and LATAM baskets. The G10 can be thought of as currencies that are attached to mature economies that are relatively stable and are not likely to see massive fluctuations over a short period of time. EMEA is the basket that encapsulates the economies that are growing but still somewhat unstable. LATAM is much more similar to EMEA economies where we see mostly growing and unstable economies.

There are some characteristics of FX markets that we need to keep in mind when we are modeling options on FX rates. There are a lot of jumps in the more risky currencies, a good example is when Argentina defaults on its sovereign debt in what has become a cyclical fashion, its Peso will jump to a much less valuable exchange rate against the US dollar for example. The variance-gamma model is a good model to use to encapsulate these jumps because it is a stochastic jump process. This model is especially good for short dated options where we can think of the option jumping into the money or out of the money.

obviously, we want to try to make money when we trade options, so if we can find options that have favorable probability densities, we would rather trade these densities over densities that are tightly centered around a mean. This can be done by extracting the risk-neutral densities from the market, and hopefully we find a mismatch.

2 Models

The model that we need to break down is the variance gamma model, where we can use a characteristic function to price an option, or we can use Monte Carlo simulation which is great for path dependant options. There are two ways we can simulate a gamma process. The first is the time changed gamma process, where we have a drift θ times a gamma process plus a gamma process times our typical Brownian Motion.

$$X^{VG}(t; , \sigma, \nu, \theta) := \theta\Gamma(t; 1, \nu) + \sigma W(\Gamma(t; 1, \nu)), \tag{1}$$

Then we will finish off the simulation by adding in the initial asset value to create jumps with respect to the random clock.

$$X(t_i) = X(t_{i-1})\theta\Delta G_i + \sigma\sqrt{\Delta G_i}Z_i \tag{2}$$

This simulation looks very similar to the normal Brownian motion that we are used to seeing, but if

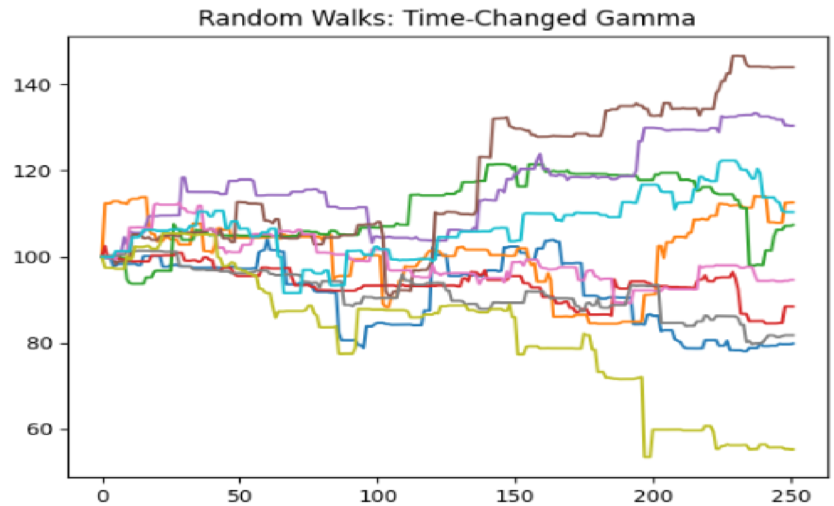


Figure 1: Time Changed Gamma Process(10 simulations)

you notice, there are much larger jumps in each individual path. Over enough simulations, we can see that the mean of the payoff will resemble that of the normally simulated Brownian Motions.

The second way to simulate is to use a difference of gammas.

$$X(t_i) = X(t_{i-1}) + \Gamma_i^+(t) - \Gamma_i^-(t).$$

$$X^{VG}(t; , \sigma, \nu, \theta) := \Gamma(t; \mu_p, \mu_p^2\nu) - \Gamma(t; \mu_p, \mu_p^2\nu) \tag{3}$$

where,

$$\mu_p := \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{1}{2} \quad \text{and} \quad \mu_p := \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{1}{2}. \tag{4}$$

We will simply feed this into the typical simulation that we have done so many times, except we'll use this is the dW_t instead of a standard normal draw for the random walks. VG as a difference between two Gamma processes:

$$X(t_i) = X(t_{i-1}) + \Gamma_i^+(t) - \Gamma_i^-(t) \tag{5}$$

parameters:

$$\theta = \text{long-run mean} \tag{6}$$
$$\sigma_0 = \text{initial volatility} \tag{7}$$
$$\nu = \text{variance of volatility} \tag{8}$$
$$\tag{9}$$

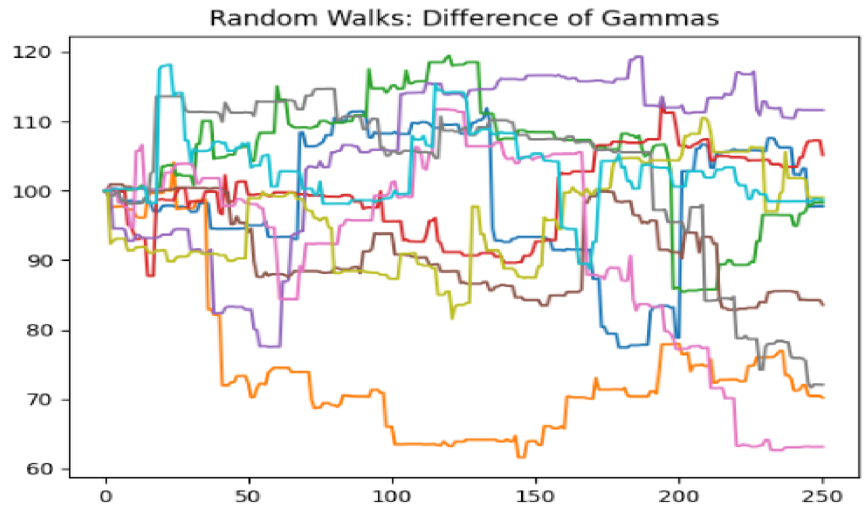


Figure 2: Difference of Gammas Process(10 simulations)

These models will be best when we want to price digital options, which are a significant part of what we need to price. This process can work well for Asian options, but for the digital options this model will work very well since we are looking at jumps.

3 Options

There are some models that will fit our different payoffs better than others. The Asian options are not going to price well with closed form solutions, however we can price a geometric mean Asian option with a closed form solutions which we will show a little later on. First, we need to cover the different payoffs of the Asian options. Fixed strikes:

$$C(T) = \max(A(0,T) - K, 0)$$
$$P(T) = \max(K - A(0,T), 0)$$

floating strikes:

$$C(T) = \max(\kappa A(0,T) - S(T), 0)$$
$$P(T) = \max(S(T) - \kappa A(0,T), 0)$$

geometric averages can be put into a closed form solution similar to Black-Scholes:

$$C_G = e^{-rT}\mathbb{E}[(G_T - K)_+] = \frac{e^{-rT}}{\sqrt{2\pi}} \int_l^{\infty} (G_T - K)e^{-x^2/2}dx$$
$$= G_T \geq K \to S_0 e^{\frac{1}{2}(r-\frac{1}{2}\sigma^2)T} e^{\frac{\sigma}{T}\int_0^T (T-t)dW_t} \geq K$$
$$C_G = S_0 e^{(b-r)T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

and for a put,

$$P_G = Ke^{-rT}\Phi(-d_2) - S_0 e^{(b-r)T}\Phi(-d_1)$$

We need these closed form solutions when we look at pulling out risk neutral densities later on. Conditional Asian options are a little bit like a barrier option but also a little bit like a digital option:

$$P = \max(K - \frac{\int_0^T S(t) I_{\{S(t)>b\}} dt}{\int_0^T I_{\{S(t)>b\}dt}}, 0)$$

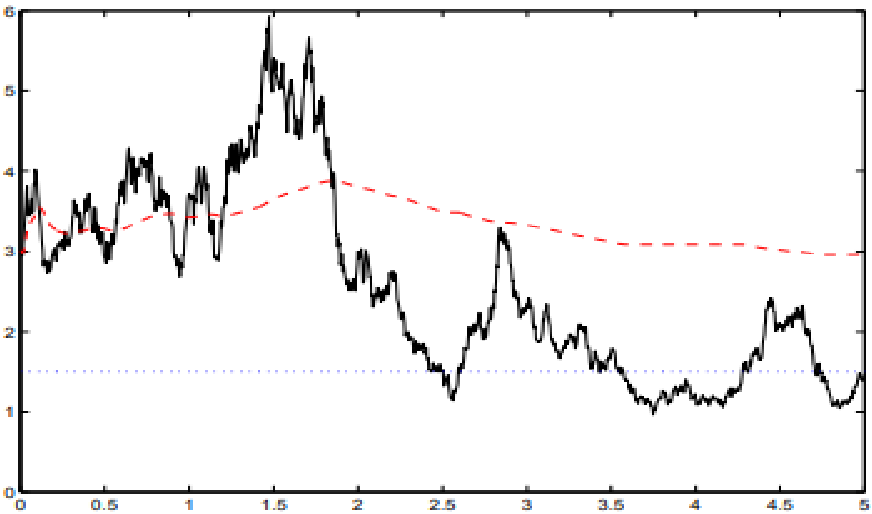


Figure 3: Conditional Asian Graph

The indicator function is acts as a barrier, and essentially cheapens the option premium given that it is conditional on the barrier. The second class of options that we need to price are the digital options. We dealt with cash-or-nothing options which are a modification of the Black-Scholes formula.

$$C = e^{-rT} \Phi(d_2)$$
$$P = e^{-rT} \Phi(-d_2)$$

where

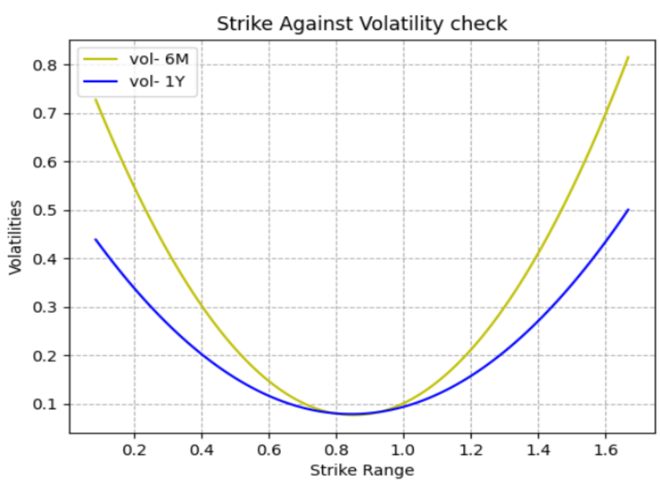
$$d_1 = \frac{\ln \frac{S}{K} + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

We can price these with the Variance Gamma model or with the Black-Scholes model. Because we are essentially pricing these options based off of a probability density, we can use the densities we extract from the market to price the options.

4 Risk-Neutral Densities

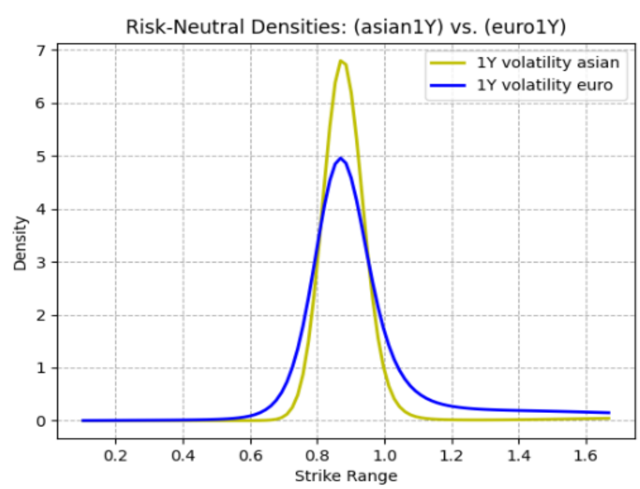
The whole idea of what this paper/project is designed on is to take advantage of mismatched densities, so extracting the most accurate densities from market prices is essential. One way of doing this is to use the Breeden-Litzenberger approach to pull out the densities. We can only do this for closed form solutions to option prices, so the Black-sholes and geometric mean Asian option formulas are the only two that we can use here.

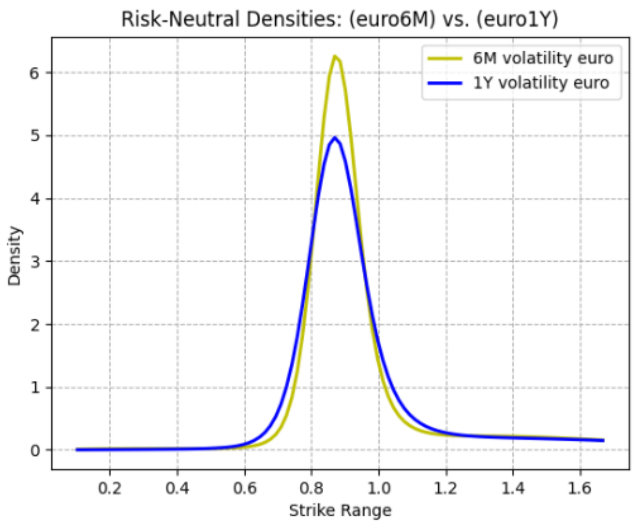
$$
\begin{aligned}
c &= e^{-\int_0^T r_u du} \int_K^{+\infty} (S_T - K)\phi(S_T)dS_T \\
\frac{\partial c}{\partial K} &= e^{-\int_0^T r_u du} \int_K^{+\infty} -\phi(S_T)dS_T \rightarrow e^{-\int_0^T r_u du}(\Phi(K) - 1) \\
\frac{\partial^2 c}{\partial K^2} &= e^{-\int_0^T r_u du}\phi(K) \\
&= e^{-\int_0^T r_u du}\frac{c(K-h) - 2c(K) + c(K+h)}{h^2}
\end{aligned}
$$



This is just a finite difference that is relatively easy to implement and gives us some probabilities that look like the following.

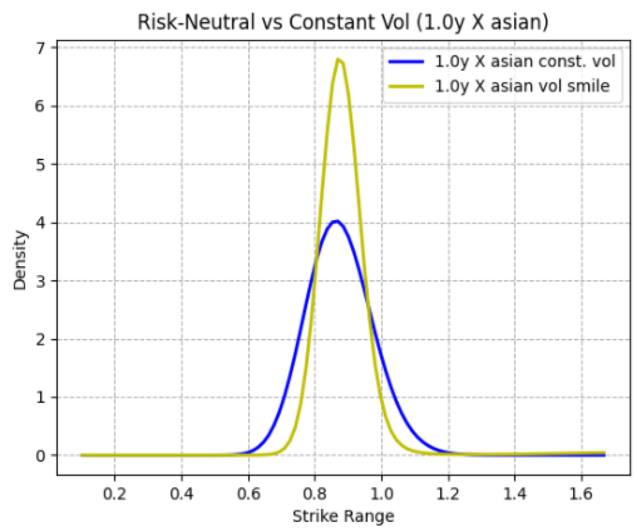
First we look at the USDEUR cross because it should be one of the more normally distributed densities of all the crosses that we look at. It has always traded slightly under an exchange rate of 1-1, has similar economies behind each currency and until recently, similar interest rates. Looking at the skew of strike vs. volatility, we see that for the USDEUR cross, there is not much of a skew to the call or the the put sides. There is a minimal skew to the put side, but noting big enough to suggest there is useful information in the differences. The USDEUR cross is probably two of the three most traded currencies, so we would expect to see a very low skew. The second density comparison is between the two main option types we are trading, where there is clearly a more tightly centered density for the Asian options than there is for the European options. Additionally, there appears to be a fatter tail too for the European option than for the Asian option. This could be because for an expiry that ends deep into the money, the payoff for a European option will be higher than that of an Asian. There is very little difference between the two densities other then the tails, which is not what we want to see. The last plot we look at is the





difference between expiries, which should in theory tell us if there is a difference of expectations for different time horizons. We can see from the plots that there is almost no difference between the two, so one way we try to take advantage of this result is to sell the shorter dated option and go long the longer dated option in the hopes that we can take advantage of potential theta decay of the shorter dated option. The last density that we can look at the difference between how we look at volatility. We can take the market implied volatility smile, or we can assume that it is constant. If we assume constant volatility, we can see that we will be paying extra premium for the what the market views as the true probability of the options that are out of the money going

into the money. Thus if we were to attempt to trade this cross, we would want to sell calls and buy puts, or if we found a dealer pricing the options based on a constant volatility we could potentially arbitrage them. This last scenario is extremely unlikely.



5 Triangle Trade

A triangle trade is defined as using two exchange rates as a function to represent a third. This lets us do something like try to isolate the profits that an oil producing country like Norway would generate in relation to the US dollar and the Brazilian Real. The Real also has an implied carry in the sense that rates are perennially higher in Brazil

$$S_{USDNOK} = S_{USDBRL} * S_{BRLNOK} \rightarrow \frac{USD}{NOK} = \frac{USD}{BRL} * \frac{BRL}{NOK}$$

than they are in the US. This will give us two potential areas to make money on the trade. so to fully take advantage of this triangle trade, we need to be clear in what direction we make our options bets, we will go short the $USDNOK$ exchange rate options, and long the $USDBRL$ and $BRLNOK$ options. Now that the theory of the trade is set up, we can look at the densities of the individual crosses and see what he payoffs may be.

The fist pair we look at is the $USDBRL$ cross, again there is an implied carry in this cross from the difference of interest rates.

We can see that trading the Asian options doesn't turn a profit when we roll forward on the trade, but the digital and European options both do quite well. It is worth pointing out that the Asian options are in the money at expiry, but the premiums are larger then the payoffs so we still lose money on the trade. The densities look like we should prefer the Asain options over the European options, and that we will want to go long the at-the-money 5YR options over the 7YR options since The longer expiry options have a much fatter tail to the call side. Overall this basket of trades made money which is good, now we can look at the next leg of the trade which is the $USDNOK$ cross.

results (5Y expiry)		
option	premium	P/L
Asian float	0.0846	-0.01804
Asian fixed	0.0938	-0.0272
digital call	0.4555	0.5445
euro call	0.1589	0.0272

Figure 4: USDBRL Cross

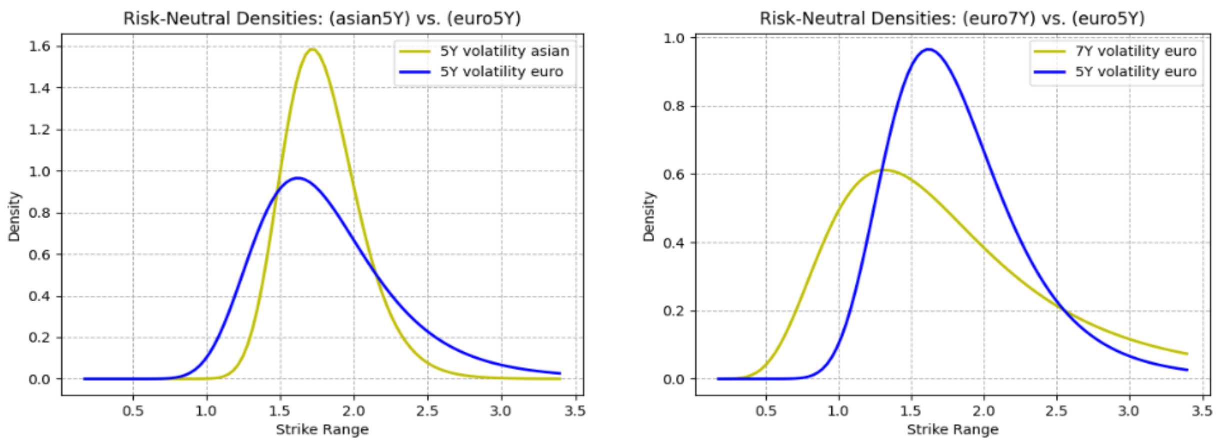


Figure 5: Market Implied Densities

Here we also expect to see a carry in two forms. The first is that Norway is a country that makes a significant amount of money drilling for and exporting oil to the rest of the world. This is very profitable, and the excess profits from this business go into the sovereign wealth fund. When Norway exports oil they need to be paid for this export, and when they get paid in US dollars, they often don't convert the money and simply hold it. Then when they need to stimulate their economy in a recession, they can simply sell the US dollars to buy back their own currency or what ever else they see fitting to prop up economic activity without printing more Krones. This keeps inflation to a minimum, which is a plus in the current economic environment. We want to go short all of these options, which means we are essentially betting on the Krone to appreciate in value against the US dollar. In this trade we lost a lot solid amount of money, but the floating strike Asain reduced some of these loses.

results (5Y expiry)		
option	premium	P/L
Asian float	0.2577	0.2230
Asian fixed	0.2901	-0.6251
digital put	0.5461	-0.5461
euro put	0.4539	-0.4539

Figure 6: USDNOK Cross

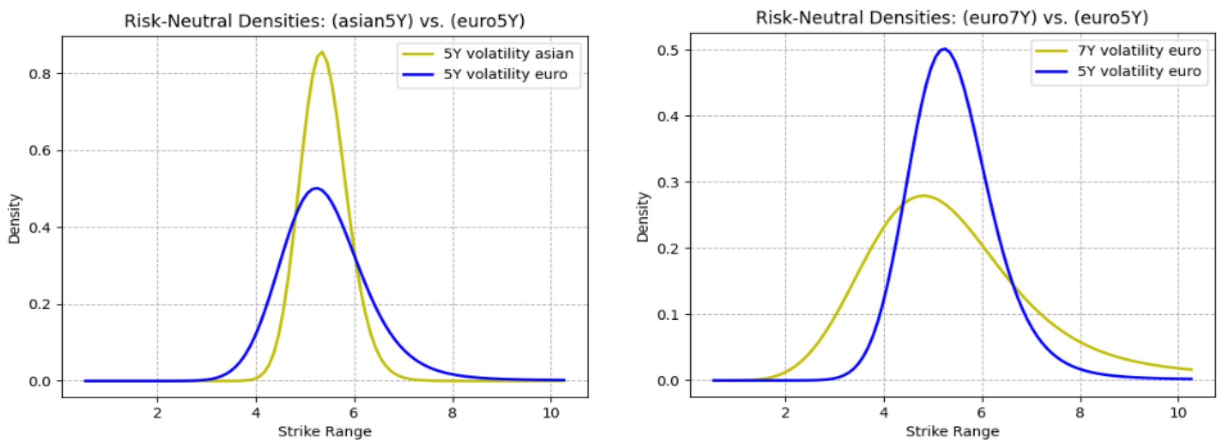


Figure 7: Market Implied Densities

The densities have a smaller fat tail skew to the upside for the different expiries, so maybe we should have gone short calls instead of puts. The reason we went short puts is because there is typically a put skew in densities, and we hoped that if we got the bets wrong it would be less costly.

The last trade is look at going long the $BRLNOK$ cross. This trade ends up being profitable since we can cover the two loses with the digital option, this trade makes money, but we see an interesting difference in the payoffs where only the digital option and the fixed strike Asian were profitable. We would have expected the European to be profitable along with the fixed strike, but it looks like the premium on the European nullified this expectation. Where we see a difference in the densities that we expect to see is the very clear skew to the upside, where over the long run the krone is likely expected to stay stringer then the Real. This gets illustrated with the closeness of the densities of the two different expiries, in conjunction with the European options being seen as more likely to go into the money than the Asian options.

results (5Y expiry)		
option	premium	P/L
Asian float	0.0342	-0.0341
Asian fixed	0.0302	0.0247
digital call	0.4154	0.5845
euro call	0.0552	-0.0130

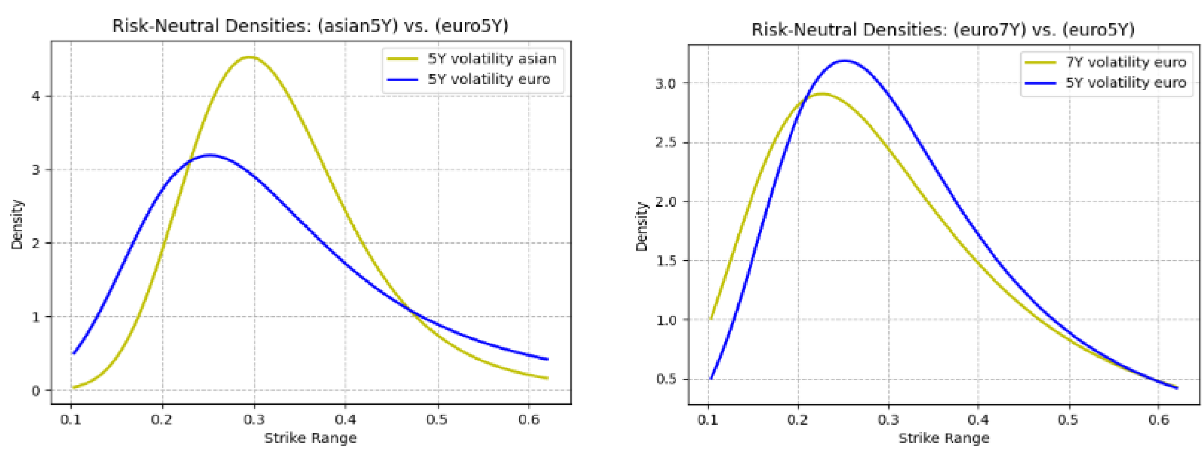


Figure 8: Market Implied Densities

6 References

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