Chapter 8

Introduction to Digital Filters

It is hard to give a formal definition of the term *filtering.* The electrical engineer often thinks of filtering as changing the frequency domain characteristics of the given (input) signal. Of course, from a purely mathematical point of view, a frequency-domain operation often has a corresponding time-domain interpretation and vice versa. However, electrical engineers are trained, by tradition, to think in the frequency domain. This way of thinking has proved its effectiveness. We have already seen this when discussing spectral analysis and its applications (in Chapters 4 through 6), and we shall see it again in this chapter and the ones to follow.

Examples of filtering operations include:

1. Noise suppression. This operation is necessary whenever the signal of interest has been contaminated by noise. Examples of signals that are typically noisy include:
   (a) Received radio signals.
   (b) Signals received by imaging sensors, such as television cameras or infrared imaging devices.
   (c) Electrical signals measured from the human body (such as brain, heart, or neurological signals).
   (d) Signals recorded on analog media, such as analog magnetic tapes.
2. Enhancement of selected frequency ranges. Examples of signal enhancement include:
   (a) Treble and bass control or graphic equalizers in audio systems. These typically serve to increase the sound level at high and low frequencies, to compensate for the lower sensitivity of the ear at those frequencies, or for special sound effects.
   (b) Enhancement of edges in images. Edge enhancement improves recognition of objects in an image, whether recognition by a human eye or by a computer. It is essentially an amplification of the high frequencies in the Fourier transform of the image: Edges are sharp transitions in the image brightness, and we know from Fourier theory that sharp transitions in a signal appear as high-frequency components in the frequency domain.
3. Bandwidth limiting. In Section 3.3 we learned about bandwidth limiting as a means of aliasing prevention in sampling. Bandwidth limiting is also useful in communication applications. A radio or a television signal transmitted over a

specific channel is required to have a limited bandwidth, to prevent interference with neighboring channels. Thus, amplitude modulation (AM) radio is limited to ± 5 kHz (in the United States) or to ± 4.5 kHz (in Europe and other countries) around the carrier frequency. Frequency modulation (FM) radio is limited to about ± 160 kHz around to the carrier frequency. Bandwidth limiting is accomplished by attenuating frequency components outside the permitted band below a specified power level (measured in dB with respect to the power level of the transmitted signal).

4. Removal or attenuation of specific frequencies. For example:
   (a) Blocking of the DC component of a signal.
   (b) Attenuation of interferences from the power line. Such interferences appear as sinusoidal signals at 50 or 60 Hz, and are common in measurement instruments designed to measure (and amplify) weak signals.
5. Special operations. Examples include:
   (a) Differentiation. Differentiation of a continuous-time signal is described in the time and frequency domains as

$$y(t) = \frac{dx(t)}{dt}, \quad Y^{\mathrm{F}}(\omega) = j\omega X^{\mathrm{F}}(\omega). \tag{8.1}$$

   (b) Integration. Integration of a continuous-time signal is described in the time and frequency domains as

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau, \quad Y^{\mathrm{F}}(\omega) = \frac{X^{\mathrm{F}}(\omega)}{j\omega} + \pi X^{\mathrm{F}}(0)\delta(\omega). \tag{8.2}$$

   (c) Hilbert transform. The continuous-time Hilbert filter has an impulse response

$$h(t) = \frac{1}{\pi t} \tag{8.3}$$

   and a frequency response

$$H^{\mathrm{F}}(\omega) = -j\,\mathrm{sign}(\omega). \tag{8.4}$$

   The Hilbert transform of a signal $x(t)$ is defined as its convolution with $h(t)$, that is, as $y(t) = \{x * h\}(t)$.[1]

These operations can be approximated by digital filters operating on the sampled input signal, using methods described in Chapters 9 and 10.

8.1 Digital and Analog Filtering

Analog filtering is performed on continuous-time signals and yields continuous-time signals. It is implemented using operational amplifiers, resistors, and capacitors. Theoretically, the frequency range of an analog filter is infinite. In practice, it is always limited, depending on the application and the technology. For example, common operational amplifiers operate up to a few hundred kilohertz. Special amplifiers operate up to a few hundred megahertz. Very high frequencies can be handled by special devices, such as microwave and surface acoustic wave (SAW) devices. Analog filters suffer from sensitivity to noise, nonlinearities, dynamic range limitations, inaccuracies due to variations in component values, lack of flexibility, and imperfect repeatability.

Digital filtering is performed on discrete-time signals and yields discrete-time signals. It is usually implemented on a computer, using operations such as addition, multiplication, and data movement. Sometimes, special-purpose hardware is used for carrying out similar operations. The frequency range is always finite, and is limited to one-half of the sampling rate. Digital filters are accurate to any desired degree (determined by the computer word length), highly linear (except for quantization effects, which will be studied in Chapter 11), flexible (especially if implemented in software), and perfectly repeatable; moreover, they do not suffer from internal noise and have a practically unlimited dynamic range if implemented in floating point. There exist, also, digital filters that are completely unlike analog filters, and enable operations that cannot be realized (or are difficult to realize) by analog means. These include pure time delays, time-varying filters, and adaptive filters. On the other hand, digital filtering requires interface to the physical world (A/D and D/A conversion); care must be taken to avoid (or minimize) aliasing, as well, and operating frequency range is sometimes limited by the available computational speed.

In this book we study only linear time-invariant (LTI) filters. Analog LTI filters are usually specified by their s-domain transfer function. The transfer function of an analog filter is rational, causal, and proper, that is,

$$H^{\mathrm{L}}(s) = \frac{b_0 s^q + b_1 s^{q-1} + \cdots + b_q}{s^p + a_1 s^{p-1} + \cdots + a_p}, \tag{8.5}$$

where $q \le p$. Except in rare cases, the filter is required to be stable. Digital LTI filters are usually specified by their z-domain transfer function. The transfer function is usually rational and causal, that is,

$$H^{\mathrm{z}}(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}. \tag{8.6}$$

Here we do not require that $q \le p$. As in the case of analog filters, usually the filter is required to be stable.

Digital filters for which $p \ge 1$ (and $a_p \ne 0$), are called *infinite impulse response* (IIR) filters, because their corresponding impulse response sequence $h[n]$ has an infinite duration—it never dies out completely. To see this, recall from (7.56) that $h[n]$ can be expressed as

$$h[n] = c_0\delta[n] + \cdots + c_{q-p}\delta[n-q+p] + \sum_{k=1}^{p} A_k \alpha_k^n, \quad n \ge 0, \tag{8.7}$$

where α_k are the poles of the transfer function $H^{\mathrm{z}}(z)$. Therefore, for $n > q - p$, $h[n]$ is a linear combination of geometric series α_k^n. Note that the impulse response $h(t)$ of a rational analog filter also has an infinite duration. The transfer function $H^{\mathrm{L}}(s)$ admits a partial fraction decomposition

$$H^{\mathrm{L}}(s) = c_0 + \sum_{k=1}^{p} \frac{A_k}{s - \lambda_k}, \tag{8.8}$$

where λ_k are the poles of $H^{\mathrm{L}}(s)$ (assuming that they are distinct); therefore,

$$h(t) = c_0\delta(t) + \sum_{k=1}^{p} A_k e^{\lambda_k t}, \quad t \ge 0. \tag{8.9}$$

Digital filters for which $p = 0$ are called *finite impulse response* (FIR) filters. Their transfer function is

$$H^{\mathrm{z}}(z) = b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}. \tag{8.10}$$

The impulse response $h[n]$ of an FIR filter is

$$h[n] = \begin{cases} b_n, & 0 \le n \le q, \\ 0, & \text{otherwise.} \end{cases} \tag{8.11}$$

As we see, the impulse response is nonzero only for a finite number of samples, hence the name *finite impulse response*. FIR filters are characteristic of the discrete time domain. Analog FIR filters are possible, but they are difficult to implement and are rarely used.[2]

8.2 Filter Specifications

Before a digital filter can be designed and implemented, we need to specify its performance requirements. A typical filter should pass certain frequencies and attenuate other frequencies; therefore, we must define exactly the frequencies in question, as well as the required gains and attenuations. There are four basic filter types, as illustrated in Figure 8.1:

1. *Low-pass filters* are designed to pass low frequencies, from zero to a certain cutoff frequency θ_0, and to block high frequencies. We encountered analog low-pass filters when we discussed antialiasing filters and reconstruction filters in Sections 3.3 and 3.4.
2. *High-pass filters* are designed to pass high frequencies, from a certain cutoff frequency θ_0 to π, and to block low frequencies.[3]
3. *Band-pass filters* are designed to pass a certain frequency range $[\theta_1, \theta_2]$, which does not include zero, and to block other frequencies. We encountered analog band-pass filters when we discussed reconstruction of band-pass signals in Section 3.6.
4. *Band-stop filters* are designed to block a certain frequency range $[\theta_1, \theta_2]$, which does not include zero, and to pass other frequencies.

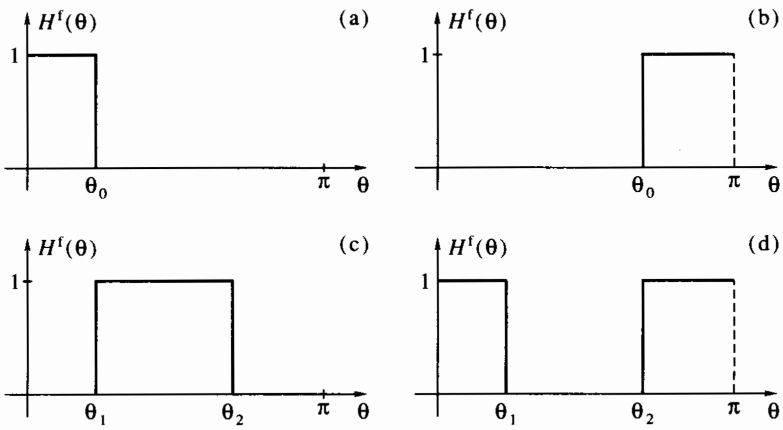

Figure 8.1 Ideal frequency responses of the four basic filter types: (a) low-pass; (b) high-pass; (c) band-pass; (d) band-stop.

The frequency responses shown in Figure 8.1 are ideal. Frequency responses of practical filters are not shaped in straight lines. The response of practical filters varies continuously as a function of the frequency: It is neither exactly 1 in the pass bands, nor exactly 0 in the stop bands. In this section we define the terms and parameters used for digital filter specifications. We assume that the filters are real, so their magnitude response is symmetric and their phase response is antisymmetric; recall properties

(2.107d, e) of the Fourier transform of a real sequence. Therefore, we shall usually show only the positive frequency range, from 0 to π.

8.2.1 Low-Pass Filter Specifications

A low-pass (LP) filter is designed to pass low frequencies, from zero to a certain cutoff frequency θ_p, with approximately unity gain. The frequency range $[0, \theta_p]$ is called the *pass band* of the filter. High frequencies, from a certain frequency θ_s up to π, are to be attenuated. The frequency range $[\theta_s, \pi]$ is called the *stop band* of the filter. The frequency range (θ_p, θ_s), between the pass band and the stop band, is called the *transition band*. The exact behavior of the frequency response in the transition band is usually of little importance.

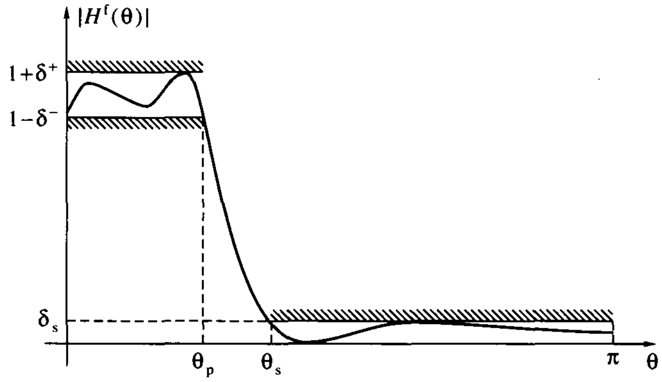

Figure 8.2 Specification of a low-pass filter.

Figure 8.2 provides a graphical description of the specifications of a low-pass filter. The hatched areas in the pass band and in the stop band indicate forbidden magnitude values in these bands. In the transition band there are no forbidden values, but it is usually required that the magnitude decrease monotonically in this band. The mathematical specification of a low-pass filter is

$$1 - \delta^- \leq |H^f(\theta)| \leq 1 + \delta^+, \quad 0 \leq \theta \leq \theta_p, \tag{8.12a}$$

$$0 \leq |H^f(\theta)| \leq \delta_s, \qquad \theta_s \leq \theta \leq \pi. \tag{8.12b}$$

The parameters δ^+ and δ^- are the positive and negative tolerances of the magnitude response in the pass band. The desired (or nominal) magnitude response in the pass band is 1. The parameter δ_s is the tolerance of the magnitude response in the stop band. The desired (or nominal) magnitude response in the stop band is 0.

The quantity $\max\{\delta^+, \delta^-\}$ is called the *pass-band ripple*. Another useful quantity is

$$A_p = \max\{20\log_{10}(1 + \delta^+), -20\log_{10}(1 - \delta^-)\}. \tag{8.13}$$

This parameter is the pass-band ripple in dB. A convenient approximation to A_p is obtained by noting that

$$\log_e(1 \pm \delta) \approx \pm\delta,$$

hence

$$20\log_{10}(1 \pm \delta) = 20\log_{10} e \cdot \log_e(1 \pm \delta) \approx \pm 8.6859\delta. \tag{8.14}$$

Substitution of (8.14) in (8.13) for δ^+ and δ^- thus gives

$$A_p \approx 8.6859 \max\{\delta^+, \delta^-\}. \tag{8.15}$$

It is common, in digital filter design, to use different pass-band tolerances for IIR and FIR filters. For IIR filters, it is common to use $\delta^+ = 0$, and denote δ^- as δ_p. For FIR filters, it is common to use $\delta^+ = \delta^-$, and denote their common value as δ_p. Thus, the value 1 is the maximum pass-band gain for IIR filters but the midrange pass-band gain for FIR filters.

The quantity δ_s is called the *stop-band attenuation*. Another useful quantity is

$$A_s = -20 \log_{10} \delta_s. \tag{8.16}$$

This parameter is the stop-band attenuation in dB.

The frequency response specification we have described concerns the magnitude only and ignores the phase. We shall discuss the phase response in Section 8.4; for now we continue to ignore it.

Example 8.1 A typical application of low-pass filters is noise attenuation. Consider, for example, a signal sampled at $f_{sam} = 20$ kHz. Suppose that the signal is band limited to 1 kHz, but discrete-time white noise is also present at the sampled signal, and the SNR is 10 dB. We wish to attenuate the noise in order to improve the SNR as much as is reasonably possible, without distorting the magnitude of the signal by more than 0.1 dB at any frequency.

Since the noise is white, its energy is distributed uniformly at all frequencies up to 10 kHz (remember that we are already in discrete time). The best we can theoretically do is pass the signal through an ideal low-pass filter having cutoff frequency 1 kHz. This will leave 10 percent of the noise energy (from zero up to 1 kHz) and remove the remainder. Since the noise energy is decreased by 10 dB, the SNR at the output of an ideal filter will be 20 dB.

Suppose now that, due to implementation limitations, the digital filter cannot have a transition bandwidth less than 200 Hz. In this case, the noise energy in the range 1kHz to 1.2 kHz will also be partly passed by the filter. The response of the filter in the transition band decreases monotonically, so we assume that the average noise gain in this band is about 0.5. Therefore, the filter now leaves about 11 percent, or -9.6 dB, of the noise energy. Thus, the SNR at the output of the filter cannot be better than 19.6 dB.

Finally, consider the contribution to the output noise energy made by the stop-band characteristics of the filter. The stop band is from 1.2 kHz to 10 kHz. Since the stop-band attenuation of a practical filter cannot be infinite, we must accept an output SNR less than 19.6 dB, say 19.5 dB. Thus, the total noise energy in the filter's output must not be greater than 11.22 percent of its input energy. Out of this, 11 percent is already lost to the pass band and the transition band, so the stop band must leave no more than 0.22 percent of the noise energy. The total noise gain in the stop band is not higher than $(8.8/10)\delta_s^2$, and this should be equal to 0.0022. Therefore, we get that $\delta_s = 0.05$, or $A_s = 26$ dB. In summary, the digital filter specifications should be

$$\theta_p = 0.1\pi, \quad \theta_s = 0.12\pi, \quad A_p = 0.1\,\text{dB}, \quad A_s = 26\,\text{dB}. \qquad \square$$

8.2.2 High-Pass Filter Specifications

Next we consider high-pass (HP) filters. A high-pass filter is designed to pass high frequencies, from a certain cutoff frequency θ_p to π, with approximately unity gain.

The frequency range $[\theta_p, \pi]$ is the pass band of the filter. Low frequencies, from 0 to a certain frequency θ_s, are to be attenuated. The frequency range $[0, \theta_s]$ is the stop band of the filter. The frequency range (θ_s, θ_p) is the transition band.

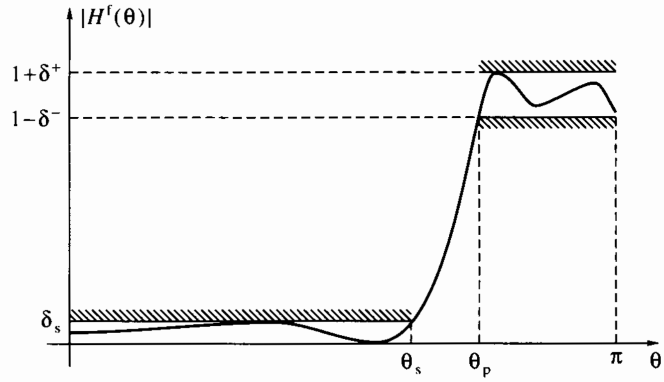

Figure 8.3 Specification of a high-pass filter.

Figure 8.3 provides a graphical description of the specifications of a high-pass filter. As before, the hatched areas in the pass band and in the stop band indicate forbidden magnitude values in these bands. The mathematical specification of a high-pass filter is

$$0 \le |H^f(\theta)| \le \delta_s, \qquad 0 \le \theta \le \theta_s, \tag{8.17a}$$

$$1 - \delta^- \le |H^f(\theta)| \le 1 + \delta^+, \quad \theta_p \le \theta \le \pi. \tag{8.17b}$$

The meaning of the parameters δ^+, δ^-, and δ_s is the same as for low-pass filters. The ripple and attenuation parameters A_p and A_s are defined as in (8.13) and (8.16), respectively. As in the case of low-pass filters, it is common to define $\delta^+ = 0$, $\delta^- = \delta_p$ for IIR filters and $\delta^+ = \delta^- = \delta_p$ for FIR filters.

Example 8.2 Electrical signals measured from the human brain are mostly in the frequency range from 0 to 30 Hz under normal conditions. The level of EEG signals (when measured by electrodes attached to the scalp) is up to about 200 μV. When the brain is stimulated by the senses, signals at other frequencies may appear, depending on the stimulus. Such signals are called *evoked potentials*. In this example we consider *brain stem auditory evoked potentials*: signals generated in the brain stem in response to sound. The level of such signals is relatively low, on the order of 1 μV. Their spectrum depends on the frequency of the sound signal but can contain energy from about 30 Hz to a few kilohertz. To analyze such signals, it is necessary to attenuate the normal brain signals (background activity). This can be accomplished by means of a high-pass filter.

Suppose that we are interested in analyzing brain stem auditory evoked potentials in the frequency range 50 to 1000 Hz. We pass the signal through an antialiasing filter and sample at 2500 Hz. The sampled signal is then passed through a digital high-pass filter. A filter suitable for this purpose should have a stop band from 0 to 30 Hz and a pass band from 50 to 1250 Hz. A reasonable value of A_s for this application is 50 dB, and a reasonable value of A_p is 0.2 dB. In summary, the filter specifications are

$$\theta_s = 0.024\pi, \quad \theta_p = 0.04\pi, \quad A_p = 0.2\,\text{dB}, \quad A_s = 50\,\text{dB}. \qquad \square$$

8.2.3 Band-Pass Filter Specifications

A band-pass (BP) filter is designed to pass signals in a certain frequency range, from $\theta_{p,1}$ to $\theta_{p,2}$. Outside this range, it must attenuate the signal below a specified level. Figure 8.4 provides a graphical description of the specifications of a band-pass filter. The mathematical specification of a band-pass filter is

$$0 \leq |H^f(\theta)| \leq \delta_{s,1}, \qquad 0 \leq \theta \leq \theta_{s,1}, \tag{8.18a}$$

$$1 - \delta^- \leq |H^f(\theta)| \leq 1 + \delta^+, \quad \theta_{p,1} \leq \theta \leq \theta_{p,2}, \tag{8.18b}$$

$$0 \leq |H^f(\theta)| \leq \delta_{s,2}, \qquad \theta_{s,2} \leq \theta \leq \pi. \tag{8.18c}$$

Note that the attenuations in the two stop bands, $\delta_{s,1}$ and $\delta_{s,2}$, are not necessarily equal. As in the case of low-pass and high-pass filters, it is common to define $\delta^+ = 0$, $\delta^- = \delta_p$ in IIR filters, and $\delta^+ = \delta^- = \delta_p$ in FIR filters. The pass-band ripple is $\max\{\delta^+, \delta^-\}$, and the corresponding dB value is

$$A_p \approx 8.6859 \max\{\delta^+, \delta^-\}.$$

There are two stop-band attenuation parameters, and their corresponding dB values are

$$A_{s,1} = -20\log_{10}\delta_{s,1}, \quad A_{s,2} = -20\log_{10}\delta_{s,2}.$$

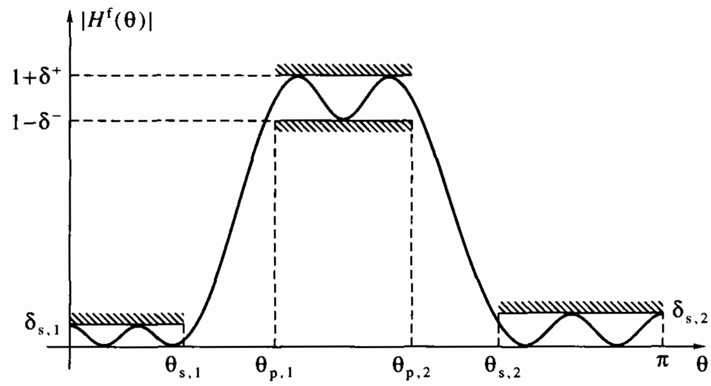

Figure 8.4 Specification of a band-pass filter.

Example 8.3 A certain BPSK communication system (see Example 3.13) is designed to share five channels, each having bandwidth 4 kHz, in a single receiver. This is accomplished by modulating each channel with a different carrier frequency, using double-side-band (DSB) modulation. The carrier frequencies are 10 kHz apart. The received signal, which contains all five channels, is demodulated such that the carrier frequencies after demodulation are 5, 15, 25, 35, 45 kHz. The spectrum of the demodulated signal is shown in Figure 8.5. Between every two adjacent channels there is a 2-kHz-wide band, called a *guard band*, that is free of any signal.

The signal shown in Figure 8.5 is sampled at $f_{sam} = 100$ kHz, using a 12-bit A/D converter. The digital signal is to be split into the five individual channels using five band-pass filters. We wish to prescribe the specifications of the five filters.

We first note that the guard bands can be used as transition bands for the filters, since no signal is expected in these bands. Therefore, the band-edge frequencies of the

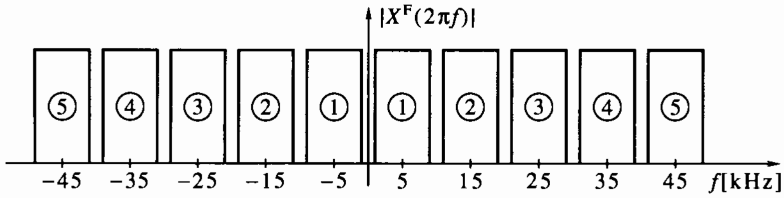

Figure 8.5 The spectrum of the signal in Example 8.3; the numbers in circles correspond to the five channels.

filters are as shown in Table 8.1. The frequencies are given in kilohertz for convenience (multiplication by $2000\pi/f_{\text{sam}}$ will convert them to the θ domain). Note, in particular, that the first filter is low pass and the fifth is high pass.

Next we observe that the dynamic range of the digital signal is 72.25 dB, since $20\log_{10}2^{12} = 72.25$. Moreover, the practical dynamic range is typically 6 dB lower, since a 12-bit A/D usually does not provide more than 11 true bits.[4] It is therefore reasonable to require stop-band attenuation of 66 dB. This guarantees that even the strongest possible signal in one channel will be below the least significant bit (hence undetectable) in the other channels.

The pass-band ripple cannot be directly inferred from the given information. However, BPSK systems are not highly sensitive to amplitude distortions. For the purpose of this example, we require that the pass-band ripple be 0.5 dB for all five filters. Table 8.1 summarizes the specifications of the filters. □

Channel	1	2	3	4	5
$f_{s,1}$ [kHz]	—	9	19	29	39
$f_{p,1}$ [kHz]	—	11	21	31	41
$f_{p,2}$ [kHz]	9	19	29	39	—
$f_{s,2}$ [kHz]	11	21	31	41	—
A_s [dB]	66	66	66	66	66
A_p [dB]	0.5	0.5	0.5	0.5	0.5

Table 8.1 The filter specifications in Example 8.3.

8.2.4 Band-Stop Filter Specifications

A band-stop (BS) filter is designed to attenuate signals in a certain frequency range, from $\theta_{s,1}$ to $\theta_{s,2}$. Outside this range, it must pass the signal within a specified tolerance. Figure 8.6 provides a graphical description of the specifications of a band-stop filter. The mathematical specification of a band-stop filter is

$$1 - \delta_1^- \le |H^f(\theta)| \le 1 + \delta_1^+, \quad 0 \le \theta \le \theta_{p,1}, \tag{8.19a}$$
$$0 \le |H^f(\theta)| \le \delta_s, \quad \theta_{s,1} \le \theta \le \theta_{s,2}, \tag{8.19b}$$
$$1 - \delta_2^- \le |H^f(\theta)| \le 1 + \delta_2^+, \quad \theta_{p,2} \le \theta \le \pi. \tag{8.19c}$$

The tolerance parameters δ_1^-, δ_2^-, δ_1^+, δ_2^+ are permitted to be different in general. There are two pass-band ripple parameters, $\max\{\delta_1^+, \delta_1^-\}$ and $\max\{\delta_2^+, \delta_2^-\}$, and their corresponding dB values are

$$A_{p,1} \approx 8.6859 \max\{\delta_1^+, \delta_1^-\}, \quad A_{p,2} \approx 8.6859 \max\{\delta_2^+, \delta_2^-\}.$$

The stop-band attenuation is δ_s and its corresponding dB value is $A_s = -20\log_{10}\delta_s$.

For IIR filters it is common to define $\delta_1^+ = \delta_2^+ = 0$, $\delta_1^- = \delta_{p,1}$, $\delta_2^- = \delta_{p,2}$; for FIR filters it is common to define $\delta_1^+ = \delta_1^- = \delta_{p,1}$, $\delta_2^+ = \delta_2^- = \delta_{p,2}$.

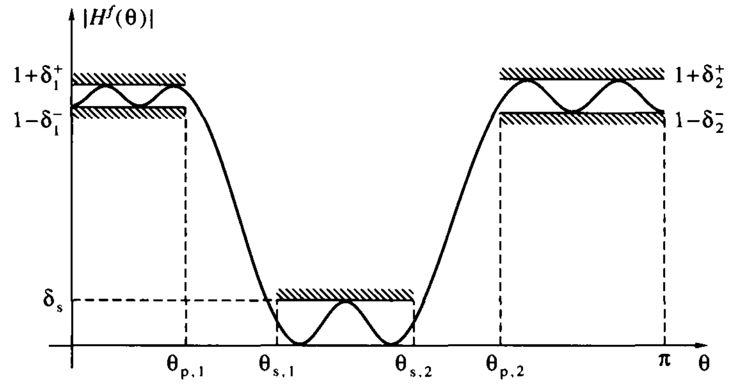

Figure 8.6 Specification of a band-stop filter.

Example 8.4 A certain physiological signal is measured by electrodes, amplified, sampled at 200 Hz, and processed digitally. Signals measured from the human body are low level, so considerable amplification is necessary to bring them to the operating level of the A/D. In such circumstances, it is difficult to completely isolate the analog circuitry from power-line interferences. Suppose that examination of the analog voltage at the A/D input reveals a power-line signal component (60 Hz in the United States) at amplitude approximately equal to that of the signal of interest. The proposed solution to this problem is to pass the digital signal through a band-stop filter, designed to attenuate the power-line interference with minimal distortion of the signal of interest. Since the power-line frequency has short-term fluctuations from its nominal value (although its long-term average value is highly stable), it is decided to use a stop band from 59 to 61 Hz. The permitted interference amplitude after filtering is required to be 1 percent or less of the amplitude of the signal of interest. This implies stop-band attenuation of 40 dB. Suppose further that the spectrum of the signal is approximately flat in the range ± 100 Hz, and we do not want the filter to cut more than 5 percent of the signal energy. Thus, the stop band and the transition band together should not span more than 5 Hz. We therefore define the two pass-band edge frequencies as 57.5 and 62.5 Hz. We define the pass-band ripple as 0.1 dB, so that the amplitude distortion of the signal in the pass band will be less than 1 percent. In summary, the filter specifications in this example are

$$\theta_{p,1} = 0.575\pi, \; \theta_{s,1} = 0.59\pi, \; \theta_{s,2} = 0.61\pi, \; \theta_{p,2} = 0.625\pi,$$
$$A_{p,1} = A_{p,2} = 0.1\,\text{dB}, \; A_s = 40\,\text{dB}. \qquad \square$$

8.2.5 Multiband Filters

The four filter types we have described are the most commonly used, but they are far from encompassing the full generality of LTI filtering. Multiband filters generalize these four types in that they allow for different gains or attenuations in different frequency bands. A *piecewise-constant multiband filter* is characterized by the following parameters:

1. A division of the frequency range $[0, \pi]$ to a finite union of intervals. Some of these intervals are pass bands, some are stop bands, and the remaining are transition bands.
2. A desired gain and a permitted tolerance for each pass band.
3. An attenuation threshold for each stop band.

Suppose that we have K_p pass bands, and let $\{[\theta_{\mathrm{p,l},k}, \theta_{\mathrm{p,h},k}],\ 1 \le k \le K_\mathrm{p}\}$ denote the corresponding frequency intervals. Similarly, suppose that we have K_s stop bands, and let $\{[\theta_{\mathrm{s,l},k}, \theta_{\mathrm{s,h},k}],\ 1 \le k \le K_\mathrm{s}\}$ denote the corresponding frequency intervals (where l and h stand for "low" and "high," respectively). Let $\{C_k,\ 1 \le k \le K_\mathrm{p}\}$ denote the desired gains in the pass band, and $\{\delta_k^-, \delta_k^+,\ 1 \le k \le K_\mathrm{p}\}$ the pass-band tolerances. Finally, let $\{\delta_{\mathrm{s},k},\ 1 \le k \le K_\mathrm{s}\}$ denote the stop-band attenuations. Then the multiband filter specification is

$$C_k - \delta_k^- \le |H^\mathrm{f}(\theta)| \le C_k + \delta_k^+, \quad \theta_{\mathrm{p,l},k} \le \theta \le \theta_{\mathrm{p,h},k}, \quad 1 \le k \le K_\mathrm{p}, \tag{8.20a}$$

$$0 \le |H^\mathrm{f}(\theta)| \le \delta_{\mathrm{s},k}, \quad \theta_{\mathrm{s,l},k} \le \theta \le \theta_{\mathrm{s,h},k}, \quad 1 \le k \le K_\mathrm{s}. \tag{8.20b}$$

Figure 8.7 illustrates the specifications of a six-band filter having three pass bands and three stop bands.

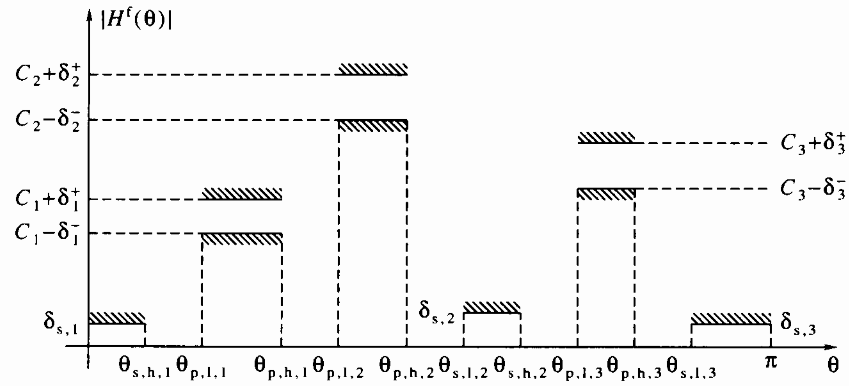

Figure 8.7 Specification of a multiband filter.

Multiband filters are not necessarily suitable for any filtering problem that may come up in a particular application. Sometimes the required gain behavior of the filter is too complex to be faithfully described by a piecewise-constant approximation. Then the engineer must rely on experience and understanding of the problem at hand. The field of closed-loop feedback control is a typical example. Filters used for closed-loop control are called *controllers*, or *compensators*. They are carefully tailored to the controlled system, their gain is continuously varying (rather than piecewise constant), and their phase behavior is of extreme importance. Consequently, compensator design for control systems is a discipline by itself, and the techniques studied in this book are of little use for it.

8.3 The Magnitude Response of Digital Filters

A digital filter designed to meet given specifications must have its magnitude response lying in the range $[1 - \delta^-, 1 + \delta^+]$ in the pass band, and in the range $[0, \delta_s]$ in the stop band. The exact magnitude response of the filter in each of these ranges is of secondary importance. For practical filters, the magnitude response in each band typically has one of two forms:

1. A monotone response, either increasing or decreasing.
2. An oscillating, or rippling, response.

A rippling response in the pass band is typically such that it reaches the tolerance limits $1 - \delta^-, 1 + \delta^+$ several times in the band. Similarly, a rippling stop-band response typically reaches the limits $0, \delta_s$ several times in the band. Such a response is called *equiripple.* In Chapter 10 we shall encounter filters that are monotone in both bands, filters that are equiripple in both bands, and filters that are monotone in one band and equiripple in the other.

If you are experienced with analog filters, you are probably familiar with asymptotic Bode diagrams of the magnitude responses of such filters. Digital filters do not have asymptotic Bode diagrams, for the following reasons:

1. In an analog filter, the frequency response of a first-order factor $(s - \lambda_k)$ is $(j\omega - \lambda_k)$, which is a rational function of ω. In a digital filter, on the other hand, the frequency response of a first-order factor $(1 - \alpha_k z^{-1})$ is $(1 - \alpha_k e^{-j\theta})$, which is not a rational function of θ.
2. It is meaningless to seek approximations as θ tends to infinity, since θ is limited to π.

Therefore, the digital filter designer usually relies on exact magnitude response plots, computed by programs such as `frqresp`, described in Section 7.6.

8.4 The Phase Response of Digital Filters

In most filtering applications, the magnitude response of the filter is of primary concern. However, the phase response may also be important in certain applications. It turns out that the phase response of practical filters cannot be made arbitrary, but is subject to certain restrictions. In this section we study the properties of the phase of digital filters. We restrict our discussion to filters that are *real, causal, stable, and rational.* We use the abbreviation RCSR for such filters. Certain results in this section hold for more general classes of filters, but we shall not need these generalizations here.

8.4.1 Phase Discontinuities

Suppose we are given an RCSR filter whose transfer function is $H^z(z)$. The frequency response of the filter can be expressed as

$$H^f(\theta) = H_r^f(\theta) + jH_i^f(\theta) = |H^f(\theta)|e^{j\psi(\theta)}, \tag{8.21}$$

where $H_r^f(\theta)$, $H_i^f(\theta)$, $|H^f(\theta)|$, $\psi(\theta)$ are, respectively, the real part, imaginary part, magnitude, and phase of the frequency response. Then

$$|H^f(\theta)| = \sqrt{[H_r^f(\theta)]^2 + [H_i^f(\theta)]^2}, \tag{8.22}$$

$$\psi(\theta) = \begin{cases} \operatorname{arctan2}\{H_\mathrm{i}^\mathrm{f}(\theta), H_\mathrm{r}^\mathrm{f}(\theta)\}, & H^\mathrm{f}(\theta) \neq 0, \\ \text{undefined}, & H^\mathrm{f}(\theta) = 0, \end{cases} \tag{8.23}$$

where the value of arctan2(y, x) is, by definition, the unique angle $\alpha \in (-\pi, \pi]$ for which[5]

$$\cos\alpha = \frac{x}{(x^2 + y^2)^{1/2}}, \quad \sin\alpha = \frac{y}{(x^2 + y^2)^{1/2}}.$$

Since $H^z(z)$ is RCSR, the frequency response $H^\mathrm{f}(\theta)$ is a continuous function of θ. As we see from (8.23), this implies continuity of $\psi(\theta)$, except in two cases:

1. At points θ_0 where $H_\mathrm{i}^\mathrm{f}(\theta_0) = 0$ and $H_\mathrm{r}^\mathrm{f}(\theta) < 0$ we have $\psi(\theta_0) = \pi$ because of our definition of the range of $\psi(\theta)$. Therefore, there will be a jump of 2π if $H_\mathrm{i}^\mathrm{f}(\theta_0^-)$ or $H_\mathrm{i}^\mathrm{f}(\theta_0^+)$ (or both) are negative, because then $\psi(\theta_0^+)$ or $\psi(\theta_0^-)$ (or both) will be $-\pi$.
2. At points θ_0 where $H^\mathrm{f}(\theta_0) = 0$ the phase is not defined, so it is not continuous either.

For rational transfer functions, the number of discontinuity points of the phase $\psi(\theta)$ in the range $-\pi < \theta \leq \pi$ is necessarily finite. This follows immediately from the following lemma.

Theorem 8.1 (lemma) If $H^z(z)$ is RCSR, the imaginary part of $H^\mathrm{f}(\theta)$ is zero only on a finite number of points in the range $-\pi < \theta \leq \pi$.

Proof We have, with $z = e^{j\theta}$,

$$\begin{aligned} H_\mathrm{i}^\mathrm{f}(\theta) &= \frac{1}{2j}[H^\mathrm{f}(\theta) - H^\mathrm{f}(-\theta)] = \frac{1}{2j}\left[\frac{b(z)}{a(z)} - \frac{b(z^{-1})}{a(z^{-1})}\right] \\ &= \frac{b(z)a(z^{-1}) - b(z^{-1})a(z)}{2ja(z)a(z^{-1})}. \end{aligned} \tag{8.24}$$

Therefore, $H_\mathrm{i}^\mathrm{f}(\theta) = 0$ if and only if the numerator on the right side is zero. The numerator is a polynomial in mixed (positive and negative) powers of z. Such a polynomial has only a finite number of zeros, and only a subset of those are on the unit circle. Therefore, the number of points in the range $-\pi < \theta \leq \pi$ for which $H_\mathrm{i}^\mathrm{f}(\theta) = 0$ is finite. In particular, the number of points for which $H_\mathrm{i}^\mathrm{f}(\theta) = 0$ and $H_\mathrm{r}^\mathrm{f}(\theta) \leq 0$, that is, the discontinuity points of the phase, is finite. □

8.4.2 Continuous-Phase Representation

As we have seen, the phase of $H^\mathrm{f}(\theta)$ is not defined when $H^\mathrm{f}(\theta) = 0$. This leads to the following question: Can we *define* the phase $\psi(\theta_0)$ at points where $H^\mathrm{f}(\theta_0) = 0$ so as to make it continuous at such points? As the following example shows, the answer is "not always."

Example 8.5 Let $H^z(z) = 1 - z^{-1}$. The corresponding frequency response is

$$\begin{aligned} H^\mathrm{f}(\theta) &= 1 - e^{-j\theta} = (e^{j0.5\theta} - e^{-j0.5\theta})e^{-j0.5\theta} = 2j\sin(0.5\theta)e^{-j0.5\theta} \\ &= 2\sin(0.5\theta)e^{j(0.5\pi - 0.5\theta)}. \end{aligned} \tag{8.25}$$

Therefore,

$$|H^\mathrm{f}(\theta)| = 2|\sin(0.5\theta)|, \quad \psi(\theta) = \begin{cases} 0.5\pi - 0.5\theta, & 0 < \theta < \pi, \\ -0.5\pi - 0.5\theta, & -\pi < \theta < 0. \end{cases} \tag{8.26}$$

In particular, at the point $\theta_0 = 0$,

$$\psi(\theta_0^-) = -0.5\pi, \quad \psi(\theta_0^+) = 0.5\pi. \tag{8.27}$$

Figure 8.8 shows the magnitude and phase responses of $H^{\mathrm{f}}(\theta)$. As we see, the phase response jumps by π radians when passing through $\theta = 0$, so in this case it is impossible to define $\psi(0)$ so as to make the phase continuous. □

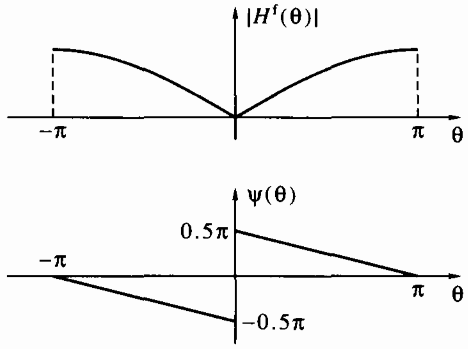

Figure 8.8 The magnitude and phase responses of $H^{\mathrm{f}}(\theta)$ in Example 8.5.

Example 8.5 is an illustration of a more general phenomenon, as the next theorem shows:

Theorem 8.2 Let $H^{\mathrm{z}}(z)$ be an RCSR transfer function. Then at any discontinuity point of the phase response, the phase jumps by either π or 2π radians.

Proof As we have seen, discontinuity of the phase can occur only at points θ_0 where either $H_{\mathrm{i}}^{\mathrm{f}}(\theta_0) = 0$ and $H_{\mathrm{r}}^{\mathrm{f}}(\theta_0) < 0$, or $H^{\mathrm{f}}(\theta_0) = 0$. We have already seen that the jump is 2π in the former case, so it only remains to examine the latter. Since $H^{\mathrm{z}}(z)$ is rational, this can happen only if $H^{\mathrm{z}}(z)$ has a zero at the point $z = e^{j\theta_0}$. Let m be the multiplicity of this zero. Then $H^{\mathrm{z}}(z)$ must have the form

$$H^{\mathrm{z}}(z) = H_1^{\mathrm{z}}(z)(1 - e^{j\theta_0}z^{-1})^m, \tag{8.28}$$

where $H_1^{\mathrm{z}}(z)$ is nonzero at $z = e^{j\theta_0}$. Therefore, as in Example 8.5,

$$\begin{aligned} H^{\mathrm{f}}(\theta) &= H_1^{\mathrm{f}}(\theta)[1 - e^{-j(\theta-\theta_0)}]^m \\ &= H_1^{\mathrm{f}}(\theta)\{2\sin[0.5(\theta - \theta_0)]\}^m e^{j[0.5m\pi - 0.5(\theta-\theta_0)m]}. \end{aligned} \tag{8.29}$$

Since $H_1^{\mathrm{f}}(\theta_0) \neq 0$, the local behavior of $\psi(\theta)$ around θ_0 is not affected by the first factor. If m is even, the sine term does not change sign, so the phase remains continuous. If m is odd, there is a sign reversal in the sine term, so there is a jump of π radians in the phase. □

A close examination of the proof of Theorem 8.2 reveals that discontinuities at which the phase jumps by π are the result of the magnitude being positive *by definition.* Suppose that we relax the definition of magnitude and permit it to be positive or negative, requiring only that it be real. As the next theorem shows, this allows us to express $H^{\mathrm{f}}(\theta)$ in terms of a modified, continuous-phase function.

Theorem 8.3 The frequency response of an RCSR transfer function $H^z(z)$ can be expressed as

$$H^f(\theta) = A(\theta)e^{j\phi(\theta)}, \tag{8.30}$$

where $A(\theta)$ is real, but not necessarily positive, and $\phi(\theta)$ is continuous.

Proof The proof is by construction. Let θ_1 be the first discontinuity point to the right of $-\pi$. Define

$$A(\theta) = |H^f(\theta)|, \quad \phi(\theta) = \psi(\theta), \quad -\pi < \theta < \theta_1,$$
$$A(\theta_1) = \lim_{\theta\uparrow\theta_1} A(\theta), \quad \phi(\theta_1) = \lim_{\theta\uparrow\theta_1} \phi(\theta).$$

The notation $\lim_{\theta\uparrow\theta_1}$ means limit as θ approaches θ_1 from below. Next let θ_2 be the first discontinuity point to the right of θ_1 and define

$$A(\theta) = (-1)^k|H^f(\theta)|, \quad \phi(\theta) = \psi(\theta) + k\pi, \quad \theta_1 < \theta < \theta_2,$$

where k is an integer (positive or negative), chosen so as to preserve continuity of $\phi(\theta)$ at θ_1; k is even in the case of 2π discontinuity, and odd in the case of π discontinuity. We continue this way until we exhaust all discontinuity points [whose number is necessarily finite, according to the lemma (Theorem 8.1)]. Each time we pass a discontinuity, we set the values of $A(\theta)$ and $\phi(\theta)$ as the limits from the left of the functions constructed up to that discontinuity. We then preserve continuity on the right either by adding an even multiple of π to $\phi(\theta)$ while leaving the sign of $A(\theta)$ unchanged or by adding an odd multiple of π to $\phi(\theta)$ while reversing the sign of $A(\theta)$. After all discontinuity points have been exhausted, $A(\theta)$ and $\phi(\theta)$ are defined on $\theta \in (-\pi, \pi)$ such that $\phi(\theta)$ is continuous and $A(\theta)$ is real, but not necessarily positive. Finally, the values of $A(-\pi)$ and $\phi(-\pi)$ are defined by continuity from the right, and those of $A(\pi)$ and $\phi(\pi)$ by continuity from the left. □

The form (8.30) is called a *continuous-phase representation* of the frequency response, $A(\theta)$ is called the *amplitude function*, and $\phi(\theta)$ the *continuous phase*. Bear in mind the difference between the magnitude $|H^f(\theta)|$, which is always real and nonnegative, and the amplitude $A(\theta)$, which can assume any real value. Continuous-phase representation is not unique, since we can replace $A(\theta)$ by $-A(\theta)$ and $\phi(\theta)$ by $\phi(\theta) + \pi$. Also, we can obviously add to $\phi(\theta)$ an arbitrary integer multiple of 2π. We can make the representation unique by imposing the additional condition

$$0 \le \phi(0) < \pi.$$

Example 8.6 The continuous-phase representation of $H^f(\theta)$ given in Example 8.5 is obviously

$$A(\theta) = 2\sin(0.5\theta), \quad \phi(\theta) = 0.5\pi - 0.5\theta. \tag{8.31}$$

The amplitude and phase functions are shown in Figure 8.9. As we see, $\phi(\theta)$ is continuous. □

8.4.3 Linear Phase

Consider the filter

$$H^z(z) = z^{-L}, \quad H^f(\theta) = e^{-j\theta L}, \tag{8.32}$$

where L is an integer. Both magnitude and amplitude functions of this filter are 1 at all frequencies, and the phase response is $\phi(\theta) = -L\theta$, which is a linear function of

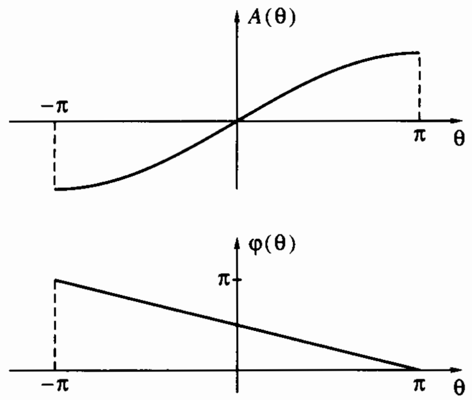

Figure 8.9 The amplitude and continuous-phase responses of $H^{\mathrm{f}}(\theta)$ in Example 8.6.

the frequency. This filter represents a pure delay; that is, the response to an input signal $x[n]$ is the signal $x[n-L]$. In particular, the filter does not distort the form of the input signal.

Are there other filters, beside the pure delay filter, that do not distort signals applied to their input except for a delay? As the next example shows, the answer is affirmative.

Example 8.7 Consider a filter whose frequency response obeys

$$H^{\mathrm{f}}(\theta) = \begin{cases} e^{-j\theta L}, & \theta_1 \le |\theta| \le \theta_2, \\ \text{arbitrary}, & \text{otherwise}, \end{cases} \tag{8.33}$$

where L is a positive integer and θ_1, θ_2 are fixed frequencies. Such a filter cannot be RCSR, but this is immaterial for our present discussion. Let $x[n]$ be a signal whose Fourier transform is nonzero only for $\theta_1 \le |\theta| \le \theta_2$. Then the signal at the output of the filter is

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H^{\mathrm{f}}(\theta)X^{\mathrm{f}}(\theta)e^{j\theta n}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{\mathrm{f}}(\theta)e^{j\theta(n-L)}d\theta = x[n-L], \tag{8.34}$$

so the filter $H^{\mathrm{f}}(\theta)$ acts as a pure delay filter for signals limited to the band $\theta_1 \le |\theta| \le \theta_2$. □

How will the result in Example 8.7 change if we let the proportionality factor of the phase be noninteger? The answer is given in the next example.

Example 8.8 Let

$$H^{\mathrm{f}}(\theta) = \begin{cases} e^{-j\theta(L+\delta)}, & \theta_1 \le |\theta| \le \theta_2, \\ \text{arbitrary}, & \text{otherwise}, \end{cases} \tag{8.35}$$

where L, θ_1, θ_2 are as in Example 8.7 and δ is a proper fraction. For a signal $x[n]$ whose bandwidth is limited to $\theta_1 \le |\theta| \le \theta_2$ we get

$$\begin{aligned} y[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{\mathrm{f}}(\theta)e^{j\theta(n-L-\delta)}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j\theta m}\right]e^{j\theta(n-L-\delta)}d\theta \\ &= \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\theta(n-m-L-\delta)}d\theta\right] = \sum_{m=-\infty}^{\infty} x[m]\operatorname{sinc}(n-m-L-\delta). \end{aligned} \tag{8.36}$$

This time, the filter acts as a *sinc interpolator*, and $y[n]$ can be interpreted as the interpolated value of the continuous-time signal $x(t)$ at time $(n - L - \delta)T$. In summary, a filter that has a pure linear-phase characteristic over a certain frequency band, such that the phase proportionality factor is noninteger, acts as an *interpolated-delay* operator for signals in that frequency band. □

A digital filter whose frequency response admits a continuous-phase representation

$$H^{\mathrm{f}}(\theta) = A(\theta)e^{-j\theta\tau_{\mathrm{p}}} \tag{8.37}$$

is said to have *linear phase*. The proportionality factor τ_{p} is called the *phase delay* of the filter and is measured in *samples* (a dimensionless number). There are no restrictions on the characteristics of the amplitude function $A(\theta)$. However, for practical filters, $A(\theta)$ will usually contain pass bands and stop bands, as we have explained in Section 8.2. In the pass bands, $A(\theta) \approx 1$ (up to the specified pass-band ripple); therefore, the filter acts as an approximate pure delay filter for signals in the pass band. In the stop bands, $A(\theta) \approx 0$ (up to the specified stop-band attenuation); therefore, the phase characteristic is immaterial in these bands. If the phase delay τ_{p} is an integer, there is a delay of τ_{p} samples; if it is fractional, there is an interpolated delay of τ_{p} samples. In any case, the output signal is almost a distortion-free copy of the input signal, as long as the input signal is in the pass band of the filter.

The concept of phase delay is defined also for filters that do not necessarily have linear phase. In general, the phase delay is defined as

$$\tau_{\mathrm{p}}(\theta) = -\frac{\phi(\theta)}{\theta}. \tag{8.38}$$

A linear-phase filter is therefore characterized by a constant phase delay.

8.4.4 Generalized Linear Phase

Suppose that the phase function is not strictly proportional to the frequency, but includes an additive constant term, as in the following example.

Example 8.9 Let

$$H^{\mathrm{f}}(\theta) = \begin{cases} e^{j\phi_0 - j\theta(L+\delta)}, & \theta_1 \le \theta \le \theta_2, \\ e^{-j\phi_0 - j\theta(L+\delta)}, & -\theta_2 \le \theta \le -\theta_1, \\ \text{arbitrary}, & \text{otherwise}, \end{cases} \tag{8.39}$$

where L is an integer, δ a proper fraction, ϕ_0 a fixed angle, and θ_1, θ_2 are fixed frequencies (again, such a filter cannot be RCSR). Consider an input signal of the form $x[n]\cos(\theta_{\mathrm{c}}n)$. Such a signal is called *amplitude modulated*, and $x[n]$ itself is the *modulating signal* or the *envelope*. The frequency θ_{c} is called the *carrier frequency*. Here we assume that

1. The modulating signal is band limited to $|\theta| \le \theta_0$ for some θ_0.
2. The following relationship holds among the frequencies of interest:

$$\theta_1 \le \theta_{\mathrm{c}} - \theta_0 < \theta_{\mathrm{c}} + \theta_0 \le \theta_2.$$

The Fourier transform of the modulated signal is given by

$$0.5X^{\mathrm{f}}(\theta - \theta_{\mathrm{c}}) + 0.5X^{\mathrm{f}}(\theta + \theta_{\mathrm{c}}).$$

Therefore, the Fourier transform of the signal $y[n]$ at the output of the filter is

$$\begin{aligned} Y^{\mathrm{f}}(\theta) &= 0.5X^{\mathrm{f}}(\theta-\theta_{\mathrm{c}})e^{j\phi_0-j\theta(L+\delta)} + 0.5X^{\mathrm{f}}(\theta+\theta_{\mathrm{c}})e^{-j\phi_0-j\theta(L+\delta)} \\ &= 0.5X^{\mathrm{f}}(\theta-\theta_{\mathrm{c}})e^{-j(\theta-\theta_{\mathrm{c}})(L+\delta)}e^{j\phi_0-j\theta_{\mathrm{c}}(L+\delta)} \\ &\quad + 0.5X^{\mathrm{f}}(\theta+\theta_{\mathrm{c}})e^{-j(\theta+\theta_{\mathrm{c}})(L+\delta)}e^{-j\phi_0+j\theta_{\mathrm{c}}(L+\delta)}. \end{aligned} \tag{8.40}$$

By the modulation property of the Fourier transform, $y[n]$ is given by

$$\begin{aligned} y[n] &= 0.5e^{j[\theta_{\mathrm{c}}n+\phi_0-\theta_{\mathrm{c}}(L+\delta)]}w[n] + 0.5e^{-j[\theta_{\mathrm{c}}n+\phi_0-\theta_{\mathrm{c}}(L+\delta)]}w[n] \\ &= w[n]\cos[\theta_{\mathrm{c}}n+\phi_0-\theta_{\mathrm{c}}(L+\delta)], \end{aligned} \tag{8.41}$$

where $w[n]$ is the inverse Fourier transform of $X^{\mathrm{f}}(\theta)e^{-j\theta(L+\delta)}$. As we saw in Example 8.8, $w[n]$ is an exact delay of $x[n]$ if $\delta = 0$, or an interpolated delay of $x[n]$ if $\delta \neq 0$. In conclusion, the filter's output is a modulated signal such that its envelope is a delayed version of the envelope of the input to the filter. The constant phase ϕ_0 affects the carrier, but not the envelope of the modulated signal at the output. □

A digital filter whose frequency response admits a continuous-phase representation

$$H^{\mathrm{f}}(\theta) = A(\theta)e^{j(\phi_0-\theta\tau_{\mathrm{g}})} \tag{8.42}$$

is said to have *generalized linear phase* (GLP). The slope τ_{g} is called the *group delay* of the filter and is measured in samples. There are no restrictions of the characteristics of the amplitude function $A(\theta)$. However, for practical filters $A(\theta)$ will usually contain pass bands and stop bands. In the pass bands, $A(\theta) \approx 1$ (up to the specified pass-band ripple); therefore, the filter acts as an approximate pure delay filter for the envelopes of modulated signals in the pass band. In the stop bands, $A(\theta) \approx 0$ (up to the specified stop-band attenuation); therefore, the phase characteristic is immaterial in these bands. If the group delay τ_{g} is an integer, there is a delay of the envelope by τ_{g} samples; if it is fractional, there is an interpolated delay of τ_{g} samples. In any case, the envelope of the output signal is almost a distortion-free copy of the envelope of the input signal, as long as the modulated signal is in the pass band of the filter.

The concept of group delay is defined also for filters that do not necessarily have generalized linear phase. In general, if the phase function $\phi(\theta)$ is differentiable, the group delay is defined as

$$\tau_{\mathrm{g}}(\theta) = -\frac{d\phi(\theta)}{d\theta}. \tag{8.43}$$

A GLP filter is therefore characterized by a constant group delay.

The group delay of a filter with frequency response $H^{\mathrm{f}}(\theta)$ can be computed as follows:

$$\tau_{\mathrm{g}}(\theta) = -\frac{d}{d\theta}\mathrm{arctan2}\{H_{\mathrm{i}}^{\mathrm{f}}(\theta), H_{\mathrm{r}}^{\mathrm{f}}(\theta)\} = \frac{\dfrac{dH_{\mathrm{r}}^{\mathrm{f}}(\theta)}{d\theta}H_{\mathrm{i}}^{\mathrm{f}}(\theta) - \dfrac{dH_{\mathrm{i}}^{\mathrm{f}}(\theta)}{d\theta}H_{\mathrm{r}}^{\mathrm{f}}(\theta)}{[H_{\mathrm{r}}^{\mathrm{f}}(\theta)]^2 + [H_{\mathrm{i}}^{\mathrm{f}}(\theta)]^2}. \tag{8.44}$$

This computation is particularly simple if the filter is FIR, since then

$$H^{\mathrm{f}}(\theta) = \sum_{n=0}^{N} h[n]e^{-j\theta n}, \quad \frac{dH^{\mathrm{f}}(\theta)}{d\theta} = -j\sum_{n=0}^{N} nh[n]e^{-j\theta n}. \tag{8.45}$$

If the filter is a rational IIR, its group delay is the difference between the group delay of the numerator and that of the denominator (Problem 8.13). Each of the two is FIR, so its group delay can be computed using (8.45). The procedure `grpdly` in Program 8.1 illustrates this computation. The input parameters of the program are in the same format as the parameters of Program 7.6 (which computes the frequency response of an RCSR filter). The program first determines if the filter is FIR. If so, it computes the group delay using (8.44) and (8.45). If not, it calls itself recursively twice, once for the numerator and once for the denominator, and subtracts the results.

8.4.5 Restrictions on GLP Filters

We now show that periodicity and realness impose restrictions on the parameters ϕ_0 and τ_g of a generalized linear-phase filter. Consider first the implications of periodicity. Since $H^f(\theta)$ is periodic with period 2π, we get from (8.42)

$$A(\theta)e^{j(\phi_0-\theta\tau_g)} = A(\theta+2\pi)e^{j(\phi_0-\theta\tau_g-2\pi\tau_g)}. \tag{8.46}$$

Therefore,

$$A(\theta) = A(\theta+2\pi)e^{-j2\pi\tau_g}. \tag{8.47}$$

But, since $A(\cdot)$ is real valued, $2\tau_g$ must be an integer. We therefore have two cases:

1. The group delay is an integer, say $\tau_g = M$. In this case we get
$$A(\theta) = A(\theta+2\pi), \tag{8.48}$$
so the amplitude function is periodic with period 2π.
2. The group delay has a fractional part 0.5, say $\tau_g = M + 0.5$. In this case we get
$$A(\theta) = -A(\theta+2\pi), \tag{8.49}$$
so the amplitude function does not have a period of 2π. However, in this case we still have
$$A(\theta) = A(\theta+4\pi), \tag{8.50}$$
so the amplitude function is periodic with period 4π.

Consider next the implications of the assumption that the filter has real impulse response. In this case, the frequency response satisfies the conjugate symmetry condition

$$H^f(-\theta) = \bar{H}^f(\theta), \tag{8.51}$$

so

$$A(-\theta)e^{j(\phi_0+\theta\tau_g)} = A(\theta)e^{-j(\phi_0-\theta\tau_g)}. \tag{8.52}$$

This gives

$$e^{j2\phi_0} = \frac{A(\theta)}{A(-\theta)}, \tag{8.53}$$

from which it follows that $e^{j2\phi_0}$ is necessarily real. As before, we have two cases:

1. $\phi_0 = 0$ and $A(\theta) = A(-\theta)$, so $A(\theta)$ is a symmetric function. In this case, the filter has constant phase delay $\tau_p = \tau_g$.
2. $\phi_0 = \pi/2$ and $A(\theta) = -A(-\theta)$, so $A(\theta)$ is an antisymmetric function.

In summary, real digital filters with generalized linear phase come in four types:

Type-I filters have integer group delay $\tau_g = M$ and initial phase $\phi_0 = 0$, so they have constant phase delay.

Type-II filters have fractional group delay $\tau_g = M + 0.5$ and initial phase $\phi_0 = 0$, so they also have constant phase delay.

Type-III filters have integer group delay $\tau_g = M$ and initial phase $\phi_0 = \pi/2$, so their phase delay is not constant.

Type-IV filters have fractional group delay $\tau_g = M + 0.5$ and initial phase $\phi_0 = \pi/2$, so their phase delay is not constant.

8.4.6 Restrictions on Causal GLP Filters

Causality imposes further restrictions on the possible form of digital GLP filters. To find these restrictions, observe from (8.42) that a GLP filter (causal or not) satisfies

$$H^{\mathrm{f}}(\theta)e^{-j(\phi_0-\theta\tau_g)} = A(\theta). \tag{8.54}$$

Consider first the case $\phi_0 = 0$. We then get from (8.54)

$$h[2\tau_g - n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H^{\mathrm{f}}(\theta)e^{j\theta(2\tau_g-n)}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\theta)e^{j\theta(\tau_g-n)}d\theta. \tag{8.55}$$

Taking the conjugate of (8.55) and using the fact that $A(\theta)$ is real, we get

$$h[2\tau_g - n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\theta)e^{j\theta(n-\tau_g)}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} H^{\mathrm{f}}(\theta)e^{j\theta n}d\theta = h[n]. \tag{8.56}$$

Now assume that the filter is causal, so $h[n] = 0$ for $n < 0$. Then, by (8.56), $h[n] = 0$ for $n > 2\tau_g$. Thus, the filter has a finite impulse response, its order N is equal to $2\tau_g$, and it satisfies the symmetry condition

$$h[n] = h[N - n]. \tag{8.57}$$

Next consider the case $\phi_0 = \pi/2$. We then get from (8.54) that

$$h[2\tau_g - n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H^{\mathrm{f}}(\theta)e^{j\theta(2\tau_g-n)}d\theta = \frac{j}{2\pi}\int_{-\pi}^{\pi} A(\theta)e^{j\theta(\tau_g-n)}d\theta. \tag{8.58}$$

Taking the conjugate of (8.58) and using the fact that $A(\theta)$ is real, we get

$$h[2\tau_g - n] = -\frac{j}{2\pi}\int_{-\pi}^{\pi} A(\theta)e^{j\theta(n-\tau_g)}d\theta = -\frac{1}{2\pi}\int_{-\pi}^{\pi} H^{\mathrm{f}}(\theta)e^{j\theta n}d\theta = -h[n]. \tag{8.59}$$

We again get, by (8.59), that $h[n] = 0$ for $n > 2\tau_g$. Thus, the filter has a finite impulse response, its order N is equal to $2\tau_g$, and it satisfies the antisymmetry condition

$$h[n] = -h[N - n]. \tag{8.60}$$

In summary, we have proved the following theorem:

Theorem 8.4 A linear phase RCSR filter is necessarily FIR. The phase or group delay of such a filter is half its order; it satisfies either the symmetry relationship (8.57) or the antisymmetry relationship (8.60). In the former case the phase delay is constant, while in the latter the group delay is constant. □

Theorem 8.4 implies that, if we look for an RCSR filter having linear phase (either exact or generalized), there is no sense in considering an IIR filter, but we must restrict ourselves to an FIR filter.[6] This property of digital FIR filters is one of the main reasons they are more commonly used than digital IIR filters.

8.4.7 Minimum-Phase Filters*

Let $H^z(z)$ be an RCSR filter. Stability implies that all poles are inside the unit circle. However, there is no constraint on the location of zeros, as far as stability is concerned.

We show that, in general, there is freedom in choosing the zero locations such that the magnitude response will remain fixed, but the phase response will vary. Then we shall discuss how this freedom can be exploited.

Let β be a zero of $H^z(z)$, so $(1-\beta z^{-1})$ is a factor of the transfer function. Then we can write

$$H^z(z) = (1-\beta z^{-1})H_0^z(z). \tag{8.61}$$

Let us replace this factor by

$$\bar{\beta} - z^{-1} = \bar{\beta}[1-\bar{\beta}^{-1}z^{-1}].$$

We then get the new transfer function

$$H_1^z(z) = \bar{\beta}[1-\bar{\beta}^{-1}z^{-1}]H_0^z(z). \tag{8.62}$$

We have

$$\bar{\beta} - e^{-j\theta} = -e^{-j\theta}(1-\bar{\beta}e^{j\theta}) = -e^{-j\theta}\overline{(1-\beta e^{-j\theta})}. \tag{8.63}$$

We get that $(\bar{\beta}-e^{-j\theta})$ and $(1-\beta e^{-j\theta})$ have the same magnitude, implying that $H^z(z)$ and $H_1^z(z)$ have the same magnitude response. However, the phase responses of the two filters are different in general. Thus, the replacement of zeros of $H^z(z)$ by their conjugate inverses is a means of changing the phase response without affecting the magnitude response. Note that, if β is complex and we replace the zero at β by a zero at $\bar{\beta}^{-1}$, we must simultaneously replace the zero at $\bar{\beta}$ by a zero at β^{-1}; otherwise $H_1^z(z)$ will not be real.

The following theorem shows how the group delay of the filter is affected by zero replacement.

Theorem 8.5 Suppose $|\beta| < 1$, so the zero at β is inside the unit circle. Then the group delay of $H_1^z(z)$ is larger than the group delay of $H^z(z)$ at all frequencies.

Proof Let

$$\beta = \beta_r + j\beta_i, \quad \zeta = \arctan2(\beta_i, \beta_r).$$

Then the phase contributed by the factor $(1-\beta z^{-1})$ is

$$\begin{aligned}\phi_\beta(\theta) &= \measuredangle\{1-(\beta_r+j\beta_i)(\cos\theta - j\sin\theta)\}\\ &= \arctan2(\beta_r\sin\theta - \beta_i\cos\theta, 1-\beta_r\cos\theta-\beta_i\sin\theta)\\ &= \arctan2\{\sin(\theta-\zeta), |\beta|^{-1}-\cos(\theta-\zeta)\}.\end{aligned} \tag{8.64}$$

The contribution of $\phi_\beta(\theta)$ to the group delay is given by

$$\begin{aligned}\tau_{g,\beta}(\theta) &= -\frac{d\phi_\beta(\theta)}{d\theta} = \frac{\sin^2(\theta-\zeta)+\cos^2(\theta-\zeta)-|\beta|^{-1}\cos(\theta-\zeta)}{\sin^2(\theta-\zeta)+\cos^2(\theta-\zeta)+|\beta|^{-2}-2|\beta|^{-1}\cos(\theta-\zeta)}\\ &= \frac{|\beta|-\cos(\theta-\zeta)}{|\beta|+|\beta|^{-1}-2\cos(\theta-\zeta)}.\end{aligned} \tag{8.65}$$

Replacing β by $\bar{\beta}^{-1}$ leaves ζ unchanged, and it also leaves the denominator of (8.65) unchanged. However, the numerator of (8.65) increases, since $|\beta|$ is replaced by $|\beta|^{-1}$, which is larger. Therefore, replacing β by $\bar{\beta}^{-1}$ increases the group delay. If β is complex, the same proof holds for the replacement of $\bar{\beta}$ by β^{-1}. □

A filter $H^z(z)$ for which the group delay is the minimum possible at all frequencies, of all filters having the same magnitude response as $H^z(z)$, is called *minimum phase.* The term *minimum group delay* is perhaps more accurate, but the former is the one in common use. Theorem 8.5 implies that an RCSR filter $H^z(z)$ is minimum phase if and only if it has no zeros outside the unit circle. This is because any zero outside the unit circle can be replaced by its conjugate inverse, thus reducing the group delay without affecting the magnitude response. A practical procedure for designing a minimum-phase filter meeting a prescribed magnitude specification is given as follows:

1. Design a filter meeting the magnitude specification, ignoring the phase.
2. Find the zeros of the filter.
3. Replace all zeros outside the unit circle by their conjugate inverses, and adjust the constant gain accordingly.

There is no need to apply the procedure to zeros *on* the unit circle, because this will not change the group delay.

Example 8.10 Consider the four filters

$$H_1^z(z) = \frac{(1-0.5z^{-1})(1-0.2z^{-1})}{1-z^{-1}+0.5z^{-2}}, \quad H_2^z(z) = \frac{0.5(1-2z^{-1})(1-0.2z^{-1})}{1-z^{-1}+0.5z^{-2}},$$
$$H_3^z(z) = \frac{0.2(1-0.5z^{-1})(1-5z^{-1})}{1-z^{-1}+0.5z^{-2}}, \quad H_4^z(z) = \frac{0.1(1-2z^{-1})(1-5z^{-1})}{1-z^{-1}+0.5z^{-2}}.$$

These filters have identical magnitude responses. The first is minimum phase, since both its zeros are inside the unit circle. The fourth is maximum phase, since both its zeros are outside the unit circle. The other two filters are mixed phase. Figure 8.10 shows the group delays of the four filters. As we see, $H_1^z(z)$ indeed has the smallest group delay at all frequencies and $H_4^z(z)$ has the largest group delay. The two other filters have group delays between the two extremes. Note that their graphs intersect, so one is not always smaller than the other. □

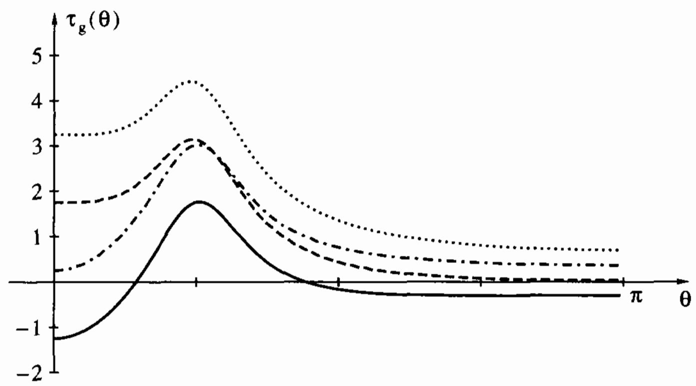

Figure 8.10 The group delays of the four filters in Example 8.10. Solid line: $H_1^z(z)$, dashed line: $H_2^z(z)$, dot-dashed line: $H_3^z(z)$, dotted line: $H_4^z(z)$.

8.4.8 All-Pass Filters*

A filter $H^z(z)$ is called *all pass* if its magnitude response is identically 1 (or, more generally, a positive constant) at all frequencies. The phase response of an all-pass filters is not restricted and is allowed to vary arbitrarily as a function of the frequency.

Example 8.11 Let

$$H^z(z) = \frac{\bar{a} - z^{-1}}{1 - az^{-1}}, \tag{8.66}$$

where $|a| < 1$; this is a stable IIR filter. We have

$$H^f(\theta) = \frac{\bar{a} - e^{-j\theta}}{1 - ae^{-j\theta}} = -e^{-j\theta}\frac{1 - \bar{a}e^{j\theta}}{1 - ae^{-j\theta}}. \tag{8.67}$$

The numerator and denominator on the right side are conjugates of each other; therefore

$$|H^f(\theta)| = 1, \tag{8.68}$$

so $H^z(z)$ is an all-pass filter. This result is not coincidental, but occurs because the zero is the inverse of the pole. □

In general, a rational filter is all pass if and only if it has the same number of poles and zeros (including multiplicities), and each zero is the conjugate inverse of a corresponding pole. In other words, the transfer function of the filter must be a product of first-order all-pass filters, that is,

$$H^z(z) = \prod_{k=1}^{p} \frac{\bar{\alpha}_k - z^{-1}}{1 - \alpha_k z^{-1}}. \tag{8.69}$$

If the filter is stable then, since all the poles are inside the unit circle, all the zeros must be outside the unit circle. See Problem 7.29 for another (but mathematically equivalent) characterization of all-pass filters.

It follows from the foregoing discussion of minimum phase filters that an RCSR filter $H^z(z)$ that is not minimum phase is related to its minimum-phase companion $H^z_{\text{mp}}(z)$ by

$$H^z(z) = H^z_{\text{mp}}(z)H^z_{\text{ap}}(z), \tag{8.70}$$

where $H^z_{\text{ap}}(z)$ is an all-pass filter whose order is equal to the total number of zeros of $H^z(z)$ outside the unit circle. These zeros are precisely the zeros of $H^z_{\text{ap}}(z)$, whereas their conjugate inverses are the poles of $H^z_{\text{ap}}(z)$.

8.5 Digital Filter Design Considerations

A typical design process of a digital filter involves four steps:

1. Specification of the filter's response, as discussed in the preceding section. The importance of this step cannot be overstated. Often a proper specification is the key to the success of the system of which the filter is a part. Therefore, this task is usually entrusted to senior engineers, who rely on experience and engineering common sense. The remaining steps are then often put in the hands of relatively junior engineers.
2. Design of the transfer function of the filter. The main goal here is to meet (or surpass) the specification with a filter of *minimum complexity*. For LTI filters, minimum complexity is usually synonymous with minimum order. Design methods are discussed in detail in Chapters 9 and 10.
3. Verification of the filter's performance by analytic means, simulations, and testing with real data when possible.
4. Implementation by hardware, software, or both. Implementation is discussed in Chapter 11.

As we have mentioned, there are two classes of LTI digital filters: IIR and FIR. Design techniques for these two classes are radically different, so we briefly discuss each of them individually.

8.5.1 IIR Filters

Analog rational filters necessarily have an infinite impulse response. Good design techniques of analog IIR filters have been known for decades. The design of digital IIR filters is largely based on analog filter design techniques. A typical design procedure of a digital IIR filter thus involves the following steps:

1. Choosing a method of transformation of a given analog filter to a digital filter having approximately the same frequency response.
2. Transforming the specifications of the digital IIR filter to equivalent specifications of an analog IIR filter such that, after the transformation from analog to digital is carried out, the digital IIR filter will meet the specifications.
3. Designing the analog IIR filter according to the transformed specifications.
4. Transforming the analog design to an equivalent digital filter.

The main advantages of this design procedure are convenience and reliability. It is convenient, since analog filter design techniques are well established and the properties of the resulting filters are well understood. It is reliable, since a good analog design (i.e., one that meets the specifications at minimum complexity) is guaranteed to yield a good digital design. The main drawback of this method is its limited generality. Design techniques of analog filters are practically limited to the four basic types (LP, HP, BP, and BS) and a few others. More general filters, such as multiband, are hard to design in the analog domain and require considerable expertise.

Beside analog-based methods, there exist design methods for digital IIR filters that are performed in the digital domain directly. Typically they belong to one of two classes:

1. Methods that are relatively simple, requiring only operations such as solutions of linear equations. Since these give rise to filters whose quality is often mediocre, they are not popular.
2. Methods that are accurate and give rise to high-quality filters, but are complicated and hard to implement; consequently they are not popular either.

Direct digital design techniques for IIR filters are not studied in this book.

8.5.2 FIR Filters

Digital FIR filters cannot be derived from analog filters, since rational analog filters cannot have a finite impulse response. So, why bother with such filters? A full answer to this question will be given in Chapter 9, when we study FIR filters in detail. Digital FIR filters have certain unique properties that are not shared by IIR filters (whether analog or digital), such as:

1. They are inherently stable.
2. They can be designed to have a linear phase or a generalized linear phase.
3. There is great flexibility in shaping their magnitude response.
4. They are convenient to implement.

These properties are highly desirable in many applications and have made FIR filters far more popular than IIR filters in digital signal processing. On the other hand, FIR filters have a major disadvantage with respect to IIR filters: The relative computational complexity of the former is higher than that of the latter. By "relative" we mean that an FIR filter meeting the *same* specifications as a given IIR filter will require many more operations per unit of time.

Design methods for FIR filters can be divided into two classes:

1. Methods that require only relatively simple calculations. Chief among those is the *windowing* method, which is based on concepts similar to those studied in Section 6.2. Another method in this category is *least-squares design*. These methods usually give good results, but are not optimal in terms of complexity.
2. Methods that rely on numerical optimization and require sophisticated software tools. Chief among those is the *equiripple* design method, which is guaranteed to meet the given specifications at a minimum complexity.

8.6 Summary and Complements

8.6.1 Summary

In this chapter we introduced the subject of digital filtering. In general, filtering means shaping the frequency-domain characteristics of a signal. The most common filtering operation is attenuation of signals in certain frequency bands and passing signals in other frequency bands with only little distortion. In the case of digital filters, the input and output signals are in discrete time. The two basic types of digital filter are infinite impulse response and finite impulse response. The former resemble analog filters in many respects, whereas the latter are unique to the discrete-time domain.

Simple filters are divided into four kinds, according to their frequency response characteristics: low pass, high pass, band pass, and band stop. Each of these kinds is characterized by its own set of specification parameters. A typical set of specification parameters includes (1) band-edge frequencies and (2) ripple and attenuation tolerances. The task of designing a filter amounts to finding a transfer function $H^z(z)$ (IIR or FIR), such that the corresponding frequency response $H^f(\theta)$ will meet or surpass the specifications, preferably at minimum complexity.

The frequency response of a rational digital filter can be written in a continuous-phase representation (8.30). If the phase function $\phi(\theta)$ in this representation is exactly linear, signals in the pass band of the filter are passed to the output almost free of distortion. If the phase function is linear up to an additive constant, the envelope of a modulated signal in the pass band is passed to the output almost free of distortion.

In the class of real, causal, stable, and rational digital filters, only FIR filters can have linear phase. Such filters come in four types: The impulse response can be either symmetric or antisymmetric, and the order can be either even or odd. Symmetric impulse response yields exact linear phase, whereas antisymmetric one yields generalized linear phase.

The phase response of a digital rational filter is not completely arbitrary, but is related to the magnitude response. For a given magnitude response, there exists a unique filter whose zeros are all inside or on the unit circle (the poles must be inside the unit circle in any case, for stability). Such a filter is called minimum phase. Any other filter having the same magnitude response has the following property: Each of its zeros is collocated either with a zero of the minimum-phase filter, or with the conjugate inverse of a zero of the minimum-phase filter.

8.6.2 Complements

1. [p. 243] Since $1/\pi t$ is discontinuous at $t = 0$ and its limits at the two sides of the discontinuity do not exist, the convolution $\{x * h\}(t)$ needs to be defined carefully. The standard definition is

$$\{x * h\}(t) = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \frac{x(t-\tau)}{\pi\tau} d\tau + \int_{\varepsilon}^{\infty} \frac{x(t-\tau)}{\pi\tau} d\tau \right]. \tag{8.71}$$

   The right side is called the *Cauchy principal value* of the integral. The problem does not arise in the discrete-time Hilbert transform.

2. [p. 245] Surface acoustic wave (SAW) filters and switched-capacitor filters are an exception to this statement: They implement z-domain transfer functions by essentially analog means.

3. [p. 245] *Analog* high-pass filters are designed to pass high frequencies, from a certain cutoff frequency ω_0 to infinity. In this sense, digital filters are fundamentally different from analog filters. A similar remark holds for band-stop filters.

4. [p. 250] A bit is typically lost due to nonlinearities, noise, etc.

5. [p. 254] The function *arctan2*, as defined here, is identical to the MATLAB function `atan2`.

6. [p. 261] We have proved Theorem 8.4 for RCSR filters. The theorem holds for real causal stable filters if we replace the rationality assumption by the less restrictive assumption that the z-transform of the filter exists on an annulus whose interior contains the unit circle. The proof of the extended version of the theorem is beyond the scope of this book. Without this assumption, the theorem is not valid. It was shown by Clements and Pease [1989] that real causal linear-phase IIR filters do exist, but their z-transform does not exist anywhere, except possibly on the unit circle itself.

8.7 MATLAB Program

Program 8.1 Group delay of a rational transfer function.

```
function D = grpdly(b,a,K,theta);
% Synopsis: D = grpdly(b,a,K,theta).
% Group delay of b(z)/a(z) on a given frequency interval.
% Input parameters:
% b, a: numerator and denominator polynomials
% K: the number of frequency response points to compute
% theta: if absent, the K points are uniformly spaced on
%        [0, pi]; if present and theta is a 1-by-2 vector,
%        its entries are taken as the end points of the
%        interval on which K evenly spaced points are
%        placed; if the size of theta is different from 2,
%        it is assumed to be a vector of frequencies for
%        which the group delay is to be computed, and K is
%        ignored.
% Output:
% D: the group delay vector.

a = reshape(a,1,length(a)); b = reshape(b,1,length(b));
if (length(a) == 1), % case of FIR
   bd = -j*(0:length(b)-1).*b;
   if (nargin == 3),
      B = frqresp(b,1,K); Bd = frqresp(bd,1,K);
   else,
      B = frqresp(b,1,K,theta); Bd = frqresp(bd,1,K,theta);
   end
   D = (real(Bd).*imag(B)-real(B).*imag(Bd))./abs(B).^2;
else % case of IIR
   if (nargin == 3), D = grpdly(b,1,K)-grpdly(a,1,K);
   else, D = grpdly(b,1,K,theta)-grpdly(a,1,K,theta); end
end
```

8.8 Problems

8.1 Repeat Example 8.1 with the following changes:

- The signal bandwidth is 250 Hz.
- The transition bandwidth is 50 Hz.
- The average noise gain in the transition band is 0.25.
- The permitted pass-band ripple is 0.05 dB.

Specify the achievable output SNR and the required filter parameters.

8.2 The digital low-pass filter $H^z(z)$ is known to meet the specifications

$$\theta_p = 0.25\pi, \quad \theta_s = 0.35\pi, \quad \delta_p = 0.02, \quad \delta_s = 0.005.$$

Define a new filter $G^z(z)$ by

$$g[n] = \begin{cases} 2h[n], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

(a) Is $G^z(z)$ LP, HP, BP, or BS? Explain.

(b) What specifications is $G^z(z)$ guaranteed to meet?

8.3 A digital filter usually has some of its zeros on the unit circle. Discuss the relationship between the locations of the zeros on the unit circle and the nature of the frequency response of the filter (LP, HB, BP, BS). Pay special attention to zeros at $z = 1$ and at $z = -1$.

8.4 A digital LTI filter has the frequency response

$$H^f(\theta) = 4\left(1 - \frac{|\theta|}{\pi}\right) e^{-j\theta}.$$

The continuous-time signal $x(t) = \cos(\pi t)$ is sampled at interval $T = 0.25$ and fed to the filter. Find the signal $y[n]$ at the output of the filter.

8.5 This problem explores the existence of zeros of a digital filter at $z = \pm 1$.

(a) Find a necessary and sufficient condition on the coefficients $h[n]$ of an FIR filter for the transfer function $H^z(z)$ to have a zero at $z = 1$. Repeat for a zero at $z = -1$.

(b) Fill the eight cells in Table 8.2 with yes or no, and explain.

Filter type	I	II	III	IV
Must have zero at $z = 1$				
Must have zero at $z = -1$				

Table 8.2 Pertaining to Problem 8.5.

8.6 If an FIR filter $H^z(z)$ has a zero at $z = 1$, its transfer function can be expressed as $H^z(z) = (1 - z^{-1})F^z(z)$; if it has a zero at $z = -1$, its transfer function can be expressed as $H^z(z) = (1 + z^{-1})F^z(z)$; if it has zeros at both locations, its transfer function can be expressed as $H^z(z) = (1 - z^{-2})F^z(z)$. In all cases, $F^z(z)$ is FIR. Use your answer to Problem 8.5 for expressing $H^z(z)$ of each of the four filter types in the applicable form. For each type, express the coefficients of $H^z(z)$ in terms of $f[n]$, the coefficients of the filter $F^z(z)$.

8.7 We are given a transfer function of a digital filter in a factored form

$$H^z(z) = b_0 \frac{\prod_{i=1}^{q}(1 - \beta_i z^{-1})}{\prod_{i=1}^{p}(1 - \alpha_i z^{-1})}.$$

Define the filter $G^z(z)$ by replacing all poles and zeros of $H^z(z)$ by their negatives, that is,

$$\alpha_i \longrightarrow -\alpha_i, \quad \beta_i \longrightarrow -\beta_i.$$

(a) Express the frequency response $G^f(\theta)$ in terms of $H^f(\theta)$.

(b) Suppose that all poles and zeros of $H^z(z)$ are on the imaginary axis. Can such a filter be low pass? high pass? band pass? band stop? Give reasons.

(c) Let

$$H^z(z) = \frac{(z-1)^2}{z^2 - 1.212436z + 0.49}.$$

Write the corresponding transfer function $G^z(z)$. Is $G^z(z)$ low pass, high pass, band pass, or band stop?

8.8 Let

$$H^z(z) = 1 - 1.5z^{-1} + 0.5z^{-2}.$$

Find $A(\theta)$ and $\phi(\theta)$ in the continuous-phase representation of $H^f(\theta)$. Hint: Factor $H^z(z)$ to its zeros.

8.9 Is it possible for a minimum-phase FIR filter to have linear phase? If so, give an example. If not, explain why.

8.10 Suppose that the FIR filter $H^z(z)$ has all its zeros on the unit circle. Show that the filter has linear phase. Hint: Write $H^z(z)$ in a factored form

$$H^z(z) = h[0](1 - z^{-1})^{N_1}(1 + z^{-1})^{N_2} \prod_{k=1}^{N_3}(1 - 2\cos\zeta_k z^{-1} + z^{-2}),$$

where N_1 is the number of zeros at $z = 1$, N_2 is the number of zeros at $z = -1$, and N_3 is the number of complex conjugate pairs of zeros on the unit circle. Then show that each of the factors has linear phase. Finally show that the product has linear phase.

8.11 A noncausal filter is called a *half-band filter* if its frequency response satisfies

$$H^f(\theta) + H^f(\theta - \pi) = c, \qquad (8.72)$$

where c is constant. The equivalent z-domain property is

$$H^z(z) + H^z(-z) = c. \qquad (8.73)$$

Such a filter is called *zero phase, half-band* if, in addition to (8.72), it is symmetric, that is, $h[n] = h[-n]$. Figure 8.11 illustrates the frequency response of a zero-phase, half-band filter.

(a) Prove that a necessary and sufficient condition for a filter to be half-band is

$$h[2n] = \begin{cases} 0.5c, & n = 0, \\ 0, & n \neq 0. \end{cases} \qquad (8.74)$$

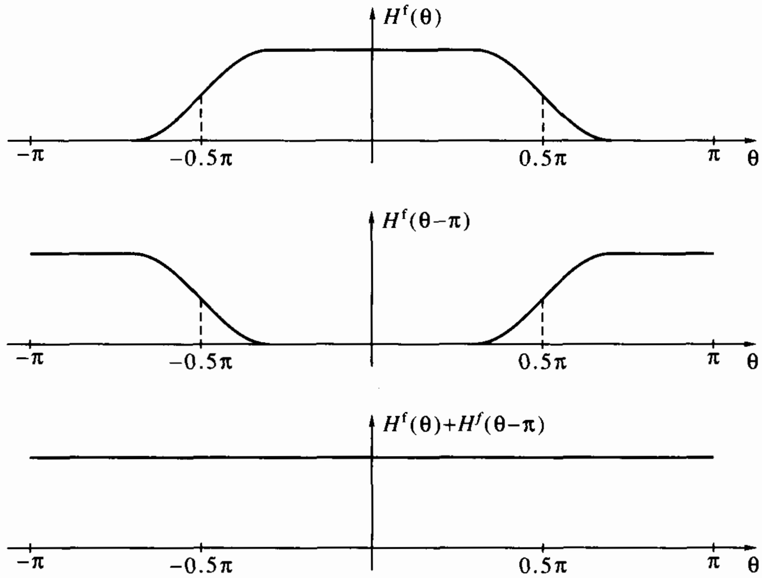

Figure 8.11 The frequency response of a half-band filter.

(b) What is $H^{\mathrm{f}}(0.5\pi)$ for a zero-phase, half-band filter?

(c) A *causal half-band filter* is a causal filter satisfying

$$H^{\mathrm{f}}(\theta) + H^{\mathrm{f}}(\theta - \pi) = ce^{-j2\theta M} \tag{8.75}$$

for some positive integer M. If, in addition to (8.75), the filter is a symmetric FIR filter (that is, $h[n] = h[N - n]$), it is called *linear phase, half-band.* What is a necessary and sufficient condition for a causal filter to be half-band?

(d) Let $h[n]$ be the impulse response of a linear-phase, half-band FIR filter of order $4M$. Show that the order of the filter can be reduced to $4M - 2$ without changing the amplitude response. What will be the group delay of the reduced-order filter?

8.12 This problem introduces the concept of *deconvolution.*

(a) The signal $x[n]$ is passed through the RCSR IIR filter $H^{\mathrm{z}}(z) = 1/a(z)$, and the signal $y[n]$ is obtained. We wish to recover the signal $x[n]$ by passing $y[n]$ through an RCSR filter $G^{\mathrm{z}}(z)$. Under what conditions is this possible, and what is $G^{\mathrm{z}}(z)$?

(b) Repeat part a when the filter is FIR, $H^{\mathrm{z}}(z) = b(z)$.

(c) Repeat part a when the filter is $H^{\mathrm{z}}(z) = b(z)/a(z)$.

The operation addressed in this problem is called deconvolution because it reverses the convolution operation of the filter.

8.13 Explain why the group delay of a rational IIR filter is the difference between the group delay of the numerator and that of the denominator.

8.14 Use MATLAB for computing the impulse responses of the four filters in Example 8.10. Plot them and observe the values of the early impulse response coefficients (the ones nearest to $n = 0$) relative to the coefficients at later times. Compare the behavior of the four filters in this respect, and reach a conclusion about the special nature of the impulse response of the minimum-phase filter $H_{\mathrm{I}}^{\mathrm{z}}(z)$.

8.15 We are given the four filters

$$H_1^z(z) = 1 + \frac{3}{16}z^{-2} - \frac{1}{64}z^{-4}, \quad H_2^z(z) = \frac{1}{4} + \frac{63}{64}z^{-2} - \frac{1}{16}z^{-4},$$
$$H_3^z(z) = -\frac{1}{16} + \frac{63}{64}z^{-2} + \frac{1}{4}z^{-4}, \quad H_4^z(z) = -\frac{1}{64} + \frac{3}{16}z^{-2} + z^{-4}.$$

(a) Compute the zeros of each of the four filters.

(b) Compute and plot the frequency response (magnitude and phase) of each of the four filters. Learn the MATLAB function `unwrap` and use it for the phase plots.

(c) Plot the group delay of each of the four filters.

(d) State your conclusions from this exercise.

8.16 Show that $H^z(z)$ in (8.69) and the filter in Problem 7.29 are mathematically equivalent in the following sense:

(a) If $H^z(z)$ in (8.69) is expanded, its numerator and denominator polynomials will be related as shown in Problem 7.29 (except for possible sign reversal).

(b) If $H^z(z)$ in Problem 7.29 is factored, its poles and zeros will satisfy the relationship shown in (8.69) (except for possible sign reversal).

8.17 Give an example of an analog FIR filter. Hint: Such a filter cannot have a rational transfer function. We encountered an analog FIR filter earlier in this book.

8.18 We are given the filter

$$H^z(z) = \frac{z^{-1} + 0.5}{1 + 0.5z^{-1}}$$

and the two input signals

$$x_1[n] = \sin(0.1\pi n), \quad x_2[n] = 3\cos(0.4\pi n).$$

Let $y_1[n]$, $y_2[n]$ be the output signals corresponding to $x_1[n]$, $x_2[n]$.

(a) What are the amplitudes of $y_1[n]$, $y_2[n]$? Answer with the minimum amount of computations.

(b) What are the phases of $y_1[n]$, $y_2[n]$ relative to those of $x_1[n]$, $x_2[n]$, respectively? Compute to four decimal digits.

(c) Is $H^z(z)$ a distortion-free filter? Explain.

8.19* The purpose of this problem is to derive a continuous-phase representation of an RCSR transfer function $H^z(z)$ from the pole-zero factorization of the transfer function. We saw in Section 7.4.2 that $H^z(z)$ can be written as

$$H^z(z) = b_{q-r}z^{p-q}\frac{\prod_{k=1}^{r}(z - \beta_k)}{\prod_{k=1}^{p}(z - \alpha_k)}. \tag{8.76}$$

(a) Divide the zeros $\{\beta_k,\ 1 \le k \le r\}$ into three sets:

- The first set, $\{\beta_{\mathrm{i},k},\ 1 \le k \le r_{\mathrm{i}}\}$, contains all the zeros inside the unit circle.
- The second set, $\{\beta_{\mathrm{u},k},\ 1 \le k \le r_{\mathrm{u}}\}$, contains all the zeros on the unit circle.
- The third set, $\{\beta_{\mathrm{o},k},\ 1 \le k \le r_{\mathrm{o}}\}$, contains all the zeros outside the unit circle.

Show that $H^z(z)$ can be written as

$$H^z(z) = b_{q-r}\prod_{k=1}^{r_o}(-\beta_{o,k})z^{-q+r_i}$$
$$\cdot\frac{\left[\prod_{k=1}^{r_i}(1-\beta_{i,k}z^{-1})\right]\left[\prod_{k=1}^{r_o}(1-\beta_{o,k}^{-1}z)\right]\left[\prod_{k=1}^{r_u}(z-\beta_{u,k})\right]}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}. \tag{8.77}$$

(b) Show that, since $|\alpha_k| < 1$, the factor $(1-\alpha_k e^{-j\theta})$ of $H^f(\theta)$ has a positive real part, hence its phase is continuous and lies in the range $(-0.5\pi, 0.5\pi)$. Since $|\beta_{i,k}| < 1$, the factor $(1-\beta_{i,k}e^{-j\theta})$ also has this property.

(c) Show that, since $|\beta_{o,k}| > 1$, the factor $(1-\beta_{o,k}^{-1}e^{j\theta})$ of $H^f(\theta)$ has a positive real part, hence its phase is continuous and lies in the range $(-0.5\pi, 0.5\pi)$.

(d) Show that, since $|\beta_{u,k}| = 1$, the factor $(e^{j\theta}-\beta_{u,k})$ of $H^f(\theta)$ can be written as a product of a real (but not necessarily positive) function of θ and a linear-phase factor.

(e) Show that the constant factor in (8.77) is real and that z^{-q+r_i} provides a linear-phase factor.

(f) Show how to combine the results of parts b through e to obtain a continuous-phase representation of $H^f(\theta)$. This provides an alternative proof of Theorem 8.3.

8.20* Discrete-time white noise is given at the input of an FIR filter. What special property does the covariance sequence of the output have? Hint: Recall Problem 2.39.

8.21* Let $\{x[n],\ 0 \le n \le N-1\}$ be a *fixed* real signal, and let $v[n]$ be a discrete-time white noise with zero mean and $\gamma_v = 1$. We feed $x[n] + v[n]$ to an LTI filter having an impulse response $h[n]$. Let $y_1[N-1]$ denote the response of the filter to $x[n]$ and $y_2[N-1]$ the response to $v[n]$, both at time $n = N-1$. Define the output signal-to-noise ratio of the filter as

$$\mathrm{SNR_o} = \frac{|y_1[N-1]|^2}{E(|y_2[N-1]|^2}.$$

We are looking for a filter $h[n]$ that will maximize $\mathrm{SNR_o}$.

(a) Show that $h[n]$ that maximizes $\mathrm{SNR_o}$ is

$$h[n] = cx[N-1-n], \quad 0 \le n \le N-1,$$

where c is an arbitrary nonzero constant. Hint: Use the Cauchy-Schwarz inequality in form (2.145).

(b) What is the output SNR of the optimal filter?

(c) Solve this problem if the signal and the noise are $\{x[n], v[n], -\infty < n \le N-1\}$. In what sense is the filter that you have found different from the one in part a? Hint: Use the Cauchy-Schwarz inequality in form (2.146).

The filter presented in this problem is called the *matched filter* for the signal $x[n]$. It is of great importance in signal detection, for example in radar and sonar applications. We shall encounter it again in Section 14.4.

8.22* This problem introduces single-side-band-modulated signals and filtering of such signals.

(a) The signal

$$x[n] = \cos(\theta_{\mathrm{m}} n) \cdot \cos(\theta_{\mathrm{c}} n), \quad \theta_{\mathrm{m}} \ll \theta_{\mathrm{c}}$$

is passed through an ideal band-pass filter whose band-edge frequencies are θ_{c} and $\theta_{\mathrm{c}} + 2\theta_{\mathrm{m}}$. What is the signal $y[n]$ at the output of the filter?

(b) How will the answer to part a change if the filter is not ideal, but has the specifications

$$\theta_{\mathrm{s},1} = \theta_{\mathrm{c}}, \quad \theta_{\mathrm{p},1} = \theta_{\mathrm{c}} + 0.5\theta_{\mathrm{m}}, \quad \theta_{\mathrm{p},2} = \theta_{\mathrm{c}} + 1.5\theta_{\mathrm{m}}, \quad \theta_{\mathrm{s},2} = \theta_{\mathrm{c}} + 2\theta_{\mathrm{m}},$$

$$A_{\mathrm{s}} = 60\,\mathrm{dB}, \quad A_{\mathrm{p}} = 0.087\,\mathrm{dB}.$$

(c) Repeat for the signal

$$x[n] = s[n] \cos\theta_{\mathrm{c}},$$

where $s[n]$ is a real band-pass signal in the frequency range $[0.5\theta_{\mathrm{m}}, 1.5\theta_{\mathrm{m}}]$.

Signals of this kind are an example of *single-side-band* (SSB) modulation. In communication, they are used for reducing the bandwidth of the modulated signal by half, while preserving the information in the modulating signal $s[n]$.