Kuramoto Oscillator

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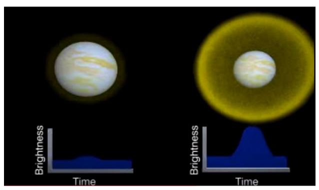
May 22, 2016

Contents

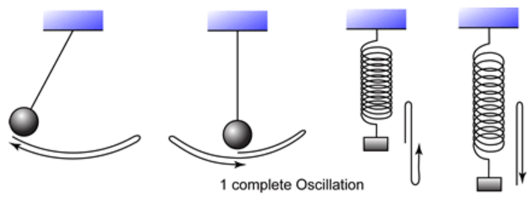
1 Introduction

1.1 Oscillators

Oscillators can be described as a repetitive motion of some measure about a central value which is often the point of equilibrium. This term is most often used in mechanical systems but it must also be noted that oscillations occur in dynamic systems too such as economic graphs, geothermal temperatures across areas and periodic "firing" of fireflies in nature.



(a) Cepheid Stars that are known to flash occasionally



(b) Oscillations

1.2 Coupling

Coupled Oscillation is a slightly more complex form of ordinary oscillators. In these models, the oscillators are connected in such a way that energy is transferred between then. This motion can very well be complex but does not have to be periodic. However, in the bigger scheme of things, every oscillator can be viewed as having a very well defined frequency of its own. Perhaps the simplest example of coupling could be a gear that transmits torque between two shafts that are not collinear. A bit more complex example can be of two pendulums joined together by an energy medium, ie a string.
As we can see in Figure 2, a pendulum only attains its maximum amplitude when the other has its lowest one. This period is achieved after sufficient time has been given to the system to attain synchronization.

1.3 Synchronisation

It should be noted that synchronization can only occur in two ways. The first being if the oscillators have some way of communicating with each

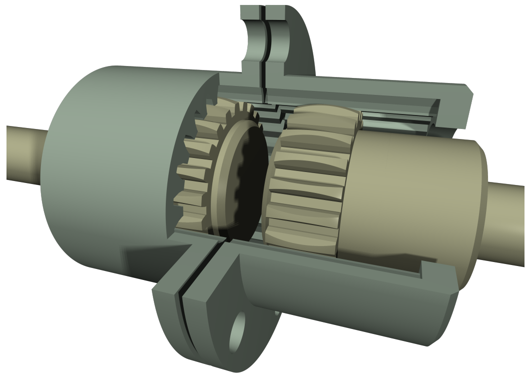


Figure 2

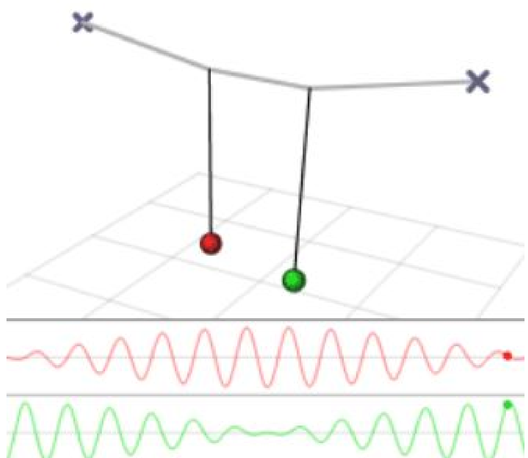


Figure 3

other or the second more rarer case where they start at the same exact time. We only focus on the first case in our study.. Communication can be achieved either by using a medium such as a string between pendulums or even just the intrinsic tendency of natural beings to produce a unison of movement(synchronization) such as fireflies flashing or humans clapping in a room. The firefly flashing, more commonly known as the firefly model, is a classic example of synchronisation. Their light patters are part of a mating display that the male fireflies use to attract the female ones and if successful, the female replies back with a characteristic flash of it own. Although, this is synchronisation on its own, there is onyl a few specie of firefly that can synchronize on a much bigger level and is usually the one seen in large groups of fireflies flashing together in unison. The following visually describes the flashes-:

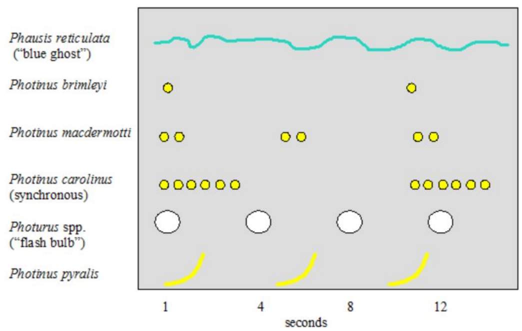


Figure 4

As we can see in the figure, all of the species fire of one after another in a synchronized fashion except the blue one which seems to be random.

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1.4 Adjacency Matrix and the Network Topology

Now we move on to define the mathematical tools being used in this project. The Adjacency matrix is a square matrix that is using to represent a finite graph. The elements of the matrix indicate whether pairs of verticies are adjacent or not, ie if they are adjacent they get assigned a value of -1 else 0. This information can be directly retrieved from the topology network graph as follows-:

As we can see, on the top is the topological network containing the vertices of the system and below lies the coupling graphs of the system over time. Over here, we can see that the couples are trying to stay as far away as possible so after the initial unrest there is always a phase difference of 2π.

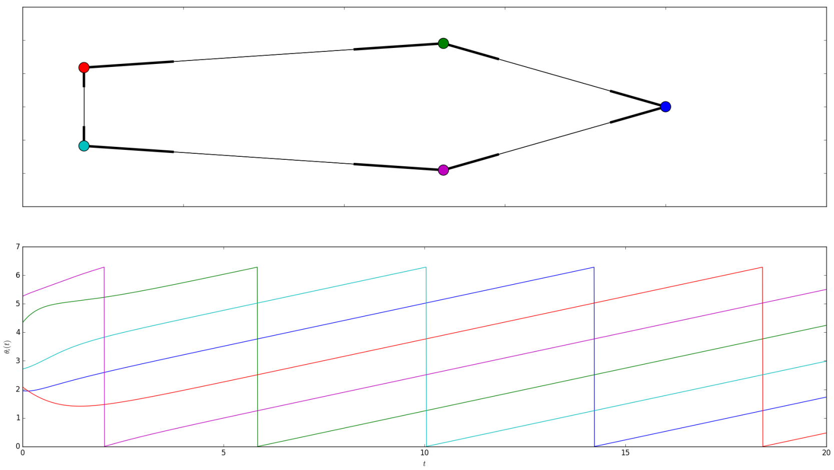

Figure 5

1.5 Goal

Our goal for this study is to investigate coupled systems with negative coupling co-efficients. Positive Coupling Coefficients can have expected looking patterns but the results get more interesting for negative ones .

In Section 2 we will introduce the Kuramoto oscillator, its equations, our implementation and do an analytic analysis of stability. In Section 3 we continue with describing the patterns we encountered when using negative coefficients. We then move on to describe a method which can be used to re-create arbitrary patterns in Section 4 before concluding in Section 5.

2 The Kuramoto oscilator

2.1 Definition

The problem of having a naive viewpoint on the subject of synchronization is that one may believe that every different coupling situation would require its own treatment. This may stem from the fact that one may be focusing too much on the differences rather then the similarities which are found in all synchronous systems. One such attempt to define these similarities in a mathematical form was proposed by Kuramoto in 1975.

$$\dot{\theta}_i = \omega_i + \sum_{j=0}^{N} A_{i,j} \sin(\theta_j - \theta_i)$$

where:

$$\begin{aligned} \theta_i &= \text{The phase of the } i^{\text{th}} \text{ oscillator} \\ \omega_i &= \text{The natural frequency of the } i^{\text{th}} \text{ oscillator} \\ A_{i,j} &= \text{The adjacency matrix of the system} \end{aligned}$$

Each oscillator has its own natural frequency which is held when kept in isolation but when coupled with other oscillators, it changes. This change is periodic and is thus represented by the periodic sin function.

2.2 Code

We used the NumPy package for computing the sin of the phase differences.

```
s.append('(%s)*np.sin(theta[%s]-theta[%s])'%(self.A[i,j],j,i))
```

3 Patterns in negative coefficient systems

For our investigation of the negative coefficient situation we consider a simplified version of the Kuramoto oscilator. Specifically we assume that all oscilators have the same basic frequency ω_i, which we call ω. We also assume that there is no external driver to the system, i.e. $b_i = 0$. This simplifies the system of equations to:

$$\dot{\theta}_i = \omega + \sum_{i=0}^{N} A_{i,j} \sin \theta_i - \theta_j$$

Furthermore we assume that all coefficients in the matrix A are either -1 or 0. We can thus see A as the negative of an adjacency matrix of a graph. Here each oscilator is considered as a node of a graph and an edge can be interpreted as A having a -1 in the appropriate place.

3.1 The 2-oscilator system

The first situation we consider is the case of only 2 oscilators. It is obvious that if the oscilators are not connected through an edge, no sycronisation occurs. Thus the only situation of interest is the case where they are connected through an edge, i.e.

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We have simulated this situation with random initial values for the oscilators. As can be seen from the typical result in Figure 6 the two oscilators lock to the same frequency but have a phase shift of π, i.e. they are in anti-phase. We observed that this behaviour occurs independent of the initial values. This behaviour can be explained since both oscilators are influencing each other with equal negative coefficients they oscilate as far away from each other as possible.

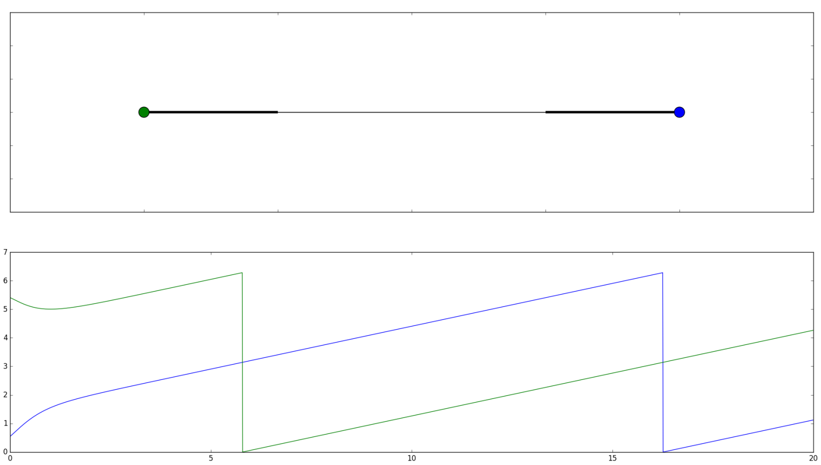

Figure 6: Simulation of a simplified 2 oscilator Kuramoto system with $\omega = 0.3$ and random initial values. A is given by the negative of the adjacency matrix of the associated graph topology which can be seen in the upper part of the figure. The behaviour of the oscilators over time can be seen in the lower part.

3.2 Oscilators on a line

Next we want to expand this behaviour to the situation of several oscilators on a line. In this sense, that refers to the situation where N oscilators are ordered linearly and each oscilator is only connected to the next and previous in the line (except for the ones at either end of the line). The associated matrices A to these situations are 0 except as two bands above and below the diagonal.

In the case of 3 oscilators the adjacency matrix is given by

$$A_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

and in the case of 10 oscilators it is given by

$$A_{10} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The typical behaviour of such systems can be seen in Figures 7 and 8 respectively. The behaviour naturally extends the behaviour of the 2 oscilator systems above: Neighbouring oscilators are in exact anti-phases. Experiments howed this behaviour to be independent of initial values and the number of oscilators, although the higher the number of oscilators, the longer the system needs to reach this state.

3.3 Oscilators on a Circle System

The final patterns we observed were patterns in the case of N oscilators on a cycle. This graph looks very similar to the oscilators-on-a-line situation except that the first and last nodes are connected via an edge as well. We have again simulated the system and in this situation we came to interesting

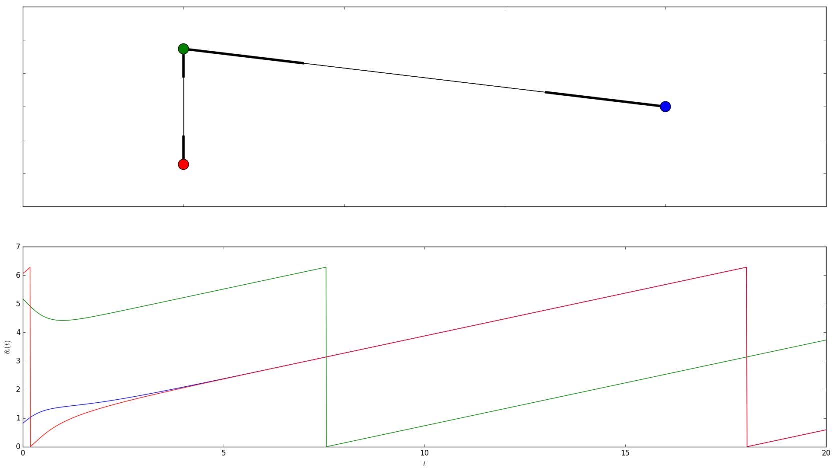

Figure 7: Simulation of a 3 oscilators-on-a-line Kuramoto oscilator system with $\omega = 0.3$ and random initial values. Again A is given by the negative of the adjacency matrix of the associated graph topology.

results. In the case of 3 oscilators on a cycle (which can be seen in Figure 9) the behaviour is obvious: The oscilators syncronise with phase-shits of exactly $\frac{2\pi}{3}$.

The behaviour becomes more interesting however if we choose a different number of oscilators, for example 6. In this case we observed 3 different situations with 3 different kinds of phase shifts: Either phase shifts of π (nodes with distance 2 were exactly in phase, see Figure 10), phase shifts of $\frac{2\pi}{3}$ (nodes with distance 3 were exactly in phase, see Figure 11) or phase shifts of $\frac{2\pi}{6}$ (see Figure 12). The last of these pattern could only be produced artificially by choosing the desired situation as initial values.

From these patterns we can deduce that the possible patterns in the general case of N oscilators-on-a-circle depends on the divisors of N. Since each oscilator wants to be as far away from its neighbours as possible, they space out equally with respect to their phase shifts. However since each oscilator has to be in phase with itself, the sum of all phase shifts around the cycle must be a multiple of 2Π. Thus for each divisor of N we get a different pattern, which is exactly what we observed.

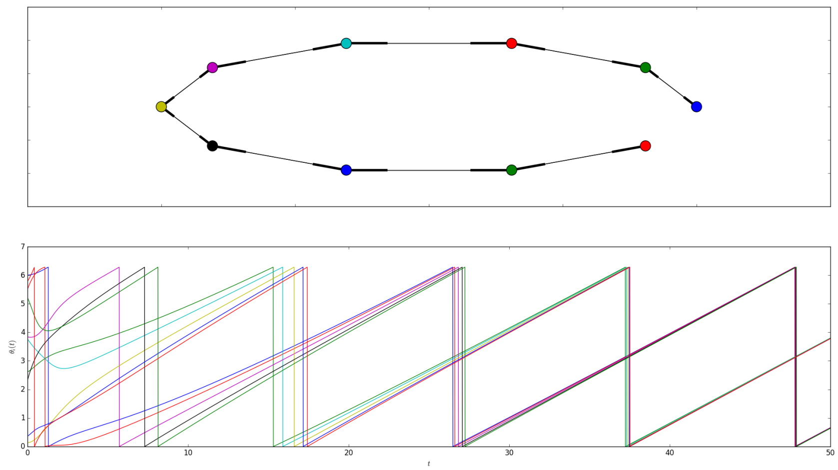

Figure 8: Simulation of a 10 oscilators-on-a-line Kuramoto oscilator system with $\omega = 0.3$ and random initial values. Again A is given by the negative of the adjacency matrix of the associated graph topology.

4 Reproducing desired patterns

4.1 Defining pattern reproduction

After making the observations in the previous section the next question that naturally occurs is if given a certain pattern, is there an associated graph topology (or associated matrix A) that is capable of reproducing it? In other words given a pattern, can we "train" a topology to create it?

In order to answer this question we first need to define "pattern" properly. We consider a pattern as the set of phase shifts between the different Kuramoto oscilators. In order to be able to observe a pattern in the first case it is neccessary for the oscilators to either have the same intrinsic frequencies ω_i or lock into the same frequency.

The reader should furthermore note that since we consider phase shifts only the order of oscilators themselves, they are invariant modulo 2π and the order of oscilators does not play a direct effect.

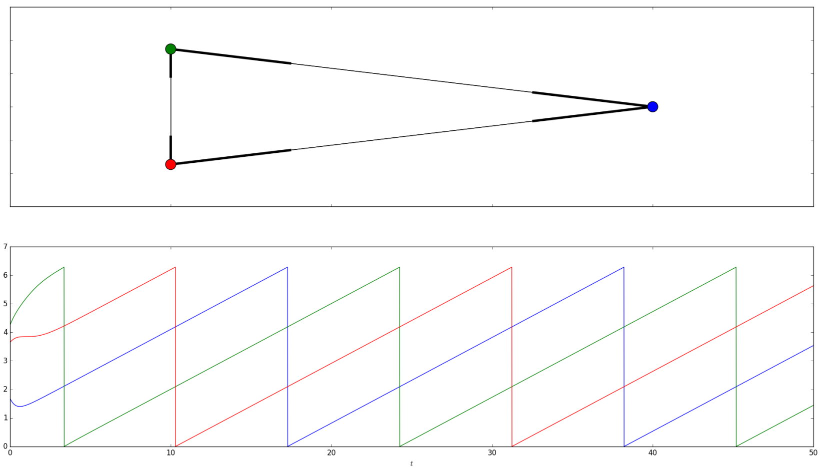

Figure 9: Simulation of a 3 oscillators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and random initial values. Again A is given by the negative of the adjacency matrix of the associated graph topology.

4.2 Simple patterns

We shall first consider the simple case of reproducing a simple pattern between 2 oscillators. If we take into account what we learned in the previous section, it is easily possible to re-create phase-shifts of $\frac{2\pi}{p}$ when p is prime.

If we set up p oscillators on a cylic graph, we can get different phase shifts between neighbouring oscillators. For each divisor $d \neq 1$ of p we can achieve phase shifts of $\frac{2\pi}{d}$ between neighbouring nodes. However since p is prime, there is only a single such divisor, namely p itself, we will always achieve an offset of $\frac{2\pi}{p}$.

There is a caveat to this situation: We now have p different oscillators in our topology instead of just two. This is not a big problem however, we can just pick 2 oscillators at random and ignore the others.

4.3 Creating More Complex Patterns

We now have the ability to create simple patterns by setting up oscillators in a cirle topology. Using the same topology we can also create offsets of the shape $\frac{a \cdot 2\pi}{p}$ where p prime and $a \in \mathbb{N}$. Instead of taking two neighbouring

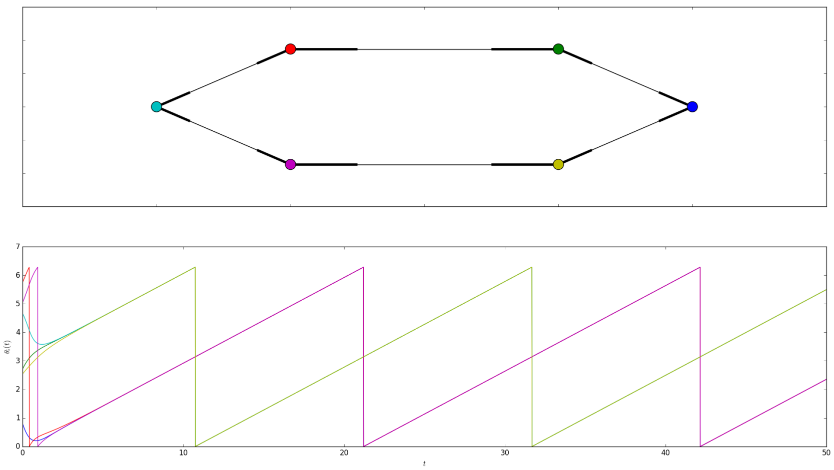

Figure 10: Simulation of a 6 oscilators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and random initial values producing a pattern of neighbouring oscilators being exactly anti-phase

nodes, we can just take two nodes of distance A on the same topology.

This introduces yet another caveat to the situation: We know the phase shift difference between two values but we can not predict which one is of higher value and which one is of lower value. Since we are working modulo 2π this is not even well defined.

But how can we expand this behaviour to reproduce any phase shifts $\frac{p \cdot 2\pi}{q}$ with $p, q \in \mathbb{N}, q \neq 0$? Turns out it is possible to "add" phase shifts by joining two cicular toplogies. Both circles will behave as they would individually and additionally the node at which they are joined will always have the same value.

If we wanted to for example reproduce the phase $\frac{8 \cdot 2\pi}{15}$ we can use one circle of size 3 and one circle of size 5. Their phase shifts of $\frac{2\pi}{5}$ and $\frac{2\pi}{5}$ add up to $\frac{8 \cdot 2\pi}{15}$, exactly what we want. The topology can bee seen in Figure 13.

There is however a slight problem still: We have no guarantee that the total phase shift is indeed the sum of phase shifts in the individual circles. Since the phase shifts could propagnate in different directions the total phase shift could actually be the difference. We can work around this however by choosing the correct initial values for the oscilators – both states are equally

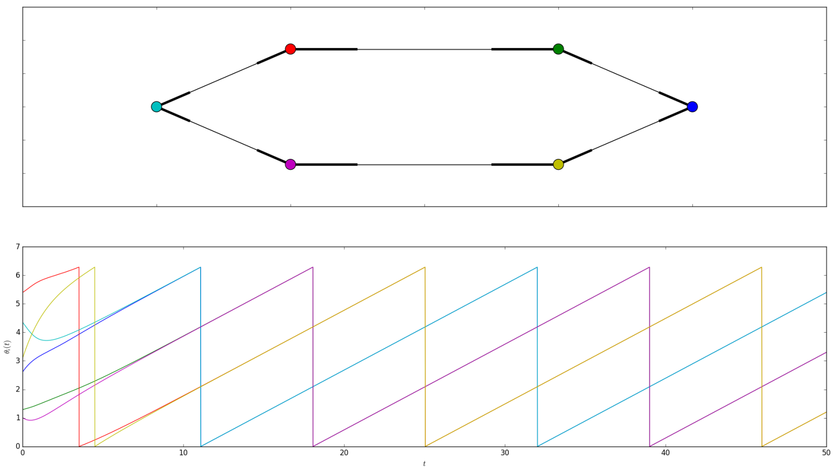

Figure 11: Simulation of a 6 oscilators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and random initial values priducing a pattern of every 3 oscilators being in sync.

stable.

Finally we can note that it is possible to further expand these pattern. Instead of just two circles, we can take more circles to produce offsets of the form

$$\sum_i \frac{q_i \cdot 2\pi}{p_i}$$

with p_i prime and $q_i \in \mathbb{N}$. In fact if we take into account that we are operating modulo 2Π we can even take $q_i \in \mathbb{Z}$. Now given any phase shift $\frac{p \cdot 2\pi}{q}$ we can obviously use the prime factors as denominators p_i and write it in this form. We can then use the procedure described above to create a topology that can produce the given phase shift.

If we have more than two oscilators, we can also continue this procedure, we simply join more circles to the node we want to define an offset relative to.

5 Conclusion

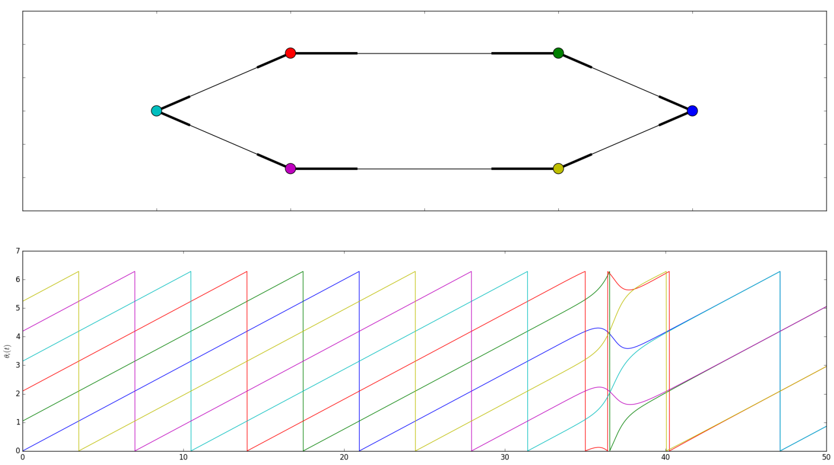

Figure 12: Simulation of a 6 oscilators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and specific initial values producing an unstable pattern of oscilators being evenly spaced out in phase shifts.

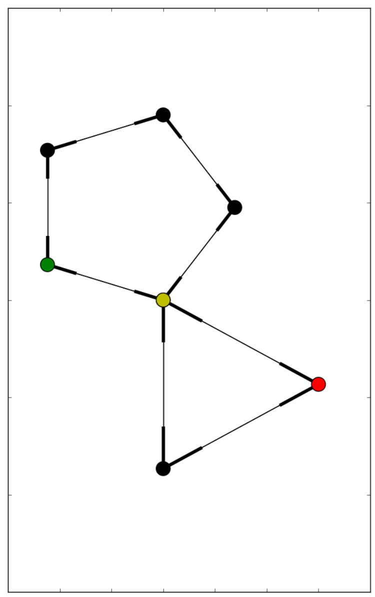

Figure 13: Graph topology used to reproduce a more complex pattern: Two cicular topologies joined at the yellow node. The green and yellow nodes will have a phase shift of $\frac{2\pi}{5}$, the yellow and red nodes will have a phase shift of $\frac{2\pi}{3}$. This gives a total phase shift of $\frac{2\pi}{5} + \frac{2\pi}{5} = \frac{8 \cdot 2\pi}{15}$.