Kuramoto Oscillator

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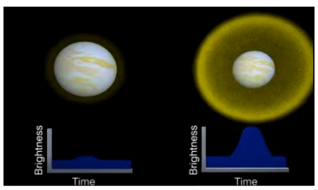
Contents

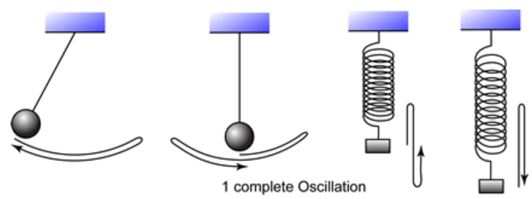
1 Introduction

1.1 Oscillators

Oscillators can be described as a repetitive motion of some measure about a central value which is often the point of equilibrium. This term is most often used in mechanical systems but it must also be noted that oscillations occur in dynamic systems too such as economic graphs, geothermal temperatures across areas and periodic "firing" of fireflies in nature.



(a) Cepheid Stars that are known to flash occasionally



(b) Oscillations

1.2 Coupling

Coupled Oscillation is a slightly more complex form of ordinary oscillators. In these models, the oscillators are connected in such a way that energy is transferred between then. This motion can very well be complex but does not have to be periodic. However, in the bigger scheme of things, every oscillator can be viewed as having a very well defined frequency of its own. Perhaps the simplest example of coupling could be a gear that transmits torque between two shafts that are not collinear. A bit more complex example can be of two pendulums joined together by an energy medium, ie a string.

As we can see in Figure 2, a pendulum only attains its maximum amplitude when the other has its lowest one. This period is achieved after sufficient time has been given to the system to attain synchronization.

1.3 Synchronisation

It should be noted that synchronization can only occur in two ways. The first being if the oscillators have some way of communicating with each other

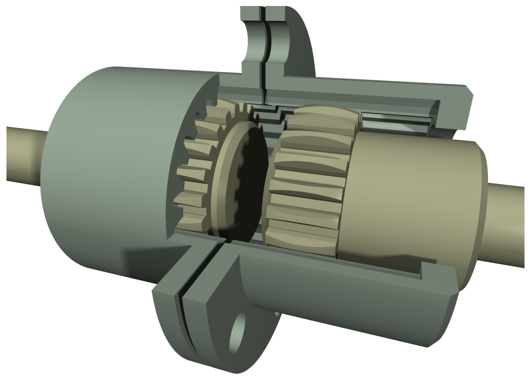


Figure 2

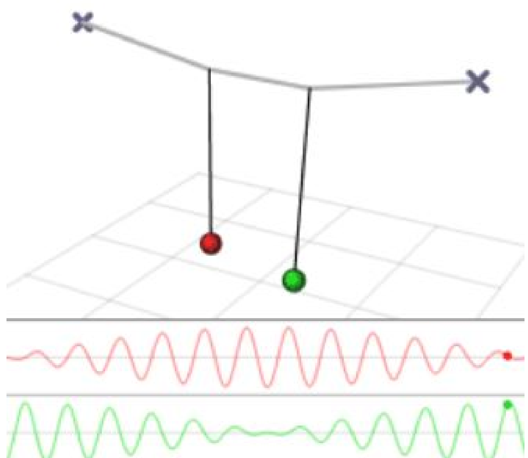


Figure 3

or the second more rarer case where they start at the same exact time. For now we will focus on the first case. Communication can be achieved either by using a medium such as a string between pendulums or even just the intrinsic tendency of natural beings to produce a unison of movement(synchronization) such as fireflies flashing or clapping in a room.

1.4 Adjacency Matrix and the Network Topology

Now we move on to define the mathematical tools being used in this project. The Adjacency matrix is a square matrix that is using to represent a finite graph. The elements of the matrix indicate whether pairs of verticies are adjacent or not, ie if they are adjacent they get assigned a value of -1 else 0. This information can be directly retrieved from the topology network graph as follows-:

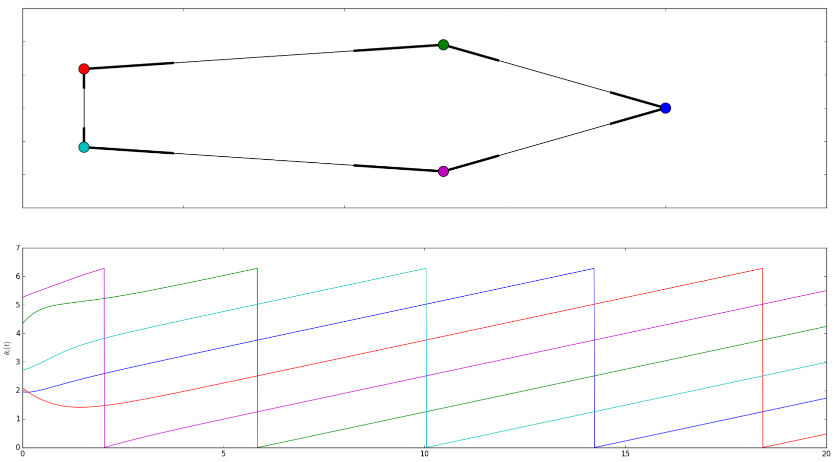


Figure 4

As we can see, on the top is the topological network containing the vertices of the system and below lies the coupling graphs of the system over time. Over here, we can see that the couples are trying to stay as far away as possible so after the initial unrest there is always a phase difference of 2π.

1.5 Goal

Our goal for this study is to investigate coupled systems with negative coupling co-efficients. Positive Coupling Coefficients can have expected looking patterns but the results get more interesting for negative ones .

In Section 2 we will introduce the Kuramoto oscillator, its equations, our implementation and do an analytic analysis of stability. In Section 3 we continue with describing the patterns we encountered when using negative coefficients. We then move on to describe a method which can be used to re-create arbitrary patterns in Section 4 before concluding in Section 5.

2 The Kuramoto oscilator

3 Patterns in negative coefficient systems

For our investigation of the negative coefficient situation we consider a simplified version of the Kuramoto oscilator. Specifically we assume that all oscilators have the same basic frequency ω_i, which we call ω. We also assume that there is no external driver to the system, i.e. $b_i = 0$. This simplifies the system of equations to:

$$\dot{\theta}_i = \omega + \sum_{i=0}^{N} A_{i,j} \sin \theta_i - \theta_j$$

Furthermore we assume that all coefficients in the matrix A are either -1 or 0. We can thus see A as the negative of an adjacency matrix of a graph. Here each oscilator is considered as a node of a graph and an edge can be interpreted as A having a -1 in the appropriate place.

3.1 The 2-oscilator system

The first situation we consider is the case of only 2 oscilators. It is obvious that if the oscilators are not connected through an edge, no sycronisation occurs. Thus the only situation of interest is the case where they are connected through an edge, i.e.

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We have simulated this situation with random initial values for the oscilators. As can be seen from the typical result in Figure 5 the two oscilators lock to the same frequency but have a phase shift of π, i.e. they are in anti-phase. We observed that this behaviour occurs independent of the initial values. This behaviour can be explained since both oscilators are influencing each other with equal negative coefficients they oscilate as far away from each other as possible.

3.2 Oscilators on a line

Next we want to expand this behaviour to the situation of several oscilators on a line. In this sense, that refers to the situation where N oscilators are ordered linearly and each oscilator is only connected to the next and previous in the line (except for the ones at either end of the line). The associated

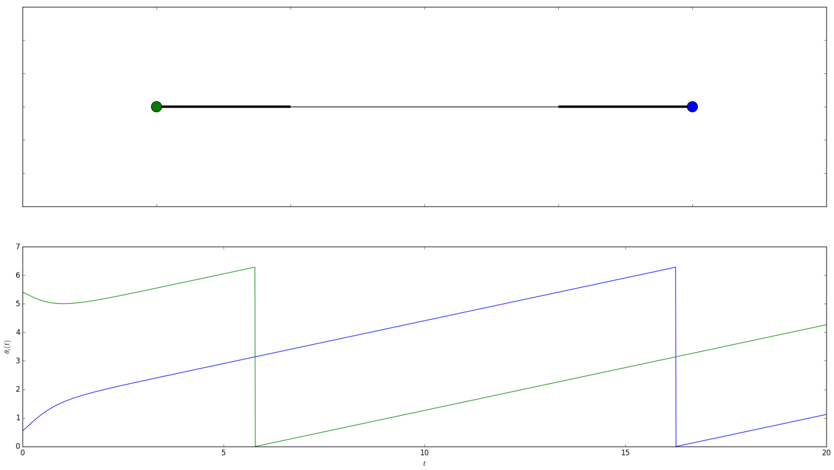


Figure 5: Simulation of a simplified 2 oscilator Kuramoto system with $\omega = 0.3$ and random initial values. A is given by the negative of the adjacency matrix of the associated graph topology which can be seen in the upper part of the figure. The behaviour of the oscilators over time can be seen in the lower part.

matrices A to these situations are 0 except as two bands above and below the diagonal.

In the case of 3 oscilators the adjacency matrix is given by

$$A_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

and in the case of 10 oscilators it is given by

$$A_{10} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The typical behaviour of such systems can be seen in Figures 6 and 7 respectively. The behaviour naturally extends the behaviour of the 2 oscilator systems above: Neighbouring oscilators are in exact anti-phases. Experiments howed this behaviour to be independent of initial values and the number of oscilators, although the higher the number of oscilators, the longer the system needs to reach this state.

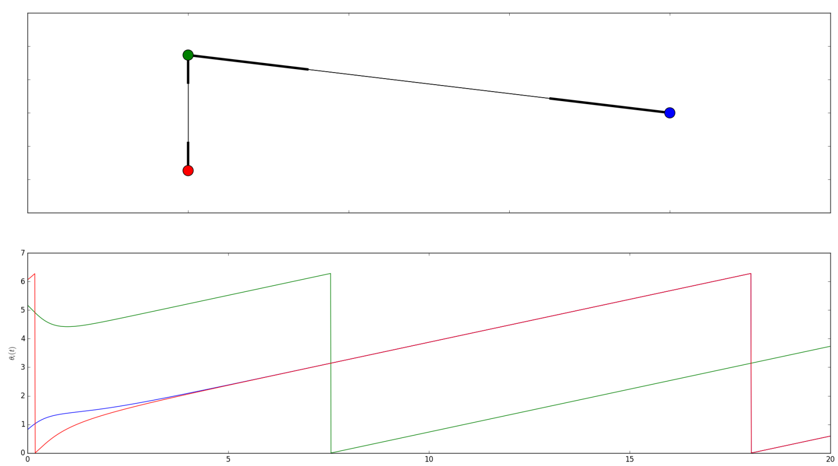


Figure 6: Simulation of a 3 oscilators-on-a-line Kuramoto oscilator system with $\omega = 0.3$ and random initial values. Again A is given by the negative of the adjacency matrix of the associated graph topology.

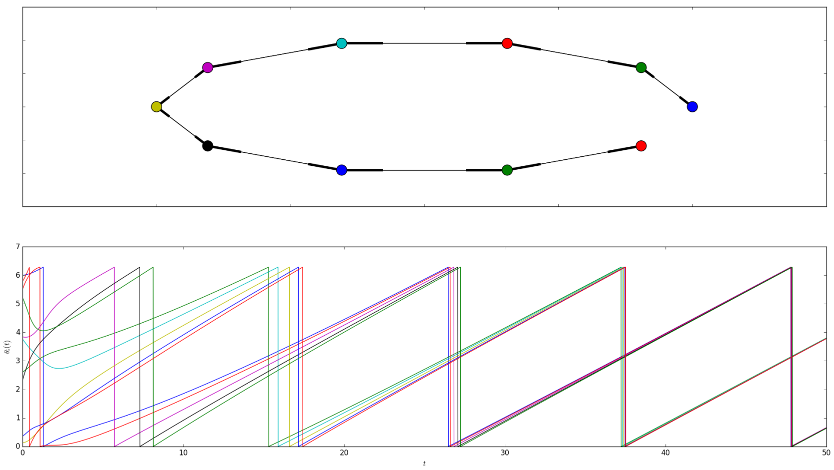


Figure 7: Simulation of a 10 oscilators-on-a-line Kuramoto oscilator system with $\omega = 0.3$ and random initial values. Again A is given by the negative of the adjacency matrix of the associated graph topology.

3.3 Oscilators on a Circle System

The final patterns we observed were patterns in the case of N oscilators on a cycle. This graph looks very similar to the oscilators-on-a-line situation except that the first and last nodes are connected via an edge as well. We have again simulated the system and in this situation we came to interesting results. In the case of 3 oscilators on a cycle (which can be seen in Figure 8) the behaviour is obvious: The oscilators syncronise with phase-shits of exactly $\frac{2\pi}{3}$.

The behaviour becomes more interesting however if we choose a different number of oscilators, for example 6. In this case we observed 3 different situations with 3 different kinds of phase shifts: Either phase shifts of π (nodes with distance 2 were exactly in phase, see Figure 9), phase shifts of $\frac{2\pi}{3}$ (nodes with distance 3 were exactly in phase, see Figure 10) or phase shifts of $\frac{2\pi}{6}$ (see Figure 11). The last of these pattern could only be produced artificially by choosing the desired situation as initial values.

From these patterns we can deduce that the possible patterns in the general case of N oscilators-on-a-circle depends on the divisors of N. Since each oscilator wants to be as far away from its neighbours as possible, they

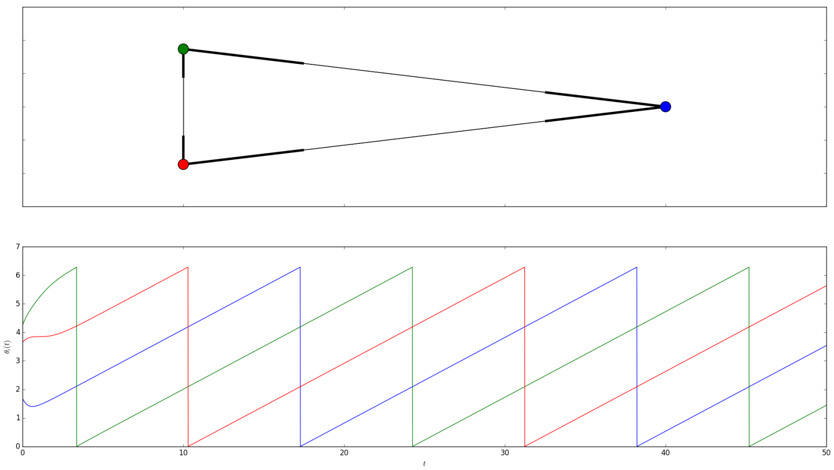


Figure 8: Simulation of a 3 oscilators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and random initial values. Again A is given by the negative of the adjacency matrix of the associated graph topology.

space out equally with respect to their phase shifts. However since each oscilator has to be in phase with itself, the sum of all phase shifts around the cycle must be a multiple of 2Π. Thus for each divisor of N we get a different pattern, which is exactly what we observed.

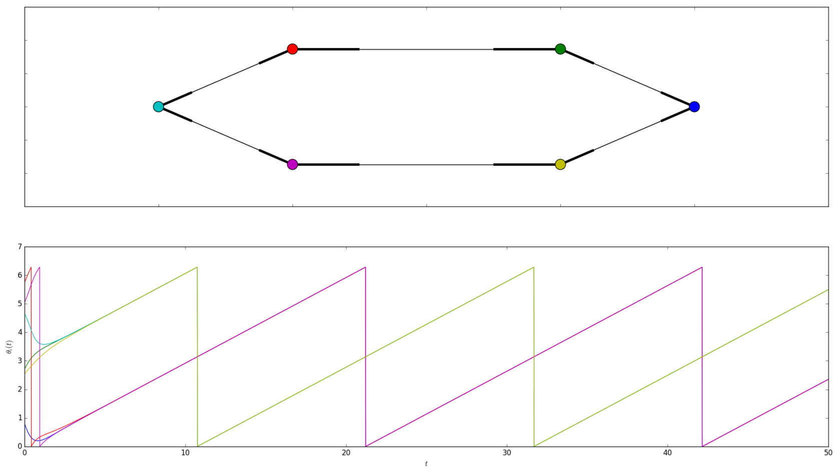


Figure 9: Simulation of a 6 oscilators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and random initial values producing a pattern of neighbouring oscilators being exactly anti-phase

4 Reproducing desired patterns

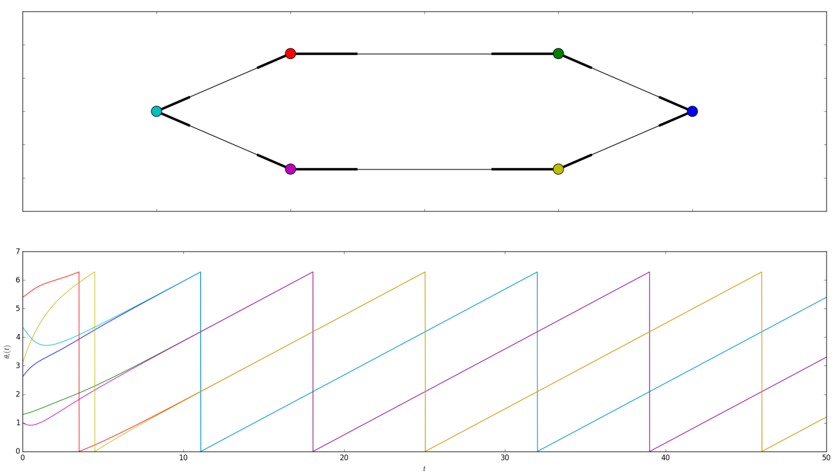


Figure 10: Simulation of a 6 oscilators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and random initial values priducing a pattern of every 3 oscilators being in sync.

5 Conclusion

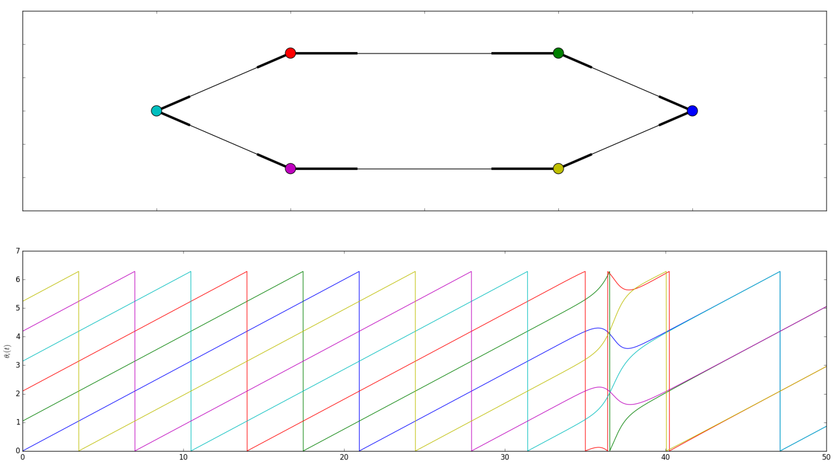


Figure 11: Simulation of a 6 oscilators-on-a-circle Kuramoto oscilator system with $\omega = 0.3$ and specific initial values producing an unstable pattern of oscilators being evenly spaced out in phase shifts.