The Optimal Capital Structure under Risks of Illiquidity and Over-indebtedness in a Double Barrier Option Framework

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Abstract

Existing dynamic capital structure models are based on single triggers determining bankruptcy, mainly over-indebtedness or illiquidity. The latter one tends to underestimate optimal capital structures by ignoring capital providers' flexibility to inject fresh money. The former one leans towards overestimation as it neglects agency conflicts between equity investors and debt holders while implying infinitely "deep pockets" of equity investors. This article incorporates both constraints, over-indebtedness and illiquidity, examining corporate debt value and optimal capital structure in a double barrier world with knock-in and knock-out options. We are first to derive closed-form solutions for all value components of a levered firm and for the optimal capital structure in such a setting. By testing our model for firms publicly listed in the US, we gain evidence that incorporating both triggers allows for capital structure estimations that are in accordance with empirical findings.

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1. Introduction

Since the famous No Magic in Leverage Theorem (Proposition 1) by Modigliani and Miller (1958) the "capital structure puzzle" Myers (1984) has bothered researchers in corporate finance. While theories like the tradeoff framework or the pecking order framework continue to compete in various research approaches, there is no dispute about the economic relevance of capital structure choices, i.e. the employment of debt, in general. Empirically estimated net benefits of debt range from approx. 5% to 10% of total firm value but are always proved to be significant.[1]

Without a doubt, the risk of financial distress or bankruptcy constitutes a major pillar in the optimal choice of capital structures. The valuations of either benefits of debt or costs of debt depend upon this risk. Tax shields generated by the tax-deductability of interests, i.e. the benefits of debt, persist as long as a company is a going concern, while direct and indirect costs, i.e. the costs of debt, are created in states of financial distress. Important for distinguishing different states and respective payoffs are well-defined triggering events (triggers), for instance based on bankruptcy legislations, debt covenants or economic rationales.

Stochastic models comparing a random variable with explicit triggers have gained wide popularity in the optimal capital structure literature over the last 25 years. Leland (1994) first provides closed-form analytical solutions to determine optimal capital structures for either an endogenously chosen bankruptcy trigger maximizing the equity value or an exogenously set bankruptcy trigger (e.g. cash flow falls below a predefined threshold) protecting the debt holders. Throughout our paper we call the first trigger *over-indebtedness* because in this case the debt burden becomes so high that equity investors have no incentive to inject additional equity as the net present value of the injection is negative. The latter of the two triggers will be called *illiquidity*, as it usually refers to a condition where an earnings or cash flow figure is not sufficient to service a debt obligation. While Leland's work offers a pioneering analytical framework and, thus, allows for the derivation of general conclusions, the model's empirical power explaining capital structure choices in reality is weak. On the one hand, a pure illiquidity trigger overestimates bankruptcy risk significantly as the inability to make payments is often overcome by capital infusions, and the breach of covenants is either tolerated by debt holders or can be solved by restructuring the debt. Only a minority of illiquid firms file for bankruptcy. On the other hand, ignoring illiquidity and triggering bankruptcy if and only if the firm is over-indebted seems too weak for two reasons: First, it is unlikely that equity investors are always able to make additional payments, i.e., as Strebulaev and Whited (2011) write they require *deep pockets*. Second, there is certainly a cost of triggering illiquidity which in turn increases the risk of over-indebtedness (see Ertan and Karolyi (2016) for an empirical analysis).

Many other authors followed Leland (1994) solving important issues in the field and by such gradually

[1] To provide some well-known examples: Korteweg (2010) finds an average net benefit of debt amounting to 5.5% of firm value while earlier an earlier study by Graham (2000) identifies average net benefits of 9.7% of firm value.

shed light into the rationale behind observed financing decisions. Leland and Toft (1996) include the maturity of debt into the standard Leland model. Goldstein et al. (2001) extend the model further by basing it on a stochastic EBIT-process and allowing for an option to increase debt (dynamic capital structure). Hackbarth et al. (2007) dive deeper into the debt structure explaining the relation of bank loans and market debt. Morellec et al. (2012) examine the impact of manager-shareholder conflicts in capital structure choices. Danis et al. (2014) show that profitability and leverage are positively correlated close to capital debt re-balancing times, thereby rebutting the puzzle of negative correlation between leverage and profitability measures in general. However, as all of these models consider only one single bankruptcy trigger the risk of bankruptcy is either over- or underestimated. Consequently, the optimal leverage, the relation between market value of debt and levered firm value, is overestimated in models solely based on an over-indebtedness trigger, and it is underestimated in models only relying on an illiquidity trigger.

Two papers differ from the usual approach. Strebulaev (2007) models a liquidity crisis where firms not able to meet their debt obligations on a cash flow basis engage in an asset sale and continue operations thereafter. Such a sale occurs with a discount, because sellers are time constrained and detach their workforce from the assets as well as because buyers may be financially constrained and less experienced in handling the assets. Titman and Tsyplakov (2007) include a similar flow based financial distress trigger and attach direct costs of equity issuance and indirect costs of reduced cash flows from issues with customers, suppliers, and employees to it. Beyond that, both models contain a second endogenous trigger where equity investors stop injecting equity because of over-indebtedness. While both papers reflect illiquidity and over-indebtedness, their numerical approaches preclude general closed-form solutions for the market value of debt and the corresponding optimal capital structure.

This article considers both triggers, illiquidity and over-indebtedness, simultaneously in an integrated model. We are first to derive closed-form solutions for all value components of a levered firm, the market values of corporate debt and equity as well as for the optimal capital structure in a double barrier option framework. Beyond that, we get to the bottom of the discrepancy between theoretical forecasts and empirical observations in context of capital structure theory as we combine both trigger boundaries into one single model. The upper boundary represents the illiquidity barrier and catches, e.g. the distinguished approaches of Couch et al. (2012) and Kim et al. (1993). Once illiquidity is triggered the firm turns into a restructuring mode as long as the illiquidity state persists. It may be able to return to a regular mode when operations improve, or it may touch the second lower barrier. This barrier represents over-indebtedness, thus includes the pioneering work of Leland (1994), and triggers the termination of the stochastic process, i.e., the liquidation of the firm. The novel combination of both triggers generates results in-between the particular single constraints. By empirically testing our model for firms publicly listed in the US, we gain evidence that incorporating both triggers explains observable capital structures significantly better than existing models do.

By extending the optimal capital structure problem towards the reflection of two boundaries at the same time, we transit from a single barrier to a double barrier option framework. The syndetic path-dependency of this class of options owe a modus operandi that is less straightforward than dealing only with a single barrier. Valuing a double barrier option and thereby a firm that faces changing payouts whenever the underlying process hits either of two well-defined boundaries illiquidity and over-indebtedness requires extensive results of the mathematical stochastic calculus. To arrive at closed-form solutions we adopt pricing formulas published by Pelsser (2000).

Reviewing bankruptcy legislation for the world's biggest economies supports the argument. The U.S. and Germany may serve as illustrative examples. The U.S. Bankruptcy Code knows two major types of bankruptcy, Chapter 7 and Chapter 11. The latter of the two allows firms filing it to continue their operations while gaining time to restructure their debt. Filing Chapter 7 is usually equivalent to stopping all operations and liquidating the firm's assets. Moreover, firms can attempt to restructure informally by direct negotiations with debt holders. This process is called 'work-out' and often results in a forbearance agreement. Chapter 11 and work-outs are comparable to entering the illiquidity state of our model while Chapter 7 is equivalent to over-indebtedness. The German Insolvency Act also contains two triggers: (impending) illiquidity (Sections 17 and 18) and over-indebtedness (Section 19). Once insolvency proceedings are opened the firm may continue its operations under own management or governed by an insolvency administrator. An insolvency court decides on whether a continuation is preferable and who should steer the firm. If insolvency proceedings prove that the companies obligations outnumber the value of its future prospects, operations will be terminated and all assets will be liquidated.

We frame our analysis as follows: section 2 introduces the model's basic framework and depicts an intuitive access. Moreover, we develop specific requirements of the triggers illiquidity and over-indebtedness and derive contingent present value factors for the different states of the framework. The section ends with determining all equity and debt value specific components including tax benefits, bankruptcy costs, and illiquidity expenses. Section 3 deals with the analytic application of the model and provides guidance for deriving optimal over-indebtedness triggers and optimal coupon payments, i.e. optimal leverage. References and comparisons to the single barrier models are made and possible extensions of our model are highlighted. The section ends with an empirical test of our model results versus the observed leverage ratios of firms of all NAICS sectors publicly listed in the US. Section 4 concludes the article.

2. The Capital Structure Model reflecting Illiquidity and Over-indebtedness (IO-Model)

2.1. Basic Framework

The assumptions we make about the nature of uncertainty are standard and we try to state them as general as possible. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ supporting a standard Brownian motion W_t, where Ω is the sample space, $\mathcal{F}$ the σ-algebra and $\mathbb{P}$ the corresponding probability measure. We denote the available information at time t, with $t \in [0, \infty)$, by the filtration $\mathcal{F}_t \subset \mathcal{F}_s$ with $0 \leq t < s$

where $\mathcal{F}_t$ describes the augmented σ-algebra generated by W_t. The market is free of arbitrage opportunities, and for each subjective probability measure $\mathbb{P}$ there exists an equivalent measure $\mathbb{Q}$ called the risk-neutral probability measure.

We consider a firm whose instantaneous revenues $(R_t)_{t\in[0,\infty)}$ follow a geometric Brownian motion under the risk-neutral pricing measure, i.e.

$$dR_t = \mu R_t dt + \sigma R_t dW_t^{\mathbb{Q}}, \tag{2.1}$$

where μ is the revenue's growth rate, σ is the corresponding volatility, and W_t is a standard Brownian motion under the risk-neutral measure. The initial value of revenue is $R_0 > 0$.

The firm faces variable costs captured by a deterministic ratio of revenues γ and deterministic fixed costs F independent of revenues. Thus, earnings before interest and taxes at time t, $EBIT_t$, are defined by

$$EBIT_t = R_t(1-\gamma) - F \quad \forall t \in [0,\infty). \tag{2.2}$$

The risk free rate is captured by r and we assume a flat corporate tax rate τ. Similar to other dynamic models (e.g., Hackbarth et al., 2007), we presuppose the unlevered cash flow to be

$$(1-\tau)\; EBIT_t \qquad \text{for all} \qquad t \in [0,\infty) \tag{2.3}$$

and ignore other cash-relevant items (e.g. depreciations, capital expenditures or changes in net working capital) for simplicity.[2]

The unlevered firm value with the information given at time t can be expressed with the help of the conditional expected value $\mathbb{E}[V_t\,|\,\mathcal{F}_t]$ and is given by[3]

$$\mathbb{E}[V_t\,|\,\mathcal{F}_t] = \int_t^\infty e^{-r(s-t)}\,(R_s(1-\gamma)-F)\,(1-\tau)\,ds \tag{2.4}$$

$$= \frac{R_t(1-\gamma)(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}. \tag{2.5}$$

Please note that we will suppress the conditional expected value notation $\mathbb{E}[\cdot\,|\,\mathcal{F}_t]$ due to readability. Whenever we will consider an expected value we deal with a conditional expected value. The corresponding σ-algebra is given by the context and indicated by R_t[4]. Thus equation 2.5 reduces to

$$V_t = \frac{R_t(1-\gamma)(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}. \tag{2.6}$$

We need to split the variable part $(R_t(1-\gamma)(1-\tau))$ and the fixed part $(F(1-\tau))$ of the cash flow in (2.6)

2 We do so without a loss of generality. The inclusion of these items in our model is simple but inflates the cash flow equation without adding further insights to our underlying research questions.

3 Throughout the whole paper we make the convenient assumptions that $r > \mu$.

4 It should be remembered here that $\mathbb{E}[V_0\,|\,\mathcal{F}_0] = \mathbb{E}[V_0]$

as the fixed part is not expected to grow with μ over time but to remain constant.

In our setting, the levered firm value $V^L(V_t)$ at time t is defined as the sum of the unlevered firm value V_t and the net benefits of debt $NB(V_t)$, where $NB(V_t)$ consists of the tax benefits of debt $TB(V_t)$, less bankruptcy costs of debt $BC(V_t)$ and illiquidity expenses of debt $IE(V_t)$. While we detail rationale and calculation of these components in section 2.4, we already provide the basic equation for the expected conditional value of the levered firm under suppressed notation,

$$V^L(V_t) = V_t + NB(V_t) \tag{2.7}$$

with

$$NB(V_t) = TB(V_t) - BC(V_t) - IE(V_t). \tag{2.8}$$

Moreover, we denote the market value of debt at time t as $D(V_t)$ and follow the classic assumption of debt being issued as a console bond with constant coupon payment C to infinity (cf. Leland (1994), Goldstein, Ju, and Leland (2001), Strebulaev (2007) et al.). Thus, we obtain for the expected conditional market value value of equity at time t,

$$E(V_t) = V^L(V_t) - D(V_t). \tag{2.9}$$

Conditionalities and payout structures underlying the value components introduced above are detailed in the subsequent sections. We present closed-form analytic solutions for all of them in section 2.4.

2.2. *Default Triggers*

Existing dynamic models in corporate finance involve only one lower boundary for the underlying stochastic process. In Leland (1994) bankruptcy is triggered if the discounted conditional expected asset value V_t falls to a certain level V_B which is endogenously derived by the investors in order to maximize their equity value (endogenous default trigger). The second type of default trigger is exogenously determined by a covenant within the debt contract or by liquidity constraints. In such a setup the firm defaults either because it violates a certain debt covenant or because firm and equity investors have no spare cash to pay their current cash obligations (i.e., redemption payments and/or interest payments).

The exogenous trigger is less often applied in literature (see e.g., Kim et al., 1993; Couch et al., 2012). Usually it is argued that it causes firms to cease their operations although the equity value is still positive. However, rationale equity investors would be ready to fund the firm as long as the market value of their investment exceeds the debt obligation. Only if the described condition is not fulfilled, equity investors will file for bankruptcy (Leland, 2006).[5] Thus, the vast majority of existing dynamic models relies on

[5] A crucial assumption for this policy is that equity investors can access external funds whenever the firm is threatened by illiquidity, i.e., they have “deep pocket”. This assumption opens the field for arguments preferring the exogenous trigger (no external funds available or if it may be costly or difficult due to timing constraints or covenants in the debt contract).

the endogenous trigger and ignores the exogenous one (see e.g., Leland and Toft, 1996; Goldstein et al., 2001; Hackbarth et al., 2007).

However, in reality we frequently observe that debtholders protect their claims with well-defined financial covenants allowing them to cancel the debt (and request a full redemption) whenever the covenant is triggered. While the option to cancel the debt is usually not exercised, the triggering event provides the opportunity to adjust (or to renegotiate if not pre-specified) the promised yield of debt and to influence strategic decisions regarding the firm (Achleitner et al., 2012). Moreover, entering this state, which we call *illiquidity state* , generates additional direct costs (e.g., lawyer or advisory expenses, discounts when selling assets) and indirect costs (e.g. loss of clients, disproportionate dilution by additionally raised equity) to the firm.

As additional covenant restrictions and liquidity constraints are ignored by traditional dynamic trade-off models it is not surprising that these models imply excessively high optimal leverage ratios compared to reality. Strebulaev (2007) emphasizes this fact and proposes the so far only known model combining both boundaries. He does not attempt to solve the model analytically and to derive general theoretical proofs but to calibrate the model for simulating firms' capital structure paths. His results are of particular importance for empirical tests of dynamic capital structure models.

We are able to model both, the exogenous covenant (B_{IS} as the illiquidity boundary and B_{LS} as the liquidity boundary) and the endogenous over-indebtedness boundary (B_{OS} from above), and to derive a closed-form analytic solution allowing us to draw general theorems regarding the choice of optimal capital structures. To the best of the authors knowledge this is the first attempt to model the optimal capital structure in a double barrier option framework[6]. We state our first model-specific assumption:

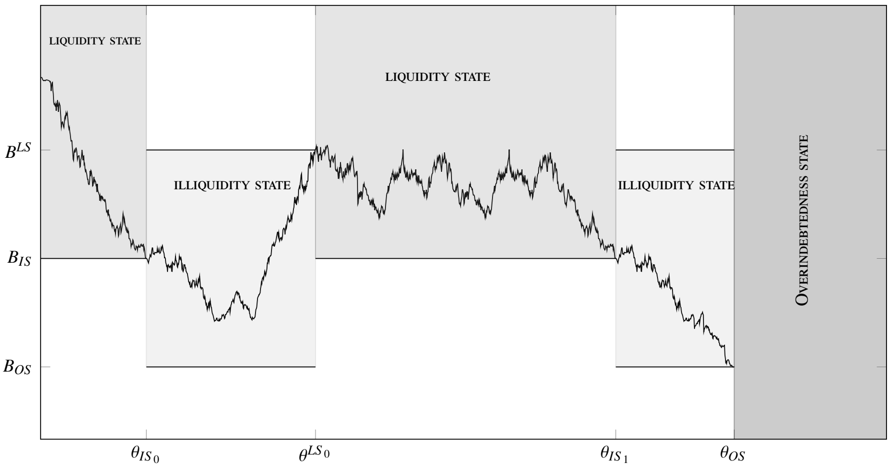
Assumption 2.1. *The stochastic revenue process of our firm* $(R_t)_{t\in[0,\infty)}$ *starts in liquidity state* LS *at* R_0 *above the illiquidity state boundary* B_{IS}. *When* R_t *hits* B_{IS} *for some* $t \in [0,\infty)$ *the firm switches into illiquidity state* IS, *and* R_s *continues facing the liquidity state boundary* B_{LS} *as well as the over-indebtedness state boundary* B_{OS} *for some* $t < s$. *The firm reenters* LS *if and only if* R_s *hits* B_{LS} *before it hits* B_{OS} *for* $t < s$. *The number of switching events between* LS *and* IS *is not restricted. Given the firm stays in* IS, *the over-indebtedness state is triggered if and only if* R_s *hits* B_{OS} *before it hits* B_{LS} *for* $t < s$. *At the time where* $R_s = B_{OS}$ *for* $t < s$ *the stochastic process* R_s *stops, i.e.* R_s *is not defined for* $t > s$.

The following figure 1 illustrates the assumption. Without loss of generality we can assume that the starting point of the process R_0 is greater than B_{IS}[7].

[6] In the following we will discuss that the exogenous boundaries can be regarded as knock-in barrier options and the endogenous as a knock-out option

[7] If $R_0 < B_{IS}$ the equity holders would invest a certain amount such that $R_0 < B_{IS}$ holds.

Figure 1: Introduction to the General Model



This figure depicts a stochastic process that starts in the *liquidity state* (LS). The process runs into *illiquidity state* (IS) at the very moment θ_{IS_0} when the illiquidity barrier B_{IS} is hit. Continuing in IS, the process reenters LS in θ^{LS_0} by hitting the boundary B_{LS}. In θ_{IS_1} the process touches B_{IS} again and falls back into IS. Finally, the process is killed in θ_{OS}, i.e. the process runs into *over-indebtedness state* (OS) and hits thus the barrier B_{OS}.

The instant of time where the process enters another state are mathematically known as stopping times[8]. Obviously, there is no need to subscript the hitting time θ_{OS} due to the simple fact that the process is killed at the precise moment when it hits B_{OS}. On the other hand, there is an obligation to subscript θ_{IS_i} and θ^{LS_i} with $i \in \mathbb{N}_0$, respectively because B_{IS} and B_{LS} could be hit countably infinite times almost surely without hitting B_{OS}. In our framework the barriers B_{IS}, B_{LS} and B_{OS} are constant in time. Please notice that there are only two possibilities: Either B_{IS} is an valid barrier, i.e. B_{IS} is *on* and this implies that B_{LS} and B_{OS} are both switched off or vice versa (cf. figure 1).

Another important prerequisite in this setting is the relation $B_{OS} \leq B_{IS} < B_{LS}$ which we prove in Lemma 2.6 after having derived explicit expressions of the boundaries.

Lemma 2.2. *The firm will enter illiquidity state (IS) if $EBIT(1-\tau) \leq \delta C$, which corresponds to $R_t \leq B_{IS}$ where $B_{IS} = (\delta C + F(1-\tau)) / ((1-\gamma)(1-\tau))$.*

The starting point of the revenue process $(R_t)_{t\in[0,\infty]}$ in *illiquidity state* (IS) is $R_{\theta_{IS}}$ which can be substituted by B_{IS}, i.e. $R_{\theta_{IS}} = B_{IS}$. We capture the consequences for a firm entering (IS) in our second model-specific assumption.

Assumption 2.3. *When the firm enters illiquidity state (IS), certain default expenses occur, e.g. due to customers that stop buying the firms' products, which we assume to be a proportion ϵ of $\mathbb{E}[V_{\theta_{IS_i}} | \mathcal{F}_{\theta_{IS_i}}]$. Moreover, as long as the firm remains in IS (i.e. $B_{OS} < R_t < B_{LS}$ with $t \geq \theta_{IS_i}$) the debtholders demand penalty interest C_{il} with $C_{il} > C$. Consequently, the covenant boundary B_{LS} for the revenue process coming from below is greater than the covenant boundary B_{IS} for the revenue process coming*

8 In the following named as hitting times. For a formal definition cf. Definition 2.8.

from above, i.e. $B_{IS} < B_{LS}$. If the firm returns from IS to liquidity state LS, the penalty interest payments will stop and the regular coupon payment C will be enforced.

Assumption 2.3 allows us to derive B_{LS} explicitly in our setting:

Lemma 2.4. *The firm will reenter liquidity state LS if $EBIT(1-\tau) = \delta C_{il}$ with $t \geq \theta_{IS_i}$, which corresponds to $R_t = B_{LS}$ where $B_{LS} = (\delta C_{il} + F(1-\tau)) / ((1-\gamma)(1-\tau))$ with $t \geq \theta_{IS_i}$.*

Note that for $\delta = 1-\tau$ the boundaries B_{IS} and B_{LS} do not only represent covenant triggers but, indeed, illiquidity triggers, i.e., the firm is not able to pay its cash obligations.

The last possibility to be detailed is when the firm runs from IS to *over-indebtedness state* (OS). In triggering over-indebtedness we follow the classic assumption of Leland (1994) which is used in many more models (e.g., Leland and Toft, 1996; Goldstein et al., 2001; Hackbarth et al., 2007; Danis et al., 2014): If the expected asset value $\mathbb{E}[V_t]$ falls to a certain level V_B where liquidating the firm is optimal, i.e., value maximizing for the equity investors, the firm will file for bankruptcy. V_B is endogenously chosen by maximizing the equity value. In section 3 we demonstrate how to derive V_B. For now we consider it a constant parameter. The difference of our setting compared to existing models is that our underlying stochastic process regards the revenue and, thus, we need to transfer the classic over-indebtedness condition $\mathbb{E}[V_t] = V_B$ to the condition $R_t = B_{OS}$. Lemma 2.5 presents the transformation.

Lemma 2.5. *The firm will file for bankruptcy if $\mathbb{E}[V_t] = V_B$ with $t \geq \theta_{IS_i}$, which corresponds to $R_t = B_{OS}$ where $B_{OS} = \left(\left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu)\right) / ((1-\gamma)(1-\tau))$ with $t \geq \theta_{IS_i}$.*

Finally, we prove the necessary relationship of our triggers in Lemma 2.6.

Lemma 2.6. *Under the assumption $\frac{\delta}{1-\tau} > 1 - \frac{\mu}{r}$ the liquidity boundary B_{LS} is strictly greater than the illiquidity boundary B_{IS}. Moreover, B_{IS} is greater than or equal to the over-indebtedness boundary B_{OS}. Thus, we have $B_{OS} < B_{IS} < B_{LS}$.*

Remark 2.7. *This implies that we allow for a negative growth rate μ. So the assumption in lemma 2.6 provides a lower boundary for δ.*

Without loss of generality let the fixed costs F be equal to zero. By assumption, whenever the firm enters illiquidity state IS from above the value of the revenues needs to equal the discounted debt obligations, i.e.

$$\frac{B_{IS}(1-\gamma)(1-\tau)}{r-\mu} \geq \frac{(1-\tau)C}{r}. \tag{2.10}$$

Some simple rearrangements and the value of B_{IS} given in lemma 2.2 show the necessary assumption made in lemma 2.6. The the right inequality given in lemma 2.6 holds since we have $C_{il} > C$ by assumption 2.3. For the left inequality $B_{OS} < B_{IS}$ the upper limit for considering over-indebtedness V_B on the part of the equity holders is simply $\frac{\delta C}{r-\mu}$. They have to subtract C on their cash flow and add in case of tax advantages τC to their cash flow in a continuous setting. This equals $\frac{\delta C}{r-\mu}$ in $t = 0$. So δ covers the tax advantage. Its lower limit is given by $1-\tau$ just simply owing that no more tax benefits can be generated in our model. For $\delta > (1 - \tau)$ the tax effect is strengthened. The same holds for the fixed term $\frac{F(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}$ which has the function of an additive term.

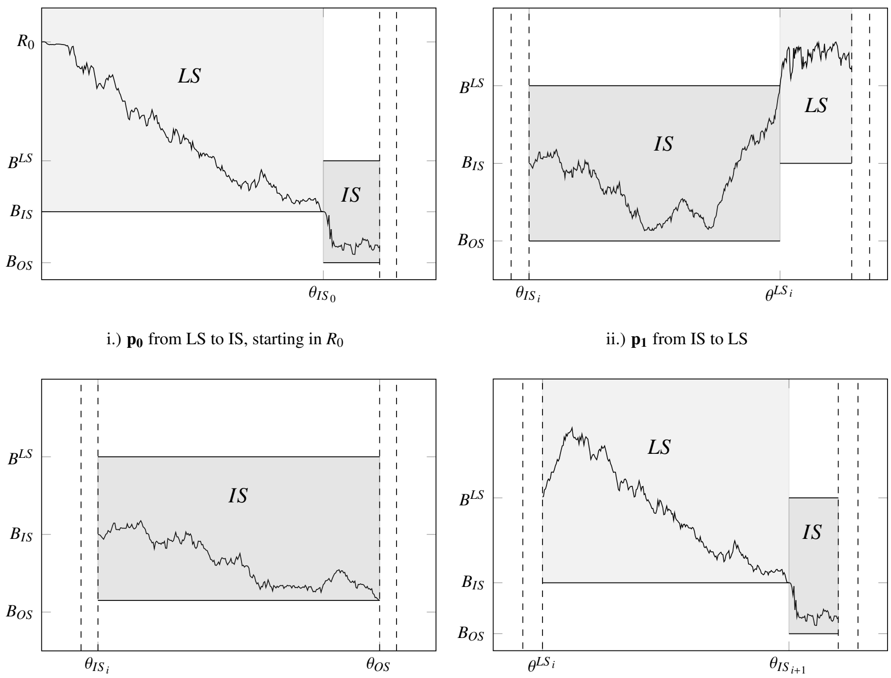
As aforementioned, LS and IS can alternate infinite times but the process will stop immediately as soon as the over-indebtedness trigger B_{OS} is hit. While the starting point of R_t in the first LS is special (R_0), the starting points of R_t for the subsequent IS and LS are repetitive ($R_{\theta_{IS}}$ and $R_{\theta^{LS}}$, respectively). This is an important feature for valuing the levered firm in section 2.4.

2.3. Contingent Present Value Factors

Before we can adapt the aforementioned framework to an optimal capital structure model, we need to derive the contingent present value factors of our defined states. Contingent present value factors, a term also applied by Couch et al. (2012)[9], reflect the present value of an asset that pays \$1 contingent on stochastic process reaching one of our specific states. In other words, contingent present value factors represent the probability of entering a certain state discounted back to today. Figure 2 illustrates our methodology.

9 There is no consistent terminology throughout literature. Exemplarily, Goldstein et al. (2001) choose a longer description, i.e. “the present value of a claim that pays \$1 contingent on firm value reaching V_B”, while Hackbarth et al. (2007) call it a “Hitting Claim Value”.

Figure 2: Contingent Present Value Factors p_0, p_1, p_2, and p_3



i.) $\mathbf{p_0}$ from LS to IS, starting in R_0 ii.) $\mathbf{p_1}$ from IS to LS

iii.) $\mathbf{p_2}$ from LS to OS iv.) $\mathbf{p_0}$ from LS to IS

R_0 is the starting point of the stochastic process. B_{LS} and B_{IS} are the well known barriers and can be regarded as knock-in barrier option. The over-indebtedness barrier B_{OS} represents a knock-out barrier option. The field on a lighter grey background LS represents the LS. In contrast the field on a darker grey background IS symbolizes IS. θ_{IS_i} represents an arbitrary point in time at which the firm runs into *illiquidity state* (IS) coming from *liquidity state* (LS). θ^{LS_i} represents an arbitrary point in time at which the firm runs into LS coming from IS. θ_{OS} is the exact point in time at which the process is killed and the over-indebtedness state is reached. The parallel dashed lines indicate that the given figure is only an excerpt of the underlying process.

The first graph sketches a firm that runs from LS into IS. This is abbreviated by p_0. The second shows the path of a firm that runs from IS into LS, denoted by p_1. The third picture represents the path of a firm that goes bankrupt entering BS, labeled with the contingent present value factor p_2 [10]. Finally, p_3 is represented in the last figure that shows again a firm running from LS to IS. The difference to the first picture is that the last represents the behavior of one path in the middle of a firm's life, while the first illustrates only a possible path development at the beginning of a firm's life. Without loss of generality the following figure comprises all possible development opportunities of a firm in our model. The following definitions provide proper mathematical techniques, starting with stopping times.

Definition 2.8 (**Hitting Times**). *Given three boundary constraints* B_{OS}, B_{IS}, B_{LS} *with* $B_{OS} \leq B_{IS} < B_{LS}$,

[10] Again note that having been in IS is a crucial prerequisite for running into BS. Obviously, the firm is bankrupt at the very moment when the stochastic process R_t equals B_{OS} for an arbitrary $t \in [0, \infty)$ (this happens if and only if $t = \theta_{OS}$ (cf. Def. 2.8).

the corresponding hitting times are defined as follows for $i \in \mathbb{N}$:

$$\begin{aligned}
\theta_{OS} &:= \inf\{t \geq 0 \,|\, R_t = B_{OS}\} \\
\theta_{IS_0} &:= \inf\{t \geq 0 \,|\, R_t = B_{IS}\} \\
\theta^{LS_0} &:= \inf\{t \geq \theta_{IS_0} \,|\, R_t = B_{LS} \;\wedge\; R_s > B_{OS} \;\textit{for all}\; s \in [\theta_{IS_0}, t]\} \\
\theta_{IS_i} &:= \inf\{t \geq \theta^{LS_{i-1}} \,|\, R_t = B_{IS} \;\wedge\; R_s > B_{OS} \;\textit{for all}\; s \in [\theta^{LS_{i-1}}, t]\} \\
\theta^{LS_i} &:= \inf\{t \geq \theta_{IS_i} \,|\, R_t = B_{LS} \;\wedge\; R_s > B_{OS} \;\textit{for all}\; s \in [\theta_{IS_i}, t]\}.
\end{aligned}$$

Remark 2.9. *Technically speaking, for the definition of θ_{IS_i} we can omit the constraint $R_s > B_{OS}$ for all $s \in [\theta^{LS_{i-1}}, t]$. So the following remains*

$$\theta_{IS_i} := \inf\{t \geq \theta^{LS_{i-1}} \,|\, R_t \leq B_{IS}\}. \tag{2.11}$$

Owing to readability we do not suppress this constraint, since we want to make sure that the above given nonempty stopping times θ_{IS_i} and θ^{LS_i} for $i \in \mathbb{N}_0$ exclude over-indebtedness.

Remark 2.10. *If $\theta_{IS_i} \leq \theta_{OS} \leq \theta^{LS_i}$, then $\theta_{IS_{i+1}} = \theta^{LS_i} = \emptyset$.*

Based on the aforementioned insights, we define the contingent present value factors p_0, p_1, p_2, p_3 as follows:

Definition 2.11 (Contingent Present Value Factors $p_0, ..., p_3$).
p_0 is the price of a knock out barrier option that pays 1 \$ in θ_{IS_0} starting in $t = 0$ (with the corresponding ordinate value R_0) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the lower-upper barrier B_{IS}, i.e. p_0 represents the discounted probability of hitting B_{IS} in θ_{IS_0}.

Analogously, p_1 is the price of 1 \$ in θ^{LS_i} starting in θ_{IS_i} for all $i \in \mathbb{N}_0$ (with the corresponding ordinate value B_{IS}) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the upper-upper barrier B_{LS} without hitting the lower barrier B_{OS}.

p_2 is the price of 1 \$ in θ_{OS} starting in θ_{IS_i} for alle $i \in \mathbb{N}_0$ (with the corresponding ordinate value B_{IS}) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the lower barrier B_{OS} without hitting the upper-upper barrier B_{LS}.

Finally, p_3 is the price of a knock out barrier option that pays 1 \$ in $\theta_{IS_{i+1}}$ starting in θ^{LS_i} for all $i \in \mathbb{N}_0$ (with the corresponding ordinate value B_{LS}) when the stochastic process $(R_t)_{t\in[0,\infty)}$ hits the lower-upper barrier B_{IS}.

As a reminder, p_0 and p_3 can be seen as assets, or more specifically as perpetual, down-and-in, cash-at-hit-or-nothing, single-barrier options which pay \$1 when the stochastic process R_t hits the barrier B_{IS} which is below the initial value of the stochastic process. p_0 and p_3 only differ with respect to its initial values which are R_0 and $R_{\theta^{LS}} = B_{LS}$, respectively. The pricing formula for such an option type is well known[11] and, thus, can be applied to

$$p_0 = \left(\frac{B_{IS}}{R_0}\right)^y \tag{2.12}$$

[11] Rubinstein and Reiner (1991) provide a very intuitive access to valuing such options. Moreover, in their compendium of exotic options (Rubinstein and Reiner, 1992) they investigate the pricing of many more option types.

and analogously to

$$p_3 = \left(\frac{B_{IS}}{B_{LS}}\right)^y, \tag{2.13}$$

where

$$a := \mu - \frac{1}{2}\sigma^2, \qquad b := \sqrt{a^2 + 2\sigma^2 \cdot r}, \qquad y := \frac{a+b}{\sigma^2}. \tag{2.14}$$

Explicitly pricing p_1 and p_2 is less trivial as we deal with perpetual, cash-at-hit-or-nothing, double barrier options. The lower barrier is the over-indebtedness boundary B_{OS} and the upper barrier is the covenant boundary B_{LS}. p_1 and p_2 differ with respect to its payout structure as the latter pays \$1 when the lower barrier is hit before the upper barrier has been hit and vice versa. Pelsser (2000) provides a pricing formulas for both structures in finite time which can be easily extended to a perpetual setting and applied to our specific problem. Thus, we have

$$p_1 = \exp\left\{\frac{a(l-x)}{\sigma^2}\right\} \frac{\sinh(\frac{b}{\sigma^2}x)}{\sinh(\frac{b}{\sigma^2}l)} \tag{2.15}$$

and analogously

$$p_2 = \exp\left\{\frac{-ax}{\sigma^2}\right\} \frac{\sinh(\frac{b}{\sigma^2}(l-x))}{\sinh(\frac{b}{\sigma^2}l)}, \tag{2.16}$$

where

$$x := \log\left(\frac{B_{IS}}{B_{OS}}\right) := \log\left(\frac{\delta C + F(1-\tau)}{V_B + \frac{F(1-\tau)}{r}(r-\mu)}\right), \tag{2.17}$$

$$l := \log\left(\frac{B_{LS}}{B_{OS}}\right) := \log\left(\frac{\delta C_{il} + F(1-\tau)}{V_B + \frac{F(1-\tau)}{r}(r-\mu)}\right), \tag{2.18}$$

and a as well as b are as defined in (2.14). Please note that x and l are functions of V_B.

2.4. Levered Firm Value and its Components

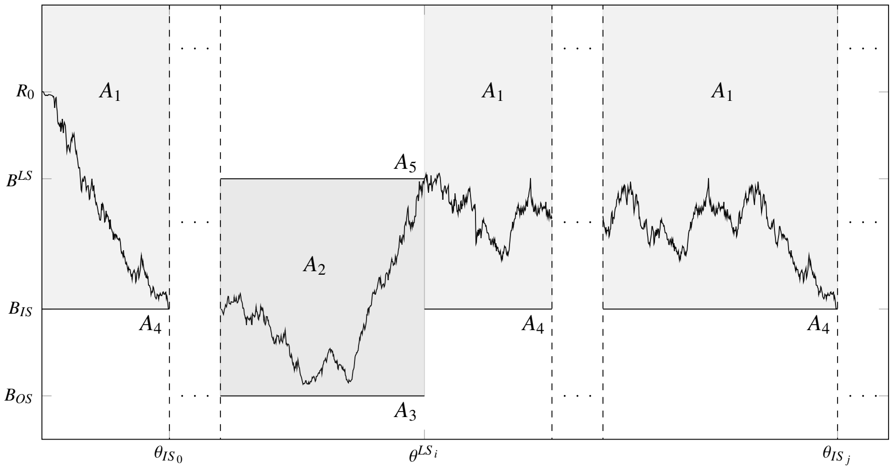
As introduced in section 2.1 the levered firm value $V^L(V)$ is the sum of the unlevered firm value V and the net benefits of debt $NB(V)$. Those net benefits are claims with deterministic payouts contingent on the states of the world discussed above. In this section we develop a systematic payout structure, form expected values of these payouts and finally value the contingent claims generated by debt financing.

The terms A_j, with $j = 1, ..., 5$, denote all payouts in our framework subject to the relationship of the revenue process $(R_t)_{t\in[0,\infty)}$ and the barriers B_{IS}, B_{LS} and B_{OS}. A_1 comprises a continuous payout in LS, i.e. $R_t > B_{IS}$ with $t \in [0, \theta_{IS_0}] \cup [\theta^{LS_i}, \theta_{IS_{i+1}}]$ and $i \in \mathbb{N}_0$. A_2 represents the continuous payout in IS that is realized if and only if the revenue process lies in the middle of the barriers B_{OS} and B_{LS} until the process hits one of them, i.e. A_2 is given if and only if $B_{OS} < R_t < B_{LS}$ with $t \in [\theta_{IS_i}, \theta^{LS_i}] \cup [\theta_{IS_j}, \theta_{OS}]$ and

$i < j \in \mathbb{N}$. In contrast to A_1 and A_2 the payouts A_3 to A_5 are point payouts, i.e. they are generated if and only if one of the barriers is hit by the revenue process. The payout A_3 occurs in the moment when the firm switches from IS to OS under the condition that $R_t = B_{OS}$ with $t = \theta_{OS}$ for an arbitrary $t \in [0, \infty)$. Analogously A_4 is generated if and only if $R_t = B_{IS}$ for all $t \in [0, \infty)$, i.e. $t = \theta_{IS_i}$ with $i \in \mathbb{N}_0$. Finally, the payout A_5 is realized if and only if $t = \theta^{LS_i}$ with $i \in \mathbb{N}_0$.

Without loss of generality Figure 3 shows all possible states of a firm that has not hit B_{OS} yet and the five possible payoffs.

Figure 3: General Payout Structure of a Stochastic Process



The figure depicts a general payout structure generated in a double barrier framework with *liquidity state* (LS) , *illiquidity state* (IS) and *over-indebtedness state* (OS). If the underlying process is in LS, the payout will equal A_1. In case of IS the generated payout is A_2.lower-upper barrier B_{IS} the payout accords with A_2. Hitting the lower boundary B_{OS} the payout accords with A_3. The same holds for the lower-upper barrier B_{IS} and the payout A_4 and the upper-upper barrier B_{LS} with the payout A_5, respectively.

comprises an arbitrary payout of R_t with $t \in [0, \theta_{IS_0}] \cup [\theta^{LS_i}, \theta_{IS_{i+1}}]$ with $i \in \mathbb{N}_0$. This is the payout in LS. A_2 represents the payout in IS that is realized if and only if the stochastic process lies in the middle of the barriers B_{OS} and B_{LS} until the process hits one of them, i.e. A_2 is given if and only if $t \in [\theta_{IS_i}, \theta^{LS_i}] \cup [\theta_{IS_j}, \theta_{OS}]$ with $i < j \in \mathbb{N}$. Note that there is no need that B_{OS} equals A_3 and B_{LS} equals A_5, respectively. The payout A_3 is given if and only if $t = \theta_{OS}$. This is equivalent to the condition that $R_t = B_{OS}$ for an arbitrary $t \in [0, \infty)$. Analogously A_4 is generated if and only if $R_t = B_{IS}$ for all $t \in [0, \infty)$, i.e. $t = \theta_{IS_i}$ with $i \in \mathbb{N}_0$. Finally, the payout A_5 is realized if and only if $t = \theta^{LS_i}$ with $i \in \mathbb{N}_0$. that this is the most general model that is considerable in a double barrier option framework. Next we derive the expected present values of the payouts A_j considering each single term from today to infinity probability-weighted and discounted by connecting the contingent value factors (CVFs) p_0 to p_3 for an infinite time horizon. Lemma 2.12 summarizes our results:

Lemma 2.12. *The expected present values of the payouts $\mathbb{E}[A_j]$, with $j = 1, ..., 5$, in $t = 0$ are products of the deterministic payout A_j and the corresponding contingent present value factor denoted by $pr^0_{A_j}$. The* *terms $pr^0_{A_j}$ are combinations of the contingent present value factors of the different states, p_0 to p_3. It follows that*

$$\mathbb{E}[A_1] = A_1 pr^0_{A_1} \text{ ,with } pr^0_{A_1} := [(1-p_0) + \frac{p_0 p_1 (1-p_3)}{1-p_1 p_3}], \tag{2.19}$$

$$\mathbb{E}[A_2] = A_2 pr^0_{A_2} \text{ ,with } pr^0_{A_2} := [\frac{p_0(1-p_1-p_2)}{1-p_1 p_3}], \tag{2.20}$$

$$\mathbb{E}[A_3] = A_3 pr^0_{A_3} \text{ ,with } pr^0_{A_3} := [\frac{p_0 p_2}{1-p_1 p_3}], \tag{2.21}$$

$$\mathbb{E}[A_4] = A_4 pr^0_{A_4} \text{ ,with } pr^0_{A_4} := [\frac{p_0}{1-p_1 p_3}], \tag{2.22}$$

$$\mathbb{E}[A_5] = A_5 pr^0_{A_5} \text{ ,with } pr^0_{A_5} := [\frac{p_0 p_1}{1-p_1 p_3}]. \tag{2.23}$$

By converting the one dimensional setup into vectorial calculus, we suppress notation to a minimum and improve readability. Therefore, let $\vec{PO}$ denote the general payout structure and $p\vec{r}_0$ the according contingent present value factors starting in $t = 0$. The first row represents the payout A_1 and the contingent present value factor $pr^0_{A_1}$, respectively. In conclusion, we have

$$\vec{PO} := \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} \qquad p\vec{r}_0 := \begin{pmatrix} pr^0_{A_1} \\ pr^0_{A_2} \\ pr^0_{A_3} \\ pr^0_{A_4} \\ pr^0_{A_5} \end{pmatrix} = \begin{pmatrix} (1-p_0) + \frac{p_0 p_1 (1-p_3)}{1-p_1 p_3} \\ \frac{p_0(1-p_1-p_2)}{1-p_1 p_3} \\ \frac{p_0 p_2}{1-p_1 p_3} \\ \frac{p_0}{1-p_1 p_3} \\ \frac{p_0 p_1}{1-p_1 p_3} \end{pmatrix}. \tag{2.24}$$

Based on Lemma 2.12 and Equation (2.24) we obtain the value components of the levered firm value from Equation (2.7) and the equity value from Equation (2.9).

We start with the value of debt defined by $D(V, C, C_{il})$. Due to readability we suppress coupon payments C and penalty coupon payments C_{il}, and simply write $D(V)$. In our setting debt promises a perpetual coupon payment C whose level remains constant unless the firm enters IS, i.e. the stochastic process R_t hits the covenant barrier B_{IS}. Thus, in LS the debt value equals $\frac{C}{r}$ (c.f. A_1). As long as the firm remains in IS it needs to pay a permanent penalty coupon C_{il} unless the firm reenters LS or declares bankruptcy, i.e. enters OS. The debt value in IS is equal to $\frac{C_{il}}{r}$ (c.f. A_2). Let V_B denote the level of the asset value at which the firm runs into bankruptcy. If bankruptcy occurs, a fraction $0 \leq \alpha \leq 1$ of value will be lost to bankruptcy costs, including direct and indirect costs. This leaves the debtholders with value $(1-\alpha)V_B$ (c.f. A_3) and the equity investors with nothing. Note that we will not take any taxes in cases of bankruptcy into consideration, such as taxes on cancellation of debt. In the very moment the firm hits the barrier B_{IS} or B_{LS} the value of the debt does not change (c.f. $A_4 = A_5 = 0$). Summarizing, we have the following payout structure $\vec{D}$ for the debt value:

$$\vec{D}^{\intercal} \quad = \begin{pmatrix} \frac{C}{r} & C_{il} & (1-\alpha)V_B & 0 & 0 \end{pmatrix}. \tag{2.25}$$

To obtain the conditional expected debt value $D(V)$ we need to multiply the payout vector $\vec{D}$ with the vector of the contingent present value factors $\vec{pr}_0$ from Equation (2.24), i.e.

$$D(V) = \vec{D}^{\intercal} \vec{pr}_0. \tag{2.26}$$

default is higher than in case of liquidity. So the debt holder will raise a penalty owing to circumstances arising from their higher number of writing-offs they are forced to make. This penalty payment equals a portion ρ of the covenant C. Thus, we have the following payout structure

Now we consider the value of tax benefits associated with the debt financing. These benefits resemble a security that pays a constant coupon equal to the tax-sheltering value of interest payments τC as long as the firm is in LS, τC_{il} in case of IS and nothing in OS. In the very moment the stochastic process hits a barrier B_{LS}, B_{IS} or B_{OS} no tax benefits are generated. As we are concerned with a continuous framework A_4 and A_5 equal zero. Thus, we have the following payout structure $\vec{TB}$:

$$\vec{TB}^{\intercal} \quad = \begin{pmatrix} \frac{\tau C}{r} & \frac{\tau C_{il}}{r} & 0 & 0 & 0 \end{pmatrix}. \tag{2.27}$$

Suppressing the expected value notation and the coupon payment, C, as well as multiplying with $\vec{pr}_0$ from Equation (2.24) yields the following value of tax benefits $TB(V)$:

$$TB(V) = \vec{TB}^{\intercal} \vec{pr}_0. \tag{2.28}$$

Bankruptcy costs $BC(V)$ occur if and only if the firm is over-indebted and files for bankruptcy. This implies that the stochastic process R_t equals B_{OS}. Thus, the unlevered firm value at θ_{OS} is represented by

$$V_B = \frac{B_{OS}(1-\gamma)(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r} \tag{2.29}$$

while αV_B reflects the bankruptcy costs if is triggered (A_3). In no other states bankruptcy costs occur leaving us with a bankruptcy cost payout structure as follows

$$\vec{BC}^{\intercal} \quad = \begin{pmatrix} 0 & 0 & \alpha V_B & 0 & 0 \end{pmatrix}. \tag{2.30}$$

In vectorial writing, we represent the value of bankruptcy costs $BC(V)$ as

$$BC(V) = \vec{BC}^{\intercal} \vec{pr}_0. \tag{2.31}$$

Finally, illiquidity expenses IE may occur whenever the firm enters IS. This can ultimately be ascribed to two key causes: on the one hand, direct costs of lawyers, banking fees and so on and on the other hand, indirect costs such as loss of investors' or customers' confidence. This will be priced with a fee in portion ϵ to the then prevailing unlevered firm value $\mathbb{E}[V_{\theta_{IS_i}}]$. Thus, we have the following payout structure for IE:

$$\vec{IE}^{\intercal} = \begin{pmatrix} 0 & 0 & 0 & \varepsilon \cdot \mathbb{E}[V_{\theta_{IS_i}}] & 0 \end{pmatrix}. \tag{2.32}$$

Again, multiplication with $\vec{pr}_0$ from Equation (2.24) yields the value of the illiquidity expenses $IE(V)$

$$IE(V) = \vec{IE}^{\intercal} \vec{pr}_0. \tag{2.33}$$

Based on Equation (2.8) we obtain the payout structure of the net benefits of debt,

$$\vec{NB}^{\intercal} = \vec{TB} - \vec{IE} - \vec{BC} \tag{2.34}$$

$$= \begin{pmatrix} \frac{\tau C}{r} \\ \frac{\tau C_{il}}{r} \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varepsilon \cdot \mathbb{E}[V_{\theta_{IS_i}}] \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \alpha V_B \\ 0 \\ 0 \end{pmatrix} \tag{2.35}$$

$$= \begin{pmatrix} \frac{\tau C}{r} & \frac{\tau C_{il}}{r} & -\alpha V_B & -\varepsilon \cdot \mathbb{E}[V_{\theta_{IS_i}}] & 0 \end{pmatrix}^{\intercal}, \tag{2.36}$$

and the corresponding conditional expected value as

$$NB(V) = \vec{NB}^{\intercal} \vec{pr}_0. \tag{2.37}$$

By adding the unlevered firm value according to Equation (2.7) we arrive at

$$V^L(V) = V + NB(V), \tag{2.38}$$

the levered firm value. Finally, we determine the value of equity as the difference of the levered firm value from Equation (2.38) and the market value of debt from Equation (2.26):

$$E(V) = V^L(V) - D(V). \tag{2.39}$$

The contingent claims of our IO-model developed in this section provide safe grounds for exploring solutions to the optimal capital structure problem in the next section.

3. Analysis of the Optimal Capital Structure in the IO-Model

In general, we are concerned with maximizing the levered firm value with respect to the coupon payments C subject to certain constraints. The classic constraint introduced by Leland (1994) is that equity investors choose V_B, the asset value where the firm files for bankruptcy, i.e. it is over-indebted, in order to maximize the equity value. We denote this optimal level of bankruptcy asset value with V_B^* which is not exogenously determined but endogenously obtained by setting the first derivative of the equity value with respect to V_B equal to zero. An additional constraint in our setting is that C_{il} needs to reflect a certain risk spread φ above the risk free rate r. Thus, our optimization problem can be formally stated as follows:

$$\begin{aligned} V^L(V, C, C_{il}) &\rightarrow \max \\ \text{s.t. } \frac{\partial E(V, C, C_{il})}{\partial V_B} &= 0 \\ C_{il} - \varphi r D(V, C, C_{il}) &= 0. \end{aligned} \tag{3.1}$$

All other parameters in our model are exogenously set and can be either observed in reality or empirically estimated. Table 1 summarizes these parameters, suggests how to determine them, and provides an idea with respect to reasonable value assumptions.

In the subsequent subsection we develop a solution to our general optimization problem outlined in Eq. (3.1) and compare the results of our IO-model to the results of pure illiquidity and over-indebtedness models. Thereafter, we discuss a possible extension to our optimization framework by endogenizing the covenant ratio δ. This allows us to investigate not only the influence of δ on the optimal solution but also whether optimal δ values may exist. Finally, we apply the IO-model to publicly listed companies in the US in order to judge whether our model may explain observed leverage ratios.

3.1. *Identification of the Optimal Bankruptcy Trigger V_B^**

This subsection investigates the optimal bankruptcy trigger V_B^* via maximizing the equity value, i.e.

$$E(V) \rightarrow \max \tag{3.2}$$

$$\Leftrightarrow \frac{\partial E(V)}{\partial V_B} = 0. \tag{3.3}$$

Technically, we calculate the first derivative of the the equity value with respect to V_B. As we face a long complex value function we present the result based on the modular principle. We benefit from this technique since the *differentiation is linear*. Additionally, beyond reducing complexity, this method allows for investigating some boundary constraints, e.g. fixed costs equal to zero $F = 0$. Our proceeding is related to the equity value function $E(V)$ (cf. Eq. (2.39)) consisting of vectors $\vec{NB}$ and $\vec{D}$, the unlevered firm value V, as well as the vector of contingent present value factors $\vec{pr}_0$ consisting of the single factors p_0 to p_3 derived in section 2.3. The place holders a, b and y of p_0 to p_3 are constants. However, the place

Table 1: Exogenous Parameters of the IO-Model

Parameter	Description	Rationale	Exemplary reasonable values
r	risk free rate	Average of 10-year Treasury rate (1/1989-7/2016) Approach similar to Leland (2004), Huang and Huang (2012)	0.05
τ	corporate tax rate	Federal corporate income tax rate in the US for bigger companies Approach similar to similar to Leland and Toft (1996), Strebulaev (2007)	0.35
R_0	initial value of the revenue process	Firm individual observable parameter	\$25 bn
μ	risk-neutral drift of the revenue process	Firm individual empirical estimation of the real drift μ_P and risk-neutral adjustment by $\mu = \mu_P - (r_A - r)$ Adjustment similar to Goldstein et al. (2001), Couch et al. (2012)	0.02
σ	volatility of the revenue process	Firm individual empirical estimation of the revenue's volatility	0.25
γ	variable cost ratio	Firm individual empirical estimation of the costs of goods sold ratio	0.70
F	fixed costs	Firm individual empirical estimation of selling, general and administrative expenses	0.00
δ	interest coverage ratio	Firm or debt tranche individual covenant defined in the debt contract. Natural lower boundary: $1 - \tau$ as this reflects illiquidity.	$1 - \tau$
φ	spread factor for illiquid firms vs. r	Estimation based on average spread between the promised yield of Caa-rated firms (highly vulnerable to nonpayment) and the risk free rate with 10 years maturity (source: Moody's)	2.50
α	bankruptcy cost ratio	Firm or industry-specific estimation based on empirical models We use findings of Glover (2016)	e.g. 0.39 (Food) 0.49 (machinery)
ϵ	illiquidity cost ratio	Firm or industry-specific estimation based on emprical models with respect to technical defaults We use findings of Ertan and Karolyi (2016)	0.04

This table contains all exogenously set parameters of the IO-model. It also provides suggestions how to observe or estimate the parameters and gives indications with respect to reasonable values.

holders x and l of p_0 to p_3 are functions of V_B (cf. Eq. (2.17)-(2.18)). We start with their first derivatives. The following holds:

$$\frac{\partial x}{\partial V_B} = \frac{-1}{V_B + F(1-\tau)r^{-1}} \tag{3.4}$$

$$\frac{\partial l}{\partial V_B} = \frac{-1}{V_B + F(1-\tau)r^{-1}} \tag{3.5}$$

$$\frac{\partial (l-x)}{\partial V_B} = 0. \tag{3.6}$$

Please note that the first derivative x' of x with respect to V_B equals the first derivative l' of l with respect to V_B. In the next step we want to calculate the derivatives of the contingent present value factors p_0 to p_3. Since p_0 is independent of V_B we obtain:

$$\frac{\partial p_0}{\partial V_B} = 0.$$

The same holds for p_3. Thus, we have

$$\frac{\partial p_3}{\partial V_B} = 0.$$

Consequently, it remains to calculate the derivatives of p_1 and p_2 which we do by applying $\frac{\partial \sinh(x)}{\partial x} = \cosh(x)$:

$$\begin{aligned}
p_1' = \frac{\partial p_1}{\partial V_B} =& 0 + e^{\frac{a}{\sigma^2}(l-x)} \frac{\frac{b}{\sigma^2} x' \sinh'(\frac{b}{\sigma^2}x)\sinh(\frac{b}{\sigma^2}l) - \frac{b}{\sigma^2} l' \sinh(\frac{b}{\sigma^2}x)\sinh'(\frac{b}{\sigma^2}l)}{\sinh^2(\frac{b}{\sigma^2}l)} \\
=& e^{\frac{a}{\sigma^2}(l-x)} \frac{b}{\sigma^2} x' \frac{\cosh(\frac{b}{\sigma^2}x)\sinh(\frac{b}{\sigma^2}l) - \sinh(\frac{b}{\sigma^2}x)\cosh(\frac{b}{\sigma^2}l)}{\sinh^2(\frac{b}{\sigma^2}l)} \\
=& e^{\frac{a}{\sigma^2}(l-x)} \frac{b}{\sigma^2} x' \frac{\sinh(\frac{b}{\sigma^2}(x-l))}{\sinh^2(\frac{b}{\sigma^2}l)} \\
=& \frac{-e^{\frac{a}{\sigma^2}(l-x)} \frac{b}{\sigma^2}}{V_B + F(1-\tau)r^{-1}} \cdot \frac{\sinh(\frac{b}{\sigma^2}(x-l))}{\sinh^2(\frac{b}{\sigma^2}l)}.
\end{aligned}$$

For the derivative of the contingent present value factor p_2 we receive the following:

$$\begin{aligned}
p_2' = \frac{\partial p_2}{\partial V_B} =& \frac{-a}{\sigma^2} x' e^{\frac{-a}{\sigma^2}(l-x)} + e^{\frac{-a}{\sigma^2}(l-x)} \frac{\frac{b}{\sigma^2}(l-x)' \sinh'(\frac{b}{\sigma^2}(l-x))\sinh(\frac{b}{\sigma^2}l) - \frac{b}{\sigma^2} l' \sinh(\frac{b}{\sigma^2}(l-x))\sinh'(\frac{b}{\sigma^2}l)}{\sinh^2(\frac{b}{\sigma^2}l)} \\
=& \frac{-a}{\sigma^2} x' e^{\frac{-a}{\sigma^2}(l-x)} + e^{\frac{-a}{\sigma^2}(l-x)} \frac{-\frac{b}{\sigma^2} l' \sinh(\frac{b}{\sigma^2}(l-x))\cosh(\frac{b}{\sigma^2}l)}{\sinh^2(\frac{b}{\sigma^2}l)} \\
=& x' e^{\frac{-a}{\sigma^2}(l-x)} [\frac{-a}{\sigma^2} - \frac{b}{\sigma^2}] \frac{\sinh(\frac{b}{\sigma^2}(l-x))\cosh(\frac{b}{\sigma^2}l)}{\sinh^2(\frac{b}{\sigma^2}l)} \\
=& \frac{e^{\frac{-a}{\sigma^2}(l-x)}}{\sigma^2[V_B + F(1-\tau)r^{-1}]} [a+b] \frac{\sinh(\frac{b}{\sigma^2}(l-x))\cosh(\frac{b}{\sigma^2}l)}{\sinh^2(\frac{b}{\sigma^2}l)}.
\end{aligned}$$

Hence, we obtain the first derivative of the vector of contingent present value factors $\vec{pr}_0$ with respect to V_B by using the product and quotient rule, i.e.

$$\frac{\partial \vec{pr}_0}{\partial V_B} = \begin{pmatrix} 1 + \frac{p_0 p_1 (1-p_3)}{(1-p_1 p_3)^2} \\ \frac{-p_0(1-p_1-p_2)}{(1-p_1 p_3)^2} \\ \frac{-p_0 p_2}{(1-p_1 p_3)^2} \\ \frac{-p_0}{(1-p_1 p_3)^2} \\ \frac{-p_0 p_1}{(1-p_1 p_3)^2} \end{pmatrix}. \tag{3.7}$$

For the derivative of the net benefit vector $\vec{NB}$ and the debt vector $\vec{D}$ we have

$$\frac{\partial \vec{NB}}{\partial V_B} = \begin{pmatrix} 0 \\ 0 \\ -\alpha \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{\partial \vec{D}}{\partial V_B} = \begin{pmatrix} 0 \\ 0 \\ 1-\alpha \\ 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

In summary, we have to solve the following equation with the help of the product rule and linearity

$$\frac{\partial E(V)}{\partial V_B} = \frac{\partial}{\partial V_B}[V + ((\vec{NB}) - (\vec{D}))\vec{pr}_0]. \tag{3.9}$$

Thus, we arrive at

$$0 \stackrel{!}{=} 1 + \frac{p_0 p_2}{1 - p_1 p_3} + (\vec{NB} - \vec{D})\vec{pr}_0{}'. \tag{3.10}$$

This implicit equation can be solved with the help of mathematical software such as Matlab and using some known methods, e.g. Newton's method.

Before we can compare the optimal bankruptcy trigger $V^*_{B_{IO}}$ from the IO-model with the optimal bankruptcy trigger in a single barrier world, such as the model of Leland (1994) (over-indebtedness) or Couch et al. (2012) (illiquidity), we need to match the assumptions. As mentioned in section 2.1 we refer to a revenue process. Thus, we have to transfer the firm's asset approach in a single barrier world into a revenue's approach in a single barrier world[12]. Furthermore, two famous bankruptcy triggers are known in literature. On the one hand bankruptcy is triggered when the firm is over-indebted. Leland (1994) investigates the implications to the optimal capital structure given this constraint. On the other hand bankruptcy can be declared when the firm is illiquid or breaks a covenant. Couch et al. (2012) base their investigations of valuing tax shields on this barrier. Adjusting the Leland model (over-indebtedness)

12 The firm's asset approach is given by the diffusion process $\frac{dV}{V} = \mu dt + \sigma dW$, where V represents the value of the firm's activities, μ the constant growth rate, σ the constant volatility, and W a standard Brownian motion. V is usually known as the *asset value* of the firm.

to the revenue process yields the following optimal bankruptcy trigger $V^*_{B_{over}}$:

$$V^*_{B_{over}} = \frac{y}{1+y}\frac{C(1-\tau)}{r} - (1 - \frac{y}{1+y})\frac{F(1-\tau)}{r}. \tag{3.11}$$

When the fixed costs F equal zero we generate the standard Leland solution. The appropriate optimal bankruptcy trigger $V^*_{B_{illiquid}}$ given illiquidity as the bankruptcy criterion with fixed costs F equal to zero resemble the standard solution given inCouch et al. (2012).

3.2. Identification of the Optimal Coupon Payment C

With the help of section 3.1 we are able to maximize our total firm value V^L given the optimal bankruptcy trigger $V^*_{B_{IO}}$. This is done by endogenizing the coupon payments C. Thus, the coupon payment is no longer fixed and considered as a constant. Rather, we compute the first derivative of the total firm value V^L subject to C. Finally, we set the first derivative of the total firm value equal to zero, i.e.

$$\frac{\partial V^L(V)}{\partial C} = 0. \tag{3.12}$$

Solving this equation for the optimal coupon C^*_{IO} maximizes the total firm value. We will now compare the firm's maximizing coupon payment C^*_{IO} in a double barrier world with the firm maximizing coupon payment C^*_{over} and $C^*_{illiquid}$ that are generated when either over-indebtedness or illiquidity are the bankruptcy triggers. The following figure illustrates the findings graphically.

Figure 4: Optimal Capital Structure under the IO-Model, pure Illiquidity Model and pure Over-indebtedness Model

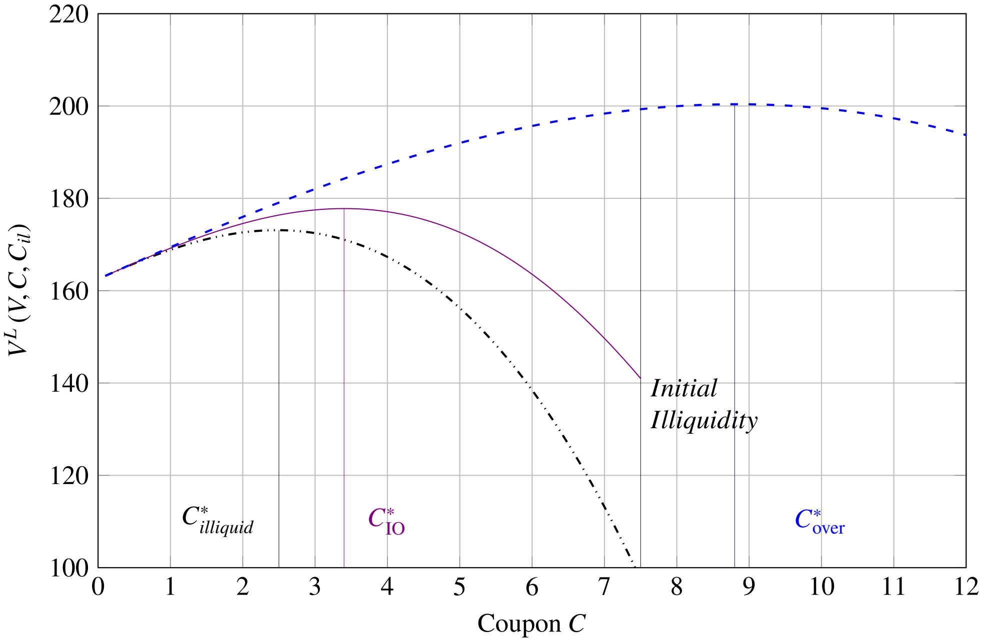


Figure 5: This figure analyzes the firm value $(V^L(V))$-maximizing choice of coupon payments C for the IO-model in comparison to the classic models of illiquidity and over-indebtedness. The blue, dashed line represents $V^L(V)$ for different C with over-indebtedness as a bankruptcy trigger. The violet, solid line represents the IO-model and the black, dashed-dotted line depicts the case of illiquidity. The chosen model parameters are as follows: $r = 0.05$, $\tau = 0.35$, $R_0 = 25$, $\mu = 0.02$, $\sigma = 0.20$, $\gamma = 0.70$, $F = 0$, $\delta = 1 - \tau$, $\epsilon = 0.00$, and $\varphi = 2.5$.

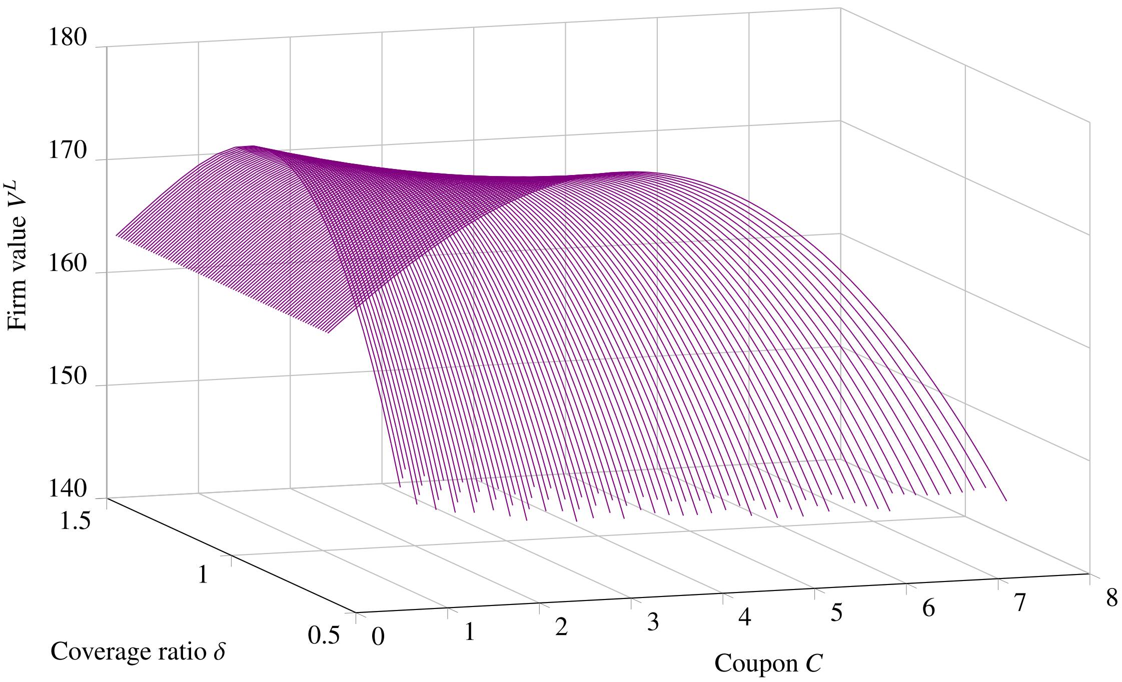
The figure shows the coupon level C subject to the total firm value V^L. Each of the three parables represent a different bankruptcy trigger. The curve on top is the function that arises if and only if over-indebtedness is the only bankruptcy trigger. Analogously, the curve on bottom is generated if and only if illiquidity creates bankruptcy. The curve in the middle combines both approaches and represents the total firm value function with respect to C of the IO-model. The figure depicts four main aspects. (i) We can observe that all three curves are concave, i.e. there exists a global maximum. (ii) In case of over-indebtedness the total firm value with respect to C is greater than in case of illiquidity. Taking both barriers into consideration provides a curve that lies in-between. (iii) The same holds true for the optimal coupon payments, i.e. $C^*_{illiquid} < C^*_{IO} < C^*_{over}$. Finally, (iv) if there is only over-indebtedness as the bankruptcy trigger, the optimal coupon payment C^*_{over} is in the area of illiquidity. Thus, optimizing the total firm value with over-indebtedness as the bankruptcy criterion provokes directly illiquidity.

As the figure shows, the double barrier approach provides solution that are in-between the rough constraints of over-indebtedness and illiquidity. This is in accordance with the intuition. Moreover, the optimal coupon payment C^*_{IO} of the IO-model is in the area of liquidity.

3.3. Extensions to the Optimization Framework - Endogenizing Debt Contract Parameters

The IO-model provides insights beyond the discussed framework where the optimal capital structure is derived with an endogenously obtained V_B^* but otherwise given parameters. For instance, it allows for analyzing some standard debt contract parameters like the covenant ratio δ. We are able to determine its impact on the optimal capital structure choice and to investigate whether an optimal δ exists. Figure 6 depicts the analysis results when C and δ can be freely chosen.

Figure 6: Levered Firm Value $V^L(V)$ in dependence of Covenant Ratio δ and Coupon Payment C



The graph depicts how changing δ and C impacts $V^L(V)$. For lower *delta* values the maximum levered firm value $V^{L,*}(V)$ is achieved with higher choices of C^* and vice versa. The global optimum is at the minimum δ of $1-\tau$. The chosen model parameters are as follows: $r = 0.05$, $\tau = 0.35$, $R_0 = 25$, $\mu = 0.02$, $\sigma = 0.20$, $\gamma = 0.70$, $F = 0$, $\epsilon = 0.00$, and $\varphi = 2.5$.

As Figure 6 reveals, a higher δ causes lower optimal choices of C^* and also reduces the optimal levered firm value $V^{L,*}(V)$. The results may surprise as we usually observe δ values between 1 and 2 in corporate debt contracts. Two reasons for the discrepancy are identified:

(i) Debtholders in our setting are risk-neutral, i.e. they are only interested in an expected net present value of zero and do not discount riskier payoff structures. We demonstrate the effect of higher δ values on the contingent present value factor, i.e. discounted probability, of the OS in Table 2. Clearly, the contingent present value factor $pr^0_{(1-\alpha)V_B}$ decreases with increasing δ. Risk-averse debtholders value this fact while risk-neutral debtholders are indifferent. Thus, we may have found an indication for risk-aversion of debtholders.

Table 2: Bankruptcy State Prices $pr^0_{(1-\alpha)V_B}$ in dependence of the covenant ratio δ

δ	C^*	$V^{L,*}$	$L^* = D(V)/V^{L,*}$	$pr^0_{(1-\alpha)V_B}$
0.65	3.40	177.79	0.37	0.21
0.75	3.00	176.56	0.35	0.20
0.85	2.70	175.59	0.32	0.20
0.95	2.40	174.80	0.30	0.18
1.05	2.20	174.15	0.29	0.18
1.15	2.00	173.59	0.27	0.17

The table illustrates how increasing δ values lead to a lower bankruptcy risk (represented by a lower contingent present value factor for the over-indebtedness state). This shows that debt holders which are not risk-neutral may actually insist on a δ greater than $1-\tau$ depending on their risk appetite.

(ii) Information are symmetrically distributed in our setting, i.e. debtholders know the true V_B where equity investors file for bankruptcy. However, in reality this information is most likely only known to the equity investors themselves. Pretending a higher V_B may result in better debt contracts. Debtholders shield themselves against such behavior with increased covenant ratios. Please note that the analysis of (ii) will be detailed in the next version of the working paper.

3.4. *Empirical Application of the IO-Model*

Finally, we test our model for firms publicly listed in the US. Our dataset, retrieved from Thomson-Reuters EIKON, is based on the logic of the Center for Research in Security Prices (CRSP). We consider all firms that have been listed on the NYSE, NASDAQ, NYSE MKT and NYSE ARCA between 1981 and 2016 including all leavers and joiners of this period. We exclude firms from finance and insurance (NAICS sector code 52) as well as firms with inconsistent data (e.g. constantly negative revenues) or not sufficient time series (less than 10 firm years). After these exclusions, our sample contains 4,845 firms and 97,001 firm-year observations with non-missing values for revenues, costs of goods sold (COGS), selling, general and administrative expenses (SGA), debt, total assets, and market capitalization.

In a first step we estimate the parameters of the stochastic revenue process, drift rate μ and standard deviation σ. Moreover, we test whether the observed revenue paths could follow a geometric Brownian motion (gBm) by applying the Jarque-Bera (JB) test for normal distribution. In total, at the 5% interval we cannot reject the null hypothesis of the JB-test postulating that the considered process is not following a gBm for 47.5% of the firms. Thus, our basic model requirement is valid for almost half of the publicly listed firms in the US. Table 3 summarizes the test results for all NAICS sectors.

We only proceed with estimating the other parameters and calculate the average observed leverage $L = D(V)/V^L(V)$ for firms where revenues follow a gBm. Please note that our model generates equivalently robust results on the full sample. It is solely for mathematical exactness that we exclude firms where the

Table 3: Normal-Distribution Test of the log-changes of R_t

		Jarque-Bera Test					
		N, Norm.-Dist.		in %, Norm.-Dist.			
NAICS Sector	**No. Of Firms (N)**	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$	μ	σ
Accommodation and Food Services	96	44	36	0.4583	0.3750	0.0264	0.1030
Administrative, Support, Waste, Remediation	105	42	31	0.4000	0.2952	0.0253	0.2175
Construction	77	33	29	0.4286	0.3766	0.0110	0.2190
Health Care and Social Assistance	103	31	27	0.3010	0.2621	0.0074	0.1824
Information	529	256	216	0.4839	0.4083	0.0199	0.1776
Manufacturing	1988	948	785	0.4769	0.3949	0.00026	0.1526
Mining, Quarrying, and Oil and Gas Extraction	264	157	126	0.5947	0.4773	0.0117	0.2622
Professional, Scientific, and Technical Services	408	221	181	0.5417	0.4436	0.0014	0.1454
Real Estate and Rental and Leasing	204	77	66	0.3775	0.3235	0.0392	0.1912
Retail Trade	242	122	107	0.5041	0.4421	0.0438	0.1270
Transportation and Warehousing	143	56	46	0.3916	0.3217	0.0188	0.1607
Utilities	103	41	33	0.3981	0.3204	0.00054	0.1780
Wholesale Trade	147	69	55	0.4694	0.3741	0.0269	0.2304
Others	110	48	41	0.4364	0.3727	0.0274	0.2014

The table depicts the results of the Jarque-Bera test for normal distribution which we apply to examine the log-changes of the stochastic process R_t. The null hypothesis of the test is that the underlying process is normally distributed. Thus, choosing a higher significance level α leads to a higher number of firms for which normal distribution is ruled out. The last two columns provide our estimations of the risk-neutral drift of the revenue process μ and its standard deviation σ.

null hypothesis of JB-test was rejected.[13]

While we can retrieve the estimates for the variable cost ratio γ and the fixed costs F from our dataset, we have to rely on other studies for the other missing parameters. We follow Glover (2016) in his industry-specific estimates of the expected bankruptcy costs α. For the illiquidity expenses ϵ and the average covenant ratio δ industry-specific estimates are not yet available. Thus, we apply the general estimates of Ertan and Karolyi (2016) to all industries. Please note that we have indexed the initial level of the stochastic process R_0 to 100 in order to make all firms comparable. Table 4 summarizes our input choices.

To conclude, we obtain the optimal leverage based on the IO-model as well as for the pure illiquidity and pure over-indebtedness model. These results are compared to the observed leverage ratios. The results are shown in Table 5.

The leverage ratios estimated by the IO-model show the lowest absolute deviation (Abs. Dev.) from the observed leverage except for the sector "Real Estate and Rental and Leasing" where the over-indebtedness model performs slightly better. The IO-estimates lie within the one standard error range for 3 of the sectors and within a two standard error range for another 3 sectors. The pure illiquidity model underestimates optimal leverage consistently in all sectors while the pure over-indebtedness model leads consistently to overestimation. None of the two models achieves results within one or two standard errors from the observed leverage. The results prove that the IO-model is a major step in explaining observed

13 Within the literature strand of optimal capital structure it is not common to perform tests to validate the underlying gBm assumption or at least it is not common to publish them. Usually, gBm is implicitly assumed throughout all empirical analyses.

Table 4: Input Parameters of the IO-Model and Observed Leverage

NAICS Sector	α	ϵ	δ	γ	F	R_0	$L = D(V)/V^L(V)$
Accommodation and Food Services	0.3890	0.04	1.00	0.6195	14.00	100	0.4594
Administrative, Support, Waste, Remediation	0.4740	0.04	1.00	0.5110	23.04	100	0.1994
Construction	0.3740	0.04	1.00	0.7220	18.83	100	0.4405
Health Care and Social Assistance	0.4740	0.04	1.00	0.2483	51.35	100	0.5497
Information	0.4740	0.04	1.00	0.3941	25.37	100	0.2927
Manufacturing	0.3970	0.04	1.00	0.6915	15.85	100	0.3071
Mining, Quarrying, and Oil and Gas Extraction	0.4630	0.04	1.00	0.5165	11.42	100	0.2535
Professional, Scientific, and Technical Services	0.4740	0.04	1.00	0.5131	33.90	100	0.2225
Real Estate and Rental and Leasing	0.4740	0.04	1.00	0.4066	14.27	100	0.5412
Retail Trade	0.4420	0.04	1.00	0.7026	19.19	100	0.2714
Transportation and Warehousing	0.4130	0.04	1.00	0.4513	27.16	100	0.4286
Utilities	0.4740	0.04	1.00	0.3518	33.02	100	0.4531
Wholesale Trade	0.4420	0.04	1.00	0.7382	23.74	100	0.2923
Others	0.4598	0.04	1.00	0.5824	23.69	100	0.2683

The table provides an overview of the chosen input parameters for each NAICS sector. For the bankruptcy costs α we follow the estimates of Glover (2016). Regarding the illiquidity expenses ϵ and the average covenant ratio δ industry-specific estimates are not yet available. Thus, we apply the general estimates of Ertan and Karolyi (2016) to all industries. The starting point of the stochastic revenue process R_0 is indexed to 100. The estimates for the variable cost ratio γ and the fixed costs F are based on all normally distributed firms of our sample from NASDAQ, NYSE, NYSE ARCA, and NYSE MKT. F has been related to the index of R_0. The leverage ratio $L = D(V)/V^L(V)$ is based on the sample as well.

Table 5: Optimal Capital Structure Estimates versus Observed Leverage for NAICS Sectors

	Observed		**Illiquidity**		**IO-Model**		**Overindebtedness**	
NAICS Sector	L	**1 Std. Err.**	L^*	**Abs. Dev.**	L^*	**Abs. Dev.**	L^*	**Abs. Dev.**
Accommodation and Food Services	0.4594	0.0752	0.1861	0.2734	0.3096^{**}	0.1498	0.7942	0.3348
Administrative, Support, Waste, Remediation	0.1994	0.0380	0.1599	0.0394	0.2985^{*}	0.0991	0.8064	0.6071
Construction	0.4405	0.0376	0.1442	0.2962	0.3658^{**}	0.0747	0.6620	0.2215
Health Care and Social Assistance	0.5497	0.0440	0.0662	0.4835	0.4627^{**}	0.0871	0.6554	0.1057
Information	0.2927	0.0160	0.0316	0.2611	0.2236	0.0691	0.8629	0.5703
Manufacturing	0.3071	0.0096	0.0472	0.2599	0.3252^{**}	0.0181	0.6746	0.3675
Mining, Quarrying, and Oil and Gas Extraction	0.2535	0.0142	0.0125	0.2410	0.2450^{***}	0.0085	0.6359	0.3824
Professional, Scientific, and Technical Services	0.2225	0.0127	0.0974	0.1251	0.2227^{***}	0.0003	0.6897	0.4672
Real Estate and Rental and Leasing	0.5412	0.0268	0.0149	0.5263	0.1584	0.3828	0.7198	0.1786
Retail Trade	0.2714	0.0170	0.0412	0.2302	0.3017^{**}	0.0303	0.6859	0.4145
Transportation and Warehousing	0.4286	0.0252	0.0452	0.3834	0.2446	0.1840	0.7138	0.2852
Utilities	0.4531	0.0259	0.0104	0.4427	0.4583^{***}	0.0051	0.6323	0.1791
Wholesale Trade	0.2923	0.0337	0.0459	0.2464	0.1329	0.1594	0.6626	0.3703
Others	0.2683	0.0242	0.0936	0.1747	0.1689	0.0994	0.6880	0.4197

This table summarizes the optimal leverage ratios $L^* = D(V)/V^{L,*}(V)$ generated by the IO-model, and for a pure illiquidity or over-indebtedness trigger. The results are compared to the observed average leverage L for all NAICS sectors. The absolute deviation towards the observed leverage is depicted for each of the three models (Abs. Dev.). $***$ = Within 1 standard error; $**$ = Within 2 standard errors; $*$ = Within 3 standard errors leverage ratios and delivers a new unique contribution to the capital structure literature.

4. Conclusion

This article establishes the first dynamic capital structure model in closed-form solutions which incorporates an illiquidity trigger and an over-indebtedness trigger in a double barrier option framework.

We start by introducing a corporate valuation framework including all relevant benefits and costs of debt which is based on a stochastic revenue process that follows a gBm. Subsequently, we explicitly define default triggers for illiquidity as well as over-indebtedness, and integrate those in our framework. From here, we carefully develop towards definitions of contingent present value factors, describing probabilities and discount factors of potential states of the world, as well as payout structures for the different corporate value components. Based on these analyses we provide closed-form valuation equations for all components of debt and equity. Moreover, we obtain solutions for the optimal capital structure choice in such a double barrier option framework. Finally, we compare our solution to the two classic cases of only considering one of the two triggers. The results we generate lie in-between and explain observed capital structure choices much better than the existing models as we demonstrate by an empirical study of the US market.

The model also provides a good base for further extensions. For instance, it is sometimes observed in reality that the stochastic process jumps whenever the illiquidity boundary is hit which is easily implementable into the existing framework. Additionally, adjustments in the payout structure can be simply executed as we provide a general framework (A_1 to A_5) for all kinds of payout. In a further effort, the idea of dynamic capital structures, where additional debt is borrowed whenever an upside trigger, i.e. the firm value increased, is hit, could be implemented via the double barrier option framework. Beyond that, further empirical studies in the field of corporate finance (e.g. regarding costs of capital or probabilities of default) can be based upon the model.

References

Achleitner, A.-K., Braun, R., Hinterramskogler, B., Tappeiner, F., 2012. Structure and determinants of financial covenants in leveraged buyouts. Review of Finance 16, 647–684.

Couch, R., Dothan, M., Wu, W., 2012. Interest tax shields: A barrier options approach. Review of Quantitative Finance and Accounting 39, 123–146.

Danis, A., Rettl, D. A., Whited, T. M., 2014. Refinancing, profitability, and capital structure. Journal of Financial Economics 114, 424–443.

Ertan, A., Karolyi, S. A., June 2016. Debt covenants and the expected cost of technical default, available at SSRN: http://ssrn.com/abstract=2795226.

Glover, B., 2016. The expected cost of default. Journal of Financial Economics 119 (2), 284–299.

Goldstein, R., Ju, N., Leland, H. E., 2001. An EBIT-based model of dynamic capital structure. Journal of Business 74 (4), 483–512.

Graham, J. R., October 2000. How big are the tax benefits of debt? The Journal of Finance 55 (5), 1901–1941.

Hackbarth, D., Hennessy, C., Leland, H. E., 2007. Can the tradeoff theory explain debt structure? Review of Financial Studies 20 (5), 1389–1428.

Huang, J.-Z., Huang, M., 2012. How much of the corporate-treasury yield spread is due to credit risk? The Review of Asset Pricing Studies 2 (2), 153–202.

Kim, J., Ramaswamy, K., Sundaresan, S., 1993. Does default risk in coupons affect the valuation of corporate bonds?: A contingent claims model. Financial Management.

Korteweg, A., December 2010. The net benefits to leverage. The Journal of Finance 65 (6), 2137–2170.

Leland, H. E., 1994. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance 49 (4), 1213–1252.

Leland, H. E., 2004. Predictions of default probabilities in structural models of debt. Journal of Investment Management 2 (2).

Leland, H. E., 2006. Structural models in corporate finance. In: Bendheim Lectures in Finance - Princeton University.

Leland, H. E., Toft, K. B., July 1996. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. Journal of Finance 51 (3), 987–1019.

Modigliani, F., Miller, M. H., June 1958. The cost of capital, corporation finance and the theory of investment. The American Economic Review 48 (3), 261–297.

Morellec, E., Nikolov, B., Schürhoff, N., may 2012. Corporate governance and capital structure dynamics. Journal of Finance 67 (3), 803–848.

Myers, S., July 1984. The capital structure puzzle. The Journal of Finance 39 (3), 575–592.

Pelsser, A., 2000. Pricing double barrier options using laplace transforms. Finance and Stochastics 4, 95–104.

Rubinstein, M., Reiner, E., 1991. Unscrambling the binary code. Risk Magazine 4 (9), 75–83.

Rubinstein, M., Reiner, E., 1992. Exotic options (research program in finance - working paper series).

Strebulaev, I. A., aug 2007. Do tests of capital structure theory mean what they say? Journal of Finance 62 (4), 1747–1787.

Strebulaev, I. A., Whited, T. M., 2011. Dynamic models and structural estfinance in corporate finance. Foundations and Trends in Finance 6 (1-2), 1–163.

Titman, S., Tsyplakov, S., 2007. A dynamic model of optimal capital structure. Review of Finance 11 (3), 401–451.

Appendices

Lemma (2.2). *The firm will enter illiquidity state (IS) if* $EBIT(1-\tau) \leq \delta C$*, which corresponds to* $R_t \leq B_{IS}$ *where* $B_{IS} = (\delta C + F(1-\tau)) \,/\, ((1-\gamma)(1-\tau))$.

Proof. We substitute Equation (2.2) into the covenant definition from above and rearrange for R_t:

$$\begin{aligned} EBIT(1-\tau) &= \delta C \\ (R_t(1-\gamma) - F)(1-\tau) &= \delta C \\ R_t &= \frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)}. \end{aligned}$$

Since the covenant definition $(1-\tau)EBIT_t = \delta C$ corresponds to $R_t = B_{IS}$, we have:

$$B_{IS} := \frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)}$$ □

Lemma (2.4). *The firm will reenter liquidity state LS if* $EBIT(1-\tau) = \delta C_{il}$ *with* $t \geq \theta_{IS_i}$*, which corresponds to* $R_t = B_{LS}$ *where* $B_{LS} = (\delta C_{il} + F(1-\tau)) \,/\, ((1-\gamma)(1-\tau))$ *with* $t \geq \theta_{IS_i}$.

Proof. We substitute Equation (2.2) into the adjusted covenant definition from above and rearrange for R_t:

$$\begin{aligned} EBIT(1-\tau) &= \delta C_{il} \\ (R_t(1-\gamma) - F)(1-\tau) &= \delta C_{il} \\ R_t &= \frac{\delta C_{il} + F(1-\tau)}{(1-\gamma)(1-\tau)}. \end{aligned}$$

Since the covenant definition $EBIT(1-\tau) = \delta C_{il}$ corresponds to $R_t = B_{LS}$, we have:

$$B_{LS} := \frac{\delta C_{il} + F(1-\tau)}{(1-\gamma)(1-\tau)}.$$ □

Lemma (2.5). *The firm will file for bankruptcy if* $\mathbb{E}[V_t] = V_B$ *with* $t \geq \theta_{IS_i}$*, which corresponds to* $R_t = B_{OS}$ *where* $B_{OS} = \left(\left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu)\right) \,/\, ((1-\gamma)(1-\tau))$ *with* $t \geq \theta_{IS_i}$.

Proof. We substitute equation (2.6) into the bankruptcy trigger definition from above and rearrange for R_t:

$$\begin{aligned} \mathbb{E}[V_t \,|\, \mathcal{F}_t] &= V_B \\ \frac{R_t(1-\gamma)(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r} &= V_B. \end{aligned}$$

Since the bankruptcy definition $\mathbb{E}[V_t] = V_B$ corresponds to $R_t = B_{OS}$, we have by simple rearrangements:

$$B_{OS} := \frac{\left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu)}{(1-\gamma)(1-\tau)}$$ □

Lemma (2.6). *The covenant boundary* B_{LS}*, upper-upper boundary to the revenue process* R_t *if the firm stays in illiquidity state (IS), is strictly greater than the covenant boundary* B_{IS}*, lower-upper boundary* *to R_t if the firm stays in liquidity state (LS). Moreover, B_{IS} is greater than or equal to the bankruptcy boundary B_{OS}, lower boundary to R_t if the firm stays in IS. Thus, we have $B_{OS} < B_{IS} < B_{LS}$.*

Proof.

$$B_{LS} > B_{IS} \tag{.1}$$

$$\frac{\delta C_{il} + F(1-\tau)}{(1-\gamma)(1-\tau)} > \frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)} \tag{.2}$$

$$C_{il} > C. \qquad \square$$

This holds since $C_{il} > C$ by definition.

$$B_{IS} > B_{OS} \tag{.3}$$

$$\frac{\delta C + F(1-\tau)}{(1-\gamma)(1-\tau)} > \frac{\left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu)}{(1-\gamma)(1-\tau)} \tag{.4}$$

$$\delta C + F(1-\tau) > \left(V_B + \frac{F(1-\tau)}{r}\right)(r-\mu) \tag{.5}$$

$$V_B < \frac{\delta C + F(1-\tau)}{r-\mu} - \frac{F(1-\tau)}{r}. \tag{.6}$$

Remark (2.10). *If $\theta_{IS_i} \leq \theta_{OS} \leq \theta^{LS_i}$, then $\theta_{IS_{i+1}} = \theta^{LS_i} = \emptyset$.*

Proof. Assume that $\theta_{IS_i} \leq \theta_{OS} \leq \theta^{LS_i}$. This yields that $\theta^{LS_i} = \emptyset$. Simply applying the definition for $\theta_{IS_{i+1}}$ we have

$$\begin{aligned}\theta_{IS_{i+1}} &= \inf\{t \geq \theta^{LS_i} \,|\, R_t = B_{IS} \ \wedge \ R_s > B_{OS} \ \forall s \in [\theta^{LS_i}, t]\} \\ &= \inf\{t \geq \theta^{LS_i} \,|\, R_t = B_{IS}\} \\ &= \emptyset.\end{aligned}$$

The last equality holds due to the simple fact that R_t for all $t \geq \theta_{OS}$ and $\theta^{LS_i} \geq \theta_{OS}$ owing to the above mentioned assumption. $\square$

Lemma (2.12). *The expected present values of the payouts $\mathbb{E}[A_j]$, with $j = 1, ..., 5$, in $t = 0$ are products of the deterministic payout A_j and the corresponding contingent present value factor denoted by $pr^0_{A_j}$. The terms $pr^0_{A_j}$ are combinations of the contingent present value factors of the different states, p_0 to p_3. It follows that*

$$\mathbb{E}[A_1] = A_1 pr^0_{A_1} \text{ ,with } pr^0_{A_1} := [(1-p_0) + \frac{p_0 p_1 (1-p_3)}{1-p_1 p_3}], \tag{.7}$$

$$\mathbb{E}[A_2] = A_2 pr^0_{A_2} \text{ ,with } pr^0_{A_2} := [\frac{p_0(1-p_1-p_2)}{1-p_1 p_3}], \tag{.8}$$

$$\mathbb{E}[A_3] = A_3 pr^0_{A_3} \text{ ,with } pr^0_{A_3} := [\frac{p_0 p_2}{1-p_1 p_3}], \tag{.9}$$

$$\mathbb{E}[A_4] = A_4 pr^0_{A_4} \text{ ,with } pr^0_{A_4} := [\frac{p_0}{1-p_1 p_3}], \tag{.10}$$

$$\mathbb{E}[A_5] = A_5 pr^0_{A_5} \text{ ,with } pr^0_{A_5} := [\frac{p_0 p_1}{1-p_1 p_3}]. \tag{.11}$$

Proof. First, we develop the payout series for A_1 where

$$\begin{aligned}
\mathbb{E}[A_1] = A_1[&(1-p_0)+ && \text{, value until the first liquidity crisis } \theta_{IS_0}\\
&p_0p_1(1-p_3)+ && \text{, value after leaving first IS } \theta^{LS_0} \text{ and until } \theta_{IS_1}\\
&p_0p_1p_3p_1(1-p_3)+ && \text{, value after } \theta^{LS_1} \text{ and until } \theta_{IS_2}\\
&...]\\
= A_1&[(1-p_0)+p_0p_1(1-p_3)\sum_{i=0}^{\infty}p_1^ip_3^i]\\
= A_1&[(1-p_0)+\frac{p_0p_1(1-p_3)}{1-p_1p_3}].
\end{aligned}$$

From now on we say $pr^0_{A_1} := (1-p_0)+\frac{p_0p_1(1-p_3)}{1-p_3p_1}$ is the contingent present value factor of the payout A_1 starting in $t=0$. Analogously, we calculate the expected value for the payout A_2.

$$\begin{aligned}
\mathbb{E}[A_2] = A_2[&p_0(1-p_1-p_2)+ && \text{, value after } \theta_{IS_0} \text{ until } \theta^{LS_0}\\
&p_0p_1p_3(1-p_1-p_2)+ && \text{, value after } \theta_{IS_1} \text{ until } \theta^{LS_1}\\
&...]\\
= A_2&[p_0(1-p_1-p_2)\sum_{i=0}^{\infty}p_1^ip_3^i]\\
= A_2&[\frac{p_0(1-p_1-p_2)}{1-p_1p_3}].
\end{aligned}$$

So the contingent present value factor of the payout A_2 starting in $t=0$ is given by $pr^0_{A_2} := \frac{p_0(1-p_1-p_2)}{1-p_1p_3}$. Analogously, we calculate the expected value for the payout A_3.

$$\begin{aligned}
\mathbb{E}[A_3] = A_3[&p_0p_2+ && \text{, going bankrupt in } [\theta_{IS_0},\theta^{LS_0}]\\
&p_0p_1p_3p_2+ && \text{, going bankrupt in } [\theta_{IS_1},\theta^{LS_1}]\\
&...]\\
= A_3&[p_0p_2\sum_{i=0}^{\infty}p_1^ip_3^i]\\
= A_3&[\frac{p_0p_2}{1-p_1p_3}].
\end{aligned}$$

From now on we say $pr^0_{A_3} := \frac{p_0p_2}{1-p_1p_3}$ is the contingent present value factor of the payout A_3 starting in $t=0$. Calculating the expected value for the payout A_4 yields

$$\begin{aligned}
\mathbb{E}[A_4] = A_4[&p_0+ && \text{, touching } B_{IS} \text{ in } \theta_{IS_0}\\
&p_0p_1p_3+ && \text{, touching } B_{IS} \text{ in } \theta_{IS_1}\\
&...]\\
= A_4&[p_0\sum_{i=0}^{\infty}p_1^ip_3^i]\\
= A_4&[\frac{p_0}{1-p_1p_3}]
\end{aligned}$$

where $pr^0_{A_4} := \frac{p_0}{1-p_1p_3}$ is the contingent present value factor of the payout A_4 starting in $t = 0$. Finally, we calculate the expected value for the payout A_5.

$$
\begin{aligned}
\mathbb{E}[A_5] = A_5[&p_0p_1+ && \text{, touching } B_{LS} \text{ in } \theta^{LS_0}\\
&p_0p_1p_3p_1+ && \text{, touching } B_{LS} \text{ in } \theta^{LS_1}\\
&...]\\
= A_5[&p_0p_1\sum_{i=0}^{\infty} p_1^i p_3^i]\\
= A_5[&\frac{p_0p_1}{1-p_1p_3}].
\end{aligned}
$$

From now on we say $pr^0_{A_5} := \frac{p_0p_1}{1-p_1p_3}$ is the contingent present value factor of the payout A_5 starting in $t = 0$. adapt the initial probability that means to divide the original probability by p_0. For the first case we have to subtract $(1 - p_0)$ initially. This is simply due to the fact that we start in θ_{IS_0}. Thus, the probability of $t \in [0, \theta_{IS_0}]$ equals 0. □