

**Section 1.2. Sample Space and Events**

1.1 Let  $X$  denote the outcome of the first roll of the die and  $Y$  the outcome of the second roll. Then  $(x, y)$  denotes the event  $\{X = x \text{ and } Y = y\}$ .

- a. Let  $U$  denote the event that the second number is twice the first; that is,  $y = 2x$ . Then  $U$  can be represented by

$$U = \{(1, 2), (2, 4), (3, 6)\}$$

Since there are 36 equally likely sample points in the experiment, the probability of  $U$  is given by

$$P[U] = 3/36 = 1/12$$

- b. Let  $V$  denote the event that the second number is greater than the first. Then  $V$  can be represented by

$$V = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\}$$

Thus, the probability of  $V$  is given by

$$P[V] = 15/36 = 5/12$$

and the probability  $q$  that the second number is not greater than the first is given by

$$q = 1 - P[V] = 7/12$$

- c. Let  $W$  denote the event that at least one number is greater than 3. If we use “na” to denote that an entry is not applicable, then  $W$  can be represented by

$$W = \left\{ \begin{array}{l} na \quad na \quad na \quad (1, 4) \quad (1, 5) \quad (1, 6) \\ na \quad na \quad na \quad (2, 4) \quad (2, 5) \quad (2, 6) \\ na \quad na \quad na \quad (3, 4) \quad (3, 5) \quad (3, 6) \\ (4, 1) \quad (4, 2) \quad (4, 3) \quad (4, 4) \quad (4, 5) \quad (4, 6) \\ (5, 1) \quad (5, 2) \quad (5, 3) \quad (5, 4) \quad (5, 5) \quad (5, 6) \\ (6, 1) \quad (6, 2) \quad (6, 3) \quad (6, 4) \quad (6, 5) \quad (6, 6) \end{array} \right\}$$

Thus, the probability of W is given by

$$P[W] = 27/36 = 3/4$$

1.2 Let  $(a, b)$  denote the event  $\{A = a \text{ and } B = b\}$ .

(a) Let X denote the event that at least one 4 appears. Then X can be represented by

$$X = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)\}$$

Thus, the probability of X is given by

$$P[X] = 11/36$$

(b) Let Y denote the event that just one 4 appears. Then Y can be represented by

$$Y = \{(1, 4), (2, 4), (3, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)\}$$

Thus, the probability of Y is given by

$$P[Y] = 10/36 = 5/18$$

(c) Let Z denote the event that the sum of the face values is 7. Then Z can be represented by

$$Z = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

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Thus, the probability of Z is given by

$$P[Z] = 6/36 = 1/6$$

- (d) Let U denote the event that one of the values is 3 and the sum of the two values is 5. Then U can be represented by

$$U = \{(2, 3), (3, 2)\}$$

Thus, the probability of U is given by

$$P[U] = 2/36 = 1/18$$

- (e) Let V denote that event that one of the values is 3 or the sum of the two values is 5. Let H denote the event that one of the values is 3 and F the event that the sum of the two values is 5. Then H and F can be represented by

$$\begin{aligned} H &= \{(1, 3), (2, 3), (4, 3), (5, 3), (6, 3), (3, 1), (3, 2), (3, 4), (3, 5), (3, 6)\} \\ F &= \{(1, 4), (2, 3), (3, 2), (4, 1)\} \end{aligned}$$

The probabilities of these events are

$$P[H] = 10/36 = 5/18$$

$$P[F] = 4/36 = 1/9$$

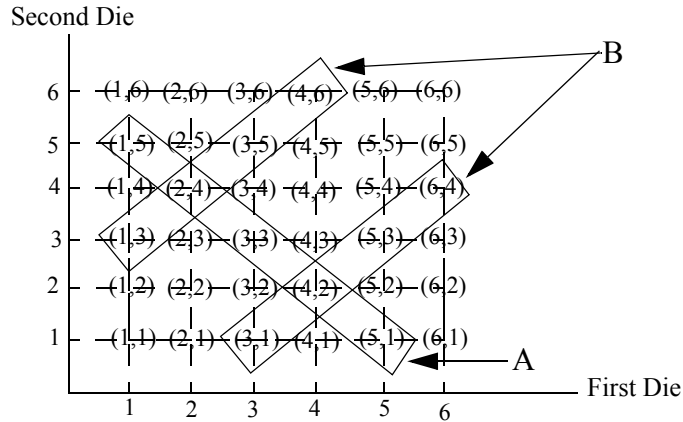
The event V is the union of events H and F; that is,  $V = H \cup F$ . Thus, we have the probability of event V is given by

$$P[V] = P[H \cup F] = P[H] + P[F] - P[H \cap F]$$

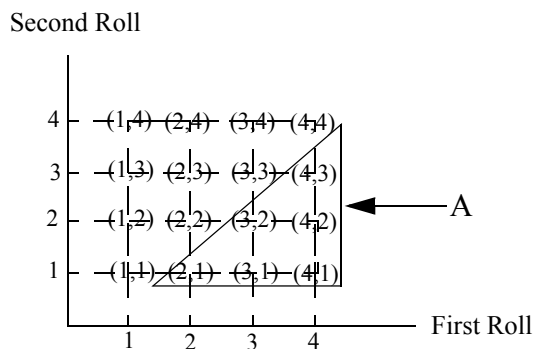
But from earlier results in part (d) we have that  $P[H \cap F] = P[U]$ . Therefore, the probability of event V is given by

$$\begin{aligned}
 P[V] &= P[H] + P[F] - P[H \cap F] = P[H] + P[F] - P[U] \\
 &= \frac{5}{18} + \frac{1}{9} - \frac{1}{18} = \frac{1}{3}
 \end{aligned}$$

- 1.3 We represent the outcomes of the experiment as a two-dimensional diagram where the horizontal axis is the outcome of the first die and the vertical axis is the outcome of the second die. The event A is the event that the sum of the two outcomes is equal to 6, and the event B is the event that the difference between the two outcomes is equal to 2.

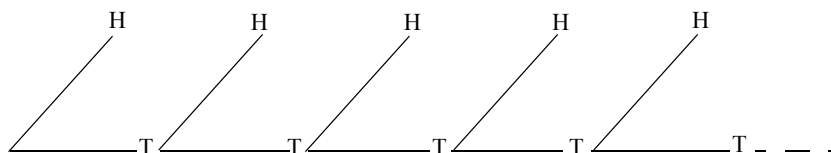


- 1.4 We represent the outcomes of the experiment as a two-dimensional diagram where the horizontal axis is the outcome of the first roll and the vertical axis is the outcome of the second roll. Let A denote the event that the outcome of the first roll is greater than the outcome of the second roll.

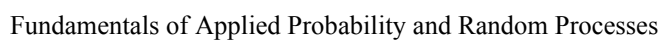


Since all the outcomes are equally likely, the probability that the outcome of the first roll is greater than the outcome of the second roll is  $P[A] = 6/16 = 3/8$ .

- 1.5 The experiment can stop after the first trial if the outcome is a head (H). If the first trial results in a tail (T), we try again and stop if a head appears or continue if a tail appears again, and so on. Thus the sample space of the experiment is as shown below.



- 1.6 The sample space for the experiment is as shown below:



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### Section 1.3. Definitions of Probability

- 1.8 Let  $M$  denote the number of literate males in millions and  $F$  the number of literate females in millions. Then we know that

$$\begin{aligned}M &= 0.75 \times 8.4 = 6.3 \\F &= 0.63 \times 8.6 = 5.418\end{aligned}$$

Thus, the number of literate people in the population is  $M + F = 11.718$ . Therefore, the percentage  $p$  of literate people in total population is

$$p = \frac{11.718}{17} \times 100 = 68.93$$

- 1.9 We are given that  $A$  and  $B$  are two independent events with  $P[A] = 0.4$  and  $P[A \cup B] = 0.7$ . Now,

$$\begin{aligned}P[A \cup B] &= P[A] + P[B] - P[A \cap B] \\&= P[A] + P[B] - P[A]P[B] = P[A] + P[B]\{1 - P[A]\}\end{aligned}$$

where the second equality follows from the fact that  $A$  and  $B$  are independent events. Thus, we have that

$$P[B] = \frac{P[A \cup B] - P[A]}{1 - P[A]} = \frac{0.7 - 0.4}{0.6} = 0.5$$

- 1.10 Recall that  $P[A \cup B]$  denotes the probability that either  $A$  occurs or  $B$  occurs or both events occur. Thus, if  $Z$  is the event that exactly one of the two events occurs, then

$$P[Z] = P[A \cup B] - P[A \cap B] = P[A] + P[B] - 2P[A \cap B]$$

Another way to see this result is by noting that  $P[A - B] = P[A] - P[A \cap B]$  is the probability that only the portion of  $A$  that is disjoint from  $B$  occurs; that is, the points

that are common to both  $A$  and  $B$  do not occur. Similarly,  $P[B - A] = P[B] - P[A \cap B]$  is the probability that only the portion of  $B$  that is disjoint from  $A$  occurs. Thus, since the events  $A - B$  and  $B - A$  are disjoint,

$$P[Z] = P[(A - B) \cup (B - A)] = P[A - B] + P[B - A] = P[A] + P[B] - 2P[A \cap B]$$

Finally we note that the event  $Z$  is given by  $Z = (A \cap \bar{B}) \cup (B \cap \bar{A})$ .

$$P[Z] = P[(A \cap \bar{B}) \cup (B \cap \bar{A})] = P[A \cap \bar{B}] + P[B \cap \bar{A}] = P[A - B] + P[B - A]$$

which yields the same result. Note that the problem specifically requires the answer in terms of  $P[A]$ ,  $P[B]$ , and  $P[A \cap B]$ . We might be tempted to solve the problem in the following manner:

$$P[Z] = P[A]\{1 - P[B]\} + P[B]\{1 - P[A]\} = P[A] + P[B] - 2P[A]P[B]$$

However, this result is correct only if  $A$  and  $B$  are independent events because the implicit assumption made here is that  $P[A \cap \bar{B}] = P[A]P[\bar{B}] = P[A]\{1 - P[B]\}$ , which implies that the events  $A$  and  $\bar{B}$  are independent, which in turn means that the events  $A$  and  $B$  are independent.

1.11 We are given two events  $A$  and  $B$  with  $P[A] = 1/4$ ,  $P[B|A] = 1/2$ , and  $P[A|B] = 1/3$ .

$$P[B|A] = \frac{P[A \cap B]}{P[A]} \Rightarrow P[A \cap B] = P[A]P[B|A] = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = 0.125$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \Rightarrow P[B] = \frac{P[A \cap B]}{P[A|B]} = \frac{1/8}{1/3} = \frac{3}{8} = 0.375$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = \frac{1}{4} + \frac{3}{8} - \frac{1}{8} = \frac{1}{2} = 0.5$$

1.12 We are given two events  $A$  and  $B$  with  $P[A] = 0.6$ ,  $P[B] = 0.7$ , and  $P[A \cap B] = p$ . We know that



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$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \leq 1$$

Thus, we have that

$$0.6 + 0.7 - p = 1.3 - p \leq 1 \Rightarrow p \geq 0.3$$

- 1.13 We are given two events  $A$  and  $B$  with  $P[A] = 0.5$ ,  $P[B] = 0.6$ , and  $P[\bar{A} \cap \bar{B}] = 0.25$ . From the De Morgan's first law we have that

$$P[\bar{A} \cap \bar{B}] = P[\overline{A \cup B}] = 1 - P[A \cup B] = 0.25$$

This means that  $P[A \cup B] = 0.75$ . But  $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ . Thus,

$$P[A \cap B] = P[A] + P[B] - P[A \cup B] = 0.5 + 0.6 - 0.75 = 0.35$$

- 1.14 We are given two events  $A$  and  $B$  with  $P[A] = 0.4$ ,  $P[B] = 0.5$ , and  $P[A \cap B] = 0.3$ .

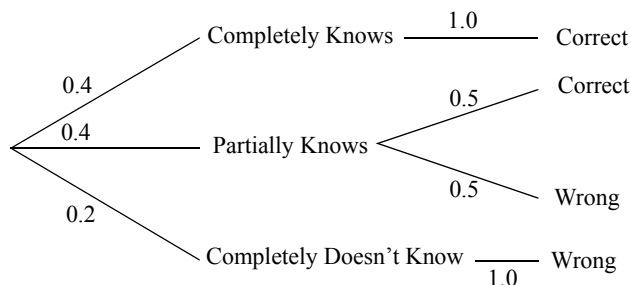
- a.  $P[A \cup B] = P[A] + P[B] - P[A \cap B] = 0.4 + 0.5 - 0.3 = 0.6$
- b.  $P[A \cap \bar{B}] = P[A] - P[A \cap B] = 0.4 - 0.3 = 0.1$
- c. From the De Morgan's second law,  $P[\bar{A} \cup \bar{B}] = P[\overline{A \cap B}] = 1 - P[A \cap B] = 0.7$ .

- 1.15 We use the tree diagram to solve the problem. First, we note that there are two cases to consider in this problem:

1. Christie does not answer questions she knows nothing about
2. She answers all questions, resorting to guesswork on those she knows nothing about

Under case 1, we assume that after Christie has narrowed down the choices to two answers, she flips a coin to guess the answer. That is, given that she can narrow the choices down to two answers, the probability of getting the correct answer is 0.5. Thus,

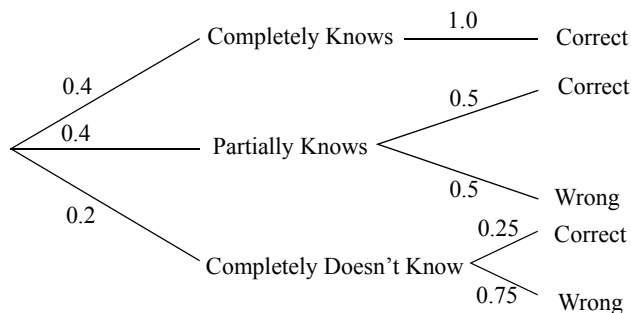
the tree diagram for this case is as follows.



Thus, the probability  $p$  that she will correctly answer a question chosen at random from the test is given by

$$p = 0.4 \times 1.0 + 0.4 \times 0.5 = 0.6$$

Under case 2, she adopts the same strategy above for those questions that she can narrow down the answer to two questions; that is, the final answer is based on flipping a fair coin. For those she knows nothing about, she is equally likely to choose any one of the four answers. Thus, the tree diagram for this case is as follows:



Thus, the probability  $p$  that she will correctly answer a question chosen at random from the test is given by

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$$p = 0.4 \times 1.0 + 0.4 \times 0.5 + 0.2 \times 0.25 = 0.65$$

1.16 We are given a box that contains 9 red balls, 6 white balls, and 5 blue balls from which 3 balls are drawn successively. The total number of balls is 20.

- a. If the balls are drawn in the order red, white, and blue, and each ball is replaced after it has been drawn, the probability of getting a ball of a particular color in any drawing does not change due to the replacement policy. Therefore, the probability of drawing a red ball is  $9/20$ , the probability of drawing a white ball is  $6/20$ , and the probability of drawing a blue ball is  $5/20$ . Since these probabilities are independent of each other, the probability  $p$  of this policy is

$$p = P[R] \times P[W] \times P[B] = \frac{9}{20} \times \frac{6}{20} \times \frac{5}{20} = \frac{27}{800} = 0.03375$$

- b. If the balls are drawn without replacement, then the probability  $q$  that they are drawn in the order red, white, and blue is given

$$q = P[R] \times P[W|R] \times P[B|RW] = \frac{9}{20} \times \frac{6}{19} \times \frac{5}{18} = \frac{3}{76} = 0.0395$$

1.17 Given that  $A$  is the set of positive even integers,  $B$  the set of positive integers that are divisible by 3, and  $C$  the set of positive odd integers. Then the following events can be described as follows:

- a.  $E_1 = A \cup B$  is the set of positive integers that are either even integers or integers that are divisible by 3.
- b.  $E_2 = A \cap B$  is the set of positive integers that are both even integers and integers that are divisible by 3
- c.  $E_3 = A \cap C$  is the set of positive integers that are both even integers and odd integers, which is the null set, since no integer can be both even and at the same time.

- d.  $E_4 = (A \cup B) \cap C$  is the set of positive integers that are odd integers and are either even integers or divisible by 3. (This reduces to the set of odd integers that are divisible by 3.)
- e.  $E_5 = A \cup (B \cap C)$  is the set of positive integers that are even or are both divisible by 3 and are odd.

1.18 Given a box that contains 4 red balls labeled  $R_1, R_2, R_3$ , and  $R_4$ ; and 3 white balls labeled  $W_1, W_2$ , and  $W_3$ . If a ball is randomly drawn from the box, then

- a.  $E_1$ , the event that the number on the ball (i.e., the subscript of the ball) is even is given by  $E_1 = \{R_2, R_4, W_2\}$
- b.  $E_2$ , the event that the color of the ball is red and its number is greater than 2 is given by  $E_2 = \{R_3, R_4\}$
- c.  $E_3$ , the event that the number on the ball is less than 3 is given by  $E_3 = \{R_1, R_2, W_1, W_2\}$
- d.  $E_4 = E_1 \cup E_3 = \{R_1, R_2, R_4, W_1, W_2\}$
- e.  $E_5 = E_1 \cup (E_2 \cap E_3) = E_1 = \{R_2, R_4, W_2\}$ , since  $E_2 \cap E_3 = \emptyset$

1.19 Given a box that contains 50 computer chips of which 8 are known to be bad. A chip is selected at random and tested.

- (a) The probability that the selected chip is bad is  $8/50 = 4/25 = 0.16$ .
- (b) Let  $X$  be the event that the first chip is bad and  $Y$  the event that the second chip is bad. If the tested chip is not put back into the box, then there are 49 chips left after the first chip has been removed and 7 of them are bad. Thus,

$$P[Y|X] = 7/49 = 1/7 = 0.1428$$

- (c) If the first chip tests good and a tested chip is not put back into the box, the probability that a second chip selected at random is bad is

$$P[Y|\bar{X}] = 8/49 = 0.1633$$

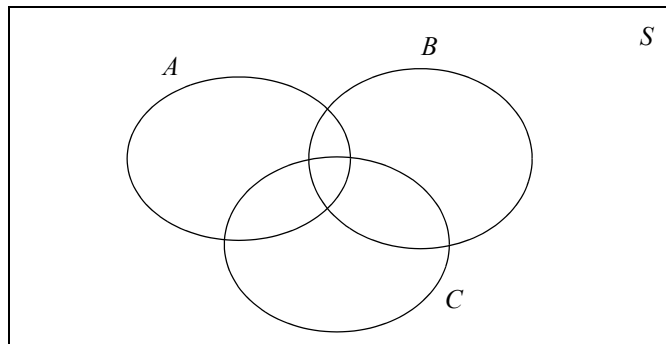
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## Section 1.5. Elementary Set Theory

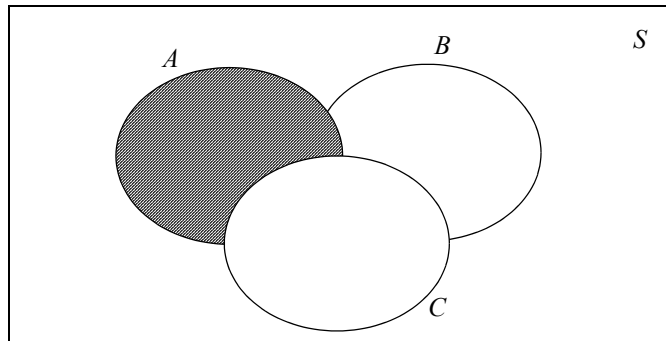
1.20  $S = \{A, B, C, D\}$ . Thus, all the possible subsets of  $S$  are as follows:

$$\begin{aligned} &\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\} \\ &\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{A, B, C, D\} \end{aligned}$$

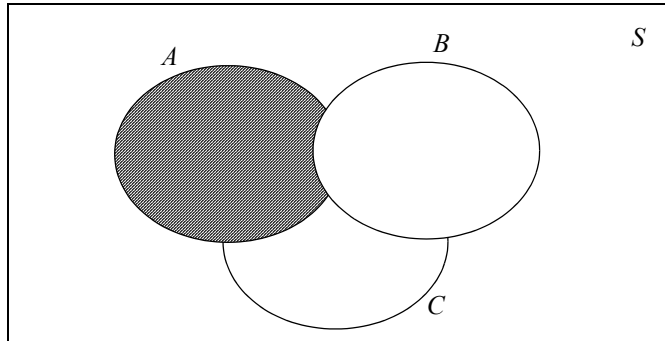
1.21 Let  $S$  denote the universal set, and assume that the three sets  $A$ ,  $B$ , and  $C$  have intersections as shown below.



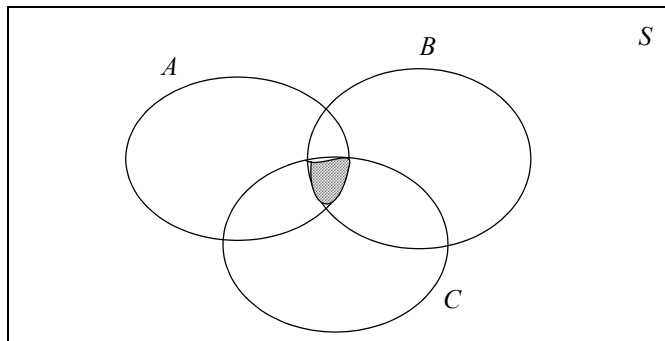
(a)  $(A \cup C) - C$ : This is the shaded area in the figure below.



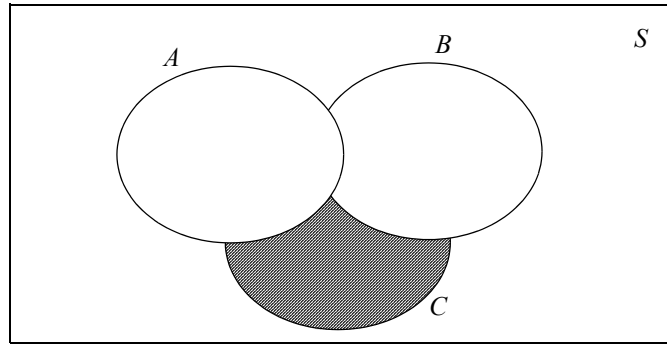
(b)  $\bar{B} \cap A$ : This is the shaded area in the figure below.



(c)  $A \cap B \cap C$ : This is the shaded area in the figure below.



(d)  $(\overline{A \cup B}) \cap C$ : This is the shaded area in the figure below.



1.22 Given:  $S = \{2, 4, 6, 8, 10, 12, 14\}$ ,  $A = \{2, 4, 8\}$ , and  $B = \{4, 6, 8, 12\}$ .

(a)  $\bar{A} = S - A = \{6, 10, 12, 14\}$

(b)  $B - A = \{6, 12\}$

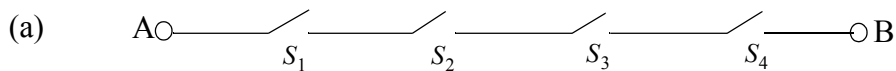
(c)  $A \cup B = \{2, 4, 6, 8, 12\}$

(d)  $A \cap B = \{4, 8\}$

(e)  $\bar{A} \cap B = \{6, 12\} = B - A$

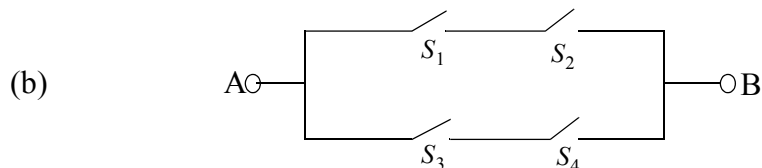
(f)  $(A \cap B) \cup (\bar{A} \cap B) = \{4, 8\} \cup \{6, 12\} = \{4, 6, 8, 12\}$

1.23  $E_k$  denotes the event that switch  $S_k$  is closed,  $k = 1, 2, 3, 4$ , and  $E_{AB}$  denotes the event that there is a closed path between nodes A and B. Then  $E_{AB}$  is given as follows:



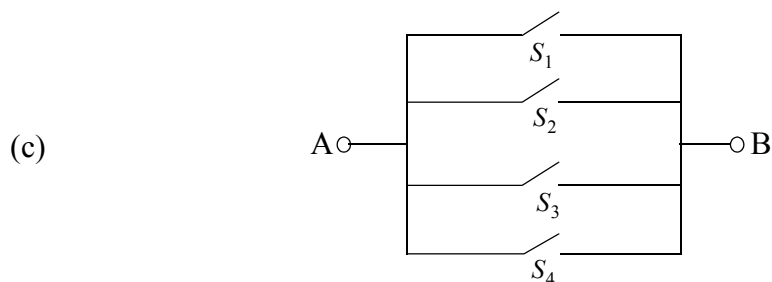
This is a serial system that requires all switches to be closed for the path to exist. Thus,

$$E_{AB} = E_1 \cap E_2 \cap E_3 \cap E_4$$



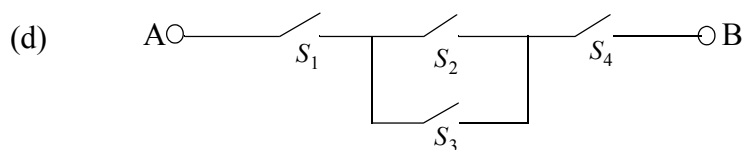
This is a combined serial-parallel system that requires that either all the switches in one serial path be closed, or those in the other serial path be closed, or all the switches in both paths be closed. Thus,

$$E_{AB} = (E_1 \cap E_2) \cup (E_3 \cap E_4)$$



This is a pure parallel system that requires at least one switch to be closed. Thus,

$$E_{AB} = E_1 \cup E_2 \cup E_3 \cup E_4$$





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This is another serial-parallel system that requires switches  $S_1$ ,  $S_4$ , and either switch  $S_2$  or switch  $S_3$  or both to be closed. Thus,

$$E_{AB} = E_1 \cap (E_2 \cup E_3) \cap E_4 = (E_1 \cap E_2 \cap E_4) \cup (E_1 \cap E_3 \cap E_4)$$

1.24 A, B, and C are three given events.

- a. The event “A occurs but neither B nor C occurs” is defined by  $A \cap \bar{B} \cap \bar{C} = A \cap (\overline{B \cup C}) = A - (\bar{B} \cup \bar{C})$ .
- b. The event “A and B occur, but not C” is defined by

$$A \cap B \cap \bar{C} = (A \cap B) - C = A \cap \{\overline{B \cup C}\} = A - (B \cup C)$$

- c. The event “A or B occurs, but not C” is defined by  $(A \cup B) \cap \bar{C} = (A \cup B) - C$ .
- d. The event “Either A occurs and not B, or B occurs and not A” is defined by  $(A \cap \bar{B}) \cup (\bar{A} \cap B)$ .

## Section 1.6. Properties of Probability

1.25 Let  $P_M$  denote the probability that Mark attends class on a given day and  $P_L$  the probability that Lisa attends class on a given day. Then we know that  $P_M = 0.65$  and  $P_L = 0.75$ . Let  $M$  denote the event that Mark attends class and  $L$  the event that Lisa attends class.

- (a) The probability  $p$  that at least one of them is in class is the complement of the probability that neither of them is in class and is given by

$$p = 1 - P[\bar{M} \cap \bar{L}] = 1 - P[\bar{M}]P[\bar{L}] = 1 - (0.35)(0.25) = 0.9125$$

where the second equality is due to the fact that the two events are independent.

- (b) The probability that exactly one of them is in class is the probability that either Mark is in class and Lisa is not or Lisa is in class and Mark is not. This is given by

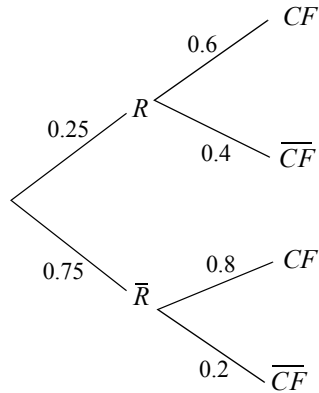
$$\begin{aligned} q &= P[(M \cap \bar{L}) \cup (L \cap \bar{M})] = P[M \cap \bar{L}] + P[L \cap \bar{M}] = P[M]P[\bar{L}] + P[L]P[\bar{M}] \\ &= (0.65)(0.25) + (0.75)(0.35) = 0.425 \end{aligned}$$

where the second equality is due to the fact that the events are mutually exclusive and the third equality is due to the independence of the events.

- (c) Let  $X$  denote the event that exactly one of them is in class. Then the probability that Mark is in class, given that only one of them is in class is given by

$$P[M|X] = \frac{P[M \cap X]}{P[X]} = \frac{P[M \cap \bar{L}]}{P[M \cap \bar{L}] + P[L \cap \bar{M}]} = \frac{(0.65)(0.25)}{(0.65)(0.25) + (0.75)(0.35)} = 0.3823$$

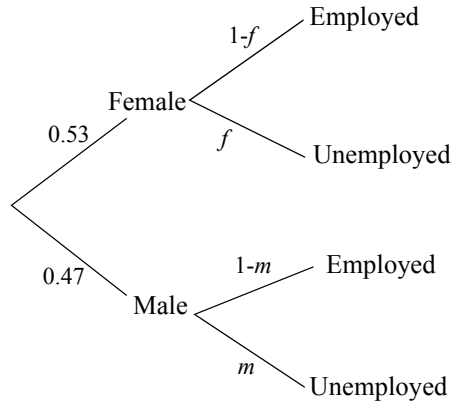
- 1.26 Let  $R$  denote the event that it rains and  $CF$  the event that the forecast is correct. We can use the following diagram to solve the problem:



The probability that the forecast on a day selected at random is correct is given by

$$\begin{aligned} P[CF] &= P[CF|R]P[R] + P[CF|\bar{R}]P[\bar{R}] \\ &= (0.6) \times (0.25) + (0.8) \times (0.75) = 0.75 \end{aligned}$$

- 1.27 Let  $m$  denote the fraction of adult males that are unemployed and  $f$  the fraction of adult females that are unemployed. We use the following tree diagram to solve the problem.



- (a) The probability that an adult chosen at random in this city is an unemployed male can be obtained from the root of the tree to the lower segment of the lower branch as  $0.47m$ . If we equate  $0.47m = 0.15$ , we obtain  $m = 0.15/0.47 = 0.32$ . Thus, the probability that a randomly chosen adult is an employed male is  $0.47 \times (1 - 0.32) = 0.32$ .
- (b) If the overall unemployment rate in the city is 22%, the probability that any adult in the city is unemployed is given by  $0.47m + 0.53f = 0.22$ . From this we obtain

$$f = (0.22 - 0.47m)/0.53 = (0.22 - 0.47 \times 0.32)/0.53 = 0.1313$$

Thus, the probability that a randomly selected adult is an employed female is  $0.53(1 - f) = 0.4604$ .

- 1.28 If three companies are randomly selected from the 100 companies without replacement, the probability that each of the three has installed WLAN is given by

$$\frac{75}{100} \times \frac{74}{99} \times \frac{73}{98} = 0.417$$

### Section 1.7. Conditional Probability

1.29 Let  $A$  denote the event that a randomly selected car is produced in factory A, and  $B$  the event that it is produced in factory B. Let  $D$  denote the event that a randomly selected car is defective. Now we are given the following:

$$P[A] = 100000/(100000 + 50000) = 2/3$$

$$P[B] = 50000/(100000 + 50000) = 1/3$$

$$P[D|A] = 0.10$$

$$P[D|B] = 0.05$$

(a) The probability of purchasing a defective car from the manufacturer is given by

$$P[D] = P[D|A]P[A] + P[D|B]P[B] = (0.1)(2/3) + (0.05)(1/3) = 0.25/3 = 0.083$$

(b) Given that a car purchased from the manufacturer is defective, the probability that it came from factory A is given by

$$\begin{aligned} P[A|D] &= \frac{P[A \cap D]}{P[D]} = \frac{P[D|A]P[A]}{P[D]} = \frac{(2/3)(0.10)}{(0.25/3)} \\ &= 0.20/0.25 = 0.80 \end{aligned}$$

1.30 Let  $X$  denote the event that there is at least one 6, and  $Y$  the event that the sum is at least 9. Then  $X$  can be represented as follows.

$$X = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 5), (6, 4), (6, 3), (6, 2), (6, 1)\}$$

Thus,  $P[X] = 11/36$  and the probability that the sum is at least 9 given that there is at least one 6 is given by

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$$P[Y|X] = \frac{P[Y \cap X]}{P[X]}$$

But the event  $Y \cap X$  can be represented by

$$Y \cap X = \{(3, 6), (4, 6), (5, 6), (6, 6), (6, 5), (6, 4), (6, 3)\}$$

Thus,  $P[Y \cap X] = 7/36$ , and we have that

$$P[Y|X] = \frac{7/36}{11/36} = \frac{7}{11} = 0.6364$$

- 1.31 Let  $F$  denote the event that Chuck is a fool and  $T$  the event that he is a thief. Then we know that  $P[F] = 0.6$ ,  $P[T] = 0.7$ , and  $P[\bar{F} \cap \bar{T}] = 0.25$ . From the De Morgan's first law,

$$P[\bar{F} \cap \bar{T}] = P[\overline{F \cup T}] = 1 - P[F \cup T] = 0.25$$

Thus,  $P[F \cup T] = 0.75$ . But  $P[F \cup T] = P[F] + P[T] - P[F \cap T] = 0.6 + 0.7 - P[F \cap T] = 0.75$ . From this we obtain

$$P[F \cap T] = 0.55$$

- (a) Now,  $P[F \cup T]$  is the probability that he is either a fool or a thief or both. Therefore, the probability that he is a fool or a thief but not both is given by

$$P[F \cup T] - P[F \cap T] = 0.20$$

Note that we can obtain the same result by noting that the probability that he is either a fool or a thief but both is the probability of the union of the events that he is a fool and not a thief, and he is a thief and not a fool. That is, the required result is

$$P[(F \cap \bar{T}) \cup (\bar{F} \cap T)] = P[F \cap \bar{T}] + P[\bar{F} \cap T]$$

where the equality is due to the fact that the events are mutually exclusive. But

$$P[F] = P[F \cap T] + P[F \cap \bar{T}] \Rightarrow P[F \cap \bar{T}] = P[F] - P[F \cap T] = 0.05$$

$$P[T] = P[F \cap T] + P[\bar{F} \cap T] \Rightarrow P[\bar{F} \cap T] = P[T] - P[F \cap T] = 0.15$$

Adding the two probabilities we obtain the previous result.

(b) The probability that he is a thief, given that he is not a fool is given by

$$P[T|\bar{F}] = \frac{P[\bar{F} \cap T]}{P[\bar{F}]} = \frac{0.15}{0.40} = \frac{3}{8} = 0.375$$

1.32 Let  $M$  denote the event that a married man votes and  $W$  the event that a married woman votes. Then we know that

$$P[M] = 0.45$$

$$P[W] = 0.40$$

$$P[W|M] = 0.60$$

(a) The probability that both a man and his wife vote is  $P[M \cap W]$ , which is given by

$$P[M \cap W] = P[W|M]P[M] = (0.60)(0.45) = 0.27$$

(b) The probability that a man votes given that his wife votes is given by

$$P[M|W] = \frac{P[M \cap W]}{P[W]} = \frac{0.27}{0.40} = 0.675$$

- 1.33 Let  $L$  denote the event that the plane is late and  $R$  the event that the forecast calls for rain. Then we know that

$$P[L|R] = 0.80$$

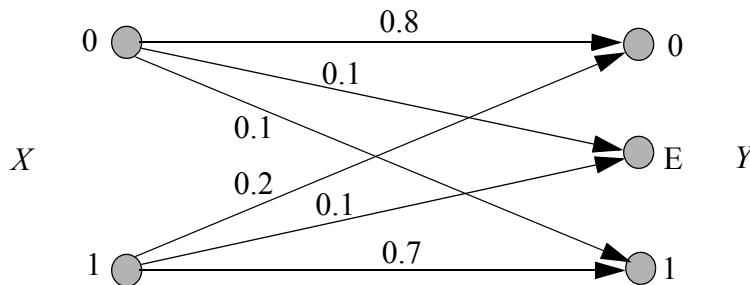
$$P[L|\bar{R}] = 0.30$$

$$P[R] = 0.40$$

Therefore, the probability that the plane will be late is given by

$$\begin{aligned} P[L] &= P[L|R]P[R] + P[L|\bar{R}]P[\bar{R}] = (0.8)(0.4) + (0.3)(0.6) \\ &= 0.50 \end{aligned}$$

- 1.34 We are given the following communication channel with the input symbol set  $X \in \{0, 1\}$  and the output symbol set  $Y \in \{0, 1, E\}$  as well as the transition (or conditional) probabilities defined by  $p_{Y|X}$  that are indicated on the directed links. Also,  $P[X = 0] = P[X = 1] = 0.5$ .



- (a) The probabilities that 0, 1, and  $E$  are received are given, respectively, by

$$\begin{aligned}
 P[Y = 0] &= P[Y = 0|X = 0]P[X = 0] + P[Y = 0|X = 1]P[X = 1] \\
 &= (0.8)(0.5) + (0.2)(0.5) = 0.5 \\
 P[Y = 1] &= P[Y = 1|X = 0]P[X = 0] + P[Y = 1|X = 1]P[X = 1] \\
 &= (0.1)(0.5) + (0.7)(0.5) = 0.4 \\
 P[Y = E] &= P[Y = E|X = 0]P[X = 0] + P[Y = E|X = 1]P[X = 1] \\
 &= (0.1)(0.5) + (0.1)(0.5) = 0.1
 \end{aligned}$$

(b) Given that 0 is received, the probability that 0 was transmitted is given by

$$\begin{aligned}
 P[X = 0|Y = 0] &= \frac{P[(X = 0) \cap (Y = 0)]}{P[Y = 0]} = \frac{P[Y = 0|X = 0]P[X = 0]}{P[Y = 0|X = 0]P[X = 0] + P[Y = 0|X = 1]P[X = 1]} \\
 &= \frac{(0.8)(0.5)}{0.5} = 0.8
 \end{aligned}$$

(c) Given that  $E$  is received, the probability that 1 was transmitted is given by

$$\begin{aligned}
 P[X = 1|Y = E] &= \frac{P[(X = 1) \cap (Y = E)]}{P[Y = E]} = \frac{P[Y = E|X = 1]P[X = 1]}{P[Y = E|X = 0]P[X = 0] + P[Y = E|X = 1]P[X = 1]} \\
 &= \frac{(0.1)(0.5)}{0.5} = 0.5
 \end{aligned}$$

(d) Given that 1 is received, the probability that 1 was transmitted is given by

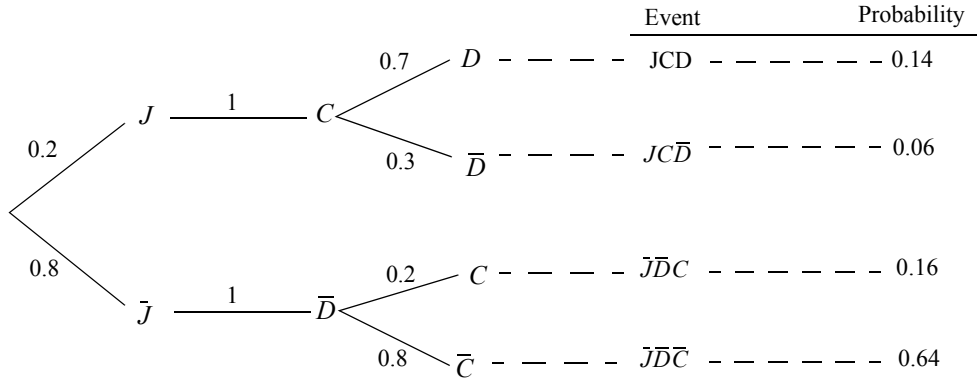
$$\begin{aligned}
 P[X = 1|Y = 1] &= \frac{P[(X = 1) \cap (Y = 1)]}{P[Y = 1]} = \frac{P[Y = 1|X = 1]P[X = 1]}{P[Y = 1|X = 0]P[X = 0] + P[Y = 1|X = 1]P[X = 1]} \\
 &= \frac{(0.7)(0.5)}{0.4} = 0.875
 \end{aligned}$$

1.35 Let  $M$  denote the event that a student is a man,  $W$  the event that a student is a woman, and  $F$  the event that a student is a foreign student. We are given that  $P[M] = 0.6$ ,  $P[W] = 0.4$ ,  $P[F|M] = 0.3$ , and  $P[F|W] = 0.2$ . The probability that a randomly selected student who is found to be a foreign student is a woman is given by



$$\begin{aligned}
 P[W|F] &= \frac{P[W \cap F]}{P[F]} = \frac{P[F|W]P[W]}{P[F|W]P[W] + P[F|M]P[M]} = \frac{(0.2)(0.4)}{(0.2)(0.4) + (0.3)(0.6)} \\
 &= \frac{4}{13} = 0.3077
 \end{aligned}$$

- 1.36 Let  $J$  denote the event that Joe is innocent,  $C$  the event that Chris testifies that Joe is innocent,  $D$  the event that Dana testifies that Joe is innocent, and  $X$  the event that Chris and Dana give conflicting testimonies. We draw the following tree diagram that describes the process:



- a. The probability that Chris and Dana give conflicting testimonies is given by

$$P[X] = P[JC\bar{D} \cup \bar{J}\bar{D}C] = P[JC\bar{D}] + P[\bar{J}\bar{D}C] = 0.06 + 0.16 = 0.22$$

- b. The probability that Joe is guilty, given that Chris and Dana give conflicting testimonies is given by

$$P[\bar{J}|X] = \frac{P[\bar{J} \cap X]}{P[X]} = \frac{P[\bar{J}\bar{D}C]}{P[X]} = \frac{0.16}{0.22} = \frac{8}{11} = 0.7273$$

1.37 Let  $A$  denote the event that a car is a brand A car,  $B$  the event that it is a brand B car,  $C$  the event that it is a brand C car, and  $R$  the event that it needs a major repair during the first year of purchase. We know that

$$\begin{aligned} P[A] &= 0.2 \\ P[B] &= 0.3 \\ P[C] &= 0.5 \\ P[R|A] &= 0.05 \\ P[R|B] &= 0.10 \\ P[R|C] &= 0.15 \end{aligned}$$

- a. The probability that a randomly selected car in the city needs a major repair during its first year of purchase is given by

$$P[R] = P[R|A]P[A] + P[R|B]P[B] + P[R|C]P[C] = (0.05)(0.2) + (0.10)(0.3) + (0.15)(0.5) = 0.115$$

- b. Given that a car in the city needs a major repair during its first year of purchase, the probability that it is a brand A car is given by

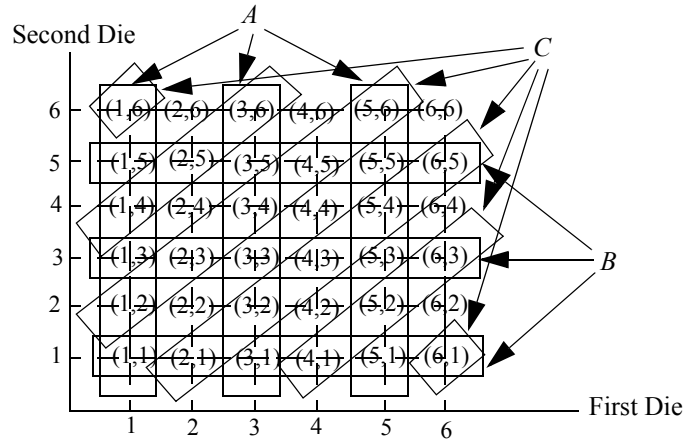
$$P[A|R] = \frac{P[A \cap R]}{P[R]} = \frac{P[R|A]P[A]}{P[R]} = \frac{(0.05)(0.2)}{0.115} = \frac{2}{23} = 0.0870$$

## Section 1.8. Independent Events

1.38 The sample space of the experiment is  $S = \{HH, HT, TH, TT\}$ . Let  $A$  denote the event that at least one coin resulted in heads and  $B$  the event that the first coin came up heads. Then  $A = \{HH, HT, TH\}$  and  $B = \{HH, HT\}$ . The probability that the first coin came up heads, given that there is at least one head is given by

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[B]}{P[A]} = \frac{1/2}{3/4} = \frac{2}{3} = 0.667$$

- 1.39 The defined events are  $A = \{\text{The first die is odd}\}$ ,  $B = \{\text{The second die is odd}\}$ , and  $C = \{\text{The sum is odd}\}$ . The sample space for the experiment is as follows:



To show that these events are pairwise independent, we proceed as follows:

$$P[A] = \frac{18}{36} = \frac{1}{2}$$

$$P[B] = \frac{18}{36} = \frac{1}{2}$$

$$P[C] = \frac{18}{36} = \frac{1}{2}$$

$$P[A \cap B] = \frac{9}{39} = \frac{1}{4} = P[A]P[B]$$

$$P[A \cap C] = \frac{9}{39} = \frac{1}{4} = P[A]P[C]$$

$$P[B \cap C] = \frac{9}{39} = \frac{1}{4} = P[B]P[C]$$

Since  $P[A \cap B] = P[A]P[B]$ , we conclude that events  $A$  and  $B$  are independent. Similarly, since  $P[A \cap C] = P[A]P[C]$ , we conclude that events  $A$  and  $C$  are independent. Finally, since  $P[B \cap C] = P[B]P[C]$ , we conclude that events  $B$  and  $C$  are independent. Thus, the

events are pairwise independent. Now,

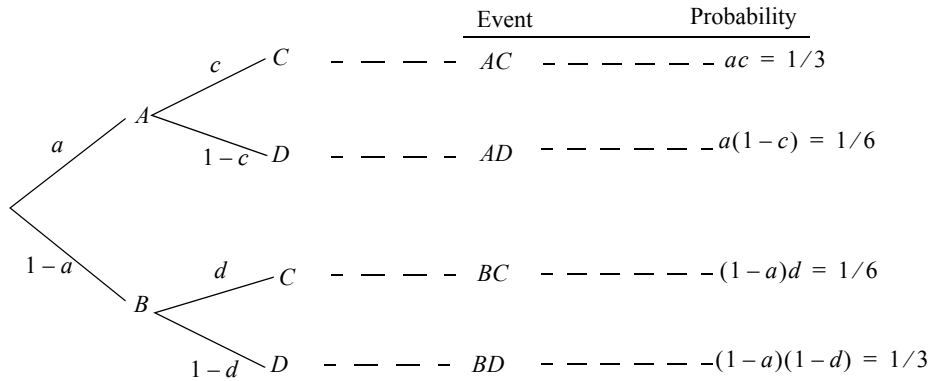
$$P[A \cap B \cap C] = P[\emptyset] = 0 \neq P[A]P[B]P[C]$$

Therefore, we conclude that the three events are not independent.

- 1.40 We are given a game that consists of two successive trials in which the first trial has outcome A or B and the second trial has outcome C or D. The probabilities of the four possible outcomes of the game are as follows:

Outcome	AC	AD	BC	BD
Probability	1/3	1/6	1/6	1/3

Let  $P[A] = a$ ,  $P[C|A] = c$ , and  $P[C|B] = d$ . Then we can represent the outcome of the experiment by the following tree diagram:



Now,

$$a(1-c) = \frac{1}{6} = a - ac = a - \frac{1}{3} \Rightarrow a = \frac{1}{2}$$

$$c = \frac{1/3}{1/2} = \frac{2}{3}$$

$$(1-a)d = \frac{1}{6} \Rightarrow d = \frac{1/6}{1/2} = \frac{1}{3}$$

---

If A and C are statistically independent, then the outcome of the second trial should be independent of the outcome of the first trial, and we should have that  $P[C|A] = P[C|B]$ ; that is, we should have that  $c = d$ . Since this is not the case here, that is, since  $P[C|A] \neq P[C|B]$ , we conclude that A and C are not independent.

1.41 Since the two events A and B are mutually exclusive, we have that

$$P[A \cap B] = P[A|B]P[B] = 0$$

Since  $P[B] > 0$ , we have that  $P[A|B] = 0$ . For A and B to be independent, we must have that  $P[A|B] = P[A]P[B]$ . Since  $P[B] > 0$ , we must have that  $P[A] = 0$ .

### Section 1.10. Combinatorial Analysis

1.42 We are given 4 married couples who bought 8 seats in a row for a football game.

- a. Since there are 8 different people, there are  $8! = 40320$  different ways that they can be seated.
- b. Since there are 4 couples and each couple can be arranged in only one way, there are  $4! = 24$  different ways that they can be seated if each couple is to sit together with the husband to the left of his wife.
- c. In this case there is no restriction on how each couple can sit next to each other: the husband can sit to the right or to the left of his wife. Thus, there are  $2!$  ways of arranging each couple. Also for each sitting arrangement of a couple, there are  $4! = 24$  ways of arranging all the couples. Therefore, the number of ways that they be seated if each couple is to sit together is given by

$$4! \times (2!)^4 = 24 \times 16 = 384$$

- d. If all the men are to sit together and all the women are to sit together, then there are 2 groups of people that can be arranged in  $2!$  ways. Within each group, there are  $4!$  ways of arranging the members. Therefore, the number of arrangements is given by

$$2! \times (4!)^2 = 2 \times 24 \times 24 = 1152$$

1.43 We have that a committee consisting of 3 electrical engineers and 3 mechanical engineers is to be formed from a group of 7 electrical engineers and 5 mechanical engineers.

- a. If any electrical engineer and any mechanical engineer can be included, the number of committees that can be formed is

$$\binom{7}{3} \times \binom{5}{3} = \frac{7!}{4!3!} \times \frac{5!}{2!3!} = 350$$

- b. If one particular electrical engineer must be on the committee, then we need to select 2 more electrical engineers on a committee. Thus, the number of committees that can be formed is

$$\binom{6}{2} \times \binom{5}{3} = \frac{6!}{4!2!} \times \frac{5!}{2!3!} = 150$$

- c. If two particular mechanical engineers cannot be on the same committee, then we consider the number of committees that include both of them, which are committees where we need to choose one mechanical engineer from the remaining 3. The number of such committees is

$$\binom{7}{3} \times \binom{3}{1} = \frac{7!}{4!3!} \times \frac{3!}{2!1!} = 105$$

Thus, the number of committees that do not include both mechanical engineers together is given by

$$\left\{ \binom{7}{3} \times \binom{5}{3} \right\} - \left\{ \binom{7}{3} \times \binom{3}{1} \right\} = 350 - 105 = 245$$

1.44 The Stirling's formula is given by  $n! = \sqrt{2\pi n} \times n^n \times e^{-n}$ .

Thus,

$$\begin{aligned} 200! &= \sqrt{400\pi} \times 200^{200} \times e^{-200} = 20\sqrt{\pi} \times 200^{200} \times e^{-200} \\ \log(200!) &= \log(20) + \frac{1}{2}\log(\pi) + 200\log(200) - 200\log(e) \\ &= 1.30103 + 0.24857 + 460.20600 - 86.85890 = 374.8967 \end{aligned}$$

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Taking the antilog we obtain

$$200! = 7.88315 \times 10^{374}$$

1.45 The number of different committees that can be formed is

$$\binom{7}{1} \times \binom{4}{1} \times \binom{5}{1} = 7 \times 4 \times 5 = 140$$

1.46 The number of ways of indiscriminately choosing two from the 50 states is

$$\binom{100}{2} = \frac{100!}{2!98!} = \frac{100 \times 99}{2} = 4950$$

(a) The probability that 2 senators chosen at random are from the same state is

$$\frac{\binom{50}{1} \times \binom{2}{2} \times \left\{ \binom{2}{0} \right\}^{49}}{\binom{100}{2}} = \frac{50}{4950} = \frac{1}{99} = 0.0101$$

where the first combination term is the number of ways of choosing the state where the 2 senators come from.

(b) The probability of randomly choosing 10 senators who are all from different states can be obtained as follows. There are  $C(50, 10)$  ways of choosing the 10 states where these 10 senators come from, and for each of these states there are  $C(2, 1)$  ways of choosing 1 senator out of the 2 senators. For the remaining 40 states, no senators are chosen. Therefore, the probability of this event is given by

$$\frac{\binom{50}{10} \times \left\{ \binom{2}{1} \right\}^{10} \times \left\{ \binom{2}{0} \right\}^{40}}{\binom{100}{10}} = \frac{50! \times 2^{10} \times 90! \times 10!}{100! \times 10! \times 40!} = 0.60766$$

1.47 We are required to form a committee of 7 people from 10 men and 12 women.

(a) The probability that the committee will consist of 3 men and 4 women is given by

$$\frac{\binom{10}{3} \times \binom{12}{4}}{\binom{22}{7}} = \frac{10! \times 12! \times 7! \times 15!}{22! \times 3! \times 7! \times 4! \times 8!} = 0.3483$$

(b) The probability that the committee will consist of all men is given by

$$\frac{\binom{10}{7} \times \binom{12}{0}}{\binom{22}{7}} = \frac{\binom{10}{7}}{\binom{22}{7}} = \frac{10! \times 7! \times 15!}{22! \times 3! \times 7!} = 0.0007$$

1.48 Five departments labeled A, B, C, D, and E, send 3 delegates each to the college's convention for a total of 15 delegates. A committee of 4 delegates is formed.

(a) The probability that department A is not represented on the committee is the probability of choosing 0 delegates from A and 4 from the other 12 delegates. This is given by

$$\frac{\binom{3}{0} \times \binom{12}{4}}{\binom{15}{4}} = \frac{\binom{12}{4}}{\binom{15}{4}} = \frac{12! \times 11! \times 4!}{15! \times 4! \times 8!} = \frac{33}{91} = 0.36264$$

(b) The probability that department A has exactly one representative on the committee is the probability that 1 delegate is chosen out of the 3 from department A and 3 delegates are chosen out of the 12 delegates from the other departments. This is given by

$$\frac{\binom{3}{1} \times \binom{12}{3}}{\binom{15}{4}} = \frac{3 \times \binom{12}{3}}{\binom{15}{4}} = \frac{3 \times 12! \times 11! \times 4!}{15! \times 3! \times 9!} = \frac{44}{91} = 0.48352$$

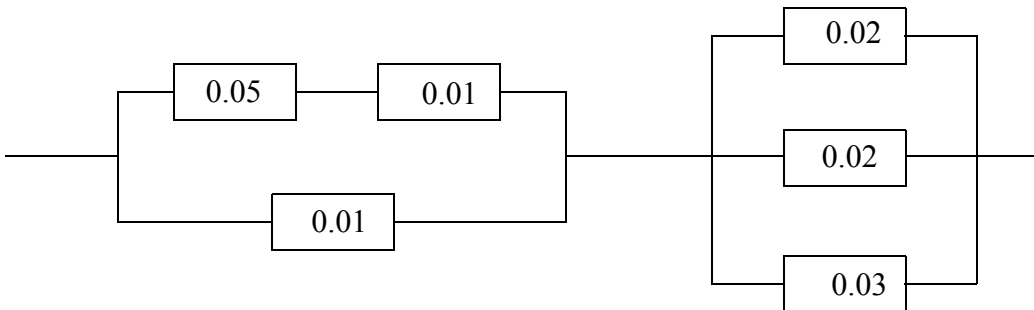


- (c) The probability that neither department A nor department C is represented on the committee is the probability that all chosen delegates are from departments B, D, and E, and is given by

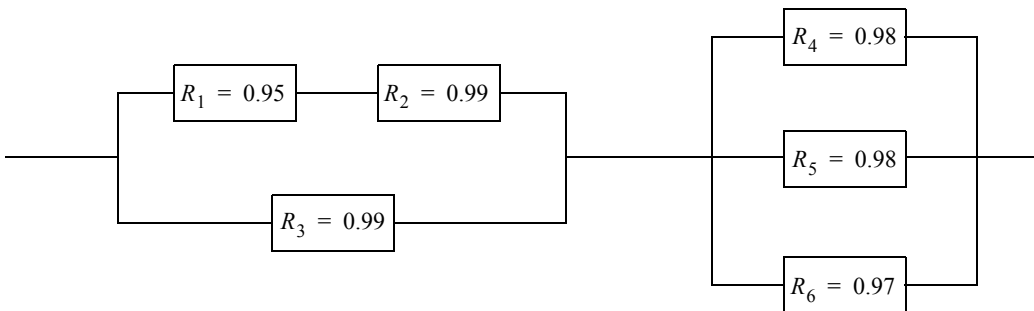
$$\frac{\left\{ \binom{3}{0} \right\}^2 \times \binom{9}{4}}{\binom{15}{4}} = \frac{\binom{9}{4}}{\binom{15}{4}} = \frac{9! \times 11! \times 4!}{15! \times 4! \times 5!} = 0.0923$$

### Section 1.11. Reliability Applications

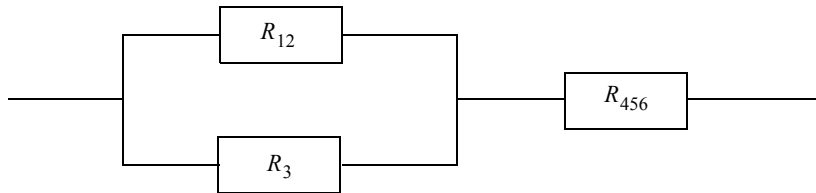
- 1.49 We are given the system shown below, where the number against each component indicates the probability that the component will independently fail within the next two years. To find the probability that the system fails within two years, we must first convert these “unreliability” numbers to reliability numbers.



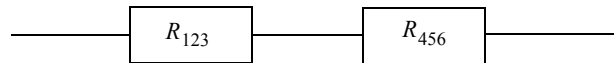
Thus, the figure is equivalent to the following:



Next, we carry out the first reduction of the system as follows, where  $R_{12} = R_1 R_2$  and  $R_{456} = 1 - (1 - R_4)(1 - R_5)(1 - R_6)$ . Note the  $R_{12} = 0.9405$  and  $R_{456} = 0.99999$ .



Next, we reduce the system to the following where  $R_{123} = 1 - (1 - R_{12})(1 - R_3) = 0.999405$ .

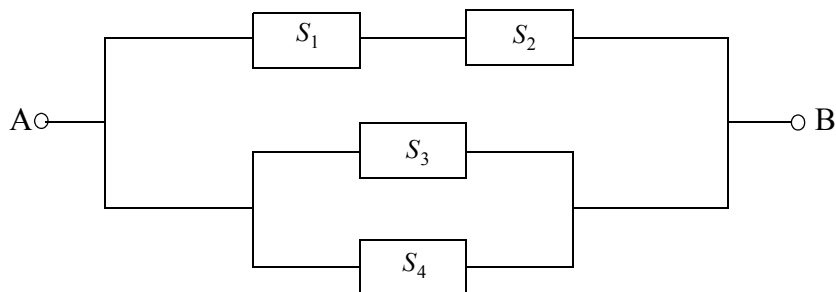


Thus, the reliability of the system is given by

$$R = R_{123} \times R_{456} = 0.999405 \times 0.99999 = 0.999395$$

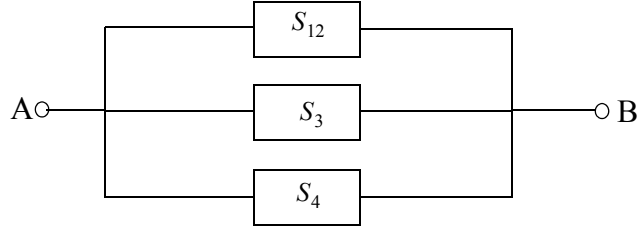
Therefore, the probability that the system fails within the next 2 years is  $1 - R = 0.0006$ .

- 1.50 In the structure shown the reliability functions of the switches  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$  are  $R_1(t)$ ,  $R_2(t)$ ,  $R_3(t)$ , and  $R_4(t)$ , respectively, and the switches are assumed to fail independently.



We start by reducing the system as shown below, where the reliability function of the

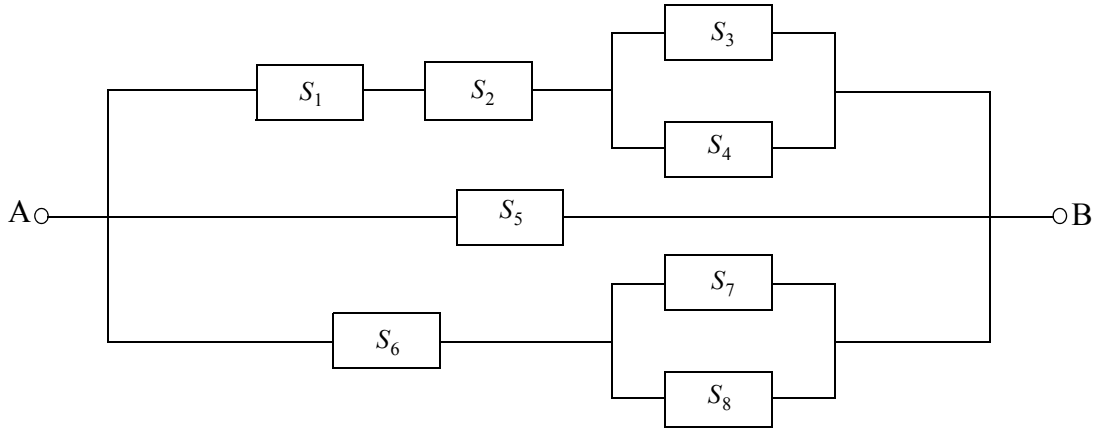
switch  $S_{12}$  is  $R_{12}(t) = R_1(t)R_2(t)$ .



Thus, the reliability function of the system is given by

$$R(t) = 1 - \{1 - R_{12}(t)\} \{1 - R_3(t)\} \{1 - R_4(t)\} = 1 - \{1 - R_1(t)R_2(t)\} \{1 - R_3(t)\} \{1 - R_4(t)\}$$

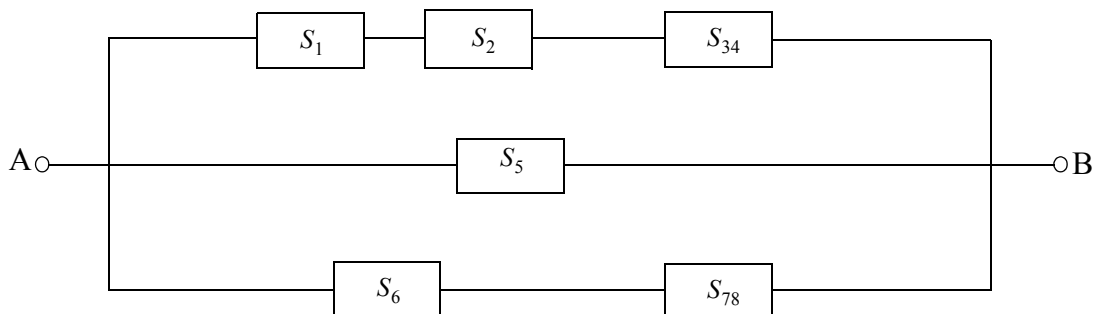
1.51 The switches labeled  $S_1, S_2, \dots, S_8$  that interconnect nodes A and B have the reliability functions  $R_1(t), R_2(t), \dots, R_8(t)$ , respectively, and are assumed to fail independently.



We start with the first level of system reduction as follows, where the switches labeled  $S_{34}$  and  $S_{78}$  have the following reliability functions, respectively:

$$R_{34}(t) = 1 - \{1 - R_3(t)\} \{1 - R_4(t)\}$$

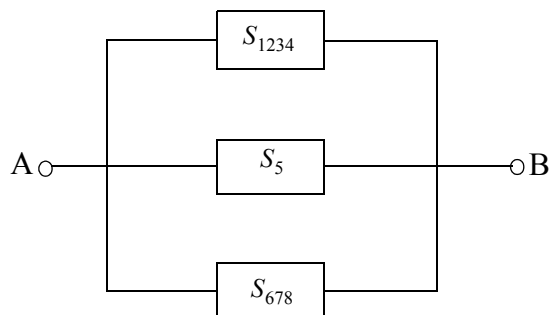
$$R_{78}(t) = 1 - \{1 - R_7(t)\} \{1 - R_8(t)\}$$



Next, we reduce the system again as shown below. The switches labeled  $S_{1234}$  and  $S_{678}$  have the following reliability functions, respectively:

$$R_{1234}(t) = R_1(t)R_2(t)R_{34}(t) = R_1(t)R_2(t)[1 - \{1 - R_3(t)\}\{1 - R_4(t)\}]$$

$$R_{678}(t) = R_6(t)R_{78}(t) = R_6(t)[1 - \{1 - R_7(t)\}\{1 - R_8(t)\}]$$

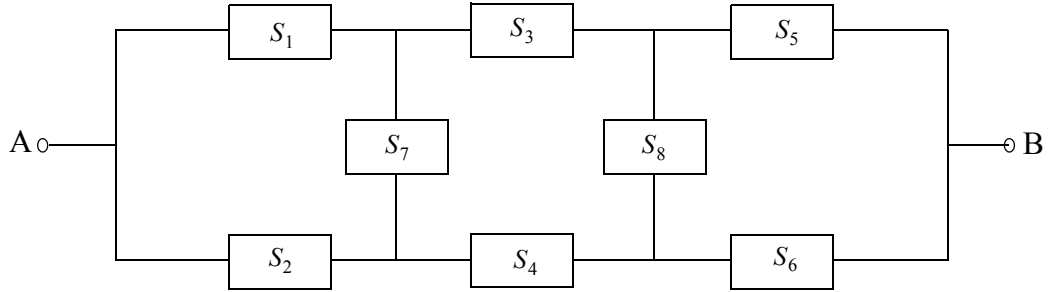


Thus, the reliability function of the system is given by

$$R(t) = 1 - \{1 - R_{1234}(t)\}\{1 - R_5(t)\}\{1 - R_{678}(t)\}$$

where  $R_{1234}(t)$  and  $R_{678}(t)$  are as previously defined.

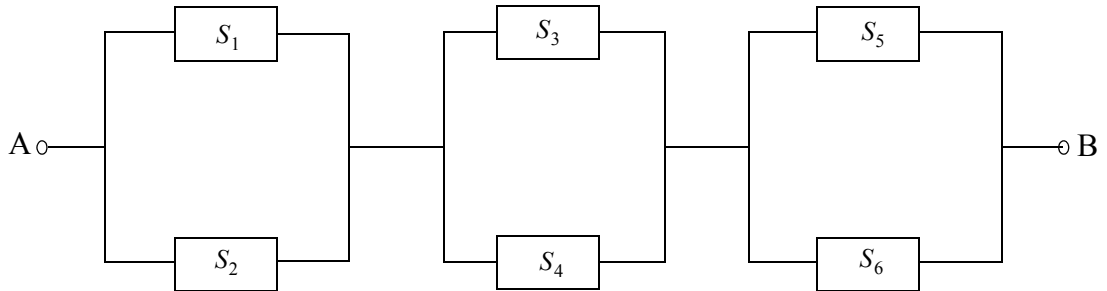
1.52 We are given the bridge network that interconnects nodes A and B. The switches labeled  $S_1, S_2, \dots, S_8$  have the reliability functions  $R_1(t), R_2(t), \dots, R_8(t)$ , respectively, and are assumed to fail independently.



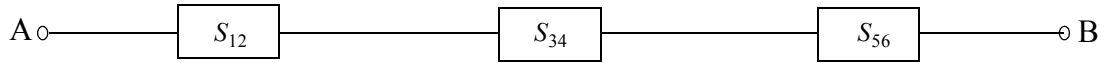
We consider the following 4 cases associated with the bridge switches  $S_7$  and  $S_8$  :

- A. Switches  $S_7$  and  $S_8$  do not fail by time  $t$ ; the probability of this event is  $P[A] = R_7(t)R_8(t)$
- B. Switch  $S_7$  fails, but switch  $S_8$  does not fail by time  $t$ ; the probability of this event is  $P[B] = \{1 - R_7(t)\}R_8(t)$
- C. Switch  $S_8$  fails, but switch  $S_7$  does not fail by time  $t$ ; the probability of this event is  $P[C] = R_7(t)\{1 - R_8(t)\}$
- D. Switches  $S_7$  and  $S_8$  fail by time  $t$ ; the probability of this event is  $P[D] = \{1 - R_7(t)\}\{1 - R_8(t)\}$

Case A is equivalent to the following system:



This in turn is equivalent to the following system:



The respective reliability functions of the switches labeled  $S_{12}$ ,  $S_{34}$ , and  $S_{56}$  are as follows:

$$R_{12}(t) = 1 - \{1 - R_1(t)\} \{1 - R_2(t)\}$$

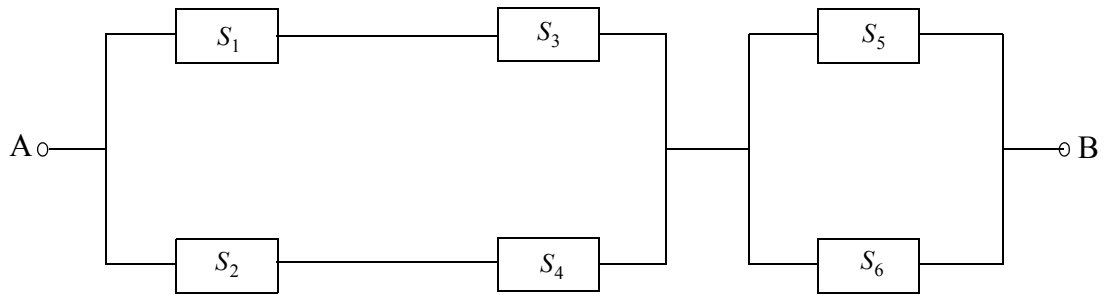
$$R_{34}(t) = 1 - \{1 - R_3(t)\} \{1 - R_4(t)\}$$

$$R_{56}(t) = 1 - \{1 - R_5(t)\} \{1 - R_6(t)\}$$

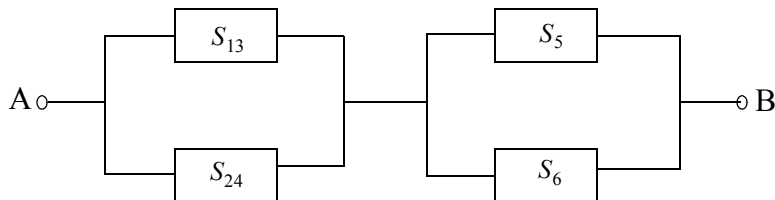
Thus, the reliability function  $R_A(t)$  for Case A is given by

$$\begin{aligned} R_A(t) &= R_{12}(t)R_{34}(t)R_{56}(t) \\ &= [1 - \{1 - R_1(t)\} \{1 - R_2(t)\}][1 - \{1 - R_3(t)\} \{1 - R_4(t)\}][1 - \{1 - R_5(t)\} \{1 - R_6(t)\}] \end{aligned}$$

Case B is equivalent to the following system:



This can be further reduced as follows:

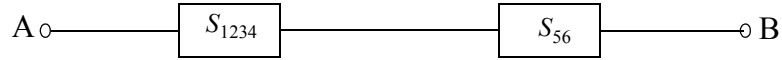


The respective reliability functions of the switches labeled  $S_{13}$  and  $S_{24}$  are as follows:

$$R_{13}(t) = R_1(t)R_3(t)$$

$$R_{24}(t) = R_2(t)R_4(t)$$

Finally we can reduce the system to the following structure:



where the reliability functions of the switches  $S_{1234}$  and  $S_{56}$  are given by

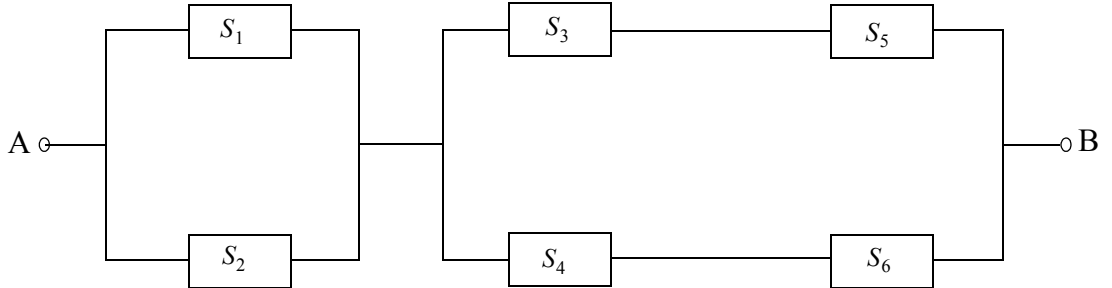
$$R_{1234}(t) = 1 - \{1 - R_{13}(t)\} \{1 - R_{24}(t)\} = 1 - \{1 - R_1(t)R_3(t)\} \{1 - R_2(t)R_4(t)\}$$

$$R_{56}(t) = 1 - \{1 - R_5(t)\} \{1 - R_6(t)\}$$

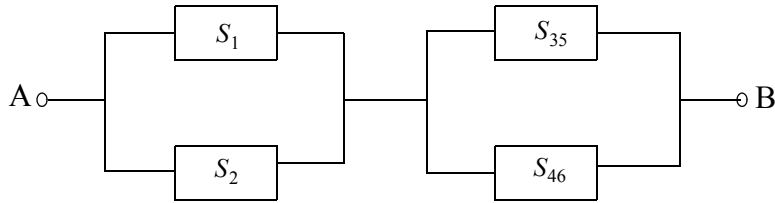
Thus, the reliability function for Case B is

$$\begin{aligned} R_B(t) &= R_{1234}(t)R_{56}(t) \\ &= [1 - \{1 - R_1(t)R_3(t)\} \{1 - R_2(t)R_4(t)\}] [1 - \{1 - R_5(t)\} \{1 - R_6(t)\}] \end{aligned}$$

Case C is equivalent to the following system:



This can be further reduced as follows:

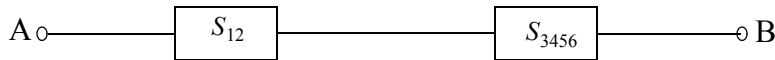


The respective reliability functions of the switches labeled  $S_{35}$  and  $S_{46}$  are as follows:

$$R_{35}(t) = R_3(t)R_5(t)$$

$$R_{46}(t) = R_4(t)R_6(t)$$

Finally we can reduce the system to the following structure:



where the reliability functions of the switches  $S_{1234}$  and  $S_{56}$  are given by

$$R_{12}(t) = 1 - \{1 - R_1(t)\} \{1 - R_2(t)\}$$

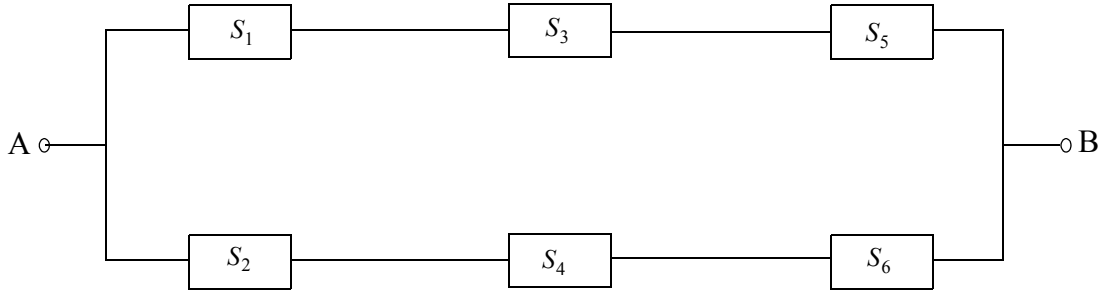
$$R_{3456}(t) = 1 - \{1 - R_{35}(t)\} \{1 - R_{46}(t)\} = 1 - \{1 - R_3(t)R_5(t)\} \{1 - R_4(t)R_6(t)\}$$

Thus, the reliability function for Case C is

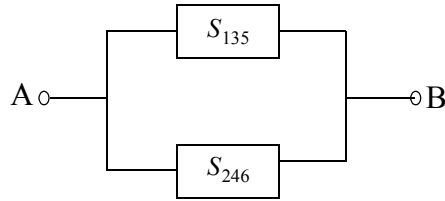
$$\begin{aligned}
 R_C(t) &= R_{12}(t)R_{3456}(t) \\
 &= [1 - \{1 - R_1(t)\} \{1 - R_2(t)\}][1 - \{1 - R_3(t)R_5(t)\} \{1 - R_4(t)R_6(t)\}]
 \end{aligned}$$

Case D is equivalent to the following system:





This can be further reduced as follows:



The respective reliability functions of the switches labeled  $S_{135}$  and  $S_{246}$  are as follows:

$$R_{135}(t) = R_1(t)R_3(t)R_5(t)$$

$$R_{246}(t) = R_2(t)R_4(t)R_6(t)$$

Thus, the reliability function for Case D is

$$\begin{aligned} R_D(t) &= 1 - \{1 - R_{135}(t)\} \{1 - R_{246}(t)\} \\ &= 1 - \{1 - R_1(t)R_3(t)R_5(t)\} \{1 - R_2(t)R_4(t)R_6(t)\} \end{aligned}$$

Finally, the reliability function of the system is given by

$$\begin{aligned} R(t) &= R_A(t)P[A] + R_B(t)P[B] + R_C(t)P[C] + R_D(t)P[D] \\ &= R_A(t)R_7(t)R_8(t) + R_B(t)\{1 - R_7(t)\}R_8(t) + R_C(t)R_7(t)\{1 - R_8(t)\} + R_D(t)\{1 - R_7(t)\}\{1 - R_8(t)\} \end{aligned}$$

where, as obtained above,

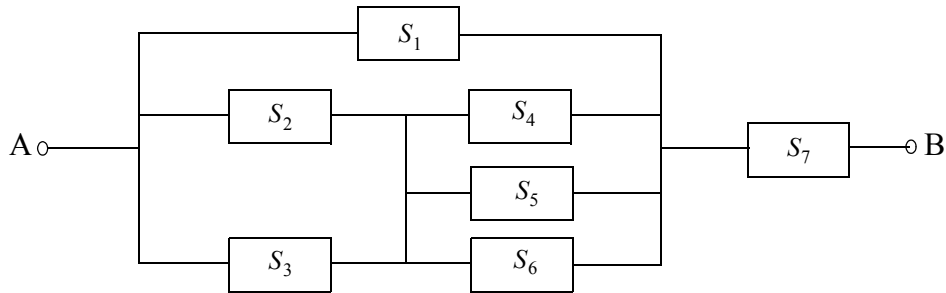
$$R_A(t) = [1 - \{1 - R_1(t)\}\{1 - R_2(t)\}][1 - \{1 - R_3(t)\}\{1 - R_4(t)\}][1 - \{1 - R_5(t)\}\{1 - R_6(t)\}]$$

$$R_B(t) = [1 - \{1 - R_1(t)R_3(t)\}\{1 - R_2(t)R_4(t)\}][1 - \{1 - R_5(t)\}\{1 - R_6(t)\}]$$

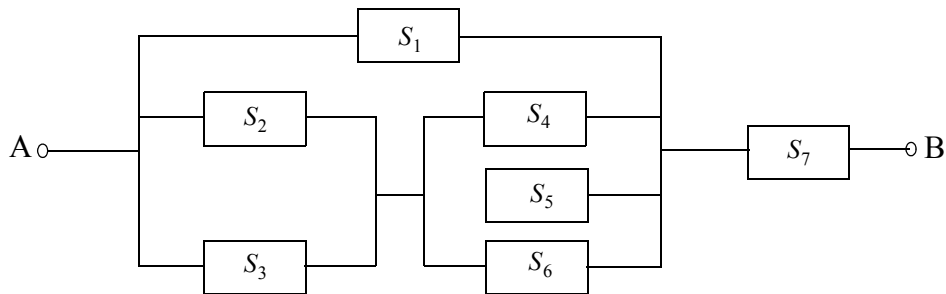
$$R_C(t) = [1 - \{1 - R_1(t)\}\{1 - R_2(t)\}][1 - \{1 - R_3(t)R_5(t)\}\{1 - R_4(t)R_6(t)\}]$$

$$R_D(t) = 1 - \{1 - R_1(t)R_3(t)R_5(t)\}\{1 - R_2(t)R_4(t)R_6(t)\}$$

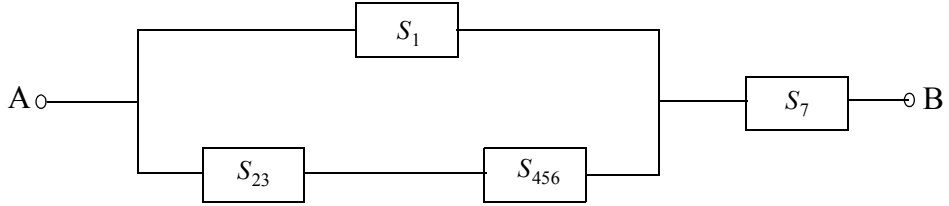
1.53 We are given the following network that interconnects nodes A and B, where the switches labeled  $S_1, S_2, \dots, S_7$  have the reliability functions  $R_1(t), R_2(t), \dots, R_7(t)$ , respectively, and fail independently.



This network is equivalent to the following system:



This can now be reduced to the following network:



The reliability functions of the switches labeled  $S_{23}$  and  $S_{456}$  are as follows:

$$R_{23}(t) = 1 - \{1 - R_2(t)\} \{1 - R_3(t)\}$$

$$R_{456}(t) = 1 - \{1 - R_4(t)\} \{1 - R_5(t)\} \{1 - R_6(t)\}$$

Thus, the reliability function of the system is given by

$$\begin{aligned}
 R(t) &= [1 - \{1 - R_1(t)\} \{1 - R_{23}(t)R_{456}(t)\}]R_7(t) \\
 &= [1 - \{1 - R_1(t)\} \{1 - [1 - \{1 - R_2(t)\} \{1 - R_3(t)\}][1 - \{1 - R_4(t)\} \{1 - R_5(t)\} \{1 - R_6(t)\}]\}]R_7(t)
 \end{aligned}$$



## Section 2.4: Distribution Functions

2.1 We are given the following function that is potentially a CDF:

$$F_X(x) = \begin{cases} 0 & -\infty < x \leq 1 \\ B\{1 - e^{-(x-1)}\} & 1 < x < \infty \end{cases}$$

(a) For the function to be a valid CDF it must satisfy the condition

$$F_X(\infty) = 1 = B\{1 - e^{-\infty}\} = 1 \Rightarrow B = 1$$

(b)  $F_X(3) = B\{1 - e^{-(3-1)}\} = 1 - e^{-2} = 0.86466$

(c)  $P[2 < X < \infty] = 1 - F_X(2) = 1 - \{1 - e^{-1}\} = e^{-1} = 0.3679$

(d)  $P[1 < X \leq 3] = F_X(3) - F_X(1) = \{1 - e^{-2}\} - \{1 - e^{-0}\} = 1 - e^{-2} = 0.86466$

2.2 The CDF of a random variable  $X$  is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 3x^2 - 2x^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

The PDF of  $X$  is given by

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} 6x - 6x^2 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

2.3 The random variable  $X$  has the CDF

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^2/2\sigma^2} & x \geq 0 \end{cases}$$

a. 
$$P[\sigma \leq X \leq 2\sigma] = F_X(2\sigma) - F_X(\sigma) = \{1 - e^{-4\sigma^2/2\sigma^2}\} - \{1 - e^{-\sigma^2/2\sigma^2}\}$$
$$= e^{-0.5} - e^{-2} = 0.4712$$

b. 
$$P[X > 3\sigma] = 1 - P[X \leq 3\sigma] = 1 - F_X(3\sigma) = e^{-9\sigma^2/2\sigma^2} = e^{-4.5} = 0.0111$$

2.4 The CDF of a random variable  $T$  is given by

$$F_T(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases}$$

a. The PDF of  $T$  is

$$f_T(t) = \frac{d}{dt}F_T(t) = \begin{cases} 2t & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

b. 
$$P[T > 0.5] = 1 - P[T \leq 0.5] = 1 - F_T(0.5) = 1 - (0.5)^2 = 1 - 0.25 = 0.75$$

c. 
$$P[0.5 < T < 0.75] = F_T(0.75) - F_T(0.5) = (0.75)^2 - (0.5)^2 = 0.5625 - 0.25 = 0.3125$$

2.5 The CDF of a continuous random variable  $X$  is given by

$$F_X(x) = \begin{cases} 0 & x \leq -\pi/2 \\ k(1 + \sin x) & -\pi/2 < x \leq \pi/2 \\ 1 & x > \pi/2 \end{cases}$$

a.  $F_X(\pi/2) = 1 = k\{1 + \sin(\pi/2)\} = 2k \Rightarrow k = \frac{1}{2}$

Alternatively,  $f_X(x) = \frac{d}{dx}F_X(x) = k\cos(x)$ ,  $-\pi/2 < x \leq \pi/2$ . Thus,

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 = \int_{-\pi/2}^{\pi/2} f_X(x)dx = \int_{-\pi/2}^{\pi/2} k\cos(x)dx = k[\sin(x)]_{-\pi/2}^{\pi/2} = 2k \Rightarrow k = \frac{1}{2}$$

b. Using the above results we obtain  $f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} \frac{\cos(x)}{2} & -\pi/2 < x \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}$

2.6 The CDF of a random variable  $X$  is given by

$$F_X(x) = \begin{cases} 0 & x \leq 2 \\ 1 - \frac{4}{x^2} & x > 2 \end{cases}$$

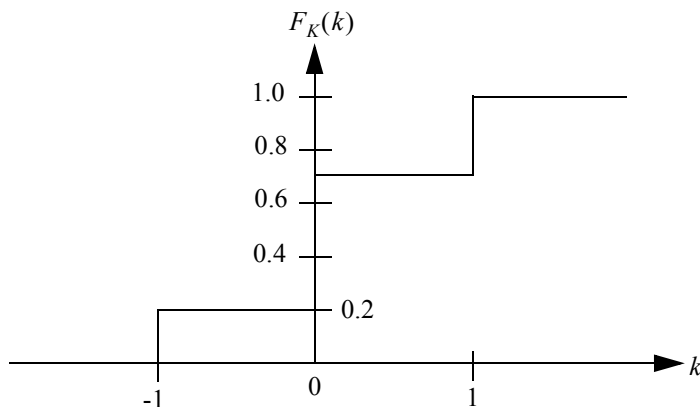
a.  $P[X < 3] = F_X(3) = 1 - \frac{4}{9} = \frac{5}{9}$

b.  $P[4 < X < 5] = F_X(5) - F_X(4) = \left\{1 - \frac{4}{25}\right\} - \left\{1 - \frac{4}{16}\right\} = \frac{9}{100} = 0.09$

2.7 The CDF of a discrete random variable  $K$  is given by

$$F_K(k) = \begin{cases} 0.0 & k < -1 \\ 0.2 & -1 \leq k < 0 \\ 0.7 & 0 \leq k < 1 \\ 1.0 & k \geq 1 \end{cases}$$

- a. The graph of  $F_K(k)$  is as follows:



- b. To find the PMF of  $K$ , we observe that it has nonzero values at those values of  $k$  where the value of the CDF changes, and its value at any such point is equal to the change in the value of the CDF. Thus,

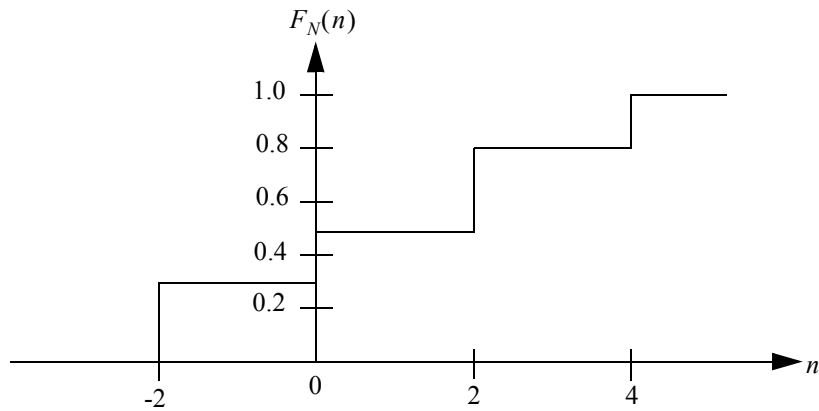
$$p_K(k) = \begin{cases} 0.2 & k = -1 \\ 0.5 & k = 0 \\ 0.3 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

2.8 The random variable  $N$  has the CDF

$$F_N(n) = \begin{cases} 0.0 & n < -2 \\ 0.3 & -2 \leq n < 0 \\ 0.5 & 0 \leq n < 2 \\ 0.8 & 2 \leq n < 4 \\ 1 & n \geq 4 \end{cases}$$

- a. The graph of  $F_N(n)$  is as follows:

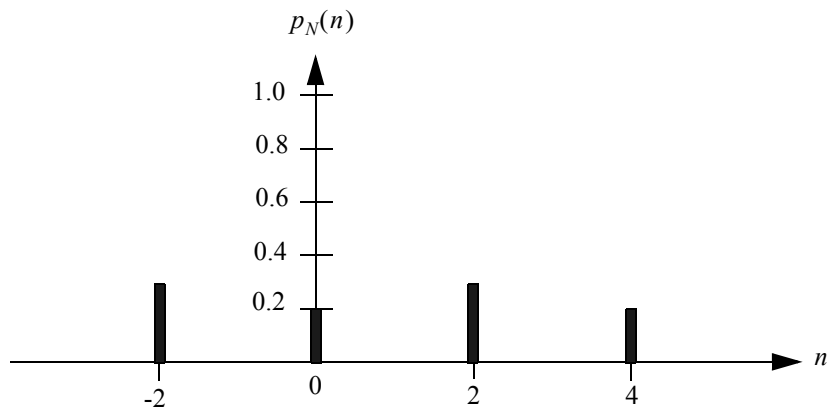




b. The PMF of  $K$  is given by

$$p_N(n) = \begin{cases} 0.3 & n = -2 \\ 0.2 & n = 0 \\ 0.3 & n = 2 \\ 0.2 & n = 4 \\ 0 & \text{otherwise} \end{cases}$$

c. The graph of  $p_N(n)$  is as follows:



2.9 The CDF of a discrete random variable  $Y$  is given by

$$F_Y(y) = \begin{cases} 0.0 & y < 2 \\ 0.3 & 2 \leq y < 4 \\ 0.8 & 4 \leq y < 6 \\ 1.0 & y \geq 6 \end{cases}$$

- a.  $P[3 < Y < 4] = F_Y(4) - F_Y(3) = 0$ . Another way to see this is to first obtain the PMF of  $Y$ , which is given by

$$p_Y(y) = \begin{cases} 0.3 & y = 2 \\ 0.5 & y = 4 \\ 0.2 & y = 6 \\ 0 & \text{otherwise} \end{cases}$$

Thus,  $P[3 < Y < 4] = p_Y(2) = 0$ .

- b.  $P[3 < Y \leq 4] = p_Y(4) = 0.5$ .

2.10 The CDF of the random variable  $Y$  is given by

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 0.50 & 0 \leq y < 2 \\ 0.75 & 2 \leq y < 3 \\ 0.90 & 3 \leq y < 5 \\ 1 & y \geq 5 \end{cases}$$

The PMF of  $Y$  is given by

$$p_Y(y) = \begin{cases} 0.50 & y = 0 \\ 0.25 & y = 2 \\ 0.15 & y = 3 \\ 0.10 & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

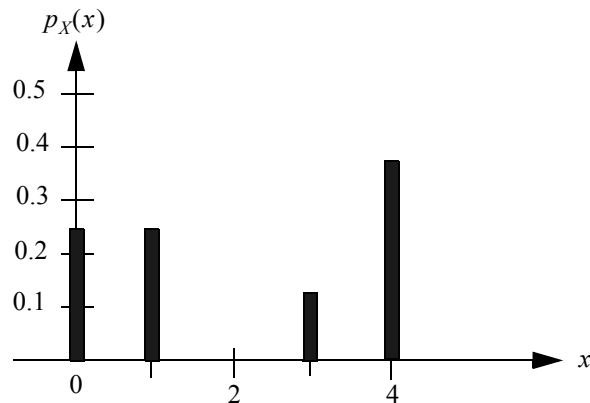
2.11 The CDF of a discrete random variable  $X$  is given as follows:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq x < 1 \\ 1/2 & 1 \leq x < 3 \\ 5/8 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

(a) The PMF of  $X$  is given by

$$p_X(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{4} & x = 1 \\ \frac{1}{8} & x = 3 \\ \frac{3}{8} & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

The graph of the PMF is as shown below.



(b)

$$P[X < 2] = p_X(0) + p_X(1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

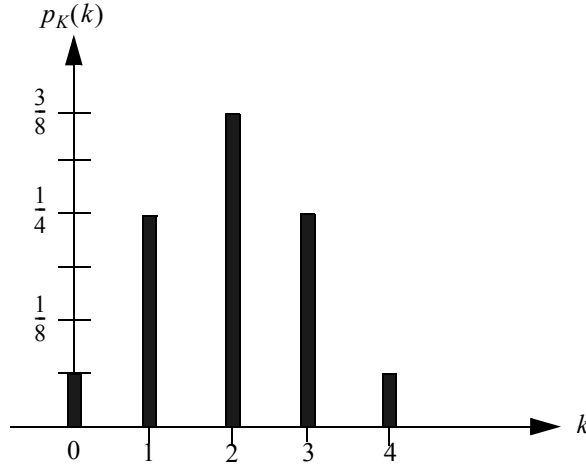
$$P[0 \leq X < 4] = p_X(0) + p_X(1) + p_X(3) = \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = \frac{5}{8}$$

## Section 2.5: Discrete Random Variables

2.12  $K$  is a random variable that denotes the number of heads in 4 flips of a fair coin. The PMF of  $K$  can be obtained as follows. For  $0 \leq k \leq 4$ ,  $p_K(k)$  is the probability that there are  $k$  heads and, therefore,  $4 - k$  tails. Since the probability of a head in any flip is  $1/2$  and the outcomes of the flips are independent, the probability of  $k$  heads in 4 flips of the coin is  $(1/2)^k(1/2)^{4-k} = (1/2)^4$ . However, to account for the possible combinations of  $k$  heads and  $4 - k$  tails, we need the combinatorial term  $C(4, k)$ . Thus, the PMF of  $K$  is  $p_K(k) = C(4, k)(1/2)^4$ ; that is,

$$p_K(k) = \begin{cases} 1/16 & k = 0 \\ 1/4 & k = 1 \\ 3/8 & k = 2 \\ 1/4 & k = 3 \\ 1/16 & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) The graph of  $p_K(k)$  is as follows:



(b)  $P[K \geq 3] = p_K(3) + p_K(4) = 5/16$

(c)  $P[2 \leq K \leq 4] = p_K(2) + p_K(3) + p_K(4) = 11/16$

2.13 Ken was watching people playing the game of poker and wanted to model the PMF of the random variable  $N$  that denotes the number of plays up to and including the play in which his friend Joe won a game. He conjectured that if  $p$  is the probability that Joe wins any game and the games are independent, then the PMF of  $N$  is given by

$$p_N(n) = p(1-p)^{n-1} \quad n = 1, 2, \dots$$

- a. For  $p_N(n)$  to be a proper PMF it must satisfy the condition  $\sum_n p_N(n) = 1$ . Thus, we evaluate the sum:

$$\sum_{n=1}^{\infty} p_N(n) = p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \left\{ \frac{1}{1-(1-p)} \right\} = \frac{p}{p} = 1$$

Thus, we conclude that  $p_N(n)$  is a proper PMF.

b. The CDF of  $N$  is given by

$$\begin{aligned}
 F_N(n) &= P[N \leq n] = \sum_{k=1}^n p_N(k) = p \sum_{k=1}^n (1-p)^{k-1} = p \sum_{k=0}^{n-1} (1-p)^k \\
 &= p \left\{ \frac{1 - (1-p)^n}{1 - (1-p)} \right\} = 1 - (1-p)^n \quad n = 1, 2, \dots
 \end{aligned}$$

2.14 Given a discrete random variable  $K$  with the following PMF:

$$p_K(k) = \begin{cases} b & k = 0 \\ 2b & k = 1 \\ 3b & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

(a)

$$\sum_k p_K(k) = 1 = b + 2b + 3b = 6b \Rightarrow b = \frac{1}{6}$$

(b)

$$\begin{aligned}
 P[K < 2] &= p_K(0) + p_K(1) = 1/2 \\
 P[K \leq 2] &= p_K(0) + p_K(1) + p_K(2) = 1 \\
 P[0 < K < 2] &= p_K(1) = 1/3
 \end{aligned}$$

(c) The CDF of  $K$  is given by

$$F_K(k) = \begin{cases} 0 & k < 0 \\ \frac{1}{6} & 0 \leq k < 1 \\ \frac{1}{2} & 1 \leq k < 2 \\ 1 & k \geq 2 \end{cases}$$

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2.15 The postulated PMF of  $K$  is

$$p_K(k) = \begin{cases} \lambda^k e^{-\lambda} / k! & k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

(a) To show that  $p_K(k)$  is a proper PMF, we must have that it sums to 1 over all values of  $k$ ; that is,

$$\sum_{k=0}^{\infty} p_K(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

Thus, we conclude that  $p_K(k)$  is a proper PMF.

(b)  $P[K > 1] = 1 - P[K \leq 1] = 1 - p_K(0) - p_K(1) = 1 - e^{-\lambda} \{1 + \lambda\}$

(c)  $P[2 \leq K \leq 4] = p_K(2) + p_K(3) + p_K(4) = e^{-\lambda} \left\{ \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \frac{\lambda^4}{24} \right\}$

2.16 Let  $X$  be the random variable that denotes the number of times we roll a fair die until the first time the number 5 appears. Since the probability that the number 5 appears in any roll is  $1/6$ , then the probability that  $X = k$  is the probability that we had no number 5 in the previous  $k-1$  rolls and the number 5 in the  $k$ th roll. Since the outcomes of the different rolls are independent,

$$P[K = k] = \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)$$

2.17 We are given the PMF of a random variable  $X$  as  $p_X(x) = b\lambda^x/x!$ ,  $x = 0, 1, 2, \dots$ , where  $\lambda > 0$ . We first evaluate the value of  $b$  as follows:

$$\sum_{x=0}^{\infty} p_X(x) = b \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = b e^{\lambda} = 1 \Rightarrow b = e^{-\lambda}$$

Thus, we have that

$$P[X = 1] = p_X(1) = \lambda e^{-\lambda}$$

$$P[X > 3] = 1 - P[X \leq 3] = 1 - \{p_X(0) + p_X(1) + p_X(2) + p_X(3)\} = 1 - e^{-\lambda} \left\{ 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} \right\}$$

2.18 A random variable  $K$  has the PMF

$$p_K(k) = \binom{5}{k} (0.1)^k (0.9)^{5-k} \quad k = 0, 1, \dots, 5$$

(a)  $P[K = 1] = 5(0.1)(0.9)^4 = 0.32805$

(b)  $P[K \geq 1] = 1 - P[K = 0] = 1 - (0.9)^5 = 0.40951$

2.19 A biased four-sided die has faces labeled 1, 2, 3, and 4. Let the random variable  $X$  denote the outcome of a roll of the die. Extensive testing of the die shows that the PMF of  $X$  is given by

$$p_X(x) = \begin{cases} 0.4 & x = 1 \\ 0.2 & x = 2 \\ 0.3 & x = 3 \\ 0.1 & x = 4 \end{cases}$$

a. The CDF of  $X$  is given by

$$F_X(x) = P[X \leq x] = \begin{cases} 0.0 & x < 1 \\ 0.4 & 1 \leq x < 2 \\ 0.6 & 2 \leq x < 3 \\ 0.9 & 3 \leq x < 4 \\ 1.0 & x \geq 4 \end{cases}$$

b.  $P[X < 3] = p_X(1) + p_X(2) = 0.6$



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c.  $P[X \geq 3] = 1 - P[X < 3] = 1 - 0.6 = 0.4$

2.20 The number  $N$  of calls arriving at a switchboard during a period of one hour has the PMF

$$p_N(n) = \frac{10^n e^{-10}}{n!} \quad n = 0, 1, \dots$$

a.  $P[N \geq 2] = 1 - P[N < 2] = 1 - \{p_N(0) + p_N(1)\} = 1 - e^{-10}\{1 + 10\} = 1 - 11e^{-10} = 0.9995$

b.  $P[N \leq 3] = p_N(0) + p_N(1) + p_N(2) + p_N(3) = e^{-10} \left\{ 1 + 10 + \frac{10^2}{2} + \frac{10^3}{6} \right\} = 0.01034$

c.  $P[3 < N \leq 6] = p_N(4) + p_N(5) + p_N(6) = e^{-10} \left\{ \frac{10^4}{24} + \frac{10^5}{120} + \frac{10^6}{720} \right\} = 0.1198$

2.21 The random variable  $K$  denotes the number of successes in  $n$  trials of an experiment and its PMF is given by

$$p_K(k) = \binom{n}{k} (0.6)^k (0.4)^{n-k} \quad k = 0, 1, \dots, n; n = 1, 2, \dots$$

a.  $P[K \geq 1, n = 5] = 1 - p_K(0)|_{n=5} = 1 - (0.4)^5 = 0.98976$

b.  $P[K \leq 1, n = 5] = [p_K(0) + p_K(1)]_{n=5} = (0.4)^5 + 5(0.6)(0.4)^4 = 0.08704$

c.  $P[1 < K < 4, n = 5] = [p_K(2) + p_K(3)]_{n=5} = 10(0.6)^2(0.4)^3 + 10(0.6)^3(0.4)^2 = 0.576$

2.22

$$p(x) = \begin{cases} \frac{2}{3}\left(\frac{1}{3}\right)^x & x = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x=0}^{\infty} p(x) = \frac{2}{3} \sum_{x=0}^{\infty} \left(\frac{1}{3}\right)^x = \frac{2}{3} \left\{ \frac{1}{1 - (1/3)} \right\} = \frac{2}{3} \left\{ \frac{3}{2} \right\} = 1$$

Thus,  $p(x)$  is a valid PMF.

## Section 2.6: Continuous Random Variables

2.23 Consider the following function:

$$g(x) = \begin{cases} a(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) For  $g(x)$  to be a valid PDF we must have that

$$\int_{x=-\infty}^{\infty} g(x)dx = 1 = \int_{x=-1}^1 a(1 - x^2)dx = a \left[ x - \frac{x^3}{3} \right]_{-1}^1 = \frac{4a}{3}$$

Thus,  $a = 3/4$ .

(b) If  $X$  is the random variable with this PDF, then

$$P[0 < X < 0.5] = \int_{x=0}^{0.5} g(x)dx = \frac{3}{4} \int_{x=0}^{0.5} (1 - x^2)dx = \frac{3}{4} \left[ x - \frac{x^3}{3} \right]_0^{0.5} = 0.34375$$

2.24 The PDF  $f_X(x)$  of a continuous random variable  $X$  is defined as follows for  $\lambda > 0$ ,

$$f_X(x) = \begin{cases} bxe^{-\lambda x} & 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) To obtain the value of  $b$  we have that  $\int_{x=-\infty}^{\infty} f_X(x)dx = 1 = b \int_{x=0}^{\infty} x e^{-\lambda x} dx$

Let  $u = x \Rightarrow du = dx$  and let  $dv = e^{-\lambda x} dx \Rightarrow v = -\frac{e^{-\lambda x}}{\lambda}$ . Thus,

$$b \int_{x=0}^{\infty} x e^{-\lambda x} dx = 1 = b \left[ -\frac{x e^{-\lambda x}}{\lambda} \right]_{x=0}^{\infty} + \int_{x=0}^{\infty} \frac{e^{-\lambda x}}{\lambda} dx = \frac{b}{\lambda} \left[ -\frac{e^{-\lambda x}}{\lambda} \right]_0^{\infty} = \frac{b}{\lambda^2}$$

This implies that  $b = \lambda^2$ .

(b) The CDF of  $X$  is given by  $F_X(x) = P[X \leq x] = \int_{w=-\infty}^x f_X(w)dw = \lambda^2 \int_{w=0}^x w e^{-\lambda w} dw$

Let  $u = w \Rightarrow du = dw$  and let  $dv = e^{-\lambda w} dw \Rightarrow v = -\frac{e^{-\lambda w}}{\lambda}$ . Thus,

$$F_X(x) = \lambda^2 \left[ -\frac{w e^{-\lambda w}}{\lambda} \right]_0^x + \int_{w=0}^x \frac{e^{-\lambda w}}{\lambda} dw = -\lambda x e^{-\lambda x} + \lambda \left[ -\frac{e^{-\lambda w}}{\lambda} \right]_0^x = 1 - e^{-\lambda x} - \lambda x e^{-\lambda x} \quad x \geq 0$$

(c)  $P[0 \leq X \leq 1/\lambda] = F_X(1/\lambda) = 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} = 0.26424$

2.25 Given the CDF

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 2x^2 - x^3 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

the PDF is

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 4x - 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

2.26 Given a random variable  $X$  with following PDF

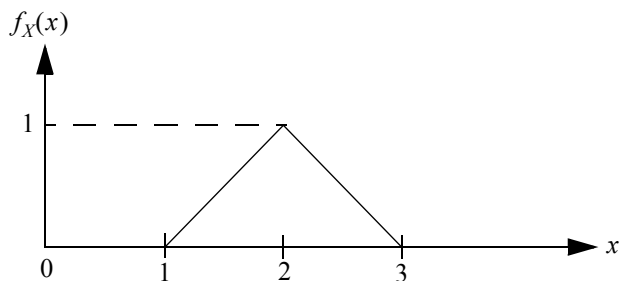
$$f_X(x) = \begin{cases} 0 & x < 1 \\ K(x-1) & 1 \leq x < 2 \\ K(3-x) & 2 \leq x < 3 \\ 0 & x \geq 3 \end{cases}$$

(a) The value of  $K$  that makes it a valid PDF can be obtained as follows:

$$\begin{aligned} \int_{x=-\infty}^{\infty} f_X(x) dx &= 1 = K \left\{ \int_{x=1}^2 (x-1) dx + \int_{x=2}^3 (3-x) dx \right\} \\ &= K \left\{ \left[ \frac{x^2}{2} - x \right]_1^2 + \left[ 3x - \frac{x^2}{2} \right]_2^3 \right\} = K \left\{ \frac{1}{2} + \frac{1}{2} \right\} = K \end{aligned}$$

Thus,  $K = 1$ .

(b) The plot of  $f_X(x)$  is as follows:



(c)

$$F_X(x) = P[X \leq x] = \begin{cases} 0 & x < 1 \\ K \int_{u=1}^x (u-1) du & 1 \leq x \leq 2 \\ K \left\{ \int_{u=1}^2 (u-1) dx + \int_{u=2}^x (3-u) du \right\} & 2 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$

$$= \begin{cases} 0 & x < 1 \\ \frac{x^2}{2} - x + \frac{1}{2} & 1 \leq x < 2 \\ 3x - \frac{x^2}{2} - \frac{7}{2} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

$$(d) P[1 \leq X \leq 2] = F_X(2) - F_X(1) = \frac{1}{2} - 0 = \frac{1}{2}$$

2.27 A random variable  $X$  has the CDF

$$F_X(x) = \begin{cases} 0 & x < -1 \\ A(1+x) & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

(a) To find the value of  $A$ , we know that  $F_X(1) = 1 \Rightarrow 2A = 1 \Rightarrow A = 1/2$ . Alternatively,

$$f_X(x) = \frac{d}{dx} F_X(x) = A, \quad -1 \leq x < 1$$

$$1 = \int_{-\infty}^{\infty} f_X(x) dx = \int_{-1}^1 A dx = 2A \Rightarrow A = \frac{1}{2}$$

$$(b) P[X > 1/4] = 1 - P[X \leq 1/4] = 1 - F_X(1/4) = 1 - A(1 + 1/4) = 1 - 5/8 = 3/8$$

$$(c) P[-0.5 \leq X \leq 0.5] = F_X(0.5) - F_X(-0.5) = \frac{1}{2} \{ (1 + 0.5) - (1 - 0.5) \} = \frac{1}{2}$$

2.28 The lifetime  $X$  of a system in weeks is given by the following PDF:

$$f_X(x) = \begin{cases} 0.25e^{-0.25x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

First we observe that the CDF of  $X$  is given by

$$F_X(x) = \int_{w=-\infty}^x f_X(w)dw = 0.25 \int_{w=0}^x e^{-0.25w} dw = -e^{-0.25w} \Big|_0^x = 1 - e^{-0.25x}$$

(a) The probability that the system will not fail within two weeks is given by

$$P[X > 2] = 1 - P[X \leq 2] = 1 - F_X(2) = e^{-0.5} = 0.6065$$

(b) Given that the system has not failed by the end of the fourth week, the probability that it will fail between the fourth and sixth weeks is given by

$$\begin{aligned} P[4 < X < 6 | X > 4] &= \frac{P[(4 < X < 6) \cap (X > 4)]}{P[X > 4]} = \frac{P[4 < X < 6]}{P[X > 4]} = \frac{F_X(6) - F_X(4)}{1 - F_X(4)} \\ &= \frac{e^{-1} - e^{-1.5}}{e^{-1}} = 1 - e^{-0.5} = 0.3935 \end{aligned}$$

2.29 The PDF of the time  $T$  until the radar fails in years is given by  $f_T(t) = 0.2e^{-0.2t}$ , where  $t \geq 0$ . Thus, the probability that the radar lasts for at least four years is given by

$$P[T \geq 4] = \int_{t=4}^{\infty} f_T(t)dt = \int_{t=4}^{\infty} 0.2e^{-0.2t} dt = -e^{-0.2t} \Big|_4^{\infty} = e^{-0.2(4)} = e^{-0.8} = 0.4493$$

2.30 The PDF of a random variable  $X$  is given by

$$f_X(x) = \begin{cases} A/x^2 & x > 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) If this is a valid PDF, then

$$\int_{x=-\infty}^{\infty} f_X(x) dx = 1 = A \int_{x=10}^{\infty} \left(\frac{1}{x^2}\right) dx = A \left[ -\frac{1}{x} \right]_{10}^{\infty} = \frac{A}{10} \Rightarrow A = 10$$

(b)

$$F_X(x) = \int_{u=-\infty}^x f_X(u) du = \int_{u=-\infty}^x \left(\frac{10}{u^2}\right) du = \left[ -\frac{10}{u} \right]_{10}^x = \begin{cases} 0 & x \leq 10 \\ 10 \left( \frac{1}{10} - \frac{1}{x} \right) & 10 < x < \infty \end{cases}$$

$$(c) P[X > 20] = 1 - P[X \leq 20] = 1 - F_X(20) = 1 - 10 \left( \frac{1}{10} - \frac{1}{20} \right) = \frac{1}{2}$$

2.31 We are given that  $f_X(x) = \begin{cases} A(3x^2 - x^3) & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

(a) For the function to be a valid PDF we have that

$$\int_{x=-\infty}^{\infty} f_X(x) dx = 1 = \int_{x=0}^3 A(3x^2 - x^3) dx = A \left[ x^3 - \frac{x^4}{4} \right]_0^3 = A \left[ 27 - \frac{81}{4} \right] = \frac{27A}{4} \Rightarrow A = \frac{4}{27}$$

(b)

$$P[1 < X < 2] = \int_{x=1}^2 f_X(x) dx = \frac{4}{27} \int_{x=1}^2 (3x^2 - x^3) dx = \frac{4}{27} \left[ x^3 - \frac{x^4}{4} \right]_1^2 = \frac{4}{27} \left[ (8 - 4) - \left( 1 - \frac{1}{4} \right) \right] = \frac{13}{27}$$

2.32 A random variable  $X$  has the PDF

$$f_X(x) = \begin{cases} k(1 - x^4) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. If this is a valid PDF, then

$$\int_{x=-\infty}^{\infty} f_X(x) dx = 1 = k \int_{x=-1}^1 (1 - x^4) dx = k \left[ x - \frac{x^5}{5} \right]_{-1}^1 = \frac{8k}{5} \Rightarrow k = \frac{5}{8}$$

b. The CDF of  $X$  is given by

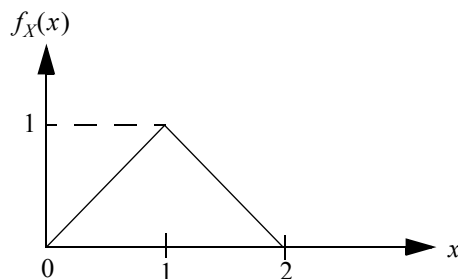
$$F_X(x) = \int_{u=-\infty}^x f_X(u) du = \begin{cases} \frac{5}{8} \int_{u=-1}^x (1-u^4) du = \frac{5}{8} \left[ u - \frac{u^5}{5} \right]_{-1}^x & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$= \begin{cases} \frac{5}{8} \left\{ x - \frac{x^5}{5} + \frac{4}{5} \right\} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

c.  $P[X < 1/2] = F_X(1/2) = \frac{5}{8} \left\{ x - \frac{x^5}{5} + \frac{4}{5} \right\} \Big|_{x=1/2} = 0.8086$

2.33 Given that  $f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$

We start by drawing the PDF, as shown below.



a. The CDF of  $X$  is given by



$$F_X(x) = \int_{u=-\infty}^x f_X(u)du = \begin{cases} 0 & x < 0 \\ \int_{u=0}^x u du = \left[\frac{u^2}{2}\right]_0^x & 0 \leq x < 1 \\ \int_{u=0}^1 u du + \int_{u=1}^x (2-u)du & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \leq x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

b.  $P[0.2 < X < 0.8] = F_X(0.8) - F_X(0.2) = \frac{0.64}{2} - \frac{0.04}{2} = 0.3$

c.  $P[0.6 < X < 1.2] = F_X(1.2) - F_X(0.6) = \left\{2.4 - \frac{1.44}{2} - 1\right\} - \frac{0.36}{2} = 0.5$

2.34  $f_X(x) = \begin{cases} Ae^{-x/20} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

a. For this to be a valid PDF we have that

$$\int_{x=-\infty}^{\infty} f_X(x)dx = 1 = A \int_{x=0}^{\infty} e^{-x/20} dx = A[-20e^{-x/20}]_0^{\infty} = 20A \Rightarrow A = \frac{1}{20}$$

b. The CDF of  $X$  is given by

$$F_X(x) = \int_{u=-\infty}^x f_X(u)du = \begin{cases} 0 & x < 0 \\ \frac{1}{20} \int_{u=0}^x e^{-u/20} du & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ 1 - e^{-x/20} & x \geq 0 \end{cases}$$

c.  $P[X \leq 10] = F_X(10) = 1 - e^{-0.5} = 0.3935$

d.  $P[16 < X < 24] = F_X(24) - F_X(16) = e^{-0.8} - e^{-1.2} = 0.1481$

2.35  $f_X(x) = \begin{cases} 0 & x < 0.5 \\ ke^{-2(x-0.5)} & x \geq 0.5 \end{cases}$

a. For the function to be a valid PDF we must have that

$$\int_{x=-\infty}^{\infty} f_X(x)dx = 1 = k \int_{x=0.5}^{\infty} e^{-2(x-0.5)} dx = ke^1 \left[ -\frac{e^{-2x}}{2} \right]_{0.5}^{\infty} = \frac{ke^1 e^{-1}}{2} = \frac{k}{2} \Rightarrow k = 2$$

b. The CDF of  $X$  is given by

$$F_X(x) = \int_{u=-\infty}^x f_X(u)du = \begin{cases} 0 & x < 0.5 \\ 2e^1 \int_{u=0.5}^x e^{-2u} du & x \geq 0.5 \end{cases}$$

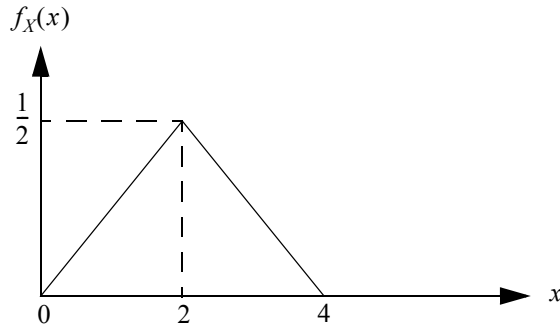
$$= \begin{cases} 0 & x < 0.5 \\ 1 - e^{-2(x-0.5)} & x \geq 0.5 \end{cases}$$

c.  $P[X \leq 1.5] = F_X(1.5) = 1 - e^{-2(1.5-0.5)} = 1 - e^{-2} = 0.8647$

d.  $P[1.2 < X < 2.4] = F_X(2.4) - F_X(1.2) = e^{-2(1.2-0.5)} - e^{-2(2.4-0.5)} = e^{-1.4} - e^{-3.8} = 0.2242$

## Section 3.2: Expected Values

3.1 We are given the triangular PDF.



We have that

$$f_X(x) = \begin{cases} \frac{x}{4} & 0 \leq x < 2 \\ 1 - \frac{x}{4} & 2 \leq x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^2 x \left( \frac{x}{4} \right) dx + \int_2^4 x \left( 1 - \frac{x}{4} \right) dx = \left[ \frac{x^3}{12} \right]_0^2 + \left[ \frac{x^2}{2} - \frac{x^3}{12} \right]_2^4 \\ &= \frac{8}{12} + \left( \frac{16}{2} - \frac{64}{12} \right) - \left( \frac{4}{2} - \frac{8}{12} \right) = 2 \\ E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^2 x^2 \left( \frac{x}{4} \right) dx + \int_2^4 x^2 \left( 1 - \frac{x}{4} \right) dx = \left[ \frac{x^4}{16} \right]_0^2 + \left[ \frac{x^3}{3} - \frac{x^4}{16} \right]_2^4 = \frac{14}{3} \\ \sigma_X^2 &= E[X^2] - (E[X])^2 = \frac{2}{3} \end{aligned}$$

- 3.2 Let  $N$  be a random variable that denotes the number of claims in one year. If the probability that a man of age 50 dies within one year is 0.02, then the expected number of claims that the company can expect from the beneficiaries of the 1000 men within one year is

$$E[N] = 1000p = (1000)(0.02) = 20$$

- 3.3 Let  $X$  be the random variable that denotes the height of a student. Then the PMF of  $X$  is given by

$$p_X(x) = \begin{cases} \frac{4}{20} & x = 5.5 \\ \frac{5}{20} & x = 5.8 \\ \frac{3}{20} & x = 6.0 \\ \frac{5}{20} & x = 6.2 \\ \frac{3}{20} & x = 6.5 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the expected height of a student selected randomly from the class

$$E[X] = 5.5\left(\frac{4}{20}\right) + 5.8\left(\frac{5}{20}\right) + 6.0\left(\frac{3}{20}\right) + 6.2\left(\frac{5}{20}\right) + 6.5\left(\frac{3}{20}\right) = 5.975$$

- 3.4 Let  $T$  denote the time it takes the machine to perform an operation. Then the PMF of  $T$  is given by

$$p_T(t) = \begin{cases} 0.60 & t = 2 \\ 0.25 & t = 4 \\ 0.15 & t = 7 \\ 0 & \text{otherwise} \end{cases}$$

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Thus, the expected time it takes the machine to perform a random operation is given by

$$E[T] = 2(0.6) + 4(0.25) + 7(0.15) = 3.25$$

- 3.5 Let  $X$  denote the time it takes the student to solve a problem. Then the PMF of  $X$  is given by

$$p_X(x) = \begin{cases} 0.1 & x = 60 \\ 0.4 & x = 45 \\ 0.5 & x = 30 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the expected time it takes the student to solve a random problem is given by

$$E[X] = 60(0.1) + 45(0.4) + 30(0.5) = 39$$

- 3.6 Let  $N$  be a random variable that denotes the amount won in a game. Then the PMF of  $N$  is given by

$$p_N(n) = \begin{cases} \frac{1}{6} & n = -3 \\ \frac{2}{6} & n = -1 \\ \frac{3}{6} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the expected winning in a game is given by

$$E[N] = 2\left(\frac{3}{6}\right) - \left(\frac{2}{6}\right) - 3\left(\frac{1}{6}\right) = \frac{1}{6}$$

- 3.7 Let  $K$  be a random variable that denotes the number of students in a van. Then the PMF of  $K$  is given by

$$p_K(k) = \begin{cases} \frac{12}{45} & k = 12 \\ \frac{15}{45} & k = 15 \\ \frac{18}{45} & k = 18 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the expected number of students in the van that carried the selected student is given by

$$E[K] = 12\left(\frac{12}{45}\right) + 15\left(\frac{15}{45}\right) + 18\left(\frac{18}{45}\right) = 16.4$$

3.8 Given the discrete random variable  $N$  whose PMF is given by

$$p_N(n) = p(1-p)^{n-1} \quad n = 1, 2, \dots$$

The expected value of  $N$  is given by

$$E[N] = \sum_n n p_N(n) = p \sum_{n=1}^{\infty} n (1-p)^{n-1}$$

Now,

$$\begin{aligned} \sum_{n=1}^{\infty} (1-p)^n &= \frac{1}{1-(1-p)} - 1 = \frac{1}{p} - 1 \\ \frac{d}{dp} \sum_{n=1}^{\infty} (1-p)^n &= \sum_{n=1}^{\infty} \frac{d}{dp} (1-p)^n = - \sum_{n=1}^{\infty} n (1-p)^{n-1} \end{aligned}$$

Thus,

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$$E[N] = -p \frac{d}{dp} \sum_{n=1}^{\infty} (1-p)^n = -p \frac{d}{dp} \left\{ \frac{1}{p} - 1 \right\} = -p \left\{ -\frac{1}{p^2} \right\} = \frac{1}{p}$$

3.9 Given a discrete random variable  $K$  whose PMF is given by

$$p_K(k) = \frac{5^k e^{-5}}{k!} \quad k = 0, 1, 2, \dots$$

The expected value of  $K$  is given by

$$E[K] = \sum_k k p_K(k) = e^{-5} \sum_{k=0}^{\infty} k \frac{5^k}{k!} = e^{-5} \sum_{k=1}^{\infty} \frac{5^k}{(k-1)!} = 5e^{-5} \sum_{k=1}^{\infty} \frac{5^{k-1}}{(k-1)!} = 5e^{-5} e^5 = 5$$

3.10 Given a continuous random variable  $X$  whose PDF is given by

$$f_X(x) = 2e^{-2x} \quad x \geq 0$$

The expected value of  $X$  is given by

$$E[X] = \int_0^{\infty} x f_X(x) dx = 2 \int_0^{\infty} x e^{-2x} dx$$

Let  $u = x \Rightarrow du = dx$ , and  $dv = e^{-2x} dx \Rightarrow v = -e^{-2x}/2$ . Thus,

$$E[X] = 2 \left\{ -\frac{x e^{-2x}}{2} \Big|_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-2x} dx \right\} = -\frac{e^{-2x}}{2} \Big|_0^{\infty} = \frac{1}{2}$$

3.11 If the random variable  $X$  represents the outcome of a single roll of a fair die, then  $p_i = 1/6, i = 1, 2, \dots, 6$ . Thus, the entropy of  $X$  is given by

$$H(X) = \sum_{i=1}^n p_i \log \frac{1}{p_i} = \sum_{i=1}^n \frac{1}{6} \log 6 = 6 \left( \frac{1}{6} \right) \log 6 = \log 6 = 2.5850$$

### Section 3.4: Moments of Random Variables and the Variance

3.12 The PMF of the random variable  $X$  is given by

$$p_X(x) = \begin{cases} p & x = 4 \\ 1-p & x = 7 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the mean  $E[X]$  and standard deviation  $\sigma_X$  of  $X$  are given by

$$E[X] = \sum_x x p_X(x) = 4p + 7(1-p) = 7 - 3p$$

$$E[X^2] = \sum_x x^2 p_X(x) = 16p + 49(1-p) = 49 - 33p$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = 49 - 33p - (7 - 3p)^2 = 9p(1-p)$$

$$\sigma_X = \sqrt{9p(1-p)} = 3\sqrt{p(1-p)}$$

3.13 The PMF of the discrete random variable  $X$  is given by

$$p_X(x) = \begin{cases} \frac{2}{5} & x = 3 \\ \frac{3}{5} & x = 6 \end{cases}$$

Thus, the mean and variance of  $X$  are given by



$$\begin{aligned}
E[X] &= \sum_x x p_X(x) = 3\left(\frac{2}{5}\right) + 6\left(\frac{3}{5}\right) = 4.8 \\
E[X^2] &= \sum_x x^2 p_X(x) = 9\left(\frac{2}{5}\right) + 36\left(\frac{3}{5}\right) = 25.2 \\
\sigma_X^2 &= E[X^2] - (E[X])^2 = 25.2 - (4.8)^2 = 2.16
\end{aligned}$$

3.14  $N$  is a random variable with the following CDF:

$$P_N(n) = \begin{cases} 0 & n < 1 \\ 0.2 & 1 \leq n < 2 \\ 0.5 & 2 \leq n < 3 \\ 0.8 & 3 \leq n < 4 \\ 1 & n \geq 4 \end{cases}$$

a. The PMF of  $N$  is given by

$$p_N(n) = \begin{cases} 0.2 & n = 1 \\ 0.3 & n = 2 \\ 0.3 & n = 3 \\ 0.2 & n = 4 \\ 0.0 & \text{otherwise} \end{cases}$$

b. The expected value of  $N$  is given by

$$E[N] = 1(0.2) + 2(0.3) + 3(0.3) + 4(0.2) = 0.2 + 0.6 + 0.9 + 0.8 = 2.5$$

c. The second moment of  $N$  is given by

$$E[N^2] = 1^2(0.2) + 2^2(0.3) + 3^2(0.3) + 4^2(0.2) = 0.2 + 1.2 + 2.7 + 3.2 = 7.3$$

Thus, the variance of  $N$  is given by

$$\sigma_N^2 = E[N^2] - (E[N])^2 = 7.3 - (2.5)^2 = 1.05$$

3.15  $X$  is a random variable that denotes the outcome of tossing a fair die once.

- a. The PMF of  $X$  is  $p_X(x) = \frac{1}{6} \quad x = 1, 2, \dots, 6$
- b. The expected value of  $X$  is  $E[X] = \frac{1}{6}\{1 + 2 + 3 + 4 + 5 + 6\} = \frac{21}{6} = 3.5$
- c. The variance of  $X$  is obtained as follows:

$$E[X^2] = \frac{1}{6}\{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\} = \frac{91}{6}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

3.16 The random variable  $X$  has the PDF  $f_X(x) = ax^3, 0 < x < 1$ .

- a. The value of  $a$  is obtained by  $\int_{-\infty}^{\infty} f_X(x)dx = 1 = a \int_0^1 x^3 dx = a \left[ \frac{x^4}{4} \right]_0^1 = \frac{a}{4} \Rightarrow a = 4$
- b. The expected value of  $X$  is given by

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx = a \int_0^1 x^4 dx = a \left[ \frac{x^5}{5} \right]_0^1 = \frac{a}{5} = \frac{4}{5} = 0.80$$

- c. The variance of  $X$  is obtained as follows:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = a \int_0^1 x^5 dx = a \left[ \frac{x^6}{6} \right]_0^1 = \frac{a}{6} = \frac{4}{6} = \frac{2}{3} = 0.667$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{2}{3} - \frac{16}{25} = \frac{2}{75} = 0.0267$$

- d. The value of  $m$  such that  $P[X \leq m] = 1/2$  is obtained as follows:

$$F_X(x) = P[X \leq x] = \int_0^x 4u^3 du = [u^4]_0^x = x^4$$

$$F_X(m) = m^4 = \frac{1}{2} \Rightarrow m^2 = \sqrt{\frac{1}{2}} = 0.7071$$

$$m = \sqrt{0.7071} = 0.8409$$

3.17 A random variable  $X$  has the CDF

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 0.5(x-1) & 1 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

a. The PDF of  $X$  is given by  $f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} 0.5 & 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$

b. The expected value of  $X$  is given by

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_1^3 0.5xdx = 0.5 \left[ \frac{x^2}{2} \right]_1^3 = 0.25[9-1] = 2$$

c. The variance of  $X$  is obtained as follows:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = 0.5 \int_1^3 x^2 dx = 0.5 \left[ \frac{x^3}{3} \right]_1^3 = \frac{0.5}{3} \{27-1\} = \frac{13}{3}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{13}{3} - 4 = \frac{13-12}{3} = \frac{1}{3}$$

3.18 Given a random variable  $X$  with the PDF  $f_X(x) = x^2/9$ ,  $0 \leq x \leq 3$ .

$$\begin{aligned}
 E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \frac{1}{9} \int_0^3 x^3 dx = \frac{1}{36} [x^4]_0^3 = \frac{81}{36} = \frac{9}{4} \\
 E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \frac{1}{9} \int_0^3 x^4 dx = \frac{1}{9} \left[ \frac{x^5}{5} \right]_0^3 = \frac{243}{45} = \frac{27}{5} \\
 \sigma_X^2 &= E[X^2] - (E[X])^2 = \frac{27}{5} - \frac{81}{16} = \frac{27\{16-15\}}{80} = \frac{27}{80} \\
 E[X^3] &= \int_{-\infty}^{\infty} x^3 f_X(x) dx = \frac{1}{9} \int_0^3 x^5 dx = \frac{1}{9} \left[ \frac{x^6}{6} \right]_0^3 = \frac{729}{54} = \frac{81}{6} = \frac{27}{2}
 \end{aligned}$$

3.19 Given the random variable  $X$  has the PDF  $f_X(x) = \lambda e^{-\lambda x}$ ,  $x \geq 0$ , the third moment of  $X$  is given by

$$E[X^3] = \int_{-\infty}^{\infty} x^3 f_X(x) dx = \lambda \int_0^{\infty} x^3 e^{-\lambda x} dx$$

Let  $u = x^3 \Rightarrow du = 3x^2 dx$ , and let  $dv = e^{-\lambda x} dx \Rightarrow v = -e^{-\lambda x}/\lambda$ . Thus,

$$E[X^3] = \lambda \left\{ -\frac{x^3 e^{-\lambda x}}{\lambda} \Big|_0^{\infty} + \frac{3}{\lambda} \int_0^{\infty} x^2 e^{-\lambda x} dx \right\} = 3 \int_0^{\infty} x^2 e^{-\lambda x} dx$$

Let  $u = x^2 \Rightarrow du = 2x dx$ , and let  $dv = e^{-\lambda x} dx \Rightarrow v = -e^{-\lambda x}/\lambda$ . Thus,

$$E[X^3] = 3 \left\{ -\frac{x^2 e^{-\lambda x}}{\lambda} \Big|_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} x e^{-\lambda x} dx \right\} = \frac{6}{\lambda} \int_0^{\infty} x e^{-\lambda x} dx$$

Let  $u = x \Rightarrow du = dx$ , and let  $dv = e^{-\lambda x} dx \Rightarrow v = -e^{-\lambda x}/\lambda$ . Thus,

$$E[X^3] = \frac{6}{\lambda} \left\{ -\frac{x e^{-\lambda x}}{\lambda} \Big|_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right\} = \frac{6}{\lambda^2} \int_0^{\infty} e^{-\lambda x} dx = \frac{6}{\lambda^2} \left[ -\frac{e^{-\lambda x}}{\lambda} \right]_0^{\infty} = \frac{6}{\lambda^3}$$

3.20  $X$  is a random variable with PDF  $f_X(x)$ , mean  $E[X]$ , and variance  $\sigma_X^2$ . We are given that  $Y = X^2$ .

$$\begin{aligned}
E[Y] &= E[X^2] = \sigma_X^2 + (E[X])^2 \\
E[Y^2] &= E[X^4] \\
\sigma_Y^2 &= E[Y^2] - (E[Y])^2 = E[X^4] - \{\sigma_X^2 + (E[X])^2\}^2 \\
&= E[X^4] - (\sigma_X^2)^2 - 2\sigma_X^2(E[X])^2 - (E[X])^4
\end{aligned}$$

3.21 The PDF of the random variable  $X$  is given by  $f_X(x) = 4x(9-x^2)/81$ ,  $0 \leq x \leq 3$ .

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf_X(x)dx = \frac{4}{81} \int_0^3 x^2(9-x^2)dx = \frac{4}{81} \int_0^3 (9x^2 - x^4)dx = \frac{4}{81} \left[ 3x^3 - \frac{x^5}{5} \right]_0^3 = \frac{8}{5} \\
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \frac{4}{81} \int_0^3 x^3(9-x^2)dx = \frac{4}{81} \int_0^3 (9x^3 - x^5)dx = \frac{4}{81} \left[ \frac{9x^4}{4} - \frac{x^6}{6} \right]_0^3 = 3 \\
\sigma_X^2 &= E[X^2] - (E[X])^2 = 3 - \frac{64}{25} = \frac{75-64}{25} = \frac{11}{25} \\
E[X^3] &= \int_{-\infty}^{\infty} x^3 f_X(x)dx = \frac{4}{81} \int_0^3 x^4(9-x^2)dx = \frac{4}{81} \int_0^3 (9x^4 - x^6)dx = \frac{4}{81} \left[ \frac{9x^5}{5} - \frac{x^7}{7} \right]_0^3 = \frac{216}{35}
\end{aligned}$$

### Section 3.5: Conditional Expectations

3.22 The PDF of  $X$  is given by  $f_X(x) = 4x(9-x^2)/81$ ,  $0 \leq x \leq 3$ . We obtain  $E[X|X \leq 2]$  as follows:

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^x f_X(u)du = \frac{4}{81} \int_0^x u(9-u^2)du = \frac{4}{81} \left[ \frac{9u^2}{2} - \frac{u^4}{4} \right]_0^x = \begin{cases} 0 & x < 0 \\ \frac{1}{81} \{ 18x^2 - x^4 \} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases} \\
E[X|X \leq 2] &= \frac{E[X]}{P[X \leq 2]} = \frac{E[X]}{F_X(2)}, \quad x \leq 2 \\
E[X] &= \int_{-\infty}^2 xf_X(x)dx = \frac{4}{81} \int_0^2 x^2(9-x^2)dx = \frac{4}{81} \int_0^2 (9x^2 - x^4)dx = \frac{4}{81} \left[ 3x^3 - \frac{x^5}{5} \right]_0^2 = \frac{352}{405} \\
E[X|X \leq 2] &= \frac{E[X]}{P[X \leq 2]} = \frac{E[X]}{F_X(2)} = \frac{(352/405)}{(56/81)} = \frac{44}{35} = 1.2571
\end{aligned}$$

3.23 The PDF of a continuous random variable  $X$  is given by

$$f_X(x) = 2e^{-2x} \quad x \geq 0$$

The conditional expected value of  $X$ , given that  $X \leq 3$  is obtained as follows:

$$F_X(x) = \int_{-\infty}^x f_X(u) du = \int_0^x 2e^{-2u} du = [-e^{-2u}]_0^x = \begin{cases} 0 & x < 0 \\ 1 - e^{-2x} & 0 \leq x < \infty \end{cases}$$

$$E[X|X \leq 3] = \frac{E[X]}{P[X \leq 3]} = \frac{\int_{-\infty}^3 xf_X(x) dx}{F_X(3)} = \frac{\int_0^3 2xe^{-2x} dx}{1 - e^{-6}} = \frac{2 \int_0^3 xe^{-2x} dx}{1 - e^{-6}}$$

Let  $u = x \Rightarrow du = dx$ , and let  $dv = e^{-2x} dx \Rightarrow v = -\frac{e^{-2x}}{2}$ . Thus,

$$2 \int_0^3 xe^{-2x} dx = 2 \left[ -\frac{xe^{-2x}}{2} \Big|_0^3 + \int_0^3 \frac{e^{-2x}}{2} dx \right] = -3e^{-6} + \left[ -\frac{e^{-2x}}{2} \right]_0^3 = \frac{1}{2} - \frac{1}{2}e^{-6} - 3e^{-6} = \frac{1 - 7e^{-6}}{2}$$

Therefore,

$$E[X|X \leq 3] = \frac{2 \int_0^3 xe^{-2x} dx}{1 - e^{-6}} = \frac{1 - 7e^{-6}}{2(1 - e^{-6})} = 0.4925$$

3.24 The PDF of  $X$  is given by

$$f_X(x) = \begin{cases} 0.1 & 30 \leq x \leq 40 \\ 0 & \text{otherwise} \end{cases}$$

The conditional expected value of  $X$ , given that  $X \leq 35$  is given by

$$E[X|X \leq 35] = \frac{\int_{30}^{35} x f_X(x) dx}{P[X \leq 35]} = \frac{\int_{30}^{35} 0.1x dx}{\int_{30}^{35} 0.1 dx} = \frac{\left[ \frac{0.1x^2}{2} \right]_{30}^{35}}{\left[ 0.1x \right]_{30}^{35}} = \frac{1}{2}[35 + 30] = 32.5$$

3.25  $N$  denotes the outcome of the toss of a fair coin. Let  $Y$  denote the event that the outcome is an even number. Then the expected value of  $N$ , given that the outcome is an even number, is given by

$$E[N|Y] = \frac{\sum_{n \in Y} n p_N(n)}{\sum_{n \in Y} p_N(n)} = \frac{\frac{1}{6}\{2 + 4 + 6\}}{\frac{1}{6} + \frac{1}{6} + \frac{1}{6}} = \frac{2 + 4 + 6}{3} = 4$$

3.26 The PDF of  $X$ , which denotes the life of a lightbulb, is given by

$$f_X(x) = 0.5e^{-0.5x} \quad x \geq 0$$

The expected value of  $X$ , given that  $X \leq 1.5$ , is given by

$$E[X|X \leq 1.5] = \frac{\int_0^{1.5} x f_X(x) dx}{P[X \leq 1.5]} = \frac{\int_0^{1.5} 0.5x e^{-0.5x} dx}{\int_0^{1.5} 0.5e^{-0.5x} dx} = \frac{0.5 \int_0^{1.5} x e^{-0.5x} dx}{1 - e^{-0.75}}$$

Let  $u = x \Rightarrow du = dx$ , and let  $dv = e^{-0.5x} dx \Rightarrow v = -\frac{e^{-0.5x}}{0.5}$ . Thus,

$$0.5 \int_0^{1.5} x e^{-0.5x} dx = 0.5 \left[ -\frac{x e^{-0.5x}}{0.5} \Big|_0^{1.5} + \int_0^{1.5} \frac{e^{-0.5x}}{0.5} dx \right] = -1.5e^{-0.75} + \left[ -\frac{e^{-0.5x}}{0.5} \right]_0^{1.5} = 2 - 3.5e^{-0.75}$$

Therefore,  $E[X|X \leq 3] = \frac{2 - 3.5e^{-0.75}}{1 - e^{-0.75}} = 0.6571$ .

### Sections 3.6 and 3.7: Chebyshev and Markov Inequalities

3.27 The PDF of  $X$  is given by  $f_X(x) = 2e^{-2x}$ ,  $x \geq 0$ . The Markov inequality states that

$$P[X \geq a] \leq \frac{E[X]}{a}$$

Now, the mean of  $X$  is given by

$$E[X] = \int_0^{\infty} 2xe^{-2x} dx = \frac{1}{2}$$

Thus,  $P[X \geq 1] \leq \frac{E[X]}{1} = \frac{1}{2}$ .

3.28 The PDF of  $X$  is given by  $f_X(x) = 2e^{-2x}$ ,  $x \geq 0$ . To obtain an upper bound for  $P[|X - E[X]| \geq 1]$  we use the Chebyshev inequality, which states that

$$P[|X - E[X]| \geq a] \leq \frac{\sigma_X^2}{a^2}$$

Now,  $E[X] = \int_0^{\infty} 2xe^{-2x} dx = \frac{1}{2}$ , and the second moment of  $X$  is given by

$$E[X^2] = \int_0^{\infty} 2x^2 e^{-2x} dx = 2 \int_0^{\infty} x^2 e^{-2x} dx$$

Let  $u = x^2 \Rightarrow du = 2x dx$ , and let  $dv = e^{-2x} dx \Rightarrow v = -e^{-2x}/2$ . Thus,

$$E[X^2] = 2 \int_0^{\infty} x^2 e^{-2x} dx = 2 \left\{ -\frac{x^2 e^{-2x}}{2} \Big|_0^{\infty} + \int_0^{\infty} x e^{-2x} dx \right\} = 2 \int_0^{\infty} x e^{-2x} dx = E[X] = \frac{1}{2}$$



Thus, the variance of  $X$  is given by  $\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ . Therefore,

$$P[|X - E[X]| \geq 1] \leq \frac{\sigma_X^2}{1} = \frac{1}{4}$$

3.29  $X$  has a mean 4 and variance 2. According to the Chebyshev inequality,

$$P[|X - E[X]| \geq a] \leq \frac{\sigma_X^2}{a^2}$$

Thus, an upper bound for  $P[|X - 4| \geq 2]$  is given by

$$P[|X - 4| \geq 2] \leq \frac{\sigma_X^2}{2^2} = \frac{2}{4} = \frac{1}{2}$$

3.30 The PDF of  $X$  is

$$f_X(x) = \begin{cases} \frac{1}{3} & 1 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

The variance of  $X$  is obtained as follows:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \frac{1}{3} \int_1^4 x dx = \frac{1}{3} \left[ \frac{x^2}{2} \right]_1^4 = \frac{5}{2} = 2.5 \\ E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \frac{1}{3} \int_1^4 x^2 dx = \frac{1}{3} \left[ \frac{x^3}{3} \right]_1^4 = 7 \\ \sigma_X^2 &= E[X^2] - (E[X])^2 = 7 - \frac{25}{4} = \frac{3}{4} = 0.75 \end{aligned}$$

Thus,  $P[|X - 2.5| \geq 2] \leq \frac{\sigma_X^2}{2^2} = \frac{(3/4)}{4} = \frac{3}{16} = 0.1875$ .



## Section 4.3: Binomial Distribution

- 4.1 The probability of a six on a toss of a die is  $p = 1/6$ . Let  $N(4)$  be a random variable that denotes the number of sixes that appear in tossing the four dice. Since the outcome of each die is independent of the outcome of any other die, the random variable  $N(4)$  has a binomial distribution. Thus, the PMF of  $N(4)$  is given by

$$p_{N(4)}(n) = \binom{4}{n} p^n (1-p)^{4-n} = \binom{4}{n} \left(\frac{1}{6}\right)^n \left(\frac{5}{6}\right)^{4-n} \quad n = 0, 1, 2, 3, 4$$

Therefore, the probability that at most one six appears is given by

$$\begin{aligned} P[N(4) \leq 1] &= p_{N(4)}(0) + p_{N(4)}(1) = \binom{4}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 + \binom{4}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 \\ &= \left(\frac{5}{6}\right)^4 + 4 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^3 = \left(\frac{5}{6}\right)^3 \left\{ \frac{5}{6} + \frac{4}{6} \right\} = \left(\frac{5}{6}\right)^3 \left(\frac{3}{2}\right) \\ &= 0.86806 \end{aligned}$$

- 4.2 Let  $K(9)$  be a random variable that denotes the number of operational components out of 9 components.  $K(9)$  has a binomial distribution with the PMF

$$p_{K(9)}(k) = \binom{9}{k} (1-p)^k p^{9-k} \quad k = 0, 1, \dots, 9$$

Let  $A$  denote the event that at least 6 of the components are operational. Then the probability of event  $A$  is given by

$$\begin{aligned}
 P[A] &= P[K(9) \geq 6] = \sum_{k=6}^9 p_{K(9)}(k) = \sum_{k=6}^9 \binom{9}{k} (1-p)^k p^{9-k} \\
 &= \frac{9!}{6!3!} (1-p)^6 p^3 + \frac{9!}{7!2!} (1-p)^7 p^2 + \frac{9!}{8!1!} (1-p)^8 p^1 + \frac{9!}{9!0!} (1-p)^9 p^0 \\
 &= 84p^3(1-p)^6 + 36p^2(1-p)^7 + 9p(1-p)^8 + (1-p)^9
 \end{aligned}$$

- 4.3 The random variable  $X$ , which denotes the number of heads that turn up, has the binomial distribution with the PMF

$$p_X(x) = \binom{3}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x} = \binom{3}{x} \left(\frac{1}{2}\right)^3 \quad x = 0, 1, 2, 3$$

Thus, the mean and variance of  $X$  are given by

$$\begin{aligned}
 E[X] &= 3p = \frac{3}{2} \\
 \sigma_X^2 &= 3p(1-p) = \frac{3}{4}
 \end{aligned}$$

- 4.4 Let  $Y(4)$  be a random variable that denotes the number of time in the 4 meeting times a week that the student is late. Then  $Y(4)$  is a binomial random variable with success probability  $p = 0.3$ , and its PMF is given by

$$p_{Y(4)}(y) = \binom{4}{y} p^y (1-p)^{4-y} = \binom{4}{y} (0.3)^y (0.7)^{4-y} \quad y = 0, 1, 2, 3, 4$$

- (a) The probability that the student is late for at least three classes in a given week is given by

$$\begin{aligned}
 P[Y(4) \geq 3] &= p_{Y(4)}(3) + p_{Y(4)}(4) = \binom{4}{3}(0.3)^3(0.7)^1 + \binom{4}{4}(0.3)^4(0.7)^0 \\
 &= 4(0.3)^3(0.7)^1 + (0.3)^4 = 0.0837
 \end{aligned}$$

(b) The probability that the student will not be late at all during a given week is given by

$$P[Y(4) = 0] = \binom{4}{0}(0.3)^0(0.7)^4 = (0.7)^4 = 0.2401$$

- 4.5 Let  $N(6)$  be a random variable that denotes the number of correct answers that John gets out of the 6 problems. Since each problem has 3 possible answers, the probability of getting a correct answer to a question by just guessing is  $p = 1/3$ . If we assume that John's performance is independent from one question to another, then  $N(6)$  is a binomially distributed random variable with the PMF

$$p_{N(6)}(n) = \binom{6}{n} p^n (1-p)^{6-n} = \binom{6}{n} \left(\frac{1}{3}\right)^n \left(\frac{2}{3}\right)^{6-n} \quad n = 0, 1, \dots, 6$$

Thus, the probability that John will get 4 or more correct answers by just guessing is given by

$$\begin{aligned}
 P[N(6) \geq 4] &= p_{N(6)}(4) + p_{N(6)}(5) + p_{N(6)}(6) \\
 &= \frac{6!}{4!2!} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + \frac{6!}{5!1!} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + \frac{6!}{6!0!} \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0 = \frac{(15 \times 2^2) + (6 \times 2) + 1}{3^6} = \frac{73}{729} \\
 &= 0.1001
 \end{aligned}$$

- 4.6 Let  $K(100)$  be a random variable that denotes the number of bits among the 100 bits that are received in error. Given that the probability of bit error is  $p = 0.001$  and that the channel treatment of each bit is independent of other bits,  $K(100)$  is a binomially distributed random variable with the PMF

$$p_{K(100)}(k) = \binom{100}{k} (0.001)^k (0.999)^{100-k} \quad k = 0, 1, \dots, 100$$

Thus, the probability that three or more bits are received in error is given by

$$\begin{aligned} P[K(100) \geq 3] &= 1 - P[K(100) < 3] = 1 - p_{K(100)}(0) - p_{K(100)}(1) - p_{K(100)}(2) \\ &= 1 - \binom{100}{0} (0.001)^0 (0.999)^{100} - \binom{100}{1} (0.001)^1 (0.999)^{99} - \binom{100}{2} (0.001)^2 (0.999)^{98} \\ &= 1 - (0.999)^{100} - 100(0.001)(0.999)^{99} - 4950(0.001)^2 (0.999)^{98} \\ &= 0.00015 \end{aligned}$$

- 4.7 Let  $N(4)$  denote the number of busy phone lines among the 4 phone lines. Since each phone line acts independently and the probability that a phone line is busy is 0.1,  $N(4)$  has a binomial distribution with PMF

$$p_{N(4)}(n) = \binom{4}{n} p^n (1-p)^{4-n} = \binom{4}{n} (0.1)^n (0.9)^{4-n} \quad n = 0, 1, 2, 3, 4$$

- a. The probability that all 4 phones are busy is given by

$$P[N(4) = 4] = p_{N(4)}(4) = (0.1)^4 = 0.0001$$

- b. The probability that 3 of the phones are busy is given by

$$P[N(4) = 3] = p_{N(4)}(3) = 4(0.1)^3 (0.9) = 0.0036$$

- 4.8 Given that each laptop has a probability of 0.10 of being defective and  $K$  is the number of defective laptops among the 8.

- a.  $K$  is a binomially distributed random variable, and its PMF is given by

$$p_K(k) = \binom{8}{k} p^k (1-p)^{8-k} = \binom{8}{k} (0.1)^k (0.9)^{8-k} \quad k = 0, 1, \dots, 8$$

- b. The probability that at most one laptop is defective out of the 8 is given by

$$P[K \geq 1] = 1 - P[K = 0] = 1 - p_K(0) = 1 - (0.9)^8 = 0.5695$$

- c. The probability that exactly one laptop is defective is given by

$$P[K = 1] = p_K(1) = 8(0.1)(0.9)^7 = 0.3826$$

- d. The expected number of defective laptops is given by  $E[K] = 8p = 0.8$ .

- 4.9 The probability that a product is defective is  $p = 0.25$ . Given that  $X$  is a random variable that denotes the number of defective products among 4 randomly selected products, the PMF of  $X$  is given by

$$p_X(x) = \binom{4}{x} p^x (1-p)^{4-x} = \binom{4}{x} (0.25)^x (0.75)^{4-x} \quad x = 0, 1, 2, 3, 4$$

Thus, the mean and variance of  $X$  are given by

$$E[X] = 4p = 1$$

$$\sigma_X^2 = 4p(1-p) = 0.75$$

- 4.10 Let  $N(5)$  be a random variable that denotes the number of heads in the 5 tosses. The PMF of  $N(5)$  is binomially distributed and is given by

$$p_{N(5)}(n) = \binom{5}{n} p^n (1-p)^{5-n} = \binom{5}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{5-n} = \binom{5}{n} \left(\frac{1}{2}\right)^5 \quad n = 0, 1, 2, 3, 4, 5$$

- 4.11 Let  $K(8)$  be a random variable that denotes the number of gadgets in a package of eight that are defective. Since the probability that a gadget is defective is 0.1 independently of other gadgets, the PMF of  $K(8)$  is given by

$$p_{K(8)}(k) = \binom{8}{k} p^k (1-p)^{8-k} = \binom{8}{k} (0.1)^k (0.9)^{8-k} \quad k = 0, 1, \dots, 8$$

Let A denote the event that the person that bought a given package will be refunded. Then

$$\begin{aligned} P[A] &= P[K(8) > 1] = 1 - P[K(8) \leq 1] = 1 - p_{K(8)}(0) - p_{K(8)}(1) = 1 - (0.9)^8 - 8(0.1)(0.9)^7 \\ &= 0.1869 \end{aligned}$$

4.12 Let  $N(12)$  denote the number of jurors among the 12 people in the jury that find the person guilty. Since each juror acts independently of other jurors and each juror has a probability  $p = 0.7$  of finding a person guilty, the PMF of  $N(12)$  is given by

$$p_{N(12)}(n) = \binom{12}{n} p^n (1-p)^{12-n} = \binom{12}{n} (0.7)^n (0.3)^{12-n} \quad n = 0, 1, \dots, 12$$

Let B denote the event that a person is convicted. Then the probability of event B is given by

$$\begin{aligned} P[B] &= P[N(12) \geq 10] = p_{N(12)}(10) + p_{N(12)}(11) + p_{N(12)}(12) \\ &= \frac{12!}{10!2!} (0.7)^{10} (0.3)^2 + \frac{12!}{11!1!} (0.7)^{11} (0.3) + (0.7)^{12} = 66(0.7)^{10} (0.3)^2 + 12((0.7)^{11} (0.3)) + (0.7)^{12} \\ &= 0.2528 \end{aligned}$$

4.13 The probability of target detection in a single scan is  $p = 0.1$ . Let  $K(n)$  denote the number of target detections in  $n$  consecutive scans. Then the PMF of  $K(n)$  is given by

$$p_{K(n)}(k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} (0.1)^k (0.9)^{n-k} \quad k = 0, 1, \dots, n$$

a. The probability that the target will be detected at least 2 times in 4 consecutive scans is given by

$$\begin{aligned} P[K(4) \geq 2] &= p_{K(4)}(2) + p_{K(4)}(3) + p_{K(4)}(4) = \frac{4!}{2!2!} (0.1)^2 (0.9)^2 + \frac{4!}{3!1!} (0.1)^3 (0.9) + (0.1)^4 \\ &= 6(0.1)^2 (0.9)^2 + 4(0.1)^3 (0.9) + (0.1)^4 = 0.04906 \end{aligned}$$



- b. The probability that the target will be detected at least once in 20 consecutive scans is given by

$$P[K(20) \geq 1] = 1 - P[K(20) = 0] = 1 - p_{K(20)}(0) = 1 - (0.9)^{20} = 0.8784$$

4.14 Since the probability that the machine makes errors in a certain operation with probability  $p$  and the fraction of errors of type A is  $a$ , the probability of a type A error is  $p_A = pa$ , and the probability of a type B is  $p_B = p(1 - a)$ . Let  $K(n)$  denote the number of errors in  $n$  operations,  $K_A(n)$  the number of type A errors in  $n$  operations, and  $K_B(n)$  the number of type B errors in  $n$  operations. Then the PMFs of  $K$ ,  $K_A(n)$ , and  $K_B(n)$  have the binomial distribution.

- a. The probability of  $k$  errors in  $n$  operations is given by

$$p_{K(n)}(k) = \binom{n}{k} p^k (1 - p)^{n - k} \quad k = 0, 1, \dots, n$$

- b. The probability of  $k_A$  type A errors in  $n$  operations is given by

$$p_{K_A(n)}(k_A) = \binom{n}{k_A} p_A^{k_A} (1 - p_A)^{n - k_A} = \binom{n}{k_A} (ap)^{k_A} (1 - ap)^{n - k_A} \quad k_A = 0, 1, \dots, n$$

- c. The probability of  $k_B$  type B errors in  $n$  operations is given by

$$p_{K_B(n)}(k_B) = \binom{n}{k_B} p_B^{k_B} (1 - p_B)^{n - k_B} = \binom{n}{k_B} \{p(1 - a)\}^{k_B} \{1 - p(1 - a)\}^{n - k_B} \quad k_B = 0, 1, \dots, n$$

- d. The probability of  $k_A$  type A errors and  $k_B$  type B errors in  $n$  operations is given by

$$P[K_A(n) = k_A, K_B(n) = k_B] = \binom{n}{k_A, k_B} (ap)^{k_A} \{p(1 - a)\}^{k_B} (1 - p)^{n - k_A - k_B} \quad n - k_A - k_B \geq 0$$

4.15 The probability that a marriage ends in divorce is 0.6, and divorces are independent of each other. The number of married couples is 10.

- a. The event that only the Arthurs and the Martins will stay married is a specific event whose probability of occurrence is the probability that these two couples remain

married while the other 8 couples get divorced. Thus, the probability of this event is  $p = (0.4)^8 \times (0.6)^2 = 0.00024$ .

- b. If  $N(10)$  denotes the number of married couples that stay married, then the probability that exactly 2 of the 10 couples will stay married is given by

$$P[N(10) = 2] = p_{N(10)}(2) = \binom{10}{2}(0.6)^2(0.4)^8 = \frac{10!}{2!8!}(0.6)^2(0.4)^8 = 0.01062$$

4.16 There are five traffic lights and each traffic light turns red independently with a probability  $p = 0.4$ .

- a.  $K$  is a random variable that denotes the number of lights at which the car stops. Then  $K$  is a binomial random variable, and its PMF is given by

$$p_K(k) = \binom{5}{k}p^k(1-p)^{5-k} = \binom{5}{k}(0.4)^k(0.6)^{5-k} \quad k = 0, 1, 2, 3, 4, 5$$

- b. The probability that the car stops at exactly two lights is

$$p_K(2) = \binom{5}{2}(0.4)^2(0.6)^3 = 10(0.4)^2(0.6)^3 = 0.3456$$

- c. The probability that the car stops at more than two lights is

$$\begin{aligned} P[K > 2] &= 1 - P[K \leq 2] = 1 - p_K(0) - p_K(1) - p_K(2) \\ &= 1 - (0.6)^5 - 5(0.4)(0.6)^4 - 10(0.4)^2(0.6)^3 = 0.31744 \end{aligned}$$

- d. The expected value of  $K$  is  $E[K] = 5p = 5(0.4) = 2.0$ .

4.17 Since the total number of students is 30, the probability that a randomly selected student is a boy is  $p_B = 18/30 = 0.6$ , and the probability that a randomly selected student is a girl is  $p_G = 1 - p_B = 0.4$ . Thus, the probability  $p$  that a randomly selected student knows the answer is

$$p = \frac{1}{3} \times 0.6 + \frac{1}{2} \times 0.4 = 0.4$$

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If  $K$  is a random variable that denotes the number of students who know the answer to a question that the teacher asks in class, then  $K$  is a binomially distributed random variable with success probability  $p$ . Therefore,

- a. The PMF of  $K$  is given by

$$p_K(k) = \binom{30}{k} p^k (1-p)^{30-k} = \binom{30}{k} (0.4)^k (0.6)^{30-k} \quad k = 0, 1, \dots, 30$$

- b. The mean of  $K$  is given by  $E[K] = 30p = 30(0.4) = 12$

- c. The variance of  $K$  is given by  $\sigma_K^2 = 30p(1-p) = 30(0.4)(0.6) = 7.2$ .

4.18 Since the balls are drawn with replacement, the probability that a red ball is drawn is given by  $p_R = 2/8 = 0.25$  and the probability that a green ball is drawn is 0.75. Let  $K$  denote the number of times a red ball is drawn in 10 trials. Then the probability that  $K = 4$  can be obtained as follows:

- a. Using the binomial distribution,

$$P[K = 4] = \binom{10}{4} p_R^4 (1-p_R)^6 = \binom{10}{4} (0.25)^4 (0.75)^6 = \frac{10}{4!6!} (0.25)^4 (0.75)^6 = 0.1460$$

- b. Using the Poisson approximation to the binomial distribution we have that

$$P[K = 4] = \frac{\lambda^4}{4!} e^{-\lambda}$$

where  $\lambda = 10p_R = 2.5$ . Thus, we obtain

$$P[K = 4] = \frac{(2.5)^4}{4!} e^{-2.5} = \frac{(2.5)^4}{24} e^{-2.5} = 0.1336$$

4.19 Given that 10 balls are randomly tossed into 5 boxes labeled  $B_1, B_2, \dots, B_5$ . The probability that a ball lands in box  $B_i$ ,  $i = 1, 2, \dots, 5$ , is  $p_i = 1/5 = 0.2$ .

- a. Thus, the probability that each box gets 2 balls is given by

$$\binom{10}{2 \quad 2 \quad 2 \quad 2 \quad 2} \{p^2\}^5 = \frac{10!}{(2!)^5} (0.2)^{10} = \frac{10!}{32} (0.2)^{10} = 0.0116$$

- b. The probability that box  $B_3$  is empty is given by

$$\binom{10}{0} p_3^0 (1-p_3)^{10} = (0.8)^{10} = 0.1074$$

- c. The probability that box  $B_2$  has 6 balls is given by

$$\binom{10}{6} p_2^6 (1-p_2)^4 = \frac{10!}{6!4!} (0.2)^6 (0.8)^4 = 0.0055$$

## Section 4.4: Geometric Distribution

4.20 Let  $K$  be the random variable that denotes the number of times that a fair die is rolled repeatedly until a 6 appears. The probability that a 6 appears on any roll is  $p = 1/6$ . Thus,  $K$  is a geometrically distributed random variable with the PMF

$$p_K(k) = p(1-p)^{k-1} = \frac{1}{6} \left(\frac{5}{6}\right)^{k-1} \quad k = 1, 2, \dots$$

- a. The probability that the experiment stops at the fourth roll is given by

$$P[K = 4] = p_K(4) = \frac{1}{6} \left(\frac{5}{6}\right)^3 = \frac{125}{1296} = 0.0964$$

- b. Let  $A$  be the event that the experiment stops at the third roll and  $B$  the event that the sum of three rolls is at least 12. Then

$$P[B|A] = \frac{P[A \cap B]}{P[A]}$$

The event  $A \cap B$  is the event that the sum of the first two rolls is at least 6 and the third roll is a 6. That is,  $A \cap B$  is the event whose sample space is

$$\left\{ \begin{array}{l} (1, 5, 6), (2, 4, 6), (2, 5, 6), (3, 3, 6), (3, 4, 6) \\ (3, 5, 6), (4, 2, 6), (4, 3, 6), (4, 4, 6), (4, 5, 6) \\ (5, 1, 6), (5, 2, 6), (5, 3, 6), (5, 4, 6), (5, 5, 6) \end{array} \right\}$$

Since the sample space of an experiment that consists of rolling a die three times contains  $6 \times 6 \times 6 = 216$  equally likely sample points, we have that  $P[A \cap B] = 15/216$ . Also, since  $P[A] = p(1-p)^2 = (1/6) \times (5/6)^2 = 25/216$ , we have that

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{(15/216)}{(25/216)} = \frac{3}{5}$$

4.21 The probability that a key opens the door on any trial is  $p = 1/6$ . Let  $K$  be a random variable that denotes the number of trials until the door is opened. The PMF of  $K$  is given by

$$p_K(k) = Ap(1-p)^{k-1} \quad k = 1, 2, \dots, 6$$

where  $A$  is the normalization factor required to make the PMF sum to 1. Specifically,

$$\sum_{k=1}^6 Ap(1-p)^{k-1} = 1 = Ap \left\{ \frac{1-(1-p)^6}{1-(1-p)} \right\} = A \{1-(1-p)^6\} \Rightarrow A = \frac{1}{1-(1-p)^6}$$

Thus, we have the following *truncated geometric distribution*:

$$p_K(k) = Ap(1-p)^{k-1} = \frac{p(1-p)^{k-1}}{1-(1-p)^6} \quad k = 1, 2, \dots, 6$$

The expected number of keys we will have to try before the door is opened is given by

$$\begin{aligned} \sum_{k=1}^6 k p_K(k) &= \frac{p}{1 - (1-p)^6} \sum_{k=1}^6 k (1-p)^{k-1} \\ &= \left[ \frac{(1/6)}{1 - (5/6)^6} \right] \left\{ 1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + 4\left(\frac{5}{6}\right)^3 + 5\left(\frac{5}{6}\right)^4 + 6\left(\frac{5}{6}\right)^5 \right\} = 2.8535 \end{aligned}$$

4.22 We are given a box containing  $R$  red balls and  $B$  blue balls in which a ball is randomly selected from the box with replacement until a blue ball is selected. First, we note that the probability of success in any trial is given by  $p = B/(R+B)$ . Let  $N$  be a random variable that denotes the number of trials until a blue ball is selected. Then the PMF of  $N$  is given by

$$p_N(n) = p(1-p)^{n-1} \quad n = 1, 2, \dots$$

a. The probability that the experiment stops after exactly  $n$  trials is

$$P[N = n] = p_N(n) = p(1-p)^{n-1} = \left(\frac{B}{R+B}\right) \left(\frac{R}{R+B}\right)^{n-1} \quad n = 1, 2, \dots$$

b. The probability that the experiment requires at least  $k$  trials before it stops is

$$P[N \geq k] = 1 - P[N < k] = 1 - \sum_{n=1}^{k-1} p(1-p)^{n-1} = 1 - \frac{p[1 - (1-p)^{k-1}]}{1 - (1-p)} = (1-p)^{k-1} = \left(\frac{R}{R+B}\right)^{k-1}$$

4.23 Let  $K$  denote the number of tries until we find a person who wears glasses. The probability of success on any try is  $p = 0.2$ . Thus, the PMF of  $K$  is given by

$$p_K(k) = p(1-p)^{k-1} = 0.2(0.8)^{k-1} \quad k = 1, 2, \dots$$

a. The probability that it takes exactly 10 tries to get a person who wears glasses is

$$P[K = 10] = 0.2(0.8)^{10-1} = 0.2(0.8)^9 = 0.02684$$

- b. The probability that it takes at least 10 tries to get a person who wears glasses is

$$P[K \geq 10] = 1 - P[K < 10] = 1 - \sum_{k=1}^9 p(1-p)^{k-1} = (1-p)^9 = (0.8)^9 = 0.1342$$

- 4.24 Since her score in any of the exams is uniformly distributed between 800 and 2200, the PDF of  $X$ , her score in any exam, is given by

$$f_X(x) = \begin{cases} \frac{1}{1400} & 800 \leq x \leq 2200 \\ 0 & \text{otherwise} \end{cases}$$

- a. The probability that she reaches her goal of scoring at least 2000 points in any exam is given by

$$p = P[X \geq 2000] = \int_{x=2000}^{2200} f_X(x) dx = \frac{1}{1400} \int_{x=2000}^{2200} dx = \frac{1}{1400} [x]_{2000}^{2200} = \frac{200}{1400} = \frac{1}{7} = 0.1428$$

- b. Let  $K$  denote the number times she will take the exam before reaching her goal. Then  $K$  is a geometrically distributed random variable whose PMF is given by

$$p_K(k) = p(1-p)^{k-1} = \frac{1}{7} \left(\frac{6}{7}\right)^{k-1} \quad k = 1, 2, \dots$$

- c. The expected number of times she will take the exam is  $E[K] = 1/p = 7$ .

## Section 4.5: Pascal Distribution

- 4.25 Let  $K$  be a random variable that denotes the number of 100-mile units Sam travels before a tire fails. Then the PMF of  $K$  is given by

$$p_K(k) = 0.05(0.95)^{k-1} \quad k = 1, 2, 3, \dots$$

We are told that Sam embarked on an 800-mile trip and took two spare tires with him on the trip.

- a. The probability that the first change of tire occurred 300 miles from his starting point is given by

$$P[K = 3] = p_K(3) = 0.05(0.95)^2 = 0.04512$$

- b. Let  $K_r$  denote the number of 100-mile units he travels until the  $r$ th tire change. Then  $K_r$  is the  $r$ th-order Pascal random variable with the PMF

$$p_{K_r}(k) = \binom{k-1}{r-1} (0.05)^r (0.95)^{k-r} \quad k = r, r+1, \dots; r = 1, 2, \dots, k$$

Thus, the probability that his second change of tire occurred 500 miles from his starting point (or 5th 100-mile unit) is given by

$$\begin{aligned} P[K_2 = 5] &= \binom{5-1}{2-1} (0.05)^2 (0.95)^3 = \binom{4}{1} (0.05)^2 (0.95)^3 = 4(0.05)^2 (0.95)^3 \\ &= 0.00857 \end{aligned}$$

- c. The probability that he completed the trip without having to change tires is given by

$$\begin{aligned} P[K > 8] &= 1 - P[K \leq 8] = 1 - \sum_{k=1}^8 0.05(0.95)^{k-1} = 1 - \frac{0.05[1 - 0.95^8]}{1 - 0.95} \\ &= 0.95^8 = 0.6634 \end{aligned}$$

- 4.26 Let  $p = 0.2$  denote the success probability, which is the probability that an applicant offered a job actually accepts the job. Let  $K_r$  be a random variable that denotes the number of candidates offered a job up to and including the  $r$ th candidate to accept the job. Then  $K_r$  is the  $r$ th-order Pascal random variable with the PMF

$$p_{K_r}(k) = \binom{k-1}{r-1} (0.2)^r (0.8)^{k-r} \quad k = r, r+1, \dots; r = 1, 2, \dots, k$$

The probability that the sixth-ranked applicant will be offered one of the 3 positions is the probability that 6th candidate is either the first or second or third person to accept a job. This probability,  $Q$ , is given by



$$\begin{aligned}
Q &= P[(K_1 = 6) \cup (K_2 = 6) \cup (K_3 = 6)] + P[K_1 = 6] + P[K_2 = 6] + P[K_3 = 6] \\
&= p(1-p)^5 + \binom{5}{1}p^2(1-p)^4 + \binom{5}{2}p^3(1-p)^3 \\
&= 0.2(0.8)^5 + 5(0.2)^2(0.8)^4 + 10(0.2)^3(0.8)^3 = 0.0463
\end{aligned}$$

4.27 Let  $K_r$  be a random variable that denotes the number of tries up to and including the try that results in the  $r$ th person who wears glasses. Since the probability of success on any try is  $p = 0.2$ , the PMF of  $K_r$  is given by

$$p_{K_r}(k) = \binom{k-1}{r-1}(0.2)^r(0.8)^{k-r} \quad k = r, r+1, \dots; r = 1, 2, \dots, k$$

- a. The probability that it takes exactly 10 tries to get the 3rd person who wears glasses is given by

$$P[K_3 = 10] = p_{K_3}(10) = \binom{10-1}{3-1}(0.2)^3(0.8)^7 = \binom{9}{2}(0.2)^3(0.8)^7 = 36(0.2)^3(0.8)^7 = 0.0604$$

- b. The probability that it takes at least 10 tries to get the 3rd person who wears glasses is given by

$$P[K_3 \geq 10] = 1 - P[K_3 < 10] = 1 - \sum_{k=3}^9 \binom{k-1}{3-1}(0.2)^3(0.8)^{k-3} = 1 - \sum_{k=3}^9 \binom{k-1}{2}(0.2)^3(0.8)^{k-3}$$

4.28 The probability of getting a head in a single toss of a biased coin is  $q$ . Let  $N_k$  be a random variable that denotes the number of tosses up to and including the toss that results in the  $k$ th head. Then  $N_k$  is a  $k$ th-order Pascal random variable with the PMF

$$p_{N_k}(n) = \binom{n-1}{k-1}q^k(1-q)^{n-k} \quad n = k, k+1, \dots; k = 1, 2, \dots, n$$

The probability that the 18th head occurs on the 30th toss is given by

$$P[N_{18} = 30] = p_{N_{18}}(30) = \binom{30-1}{18-1}q^{18}(1-q)^{12} = \binom{29}{17}q^{18}(1-q)^{12} = 51895935q^{18}(1-q)^{12}$$

4.29 If we define success as the situation in which Pete gives away a book, then the probability of success at each door Pete visits is given by  $p = 0.75 \times 0.5 = 0.375$ . Let  $X_k$  be a random variable that denotes the number of doors Pete visits up to and including the door where he has his  $k$ th success. Then  $X_k$  is a  $k$ th-order Pascal random variable with the PMF

$$p_{X_k}(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} \quad x = k, k+1, \dots; k = 1, 2, \dots, x$$

- a. The probability that Pete gives away his first book at the third house he visits is given by

$$P[X_1 = 3] = p_{X_1}(3) = \binom{3-1}{1-1} p^1 (1-p)^2 = \binom{2}{0} p (1-p)^2 = p(1-p)^2 = (0.375)(0.625)^2 = 0.1465$$

- b. The probability that he gives away his second book to the fifth family he visits is

$$P[X_2 = 5] = p_{X_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^3 = \binom{4}{1} (0.375)^2 (0.625)^3 = 4(0.375)^2 (0.625)^3 = 0.1373$$

- c. Since the outcomes of the visits are independent, the event that he gives away the fifth book to the eleventh family he visits, given that he has given away exactly four books to the first eight families he visited, is equivalent to the event that he encounters failures at the 9th door and the 10th door but success at the 11th door. Thus, the probability of this event is  $q = p(1-p)^2 = (0.375)(0.625)^2 = 0.1465$ .

- d. Given that he did not give away the second book at the second house, the probability that he will give it out at the fifth house is given by

$$\begin{aligned} P[X_2 = 5 | X_2 > 2] &= \frac{P[(X_2 = 5) \cap (X_2 > 2)]}{P[X_2 > 2]} = \frac{P[X_2 = 5]}{P[X_2 > 2]} = \frac{p_{X_2}(5)}{1 - p_{X_2}(2)} = \frac{\binom{5-1}{2-1} p^2 (1-p)^3}{1 - p^2 (1-p)^0} \\ &= \frac{4p^2 (1-p)^3}{1 - p^2} = \frac{4p^2 (1-p)^2}{1 + p} = \frac{4(0.375)^2 (0.625)^2}{1.375} = 0.1598 \end{aligned}$$

- 4.30 Let  $X_k$  be a random variable that denotes the number of customers up to and including the customer that received the  $k$ th coupon. If  $p = 0.3$  is the probability that a customer receives a coupon, then  $X_k$  is a  $k$ th-order Pascal random variable with the PMF

$$p_{X_k}(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} \quad x = k, k+1, \dots; k = 1, 2, \dots, x$$

Thus, the probability that on a particular day the third coupon was given to the eighth customer is given by

$$P[X_3 = 8] = p_{X_3}(8) = \binom{8-1}{3-1} p^3 (1-p)^{8-3} = \binom{7}{2} (0.3)^3 (0.7)^5 = 21(0.3)^3 (0.7)^5 = 0.0953$$

- 4.31 The probability of success in any sale is  $p = 0.6$ . Let  $X_k$  denote the number of calls up to and including the  $k$ th success. The PMF of  $X_k$  is given by

$$p_{X_k}(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k} = \binom{x-1}{k-1} (0.6)^k (0.4)^{x-k} \quad x = k, k+1, \dots; k = 1, 2, \dots, x$$

- a. The probability that she earned her third dollar on the sixth call she made is given by

$$P[X_3 = 6] = p_{X_3}(6) = \binom{6-1}{3-1} p^3 (1-p)^{6-3} = \binom{5}{2} (0.6)^3 (0.4)^3 = 10(0.24)^3 = 0.13824$$

- b. If she made 6 calls per hour, then in 2 hours she made 12 calls. Therefore, the probability that she earned \$8 in two hours is given by the binomial distribution of 8 successes in 12 trials, which is

$$\binom{12}{8} p^8 (1-p)^4 = 495(0.6)^8 (0.4)^4 = 0.2128$$

## Section 4.6: Hypergeometric Distribution

- 4.32 Given a list that contains the names of 4 girls and 6 boys, let the random variable  $G$  denote the number of girls among 5 students that are randomly selected from the list. Then the probability that the 5 students selected will consist of 2 girls and 3 boys is given by

$$P[G = 2] = \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}} = 0.4762$$

4.33 Let  $M$  denote the number of Massachusetts senators among the group of 20 senators randomly chosen. To see  $M$  as a hypergeometric random variable, we imagine the senators being grouped into two: one group of 2 from Massachusetts and one group of 98 from other states. Thus,

- a. The probability that the two Massachusetts senators are among those chosen is

$$P[M = 2] = \frac{\binom{2}{2}\binom{98}{18}}{\binom{100}{20}} = 0.03838$$

- b. The probability that neither of the two Massachusetts senators is among those selected is

$$P[M = 0] = \frac{\binom{2}{0}\binom{98}{20}}{\binom{100}{20}} = 0.63838$$

4.34 By memorizing only 8 out of 12 problems, Alex has partitioned the problems into two sets: the set of problems he knows and the set of problems he does not know. Let  $K$  be a random variable that denotes the number of problems that Alex gets correctly. Then  $K$  is a hypergeometric random variable; thus, the probability that Alex is able to solve 4 or more problems correctly in the exam is given by

$$\begin{aligned}
 P[K \geq 4] &= \sum_{k=4}^6 \left\{ \frac{\binom{8}{k} \binom{4}{6-k}}{\binom{12}{6}} \right\} = \frac{\binom{8}{4} \binom{4}{2} + \binom{8}{5} \binom{4}{1} + \binom{8}{6} \binom{4}{0}}{\binom{12}{6}} = \frac{420 + 224 + 28}{924} = \frac{672}{924} \\
 &= 0.7273
 \end{aligned}$$

4.35 The total number of students is 30. Let  $N$  be a random variable that denotes the number of girls among a group of 15 students randomly selected to represent the class in a competition.

a. The probability that 8 girls are in the group is given by

$$P[N = 8] = \frac{\binom{12}{8} \binom{18}{7}}{\binom{30}{15}} = 0.10155$$

b. The probability that a randomly selected student is a boy is  $p = 18/30 = 0.6$ . Thus, the expected number of boys in the selected group is  $15p = 15(0.6) = 9$ .

4.36 The probability of randomly selecting 4 gloves that consist of 2 right gloves and 2 left gloves from a drawer containing 10 left gloves and 12 right gloves is given by

$$\frac{\binom{10}{2} \binom{12}{2}}{\binom{22}{4}} = 0.4060$$

## Section 4.7: Poisson Distribution

4.37 Let  $N$  denote the number of cars that arrive at the gas station. Since  $N$  is a Poisson random variable with a mean of  $\lambda = 50/60 = 5/6$  cars per minute, the PMF is given by

$$p_N(n) = \frac{(5/6)^n}{n!} e^{-(5/6)} \quad n = 0, 1, 2, \dots$$

Since it takes exactly 1 minute to service a car, a waiting line occurs when at least 1 other car arrives within the 1-minute interval it takes to finish serving the current car receiving service. Thus, the probability that a waiting line will occur at the station is given by

$$P[N > 0] = 1 - P[N = 0] = 1 - e^{-(5/6)} = 0.5654$$

4.38 Let  $K$  denote the number of traffic tickets that the traffic officer gives out on any day. Then the PMF of  $K$  is given by

$$p_K(k) = \frac{7^k e^{-7}}{k!} \quad k = 0, 1, 2, \dots$$

a. The probability that on one particular day the officer gave out no ticket is given by

$$P[K = 0] = p_K(0) = e^{-7} = 0.0009$$

b. The probability that she gives out fewer than 4 tickets on that day is given by

$$P[K < 4] = p_K(0) + p_K(1) + p_K(2) + p_K(3) = e^{-7} \left\{ 1 + 7 + \frac{49}{2} + \frac{343}{6} \right\} = 0.0818$$

4.39 Let  $M$  denote the number of particles emitted per second. Since  $M$  has a Poisson distribution with a mean of 10, its PMF is given by

$$p_M(m) = \frac{10^m e^{-10}}{m!} \quad m = 0, 1, 2, \dots$$

a. The probability of at most 3 particles in one second is given by

$$P[M \leq 3] = p_M(0) + p_M(1) + p_M(2) + p_M(3) = e^{-10} \left\{ 1 + 10 + \frac{100}{2} + \frac{1000}{6} \right\} = 0.01034$$

b. The probability of more than 1 particle in one second is given by

$$P[M > 1] = 1 - P[M \leq 1] = 1 - p_M(0) - p_M(1) = 1 - e^{-10} \{ 1 + 10 \} = 0.9995$$

- 
- 4.40 Let  $K$  denote the number of cars that arrive at the window over a 20-minute period. Since  $K$  is a Poisson random variable with a mean of 4, its PMF is given by

$$p_K(k) = \frac{4^k e^{-4}}{k!} \quad k = 0, 1, 2, \dots$$

The probability that more than three cars will arrive during any 20-minute period is

$$\begin{aligned} P[K > 3] &= 1 - P[K \leq 3] = 1 - \{p_K(0) + p_K(1) + p_K(2) + p_K(3)\} \\ &= 1 - e^{-4} \left\{ 1 + 4 + \frac{16}{2} + \frac{64}{6} \right\} = 0.5665 \end{aligned}$$

- 4.41 Let  $N$  denote the number of phone calls that arrive in 1 hour. Since  $N$  has a Poisson distribution with a mean of 4, its PMF is given by

$$p_N(n) = \frac{4^n e^{-4}}{n!} \quad n = 0, 1, 2, \dots$$

- a. The probability that no phone calls arrive in a given hour is

$$P[N = 0] = p_N(0) = e^{-4} = 0.0183$$

- b. The probability that more than 2 calls arrive within a given hour is given by

$$P[N > 2] = 1 - P[N \leq 2] = 1 - \{p_N(0) + p_N(1) + p_N(2)\} = 1 - e^{-4} \left\{ 1 + 4 + \frac{16}{2} \right\} = 0.7619$$

- 4.42 Let  $K$  denote the number of typing mistakes on a given page. Since  $K$  has a Poisson distribution with a mean of 3, its PMF is given by

$$p_K(k) = \frac{3^k e^{-3}}{k!} \quad k = 0, 1, 2, \dots$$

- a. The probability that there are exactly 7 mistakes on a given page is

$$P[K = 7] = p_K(7) = \frac{3^7 e^{-3}}{7!} = 0.0216$$

- b. The probability that there are fewer than 4 mistakes on a given page is

$$P[K < 4] = p_K(0) + p_K(1) + p_K(2) + p_K(3) = e^{-3} \left\{ 1 + 3 + \frac{9}{2} + \frac{27}{6} \right\} = 0.6472$$

- c. The probability that there is no mistake on a given page is

$$P[K = 0] = p_K(0) = e^{-3} = 0.0498$$

## Section 4.8: Exponential Distribution

4.43 The PDF of the random variable  $T$  is given by

$$f_T(t) = ke^{-4t} \quad t \geq 0$$

- a. To find the value of  $k$  we know that

$$\int_0^{\infty} f_T(t) dt = k \int_0^{\infty} e^{-4t} dt = 1 \Rightarrow k \left[ -\frac{e^{-4t}}{4} \right]_0^{\infty} = \frac{k}{4} = 1 \Rightarrow k = 4$$

- b. The expected value of  $T$  is given by  $E[T] = \frac{1}{4}$ .

- c.  $P[T < 1] = \int_0^1 f_T(t) dt = \int_0^1 4e^{-4t} dt = [-e^{-4t}]_0^1 = 1 - e^{-4} = 0.9817$

4.44 Given the lifetime  $X$  of a system in weeks with the PDF

$$f_X(x) = \begin{cases} 0.25e^{-0.25x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- a.  $E[X] = 1/0.25 = 4$

- b.  $F_X(x) = 1 - e^{-0.25x} \quad x \geq 0$



c.  $\sigma_X^2 = 1/(0.25)^2 = 16$

d.  $P[X > 2] = 1 - F_X(2) = e^{-0.25(2)} = e^{-0.5} = 0.6065$

- e. Because of the forgetfulness property of the exponential distribution, the probability that the system will fail between the fourth and sixth weeks, given that it has not failed by the end of the fourth week, is simply the probability that it will fail within 2 weeks. That is,

$$P[4 < X < 6 | X > 4] = P[X \leq 2] = F_X(2) = 1 - e^{-0.5} = 0.3935$$

4.45 The PDF of  $T$ , the time in hours between bus arrivals at a bus station, is given by

$$f_T(t) = 2e^{-2t} \quad t \geq 0$$

a.  $E[T] = 1/2 = 0.5$

b.  $\sigma_X^2 = 1/(2)^2 = 1/4 = 0.25$

c.  $P[T > 1] = 1 - P[T \leq 1] = 1 - F_T(1) = e^{-2(1)} = e^{-2} = 0.1353$

4.46 Given that the PDF of the times  $T$  in minutes between successive bus arrivals at a suburban bus stop is given by

$$f_T(t) = 0.1e^{-0.1t} \quad t \geq 0$$

If a turtle that requires 15 minutes to cross the street starts crossing the street at the bus station immediately after a bus just left the station, the probability that the turtle will not be on the road when the next bus arrives is the probability that no bus arrives within the time it takes the turtle to cross the street. This probability,  $p$ , is given by

$$p = \int_{15}^{\infty} f_T(t) dt = \int_{15}^{\infty} 0.1e^{-0.1t} dt = [-e^{-0.1t}]_{15}^{\infty} = e^{-1.5} = 0.2231$$

4.47 Given that the PDF of the times  $T$  in minutes between successive bus arrivals at a suburban bus stop is given by

$$f_T(t) = 0.2e^{-0.2t} \quad t \geq 0$$

If an ant that requires 10 minutes to cross the street starts crossing the street at the bus station immediately after a bus has left the station has survived 8 minutes since its journey, then we obtain the following:

- a. The probability  $p$  that the ant will completely cross the road before the next bus arrives is the probability that no bus arrives within the remaining 2 minutes of its journey. Because of the forgetfulness property of the exponential distribution, the PDF of the time until the next bus arrives is still exponentially distributed with the same mean as  $T$ . Thus,  $p$  is given by

$$p = P[T > 2] = 1 - F_T(2) = e^{-0.2(2)} = e^{-0.4} = 0.6703$$

- b. Because of the forgetfulness property of the exponential distribution, the expected time in minutes until the next bus arrives is the expected value of  $T$ , which is

$$E[T] = 1/0.2 = 5$$

4.48 The PDF of the times  $X$  between telephone calls that arrive at a switchboard is given by

$$f_X(x) = \frac{1}{30}e^{-x/30} \quad x \geq 0$$

Given that a call has just arrived, the probability that it takes at least 2 hours (or 120 minutes) before the next call arrives is given by

$$P[X \geq 2] = 1 - P[X < 120] = 1 - F_X(120) = e^{-120/30} = e^{-4} = 0.0183$$

4.49 The PDF of durations  $T$  of calls to a radio talk show is given by

$$f_T(t) = \frac{1}{3}e^{-t/3} \quad t \geq 0$$

- 
- a. The probability that a call will last less than 2 minutes is given by

$$P[T < 2] = F_T(2) = 1 - e^{-2/3} = 0.4866$$

- b. The probability that a call will last longer than 4 minutes is given by

$$P[T > 4] = 1 - P[T \leq 4] = 1 - F_T(4) = e^{-4/3} = 0.2636$$

- c. Because of the forgetfulness property of the exponential distribution, the probability that a call will last at least another 4 minutes, given that it has already lasted 4 minutes, is the probability that a call will last at least 4 minutes. This is given by

$$P[T \geq 8 | T > 4] = P[T \geq 4] = 1 - F_T(4) = e^{-4/3} = 0.2636$$

- d. Because of the forgetfulness property of the exponential distribution, the expected remaining time until a call ends, given that it has already lasted 4 minutes, is the mean length of a call, which is  $E[T] = 3$  minutes.

4.50 Let  $X$  denote the life of a battery in weeks. Then the PDF of  $X$  is given by

$$f_X(t) = \frac{1}{4}e^{-x/4} \quad x \geq 0$$

- a. The probability that the battery life exceeds 2 weeks is given by

$$P[X > 2] = 1 - P[X \leq 2] = 1 - F_X(2) = e^{-2/4} = e^{-0.5} = 0.6065$$

- b. Because of the forgetfulness property of the exponential distribution, the probability that the battery will last at least another 5 weeks, given that it has already lasted 6 weeks, is the probability that a battery will last at least 5 weeks. That is,

$$P[X \geq 11 | X > 6] = P[X \geq 5] = 1 - F_X(5) = e^{-5/4} = 0.2865$$

4.51 The PDF of the times  $T$  in weeks between employee strikes is given by

$$f_T(t) = 0.02e^{-0.02t} \quad t \geq 0$$

- a. The expected time between strikes at the company is given by  $E[T] = 1/0.02 = 50$  weeks.

$$P[T \leq t | T < 40] = \frac{F_T(t)}{F_T(40)} \quad 0 \leq t < 40$$

b.

$$= \frac{1 - e^{-0.02t}}{1 - e^{-0.02(40)}} = \frac{1 - e^{-0.02t}}{1 - e^{-0.8}} = 1.8160(1 - e^{-0.02t}) \quad 0 \leq t < 40$$

c.  $P[40 < T < 60] = F_T(60) - F_T(40) = e^{-0.8} - e^{-1.2} = 0.1481$ .

4.52 The hazard function of the random variable  $X$  is given by  $h_X(x) = 0.05$ . Thus,

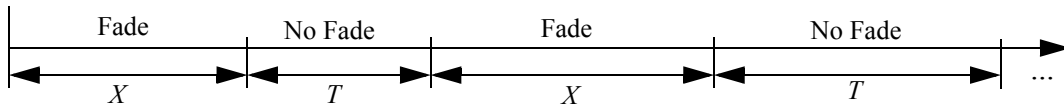
$$h_X(x) = 0.05 = \frac{f_X(x)}{1 - F_X(x)} \Rightarrow 1 - F_X(x) = \exp\left\{-\int_0^x h_X(t)dt\right\} = \exp\{-[0.5t]_0^x\} = e^{-0.5x}$$

This implies that  $F_X(x) = 1 - e^{-0.5x}$ ; that is,  $X$  is an exponentially distributed random variable with the PDF

$$f_X(x) = \frac{d}{dx}F_X(x) = 0.5e^{-0.5x} \quad x \geq 0$$

## Section 4.9: Erlang Distribution

- 4.53 The random variable  $X$  denotes the duration of a fade, and the random variable  $T$  denotes the intervals between fades. Thus, the cycle of events on the channel can be represented as follows:



The PDF of  $X$  is given by

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

Since  $T$  is an Erlang random variable with PDF

$$f_T(t) = \frac{\mu^4 t^3 e^{-\mu t}}{3!} \quad t \geq 0$$

we observe that it is a 4th-order Erlang random variable. Thus, its expected value is

$$E[T] = \frac{4}{\mu}$$

Let  $Y$  denote the duration of one cycle of fade-no fade condition on the channel. Then  $Y = X + T$ , and  $E[Y] = E[X] + E[T]$ . Thus, the probability  $p$  that the channel is in the fade state at a randomly selected instant is given by

$$p = \frac{E[X]}{E[Y]} = \frac{E[X]}{E[X] + E[T]} = \frac{1/\lambda}{(1/\lambda) + (4/\mu)} = \frac{\mu}{\mu + 4\lambda}$$

4.54 The random variable  $X$ , which denotes the interval between two consecutive events, has the PDF

$$f_X(x) = 4x^2 e^{-2x} \quad x \geq 0$$

This means that  $X$  is a 3rd-order Erlang random variable, and the parameter of the underlying exponential distribution is  $\lambda = 2$ . Thus,

- a. The expected value of  $X$  is  $E[X] = 3/\lambda = 3/2 = 1.5$ .
- b. The expected value of the interval between the 11th and 13th events is the length of 2 interarrival times, which is  $2E[X] = 3$ .

- c. The probability that  $X \leq 6$  is given by

$$\begin{aligned} P[X \leq 6] &= F_X(6) = 1 - \sum_{k=0}^2 \frac{(2x)^k e^{-2x}}{k!} \Big|_{x=6} = 1 - e^{-12} - 12e^{-12} - \frac{(12)^2}{4} e^{-12} \\ &= 1 - e^{-12} \{1 + 12 + 36\} = 0.9997 \end{aligned}$$

- 4.55 Let  $N$  denote the number of students that arrive in one hour. Then the PMF of  $N$  is given by

$$p_N(n) = \frac{5^n e^{-5}}{n!} \quad n = 0, 1, \dots, \infty$$

- a. Since the number of arrivals is a Poisson random variable with a mean of  $\lambda = 5$ , the intervals  $X$  between arrivals have the exponential distribution with the PDF

$$f_X(x) = 5e^{-5x} \quad x \geq 0$$

where  $E[X] = 1/5$  hours or 12 minutes. Thus, given that there is currently no student in the lounge, the expected waiting time until the VCR is turned on is the mean time between 5 arrivals, which is mean time of a 5th-order Erlang random variable  $X_5$  and is given by  $E[X_5] = 5E[X] = 1$  hour.

- b. Given that there is currently no student in the lounge, the probability that the VCR is not turned on within one hour from now is given by

$$\begin{aligned} P[X_5 > 1] &= 1 - P[X_5 \leq 1] = 1 - F_{X_5}(1) = \sum_{k=0}^4 \frac{[5(1)]^k e^{-5(1)}}{k!} = \sum_{k=0}^4 \frac{5^k e^{-5}}{k!} \\ &= e^{-5} \left\{ 1 + 5 + \frac{25}{2} + \frac{125}{6} + \frac{625}{24} \right\} = 0.4405 \end{aligned}$$

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## Section 4.10: Uniform Distribution

4.56 The PDF of the time  $T$  minutes that it takes Jack to install a muffler is given by

$$f_T(t) = \begin{cases} 0.05 & 10 \leq t \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

That is,  $T$  is a uniformly distributed random variable.

- a. The expected value of  $T$  is given by  $E[T] = (10 + 30)/2 = 20$
- b. The variance of  $T$  is given by  $\sigma_X^2 = (30 - 10)^2/12 = 100/3 = 33.33$ .

4.57 Since the random variable  $X$  is uniformly distributed between 0 and 10, its PDF is given by

$$f_X(x) = \begin{cases} 0.1 & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$$

And its mean, variance and standard deviation are given by

$$\begin{aligned} E[X] &= \frac{10 + 0}{2} = 5 \\ \sigma_X^2 &= \frac{(10 - 0)^2}{12} = \frac{100}{12} \\ \sigma_X &= \sqrt{\frac{100}{12}} = 2.8867 \end{aligned}$$

Thus, the probability that  $X$  lies between  $\sigma_X$  and  $E[X]$  is given by

$$P[\sigma_X < X < E[X]] = P[2.8867 < X < 5] = \int_{2.8867}^5 f_X(x) dx = \int_{2.8867}^5 0.1 dx = [0.1x]_{2.8867}^5 = 0.2113$$

4.58 Since the random variable  $X$  is uniformly distributed between 3 and 15, its PDF is given by

$$f_X(x) = \begin{cases} \frac{1}{12} & 3 < x < 15 \\ 0 & \text{otherwise} \end{cases}$$

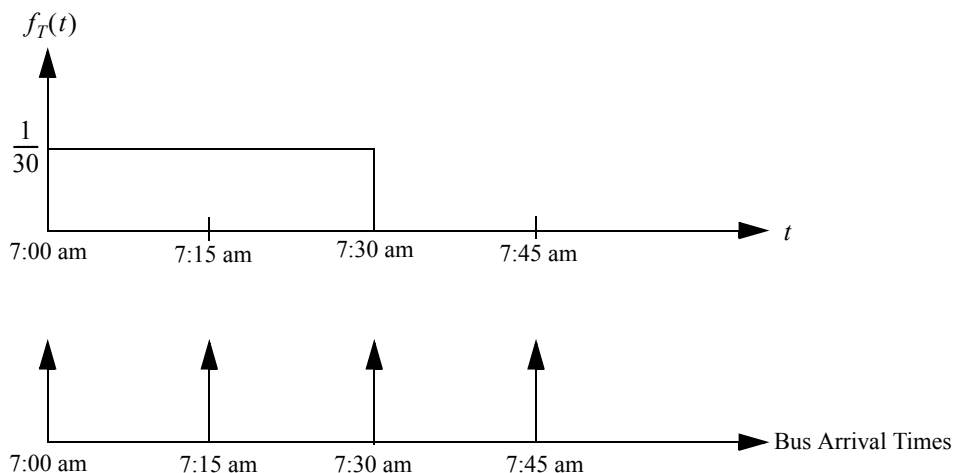
- a. The expected value of  $X$  is given by  $E[X] = (3 + 15)/2 = 9$
- b. The variance of  $X$  is given by  $\sigma_X^2 = (15 - 3)^2/12 = 12$
- c. The probability that  $X$  lies between 5 and 10 is

$$P[5 < X < 10] = \int_5^{10} f_X(x) dx = \int_5^{10} \frac{1}{12} dx = \left[ \frac{x}{12} \right]_5^{10} = \frac{5}{12}$$

- d. The probability that  $X$  is less than 6 is given by

$$P[X < 6] = \int_3^6 f_X(x) dx = \int_3^6 \frac{1}{12} dx = \left[ \frac{x}{12} \right]_3^6 = \frac{3}{12} = \frac{1}{4}$$

4.59 Let the random variable  $T$  denote the time that Joe arrives at the bus stop. The figure below shows the PDF of  $T$  as well as part of the bus arrival times.





- a. To wait less than 5 minutes for the bus, Joe must arrive between 7:10 am and 7:15 am or between 7:25 am and 7:30 am. Thus, if  $A$  is the event that Joe waits less than 5 minutes for the bus, then the probability of this event is given by

$$\begin{aligned} P[A] &= P[(10 < T < 15) \cup (25 < T < 30)] = P[10 < T < 15] + P[25 < T < 30] \\ &= \int_{10}^{15} f_X(x) dx + \int_{25}^{30} f_X(x) dx = \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{30} \{ [x]_{10}^{15} + [x]_{25}^{30} \} = \frac{10}{30} \\ &= \frac{1}{3} \end{aligned}$$

- b. To wait more than 10 minutes for a bus, Joe must arrive between 7:00 am and 7:05 am, or between 7:15 am and 7:20 am. Thus, if  $B$  is the event that Joe waits more than 10 minutes for the bus, then the probability of this event is given by

$$\begin{aligned} P[B] &= P[(0 < T < 5) \cup (15 < T < 20)] = P[0 < T < 5] + P[15 < T < 20] \\ &= \int_0^5 f_X(x) dx + \int_{15}^{20} f_X(x) dx = \int_0^5 \frac{1}{30} dx + \int_{15}^{20} \frac{1}{30} dx = \frac{1}{30} \{ [x]_0^5 + [x]_{15}^{20} \} = \frac{10}{30} \\ &= \frac{1}{3} \end{aligned}$$

4.60 Let the random variable  $X$  denote the time it takes a teller to serve a customer. Then the PDF of  $X$  is given by

$$f_X(x) = \begin{cases} \frac{1}{4} & 2 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Given that a customer has just stepped up to the window and you are next in line,

- The expected time you will wait before it is your turn to be served is the expected time of  $X$ , which is  $E[X] = (2 + 6)/2 = 4$  minutes.
- The probability that you wait less than 1 minute before being served is the probability that it takes less than 1 minute to serve a customer, which is 0, since the service time lies between 2 and 6 minutes.

- c. The probability that you wait between 3 and 5 minutes before being served is given by

$$P[3 < X < 5] = \int_3^5 f_X(x) dx = \int_3^5 \frac{1}{4} dx = \frac{1}{4} [x]_3^5 = \frac{1}{4} (2) = \frac{1}{2}$$

## Section 4.11: Normal Distribution

4.61 Let  $X$  denote the weights of students. Since the number of observations  $N = 200$ , we know that  $X$  is approximately normally distributed. Given that  $\mu_X = 140$  and  $\sigma_X = 10$ , we have that

- a. The fraction of students that weigh between 110 and 145 lbs is given by

$$\begin{aligned} P[110 < X < 145] &= F_X(145) - F_X(110) = \Phi\left(\frac{145 - 140}{10}\right) - \Phi\left(\frac{110 - 140}{10}\right) \\ &= \Phi(0.5) - \Phi(-3) = \Phi(0.5) - \{1 - \Phi(3)\} = \Phi(0.5) + \Phi(3) - 1 \\ &= 0.6915 + 0.9987 - 1 = 0.6902 \end{aligned}$$

Therefore, the expected number of students that weigh between 110 and 145 lbs is given by  $0.6902 \times 200 = 138.04$ .

- b. The fraction of students that weigh less than 120 lbs is given by

$$\begin{aligned} P[X < 120] &= F_X(120) = \Phi\left(\frac{120 - 140}{10}\right) = \Phi(-2) = 1 - \Phi(2) \\ &= 1 - 0.9772 = 0.0228 \end{aligned}$$

Therefore, the expected number of students that weigh less than 120 lbs is given by  $0.0228 \times 200 = 4.56$ .

- c. The fraction of students that weigh more than 170 lbs is given by

$$\begin{aligned} P[X > 170] &= 1 - P[X \leq 170] = 1 - F_X(170) = 1 - \Phi\left(\frac{170 - 140}{10}\right) = 1 - \Phi(3) \\ &= 1 - 0.9987 = 0.0013 \end{aligned}$$

Therefore, the expected number of students that weigh more than 170 lbs is given by  $0.0013 \times 200 = 0.26$ .

4.62 The random variable  $X$  is normally distributed with mean  $\mu_X = 70$  and standard deviation  $\sigma_X = 10$ .

- a. 
$$P[X > 50] = 1 - P[X \leq 50] = 1 - F_X(50) = 1 - \Phi\left(\frac{50 - 70}{10}\right) = 1 - \Phi(-2)$$
$$= 1 - \{1 - \Phi(2)\} = \Phi(2) = 0.9772$$
- b. 
$$P[X < 60] = F_X(60) = \Phi\left(\frac{60 - 70}{10}\right) = \Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$
- c. 
$$P[60 < X < 90] = F_X(90) - F_X(60) = \Phi\left(\frac{90 - 70}{10}\right) - \Phi\left(\frac{60 - 70}{10}\right) = \Phi(2) - \Phi(-1)$$
$$= \Phi(2) - \{1 - \Phi(1)\} = \Phi(2) + \Phi(1) - 1 = 0.9772 + 0.8413 - 1 = 0.8185$$

4.63 Let  $K$  be a random variable that denotes the number of heads in 12 tosses of a fair coin. Then the probability of success is  $p = 1/2$ , and the PMF of  $K$  is given by

$$p_K(k) = \binom{12}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{12-k} = \binom{12}{k} \left(\frac{1}{2}\right)^{12} \quad k = 0, 1, \dots, 12$$

- a. Using the direct method, the probability of getting between 4 and 8 heads is given by

$$P[4 \leq K \leq 8] = \sum_{k=4}^8 \binom{12}{k} \left(\frac{1}{2}\right)^{12} = \left(\frac{1}{2}\right)^{12} \left\{ \binom{12}{4} + \binom{12}{5} + \binom{12}{6} + \binom{12}{7} + \binom{12}{8} \right\} = \frac{3498}{4096} = 0.8540$$

- b. Using the normal approximation to the binomial distribution, we have that

$$\begin{aligned} n &= 12 \\ p &= q = 0.5 \\ E[K] &= np = 6 \\ \sigma_K^2 &= np(1-p) \end{aligned}$$

Therefore, the approximate value of the probability of getting between 4 and 8 heads is given by

$$\begin{aligned} P[4 \leq K \leq 8] &= F_K(8) - F_K(4) = \Phi\left(\frac{8-6}{\sqrt{3}}\right) - \Phi\left(\frac{4-6}{\sqrt{3}}\right) = \Phi\left(\frac{2}{\sqrt{3}}\right) - \Phi\left(\frac{-2}{\sqrt{3}}\right) \\ &= \Phi\left(\frac{2}{\sqrt{3}}\right) - \left\{1 - \Phi\left(\frac{2}{\sqrt{3}}\right)\right\} = 2\Phi\left(\frac{2}{\sqrt{3}}\right) - 1 = 2\Phi(1.15) - 1 = 2(0.9394) - 1 = 0.8788 \end{aligned}$$

4.64 The random variable  $X$  is approximately normally distributed with mean  $\mu_X$  and standard deviation  $\sigma_X$ .

a. The fraction of the class with the letter grade A is given by

$$\begin{aligned} P[A] &= P[\mu_X + \sigma_X < X] = 1 - P[X \leq \mu_X + \sigma_X] = 1 - F_X(\mu_X + \sigma_X) = 1 - \Phi\left(\frac{\mu_X + \sigma_X - \mu_X}{\sigma_X}\right) \\ &= 1 - \Phi(1) = 1 - 0.8413 = 0.1587 \end{aligned}$$

b. The fraction of the class with the letter grade B is given by

$$\begin{aligned} P[B] &= P[\mu_X < X < \mu_X + \sigma_X] = F_X(\mu_X + \sigma_X) - F_X(\mu_X) = \Phi\left(\frac{\mu_X + \sigma_X - \mu_X}{\sigma_X}\right) - \Phi\left(\frac{\mu_X - \mu_X}{\sigma_X}\right) \\ &= \Phi(1) - \Phi(0) = 0.8413 - 0.5000 = 0.3413 \end{aligned}$$

c. The fraction of the class with the letter grade C is given by

$$\begin{aligned} P[C] &= P[\mu_X - \sigma_X < X < \mu_X] = F_X(\mu_X) - F_X(\mu_X - \sigma_X) = \Phi\left(\frac{\mu_X - \mu_X}{\sigma_X}\right) - \Phi\left(\frac{\mu_X - \sigma_X - \mu_X}{\sigma_X}\right) \\ &= \Phi(0) - \Phi(-1) = \Phi(0) - \{1 - \Phi(1)\} = \Phi(0) + \Phi(1) - 1 = 0.5 + 0.8413 - 1 = 0.3413 \end{aligned}$$

d. The fraction of the class with the letter grade D is given by

$$\begin{aligned} P[D] &= P[\mu_X - 2\sigma_X < X < \mu_X - \sigma_X] = F_X(\mu_X - \sigma_X) - F_X(\mu_X - 2\sigma_X) \\ &= \Phi\left(\frac{\mu_X - \sigma_X - \mu_X}{\sigma_X}\right) - \Phi\left(\frac{\mu_X - 2\sigma_X - \mu_X}{\sigma_X}\right) = \Phi(-1) - \Phi(-2) = \{1 - \Phi(1)\} - \{1 - \Phi(2)\} \\ &= \Phi(2) - \Phi(1) = 0.9772 - 0.8413 = 0.1359 \end{aligned}$$

- 
- e. The fraction of the class with the letter grade F is given by

$$\begin{aligned} P[F] &= P[X < \mu_X - 2\sigma_X] = F_X(\mu_X - 2\sigma_X) = \Phi\left(\frac{\mu_X - 2\sigma_X - \mu_X}{\sigma_X}\right) = \Phi(-2) = 1 - \Phi(2) \\ &= 1 - 0.9772 = 0.0228 \end{aligned}$$

- 4.65 The random variable  $X$  is a normal random variable with zero mean and standard deviation  $\sigma_X$ . The probability  $P[|X| \leq 2\sigma_X]$  is given by

$$\begin{aligned} P[|X| \leq 2\sigma_X] &= P[-2\sigma_X < X < 2\sigma_X] = F_X(2\sigma_X) - F_X(-2\sigma_X) = \Phi\left(\frac{2\sigma_X - 0}{\sigma_X}\right) - \Phi\left(\frac{-2\sigma_X - 0}{\sigma_X}\right) \\ &= \Phi(2) - \Phi(-2) = \Phi(2) - \{1 - \Phi(2)\} = 2\Phi(2) - 1 = 2(0.9772) - 1 \\ &= 0.9544 \end{aligned}$$

- 4.66 Let the random variable  $X$  denote the annual rainfall in inches. Given that  $\mu_X = 40$  and  $\sigma_X = 4$ , the probability that the rainfall in a given year is between 30 and 48 inches is given by

$$\begin{aligned} P[30 < X < 48] &= F_X(48) - F_X(30) = \Phi\left(\frac{48 - 40}{4}\right) - \Phi\left(\frac{30 - 40}{4}\right) = \Phi(2) - \Phi(-2.5) \\ &= \Phi(2) - \{1 - \Phi(2.5)\} = \Phi(2) + \Phi(2.5) - 1 = 0.9772 + 0.9938 - 1 \\ &= 0.9710 \end{aligned}$$



## Section 5.3: Bivariate Discrete Random Variables

$$5.1 \quad p_{XY}(x, y) = \begin{cases} kxy & x = 1, 2, 3; y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

a. To determine the value of  $k$ , we have that

$$\sum_x \sum_y p_{XY}(x, y) = 1 = k \sum_{x=1}^3 \sum_{y=1}^3 xy = k \sum_{x=1}^3 x(1 + 2 + 3) = 6k \sum_{x=1}^3 x = 36k \Rightarrow k = \frac{1}{36}$$

b. The marginal PMFs of  $X$  and  $Y$  are given by

$$\begin{aligned} p_X(x) &= \sum_y p_{XY}(x, y) = \frac{1}{36} \sum_{y=1}^3 xy = \frac{x}{36} \{1 + 2 + 3\} = \frac{6x}{36} \\ &= \frac{x}{6} \quad x = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} p_Y(y) &= \sum_x p_{XY}(x, y) = \frac{1}{36} \sum_{x=1}^3 xy = \frac{y}{36} \{1 + 2 + 3\} = \frac{6y}{36} \\ &= \frac{y}{6} \quad y = 1, 2, 3 \end{aligned}$$

c. Observe that  $p_{XY}(x, y) = p_X(x)p_Y(y)$ , which implies that  $X$  and  $Y$  are independent random variables. Thus,

$$\begin{aligned} P[1 \leq X \leq 2, Y \leq 2] &= P[1 \leq X \leq 2]P[Y \leq 2] = \{p_X(1) + p_X(2)\} \times \{p_Y(1) + p_Y(2)\} \\ &= \frac{1}{36} \{1 + 2\} \{1 + 2\} = \frac{9}{36} = \frac{1}{4} \end{aligned}$$

- 5.2 The random variable  $X$  denotes the number of heads in the first two of three tosses of a fair coin, and the random variable  $Y$  denotes the number of heads in the third toss. Let  $S$  denote the sample space of the experiment. Then  $S$ ,  $X$ , and  $Y$  are given as follows:

S	X	Y
HHH	2	1
HHT	2	0
HTH	1	1
HTT	1	0
THH	1	1
THT	1	0
TTH	0	1
TTT	0	0

Thus, the joint PMF  $p_{XY}(x, y)$  of  $X$  and  $Y$  is given by

$$p_{XY}(x, y) = \begin{cases} \frac{1}{8} & x = 0, y = 0 \\ \frac{1}{8} & x = 0, y = 1 \\ \frac{1}{4} & x = 1, y = 0 \\ \frac{1}{4} & x = 1, y = 1 \\ \frac{1}{8} & x = 2, y = 0 \\ \frac{1}{8} & x = 2, y = 1 \\ 0 & \text{otherwise} \end{cases}$$

- 5.3 The joint PMF of two random variables  $X$  and  $Y$  is given by



$$p_{XY}(x, y) = \begin{cases} 0.10 & x = 1, y = 1 \\ 0.35 & x = 2, y = 2 \\ 0.05 & x = 3, y = 3 \\ 0.50 & x = 4, y = 4 \\ 0 & \text{otherwise} \end{cases}$$

a. The joint CDF  $F_{XY}(x, y)$  is obtained as follows:

$$F_{XY}(x, y) = P[X \leq x, Y \leq y] = \sum_{u \leq x} \sum_{v \leq y} p_{XY}(u, v)$$

$$F_{XY}(1, 1) = \sum_{u \leq 1} \sum_{v \leq 1} p_{XY}(u, v) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(1, 2) = \sum_{u \leq 1} \sum_{v \leq 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(1, 3) = \sum_{u \leq 1} \sum_{v \leq 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(1, 4) = \sum_{u \leq 1} \sum_{v \leq 4} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) + p_{XY}(1, 4) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(2, 1) = \sum_{u \leq 2} \sum_{v \leq 1} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 1) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(2, 2) = \sum_{u \leq 2} \sum_{v \leq 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(2, 1) + p_{XY}(2, 2) = 0.45$$

$$\begin{aligned} F_{XY}(2, 3) &= \sum_{u \leq 2} \sum_{v \leq 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) + p_{XY}(2, 1) + p_{XY}(2, 2) + p_{XY}(2, 3) \\ &= p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45 \end{aligned}$$

$$\begin{aligned} F_{XY}(2, 4) &= \sum_{u \leq 2} \sum_{v \leq 4} p_{XY}(u, v) \\ &= p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) + p_{XY}(1, 4) + p_{XY}(2, 1) + p_{XY}(2, 2) + p_{XY}(2, 3) + p_{XY}(2, 4) \\ &= p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45 \end{aligned}$$

$$F_{XY}(3, 1) = \sum_{u \leq 3} \sum_{v \leq 1} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(3, 2) = \sum_{u \leq 3} \sum_{v \leq 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45$$

$$F_{XY}(3, 3) = \sum_{u \leq 3} \sum_{v \leq 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) + p_{XY}(3, 3) = 0.50$$

$$F_{XY}(3, 4) = \sum_{u \leq 3} \sum_{v \leq 4} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) + p_{XY}(3, 3) = 0.50$$

$$F_{XY}(4, 1) = \sum_{u \leq 4} \sum_{v \leq 1} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) + p_{XY}(4, 1) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(4, 2) = \sum_{u \leq 4} \sum_{v \leq 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45$$

$$F_{XY}(4, 3) = \sum_{u \leq 4} \sum_{v \leq 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) + p_{XY}(3, 3) = 0.50$$

Thus, the joint CDF of  $X$  and  $Y$  is given by

$$F_{XY}(x, y) = \begin{cases} 0.00 & x < 1, y < 1 \\ 0.10 & x = 1, y = 1 \\ 0.10 & x = 1, y = 2 \\ 0.10 & x = 1, y = 3 \\ 0.10 & x = 1, y = 4 \\ 0.10 & x = 2, y = 1 \\ 0.45 & x = 2, y = 2 \\ 0.45 & x = 2, y = 3 \\ 0.45 & x = 2, y = 4 \\ 0.10 & x = 3, y = 1 \\ 0.45 & x = 3, y = 2 \\ 0.50 & x = 3, y = 3 \\ 0.50 & x = 3, y = 4 \\ 0.10 & x = 4, y = 1 \\ 0.45 & x = 4, y = 2 \\ 0.50 & x = 4, y = 3 \\ 1.00 & x = 4, y = 4 \end{cases}$$

b. 
$$P[1 \leq X \leq 2, Y \leq 2] = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(2, 1) + p_{XY}(2, 2) = p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45$$

5.4 The joint CDF of  $X$  and  $Y$  is given by

$$F_{XY}(x, y) = \begin{cases} 1/12 & x = 0, y = 0 \\ 1/3 & x = 0, y = 1 \\ 2/3 & x = 0, y = 2 \\ 1/6 & x = 1, y = 0 \\ 7/12 & x = 1, y = 1 \\ 1 & x = 1, y = 2 \end{cases}$$

We first find the joint PMF of  $X$  and  $Y$  as follows:

$$F_{XY}(0, 0) = \frac{1}{12} = \sum_{u \leq 0} \sum_{v \leq 0} p_{XY}(u, v) = p_{XY}(0, 0) \Rightarrow p_{XY}(0, 0) = \frac{1}{12}$$

$$F_{XY}(0, 1) = \frac{1}{3} = \sum_{u \leq 0} \sum_{v \leq 1} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(0, 1) \Rightarrow p_{XY}(0, 1) = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

$$F_{XY}(0, 2) = \frac{2}{3} = \sum_{u \leq 0} \sum_{v \leq 2} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(0, 1) + p_{XY}(0, 2) = \frac{1}{3} + p_{XY}(0, 2)$$

$$p_{XY}(0, 2) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$F_{XY}(1, 0) = \frac{1}{6} = \sum_{u \leq 1} \sum_{v \leq 0} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(1, 0) = \frac{1}{12} + p_{XY}(1, 0) \Rightarrow p_{XY}(1, 0) = \frac{1}{12}$$

$$F_{XY}(1, 1) = \frac{7}{12} = \sum_{u \leq 1} \sum_{v \leq 1} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(0, 1) + p_{XY}(1, 0) + p_{XY}(1, 1) = \frac{5}{12} + p_{XY}(1, 1)$$

$$p_{XY}(1, 1) = \frac{7}{12} - \frac{5}{12} = \frac{1}{6}$$

$$F_{XY}(1, 2) = 1 = \sum_{u \leq 1} \sum_{v \leq 2} p_{XY}(u, v) = F_{XY}(1, 1) + p_{XY}(0, 2) + p_{XY}(1, 2) = \frac{11}{12} + p_{XY}(1, 2)$$

$$p_{XY}(1, 2) = \frac{1}{12}$$

**a.**  $P[0 < X < 2, 0 < Y < 2] = p_{XY}(1, 1) = 1/6$

**b.** To obtain the marginal CDFs of  $X$  and  $Y$ , we first obtain their marginal PMFs:

$$\begin{aligned}
p_X(x) &= \sum_y p_{XY}(x, y) = \begin{cases} p_{XY}(0, 0) + p_{XY}(0, 1) + p_{XY}(0, 2) & x = 0 \\ p_{XY}(1, 0) + p_{XY}(1, 1) + p_{XY}(1, 2) & x = 1 \end{cases} \\
&= \begin{cases} \frac{2}{3} & x = 0 \\ \frac{1}{3} & x = 1 \end{cases} \\
p_Y(y) &= \sum_x p_{XY}(x, y) = \begin{cases} p_{XY}(0, 0) + p_{XY}(1, 0) & y = 0 \\ p_{XY}(0, 1) + p_{XY}(1, 1) & y = 1 \\ p_{XY}(0, 2) + p_{XY}(1, 2) & y = 2 \end{cases} \\
&= \begin{cases} \frac{1}{6} & y = 0 \\ \frac{5}{12} & y = 1 \\ \frac{5}{12} & y = 2 \end{cases}
\end{aligned}$$

Thus, the marginal CDFs are given by

$$\begin{aligned}
F_X(x) &= \sum_{u \leq x} p_X(u) = \begin{cases} 0 & x < 0 \\ \frac{2}{3} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \\
F_Y(y) &= \sum_{v \leq y} p_Y(v) = \begin{cases} 0 & y < 0 \\ \frac{1}{6} & 0 \leq y < 1 \\ \frac{7}{12} & 1 \leq y < 2 \\ 1 & y \geq 2 \end{cases}
\end{aligned}$$

c.  $P[X = 1, Y = 1] = p_{XY}(1, 1) = 1/6$

5.5 The joint PMF of  $X$  and  $Y$  is given by

$$p_{XY}(x, y) = \begin{cases} 1/12 & x = 1, y = 1 \\ 1/6 & x = 1, y = 2 \\ 1/12 & x = 1, y = 3 \\ 1/6 & x = 2, y = 1 \\ 1/4 & x = 2, y = 2 \\ 1/12 & x = 2, y = 3 \\ 1/12 & x = 3, y = 1 \\ 1/12 & x = 3, y = 2 \\ 0 & \text{otherwise} \end{cases}$$

a. The marginal PMFs of  $X$  and  $Y$  are given by

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) & x = 1 \\ p_{XY}(2, 1) + p_{XY}(2, 2) + p_{XY}(2, 3) & x = 2 \\ p_{XY}(3, 1) + p_{XY}(3, 2) + p_{XY}(3, 3) & x = 3 \end{cases}$$

$$= \begin{cases} 1/3 & x = 1 \\ 1/2 & x = 2 \\ 1/6 & x = 3 \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) & y = 1 \\ p_{XY}(1, 2) + p_{XY}(2, 2) + p_{XY}(3, 2) & y = 2 \\ p_{XY}(1, 3) + p_{XY}(2, 3) + p_{XY}(3, 3) & y = 3 \end{cases}$$

$$= \begin{cases} 1/3 & y = 1 \\ 1/2 & y = 2 \\ 1/6 & y = 3 \end{cases}$$

b.  $P[X < 2.5] = p_X(1) + p_X(2) = 5/6$

c. The probability that  $Y$  is odd is  $p_Y(1) + p_Y(3) = 0.5$ .

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5.6 The joint PMF of  $X$  and  $Y$  is given by

$$p_{XY}(x, y) = \begin{cases} 0.2 & x = 1, y = 1 \\ 0.1 & x = 1, y = 2 \\ 0.1 & x = 2, y = 1 \\ 0.2 & x = 2, y = 2 \\ 0.1 & x = 3, y = 1 \\ 0.3 & x = 3, y = 2 \end{cases}$$

a. The marginal PMFs of  $X$  and  $Y$  are given by

$$\begin{aligned} p_X(x) &= \sum_y p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(1, 2) & x = 1 \\ p_{XY}(2, 1) + p_{XY}(2, 2) & x = 2 \\ p_{XY}(3, 1) + p_{XY}(3, 2) & x = 3 \end{cases} \\ &= \begin{cases} 0.3 & x = 1 \\ 0.3 & x = 2 \\ 0.4 & x = 3 \end{cases} \\ p_Y(y) &= \sum_x p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) & y = 1 \\ p_{XY}(1, 2) + p_{XY}(2, 2) + p_{XY}(3, 2) & y = 2 \end{cases} \\ &= \begin{cases} 0.4 & y = 1 \\ 0.6 & y = 2 \end{cases} \end{aligned}$$

b. The conditional PMF of  $X$  given  $Y$  is given by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

c. To test whether  $X$  and  $Y$  are independent, we proceed as follows:

$$P[X|Y=1] = p_{X|Y}(x|1) = \frac{p_{XY}(x, 1)}{p_Y(1)} = \frac{p_{XY}(x, 1)}{0.4} = \begin{cases} \frac{0.2}{0.4} & x = 1 \\ \frac{0.1}{0.4} & x = 2 \\ \frac{0.1}{0.4} & x = 3 \end{cases}$$

$$= \begin{cases} 0.50 & x = 1 \\ 0.25 & x = 2 \\ 0.25 & x = 3 \end{cases}$$

$$P[X|Y=2] = p_{X|Y}(x|2) = \frac{p_{XY}(x, 2)}{p_Y(2)} = \frac{p_{XY}(x, 2)}{0.6} = \begin{cases} \frac{0.1}{0.6} & x = 1 \\ \frac{0.2}{0.6} & x = 2 \\ \frac{0.3}{0.6} & x = 3 \end{cases}$$

$$= \begin{cases} 1/6 & x = 1 \\ 1/3 & x = 2 \\ 1/2 & x = 3 \end{cases}$$

Since  $p_{X|Y}(x|1) \neq p_{X|Y}(x|2)$ , we conclude that  $X$  and  $Y$  are not independent. Note also that we could have tested for independence by seeing that  $p_X(x)p_Y(y) \neq p_{XY}(x, y)$ . Thus, either way we have shown that  $X$  and  $Y$  are not independent.

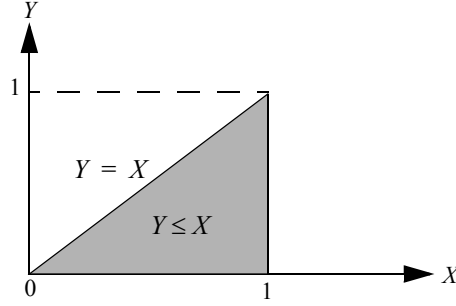
## Section 5.4: Bivariate Continuous Random Variables

5.7 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} kx & 0 < y \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



- a. To determine the value of the constant  $k$ , we must carefully define the ranges of the values of  $X$  and  $Y$  as follows:



$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = \int_{x=0}^1 \int_{y=0}^x kx dy dx = \int_{x=0}^1 kx^2 dx = k \left[ \frac{x^3}{3} \right]_0^1 = \frac{k}{3}$$

which implies that  $k = 3$ . Note that we can also obtain  $k$  by reversing the order of integration as follows:

$$1 = \int_{y=0}^1 \int_{x=y}^1 kx dy dx = k \int_{y=0}^1 \left[ \frac{x^2}{2} \right]_y^1 dy = \frac{k}{2} \int_{y=0}^1 [1 - y^2] dy = \frac{k}{2} \left[ y - \frac{y^3}{3} \right]_0^1 = \frac{k}{2} \left[ 1 - \frac{1}{3} \right] = \frac{k}{3}$$

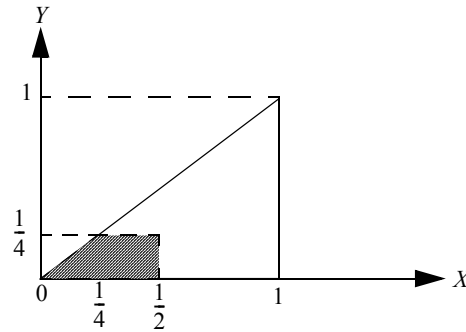
which gives the same result  $k = 3$ .

- b. The marginal PDFs of  $X$  and  $Y$  are given as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_{y=0}^x kx dy = kx^2 = 3x^2 \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \int_{x=y}^1 kx dx = k \left[ \frac{x^2}{2} \right]_y^1 = \frac{3}{2} [1 - y^2] \quad 0 < y \leq 1$$

- c. To evaluate  $P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right]$ , we need to find the region of integration as follows:



Thus, we have that

$$\begin{aligned}
 P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right] &= \int_{x=0}^{\frac{1}{4}} \int_{y=0}^x kx dy dx + \int_{x=\frac{1}{4}}^{\frac{1}{2}} \int_{y=0}^{\frac{1}{4}} kx dy dx = \int_{x=0}^{\frac{1}{4}} kx^2 dx + \int_{x=\frac{1}{4}}^{\frac{1}{2}} \frac{kx}{4} dx \\
 &= k \left\{ \left[ \frac{x^3}{3} \right]_0^{\frac{1}{4}} + \left[ \frac{x^2}{8} \right]_{\frac{1}{4}}^{\frac{1}{2}} \right\} = k \left\{ \frac{1}{3} \left( \frac{1}{64} \right) + \frac{1}{8} \left( \frac{1}{4} - \frac{1}{16} \right) \right\} = \frac{11}{128}
 \end{aligned}$$

5.8 The joint CDF of  $X$  and  $Y$  is given by

$$F_{XY}(x, y) = \begin{cases} 1 - e^{-ax} - e^{-by} + e^{-(ax+by)} & x \geq 0; y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- a. To find the marginal PDFs of  $X$  and  $Y$ , we first obtain the joint PDF and then obtain the marginal PDFs as follows:

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y) = abe^{-(ax+by)} \quad x \geq 0; y \geq 0$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_0^{\infty} abe^{-(ax+by)} dy = ae^{-ax} [-e^{-by}]_0^{\infty} = ae^{-ax} \quad x \geq 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \int_0^{\infty} abe^{-(ax+by)} dx = be^{-by} [-e^{-ax}]_0^{\infty} = be^{-by} \quad y \geq 0$$

- b. Observe that  $f_X(x)f_Y(y) = ae^{-ax}be^{-by} = abe^{-(ax+by)} = f_{XY}(x, y)$ . That is, the product of the marginal PDFs is equal to the joint PDF. Therefore, we conclude that  $X$  and  $Y$  are independent random variables.

5.9 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} ke^{-(2x+3y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- a. For  $f_{XY}(x, y)$  to be a true joint PDF, we must have that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1 = k \int_0^{\infty} e^{-2x} dx \int_0^{\infty} e^{-3y} dy = k \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) = \frac{k}{6}$$

Thus,  $k = 6$ .

- b. The marginal PDFs of  $X$  and  $Y$  are given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_0^{\infty} 6e^{-(2x+3y)} dy = 6e^{-2x} \int_0^{\infty} e^{-3y} dy \\ &= 2e^{-2x} \quad x \geq 0 \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \int_0^{\infty} 6e^{-(2x+3y)} dx = 6e^{-3y} \int_0^{\infty} e^{-2x} dx \\ &= 3e^{-3y} \quad y \geq 0 \end{aligned}$$

Another way to obtain the marginal PDFs is by noting that the joint PDF is of the form  $f_{XY}(x, y) = k \times a(x) \times b(y)$  in the rectangular region  $0 \leq x \leq \infty, 0 \leq y \leq \infty$ , where  $k$  is a constant,  $a(x)$  is the  $x$ -factor and  $b(y)$  is the  $y$ -factor. Therefore, we must have that

$$\begin{aligned} f_X(x) &= Ae^{-2x} \quad x \geq 0 \\ f_Y(y) &= Be^{-3y} \quad y \geq 0 \\ 6 &= AB \end{aligned}$$

To find the values of A and B we note that

$$\int_0^{\infty} A e^{-2x} dx = 1 = A \left[ \frac{-e^{-2x}}{2} \right]_0^{\infty} = \frac{A}{2} \Rightarrow A = 2$$

Thus,  $B = 6/2 = 3$ . From these we obtain  $f_X(x) = 2e^{-2x}, x \geq 0$ , and  $f_Y(y) = 3e^{-3y}, y \geq 0$ .

c.

$$\begin{aligned} P[X < 1, Y < 0.5] &= \int_{x=0}^1 \int_{y=0}^{0.5} f_{XY}(x, y) dy dx = \int_{x=0}^1 2e^{-2x} dx \int_{y=0}^{0.5} 3e^{-3y} dy = [-e^{-2x}]_0^1 [-e^{-3y}]_0^{0.5} \\ &= (1 - e^{-2})(1 - e^{-1.5}) = 0.6717 \end{aligned}$$

5.10 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} k(1 - x^2 y) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. To determine the value of the constant  $k$ , we know that

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy &= 1 = k \int_{y=0}^1 \int_{x=0}^1 (1 - x^2 y) dx dy = k \int_{y=0}^1 \left[ x - \frac{x^3 y}{3} \right]_0^1 dy = k \int_{y=0}^1 \left[ 1 - \frac{y}{3} \right] dy \\ &= k \left[ y - \frac{y^2}{6} \right]_0^1 = k \left[ 1 - \frac{1}{6} \right] = \frac{5k}{6} \\ k &= 6/5 = 1.2 \end{aligned}$$

b. To find the conditional PDFs, we must first find the marginal PDFs, which are given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dy = k \int_0^1 (1 - x^2 y) dy = k \left[ y - \frac{x^2 y^2}{2} \right]_{y=0}^1 = 1.2 \left( 1 - \frac{x^2}{2} \right) & 0 \leq x \leq 1 \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dx = k \int_0^1 (1 - x^2 y) dx = k \left[ x - \frac{x^3 y}{3} \right]_{x=0}^1 = 1.2 \left( 1 - \frac{y}{3} \right) & 0 \leq y \leq 1 \end{aligned}$$

Thus, the conditional PDFs of  $X$  given  $Y$ ,  $f_{X|Y}(x|y)$ , and  $Y$  given  $X$ ,  $f_{Y|X}(y|x)$ , are given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{1.2(1-x^2y)}{1.2\left(1-\frac{y}{3}\right)} = \frac{3(1-x^2y)}{3-y} \quad 0 \leq x \leq 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{1.2(1-x^2y)}{1.2\left(1-\frac{x^2}{2}\right)} = \frac{2(1-x^2y)}{2-x^2} \quad 0 \leq y \leq 1$$

5.11 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad 0 < x < 1, \quad 0 < y < 2$$

a. To find the CDF  $F_X(x)$  of  $X$ , we first obtain its PDF as follows:

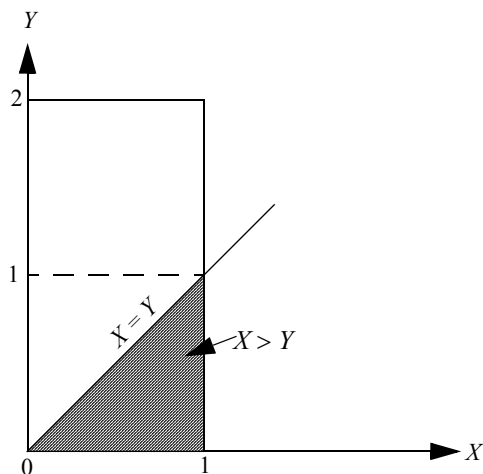
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \frac{6}{7} \int_0^2 \left(x^2 + \frac{xy}{2}\right) dy = \frac{6}{7} \left[x^2 y + \frac{xy^2}{4}\right]_0^2 = \frac{6}{7}(2x^2 + x) \quad 0 < x < 1$$

Thus, the CDF of  $X$  is given by

$$F_X(x) = \int_{-\infty}^x f_X(u) du = \frac{6}{7} \int_0^x (2u^2 + u) du = \frac{6}{7} \left[\frac{2u^3}{3} + \frac{u^2}{2}\right]_0^x$$

$$= \begin{cases} 0 & x < 0 \\ \frac{6}{7} \left(\frac{2x^3}{3} + \frac{x^2}{2}\right) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

b. To find  $P[X > Y]$ , we proceed as follows. First, we need to define the region of integration through the following figure.



Thus,

$$\begin{aligned}
 P[X > Y] &= \int_{x=0}^1 \int_{y=0}^x f_{XY}(x, y) dy dx = \frac{6}{7} \int_{x=0}^1 \int_{y=0}^x \left( x^2 + \frac{xy}{2} \right) dy dx = \frac{6}{7} \int_{x=0}^1 \left[ x^2 y + \frac{xy^2}{4} \right]_{y=0}^x dx \\
 &= \frac{6}{7} \int_{x=0}^1 \left[ x^3 + \frac{x^3}{4} \right] dx = \left( \frac{6}{7} \times \frac{5}{4} \right) \int_{x=0}^1 x^3 dx = \left( \frac{6}{7} \times \frac{5}{4} \right) \left[ \frac{x^4}{4} \right]_0^1 \\
 &= \frac{15}{56}
 \end{aligned}$$

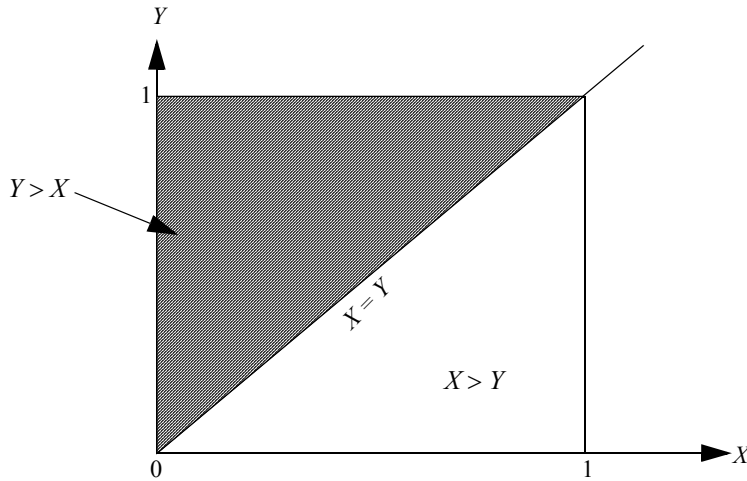
c.

$$\begin{aligned}
 P\left[Y > \frac{1}{2} \mid X < \frac{1}{2}\right] &= \frac{P\left[Y > \frac{1}{2}, X < \frac{1}{2}\right]}{P\left[X < \frac{1}{2}\right]} = \frac{P\left[Y > \frac{1}{2}, X < \frac{1}{2}\right]}{F_X\left(\frac{1}{2}\right)} = \frac{\frac{6}{7} \int_{x=0}^{1/2} \int_{y=1/2}^2 \left( x^2 + \frac{xy}{2} \right) dy dx}{\left(\frac{5}{28}\right)} = \frac{(138/896)}{(5/28)} \\
 &= 0.8625
 \end{aligned}$$

5.12 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} ke^{-(x+y)} & x \geq 0, y \geq x \\ 0 & \text{otherwise} \end{cases}$$

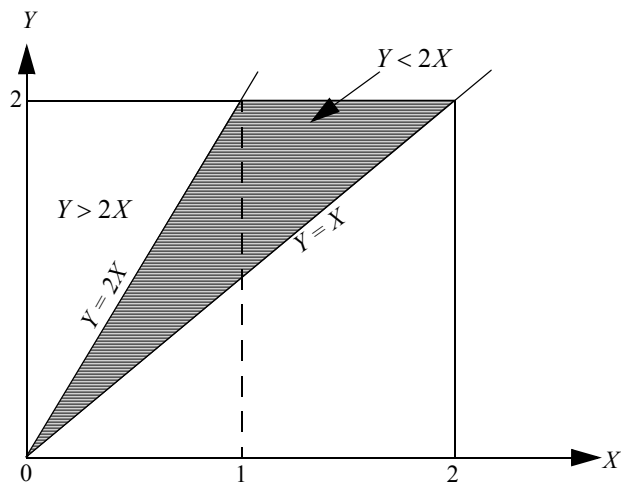
- a. The value of the constant  $k$  can be obtained as follows. First, we determine the region of integration via the following figure:



Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy &= 1 = k \int_{y=0}^{\infty} \int_{x=0}^y e^{-(x+y)} dx dy = k \int_{y=0}^{\infty} e^{-y} [-e^{-x}]_0^y dy = k \int_{y=0}^{\infty} e^{-y} (1 - e^{-y}) dy \\ &= k \left[ -e^{-y} + \frac{e^{-2y}}{2} \right]_0^{\infty} = k \left( 1 - \frac{1}{2} \right) = \frac{k}{2} \\ k &= 2 \end{aligned}$$

- b. To find  $P[Y < 2X]$ , we graph the region of integration as shown in the figure below:



Thus,

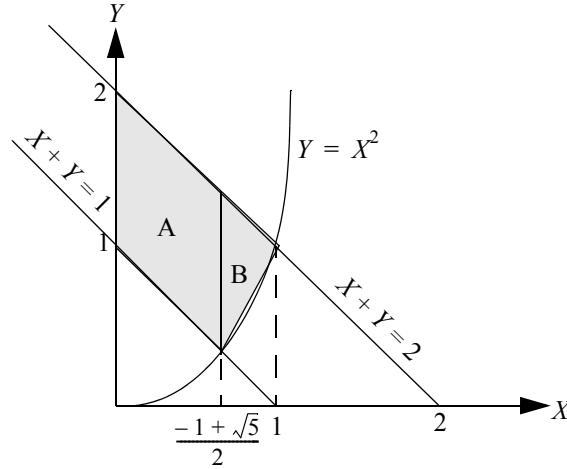
$$\begin{aligned}
 P[Y < 2X] &= k \int_{x=0}^{\infty} \int_{y=x}^{2x} e^{-(x+y)} dy dx = k \int_{x=0}^{\infty} e^{-x} [-e^{-y}]_x^{2x} dx = k \int_{x=0}^{\infty} e^{-x} [e^{-x} - e^{-2x}] dx \\
 &= k \left[ -\frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} \right]_0^{\infty} = k \left( \frac{1}{2} - \frac{1}{3} \right) = 2 \left( \frac{1}{6} \right) = \frac{1}{3}
 \end{aligned}$$

5.13 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \frac{6x}{7} \quad 1 \leq x + y \leq 2, x \geq 0, y \geq 0$$

- a. To obtain the integral that expresses the  $P[Y > X^2]$ , we must show the region of integration, as illustrated in the figure below.

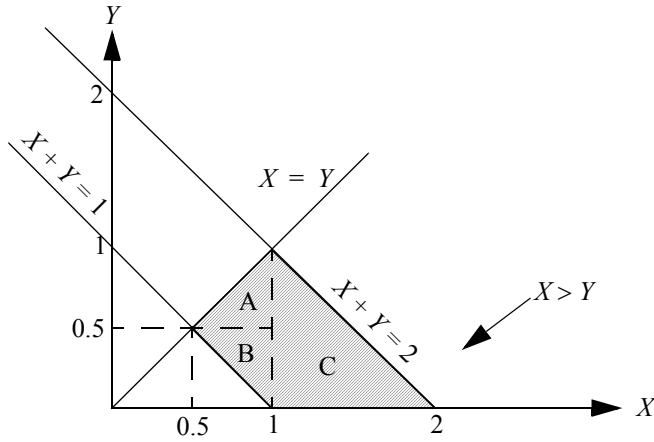




From the figure we see that the region of integration is the shaded region, which has been partitioned into two areas labeled A and B. Area A is bounded by the line  $x = 0$ , which is the  $y$ -axis; the line  $x = (-1 + \sqrt{5})/2$ , which is the feasible solution to the simultaneous equations  $x + y = 1$  and  $y = x^2$ ; the line  $x + y = 1$ ; and the line  $x + y = 2$ . Similarly, area B is bounded by the curve  $y = x^2$ , the line  $x + y = 2$ , and the line  $x = (-1 + \sqrt{5})/2$ . Thus, the desired result is given by

$$\begin{aligned}
 P[Y > X^2] &= \int_A f_{XY}(x, y) dx dy + \int_B f_{XY}(x, y) dx dy \\
 &= \int_{x=0}^{(-1+\sqrt{5})/2} \int_{y=1-x}^{2-x} \frac{6x}{7} dy dx + \int_{x=(-1+\sqrt{5})/2}^1 \int_{y=x^2}^{2-x} \frac{6x}{7} dy dx
 \end{aligned}$$

- b. To obtain the exact value of  $P[X > Y]$ , we note that the new region of integration is as shown below.



Thus, we obtain three areas labeled A, B, and C; and the desired result is as follows:

$$\begin{aligned}
 P[X > Y] &= \int_A f_{XY}(x, y) dx dy + \int_B f_{XY}(x, y) dx dy + \int_C f_{XY}(x, y) dx dy \\
 &= \int_{x=0.5}^1 \int_{y=0.5}^x \frac{6x}{7} dy dx + \int_{x=0.5}^1 \int_{y=1-x}^{0.5} \frac{6x}{7} dy dx + \int_{x=1}^2 \int_{y=0}^{2-x} \frac{6x}{7} dy dx \\
 &= \frac{6}{7} \int_{x=0.5}^1 x[x-0.5] dx + \frac{6}{7} \int_{x=0.5}^1 x[x-0.5] dx + \frac{6}{7} \int_{x=1}^2 x[2-x] dx \\
 &= \frac{6}{7} \left\{ \left[ \frac{x^3}{3} - \frac{0.5x^2}{2} \right]_{0.5}^1 + \left[ \frac{x^3}{3} - \frac{0.5x^2}{2} \right]_{0.5}^1 + \left[ x^2 - \frac{x^3}{3} \right]_1^2 \right\} = \frac{6}{7} \left\{ \frac{5}{48} + \frac{5}{48} + \frac{2}{3} \right\} = \frac{6}{7} \left\{ \frac{7}{8} \right\} = \frac{3}{4}
 \end{aligned}$$

5.14 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} \frac{1}{2} e^{-2y} & 0 \leq x \leq 4, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The marginal PDFs of  $X$  and  $Y$  are given by

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_0^{\infty} \frac{1}{2} e^{-2y} dy = \frac{1}{2} \left[ -\frac{e^{-2y}}{2} \right]_0^{\infty} \\
&= \frac{1}{4} \quad 0 \leq x \leq 4 \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \frac{1}{2} e^{-2y} \int_0^4 dx = 2e^{-2y} \quad y \geq 0
\end{aligned}$$

Note that the joint PDF is completely separable in the form  $f_{XY}(x, y) = k \times a(x) \times b(y)$  in the rectangular region  $0 \leq x \leq 4, 0 \leq y \leq \infty$ , where  $k$  is a constant,  $a(x)$  is the  $x$ -factor and  $b(y)$  is the  $y$ -factor. Therefore, we must have that

$$\begin{aligned}
f_X(x) &= A \quad 0 \leq x \leq 4 \\
f_Y(y) &= B e^{-2y} \quad y \geq 0 \\
\frac{1}{2} &= AB
\end{aligned}$$

To find the values of  $A$  and  $B$  we note that

$$\int_0^{\infty} B e^{-2y} dy = 1 = B \left[ \frac{-e^{-2y}}{2} \right]_0^{\infty} = \frac{B}{2} \Rightarrow B = 2$$

Thus,  $A = 0.5/2 = 1/4$  as before.

## Section 5.6: Conditional Distributions

- 5.15 Let  $S$  denote the sample space of the experiment,  $R$  the event that a red ball is drawn, and  $G$  the event that a green ball is drawn. Since the experiment is performed with replacement, the probability of a sample point in  $S$  is product of the probabilities of those events. More importantly, since  $P[R] = 3/5 = 0.6$  and  $P[G] = 2/5 = 0.4$ , we obtain the following values for the probabilities of the sample points in  $S$  together with their corresponding values of  $X$  and  $Y$ :

S	P[S]	X	Y
RR	0.36	1	1
RG	0.24	1	0
GR	0.24	0	1
GG	0.16	0	0

- a. The joint PMF  $p_{XY}(x, y)$  of  $X$  and  $Y$  is given by

$$p_{XY}(x, y) = \begin{cases} 0.16 & x = 0, y = 0 \\ 0.24 & x = 0, y = 1 \\ 0.24 & x = 1, y = 0 \\ 0.36 & x = 1, y = 1 \\ 0 & \text{otherwise} \end{cases}$$

- b. The conditional PMF of  $X$  given  $Y$  is given by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

But the marginal PMF of  $Y$  is given by

$$\begin{aligned} p_Y(y) &= \sum_x p_{XY}(x, y) = \begin{cases} 0.16 + 0.24 & y = 0 \\ 0.24 + 0.36 & y = 1 \end{cases} \\ &= \begin{cases} 0.4 & y = 0 \\ 0.6 & y = 1 \end{cases} \end{aligned}$$

Thus, we obtain the following result:

$$p_{X|Y}(x|0) = \begin{cases} 0.4 & x = 0 \\ 0.6 & x = 1 \end{cases}$$

$$p_{X|Y}(x|1) = \begin{cases} 0.4 & x = 0 \\ 0.6 & x = 1 \end{cases}$$

5.16 The joint PDF of  $X$  and  $Y$  is  $f_{XY}(x, y) = 2e^{-(x+2y)}$ ,  $x \geq 0, y \geq 0$ . To find the conditional expectation of  $X$  given  $Y$  and  $Y$  given  $X$ , we proceed as follows:

$$E[X|Y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$$

$$E[Y|X] = \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy$$

Now,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_0^{\infty} 2e^{-(x+2y)}dy = e^{-x}[-e^{-2y}]_0^{\infty} = e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \int_0^{\infty} 2e^{-(x+2y)}dx = 2e^{-2y}[-e^{-x}]_0^{\infty} = 2e^{-2y}$$

Since  $f_X(x)f_Y(y) = (e^{-x})(2e^{-2y}) = 2e^{-(x+2y)} = f_{XY}(x, y)$ , we conclude that  $X$  and  $Y$  are independent random variables. Therefore,

$$E[X|Y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx = \int_0^{\infty} xf_X(x)dx = E[X] = 1$$

$$E[Y|X] = \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy = \int_0^{\infty} yf_Y(y)dy = E[Y] = \frac{1}{2}$$

5.17 Let  $S$  denote the sample space of the experiment,  $H$  the event that a toss came up heads, and  $T$  the event that a toss came up tails. Since the experiment is performed with replacement, the probability of a sample point in  $S$  is product of the probabilities of those events. Thus  $S$ ,  $X$ , and  $Y$  are given as follows:

S	P[S]	X	Y
HHHH	1/16	2	2
HHHT	1/16	2	1
HHTH	1/16	2	1
HHTT	1/16	2	0
HTHH	1/16	1	2
HTHT	1/16	1	1
HTTH	1/16	1	1
HTTT	1/16	0	0
THHH	1/16	1	2
THHT	1/16	1	1
THTH	1/16	1	1
TTHH	1/16	0	2
THTT	1/16	1	0
TTHT	1/16	0	1
TTTH	1/16	0	1
TTTT	1/16	0	0

- 
- a. The joint PMF  $p_{XY}(x, y)$  of  $X$  and  $Y$  is given by

$$p_{XY}(x, y) = \begin{cases} \frac{1}{16} & x = 0, y = 0 \\ \frac{1}{8} & x = 0, y = 1 \\ \frac{1}{16} & x = 0, y = 2 \\ \frac{1}{8} & x = 1, y = 0 \\ \frac{1}{4} & x = 1, y = 1 \\ \frac{1}{8} & x = 1, y = 2 \\ \frac{1}{16} & x = 2, y = 0 \\ \frac{1}{8} & x = 2, y = 1 \\ \frac{1}{16} & x = 2, y = 2 \\ 0 & \text{otherwise} \end{cases}$$

- b. To show that  $X$  and  $Y$  are independent random variables, we proceed as follows:

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & x = 0 \\ \frac{1}{8} + \frac{1}{4} + \frac{1}{8} & x = 1 \\ \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & x = 2 \end{cases}$$

$$= \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & y = 0 \\ \frac{1}{8} + \frac{1}{4} + \frac{1}{8} & y = 1 \\ \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & y = 2 \end{cases}$$

$$= \begin{cases} \frac{1}{4} & y = 0 \\ \frac{1}{2} & y = 1 \\ \frac{1}{4} & y = 2 \end{cases}$$

Now,



$$p_X(x)p_Y(y) = \begin{cases} \frac{1}{16} & x = 0, y = 0 \\ \frac{1}{8} & x = 0, y = 1 \\ \frac{1}{16} & x = 0, y = 2 \\ \frac{1}{8} & x = 1, y = 0 \\ \frac{1}{4} & x = 1, y = 1 \\ \frac{1}{8} & x = 1, y = 2 \\ \frac{1}{16} & x = 2, y = 0 \\ \frac{1}{8} & x = 2, y = 1 \\ \frac{1}{16} & x = 2, y = 2 \\ 0 & \text{otherwise} \end{cases}$$

Since  $p_X(x)p_Y(y) = p_{XY}(x, y)$ , we conclude that  $X$  and  $Y$  are independent.

5.18 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = xye^{-y^2/4} \quad 0 \leq x \leq 1, y \geq 0$$

a. The marginal PDFs of  $X$  and  $Y$  are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = x \int_0^{\infty} ye^{-y^2/4} dy$$

Let  $z = y^2/4 \Rightarrow dz = ydy/2 \Rightarrow ydy = 2dz$ . Thus,

$$f_X(x) = x \int_0^{\infty} y e^{-y^2/4} dy = x \int_0^{\infty} 2e^{-z} dz = 2x[-e^{-z}]_0^{\infty} = 2x \quad 0 \leq x \leq 1$$

Similarly, we obtain

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = y e^{-y^2/4} \int_0^1 x dx = y e^{-y^2/4} \left[ \frac{x^2}{2} \right]_0^1 = \frac{1}{2} y e^{-y^2/4} \quad y \geq 0$$

- b. To determine if  $X$  and  $Y$  are independent we observe that

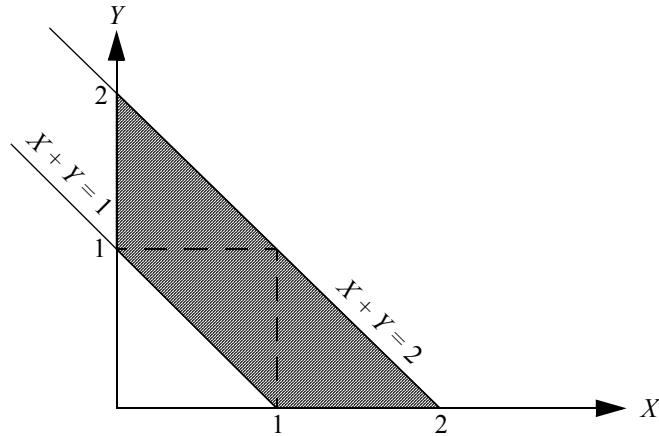
$$f_X(x)f_Y(y) = \{2x\} \left\{ \frac{1}{2} y e^{-y^2/4} \right\} = x y e^{-y^2/4} = f_{XY}(x, y)$$

Thus, we conclude that  $X$  and  $Y$  are independent. Note that this can also be observed from the fact that  $f_{XY}(x, y)$  can be separated into an  $x$ -function and a  $y$ -function and the region of interest is rectangular.

5.19 The joint PDF of random variables  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \frac{6}{7}x \quad 1 \leq x + y \leq 2, x \geq 0, y \geq 0$$

The region of interest is as shown in the following figure:



- a. The marginal PDFs of  $X$  and  $Y$  are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \begin{cases} \int_{y=1-x}^{2-x} \frac{6}{7}x dy & 0 \leq x < 1 \\ \int_{y=0}^{2-x} \frac{6}{7}x dy & 1 \leq x < 2 \end{cases}$$

$$= \begin{cases} \frac{6}{7}x & 0 \leq x < 1 \\ \frac{6}{7}x(2-x) & 1 \leq x < 2 \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \begin{cases} \int_{x=1-y}^{2-y} \frac{6}{7} x dx & 0 \leq y < 1 \\ \int_{x=0}^{2-y} \frac{6}{7} x dx & 1 \leq y < 2 \end{cases}$$

$$= \begin{cases} \frac{3}{7}(3-2y) & 0 \leq y < 1 \\ \frac{3}{7}(2-y)^2 & 1 \leq y < 2 \end{cases}$$

- b. From the above results we see that  $f_X(x)f_Y(y) \neq f_{XY}(x, y)$ . Therefore, we conclude that  $X$  and  $Y$  are not independent.

## Section 5.7: Covariance and Correlation Coefficient

5.20 The joint PMF of  $X$  and  $Y$  is given by

$$p_{XY}(x, y) = \begin{cases} 0 & x = -1, y = 0 \\ \frac{1}{3} & x = -1, y = 1 \\ \frac{1}{3} & x = 0, y = 0 \\ 0 & x = 0, y = 1 \\ 0 & x = 1, y = 0 \\ \frac{1}{3} & x = 1, y = 1 \end{cases}$$

- a. To determine if  $X$  and  $Y$  are independent we first find their marginal PMFs as follows:

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} \sum_y p_{XY}(-1, y) & x = -1 \\ \sum_y p_{XY}(0, y) & x = 0 \\ \sum_y p_{XY}(1, y) & x = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{3} & x = -1 \\ \frac{1}{3} & x = 0 \\ \frac{1}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} \sum_x p_{XY}(x, 0) & y = 0 \\ \sum_x p_{XY}(x, 1) & y = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{3} & y = 0 \\ \frac{2}{3} & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

- b. From the results we observe that  $p_X(x)p_Y(y) \neq p_{XY}(x, y)$ . Therefore, we conclude that  $X$  and  $Y$  are not independent.
- c. The covariance of  $X$  and  $Y$  can be obtained as follows:

$$E[X] = \frac{1}{3}\{-1 + 0 + 1\} = 0$$

$$E[Y] = \frac{1}{3}(0) + \frac{2}{3}(1) = \frac{2}{3}$$

$$E[XY] = \sum_x \sum_y xy p_{XY}(x, y) = \frac{1}{3}\{(-1)(1) + (0)(0) + (1)(1)\} = 0$$

$$\text{Cov}(X, Y) = \sigma_{XY} = E[XY] - E[X]E[Y] = 0$$

5.21 Two events A and B are such that  $P[A] = 1/4$ ,  $P[B|A] = 1/2$ , and  $P[A|B] = 1/4$ . The random variable  $X$  has value  $X = 1$  if event A occurs and  $X = 0$  if event A does not occur. Similarly, the random variable  $Y$  has value  $Y = 1$  if event B occurs and  $Y = 0$  if event B does not occur.

First, we find  $P[B]$  and the PMFs of  $X$  and  $Y$ .

$$P[B|A] = \frac{P[AB]}{P[A]} \Rightarrow P[AB] = P[B|A]P[A] = \frac{1}{8}$$

$$P[A|B] = \frac{P[AB]}{P[B]} \Rightarrow P[B] = \frac{P[AB]}{P[A|B]} = \frac{1}{2}$$

Note that because  $P[A|B] = P[A]$  and  $P[B|A] = P[B]$  events A and B are independent. Thus, the random variables  $X$  and  $Y$  are independent. The PMFs of  $X$  and  $Y$  are given by

$$p_X(x) = \begin{cases} 1 - P[A] & x = 0 \\ P[A] & x = 1 \end{cases} = \begin{cases} \frac{3}{4} & x = 0 \\ \frac{1}{4} & x = 1 \end{cases}$$

$$p_Y(y) = \begin{cases} 1 - P[B] & y = 0 \\ P[B] & y = 1 \end{cases} = \begin{cases} \frac{1}{2} & y = 0 \\ \frac{1}{2} & y = 1 \end{cases}$$

a. The mean and variance of  $X$  are given by

$$E[X] = \frac{3}{4}(0) + \frac{1}{4}(1) = \frac{1}{4}$$

$$E[X^2] = \frac{3}{4}(0^2) + \frac{1}{4}(1^2) = \frac{1}{4}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

- b. The mean and variance of  $Y$  are given by

$$E[Y] = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$$

$$E[Y^2] = \frac{1}{2}(0^2) + \frac{1}{2}(1^2) = \frac{1}{2}$$

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

- c. As stated earlier,  $X$  and  $Y$  are independent because the events  $A$  and  $B$  are independent. Thus,  $\sigma_{XY} = 0$  and  $\rho_{XY} = 0$ , which means that  $X$  and  $Y$  are uncorrelated.

5.22 The random variable  $X$  denotes the number of 1's that appear in three tosses of a fair die, and  $Y$  denotes the number of 3's. Let  $A$  denote the event that an outcome of the toss is neither 1 nor 3. Then the sample space of the experiment and the values of  $X$  and  $Y$  are shown in the following table.

S	P[S]	$X$	$Y$
111	$(1/6)^3$	3	0
113	$(1/6)^3$	2	1
11A	$(1/6)^2(2/3)$	2	0
1A1	$(1/6)^2(2/3)$	2	0
131	$(1/6)^3$	2	1
1AA	$(1/6)(2/3)^2$	1	0

S	P[S]	X	Y
1A3	$(1/6)^2(2/3)$	1	1
133	$(1/6)^3$	1	2
13A	$(1/6)^2(2/3)$	1	1
333	$(1/6)^3$	0	3
33A	$(1/6)^2(2/3)$	0	2
331	$(1/6)^3$	1	2
3A3	$(1/6)^2(2/3)$	0	2
313	$(1/6)^3$	1	2
3AA	$(1/6)(2/3)^2$	0	1
3A1	$(1/6)^2(2/3)$	1	1
311	$(1/6)^3$	2	1
31A	$(1/6)^2(2/3)$	1	1
AAA	$(2/3)^3$	0	0
AA1	$(2/3)^2(1/6)$	1	0
AA3	$(2/3)^2(1/6)$	0	1
A1A	$(2/3)^2(1/6)$	1	0
A3A	$(2/3)^2(1/6)$	0	1
A11	$(1/6)^2(2/3)$	2	0
A13	$(1/6)^2(2/3)$	1	1
A33	$(1/6)^2(2/3)$	0	2
A31	$(1/6)^2(2/3)$	1	1



The PMFs of  $X$  and  $Y$  are given by

$$p_X(x) = \begin{cases} (1/6)^3 + 3(1/6)^2(2/3) + 3(1/6)(2/3)^2 + (2/3)^3 & x = 0 \\ 3(1/6)^3 + 4(1/6)^2(2/3) + 3(1/6)(2/3)^2 & x = 1 \\ 3(1/6)^3 + 3(1/6)^2(2/3) & x = 2 \\ (1/6)^3 & x = 3 \end{cases}$$

$$= \begin{cases} \frac{125}{216} & x = 0 \\ \frac{75}{216} & x = 1 \\ \frac{15}{216} & x = 2 \\ \frac{1}{216} & x = 3 \end{cases}$$

$$p_Y(y) = \begin{cases} (1/6)^3 + 3(1/6)^2(2/3) + 3(1/6)(2/3)^2 + (2/3)^3 & y = 0 \\ 3(1/6)^3 + 4(1/6)^2(2/3) + 3(1/6)(2/3)^2 & y = 1 \\ 3(1/6)^3 + 3(1/6)^2(2/3) & y = 2 \\ (1/6)^3 & y = 3 \end{cases}$$

$$= \begin{cases} \frac{125}{216} & y = 0 \\ \frac{75}{216} & y = 1 \\ \frac{15}{216} & y = 2 \\ \frac{1}{216} & y = 3 \end{cases}$$

Finally, the joint PMF of  $X$  and  $Y$  is given by

$$p_{XY}(x, y) = \begin{cases} \frac{64}{216} & x = 0, y = 0 \\ \frac{48}{216} & x = 0, y = 1 \\ \frac{12}{216} & x = 0, y = 2 \\ \frac{1}{216} & x = 0, y = 3 \\ \frac{48}{216} & x = 1, y = 0 \\ \frac{24}{216} & x = 1, y = 1 \\ \frac{3}{216} & x = 1, y = 2 \\ \frac{12}{216} & x = 2, y = 0 \\ \frac{3}{216} & x = 2, y = 1 \\ \frac{1}{216} & x = 3, y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the correlation coefficient of  $X$  and  $Y$ ,  $\rho_{XY}$ , can be obtained as follows:

$$\begin{aligned}
E[X] &= E[Y] = \frac{0(125) + 1(75) + 2(15) + 3(1)}{216} = \frac{1}{2} \\
E[X^2] &= E[Y^2] = \frac{0^2(125) + 1^2(75) + 2^2(15) + 3^2(1)}{216} = \frac{2}{3} \\
\sigma_X^2 &= \sigma_Y^2 = E[X^2] - (E[X])^2 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12} \\
E[XY] &= \frac{(1)(1)(24) + (1)(2)(3)}{216} = \frac{1}{6} \\
\sigma_{XY} &= E[XY] - E[X]E[Y] = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12} \\
\rho_{XY} &= \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{-(1/12)}{(5/12)} = -0.2
\end{aligned}$$

## Section 5.9: Multinomial Distributions

5.23 Let  $p_A$  denote the probability that a chip is from supplier A,  $p_B$  the probability that it is from supplier B and  $p_C$  the probability that it is from supplier C. Now,

$$p_A = \frac{10}{40} = 0.25$$

$$p_B = \frac{16}{40} = 0.40$$

$$p_C = \frac{14}{40} = 0.35$$

Let  $K$  be a random variable that denotes the number of times that a chip from supplier B is drawn in 20 trials. Then  $K$  is a binomially distributed random variable with the PMF

$$p_K(k) = \binom{20}{k} p_B^k (1 - p_B)^{20-k} \quad k = 0, 1, \dots, 20$$

Thus, the probability that a chip from vendor B is drawn 9 times in 20 trials is given by

$$p_K(9) = \binom{20}{9}(0.4)^9(0.6)^{11} = 0.1597$$

5.24 With reference to the previous problem, the probability  $p$  that a chip from vendor A is drawn 5 times and a chip from vendor C is drawn 6 times in the 20 trials is given by

$$p = \binom{20}{5 \ 9 \ 6}(0.25)^5(0.4)^9(0.35)^6 = \frac{20!}{5!9!6!}(0.25)^5(0.4)^9(0.35)^6 = 0.0365$$

5.25 Let  $p_E$  denote the probability that a professor is rated excellent,  $p_G$  the probability that a professor is rated good,  $p_F$  the probability that a professor is rated fair, and  $p_B$  the probability that a professor is rated bad. Then we are given that

$$p_E = 0.2$$

$$p_G = 0.5$$

$$p_F = 0.2$$

$$p_B = 0.1$$

Given that 12 professors are randomly selected from the college,

a. the probability  $P_1$  that 6 are excellent, 4 are good, 1 is fair, and 1 is bad is given by

$$P_1 = \binom{12}{6 \ 4 \ 1 \ 1}(0.2)^6(0.5)^4(0.2)^1(0.1)^1 = \frac{12!}{6!4!1!1!}(0.2)^6(0.5)^4(0.2)^1(0.1)^1 = 0.0022$$

b. the probability  $P_2$  that 6 are excellent, 4 are good, and 2 are fair is given by

$$P_2 = \binom{12}{6 \ 4 \ 2 \ 0}(0.2)^6(0.5)^4(0.2)^2(0.1)^0 = \frac{12!}{6!4!2!0!}(0.2)^6(0.5)^4(0.2)^2(0.1)^0 = 0.0022$$

c. the probability  $P_3$  that 6 are excellent and 6 are good is given by

$$P_3 = \binom{12}{6 \ 6 \ 0 \ 0} (0.2)^6 (0.5)^6 (0.2)^0 (0.1)^0 = \frac{12!}{6!6!} (0.2)^6 (0.5)^6 = \frac{12!}{6!6!} (0.1)^6 = 0.000924$$

- d. the probability  $P_4$  that 4 are excellent and 3 are good is the probability that 4 are excellent, 3 are good, and 5 are either bad or fair with a probability of 0.3, and this is given by

$$P_4 = \binom{12}{4 \ 3 \ 5} (0.2)^4 (0.5)^3 (0.3)^5 = \frac{12!}{4!3!5!} (0.2)^4 (0.5)^3 (0.3)^5 = 0.0135$$

- e. the probability  $P_5$  that 4 are bad is the probability that 4 are bad and 8 are not bad, which is given by the following binomial distribution:

$$P_5 = \binom{12}{4} (0.1)^4 (0.9)^8 = \frac{12!}{4!8!} (0.1)^4 (0.9)^8 = 0.0213$$

- f. the probability  $P_6$  that none is bad is the probability that all 12 are not bad with probability 0.9, which is given by the binomial distribution

$$P_6 = \binom{12}{0} (0.1)^0 (0.9)^{12} = (0.9)^{12} = 0.2824$$

- 5.26 Let  $p_G$  denote the probability that a toaster is good,  $p_F$  the probability that it is fair,  $p_B$  the probability that it burns the toast, and  $p_C$  the probability that it can catch fire. We are given that

$$p_G = 0.50$$

$$p_F = 0.35$$

$$p_B = 0.10$$

$$p_C = 0.05$$

Given that a store has 40 of these toasters in stock, then

- a. the probability  $P_1$  that 30 are good, 5 are fair, 3 burn the toast, and 2 catch fire is given by

$$P_1 = \binom{40}{30 \ 5 \ 3 \ 2} (0.50)^{30} (0.35)^5 (0.10)^3 (0.05)^2 = \frac{40!}{30!5!3!2!} (0.50)^{30} (0.35)^5 (0.10)^3 (0.05)^2 = 0.000026$$

- b. the probability  $P_2$  that 30 are good and 4 are fair is the probability that 30 are good, 4 are fair, and 6 are either bad or can catch fire, which is given by

$$P_2 = \binom{40}{30 \ 4 \ 6} (0.50)^{30} (0.35)^4 (0.15)^6 = \frac{40!}{30!4!6!} (0.50)^{30} (0.35)^4 (0.15)^6 = 0.000028$$

- c. the probability  $P_3$  that none catches fire is given by the binomial distribution

$$P_3 = \binom{40}{0} (0.05)^0 (0.95)^{40} = (0.95)^{40} = 0.1285$$

- d. the probability  $P_4$  that none burns the toast and none catches fire is given by

$$P_4 = \binom{40}{0} (0.15)^0 (0.85)^{40} = (0.85)^{40} = 0.0015$$

5.27 Let  $p_B$  denote the probability that a candy goes to a boy,  $p_G$  the probability that a candy goes to a girl, and  $p_A$  the probability that it goes to an adult. Then we know that

$$p_B = \frac{8}{20} = 0.40$$

$$p_G = \frac{7}{20} = 0.35$$

$$p_A = \frac{5}{20} = 0.25$$

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Given that 10 pieces of candy are given out at random to the group, we have that

- a. the probability  $P_1$  that 4 pieces go to the girls and 2 go to the adults is the probability that 4 pieces go to the boys, 4 go to the girls, and 2 go to the adults, which is given by

$$P_1 = \binom{10}{4 \ 4 \ 2} (0.40)^4 (0.35)^4 (0.25)^2 = \frac{10!}{4!4!2!} (0.40)^4 (0.35)^4 (0.25)^2 = 0.0756$$

- b. The probability  $P_2$  that 5 pieces go to the boys is the probability that 5 pieces go to the boys and the other 5 pieces go to either the girls or the adults, which is given by

$$P_2 = \binom{10}{5} (0.40)^5 (0.60)^5 = \frac{10!}{5!5!} (0.40)^5 (0.60)^5 = 0.2006$$





## Section 6.2: Functions of One Random Variable

6.1 Given that  $X$  is a random variable and  $Y = aX - b$ , where  $a$  and  $b$  are constants, then

$$F_Y(y) = P[Y \leq y] = P[aX - b \leq y] = P\left[X \leq \frac{y+b}{a}\right] = F_X\left(\frac{y+b}{a}\right)$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{|a|}f_X\left(\frac{y+b}{a}\right)$$

$$E[Y] = E[aX - b] = aE[X] - b$$

$$\sigma_Y^2 = E[(Y - E[Y])^2] = E[(aX - aE[X])^2] = E[a^2(X - E[X])^2] = a^2\sigma_X^2$$

6.2 Given the random variable  $X$  whose PDF,  $f_X(x)$ , is known and the function  $Y = aX^2$ , where  $a > 0$  is a constant.

a. We find the PDF of  $Y$  as follows:

$$F_Y(y) = P[Y \leq y] = P[aX^2 \leq y] = P\left[X^2 \leq \frac{y}{a}\right] = P\left[|X| \leq \sqrt{\frac{y}{a}}\right] = P\left[-\sqrt{\frac{y}{a}} < X < \sqrt{\frac{y}{a}}\right] = F_X\left(\sqrt{\frac{y}{a}}\right) - F_X\left(-\sqrt{\frac{y}{a}}\right)$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{2\sqrt{ay}}\left\{f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right\}, \quad y > 0$$

b. When  $f_X(x)$  is an even function, we obtain

$$f_Y(y) = \frac{2f_X\left(\sqrt{\frac{y}{a}}\right)}{2\sqrt{ay}} = \frac{f_X\left(\sqrt{\frac{y}{a}}\right)}{\sqrt{ay}}, \quad y > 0$$

6.3 Given that  $Y = aX^2$ , where  $a > 0$  is a constant, and the mean and other moments of  $X$  are known.

a.  $E[Y] = E[aX^2] = aE[X^2] = a\{\sigma_X^2 + (E[X])^2\}$

$$\text{b. } \sigma_Y^2 = E[Y^2] - (E[Y])^2 = a^2 E[X^4] - a^2 \{ \sigma_X^2 + (E[X])^2 \}^2 = a^2 [E[X^4] - \{ \sigma_X^2 + (E[X])^2 \}^2]$$

6.4 Given that  $Y = |X|$  and the PDF of  $X$ ,  $f_X(x)$ , is known, we have that

$$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[|X| \leq y] = P[-y \leq X \leq y] = F_X(y) - F_X(-y) \\ f_Y(y) &= \frac{d}{dy} F_Y(y) = f_X(y) + f_X(-y) \end{aligned}$$

6.5 The PDF of  $X$  is given by

$$f_X(x) = \begin{cases} \frac{1}{3} & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

If we define  $Y = 2X + 3$ , then

$$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[2X + 3 \leq y] = P\left[X \leq \frac{y-3}{2}\right] = F_X\left(\frac{y-3}{2}\right) \\ f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{1}{2} f_X\left(\frac{y-3}{2}\right) = \begin{cases} \frac{1}{6} & 1 < y < 7 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where the limits are obtained by solving the equations  $\frac{y-3}{2} = -1$  and  $\frac{y-3}{2} = 2$ .

6.6 Given that  $Y = a^X$ , where  $a > 0$  is a constant, and the PDF of  $X$ ,  $f_X(x)$ , then

a. The PDF of  $Y$  is given by

$$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[a^X \leq y] = P[X \ln a \leq \ln y] = P\left[X \leq \frac{\ln y}{\ln a}\right] = F_X\left(\frac{\ln y}{\ln a}\right) \\ f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{1}{y \ln a} f_X\left(\frac{\ln y}{\ln a}\right), \quad y > 0 \end{aligned}$$

- 
- b. When  $a = e$ ,  $\ln a = \ln e = 1$  and the PDF of  $Y$  becomes

$$f_Y(y) = \frac{1}{y} f_X(\ln y) \quad y > 0$$

Finally, if the PDF of  $X$  is given by

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

then we obtain

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

where the limits follow from solutions to the equation  $\ln y = 0 \Rightarrow y = e^0 = 1$  and the equation  $\ln y = 1 \Rightarrow y = e^1 = e$ .

- 6.7 Given that  $Y = \ln X$ , where the PDF of  $X$ ,  $f_X(x)$ , is known, then the PDF of  $Y$  can be obtained as follows:

$$F_Y(y) = P[Y \leq y] = P[\ln X \leq y] = P[X \leq e^y] = F_X(e^y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = e^y f_X(e^y)$$

## Section 6.4: Sums of Random Variables

- 6.8 Given 2 independent random variables  $X$  and  $Y$  with the following PDFs:

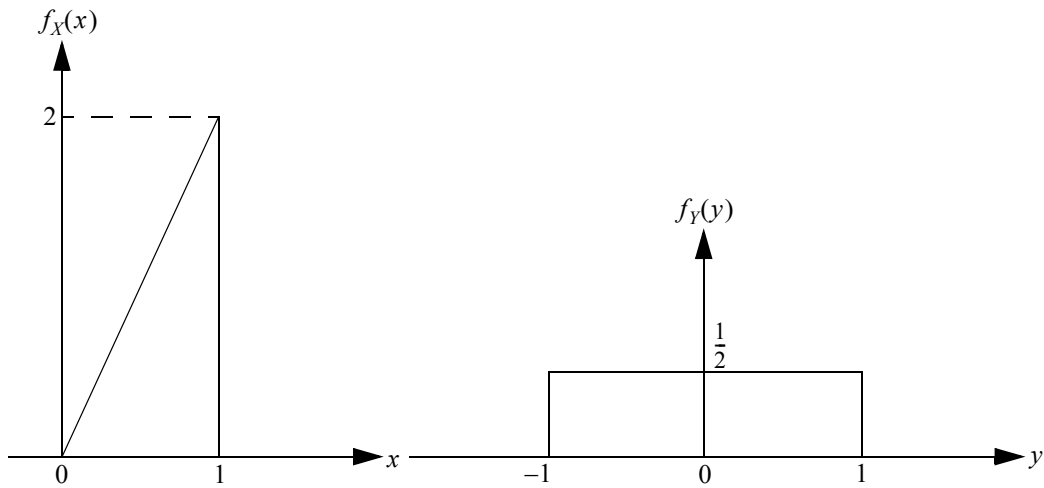
$$f_X(x) = 2x \quad 0 \leq x \leq 1$$

$$f_Y(y) = \frac{1}{2} \quad -1 \leq y \leq 1$$

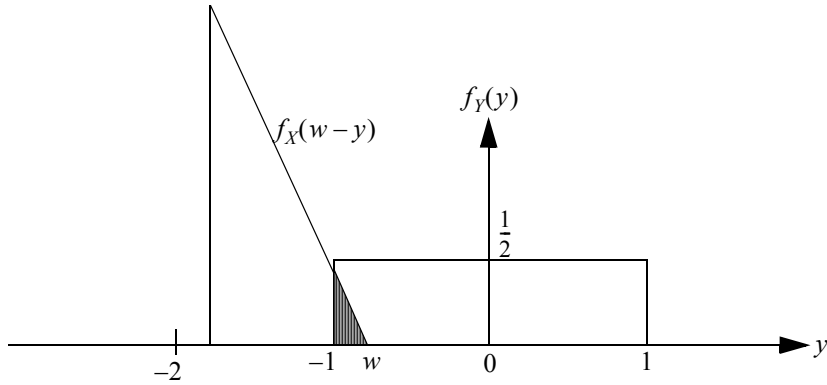
The random variable  $W$  is defined as  $W = X + Y$ . Since  $X$  and  $Y$  are independent, the PDF of  $W$  is given by

$$f_W(w) = f_X(w) * f_Y(w) = \int_{-\infty}^{\infty} f_X(w-y)f_Y(y)dy$$

To evaluate the integral we carry out the following convolution operations:

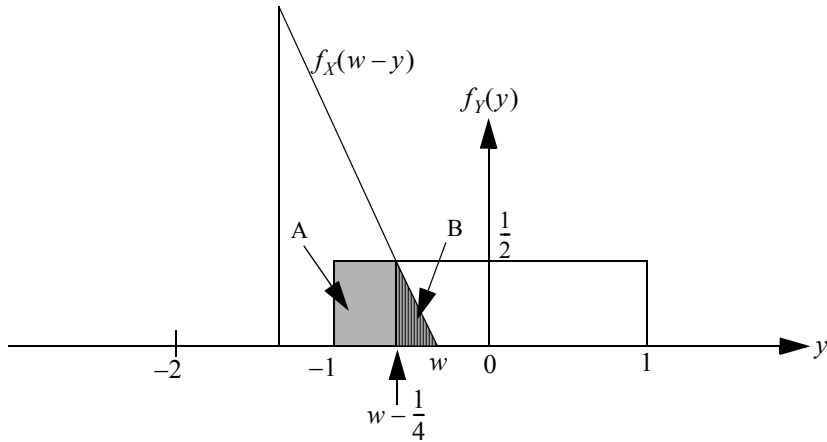


Case 1: In the range  $-1 \leq w \leq -3/4$ , we have the convolution integral as the shaded area:



Thus, in this case,  $f_W(w) = \frac{1}{2} \{w - (-1)\} \{2[w - (-1)]\} = (w + 1)^2$

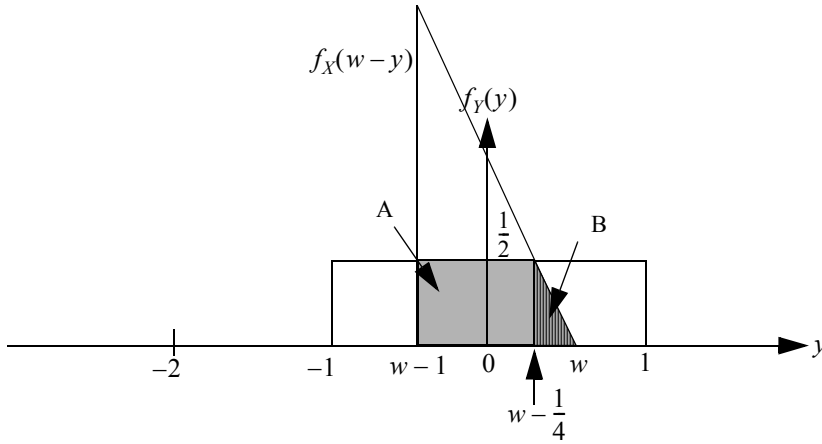
Case 2: In the range  $-\frac{3}{4} \leq w \leq 0$ , the integral is the following shaded area:



In this case, we have that

$$f_W(w) = A + B = \frac{1}{2} \left\{ w - \frac{1}{4} - (-1) \right\} + \frac{1}{2} \left\{ w - \left( w - \frac{1}{4} \right) \right\} \left( \frac{1}{2} \right) = \frac{1}{2} \left( w + \frac{3}{4} \right) + \frac{1}{16} = \frac{7}{16} + \frac{w}{2}$$

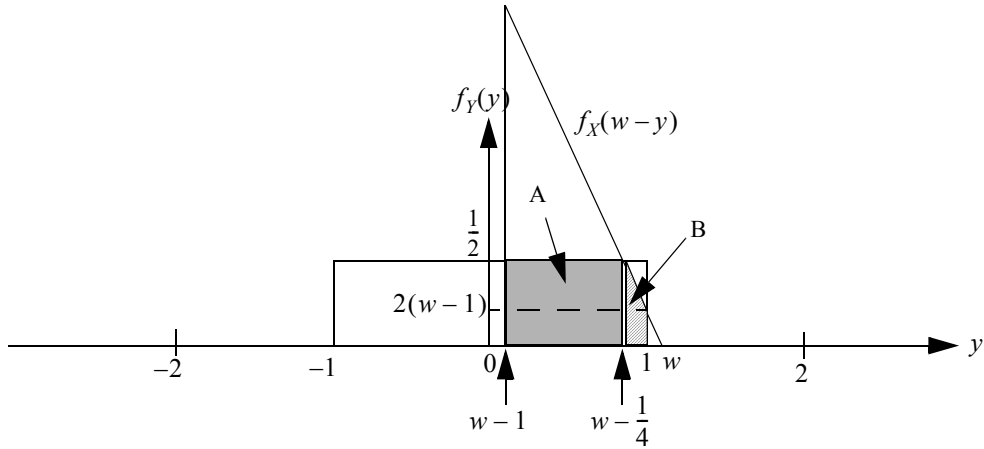
Case 3: In the range  $0 \leq w \leq 1$ , we have the convolution integral as the shaded area:



In this case, we have that

$$f_W(w) = A + B = \frac{1}{2} \left\{ w - \frac{1}{4} - (w-1) \right\} + \frac{1}{2} \left\{ w - \left( w - \frac{1}{4} \right) \right\} \left( \frac{1}{2} \right) = \frac{1}{2} \left( \frac{3}{4} \right) + \frac{1}{16} = \frac{7}{16}$$

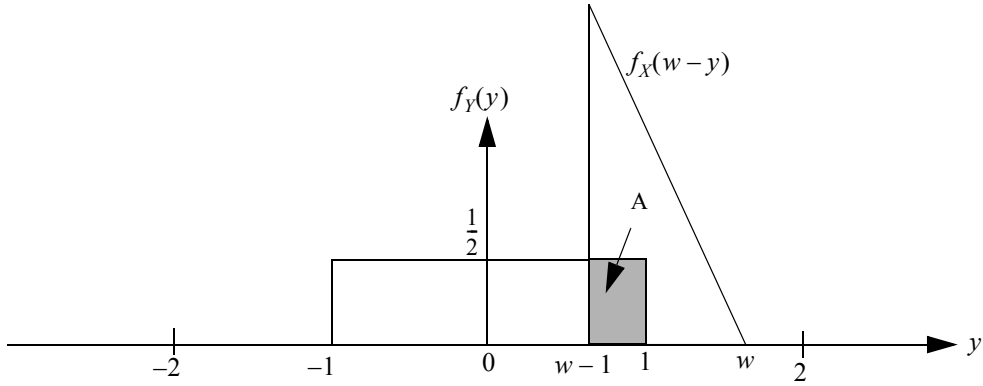
Case 4: In the range  $1 \leq w \leq \frac{5}{4}$ , we have the convolution integral as the shaded area:



In this case, since B is a trapezoid, we have that

$$\begin{aligned}
 f_w(w) &= A + B = \frac{1}{2} \left\{ w - \frac{1}{4} - (w-1) \right\} + \frac{1}{2} \left\{ \frac{1}{2} + 2(w-1) \right\} \left( 1 - \left( w - \frac{1}{4} \right) \right) \\
 &= \frac{3}{8} + \frac{1}{16} - (w-1)^2 = \frac{7}{16} - (w-1)^2
 \end{aligned}$$

Case 5: In the range  $\frac{5}{4} \leq w \leq 2$ , we have the convolution integral as the shaded area:



In this case, we have that

$$f_W(w) = A = \frac{1}{2} \{1 - (w - 1)\} = 1 - \frac{w}{2}$$

Thus, the PDF of  $W$  is given by

$$f_W(w) = \begin{cases} (w+1)^2 & -1 \leq w \leq -\frac{3}{4} \\ \frac{7}{16} + \frac{w}{2} & -\frac{3}{4} \leq w \leq 0 \\ \frac{7}{16} & 0 \leq w \leq 1 \\ \frac{7}{16} - (w-1)^2 & 1 \leq w \leq \frac{5}{4} \\ 1 - \frac{w}{2} & \frac{5}{4} \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

## 6.9 $X$ and $Y$ are two independent random variables with PDFs



$$\begin{aligned} f_X(x) &= 4e^{-4x} & x \geq 0 \\ f_Y(y) &= 2e^{-2y} & y \geq 0 \end{aligned}$$

We define the random variable  $U = X + Y$ .

- a. Since  $X$  and  $Y$  are independent, we can obtain the PDF of  $U$  as follows:

$$f_U(u) = f_X(u) * f_Y(u) = \int_{-\infty}^{\infty} f_X(u-y)f_Y(y)dy$$

Since  $f_X(x) = 0$  for  $x < 0$ , we must have that  $u - y \geq 0 \Rightarrow y \leq u$ . Thus,

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_X(u-y)f_Y(y)dy = \int_0^u 8e^{-4(u-y)}e^{-2y}dy = 8e^{-4u} \int_0^u e^{2y}dy = 8e^{-4u} \left[ \frac{e^{2y}}{2} \right]_0^u \\ &= 4e^{-4u} \{ e^{2u} - 1 \} = 4 \{ e^{-2u} - e^{-4u} \} & u \geq 0 \end{aligned}$$

- b. The probability that  $U$  is greater than 0.2 is given by

$$\begin{aligned} P[U > 0.2] &= \int_{0.2}^{\infty} f_U(u)du = 4 \int_{0.2}^{\infty} \{ e^{-2u} - e^{-4u} \} du = 4 \left[ -\frac{e^{-2u}}{2} + \frac{e^{-4u}}{4} \right]_{0.2}^{\infty} \\ &= 2e^{-0.4} - e^{-0.8} = 0.8913 \end{aligned}$$

6.10 The random variable  $X$  denotes the number that appears on first die, and  $Y$  denotes the number that appears on the second die. The PMFs of  $X$  and  $Y$  are given by

$$p_X(x) = \frac{1}{6} \quad x = 1, 2, \dots, 6$$

$$p_Y(y) = \frac{1}{6} \quad y = 1, 2, \dots, 6$$

$$E[X] = E[Y] = \frac{1}{6} \{ 1 + 2 + 3 + 4 + 5 + 6 \} = \frac{7}{2}$$

Let  $U = X + Y$ . Then the expected value of  $U$  is  $E[U] = E[X] + E[Y] = 7$ .

- 6.11 The random variable  $X$  denotes the sum of the outcomes of two tosses of a fair coin, where a count of 1 is recorded when the outcome is “heads” and a count of 0 is recorded when the outcome is “tails.” To find the expected value of  $X$ , we construct the sample space  $S$  of the experiment as follows:

<b>S</b>	<b>P[S]</b>	<b>X</b>
HH	0.25	2
HT	0.25	1
TH	0.25	1
TT	0.25	0

Thus, the PMF of  $X$  is as follows:

$$p_X(x) = \begin{cases} 0.25 & x = 0 \\ 0.50 & x = 1 \\ 0.25 & x = 2 \\ 0.00 & \text{otherwise} \end{cases}$$

The expected value of  $X$  is  $E[X] = 0.25(0) + 0.50(1) + 0.25(2) = 1$ .

- 6.12 We are required to select 4 students at random from a class of 10 boys and 12 girls. The random variable  $X$  denotes the number of boys selected, and the random variable  $Y$  denotes the number of girls selected.

The PMF of  $X$ ,  $p_X(x)$ , which is the probability of selecting  $x$  boys and hence  $4 - x$  girls, is given by

$$p_X(x) = \frac{\binom{10}{x} \binom{12}{4-x}}{\binom{22}{4}} \quad x = 0, 1, 2, 3, 4$$

The sample space of the experiment and the values of  $X$  and  $Y$  are shown in the following table.

X	Y	P[X]	X-Y
0	4	0.0677	-4
1	3	0.3007	-2
2	2	0.4060	0
3	1	0.1969	2
4	0	0.0287	4

Let  $U = X - Y$ . Then from the above table we see that the PMF of  $U$  is given by

$$p_U(u) = \begin{cases} 0.0677 & u = -4 \\ 0.3007 & u = -2 \\ 0.4060 & u = 0 \\ 0.1969 & u = 2 \\ 0.0287 & u = 4 \end{cases}$$

Thus, the expected value of  $U$  is

$$E[U] = -4(0.0677) - 2(0.3007) + 0(0.4060) + 2(0.1969) + 4(0.0287) = -0.3636$$

Note that we can also obtain the same result by noting that  $E[U] = E[X] - E[Y]$ . The PMF of  $Y$  is simply given by  $p_Y(y) = p_X(4 - y)$ . That is,

$$\begin{aligned} E[X] &= 0p_X(0) + 1p_X(1) + 2p_X(2) + 3p_X(3) + 4p_X(4) \\ &= 0(0.0677) + 1(0.3007) + 2(0.4060) + 3(0.1969) + 4(0.0287) = 1.8182 \\ E[Y] &= 0p_Y(0) + 1p_Y(1) + 2p_Y(2) + 3p_Y(3) + 4p_Y(4) \\ &= 0(0.0287) + 1(0.1969) + 2(0.4060) + 3(0.3007) + 4(0.0677) = 2.1818 \\ E[X - Y] &= E[X] - E[Y] = -0.3636 \end{aligned}$$

- 6.13 Let  $p$  denote the probability that a ball is put in a tagged box. Then  $p = 1/5$ . Let  $X_k$  be a random variable that has the value of 1 if the  $k$ th box contains no ball and 0 otherwise. Then the PMF of  $X_k$  is given by

$$p_{X_k}(x) = \begin{cases} (4/5)^8 & x = 1 \\ 1 - (4/5)^8 & x = 0 \end{cases}$$

Thus,  $E[X_k] = (1)(4/5)^8 + (0)\{1 - (4/5)^8\} = (4/5)^8$ . The number of empty boxes is given by  $X = X_1 + X_2 + X_3 + X_4 + X_5$ . Thus, the expected value of  $X$  is given by

$$E[X] = E[X_1 + X_2 + X_3 + X_4 + X_5] = 5E[X_1] = 5(4/5)^8 = 0.8388$$

- 6.14 Let  $p_A$  denote the probability that coin A comes up heads and  $p_B$  the probability that coin B comes up heads. Then we have that  $p_A = 1/4$  and  $p_B = 1/2$ . Since  $X$  denotes the number of heads resulting from 4 tosses of coin A, and  $Y$  denotes the number of heads resulting from 4 tosses of coin B, the PMFs of  $X$  and  $Y$  are given by

$$p_X(x) = \binom{4}{x} p_A^x (1 - p_A)^{4-x} = \begin{cases} 0.3164 & x = 0 \\ 0.4219 & x = 1 \\ 0.2109 & x = 2 \\ 0.0469 & x = 3 \\ 0.0039 & x = 4 \end{cases}$$

$$p_Y(y) = \binom{4}{y} \left(\frac{1}{2}\right)^4 = \begin{cases} 0.0625 & y = 0 \\ 0.2500 & y = 1 \\ 0.3750 & y = 2 \\ 0.2500 & y = 3 \\ 0.0625 & y = 4 \end{cases}$$

Since  $X$  and  $Y$  are independent, the joint PMF  $p_{XY}(x, y) = p_X(x)p_Y(y)$ . Thus,

- a. The probability that  $X = Y$  is given by

$$\begin{aligned}
P[X = y] &= P[X = 0, Y = 0] + P[X = 1, Y = 1] + P[X = 2, Y = 2] + P[X = 3, Y = 3] + P[X = 4, Y = 4] \\
&= p_X(0)p_Y(0) + p_X(1)p_Y(1) + p_X(2)p_Y(2) + p_X(3)p_Y(3) + p_X(4)p_Y(4) = 0.2163
\end{aligned}$$

b. The event  $(X > Y)$  is given by

$$\begin{aligned}
(X > Y) &= (X = 1, Y = 0) \cup (X = 2, Y = 0) \cup (X = 2, Y = 1) \cup (X = 3, Y = 0) \cup (X = 3, Y = 1) \cup \\
&(X = 3, Y = 2) \cup (X = 4, Y = 0) \cup (X = 4, Y = 1) \cup (X = 4, Y = 2) \cup (X = 4, Y = 3)
\end{aligned}$$

Since these events are mutually exclusive, the probability that  $(X > Y)$  is the sum of the probabilities of these events. Since the CDF of  $Y$ ,  $F_Y(y)$ , is defined by

$$F_Y(y) = P[Y \leq y] = \sum_{k=0}^y p_Y(k)$$

we have that

$$\begin{aligned}
F_Y(0) &= p_Y(0) = 0.0625 \\
F_Y(1) &= p_Y(0) + p_Y(1) = 0.3125 \\
F_Y(2) &= p_Y(0) + p_Y(1) + p_Y(2) = 0.6875 \\
F_Y(3) &= p_Y(0) + p_Y(1) + p_Y(2) + p_Y(3) = 0.9375
\end{aligned}$$

Thus,

$$P[X > Y] = p_X(1)F_Y(0) + p_X(2)F_Y(1) + p_X(3)F_Y(2) + p_X(4)F_Y(3) = 0.1282$$

c. The event that  $X + Y \leq 4$  is given by

$$\begin{aligned}
(X + Y \leq 4) &= (X = 0, Y = 0) \cup (X = 0, Y = 1) \cup (X = 0, Y = 2) \cup (X = 0, Y = 3) \cup (X = 0, Y = 4) \cup \\
&(X = 1, Y = 0) \cup (X = 1, Y = 1) \cup (X = 1, Y = 2) \cup (X = 1, Y = 3) \cup \\
&(X = 2, Y = 0) \cup (X = 2, Y = 1) \cup (X = 2, Y = 2) \cup (X = 3, Y = 0) \cup (X = 3, Y = 1) \cup (X = 4, Y = 0)
\end{aligned}$$

Since these events are mutually exclusive, the probability that  $X + Y \leq 4$  is the sum of the probabilities of the events; that is,

$$\begin{aligned} P[X + Y \leq 4] &= p_X(0) + p_X(1)\{1 - p_Y(4)\} + p_X(2)\{1 - p_Y(3) - p_Y(4)\} + p_X(3)\{p_Y(0) + p_Y(1)\} + p_X(4)p_Y(0) \\ &= 0.8718 \end{aligned}$$

6.15 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = 4xy \quad 0 < x < 1, \quad 0 < y < 1$$

Since the joint PDF is separable and the region of the PDF is rectangular, we conclude that  $X$  and  $Y$  are independent and their marginal PDFs are given by

$$\begin{aligned} f_X(x) &= Ax & 0 < x < 1 \\ f_Y(y) &= By & 0 < y < 1 \\ 4 &= AB \end{aligned}$$

We can obtain the parameter  $A$  as follows:

$$\int_{x=0}^1 f_X(x) dx = 1 = \int_{x=0}^1 Ax dx = A \left[ \frac{x^2}{2} \right]_0^1 = \frac{A}{2} \Rightarrow A = 2$$

Thus,  $B = 2$  and the marginal PDFs become

$$\begin{aligned} f_X(x) &= 2x & 0 < x < 1 \\ f_Y(y) &= 2y & 0 < y < 1 \end{aligned}$$

Note that the marginal PDFs can also be obtained by the traditional method of integrating the joint PDF over  $x$  and  $y$  and the independence of  $X$  and  $Y$  can be established by showing that the product of the marginal PDFs is equal to the joint PDF. Since the random variables are independent, the PDF of  $U = X + Y$  is given by

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$$f_U(u) = f_X(u) * f_Y(u) = \int_{-\infty}^{\infty} f_U(u-y)f_Y(y)dy$$

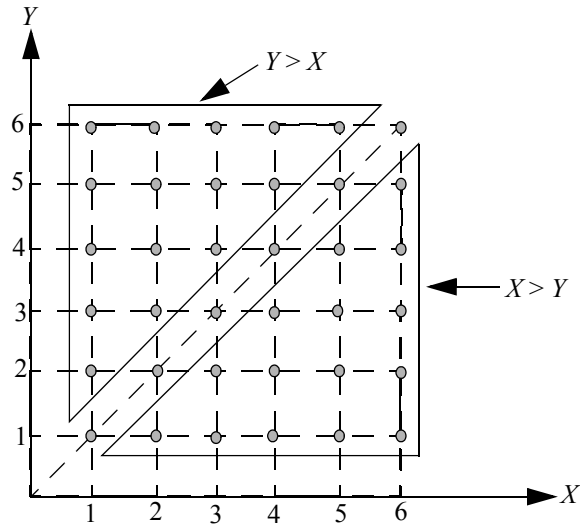
Since  $f_X(x)$  is defined to be nonzero only over the range  $0 < x < 1$ , we have that  $0 < u - y < 1$ , which implies that  $0 < y < u$  and  $u - 1 < y < 1$ . Thus, we obtain

$$f_U(u) = \begin{cases} \int_0^u 4(u-y)ydy & 0 < u < 1 \\ \int_{u-1}^1 4(u-y)ydy & 1 < u < 2 \end{cases}$$

$$= \begin{cases} \frac{2u^3}{3} & 0 < u < 1 \\ \frac{2}{3}(6u - u^3 - 4) & 1 < u < 2 \end{cases}$$

### Sections 6.4 and 6.5: Maximum and Minimum of Independent Random Variables

6.16 Let  $X$  denote the number that appears on the first die and  $Y$  the number that appears on the second die. Then the sample space of the experiment is given by



- a. Let  $W = \max(X, Y)$ . Then the PMF of  $W$  is given by

$$p_W(w) = \begin{cases} 1/36 & w = 1 \\ 3/36 & w = 2 \\ 5/36 & w = 3 \\ 7/36 & w = 4 \\ 9/36 & w = 5 \\ 11/36 & w = 6 \end{cases}$$

Thus, the expected value of  $W$  is given by

$$E[W] = \frac{1}{36} \{1(1) + 2(3) + 3(5) + 4(7) + 5(9) + 6(11)\} = \frac{161}{36}$$

- b. Let  $V = \min(X, Y)$ . Then the PMF of  $V$  is given by

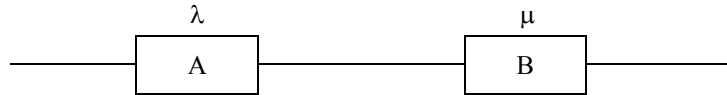


$$p_V(v) = \begin{cases} 11/36 & v = 1 \\ 9/36 & v = 2 \\ 7/36 & v = 3 \\ 5/36 & v = 4 \\ 3/36 & v = 5 \\ 1/36 & v = 6 \end{cases}$$

Thus, the expected value of  $V$  is given by

$$E[V] = \frac{1}{36} \{1(11) + 2(9) + 3(7) + 4(5) + 5(3) + 6(1)\} = \frac{91}{36}$$

6.17 The system arrangement of A and B is as shown below.



Let the random variable  $U$  denote the lifetime of A, and let the random variable  $V$  denote the lifetime of B. Then we know that the PDFs of  $U$  and  $V$  are given by

$$\begin{aligned} f_U(u) &= \lambda e^{-\lambda u} & E[U] &= \frac{1}{\lambda} = 200 \Rightarrow \lambda = \frac{1}{200}, u \geq 0 \\ f_V(v) &= \mu e^{-\mu v} & E[V] &= \frac{1}{\mu} = 400 \Rightarrow \mu = \frac{1}{400}, v \geq 0 \end{aligned}$$

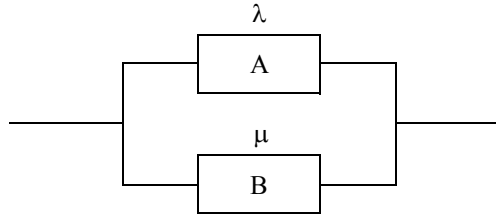
The time,  $X$ , until the system fails is given by  $X = \min(U, V)$ . If we assume that A and B fail independently, then  $U$  and  $V$  are independent and the PDF of  $X$  can be obtained as follows:

$$\begin{aligned}
 F_X(x) &= P[X \leq x] = P[\min(U, V) \leq x] = P[(U \leq x) \cup (V \leq x)] \\
 &= P[U \leq x] + P[V \leq x] - P[U \leq x, V \leq x] = F_U(x) + F_V(x) - F_{UV}(x, x) \\
 &= F_U(x) + F_V(x) - F_U(x)F_V(x) \\
 f_X(x) &= \frac{d}{dx}F_X(x) = f_U(x) + f_V(x) - f_U(x)F_V(x) - F_U(x)f_V(x) \\
 &= f_U(x)\{1 - F_V(x)\} + f_V(x)\{1 - F_U(x)\}
 \end{aligned}$$

Since  $F_U(x) = 1 - e^{-\lambda x}$  and  $F_V(x) = 1 - e^{-\mu x}$ , we obtain

$$\begin{aligned}
 f_X(x) &= \lambda e^{-\lambda x} e^{-\mu x} + \mu e^{-\mu x} e^{-\lambda x} = (\lambda + \mu) e^{-(\lambda + \mu)x} \\
 &= \left( \frac{1}{200} + \frac{1}{400} \right) e^{-\left( \frac{1}{200} + \frac{1}{400} \right)x} = \frac{3}{400} e^{-(3x/400)} \quad x \geq 0
 \end{aligned}$$

6.18 The system arrangement of components A and B is as shown below.



Let the random variable  $U$  denote the lifetime of A, and let the random variable  $V$  denote the lifetime of B. Then we know that the PDFs of  $U$  and  $V$  are given by

$$\begin{aligned}
 f_U(u) &= \lambda e^{-\lambda u} & E[U] &= \frac{1}{\lambda} = 200 \Rightarrow \lambda = \frac{1}{200}, u \geq 0 \\
 f_V(v) &= \mu e^{-\mu v} & E[V] &= \frac{1}{\mu} = 400 \Rightarrow \mu = \frac{1}{400}, v \geq 0
 \end{aligned}$$

The time,  $Y$ , until the system fails is given by  $Y = \max(U, V)$ . If we assume that A and B fail independently, then  $U$  and  $V$  are independent and the PDF of  $Y$  can be obtained as follows:

$$F_Y(y) = P[Y \leq y] = P[\max(U, V) \leq y] = P[(U \leq y) \cap (V \leq y)] = F_{UV}(y, y) = F_U(y)F_V(y)$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = f_U(y)F_V(y) + F_U(y)f_V(y)$$

Since  $F_U(y) = 1 - e^{-\lambda y}$  and  $F_V(y) = 1 - e^{-\mu y}$ , we obtain

$$f_Y(y) = \lambda e^{-\lambda y} \{1 - e^{-\mu y}\} + \mu e^{-\mu y} \{1 - e^{-\lambda y}\} = \lambda e^{-\lambda y} + \mu e^{-\mu y} - (\lambda + \mu)e^{-(\lambda + \mu)y}$$

$$= \frac{1}{200}e^{-y/200} + \frac{1}{400}e^{-y/400} - \frac{3}{400}e^{-3y/400} \quad y \geq 0$$

6.19 The PDF of  $X_k$ , is given by

$$f_{X_k}(x) = \lambda e^{-\lambda x} \quad k = 1, 2, \dots, 5; x \geq 0$$

Then since the random variables  $X_1, X_2, X_3, X_4, X_5$  are independent, we have that

$$P[\max(X_1, X_2, X_3, X_4, X_5) \leq a] = P[X_1 \leq a, X_2 \leq a, \dots, X_5 \leq a]$$

$$= F_{X_1 X_2 X_3 X_4 X_5}(a, a, a, a, a) = F_{X_1}(a)F_{X_2}(a)F_{X_3}(a)F_{X_4}(a)F_{X_5}(a)$$

$$= [1 - e^{-\lambda a}]^5$$

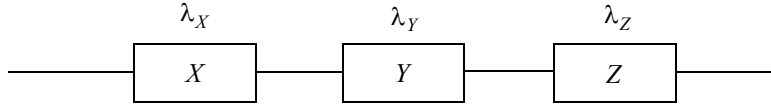
6.20 The PDFs of the lifetimes of the three components  $X$ ,  $Y$ , and  $Z$  are given by

$$f_X(x) = \lambda_X e^{-\lambda_X x} \quad x \geq 0$$

$$f_Y(y) = \lambda_Y e^{-\lambda_Y y} \quad y \geq 0$$

$$f_Z(z) = \lambda_Z e^{-\lambda_Z z} \quad z \geq 0$$

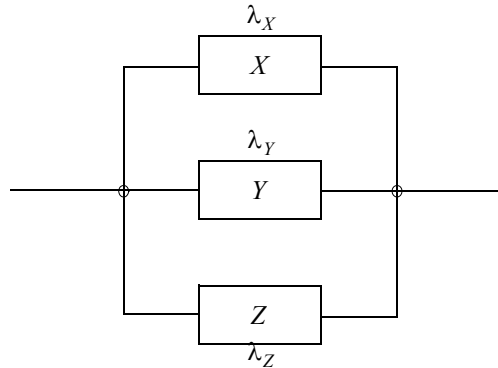
- a. When the components are connected in series, the time  $W$  until the system fails is given by  $W = \min(X, Y, Z)$ , and the arrangement is as shown in the figure below.



The PDF of  $W$  can be obtained as follows:

$$\begin{aligned}
 F_W(w) &= P[W \leq w] = P[\min(X, Y, Z) \leq w] = P[(X \leq w) \cup (Y \leq w) \cup (Z \leq w)] \\
 &= F_X(w) + F_Y(w) + F_Z(w) - F_X(w)F_Y(w) - F_X(w)F_Z(w) - F_Y(w)F_Z(w) + F_X(w)F_Y(w)F_Z(w) \\
 f_W(w) &= \frac{d}{dw}F_W(w) \\
 &= f_X(w)\{1 - F_Y(w) - F_Z(w) + F_Y(w)F_Z(w)\} + f_Y(w)\{1 - F_X(w) - F_Z(w) + F_X(w)F_Z(w)\} + \\
 &\quad f_Z(w)\{1 - F_X(w) - F_Y(w) + F_X(w)F_Y(w)\} \\
 &= (\lambda_X + \lambda_Y + \lambda_Z)e^{-(\lambda_X + \lambda_Y + \lambda_Z)w} \quad w \geq 0
 \end{aligned}$$

- b. When the components are connected in parallel, the time  $W$  until the system fails is given by  $W = \max(X, Y, Z)$ , and the arrangement is as shown in the figure below.



$$F_W(w) = P[W \leq w] = P[\max(X, Y, Z) \leq w] = P[(X \leq w) \cap (Y \leq w) \cap (Z \leq w)] = F_X(w)F_Y(w)F_Z(w)$$

$$\begin{aligned} f_W(w) &= \frac{d}{dw}F_W(w) = f_X(w)F_Y(w)F_Z(w) + f_Y(w)F_X(w)F_Z(w) + f_Z(w)F_X(w)F_Y(w) \\ &= \lambda_X e^{-\lambda_X w} \{1 - e^{-\lambda_Y w} - e^{-\lambda_Z w}\} + \lambda_Y e^{-\lambda_Y w} \{1 - e^{-\lambda_X w} - e^{-\lambda_Z w}\} + \lambda_Z e^{-\lambda_Z w} \{1 - e^{-\lambda_X w} - e^{-\lambda_Y w}\} + \\ &\quad (\lambda_X + \lambda_Y + \lambda_Z) e^{-(\lambda_X + \lambda_Y + \lambda_Z)w} \quad w \geq 0 \end{aligned}$$

- c. When the components are connected in a backup mode with  $X$  used first, then  $Y$  and then  $Z$ , the time  $W$  until the system fails is given by  $W = X + Y + Z$ , and the PDF of  $W$  is given by  $f_W(w) = f_X(w) * f_Y(w) * f_Z(w)$ . Let  $S = X + Y$ . Then  $f_S(s) = f_X(s) * f_Y(s)$ ,  $W = S + Z$ , and  $f_W(w) = f_S(w) * f_Z(w)$ . Now

$$\begin{aligned} f_S(s) &= f_X(s) * f_Y(s) = \int_0^s f_X(s-y)f_Y(y)dy = \int_0^s \lambda_X \lambda_Y e^{-\lambda_X(s-y)} e^{-\lambda_Y y} dy \\ &= \frac{\lambda_X \lambda_Y}{\lambda_Y - \lambda_X} \{e^{-\lambda_X s} - e^{-\lambda_Y s}\} \quad s \geq 0 \end{aligned}$$

Thus, the PDF of  $W$  becomes

$$\begin{aligned} f_W(w) &= f_S(w) * f_Z(w) = \int_0^w f_S(w-z)f_Z(z)dz = \frac{\lambda_X \lambda_Y \lambda_Z}{\lambda_Y - \lambda_X} \int_0^w \{e^{-\lambda_X(w-z)} - e^{-\lambda_Y(w-z)}\} e^{-\lambda_Z z} dz \\ &= \frac{\lambda_X \lambda_Y \lambda_Z}{(\lambda_X - \lambda_Y)(\lambda_Y - \lambda_Z)(\lambda_Z - \lambda_X)} \{\lambda_X (e^{-\lambda_Y w} - e^{-\lambda_Z w}) + \lambda_Y (e^{-\lambda_Z w} - e^{-\lambda_X w}) + \lambda_Z (e^{-\lambda_X w} - e^{-\lambda_Y w})\}, \quad w \geq 0 \end{aligned}$$

## Section 6.8: Two Functions of Two Random Variables

6.21 Given two independent random variables,  $X$  and  $Y$ , with variances  $\sigma_X^2 = 9$  and  $\sigma_Y^2 = 25$ , respectively, and two random variables  $U$  and  $V$  defined as follows:

$$U = 2X + 3Y \Rightarrow \mu_U = 2\mu_X + 3\mu_Y$$

$$V = 4X - 2Y \Rightarrow \mu_V = 4\mu_X - 2\mu_Y$$

- a. The variances of  $U$  and  $V$  are given by

$$\sigma_U^2 = 2^2 \sigma_X^2 + 3^2 \sigma_Y^2 = 4(9) + 9(25) = 261$$

$$\sigma_V^2 = 4^2 \sigma_X^2 + 2^2 \sigma_Y^2 = 16(9) + 4(25) = 244$$

b. The correlation coefficient of  $U$  and  $V$  is obtained as follows:

$$\begin{aligned} \sigma_{UV} &= E[UV] - \mu_U \mu_V = E[UV] - \{2\mu_X + 3\mu_Y\} \{4\mu_X - 2\mu_Y\} = E[UV] - 8\mu_X^2 - 8\mu_X \mu_Y + 6\mu_Y^2 \\ E[UV] &= E[(2X + 3Y)(4X - 2Y)] = E[8X^2 + 8XY - 6Y^2] = 8E[X^2] + 8E[X]E[Y] - 6E[Y^2] \\ &= 8\{\sigma_X^2 + \mu_X^2\} + 8\mu_X \mu_Y - 6\{\sigma_Y^2 + \mu_Y^2\} \end{aligned}$$

Thus,

$$\begin{aligned} \sigma_{UV} &= 8\sigma_X^2 - 6\sigma_Y^2 \\ \rho_{UV} &= \frac{\sigma_{UV}}{\sigma_U \sigma_V} = \frac{8\sigma_X^2 - 6\sigma_Y^2}{\sqrt{(4\sigma_X^2 + 9\sigma_Y^2)(16\sigma_X^2 + 4\sigma_Y^2)}} = \frac{8(9) - 6(25)}{\sqrt{(261)(244)}} = \frac{-78}{252.36} = -0.31 \end{aligned}$$

c. The joint PDF of  $U$  and  $V$  in terms of  $f_{XY}(x, y)$  can be obtained as follows. First, the solution to the equations  $U = 2X + 3Y$  and  $V = 4X - 2Y$  is

$$\begin{aligned} X &= \frac{2U + 3V}{16} \\ Y &= \frac{2U - V}{8} \end{aligned}$$

The Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -16$$

Thus, we obtain

$$f_{UV}(u, v) = \frac{f_{XY}\left(\frac{2u+3v}{16}, \frac{2u-v}{8}\right)}{|-16|} = \frac{1}{16}f_{XY}\left(\frac{2u+3v}{16}, \frac{2u-v}{8}\right)$$

6.22 The random variables  $X$  and  $Y$  have zero mean and variances  $\sigma_X^2 = 16$  and  $\sigma_Y^2 = 36$ , and their correlation coefficient is 0.5.

- a. Let  $U = X + Y \Rightarrow \mu_U = 0$ . Thus, the variance of  $U$  is  $\sigma_U^2 = E[U^2]$ . Now, the second moment of  $U$  is given by

$$\begin{aligned} E[U^2] &= E[(X + Y)^2] = E[X^2 + 2XY + Y^2] = E[X^2] + 2E[XY] + E[Y^2] = \sigma_X^2 + 2E[XY] + \sigma_Y^2 \\ E[XY] &= \rho_{XY}\sigma_X\sigma_Y \end{aligned}$$

Thus, the variance of  $U$  is given by

$$\sigma_U^2 = E[U^2] = \sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2 = 16 + 2(0.5)(4)(6) + 36 = 76$$

- b. Let  $V = X - Y \Rightarrow \mu_V = 0$ . Thus, the variance of  $V$  is  $\sigma_V^2 = E[V^2]$ . Now, the second moment of  $V$  is given by

$$\begin{aligned} E[V^2] &= E[(X - Y)^2] = E[X^2 - 2XY + Y^2] = E[X^2] - 2E[XY] + E[Y^2] = \sigma_X^2 - 2E[XY] + \sigma_Y^2 \\ E[XY] &= \rho_{XY}\sigma_X\sigma_Y \end{aligned}$$

Thus, the variance of  $V$  is given by

$$\sigma_V^2 = E[V^2] = \sigma_X^2 - 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2 = 16 - 2(0.5)(4)(6) + 36 = 28$$

6.23 The joint PDF of two continuous random variables  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

The random variable  $W$  is defined by  $W = X/Y$ . To find the PDF of  $W$ , we first find the marginal PDFs of  $X$  and  $Y$ . We note that the joint PDF is separable into an  $x$ -factor and a  $y$ -factor and the region over which the joint PDF is defined is rectangular. Thus, the marginal PDFs are given by

$$\begin{aligned} f_X(x) &= Ae^{-x} & x \geq 0 \\ f_Y(y) &= Be^{-y} & y \geq 0 \\ AB &= 1 \end{aligned}$$

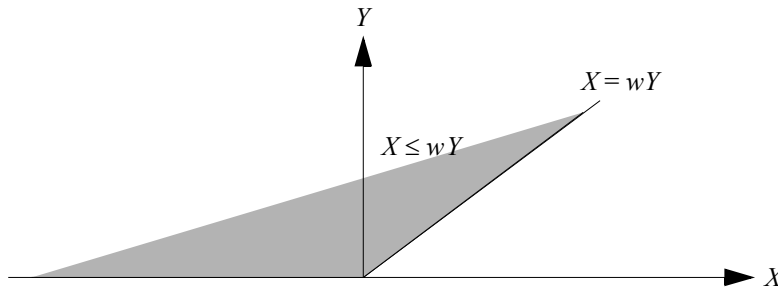
Now,

$$\int_0^{\infty} f_X(x) dx = 1 = \int_0^{\infty} Ae^{-x} dx = A[-e^{-x}]_0^{\infty} = A$$

Thus,  $A = B = 1$ . Note that the independence of  $X$  and  $Y$  can also be established in the traditional way by obtaining the marginal PDFs through integrating over all  $x$  and all  $y$  and observing that their product is equal to the joint PDF. The CDF of  $W$  is given by

$$\begin{aligned} F_W(w) &= P[W \leq w] = P\left[\frac{X}{Y} \leq w\right] = P\left[\left(\frac{X}{Y} \leq w, Y > 0\right) \cup \left(\frac{X}{Y} \leq w, Y < 0\right)\right] \\ &= P\left[\frac{X}{Y} \leq w, Y > 0\right] + P\left[\frac{X}{Y} \leq w, Y < 0\right] = P[X \leq wY, Y > 0] + P[X \geq wY, Y < 0] \end{aligned}$$

Since  $f_{XY}(x, y) = 0$  when  $y < 0$ , we obtain the following region of integration:





Thus,

$$\begin{aligned}
 F_W(w) &= \int_{y=0}^{\infty} \int_{x=0}^{wy} f_{XY}(x, y) dx dy = \int_{y=0}^{\infty} \int_{x=0}^{wy} e^{-(x+y)} dx dy = \int_{y=0}^{\infty} e^{-y} [-e^{-x}]_0^{wy} dy \\
 &= \int_{y=0}^{\infty} e^{-y} [1 - e^{-wy}] dy = \left[ -e^{-y} + \frac{e^{-y(w+1)}}{w+1} \right]_0^{\infty} = 1 - \frac{1}{w+1} = \frac{w}{w+1} \\
 f_W(w) &= \frac{d}{dw} F_W(w) = \frac{1}{(w+1)^2}, \quad 0 < w < \infty
 \end{aligned}$$

6.24  $X$  and  $Y$  are given as two independent random variables that are uniformly distributed between 0 and 1. Thus, their PDF is given by

$$f_X(x) = f_Y(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Since  $X$  and  $Y$  are independent, their joint PDF is  $f_{XY}(x, y) = f_X(x)f_Y(y)$ . Given that  $Z = XY$ , we define an auxilliary variable  $W = X$ . Thus, the solution to the equations

$$\begin{aligned}
 Z &= XY \\
 W &= X
 \end{aligned}$$

is  $X = W, Y = Z/W$ . The Jacobian of the transformation is

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -w$$

Thus, the joint PDF of  $Z$  and  $W$  is given by

$$f_{ZW}(z, w) = \frac{f_{XY}(x, y)}{|J(x, y)|} = \frac{1}{|w|} f_{XY}(w, z/w) = \frac{1}{|w|} f_X(w) f_Y(z/w) = \begin{cases} \frac{1}{w} & 0 < z \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Finally, the PDF of  $Z$  is given by

$$f_Z(z) = \int_{w=z}^1 f_{ZW}(z, w) dw = \int_{w=z}^1 \frac{1}{w} dw = \begin{cases} -\ln z & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

6.25  $X$  and  $Y$  are independent and identically distributed geometric random variables with success parameter  $p$ . Thus, their PMFs are given by

$$\begin{aligned} p_X(x) &= p(1-p)^{x-1} & x \geq 1 \\ p_Y(y) &= p(1-p)^{y-1} & y \geq 1 \end{aligned}$$

Since  $X$  and  $Y$  are independent random variables, their joint PMF is given by

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

Because  $X$  and  $Y$  are independent, the PMF of the random variable  $S = X + Y$  is the convolution of the PMFs of  $X$  and  $Y$ . That is,

$$p_S(s) = P[S = s] = \sum_{y=-\infty}^{\infty} p_{XY}(s-y, y) = \sum_{y=-\infty}^{\infty} p_X(s-y)p_Y(y)$$

To find the limits of the summation, we note that  $x \geq 1 \Rightarrow s - y \geq 1 \Rightarrow y \leq s - 1$ . And since we are given that  $y \geq 1$ , the PMF of  $S$  becomes

$$\begin{aligned}
p_S(s) &= \sum_{y=1}^{s-1} p_X(s-y)p_Y(y) = \sum_{y=1}^{s-1} p^2(1-p)^{s-y-1}(1-p)^{y-1} = p^2 \sum_{y=1}^{s-1} (1-p)^{s-2} \\
&= (s-1)p^2(1-p)^{s-2} \quad s \geq 2
\end{aligned}$$

Note that  $S$  is a second-order Pascal random variable.

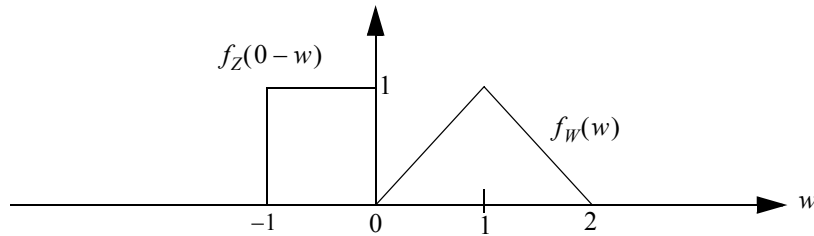
- 6.26 Three independent continuous random variables  $X$ ,  $Y$ , and  $Z$  are uniformly distributed between 0 and 1. The random variable  $S = X + Y + Z$ . Let the random variable  $W$  be defined as the sum of  $X$  and  $Y$ ; that is,  $W = X + Y$ , and thus  $S = W + Z$ . From Example 6.3, the PDF of  $W$  is given by the triangular function

$$f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2-w & 1 < w \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

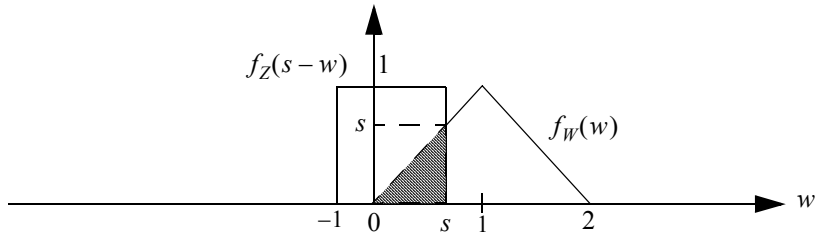
Thus, the PDF of  $S$  is given by

$$f_S(s) = f_W(s) * f_Z(s) = \int_0^s f_W(s-z)f_Z(z)dz = \int_0^s f_W(w)f_Z(s-w)dw$$

To determine the PDF of  $S$  we perform the convolution operation as follows:



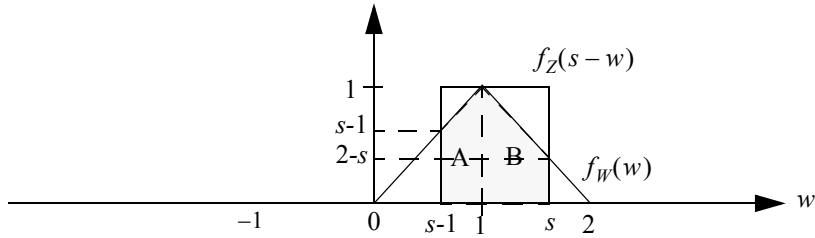
Thus, when  $s < 0$ , there is no overlap between the two PDFs. When  $0 \leq s \leq 1$ , the area overlapped by the two PDFs is shown below.



Thus,

$$f_S(s) = \int_0^s f_W(w)f_Z(s-w)dw = \frac{1}{2}s^2 \quad 0 \leq s \leq 1$$

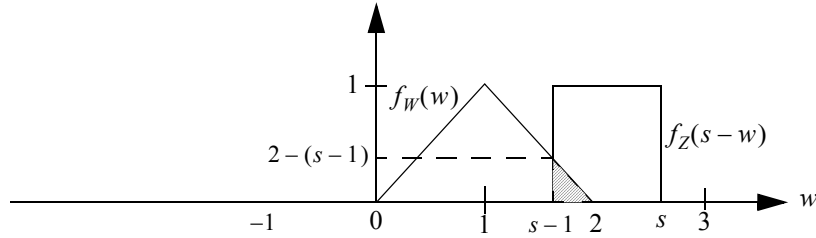
Similarly, when  $1 \leq s \leq 2$ , we have the following situation:



Since the area of interest is the sum of the areas of 2 trapezoids labeled A and B, and the area of a trapezoid is given by  $\frac{1}{2}(\text{sum of parallel sides}) \times \text{height}$ , we have that

$$\begin{aligned} f_S(s) &= \int_0^s f_W(w)f_Z(s-w)dw = \frac{1}{2}\{1 + (s-1)\}\{1 - (s-1)\} + \frac{1}{2}\{1 + (2-s)\}\{s-1\} \\ &= \frac{1}{2}\{s(2-s) + (3-s)(s-1)\} = \frac{1}{2}(6s - 2s^2 - 3) \quad 1 \leq s \leq 2 \end{aligned}$$

Finally, when  $2 \leq s \leq 3$ , we have the following situation:



For this case, we have that

$$f_S(s) = \int_0^s f_W(w)f_Z(s-w)dw = \frac{1}{2}\{2-(s-1)\}\{2-(s-1)\} = \frac{1}{2}(3-s)^2 \quad 2 \leq s \leq 3$$

Thus, we obtain the PDF of  $S$  as

$$f_S(s) = \begin{cases} \frac{s^2}{2} & 0 \leq s \leq 1 \\ \frac{1}{2}(6s - 2s^2 - 3) & 1 \leq s \leq 2 \\ \frac{1}{2}(3-s)^2 & 2 \leq s \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

6.27 Given that  $X$  and  $Y$  are two continuous random variables with the joint PDF  $f_{XY}(x, y)$ , and the functions  $U$  and  $W$  are given by

$$U = 2X + 3Y$$

$$W = X + 2Y$$

The solution to the above equations is  $X = 2U - 3W$ ,  $Y = 2W - U$ . The Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$$

Thus, the joint PDF  $f_{UW}(u, w)$  is given by

$$f_{UW}(u, w) = \frac{f_{XY}(2u - 3w, 2w - u)}{|1|} = f_{XY}(2u - 3w, 2w - u)$$

6.28 Given the random variables  $X$  and  $Y$  and their joint PDF  $f_{XY}(x, y)$ , and the functions  $U = X^2 + Y^2$  and  $W = X^2 - Y^2$ . The solutions to the equations are

$$X = \pm \sqrt{\frac{U+W}{2}}$$

$$Y = \pm \sqrt{\frac{U-W}{2}}$$

The Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -8xy$$

Thus, the joint PDF  $f_{UW}(u, w)$  is given by

$$f_{UW}(u, w) = \frac{f_{XY}\left(\sqrt{\frac{u+w}{2}}, \sqrt{\frac{u-w}{2}}\right)}{4\sqrt{u^2-w^2}} + \frac{f_{XY}\left(\sqrt{\frac{u+w}{2}}, -\sqrt{\frac{u-w}{2}}\right)}{4\sqrt{u^2-w^2}} \\ + \frac{f_{XY}\left(-\sqrt{\frac{u+w}{2}}, \sqrt{\frac{u-w}{2}}\right)}{4\sqrt{u^2-w^2}} + \frac{f_{XY}\left(-\sqrt{\frac{u+w}{2}}, -\sqrt{\frac{u-w}{2}}\right)}{4\sqrt{u^2-w^2}}$$

6.29  $X$  and  $Y$  are independent normal random variables, where  $X = N(\mu_X; \sigma_X^2)$  and  $Y = N(\mu_Y; \sigma_Y^2)$ . The random variables  $U$  and  $W$  are given by

$$U = X + Y \\ W = X - Y$$

The solution to the equations is

$$X = \frac{U+W}{2} \\ Y = \frac{U-W}{2}$$

The Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Thus, the joint PDF  $f_{UW}(u, w)$  is given by

$$f_{UW}(u, w) = \frac{f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right)}{|-2|} = \frac{1}{2}f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right)$$

Since  $X$  and  $Y$  are independent and their marginal PDFs and their joint PDF are given by

$$f_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}$$

$$f_Y(y) = \frac{1}{\sigma_Y \sqrt{2\pi}} e^{-(y-\mu_Y)^2/2\sigma_Y^2}$$

$$f_{XY}(x, y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\{(x-\mu_X)^2/2\sigma_X^2 + (y-\mu_Y)^2/2\sigma_Y^2\}}$$

Thus,  $f_{UW}(u, w)$  is given by

$$f_{UW}(u, w) = \frac{1}{2}f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right) = \frac{1}{4\pi\sigma_X\sigma_Y} \exp\left[-\left\{\left(\frac{u+w-2\mu_X}{2\sqrt{2}\sigma_X}\right)^2 + \left(\frac{u-w-2\mu_Y}{2\sqrt{2}\sigma_Y}\right)^2\right\}\right]$$

### Section 6.10: The Central Limit Theorem

6.30 Let  $X_1, X_2, \dots, X_{30}$  be random variables that denote the outcomes of the rolls of the dice. Then the mean and variance of  $X_k$ ,  $k = 1, 2, \dots, 30$ , are

$$E[X_k] = \frac{1}{6}\{1 + 2 + 3 + 4 + 5 + 6\} = \frac{21}{6} = 3.5$$

$$\sigma_{X_k}^2 = \frac{1}{6}\{(1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 + (4 - 3.5)^2 + (5 - 3.5)^2 + (6 - 3.5)^2\} = \frac{35}{12}$$

Let the random variable  $S$  denote the sum of these outcomes; that is

$$S = X_1 + X_2 + \dots + X_{30}$$

Since these outcomes and hence the random variables are independent, we have that



$$\begin{aligned}
E[S] &= 30E[X_k] = \frac{(30)(21)}{6} = 105 \\
\sigma_S^2 &= 30\sigma_{X_k}^2 = \frac{(30)(35)}{12} = 87.5 \\
\sigma_S &= \sqrt{87.5} = 9.354
\end{aligned}$$

Finally, because the number of observations is 30, we can apply the central limit theorem as follows:

$$\begin{aligned}
P[95 < S < 125] &= F_S(125) - F_S(95) = \Phi\left(\frac{125 - 105}{9.354}\right) - \Phi\left(\frac{95 - 105}{9.354}\right) = \Phi(2.14) - \Phi(-1.07) \\
&= \Phi(2.14) - \{1 - \Phi(1.07)\} = \Phi(2.14) + \Phi(1.07) - 1 = 0.9838 + 0.8577 - 1 \\
&= 0.8415
\end{aligned}$$

6.31  $X_1, X_2, \dots, X_{35}$  are independent random variables each of which is uniformly distributed between 0 and 1. Thus, the mean and variance of  $X_k$ ,  $k = 1, 2, \dots, 35$  are given by

$$\begin{aligned}
E[X_k] &= \frac{1+0}{2} = 0.5 \\
\sigma_{X_k}^2 &= \frac{(1-0)^2}{12} = \frac{1}{12}
\end{aligned}$$

Given that  $S = X_1 + X_2 + \dots + X_{35}$ , the mean, variance, and standard deviation of  $S$  are given by

$$\begin{aligned}
E[S] &= 35E[X_k] = (35)(0.5) = 17.5 \\
\sigma_S^2 &= 35\sigma_{X_k}^2 = \frac{35}{12} \\
\sigma_S &= \sqrt{\frac{35}{12}} = 1.708
\end{aligned}$$

Since the number of observations is 35, we can apply the central limit theorem as follows:

$$\begin{aligned}
 P[S > 22] &= 1 - P[S \leq 22] = 1 - F_S(22) = 1 - \Phi\left(\frac{22 - 17.5}{1.708}\right) = 1 - \Phi(2.63) \\
 &= 1 - 0.9957 = 0.0043
 \end{aligned}$$

6.32 Let  $X_1, X_2, \dots, X_{40}$  be random variables that denote the experimental values of  $X$ . Thus, the mean and variance of  $X_k$ ,  $k = 1, 2, \dots, 40$  are given by

$$\begin{aligned}
 E[X_k] &= \frac{1+2}{2} = 1.5 \\
 \sigma_{X_k}^2 &= \frac{(2-1)^2}{12} = \frac{1}{12}
 \end{aligned}$$

Given that  $S = X_1 + X_2 + \dots + X_{40}$ , the mean, variance, and standard deviation of  $S$  are given by

$$\begin{aligned}
 E[S] &= 40E[X_k] = (40)(1.5) = 60 \\
 \sigma_S^2 &= 40\sigma_{X_k}^2 = \frac{40}{12} = \frac{10}{3} \\
 \sigma_S &= \sqrt{\frac{10}{3}} = 1.826
 \end{aligned}$$

Using the central limit theorem, we have that

$$\begin{aligned}
 P[55 < S < 65] &= F_S(65) - F_S(55) = \Phi\left(\frac{65 - 60}{1.826}\right) - \Phi\left(\frac{55 - 60}{1.826}\right) = \Phi(2.74) - \Phi(-2.74) \\
 &= \Phi(2.74) - \{1 - \Phi(2.74)\} = 2\Phi(2.74) - 1 = 2(0.9969) - 1 \\
 &= 0.9938
 \end{aligned}$$

6.33 The probability  $p$  that the number 4 appears in any toss of a fair die is  $p = 1/6$ . Thus, the number of times  $K$  that the number 4 appears in 600 tosses of the die is a binomially distributed random variable whose PMF is given by

$$p_K(k) = \binom{600}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{600-k} \quad k = 0, 1, 2, \dots, 600$$

The probability that the number appears 100 times is  $p_K(100)$ , which is given by

$$p_K(100) = \binom{600}{100} \left(\frac{1}{6}\right)^{100} \left(\frac{5}{6}\right)^{500} = \frac{600!}{100!500!} \left(\frac{1}{6}\right)^{100} \left(\frac{5}{6}\right)^{500}$$

- a. Using the Stirling's formula,  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n = \sqrt{2\pi n} n^n e^{-n}$ , we have that

$$\begin{aligned} p_K(100) &= \frac{\{\sqrt{1200\pi} (600^{600}) (e^{-600})\} (5^{500})}{(6^{600}) \{\sqrt{200\pi} (100^{100}) (e^{-100})\} \{\sqrt{1000\pi} (500^{500}) (e^{-500})\}} \\ &= \frac{\sqrt{1200\pi}}{\sqrt{200\pi} \sqrt{1000\pi}} = \sqrt{\frac{3}{500\pi}} \\ &= 0.0437 \end{aligned}$$

- b. Using the Poisson approximation to the binomial distribution, we have that

$$\begin{aligned} \lambda &= np = (600) \left(\frac{1}{6}\right) = 100 \\ p_K(100) &\approx \frac{\lambda^{100} e^{-\lambda}}{100!} = \frac{100^{100} e^{-100}}{100!} = \frac{100^{100} e^{-100}}{\sqrt{200\pi} (100^{100}) (e^{-100})} = \frac{1}{\sqrt{200\pi}} \\ &= 0.0399 \end{aligned}$$

where we have used the Stirling's formula to evaluate  $100!$ .

- c. Using the central limit theorem, we have that

$$E[K] = np = 100$$

$$\sigma_K^2 = np(1-p) = \frac{500}{6}$$

$$\sigma_K = \sqrt{\frac{500}{6}} = 9.129$$

$$\begin{aligned} p_K(100) &\approx P[99.5 < K < 100.5] = F_K(100.5) - F_K(99.5) = \Phi\left(\frac{100.5 - 100}{9.129}\right) - \Phi\left(\frac{99.5 - 100}{9.129}\right) \\ &= \Phi(0.05) - \Phi(-0.05) = \Phi(0.05) - \{1 - \Phi(0.05)\} = 2\Phi(0.05) - 1 = 2(0.5199) - 1 \\ &= 0.0398 \end{aligned}$$

## Section 6.11: Order Statistics

6.34 A machine has 7 identical components that operate independently with respective lifetimes  $X_1, X_2, \dots, X_7$  hours, and their common PDF and CDF are  $f_X(x)$  and  $F_X(x)$ , respectively. We are required to find the probability that the machine lasts at most 5 hours under the following conditions:

- a. Let  $Y$  be a random variable that denotes the time until all components have failed. Then  $Y = \max(X_1, X_2, \dots, X_7)$ . Since the  $X_k$  are independent, we have that the probability that the machine lasts at most 5 hours is

$$\begin{aligned} P[Y \leq 5] &= P[\max(X_1, X_2, \dots, X_7) \leq 5] = P[X_1 \leq 5, X_2 \leq 5, \dots, X_7 \leq 5] \\ &= P[X_1 \leq 5]P[X_2 \leq 5] \dots P[X_7 \leq 5] = F_{X_1}(5) \times F_{X_2}(5) \times \dots \times F_{X_7}(5) \\ &= [F_X(5)]^7 \end{aligned}$$

- b. Let  $U$  be a random variable that denotes the time the first component fails. Now, if the machine lasts at most 5 hours, then one component lasts at most 5 hours, whereas the other 6 components last at least 5 hours. Since the components behave independently, we have that

$$P[U \leq 5] = \binom{7}{1} F_X(5) [1 - F_X(5)]^6 = 7F_X(5) [1 - F_X(5)]^6$$

- c. Let  $V$  denote the time until the 6th component failure occurs. Since the machine lasts at most 5 hours, 6 of the 7 components lasted at most 5 hours, which means that

$$P[V \leq 5] = \binom{7}{6} [F_X(5)]^6 [1 - F_X(5)] = 7[F_X(5)]^6 [1 - F_X(5)]$$

- 6.35 A machine needs 4 out of its 6 identical and independent components to operate and  $X_1, X_2, \dots, X_6$  denote the respective lifetimes of the components. Given that each component's lifetime is exponentially distributed with a mean of  $1/\lambda$  hours, the PDF and CDF of the lifetime of a component are given by

$$\begin{aligned} f_X(x) &= \lambda e^{-\lambda x} \\ F_X(x) &= 1 - e^{-\lambda x} \end{aligned}$$

- a. Let  $Y$  denote the lifetime of the machine. Since the machine needs at least four of the components to operate,  $Y$  is the time until 3rd failure occurs, and its CDF is given by

$$\begin{aligned} F_Y(y) &= P[Y \leq y] = \binom{6}{3} [F_X(y)]^3 [1 - F_X(y)]^3 = 20[F_X(y)]^3 [1 - F_X(y)]^3 \\ &= 20e^{-3\lambda y} [1 - e^{-\lambda y}]^3 = 20[e^{-\lambda y} - e^{-2\lambda y}]^3 \end{aligned}$$

- b. The PDF of  $Y$  is given by

$$f_Y(y) = \frac{d}{dy} F_Y(y) = 60[e^{-\lambda y} - e^{-2\lambda y}]^2 \{-\lambda e^{-\lambda y} + 2\lambda e^{-2\lambda y}\} = 60\{2\lambda e^{-2\lambda y} - \lambda e^{-\lambda y}\} [e^{-\lambda y} - e^{-2\lambda y}]^2, \quad y \geq 0$$

- 6.36 The random variables  $X_1, X_2, \dots, X_6$  are independent and identically distributed with the common PDF  $f_X(x)$  and common CDF  $F_X(x)$ . Let  $Y_k$  denote the  $k$ th largest of the random variables  $X_1, X_2, \dots, X_6$ . From Section 6.11 we know that the CDF and PDF of  $Y_k$  are given by

$$F_{Y_k}(y) = \sum_{l=0}^{k-1} \binom{n}{n-l} [F_X(y)]^{n-l} [1 - F_X(y)]^l$$

$$f_{Y_k}(y) = \frac{n!}{(k-1)!(n-k)!} f_X(y) [1 - F_X(y)]^{k-1} [F_X(y)]^{n-k}, \quad y \geq 0$$

- a. The CDF and PDF of the 2nd largest random variable are obtained by substituting  $n = 6$  and  $k = 2$ :

$$F_{Y_2}(y) = \sum_{l=0}^1 \binom{6}{6-l} [F_X(y)]^{6-l} [1 - F_X(y)]^l = [F_X(y)]^6 + 6[F_X(y)]^5 [1 - F_X(y)]$$

$$f_{Y_2}(y) = 30f_X(y)[1 - F_X(y)][F_X(y)]^4, \quad y \geq 0$$

- b. The CDF and PDF of the maximum random variable are obtained by substituting  $n = 6$  and  $k = 1$ :

$$F_{Y_1}(y) = [F_X(y)]^6$$

$$f_{Y_1}(y) = 6f_X(y)[F_X(y)]^5, \quad y \geq 0$$

- c. The CDF and PDF of the minimum random variable are obtained by substituting  $n = 6$  and  $k = 6$ :

$$\begin{aligned} F_{Y_6}(y) &= \sum_{l=0}^5 \binom{6}{6-l} [F_X(y)]^{6-l} [1 - F_X(y)]^l \\ &= [F_X(y)]^6 + 6[F_X(y)]^5 [1 - F_X(y)] + 15[F_X(y)]^4 [1 - F_X(y)]^2 + 20[F_X(y)]^3 [1 - F_X(y)]^3 + \\ &\quad 15[F_X(y)]^2 [1 - F_X(y)]^4 + 6F_X(y) [1 - F_X(y)]^5 \\ f_{Y_6}(y) &= 6f_X(y) [1 - F_X(y)]^5 \quad (y \geq 0) \end{aligned}$$

## Section 7.2: Characteristic Functions

7.1 We are given a random variable  $X$  with the following PDF:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

The characteristic function is given by

$$\Phi_X(w) = \int_{-\infty}^{\infty} e^{jwx} f_X(x) dx = \int_a^b \frac{e^{jwx}}{b-a} dx = \left[ \frac{e^{jwx}}{jw(b-a)} \right]_a^b = \frac{e^{jwb} - e^{jwa}}{jw(b-a)}$$

7.2 Given a random variable  $Y$  with the following PDF

$$f_Y(y) = \begin{cases} 3e^{-3y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The characteristic function is given by

$$\Phi_Y(w) = \int_{-\infty}^{\infty} e^{jwy} f_Y(y) dy = \int_0^{\infty} e^{jwy} 3e^{-3y} dy = 3 \int_0^{\infty} e^{-(3-jw)y} dy = 3 \left[ \frac{-e^{-(3-jw)y}}{3-jw} \right]_0^{\infty} = \frac{3}{3-jw}$$

7.3 Given the random variable  $X$  with the following PDF

$$f_X(x) = \begin{cases} \frac{x+3}{9} & -3 \leq x < 0 \\ \frac{3-x}{9} & 0 \leq x < 3 \\ 0 & \text{otherwise} \end{cases}$$

The characteristic function is given by

$$\begin{aligned}
 \Phi_X(w) &= \int_{-\infty}^{\infty} e^{jwx} f_X(x) dx = \int_{-3}^0 \frac{e^{jwx}(x+3)}{9} dx + \int_0^3 \frac{e^{jwx}(3-x)}{9} dx \\
 &= \frac{1}{9} \left\{ \int_{-3}^0 x e^{jwx} dx + 3 \int_{-3}^0 e^{jwx} dx + 3 \int_0^3 e^{jwx} dx - \int_0^3 x e^{jwx} dx \right\} \\
 &= \frac{1}{9} \left\{ \int_{-3}^0 x e^{jwx} dx - \int_0^3 x e^{jwx} dx + 3 \int_{-3}^3 e^{jwx} dx \right\} = \frac{1}{9} \left\{ \int_{-3}^0 x e^{jwx} dx - \int_0^3 x e^{jwx} dx + \frac{6 \sin 3w}{w} \right\}
 \end{aligned}$$

Let  $u = x \Rightarrow du = dx$ , and  $dv = e^{jwx} dx \Rightarrow v = e^{jwx}/jw$ . Thus,

$$\begin{aligned}
 \int_{-3}^0 x e^{jwx} dx - \int_0^3 x e^{jwx} dx &= \left[ \frac{x e^{jwx}}{jw} \right]_{-3}^0 - \frac{1}{jw} \int_{-3}^0 e^{jwx} dx - \left[ \frac{x e^{jwx}}{jw} \right]_0^3 + \frac{1}{jw} \int_0^3 e^{jwx} dx \\
 &= \frac{3 e^{-3jw}}{jw} + \frac{1}{w^2} [1 - e^{-3jw}] - \frac{3 e^{3jw}}{jw} - \frac{1}{w^2} [e^{3jw} - 1] \\
 &= \frac{2}{w^2} - \frac{3(e^{3jw} - e^{-3jw})}{jw} - \frac{\{e^{3jw} + e^{-3jw}\}}{w^2} = \frac{2}{w^2} - \frac{6(e^{3jw} - e^{-3jw})}{2jw} - \frac{2\{e^{3jw} + e^{-3jw}\}}{2w^2} \\
 &= \frac{2}{w^2} - \frac{6}{w} \sin 3w - \frac{2}{w^2} \cos 3w
 \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
 \Phi_X(w) &= \frac{1}{9} \left\{ \int_{-3}^0 x e^{jwx} dx - \int_0^3 x e^{jwx} dx + \frac{6 \sin 3w}{w} \right\} = \frac{1}{9} \left\{ \frac{2}{w^2} - \frac{6}{w} \sin 3w - \frac{2}{w^2} \cos 3w + \frac{6 \sin 3w}{w} \right\} \\
 &= \frac{2}{9w^2} \{1 - \cos 3w\}
 \end{aligned}$$

## Section 7.3: s-Transforms

7.4 The condition under which a function  $Y(s)$  can be the s-transform of a PDF is that  $Y(0) = 1$ .



- a. Given the function  $A(s) = \frac{1 - e^{-5s}}{s}$ , we have that  $A(0) = \frac{0}{0}$ . Therefore, using L'Hopital's rule we obtain

$$A(0) = \left. \frac{\frac{d}{ds}(1 - e^{-5s})}{\frac{ds}{ds}} \right|_{s=0} = \left. \frac{5e^{-5s}}{1} \right|_{s=0} = 5 \neq 1$$

Thus,  $A(s)$  is not a valid s-transform of a PDF.

- b. Given the function  $B(s) = \frac{7}{4 + 3s}$ , we have that  $B(0) = \frac{7}{4} \neq 1$ , which means that  $B(s)$  is not a valid s-transform of a PDF.
- c. Given the function  $C(s) = \frac{5}{5 + 3s}$ , we have that  $C(0) = \frac{5}{5} = 1$ , which means that  $C(s)$  is a valid s-transform of a PDF.

7.5 Given the s-transform of the PDF of the random variable  $Y$

$$M_Y(s) = \frac{K}{s + 2}$$

- a. The value of  $K$  that makes the function a valid s-transform of a PDF can be obtained as follows:

$$M_Y(0) = \frac{K}{2} = 1 \Rightarrow K = 2$$

- b. To obtain  $E[Y^2]$  we proceed as follows:

$$E[Y^2] = (-1)^2 \left[ \frac{d^2}{ds^2} M_Y(s) \right]_{s=0} = \left[ \frac{2K}{(s+2)^3} \right]_{s=0} = \frac{1}{2}$$

7.6  $X$  and  $Y$  are independent random variables with the PDFs

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \mu e^{-\mu y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

And the random variable  $R$  is defined by  $R = X + Y$ . First we note that

$$M_X(s) = \frac{\lambda}{s + \lambda}$$

$$M_Y(s) = \frac{\mu}{s + \mu}$$

a.  $M_R(s) = M_X(s)M_Y(s) = \left(\frac{\lambda}{s + \lambda}\right)\left(\frac{\mu}{s + \mu}\right)$

b.  $E[R] = E[X] + E[Y] = \frac{1}{\lambda} + \frac{1}{\mu}$

c.  $\sigma_R^2 = \sigma_X^2 + \sigma_Y^2 = \frac{1}{\lambda^2} + \frac{1}{\mu^2}$

7.7 The random variable  $X$  has the following PDF:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The moments of  $X$  that we will need are given by

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 2x^2 dx = \left[ \frac{2x^3}{3} \right]_0^1 = \frac{2}{3}$$

$$E[X^3] = \int_{-\infty}^{\infty} x^3 f_X(x) dx = \int_0^1 2x^4 dx = \left[ \frac{2x^5}{5} \right]_0^1 = \frac{2}{5}$$

Since we are required to determine the numerical values of the derivatives of an s-transform, we do not have to explicitly find  $M_X(s)$ . Instead we proceed as follows:

- a. To obtain  $\left[\frac{d}{ds}\{M_X(s)\}^3\right]_{s=0}$ , we realize that this is the negative of the sum of 3 independent and identically distributed random variables  $X_1, X_2, X_3$  that have the same distribution as  $X$ ; that is

$$\left[\frac{d}{ds}\{M_X(s)\}^3\right]_{s=0} = (-1)\{E[X_1] + E[X_2] + E[X_3]\} = -3E[X] = -2$$

- b. To obtain  $\left[\frac{d^3}{ds^3}M_X(s)\right]_{s=0}$ , we realize that it is related to the third moment of  $X$  as follows:

$$\left[\frac{d^3}{ds^3}M_X(s)\right]_{s=0} = (-1)^3 E[X^3] = -\frac{2}{5}$$

7.8 The s-transform of the PDF of the random variable  $X$  is given by

$$M_X(s) = \frac{\lambda^6}{(s + \lambda)^6} = \left(\frac{\lambda}{s + \lambda}\right)^6$$

Let  $Y$  be the random variable whose PDF  $f_Y(y)$  has the s-transform  $M_Y(s) = \lambda/(s + \lambda)$ . Then  $E[Y] = 1/\lambda$  and  $\sigma_Y^2 = 1/\lambda^2$ ; and we have that  $X$  is the sum of 6 independent and identically distributed random variables  $Y_1, Y_2, \dots, Y_6$  whose common PDF is  $f_Y(y)$ . That is,  $X = Y_1 + Y_2 + \dots + Y_6$ . Thus,

- a.  $E[X] = 6E[Y] = 6/\lambda$   
b.  $\sigma_X^2 = 6\sigma_Y^2 = 6/\lambda^2$

7.9 The s-transform of the PDF of the random variable  $X$  is given as  $M_X(s)$ . Given that  $Y = aX + b$ , the s-transform of the PDF of  $Y$  is given by

$$M_Y(s) = E[e^{-sY}] = E[e^{-s(aX+b)}] = E[e^{-saX}e^{-sb}] = e^{-sb}E[e^{-saX}] = e^{-sb}M_X(as)$$

7.10 The PDFs of  $X$  and  $Y$  are given as follows:

$$f_X(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 0.5 & 2 < y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Then we have that

$$E[X] = \frac{1+0}{2} = 0.5$$

$$\sigma_X^2 = \frac{(1-0)^2}{12} = \frac{1}{12}$$

$$E[Y] = \frac{4+2}{2} = 3$$

$$\sigma_Y^2 = \frac{(4-2)^2}{12} = \frac{1}{3}$$

Given that

$$L(s) = [M_X(s)]^3 [M_Y(s)]^2$$

we observe that  $L$  is the sum of 5 independent random variables, 3 of which are identically distributed as  $X$  and 2 are identically distributed as  $Y$ . That is,

$$L = X_1 + X_2 + X_3 + Y_1 + Y_2$$

Since the quantity

$$\left[ \frac{d^2}{ds^2} L(s) \right]_{s=0} - \left\{ \left[ \frac{d}{ds} L(s) \right]_{s=0} \right\}^2 = E[L^2] - (E[L])^2 = \sigma_L^2$$

we obtain

$$\left[ \frac{d^2}{ds^2} L(s) \right]_{s=0} - \left\{ \left[ \frac{d}{ds} L(s) \right]_{s=0} \right\}^2 = \sigma_L^2 = 3\sigma_X^2 + 2\sigma_Y^2 = \frac{3}{12} + \frac{2}{3} = \frac{11}{12}$$

## Section 7.4: z-Transforms

7.11 The z-transform of the PMF of  $X$  is given by

$$G_X(z) = \frac{1 + z^2 + z^4}{3}$$

a.  $E[X] = \frac{d}{dz} G_X(z) \Big|_{z=1} = \left[ \frac{2z + 4z^3}{3} \right]_{z=1} = \frac{6}{3} = 2$

b.

$$G_X(z) = \sum_k z^k p_X(k) \Rightarrow p_X(k) = \begin{cases} \frac{1}{3} & k = 0 \\ \frac{1}{3} & k = 2 \\ \frac{1}{3} & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$p_X(E[X]) = p_X(2) = \frac{1}{3}$$

7.12 The z-transform of the PMF of  $X$  is given by

$$G_X(z) = A(1 + 3z)^3$$

We first find the value of  $A$  as follows:

$$G_X(1) = 1 = A(1+3)^3 = 64A \Rightarrow A = \frac{1}{64}$$

Thus,

$$\begin{aligned}\frac{d}{dz}G_X(z) &= 3A(3)(1+3z)^2 = 9A(1+3z)^2 \\ \frac{d^2}{dz^2}G_X(z) &= 9A(2)(3)(1+3z) = 54A(1+3z) = 54A + 162Az \\ \frac{d^3}{dz^3}G_X(z) &= 162A\end{aligned}$$

Now,

$$\begin{aligned}\left[\frac{d^3}{dz^3}G_X(z)\right]_{z=1} &= \left[\frac{d^3}{dz^3}\sum_{k=0}^{\infty} z^k p_X(k)\right]_{z=1} = \left[\sum_{k=0}^{\infty} k(k-1)(k-2)z^{k-3} p_X(k)\right]_{z=1} = \sum_{k=0}^{\infty} k(k-1)(k-2)p_X(k) \\ &= \sum_{k=0}^{\infty} \{k^3 - 3k^2 + 2k\} p_X(k) = E[X^3] - 3E[X^2] + 2E[X] \\ E[X^3] &= \left[\frac{d^3}{dz^3}G_X(z)\right]_{z=1} + 3E[X^2] - 2E[X]\end{aligned}$$

But we know that

$$\begin{aligned}E[X^2] &= \left[\frac{d^2}{dz^2}G_X(z)\right]_{z=1} + \left[\frac{d}{dz}G_X(z)\right]_{z=1} \\ E[X] &= \left[\frac{d}{dz}G_X(z)\right]_{z=1}\end{aligned}$$

- a. Thus,  $E[X^3]$  is given by

$$\begin{aligned}
 E[X^3] &= \left[ \frac{d^3}{dz^3} G_X(z) \right]_{z=1} + 3 \left\{ \left[ \frac{d^2}{dz^2} G_X(z) \right]_{z=1} + \left[ \frac{d}{dz} G_X(z) \right]_{z=1} \right\} - 2 \left[ \frac{d}{dz} G_X(z) \right]_{z=1} \\
 &= \left[ \frac{d^3}{dz^3} G_X(z) \right]_{z=1} + 3 \left[ \frac{d^2}{dz^2} G_X(z) \right]_{z=1} + \left[ \frac{d}{dz} G_X(z) \right]_{z=1} \\
 &= 162A + 3 \{ 54A + 162Az \}_{z=1} + \{ 9A(1+3z)^2 \}_{z=1} = 954A = \frac{954}{64} = 14.91
 \end{aligned}$$

- b. To obtain  $p_X(2)$ , we observe that it is the coefficient of  $z^2$  in  $G_X(z)$ , which is given by

$$G_X(z) = A(1+3z)^3 = A\{1+9z+27z^2+27z^3\} \Rightarrow p_X(2) = 27A = \frac{27}{64}$$

7.13 Given the z-transform of the PMF of the random variable  $K$

$$G_K(z) = \frac{A(14+5z-3z^2)}{(2-z)}$$

- a. To find the value of  $A$ , we note that

$$G_K(1) = 1 = \frac{A(14+5-3)}{(2-1)} = 16A \Rightarrow A = \frac{1}{16}$$

- b. To find  $p_K(1)$  we note that we can express the z-transform as follows:

$$\begin{aligned}
G_K(z) &= \frac{A(14 + 5z - 3z^2)}{2\left(1 - \frac{z}{2}\right)} = \frac{1}{32}(14 + 5z - 3z^2) \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k \\
&= \frac{1}{32}(14 + 5z - 3z^2) \left\{ 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \right\} \\
&= \frac{1}{32} \{ 14 + 12z + 3z^2 + 1.5z^3 + \dots \}
\end{aligned}$$

Since  $p_K(1)$  is the coefficient of  $z$  in the above polynomial, we have that

$$p_K(1) = \frac{12}{32} = \frac{3}{8}$$

7.14 To see if the function  $C(z) = z^2 + 2z - 2$  is or is not a valid  $z$ -transform of the PMF of a random variable, we apply 2 tests: First, we evaluate it at the point  $z = 1$ . Next, we observe the signs and magnitudes of the coefficients of  $z$ . Thus, we proceed as follows:

$$C(1) = 1 + 2 - 2 = 1$$

This means that the function has passed the first test and is a potential  $z$ -transform of the PMF of a random variable  $X$ , say. However, since the coefficients of  $z^0 = 1$ ,  $z^1 = z$ , and  $z^2$  are the probabilities that  $X = 0$ ,  $X = 1$ , and  $X = 2$ , respectively, and since probability must lie between 0 and 1, we conclude that  $C(z)$  cannot be the  $z$ -transform of a PMF for the following reasons. First, the constant term, which is supposed to be the coefficient of  $z^0$  and thus the probability that  $X = 0$ , is negative. Secondly, the coefficient of  $z$ , which is supposed to be the probability that  $X = 1$ , is greater than 1.

7.15 Given the function  $D(z) = \frac{1}{2-z}$ .

- a. First, we evaluate the function at  $z = 1$ :  $D(1) = \frac{1}{2-1} = 1$ . Next, we express the function as the following polynomial



$$\begin{aligned}
 D(z) &= \frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k = \frac{1}{2} \left\{ 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \right\} \\
 &= \frac{1}{2} \left\{ 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} + \dots \right\}
 \end{aligned}$$

Since  $D(1) = 1$  and the coefficients of the powers of  $z$  are no less than 0 and no greater than 1, we conclude that  $D(z)$  is a valid z-transform of the PMF of a random variable.

- b. From the above polynomial expression for  $D(z)$  we observe that the coefficient of  $z^k$  is given by  $\frac{1}{2} \left(\frac{1}{2}\right)^k$ ,  $k = 0, 1, \dots$ . Thus, we conclude that the PMF that has the z-transform is

$$p_K(k) = \frac{1}{2} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1} \quad k = 0, 1, 2, \dots$$

7.16 The z-transform of the PMF of  $N$  is given by

$$G_N(z) = 0.5z^5 + 0.3z^7 + 0.2z^{10}$$

- a. From the coefficients of the powers of  $z$  in the above function we conclude that the PMF of  $N$  is

$$p_N(n) = \begin{cases} 0.5 & n = 5 \\ 0.3 & n = 7 \\ 0.2 & n = 10 \\ 0 & \text{otherwise} \end{cases}$$

- b.  $E[N] = \left[ \frac{d}{dz} G_N(z) \right]_{z=1} = [2.5z^4 + 2.1z^6 + 2z^9]_{z=1} = 2.5 + 2.1 + 2 = 6.6$

- c. To find  $\sigma_N^2$ , we first find the 2nd moment of  $N$  as follows:

$$E[N^2] = \left[ \frac{d^2}{dz^2} G_N(z) \right]_{z=1} + \left[ \frac{d}{dz} G_N(z) \right]_{z=1} = [10z^3 + 12.6z^5 + 18z^8]_{z=1} + 6.6 = 47.2$$

$$\text{Thus, } \sigma_N^2 = E[N^2] - (E[N])^2 = 47.2 - (6.6)^2 = 47.2 - 43.56 = 3.64.$$

7.17 The z-transform of the PMF of  $X$  is given by

$$G_X(z) = \left[ \frac{zp}{1 - z(1-p)} \right]^6$$

Let  $Y$  be a random variable whose PMF,  $p_Y(y)$ , has the z-transform

$$G_Y(z) = \frac{zp}{1 - z(1-p)}$$

Then  $X$  is the sum of 6 independent and identically distributed random variables whose PMF is the same as  $Y$ , as follows:  $X = Y_1 + Y_2 + \dots + Y_6$ . Now,  $Y$  is a geometrically distributed random variable with the PMF, mean, and variance as follows:

$$p_Y(y) = \begin{cases} p(1-p)^{y-1} & y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = 1/p$$

$$\sigma_Y^2 = (1-p)/p^2$$

- a.  $E[X] = 6E[Y] = 6/p$   
b.  $\sigma_X^2 = 6\sigma_Y^2 = 6(1-p)/p^2$

7.18 The z-transform of the PMF of  $X$  is given as  $G_X(z)$ . We define the random variable  $Y = aX + b$ . Thus, the z-transform of the PMF of  $Y$  is given by

$$G_Y(z) = E[z^Y] = E[z^{aX+b}] = E[z^{aX} z^b] = z^b E[z^{aX}] = z^b E[(z^a)^X] = z^b G_X(z^a)$$

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## Section 7.5: Random Sum of Random Variables

7.19 The number of families  $X$  that arrive over the period of 1 hour is found to be a Poisson random variable with rate  $\lambda$ . Thus, the PMF of  $X$  and its z-transform are given by

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, \dots$$
$$G_X(z) = e^{\lambda(z-1)}$$

The z-transform of the PMF of  $N$ , the number of people in an arriving family, is given by

$$G_N(z) = \frac{1}{2}z + \frac{1}{3}z^2 + \frac{1}{6}z^3$$

- a. Let  $N_k$  denote the number of people in the  $k$ th family to arrive at the restaurant. If we define  $M_x$  as the number of people in the restaurant when  $X = x$  families have arrived, then we have that

$$M_x = N_1 + N_2 + \dots + N_x$$

Since the  $N_k$  are independent and identically distributed, the z-transform of the PMF of  $M_x$  is

$$G_{M_x}(z) = [G_N(z)]^x$$

Thus, the z-transform of the PMF of  $M$ , the total number of people arriving at the restaurant in an arbitrary hour, is given by

$$G_M(z) = \sum_{x=0}^{\infty} G_{M_x}(z) p_X(x) = \sum_{x=0}^{\infty} [G_N(z)]^x p_X(x) = E[[G_N(z)]^X] = G_X(G_N(z))$$
$$= \exp \left\{ \lambda \left[ \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{6} - 1 \right] \right\}$$

- b. Let  $M_i$  denote the number of people that arrive in the  $i$ th hour,  $i = 1, 2, 3$ . Since the  $M_i$  are identically distributed, we have that the expected number,  $E[Y]$ , of the total number of people that arrive at the restaurant over a three-hour period is given by

$$E[Y] = E[M_1] + E[M_2] + E[M_3] = 3E[M] = 3E[N]E[X]$$

$$E[X] = \lambda$$

$$E[N] = \left[ \frac{d}{dz} G_N(z) \right]_{z=1} = \left[ \frac{1}{2} + \frac{2z}{3} + \frac{3z^2}{6} \right]_{z=1} = \frac{5}{3}$$

$$\text{Thus, } E[Y] = 3\left(\frac{5}{3}\right)\lambda = 5\lambda.$$

7.20 Given that the PMF of the number of customers,  $K$ , that shop at the neighborhood store in a day is

$$p_K(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad k = 0, 1, 2, \dots$$

and that the PMF of the number of items  $N$  that each customer purchases is

$$p_N(n) = \begin{cases} 1/4 & n = 0 \\ 1/4 & n = 1 \\ 1/3 & n = 2 \\ 1/6 & n = 3 \end{cases}$$

where  $K$  and  $N$  are independent random variables. The z-transforms of the PMFs of  $K$  and  $N$  are given respectively by

$$G_K(z) = e^{\lambda(z-1)}$$

$$G_N(z) = \frac{1}{4} + \frac{1}{4}z + \frac{1}{3}z^2 + \frac{1}{6}z^3$$

Let  $N_i$  denote the number of items bought by the  $i$ th customer,  $i = 1, 2, \dots$ , and  $Y_k$  the total number of items given that  $k$  customers arrived at the store that day. Then

$$Y_k = N_1 + N_2 + \dots + N_k$$

Since the  $N_i$  are independent and identically distributed, the z-transform of the PMF of  $Y_k$  is given by

$$G_{Y_k}(z) = [G_N(z)]^k$$

Thus, the z-transform of the PMF of  $Y$  is given by

$$\begin{aligned} G_Y(z) &= \sum_{k=0}^{\infty} G_{Y_k}(z) p_K(k) = \sum_{k=0}^{\infty} [G_N(z)]^k p_K(k) = E[[G_N(z)]^K] = G_K(G_N(z)) \\ &= \exp \left\{ \lambda \left( \frac{1}{4} + \frac{1}{4}z + \frac{1}{3}z^2 + \frac{1}{6}z^3 - 1 \right) \right\} = \exp \left\{ \lambda \left( \frac{1}{4}z + \frac{1}{3}z^2 + \frac{1}{6}z^3 - \frac{3}{4} \right) \right\} \end{aligned}$$

7.21 The PDF of the weight  $W$  of a book is given by

$$f_W(w) = \begin{cases} 1/4 & 1 \leq w \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

The PMF of the number  $K$  of books in any carton is given by

$$p_K(k) = \begin{cases} 1/4 & k = 8 \\ 1/4 & k = 9 \\ 1/3 & k = 10 \\ 1/6 & k = 12 \end{cases}$$

The s-transform of the PDF of  $W$  and the z-transform of the PMF of  $K$  are given by

$$M_W(s) = \frac{e^{-s} - e^{-5s}}{4s}$$

$$G_K(z) = \frac{1}{4}z^8 + \frac{1}{4}z^9 + \frac{1}{3}z^{10} + \frac{1}{6}z^{12}$$

- a. Given that  $X$  is the weight of a randomly selected carton, let  $X_k$  be the weight of a carton that contains  $k$  books, and let  $W_i$  be the weight of the  $i$ th book in the carton. Then

$$X_k = W_1 + W_2 + \dots + W_k$$

Since the  $W_i$  are independent and identically distributed, the z-transform of the PMF of  $X_k$  is given by

$$M_{X_k}(s) = [M_W(s)]^k$$

Thus, the s-transform of the PDF of  $X$  is given by

$$\begin{aligned} M_X(s) &= \sum_{k=0}^{\infty} G_{X_k}(s) p_K(k) = \sum_{k=0}^{\infty} [M_W(s)]^k p_K(k) = E[[M_W(s)]^K] = G_K(M_W(s)) \\ &= \left( \frac{1}{4}z^8 + \frac{1}{4}z^9 + \frac{1}{3}z^{10} + \frac{1}{6}z^{12} \right) \Bigg|_{z = \frac{e^{-s} - e^{-5s}}{4s}} \end{aligned}$$

- b. The expected value of  $X$  is given by  $E[X] = E[K]E[W]$ , where  $E[K]$  and  $E[W]$  are given by

$$\begin{aligned} E[K] &= \left[ \frac{d}{dz} G_K(z) \right]_{z=1} = \left[ 2z^7 + \frac{9}{4}z^8 + \frac{10}{3}z^9 + 2z^{11} \right]_{z=1} = \frac{115}{12} \\ E[W] &= \frac{5+1}{2} = 3 \end{aligned}$$

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Thus,  $E[X] = 3\left(\frac{115}{12}\right) = 28.75$ .

c. The variance of  $X$  is given by

$$\sigma_X^2 = E[K]\sigma_W^2 + (E[W])^2\sigma_K^2$$

Now,

$$\sigma_W^2 = \frac{(5-1)^2}{12} = \frac{16}{12} = \frac{4}{3}$$

$$E[K^2] = 8^2\left(\frac{1}{4}\right) + 9^2\left(\frac{1}{4}\right) + 10^2\left(\frac{1}{3}\right) + 12^2\left(\frac{1}{6}\right) = \frac{64}{4} + \frac{81}{4} + \frac{100}{3} + \frac{144}{6} = \frac{1123}{12}$$

$$\sigma_K^2 = E[K^2] - (E[K])^2 = \frac{1123}{12} - \left(\frac{115}{12}\right)^2 = \frac{251}{144} = 1.743$$

Thus,

$$\sigma_X^2 = \left(\frac{115}{12}\right)\left(\frac{4}{3}\right) + 9\left(\frac{251}{144}\right) = \frac{115}{9} + \frac{251}{16} = 28.4653$$





## Section 8.3: Mean, Autocorrelation Function, and Autocovariance Function

8.1 Since the function

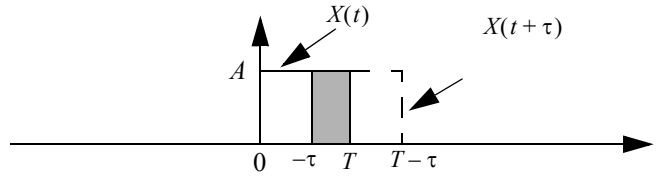
$$X(t) = A \quad 0 \leq t \leq T$$

is an aperiodic function, its autocorrelation function is given by

$$R_{XX}(t, t + \tau) = \int_{-\infty}^{\infty} X(t)X(t + \tau)dt$$

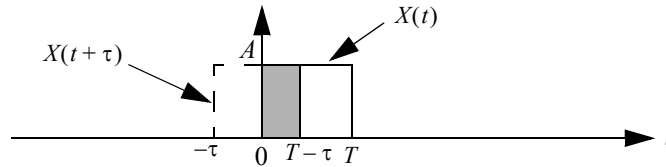
This is essentially a convolution integral that can be evaluated as follows:

(a) When  $-T \leq \tau < 0$ , we obtain the following arrangement:



Thus,  $R_{XX}(t, t + \tau) = \int_{-\infty}^{\infty} X(t)X(t + \tau)dt = A^2(T - \tau)$ , the area of the shaded portion above.

(b) When  $0 \leq \tau < T$ , we obtain the following arrangement



Thus,  $R_{XX}(t, t + \tau) = \int_{-\infty}^{\infty} X(t)X(t + \tau)dt = A^2(T - \tau)$ . From these two results we have that

$$R_{XX}(t, t + \tau) = \begin{cases} A^2(T - \tau) & |\tau| < T \\ 0 & \text{otherwise} \end{cases}$$

8.2 Since the function  $X(t) = A \sin(\omega t + \phi)$  is periodic with period  $T = 2\pi/\omega$ , its autocorrelation function is given by

$$\begin{aligned} R_{XX}(t, t + \tau) &= \frac{1}{2T} \int_{-T}^T X(t)X(t + \tau)dt = \frac{1}{2T} \int_{-T}^T A \sin(\omega t + \phi) A \sin(\omega t + \omega \tau + \phi) dt \\ &= \frac{A^2}{2T} \int_{-T}^T \sin(\omega t + \phi) \sin(\omega t + \omega \tau + \phi) dt = \frac{A^2}{4T} \int_{-T}^T \{ \cos(\omega \tau) - \cos(2\omega t + \omega \tau + 2\phi) \} dt \\ &= \frac{\omega A^2}{8\pi} \left[ t \cos(\omega \tau) - \frac{\sin(2\omega t + \omega \tau + 2\phi)}{2\omega} \right]_{t=-\frac{2\pi}{\omega}}^{\frac{2\pi}{\omega}} \\ &= \frac{\omega A^2}{8\pi} \left[ \frac{4\pi}{\omega} \cos(\omega \tau) - \left\{ \frac{\sin(\omega \tau + 2\phi)}{2\omega} - \frac{\sin(\omega \tau + 2\phi)}{2\omega} \right\} \right] \\ &= \frac{A^2}{2} \cos(\omega \tau) \end{aligned}$$

8.3 Given that

$$X(t) = Y \cos(2\pi t)$$

$$f_Y(y) = \begin{cases} \frac{1}{2} & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \frac{2+0}{2} = 1$$

$$\sigma_Y^2 = \frac{(2-0)^2}{12} = \frac{1}{3}$$

(a) The mean of  $Y$  is  $E[X(t)] = E[Y \cos(2\pi t)] = E[Y] \cos(2\pi t) = \cos(2\pi t)$ .

(b) The autocorrelation function is given by

$$\begin{aligned}
R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[Y \cos(2\pi t) Y \cos(2\pi t + 2\pi\tau)] \\
&= E[Y^2] \cos(2\pi t) \cos(2\pi t + 2\pi\tau) = \{\sigma_Y^2 + (E[Y])^2\} \cos(2\pi t) \cos(2\pi t + 2\pi\tau) \\
&= \frac{4}{3} \cos(2\pi t) \cos(2\pi t + 2\pi\tau) = \frac{2}{3} \{\cos(2\pi\tau) + \cos(4\pi t + 2\pi\tau)\}
\end{aligned}$$

8.4 We are given that  $w$  is a constant,  $Y(t)$  and  $\Theta$  are statistically independent,  $\Theta$  is uniformly distributed between 0 and  $2\pi$ , and a sample function  $X(t)$  of a stationary random process  $Y(t)$ . Thus, we have that

$$\begin{aligned}
X(t) &= Y(t) \sin(wt + \Theta) \\
f_{\Theta}(\theta) &= \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \\
E[\Theta] &= \frac{0 + 2\pi}{2} = \pi \\
\sigma_{\Theta}^2 &= \frac{(2\pi - 0)^2}{12} = \frac{\pi^2}{3}
\end{aligned}$$

The autocorrelation function of  $X(t)$  is given by

$$\begin{aligned}
R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[Y(t) \sin(wt + \Theta) Y(t + \tau) \sin(wt + w\tau + \Theta)] \\
&= E[Y(t)Y(t + \tau)] E[\sin(wt + \Theta) \sin(wt + w\tau + \Theta)] = R_{YY}(\tau) E[\sin(wt + \Theta) \sin(wt + w\tau + \Theta)] \\
&= R_{YY}(\tau) E\left[\frac{\cos(w\tau) - \cos(2wt + w\tau + 2\Theta)}{2}\right] = \frac{1}{2} R_{YY}(\tau) \{\cos(w\tau) - E[\cos(2wt + w\tau + 2\Theta)]\}
\end{aligned}$$

But

$$\begin{aligned}
E[\cos(2wt + w\tau + 2\Theta)] &= \int_{-\infty}^{\infty} \cos(2wt + w\tau + 2\theta) f_{\Theta}(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos(2wt + w\tau + 2\theta) d\theta \\
&= \frac{1}{4\pi} [\sin(2wt + w\tau + 2\theta)]_0^{2\pi} = 0
\end{aligned}$$

Therefore,  $R_{XX}(t, t + \tau) = \frac{1}{2}R_{YY}(\tau)\cos(w\tau)$ .

8.5 Given that the sample function  $X(t)$  of a stationary random process  $Y(t)$  is given by

$$X(t) = Y(t)\sin(wt + \Theta)$$

where  $w$  is a constant,  $Y(t)$  and  $\Theta$  are statistically independent, and  $\Theta$  is uniformly distributed between 0 and  $2\pi$ , we have that

$$\begin{aligned} f_{\Theta}(\theta) &= \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \\ E[X(t)] &= \mu_X(t) = E[Y(t)\sin(wt + \Theta)] = E[Y(t)]E[\sin(wt + \Theta)] \\ E[\sin(wt + \Theta)] &= \int_{-\infty}^{\infty} \sin(wt + \theta)f_{\Theta}(\theta)d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sin(wt + \theta)d\theta \\ &= \frac{1}{2\pi} [-\cos(wt + \theta)]_0^{2\pi} = 0 \end{aligned}$$

Thus,  $\mu_X(t) = 0$  and the autocovariance function of  $X(t)$  is given by

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) - \mu_X(t)\mu_X(t + \tau) = R_{XX}(t, t + \tau)$$

From Problem 8.4, we know that  $R_{XX}(t, t + \tau) = \frac{1}{2}R_{YY}(\tau)\cos(w\tau)$ . Therefore,

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) = \frac{1}{2}R_{YY}(\tau)\cos(w\tau)$$

8.6 The random process  $X(t)$  is given by

$$X(t) = A\cos(wt) + B\sin(wt)$$

where  $w$  is a constant, and  $A$  and  $B$  are independent standard normal random variables (i.e., zero mean and variance of 1). Thus, we have that

$$\begin{aligned}
\mu_A &= \mu_B = 0 \\
\sigma_A^2 &= \sigma_B^2 = 1 \Rightarrow E[A^2] = E[B^2] = 1 \\
C_{XX}(t, t + \tau) &= R_{XX}(t, t + \tau) - \mu_X(t)\mu_X(t + \tau) \\
\mu_X(t) &= E[X(t)] = E[A \cos(wt) + B \sin(wt)] = E[A \cos(wt)] + E[B \sin(wt)] \\
&= E[A] \cos(wt) + E[B] \sin(wt) = 0 \\
R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[\{A \cos(wt) + B \sin(wt)\} \{A \cos(wt + w\tau) + B \sin(wt + w\tau)\}] \\
&= E[A^2] \cos(wt) \cos(wt + w\tau) + E[A]E[B] \{ \cos(wt) \sin(wt + w\tau) + \sin(wt) \cos(wt + w\tau) \} + \\
&\quad E[B^2] \sin(wt) \sin(wt + w\tau) \\
&= \cos(wt) \cos(wt + w\tau) + \sin(wt) \sin(wt + w\tau) = \cos(-w\tau) = \cos(w\tau)
\end{aligned}$$

Thus, the autocovariance function of  $X(t)$  is  $C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) = \cos(w\tau)$ .

8.7  $Y$  is a random variable that is uniformly distributed between 0 and 2. Thus,

$$\begin{aligned}
f_Y(y) &= \begin{cases} \frac{1}{2} & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \\
E[Y] &= \frac{2+0}{2} = 1 \\
\sigma_Y^2 &= \frac{(2-0)^2}{12} = \frac{1}{3}
\end{aligned}$$

If we define  $X(t) = Y \cos(2\pi t)$ , then the autocovariance function of  $X(t)$  is given by

$$\begin{aligned}
C_{XX}(t, t + \tau) &= R_{XX}(t, t + \tau) - \mu_X(t)\mu_X(t + \tau) \\
\mu_X(t) &= E[X(t)] = E[Y \cos(2\pi t)] = E[Y] \cos(2\pi t) = \cos(2\pi t) \\
\mu_X(t)\mu_X(t + \tau) &= \cos(2\pi t) \cos(2\pi t + 2\pi\tau)
\end{aligned}$$

The autocorrelation function is given by

$$\begin{aligned} R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[Y \cos(2\pi t) Y \cos(2\pi t + 2\pi\tau)] \\ &= E[Y^2] \cos(2\pi t) \cos(2\pi t + 2\pi\tau) = \{\sigma_Y^2 + (E[Y])^2\} \cos(2\pi t) \cos(2\pi t + 2\pi\tau) \\ &= \frac{4}{3} \cos(2\pi t) \cos(2\pi t + 2\pi\tau) \end{aligned}$$

Thus, we obtain

$$\begin{aligned} C_{XX}(t, t + \tau) &= R_{XX}(t, t + \tau) - \mu_X(t)\mu_X(t + \tau) = \frac{4}{3} \cos(2\pi t) \cos(2\pi t + 2\pi\tau) - \cos(2\pi t) \cos(2\pi t + 2\pi\tau) \\ &= \frac{1}{3} \cos(2\pi t) \cos(2\pi t + 2\pi\tau) = \frac{1}{6} \{ \cos(2\pi\tau) + \cos(4\pi t + 2\pi\tau) \} \end{aligned}$$

8.8  $X(t)$  is given by

$$X(t) = A \cos(t) + (B + 1) \sin(t) \quad -\infty < t < \infty$$

where  $A$  and  $B$  are independent random variables with  $E[A] = E[B] = 0$  and  $E[A^2] = E[B^2] = 1$ . The autocovariance function of  $X(t)$  can be obtained as follows:

$$\begin{aligned} \mu_X(t) &= E[X(t)] = E[A \cos(t) + (B + 1) \sin(t)] = E[A] \cos(t) + E[B] \sin(t) + \sin(t) = \sin(t) \\ \mu_X(t)\mu_X(t + \tau) &= \sin(t) \sin(t + \tau) \\ R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[\{A \cos(t) + (B + 1) \sin(t)\} \{A \cos(t + \tau) + (B + 1) \sin(t + \tau)\}] \\ &= E[A^2] \cos(t) \cos(t + \tau) + E[B^2] \sin(t) \sin(t + \tau) + \sin(t) \sin(t + \tau) \\ &= \cos(t) \cos(t + \tau) + 2 \sin(t) \sin(t + \tau) = \cos(\tau) + \sin(t) \sin(t + \tau) \\ C_{XX}(t, t + \tau) &= R_{XX}(t, t + \tau) - \mu_X(t)\mu_X(t + \tau) = \cos(t) \cos(t + \tau) + \sin(t) \sin(t + \tau) \\ &= \cos(\tau) \end{aligned}$$

8.9 For any random process  $X(t)$ , autocovariance function is given by

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) - \mu_X(t)\mu_X(t + \tau)$$

If  $X(t)$  is a zero-mean wide-sense stationary process, then  $\mu_X(t) = 0$ , and we have that  $C_{XX}(t, t + \tau) = R_{XX}(\tau)$ . This means that if  $R_{XX}$  is the autocorrelation matrix, then  $C_{XX} = R_{XX}$ . Since  $R_{XX}$  is a symmetric matrix, we have that

$$\mathbf{C}_{\mathbf{XX}} = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.2 \\ 0.8 & 1 & 0.6 & 0.4 \\ 0.4 & 0.6 & 1 & 0.6 \\ 0.2 & 0.4 & 0.6 & 1 \end{bmatrix}$$

8.10 The random process  $X(t)$  is defined by

$$X(t) = A + e^{-B|t|}$$

where  $A$  and  $B$  are independent random variables with the following PDFs

$$f_A(a) = \begin{cases} \frac{1}{2} & -1 \leq a \leq 1 \\ \text{otherwise} & \end{cases}$$

$$f_B(b) = \begin{cases} \frac{1}{2} & 0 \leq b \leq 2 \\ \text{otherwise} & \end{cases}$$

$$E[A] = \frac{-1+1}{2} = 0$$

$$E[B] = \frac{0+2}{2} = 1$$

$$\sigma_A^2 = \frac{\{1-(1)\}^2}{2} = \frac{1}{3} = E[A^2]$$

$$\sigma_B^2 = \frac{(2-0)^2}{12} = \frac{1}{3} \Rightarrow E[B^2] = \frac{4}{3}$$

a. The mean of  $X(t)$  is given by

$$\begin{aligned}
E[X(t)] &= E[A + e^{-B|t|}] = E[A] + E[e^{-B|t|}] = E[e^{-B|t|}] \\
&= \int_{-\infty}^{\infty} e^{-b|t|} f_B(b) db = \frac{1}{2} \int_0^2 e^{-b|t|} db = \frac{1}{2} \left[ -\frac{e^{-b|t|}}{|t|} \right]_{b=0}^2 \\
&= \frac{1}{2|t|} \{1 - e^{-2|t|}\} \quad t > 0
\end{aligned}$$

b. The autocorrelation function of  $X(t)$  is given by

$$\begin{aligned}
R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[\{A + e^{-B|t|}\} \{A + e^{-B|t + \tau|}\}] \\
&= E[A^2 + Ae^{-B|t + \tau|} + Ae^{-B|t|} + e^{-B\{|t| + |t + \tau|\}}] \\
&= E[A^2] + E[A]E[e^{-B|t + \tau|}] + E[A]E[e^{-B|t|}] + E[e^{-B\{|t| + |t + \tau|\}}] \\
&= E[A^2] + E[e^{-B\{|t| + |t + \tau|\}}] \\
&= \frac{1}{3} + \frac{1 - e^{-2\{|t| + |t + \tau|\}}}{2\{|t| + |t + \tau|\}}
\end{aligned}$$

8.11 Given that the autocorrelation function of  $X(t)$  is given by  $R_{XX}(\tau) = e^{-2|\tau|}$ , and the random process  $Y(t)$  is defined as follows:

$$Y(t) = \int_0^t X^2(u) du$$

The expected value of  $Y(t)$  is given by

$$E[Y(t)] = E\left[\int_0^t X^2(u) du\right]$$

Interchanging expectation and integration we obtain

$$E[Y(t)] = \int_0^t E[X^2(u)] du = \int_0^t R_{XX}(0) du = \int_0^t du = t$$



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## Section 8.4: Crosscorrelation and Crosscovariance Functions

8.12  $X(t)$  and  $Y(t)$  are 2 zero-mean and wide-sense stationary processes, and the random process  $Z(t) = X(t) + Y(t)$ . The autocorrelation function of  $Z(t)$  is given by

$$\begin{aligned} R_{ZZ}(t, t + \tau) &= E[Z(t)Z(t + \tau)] = E[\{X(t) + Y(t)\}\{X(t + \tau) + Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau) + X(t)Y(t + \tau) + Y(t)X(t + \tau) + Y(t)Y(t + \tau)] \\ &= E[X(t)X(t + \tau)] + E[X(t)Y(t + \tau)] + E[Y(t)X(t + \tau)] + E[Y(t)Y(t + \tau)] \\ &= R_{XX}(\tau) + R_{XY}(t, t + \tau) + R_{YX}(t, t + \tau) + R_{YY}(\tau) \end{aligned}$$

- a. If  $X(t)$  and  $Y(t)$  are jointly wide-sense stationary, then  $R_{XY}(t, t + \tau) = R_{XY}(\tau)$  and  $R_{YX}(t, t + \tau) = R_{YX}(\tau)$ . Thus, the autocorrelation function of  $Z(t)$  becomes

$$R_{ZZ}(t, t + \tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

- b. If  $X(t)$  and  $Y(t)$  are orthogonal, then  $R_{XY}(t, t + \tau) = R_{YX}(t, t + \tau) = 0$  and  $R_{ZZ}(t, t + \tau)$  becomes

$$R_{ZZ}(t, t + \tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

8.13  $X(t)$  and  $Y(t)$  are defined as follows:

$$\begin{aligned} X(t) &= A \cos(\omega t) + B \sin(\omega t) \\ Y(t) &= B \cos(\omega t) - A \sin(\omega t) \end{aligned}$$

where  $\omega$  is a constant, and  $A$  and  $B$  zero-mean and uncorrelated random variables with variances  $\sigma_A^2 = \sigma_B^2 = \sigma^2$ . The crosscorrelation function  $R_{XY}(t, t + \tau)$  is given by

$$\begin{aligned} R_{XY}(t, t + \tau) &= E[X(t)Y(t + \tau)] = E[\{A \cos(\omega t) + B \sin(\omega t)\}\{B \cos(\omega t + \omega \tau) - A \sin(\omega t + \omega \tau)\}] \\ &= E[AB] \cos(\omega t) \cos(\omega t + \omega \tau) - E[A^2] \cos(\omega t) \sin(\omega t + \omega \tau) + E[B^2] \sin(\omega t) \cos(\omega t + \omega \tau) - \\ &\quad E[AB] \sin(\omega t) \sin(\omega t + \omega \tau) \end{aligned}$$

Since  $A$  and  $B$  are uncorrelated, we have that  $\text{Cov}(A, B) = E[AB] - E[A]E[B] = 0$ . Since  $E[A] = E[B] = 0$ , we have that  $E[AB] = 0$ , and the crosscorrelation function of  $X(t)$  and  $Y(t)$  becomes

$$\begin{aligned} R_{XY}(t, t + \tau) &= E[B^2] \sin(wt) \cos(wt + w\tau) - E[A^2] \cos(wt) \sin(wt + w\tau) \\ &= \sigma^2 \{ \sin(wt) \cos(wt + w\tau) - \cos(wt) \sin(wt + w\tau) \} = \sigma^2 \sin(wt - \{wt + w\tau\}) \\ &= \sigma^2 \sin(-w\tau) = \sigma^2 \sin(w\tau) \end{aligned}$$

8.14  $X(t)$  and  $Y(t)$  are defined as follows:

$$\begin{aligned} X(t) &= A \cos(wt + \Theta) \\ Y(t) &= B \sin(wt + \Theta) \end{aligned}$$

where  $w$ ,  $A$ , and  $B$  are constants, and  $\Theta$  is a random variable with the PDF

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

a. The autocorrelation function of  $X(t)$ ,  $R_{XX}(t, t + \tau)$ , is given by

$$\begin{aligned} R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[A \cos(wt + \Theta)A \cos(wt + w\tau + \Theta)] \\ &= A^2 E[\cos(wt + \Theta) \cos(wt + w\tau + \Theta)] = A^2 E\left[\frac{\cos(-w\tau) + \cos(2wt + w\tau + 2\Theta)}{2}\right] \\ &= \frac{A^2}{2} \left\{ \cos(w\tau) + \int_0^{2\pi} \cos(2wt + w\tau + 2\theta) f_{\Theta}(\theta) d\theta \right\} \\ &= \frac{A^2}{2} \left\{ \cos(w\tau) + \frac{1}{2\pi} \int_0^{2\pi} \cos(2wt + w\tau + 2\theta) d\theta \right\} = \frac{A^2}{2} \left\{ \cos(w\tau) + \frac{1}{2\pi} \left[ \frac{\sin(2wt + w\tau + 2\theta)}{2} \right]_0^{2\pi} \right\} \\ &= \frac{A^2}{2} \cos(w\tau) \end{aligned}$$

Since  $R_{XX}(t, t + \tau)$  is a function of  $\tau$  only, we conclude that  $X(t)$  is a wide-sense stationary process.

- b. The autocorrelation function of  $Y(t)$ ,  $R_{YY}(t, t + \tau)$ , is given by

$$\begin{aligned}
 R_{YY}(t, t + \tau) &= E[Y(t)Y(t + \tau)] = E[B \sin(\omega t + \Theta)B \sin(\omega t + \omega \tau + \Theta)] \\
 &= B^2 E[\sin(\omega t + \Theta) \sin(\omega t + \omega \tau + \Theta)] = B^2 E\left[\frac{\cos(-\omega \tau) - \cos(2\omega t + \omega \tau + 2\Theta)}{2}\right] \\
 &= \frac{B^2}{2} \left\{ \cos(\omega \tau) - \int_0^{2\pi} \cos(2\omega t + \omega \tau + 2\theta) f_\Theta(\theta) d\theta \right\} \\
 &= \frac{B^2}{2} \left\{ \cos(\omega \tau) - \frac{1}{2\pi} \int_0^{2\pi} \cos(2\omega t + \omega \tau + 2\theta) d\theta \right\} = \frac{B^2}{2} \left\{ \cos(\omega \tau) - \frac{1}{2\pi} \left[ \frac{\sin(2\omega t + \omega \tau + 2\theta)}{2} \right]_0^{2\pi} \right\} \\
 &= \frac{B^2}{2} \cos(\omega \tau)
 \end{aligned}$$

Since  $R_{YY}(t, t + \tau)$  is independent of  $t$  and is a function of  $\tau$  only, we conclude that  $Y(t)$  is a wide-sense stationary process.

- c. The crosscorrelation function of  $X(t)$  and  $Y(t)$ ,  $R_{XY}(t, t + \tau)$ , is given by

$$\begin{aligned}
 R_{XY}(t, t + \tau) &= E[X(t)Y(t + \tau)] = E[A \cos(\omega t + \Theta)B \sin(\omega t + \omega \tau + \Theta)] \\
 &= ABE[\cos(\omega t + \Theta) \sin(\omega t + \omega \tau + \Theta)] = ABE\left[\frac{\sin(2\omega t + \omega \tau + 2\Theta) - \sin(-\omega \tau)}{2}\right] \\
 &= \frac{AB}{2} \left\{ \sin(\omega \tau) + \int_0^{2\pi} \sin(2\omega t + \omega \tau + 2\theta) f_\Theta(\theta) d\theta \right\} \\
 &= \frac{AB}{2} \left\{ \sin(\omega \tau) + \frac{1}{2\pi} \int_0^{2\pi} \sin(2\omega t + \omega \tau + 2\theta) d\theta \right\} = \frac{AB}{2} \left\{ \sin(\omega \tau) - \frac{1}{2\pi} \left[ \frac{\cos(2\omega t + \omega \tau + 2\theta)}{2} \right]_0^{2\pi} \right\} \\
 &= \frac{AB}{2} \sin(\omega \tau)
 \end{aligned}$$

Since  $R_{XY}(t, t + \tau)$  is independent of  $t$  and is a function of  $\tau$  only, we conclude that  $X(t)$  and  $Y(t)$  are jointly wide-sense stationary.

## Section 8.5: Wide-sense Stationary Processes

8.15  $X(t)$  and  $Y(t)$  are defined as follows:

$$\begin{aligned} X(t) &= A \cos(w_1 t + \Theta) \\ Y(t) &= B \sin(w_2 t + \Phi) \end{aligned}$$

where  $w_1$ ,  $w_2$ ,  $A$ , and  $B$  are constants, and  $\Theta$  and  $\Phi$  are statistically independent random variables, each of which has the PDF

$$f_{\Theta}(\theta) = f_{\Phi}(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

a. The crosscorrelation function  $R_{XY}(t, t + \tau)$  is given by

$$\begin{aligned} R_{XY}(t, t + \tau) &= E[X(t)Y(t + \tau)] = E[A \cos(w_1 t + \Theta)B \sin(w_2 t + w_2 \tau + \Phi)] \\ &= ABE[\cos(w_1 t + \Theta)\sin(w_2 t + w_2 \tau + \Phi)] \\ &= ABE\left[\frac{\sin(w_1 t + w_2 t + w_2 \tau + \Theta + \Phi) - \sin(w_1 t - w_2 t - w_2 \tau + \Theta - \Phi)}{2}\right] \\ &= \frac{AB}{2}\{E[\sin(w_1 t + w_2 t + w_2 \tau + \Theta + \Phi)] - E[\sin(w_1 t - w_2 t - w_2 \tau + \Theta - \Phi)]\} \end{aligned}$$

Now, since  $\Theta$  and  $\Phi$  are statistically independent random variables, their joint PDF is the product of their marginal PDFs. Thus,

$$\begin{aligned}
E[\sin(w_1 t + w_2 t + w_2 \tau + \Theta + \Phi)] &= \int_0^{2\pi} \int_0^{2\pi} \sin(\{w_1 + w_2\}t + w_2 \tau + \theta + \phi) f_\Theta(\theta) f_\Phi(\phi) d\theta d\phi \\
&= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sin(\{w_1 + w_2\}t + w_2 \tau + \theta + \phi) d\theta d\phi \\
&= 0 \\
E[\sin(w_1 t - w_2 t - w_2 \tau + \Theta - \Phi)] &= \int_0^{2\pi} \int_0^{2\pi} \sin(\{w_1 - w_2\}t - w_2 \tau + \theta - \phi) f_\Theta(\theta) f_\Phi(\phi) d\theta d\phi \\
&= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sin(\{w_1 + w_2\}t + w_2 \tau + \theta + \phi) d\theta d\phi \\
&= 0
\end{aligned}$$

This implies that  $R_{XY}(t, t + \tau) = 0$ , which shows that  $X(t)$  and  $Y(t)$  are jointly wide-sense stationary.

- b. If  $\Theta = \Phi$ , then we have that

$$\begin{aligned}
R_{XY}(t, t + \tau) &= \frac{AB}{2} \{E[\sin(w_1 t + w_2 t + w_2 \tau + 2\Theta)] - \sin(w_1 t - w_2 t - w_2 \tau)\} \\
&= -\frac{AB}{2} \sin(w_1 t - w_2 t - w_2 \tau)
\end{aligned}$$

Since  $R_{XY}(t, t + \tau)$  is not a function of  $\tau$  alone, we conclude that  $X(t)$  and  $Y(t)$  are not jointly wide-sense stationary.

- c. From the result in part (b) above, we can see that when  $\Theta = \Phi$ , the condition under which  $X(t)$  and  $Y(t)$  are jointly wide-sense stationary is that  $w_1 = w_2$ .

8.16 We are required to determine if the following matrices can be autocorrelation matrices of a zero-mean wide-sense stationary random process  $X(t)$ .

a. 
$$G = \begin{bmatrix} 1 & 1.2 & 0.4 & 1 \\ 1.2 & 1 & 0.6 & 0.9 \\ 0.4 & 0.6 & 1 & 1.3 \\ 1 & 0.9 & 1.3 & 1 \end{bmatrix}$$

Since the diagonal elements are supposed to be  $R_{XX}(0)$ , their value puts an upper bound on the other entries because we know that for a wide-sense stationary process  $X(t)$ ,  $R_{XX}(\tau) \leq R_{XX}(0)$  for all  $\tau \neq 0$ . Thus, although  $G$  is a symmetric matrix, it contains off-diagonal elements whose values are larger than the value of the diagonal elements. Therefore,  $G$  cannot be the autocorrelation matrix of a wide-sense stationary process.

b. 
$$H = \begin{bmatrix} 2 & 1.2 & 0.4 & 1 \\ 1.2 & 2 & 0.6 & 0.9 \\ 0.4 & 0.6 & 2 & 1.3 \\ 1 & 0.9 & 1.3 & 2 \end{bmatrix}$$

$H$  is a symmetric matrix and the diagonal elements that are supposed to be the value of  $R_{XX}(0)$  have the same value that is the largest value in the matrix. Therefore,  $H$  can be the autocorrelation matrix of a wide-sense stationary process.

c. 
$$K = \begin{bmatrix} 1 & 0.7 & 0.4 & 0.8 \\ 0.5 & 1 & 0.6 & 0.9 \\ 0.4 & 0.6 & 1 & 0.3 \\ 0.1 & 0.9 & 0.3 & 1 \end{bmatrix}$$

The fact that  $K$  is not a symmetric matrix means that it cannot be the autocorrelation function of a wide-sense stationary process.

8.17  $X(t)$  and  $Y(t)$  are jointly stationary random processes that are defined as follows:

$$\begin{aligned} X(t) &= 2 \cos(5t + \Phi) \\ Y(t) &= 10 \sin(5t + \Phi) \end{aligned}$$

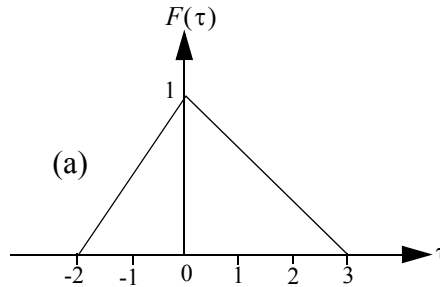
where  $\Phi$  is a random variable with the PDF

$$f_{\Phi}(\phi) = \begin{cases} \frac{1}{2\pi} & 0 \leq \phi \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Thus, the crosscorrelation functions  $R_{XY}(\tau)$  and  $R_{YX}(\tau)$  are given by

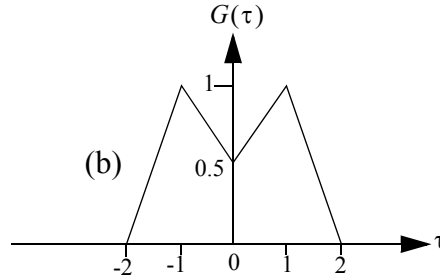
$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = E[2\cos(5t+\Phi)10\sin(5t+5\tau+\Phi)] = 20E[\sin(5t+5\tau+\Phi)\cos(5t+\Phi)] \\ &= 20E\left[\frac{\sin(10t+5\tau+2\Phi) + \sin(5\tau)}{2}\right] = 10E[\sin(10t+5\tau+2\Phi)] + 10\sin(5\tau) \\ &= 10\sin(5\tau) + 10\int_0^{2\pi} \sin(10t+5\tau+2\phi)f_{\Phi}(\phi)d\phi \\ &= 10\sin(5\tau) + \frac{5}{\pi}\int_0^{2\pi} \sin(10t+5\tau+2\phi)d\phi \\ &= 10\sin(5\tau) \\ R_{YX}(\tau) &= E[Y(t)X(t+\tau)] = E[10\sin(5t+\Phi)2\cos(5t+5\tau+\Phi)] = 20E[\sin(5t+\Phi)\cos(5t+5\tau+\Phi)] \\ &= 20E\left[\frac{\sin(10t+5\tau+2\Phi) + \sin(-5\tau)}{2}\right] = 10E[\sin(10t+5\tau+2\Phi)] - 10\sin(5\tau) \\ &= -10\sin(5\tau) \end{aligned}$$

8.18 Consider the function  $F(\tau)$ .



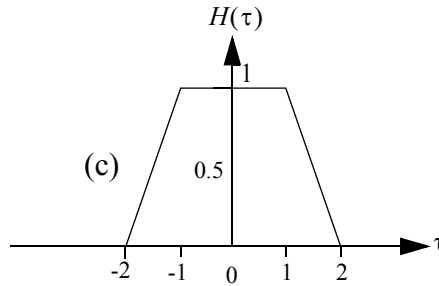
Because  $F(\tau)$  is not an even function, it cannot be the autocorrelation function of a wide-sense stationary process.

(b) Consider the function  $G(\tau)$ .



Although  $G(\tau)$  is an even function,  $G(0)$  is not the largest value of the function. In particular,  $G(0) < G(1)$ . Since the autocorrelation function  $R_{XX}(\tau)$  of a wide-sense stationary process  $X(t)$  has the property that  $R_{XX}(\tau) \leq R_{XX}(0)$  for all  $\tau \neq 0$ , we conclude that  $G(\tau)$  cannot be the autocorrelation function of a wide-sense stationary process.

(c) Consider the function  $H(\tau)$ .



Since  $H(\tau)$  is an even function and  $H(\tau) \leq H(0)$  for all  $\tau \neq 0$ , we conclude that it can be the autocorrelation function of a wide-sense stationary process.

8.19 The random process  $Y(t)$  is given by

$$Y(t) = A \cos(Wt + \Phi)$$



where  $A$ ,  $W$  and  $\phi$  are independent random variables that are characterized as follows:

$$\begin{aligned} E[A] &= 3 \\ \sigma_A^2 &= 9 \\ E[A^2] &= \sigma_A^2 + (E[A])^2 = 18 \\ f_\Phi(\phi) &= \begin{cases} \frac{1}{2\pi} & -\pi \leq \phi \leq \pi \\ 0 & \text{otherwise} \end{cases} \\ f_W(w) &= \begin{cases} \frac{1}{12} & -6 \leq w \leq 6 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The autocorrelation function of  $Y(t)$  is given by

$$\begin{aligned} R_{YY}(t, t + \tau) &= E[Y(t)Y(t + \tau)] = E[A \cos(Wt + \Phi)A \cos(Wt + W\tau + \Phi)] \\ &= E[A^2 \cos(Wt + \Phi) \cos(Wt + W\tau + \Phi)] = E\left[A^2 \frac{\cos(-W\tau) + \cos(2Wt + W\tau + 2\Phi)}{2}\right] \\ &= \frac{1}{2}E[A^2] \{E[\cos(W\tau)] + E[\cos(2Wt + W\tau + 2\Phi)]\} \\ &= 9 \{E[\cos(W\tau)] + E[\cos(2Wt + W\tau + 2\Phi)]\} \\ E[\cos(W\tau)] &= \int_{-6}^6 \cos(w\tau) f_W(w) dw = \frac{1}{12} \int_{-6}^6 \cos(w\tau) dw = \frac{1}{12} \left[ \frac{\sin(w\tau)}{\tau} \right]_{w=-6}^6 = \frac{\sin(w\tau) - \sin(-6\tau)}{12\tau} \\ &= \frac{\sin(6\tau)}{6\tau} \end{aligned}$$

Similarly,

$$\begin{aligned}
E[\cos(2Wt + W\tau + 2\Phi)] &= \int_{w=-6}^6 \int_{\phi=-\pi}^{\pi} \cos(2wt + w\tau + 2\phi) f_{\Phi, W}(\phi, w) d\phi dw \\
&= \int_{w=-6}^6 \int_{\phi=-\pi}^{\pi} \cos(2wt + w\tau + 2\phi) f_{\Phi}(\phi) f_W(w) d\phi dw \\
&= \frac{1}{24\pi} \int_{w=-6}^6 \int_{\phi=-\pi}^{\pi} \cos(2wt + w\tau + 2\phi) d\phi dw = \frac{1}{24\pi} \int_{-6}^6 \left[ \frac{\sin(2wt + w\tau + 2\phi)}{2} \right]_{-\pi}^{\pi} dw \\
&= \frac{1}{48\pi} \int_{-6}^6 [\sin(2wt + w\tau + 2\pi) - \sin(2wt + w\tau - 2\pi)] dw \\
&= \frac{1}{48\pi} \int_{-6}^6 [\sin(2wt + w\tau) - \sin(2wt + w\tau)] dw = 0
\end{aligned}$$

Thus, we obtain

$$R_{YY}(t, t + \tau) = 9 \frac{\sin(6\tau)}{6\tau} = \frac{3\sin(6\tau)}{2\tau}$$

Since  $R_{YY}(t, t + \tau)$  is independent of  $t$ , we conclude that the process  $Y(t)$  is stationary in the wide sense.

8.20 The random process  $X(t)$  is given by

$$X(t) = A \cos(t) + (B + 1) \sin(t) \quad -\infty < t < \infty$$

where  $A$  and  $B$  are independent random variables with  $E[A] = E[B] = 0$  and  $E[A^2] = E[B^2] = 1$ . The autocorrelation function of  $X(t)$  is given by

$$\begin{aligned}
R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[\{A \cos(t) + (B + 1) \sin(t)\} \{A \cos(t + \tau) + (B + 1) \sin(t + \tau)\}] \\
&= E[A^2] \cos(t) \cos(t + \tau) + E[B^2] \sin(t) \sin(t + \tau) + \sin(t) \sin(t + \tau) \\
&= \cos(t) \cos(t + \tau) + 2 \sin(t) \sin(t + \tau) = \cos(\tau) + \sin(t) \sin(t + \tau)
\end{aligned}$$

Since  $R_{XX}(t, t + \tau)$  is not independent of  $t$ , we conclude that  $X(t)$  is not a wide-sense stationary process.

8.21 The autocorrelation function of  $X(t)$  is given by

$$R_{XX}(\tau) = \frac{16\tau^2 + 28}{\tau^2 + 1} = \frac{16(\tau^2 + 1) + 12}{\tau^2 + 1} = 16 + \frac{12}{\tau^2 + 1}$$

(a)  $E[X^2(t)] = R_{XX}(0) = 16 + 12 = 28$

(b)  $E[X(t)] = \pm \sqrt{\lim_{|\tau| \rightarrow \infty} R_{XX}(\tau)} = \pm \sqrt{16 + 0} = \pm 4$

(c)  $\sigma_{X(t)}^2 = E[X^2(t)] - (E[X(t)])^2 = 28 - 16 = 12$

8.22 Given that the wide-sense stationary random process  $X(t)$  has an average power  $E[X^2(t)] = 11$ .

- a. If  $R_{XX}(\tau) = \frac{11 \sin(2\tau)}{1 + \tau^2}$ , then the average power is  $E[X^2(t)] = R_{XX}(0) = 0 \neq 11$ . This means that this function cannot be the autocorrelation function of  $X(t)$ . Note also that the given function is not an even function, which further disqualifies it as a valid autocorrelation function of a wide-sense stationary process.
- b. If  $R_{XX}(\tau) = \frac{11\tau}{1 + 3\tau^2 + 4\tau^4}$ , then the average power is  $E[X^2(t)] = R_{XX}(0) = 0 \neq 11$ . This means that this function cannot be the autocorrelation function of  $X(t)$ . As in the previous case, the fact that the given function is not an even function further disqualifies it as a valid autocorrelation function of a wide-sense stationary process.
- c. If  $R_{XX}(\tau) = \frac{\tau^2 + 44}{\tau^2 + 4}$ , then the average power is  $E[X^2(t)] = R_{XX}(0) = 11$ . Since, in addition, the given function is an even function, we conclude that the function can be the autocorrelation function of  $X(t)$ .

- d. If  $R_{XX}(\tau) = \frac{11 \cos(\tau)}{1 + 3\tau^2 + 4\tau^4}$ , then the average power is  $E[X^2(t)] = R_{XX}(0) = 11$ . Since, in addition, the given function is an even function, we conclude that the function can be the autocorrelation function of  $X(t)$ .
- e. If  $R_{XX}(\tau) = \frac{11\tau^2}{1 + 3\tau^2 + 4\tau^4}$ , then the average power is  $E[X^2(t)] = R_{XX}(0) = 0 \neq 11$ . Thus, although the given function is an even function, it cannot be the autocorrelation function of  $X(t)$ .

8.23 The random process  $X(t)$  has the autocorrelation function

$$R_{XX}(\tau) = 36 + \frac{4}{1 + \tau^2}$$

- (a)  $E[X(t)] = \pm \sqrt{\lim_{|\tau| \rightarrow \infty} R_{XX}(\tau)} = \pm \sqrt{36 + 0} = \pm 6$
- (b)  $E[X^2(t)] = R_{XX}(0) = 36 + 4 = 40$
- (c)  $\sigma_{X(t)}^2 = E[X^2(t)] - (E[X(t)])^2 = 40 - 36 = 4$

8.24 Given that

$$X(t) = Q + N(t)$$

where  $Q$  is a deterministic quantity and  $N(t)$  is a wide-sense stationary noise process.

- a. The mean of  $X(t)$  is  $E[X(t)] = E[Q + N(t)] = E[Q] + E[N(t)] = Q + 0 = Q$
- b. The autocorrelation function of  $X(t)$  is

$$\begin{aligned} R_{XX}(t, t + \tau) &= E[X(t)X(t + \tau)] = E[\{Q + N(t)\}\{Q + N(t + \tau)\}] \\ &= E[Q^2 + QN(t + \tau) + N(t)Q + N(t)N(t + \tau)] \\ &= E[Q^2] + Q\{E[N(t)] + E[N(t + \tau)]\} + E[N(t)N(t + \tau)] \\ &= Q^2 + R_{NN}(\tau) \end{aligned}$$

- c. The autocovariance function of  $X(t)$  is

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) - \mu_X(t)\mu_X(t + \tau) = Q^2 + R_{NN}(\tau) - Q^2 = R_{NN}(\tau)$$

8.25  $X(t)$  and  $Y(t)$  are independent random processes with the following autocorrelation functions and means:

$$\begin{aligned} R_{XX}(\tau) &= e^{-|\tau|} \\ R_{YY}(\tau) &= \cos(2\pi\tau) \\ \mu_X(t) &= \mu_Y(t) = 0 \end{aligned}$$

- a. The autocorrelation function of the process  $U(t) = X(t) + Y(t)$  is given by

$$\begin{aligned} R_{UU}(t, t + \tau) &= E[U(t)U(t + \tau)] = E[\{X(t) + Y(t)\}\{X(t + \tau) + Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau)] + E[X(t)]E[Y(t + \tau)] + E[Y(t)]E[X(t + \tau)] + E[Y(t)Y(t + \tau)] \\ &= R_{XX}(\tau) + R_{YY}(\tau) = e^{-|\tau|} + \cos(2\pi\tau) \end{aligned}$$

- b. The autocorrelation function of the process  $V(t) = X(t) - Y(t)$  is given by

$$\begin{aligned} R_{VV}(t, t + \tau) &= E[V(t)V(t + \tau)] = E[\{X(t) - Y(t)\}\{X(t + \tau) - Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau)] - E[X(t)]E[Y(t + \tau)] - E[Y(t)]E[X(t + \tau)] + E[Y(t)Y(t + \tau)] \\ &= R_{XX}(\tau) + R_{YY}(\tau) = e^{-|\tau|} + \cos(2\pi\tau) \end{aligned}$$

- c. The crosscorrelation function of  $U(t)$  and  $V(t)$  is given by

$$\begin{aligned} R_{UV}(t, t + \tau) &= E[U(t)V(t + \tau)] = E[\{X(t) + Y(t)\}\{X(t + \tau) - Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau)] - E[X(t)]E[Y(t + \tau)] + E[Y(t)]E[X(t + \tau)] - E[Y(t)Y(t + \tau)] \\ &= R_{XX}(\tau) - R_{YY}(\tau) = e^{-|\tau|} - \cos(2\pi\tau) \end{aligned}$$

## Section 8.6: Ergodic Random Processes

8.26 A random process  $Y(t)$  is given by  $Y(t) = A \cos(\omega t + \Phi)$ , where  $\omega$  is a constant, and  $A$  and  $\Phi$  are independent random variables. Given that

$$E[A] = 3, \sigma_A^2 = 9$$

$$f_\Phi(\phi) = \begin{cases} \frac{1}{2\pi} & -\pi \leq \phi \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

The ensemble average of  $Y(t)$  is given by

$$\begin{aligned} E[Y(t)] &= E[A \cos(\omega t + \Phi)] = E[A]E[\cos(\omega t + \Phi)] = 3 \int_{-\pi}^{\pi} \cos(\omega t + \phi) f_\Phi(\phi) d\phi \\ &= \frac{3}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \phi) d\phi = \frac{3}{2\pi} [\sin(\omega t + \phi)]_{-\pi}^{\pi} = \frac{3}{2\pi} [\sin(\omega t + \pi) - \sin(\omega t - \pi)] \\ &= \frac{3}{2\pi} [-\sin(\omega t) + \sin(\omega t)] = 0 \end{aligned}$$

Similarly, the time average of  $Y(t)$  is given by

$$\begin{aligned} \overline{Y(t)} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Y(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A \cos(\omega t + \Phi) dt = \lim_{T \rightarrow \infty} \frac{A}{2T} \left[ \frac{\sin(\omega t + \Phi)}{\omega} \right]_{-T}^T \\ &= \lim_{T \rightarrow \infty} \frac{A}{2\omega T} [\sin(\omega T + \Phi) - \sin(-\omega T + \Phi)] = \lim_{T \rightarrow \infty} \frac{A}{\omega T} [\sin(\omega T) \cos(\Phi)] \\ &= A \cos(\Phi) \lim_{T \rightarrow \infty} \left[ \frac{\sin(\omega T)}{\omega T} \right] = A \cos(\Phi) \lim_{T \rightarrow \infty} [\text{sinc}(\omega T)] = 0 \end{aligned}$$

where  $\text{sinc}(x) = \sin(x)/x$ . Thus, since the ensemble average of  $Y(t)$  is equal to its time average, we conclude that the process is a mean-ergodic process.

8.27 A random process  $X(t)$  is given by  $X(t) = A$ , where  $A$  is a random variable with a finite mean of  $\mu_A$  and finite variance  $\sigma_A^2$ . The ensemble average of  $X(t)$  is given by

$$E[X(t)] = E[A] = \mu_A$$

The time average of  $X(t)$  is given by

$$\overline{X(t)} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A dt = \lim_{T \rightarrow \infty} \frac{2AT}{2T} = A \neq \mu_A$$

Thus, we conclude that  $X(t)$  is not a mean-ergodic process.

## Section 8.7: Power Spectral Density

8.28  $V(t)$  and  $W(t)$  are zero-mean wide-sense stationary random processes. The random process  $M(t)$  is defined as follows:

$$M(t) = V(t) + W(t)$$

- a. Given that  $V(t)$  and  $W(t)$  are jointly wide-sense stationary, then

$$\begin{aligned} R_{MM}(t, t + \tau) &= E[M(t)M(t + \tau)] = E[\{V(t) + W(t)\}\{V(t + \tau) + W(t + \tau)\}] \\ &= E[V(t)V(t + \tau) + V(t)W(t + \tau) + W(t)V(t + \tau) + W(t)W(t + \tau)] \\ &= R_{VV}(\tau) + R_{VW}(\tau) + R_{WV}(\tau) + R_{WW}(\tau) = R_{MM}(\tau) \end{aligned}$$

$$S_{MM}(w) = \int_{-\infty}^{\infty} R_{MM}(\tau) e^{-jw\tau} d\tau = S_{VV}(w) + S_{VW}(w) + S_{WV}(w) + S_{WW}(w)$$

- b. Given that  $V(t)$  and  $W(t)$  are orthogonal, then  $R_{WV}(\tau) = R_{VW}(\tau) = 0$ , which means that

$$\begin{aligned} R_{MM}(t, t + \tau) &= R_{VV}(\tau) + R_{WW}(\tau) = R_{MM}(\tau) \\ S_{MM}(w) &= \int_{-\infty}^{\infty} R_{MM}(\tau) e^{-jw\tau} d\tau = S_{VV}(w) + S_{WW}(w) \end{aligned}$$

8.29 A stationary random process  $X(t)$  has an autocorrelation function  $R_{XX}(\tau) = 2e^{-|\tau|} + 4e^{-4|\tau|}$ . The power spectral density of the process is given by

$$\begin{aligned}
S_{XX}(w) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-jw\tau} d\tau = \int_{-\infty}^0 2e^{\tau} e^{-jw\tau} d\tau + \int_0^{\infty} 2e^{-\tau} e^{-jw\tau} d\tau + \int_{-\infty}^0 4e^{4\tau} e^{-jw\tau} d\tau + \int_0^{\infty} 4e^{-4\tau} e^{-jw\tau} d\tau \\
&= 2 \int_{-\infty}^0 e^{(1-jw)\tau} d\tau + 2 \int_0^{\infty} e^{-(1+jw)\tau} d\tau + 4 \int_{-\infty}^0 e^{(4-jw)\tau} d\tau + 4 \int_0^{\infty} e^{-(4+jw)\tau} d\tau \\
&= \frac{2}{1-jw} [e^{(1-jw)\tau}]_{-\infty}^0 + \frac{2}{1+jw} [-e^{-(1+jw)\tau}]_0^{\infty} + \frac{4}{4-jw} [e^{(4-jw)\tau}]_{-\infty}^0 + \frac{4}{4+jw} [-e^{-(4+jw)\tau}]_0^{\infty} \\
&= \frac{2}{1-jw} + \frac{2}{1+jw} + \frac{4}{4-jw} + \frac{4}{4+jw} \\
&= \frac{4}{1+w^2} + \frac{32}{16+w^2}
\end{aligned}$$

8.30  $X(t)$  has a power spectral density given by

$$S_{XX}(w) = \begin{cases} 4 - \frac{w^2}{9} & |w| \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

The average power and the autocorrelation function of the process are given by

$$\begin{aligned}
E[X^2(t)] &= R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) dw = \frac{1}{2\pi} \int_{-6}^6 \left(4 - \frac{w^2}{9}\right) dw = \frac{1}{2\pi} \left[4w - \frac{w^3}{27}\right]_{-6}^6 \\
&= \frac{16}{\pi} \\
R_{XX}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(w) e^{jw\tau} dw = \frac{1}{2\pi} \int_{-6}^6 \left(4 - \frac{w^2}{9}\right) e^{jw\tau} dw = \frac{1}{2\pi} \left\{ \left[ \frac{4e^{jw\tau}}{j\tau} \right]_{-6}^6 - \frac{1}{9} \int_{-6}^6 w^2 e^{jw\tau} dw \right\} \\
&= \frac{4}{\pi\tau} \left[ \frac{e^{j6\tau} - e^{-j6\tau}}{2j} \right] - \frac{1}{18\pi} \int_{-6}^6 w^2 e^{jw\tau} dw = \frac{4}{\pi\tau} \sin(6\tau) - \frac{1}{18\pi} \int_{-6}^6 w^2 e^{jw\tau} dw
\end{aligned}$$

Let  $u = w^2 \Rightarrow du = 2w dw$ , and let  $dv = e^{jw\tau} dw \Rightarrow v = e^{jw\tau}/j\tau$ . Thus,



$$\begin{aligned}
\int_{-6}^6 w^2 e^{jw\tau} dw &= \left[ \frac{w^2 e^{jw\tau}}{j\tau} \right]_{-6}^6 - \int_{-6}^6 \frac{2w e^{jw\tau}}{j\tau} dw = \frac{72}{\tau} \left\{ \frac{e^{j6\tau} - e^{-j6\tau}}{2j} \right\} - \frac{2}{j\tau} \int_{-6}^6 w e^{jw\tau} dw \\
&= \frac{72}{\tau} \sin(6\tau) - \frac{2}{j\tau} \int_{-6}^6 w e^{jw\tau} dw
\end{aligned}$$

Let  $u = w \Rightarrow du = dw$ , and let  $dv = e^{jw\tau} dw \Rightarrow v = e^{jw\tau}/j\tau$ . Thus,

$$\begin{aligned}
\int_{-6}^6 w e^{jw\tau} dw &= \left[ \frac{w e^{jw\tau}}{j\tau} \right]_{-6}^6 - \int_{-6}^6 \frac{e^{jw\tau}}{j\tau} dw = \frac{12}{j\tau} \left\{ \frac{e^{j6\tau} + e^{-j6\tau}}{2} \right\} - \frac{1}{j\tau} \left[ \frac{e^{jw\tau}}{j\tau} \right]_{-6}^6 \\
&= \frac{12}{j\tau} \cos(6\tau) + \frac{2j}{\tau^2} \left\{ \frac{e^{j6\tau} - e^{-j6\tau}}{2j} \right\} = \frac{12}{j\tau} \cos(6\tau) + \frac{2j}{\tau^2} \sin(6\tau)
\end{aligned}$$

From these results we obtain

$$\int_{-6}^6 w^2 e^{jw\tau} dw = \frac{72}{\tau} \sin(6\tau) - \frac{2}{j\tau} \left\{ \frac{12}{j\tau} \cos(6\tau) + \frac{2j}{\tau^2} \sin(6\tau) \right\} = \frac{72}{\tau} \sin(6\tau) + \frac{24}{\tau^2} \cos(6\tau) - \frac{4}{\tau^3} \sin(6\tau)$$

Therefore,

$$\begin{aligned}
R_{XX}(\tau) &= \frac{4}{\pi\tau} \sin(6\tau) - \frac{1}{18\pi} \int_{-6}^6 w^2 e^{jw\tau} dw \\
&= \frac{4}{\pi\tau} \sin(6\tau) - \frac{1}{18\pi} \left\{ \frac{72}{\tau} \sin(6\tau) + \frac{24}{\tau^2} \cos(6\tau) - \frac{4}{\tau^3} \sin(6\tau) \right\} \\
&= \frac{4}{\pi\tau} \sin(6\tau) - \frac{4}{\pi\tau} \sin(6\tau) - \frac{4}{3\pi\tau^2} \cos(6\tau) + \frac{2}{9\pi\tau^3} \sin(6\tau) \\
&= \frac{2}{9\pi\tau^3} \{ \sin(6\tau) - 6\tau \cos(6\tau) \}
\end{aligned}$$

8.31 A random process  $Y(t)$  has the power spectral density

$$S_{YY}(w) = \frac{9}{w^2 + 64}$$

To find the average power in the process and the autocorrelation function of the process we proceed as follows: From Table 8.1 we observe that

$$e^{-a|\tau|} \leftrightarrow \frac{2a}{a^2 + w^2}$$

Now,

$$\frac{9}{w^2 + 64} = \frac{9}{w^2 + 64} = \frac{9}{16} \left\{ \frac{2(8)}{w^2 + 64} \right\}$$

Thus,  $a = 8$  and we get

$$R_{YY}(\tau) = \frac{9}{16} e^{-8|\tau|}$$
$$E[Y^2(t)] = R_{YY}(0) = \frac{9}{16}$$

8.32 A random process  $Z(t)$  has the autocorrelation function given by

$$R_{ZZ}(\tau) = \begin{cases} 1 + \frac{\tau}{\tau_0} & -\tau_0 \leq \tau \leq 0 \\ 1 - \frac{\tau}{\tau_0} & 0 \leq \tau \leq \tau_0 \\ 0 & \text{otherwise} \end{cases}$$

where  $\tau_0$  is a constant. The power spectral density of the process is given by

$$\begin{aligned}
S_{XX}(w) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-jw\tau} d\tau = \int_{-\tau_0}^0 \left(1 + \frac{\tau}{\tau_0}\right) e^{-jw\tau} d\tau + \int_0^{\tau_0} \left(1 - \frac{\tau}{\tau_0}\right) e^{-jw\tau} d\tau \\
&= \int_{-\tau_0}^0 e^{-jw\tau} d\tau + \frac{1}{\tau_0} \int_{-\tau_0}^0 \tau e^{-jw\tau} d\tau + \int_0^{\tau_0} e^{-jw\tau} d\tau - \frac{1}{\tau_0} \int_0^{\tau_0} \tau e^{-jw\tau} d\tau \\
&= \left[ -\frac{e^{-jw\tau}}{jw} \right]_{-\tau_0}^0 + \frac{1}{\tau_0} \int_{-\tau_0}^0 \tau e^{-jw\tau} d\tau - \frac{1}{\tau_0} \int_0^{\tau_0} \tau e^{-jw\tau} d\tau = \frac{2}{w} \left\{ \frac{e^{jw\tau_0} - e^{-jw\tau_0}}{2j} \right\} + \frac{1}{\tau_0} \int_{-\tau_0}^0 \tau e^{-jw\tau} d\tau - \frac{1}{\tau_0} \int_0^{\tau_0} \tau e^{-jw\tau} d\tau \\
&= \frac{2}{w} \sin(w\tau_0) + \frac{1}{\tau_0} \int_{-\tau_0}^0 \tau e^{-jw\tau} d\tau - \frac{1}{\tau_0} \int_0^{\tau_0} \tau e^{-jw\tau} d\tau
\end{aligned}$$

Let  $\tau = u \Rightarrow d\tau = du$ , and let  $dv = e^{-jw\tau} d\tau \Rightarrow v = -e^{-jw\tau}/jw$ . Thus,

$$\begin{aligned}
\int_{-\tau_0}^0 \tau e^{-jw\tau} d\tau &= \left[ -\frac{\tau e^{-jw\tau}}{jw} \right]_{-\tau_0}^0 + \frac{1}{jw} \int_{-\tau_0}^0 e^{-jw\tau} d\tau = -\left[ \frac{\tau e^{-jw\tau}}{jw} \right]_{-\tau_0}^0 - \frac{1}{jw} \left[ \frac{e^{-jw\tau}}{jw} \right]_{-\tau_0}^0 \\
&= -\frac{1}{jw} \{0 - (-\tau_0) e^{jw\tau_0}\} + \frac{1}{w^2} \{1 - e^{jw\tau_0}\} = \frac{1}{w^2} - \frac{\tau_0 e^{jw\tau_0}}{jw} - \frac{e^{jw\tau_0}}{w^2} \\
\int_0^{\tau_0} \tau e^{-jw\tau} d\tau &= \left[ -\frac{\tau e^{-jw\tau}}{jw} \right]_0^{\tau_0} + \frac{1}{jw} \int_0^{\tau_0} e^{-jw\tau} d\tau = -\left[ \frac{\tau e^{-jw\tau}}{jw} \right]_0^{\tau_0} - \frac{1}{jw} \left[ \frac{e^{-jw\tau}}{jw} \right]_0^{\tau_0} \\
&= -\frac{1}{jw} \{\tau_0 e^{-jw\tau_0}\} + \frac{1}{w^2} \{e^{-jw\tau_0} - 1\} = \frac{e^{-jw\tau_0}}{w^2} - \frac{1}{w^2} - \frac{\tau_0 e^{-jw\tau_0}}{jw}
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_{XX}(w) &= \frac{2}{w} \sin(w\tau_0) + \frac{1}{\tau_0} \int_{-\tau_0}^0 \tau e^{-jw\tau} d\tau - \frac{1}{\tau_0} \int_0^{\tau_0} \tau e^{-jw\tau} d\tau \\
&= \frac{2}{w} \sin(w\tau_0) + \frac{1}{\tau_0} \left\{ \frac{1}{w^2} - \frac{\tau_0 e^{jw\tau_0}}{jw} - \frac{e^{jw\tau_0}}{w^2} \right\} - \frac{1}{\tau_0} \left\{ \frac{e^{-jw\tau_0}}{w^2} - \frac{1}{w^2} - \frac{\tau_0 e^{-jw\tau_0}}{jw} \right\} \\
&= \frac{2}{w} \sin(w\tau_0) + \frac{2}{w^2 \tau_0} - \frac{2(e^{jw\tau_0} + e^{-jw\tau_0})}{2w^2 \tau_0} - \frac{2(e^{jw\tau_0} - e^{-jw\tau_0})}{2jw} \\
&= \frac{2}{w} \sin(w\tau_0) + \frac{2}{w^2 \tau_0} - \frac{2}{w^2 \tau_0} \cos(w\tau_0) - \frac{2}{w} \sin(w\tau_0) = \frac{2}{w^2 \tau_0} - \frac{2}{w^2 \tau_0} \cos(w\tau_0) \\
&= \frac{2}{w^2 \tau_0} \{1 - \cos(w\tau_0)\} = \frac{2}{w^2 \tau_0} \left\{ 2 \left[ \sin\left(\frac{w\tau_0}{2}\right) \right]^2 \right\} = \frac{4\tau_0 [\sin(w\tau_0/2)]^2}{(w\tau_0)^2} = \tau_0 \left[ \frac{\sin(w\tau_0/2)}{(w\tau_0/2)} \right]^2
\end{aligned}$$

8.33 We are required to give reasons why the functions given below can or cannot be the power spectral density of a wide-sense stationary random process.

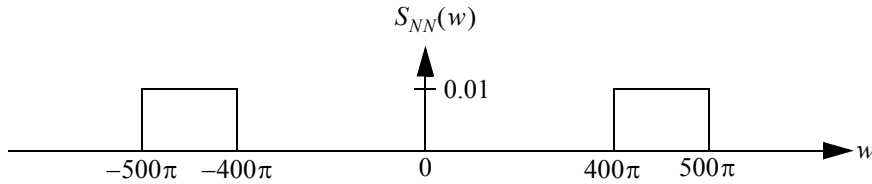
- a. The function  $S_{XX}(w) = \frac{\sin(w)}{w}$  is an even function. However, in addition to being an even function, a valid power spectral density should satisfy the condition  $S_{XX}(w) \geq 0$ . Since the above function can take both positive and negative values, we conclude that it cannot be the power spectral density of a wide-sense stationary process.
- b. The function  $S_{XX}(w) = \frac{\cos(w)}{w}$  is not an even function and takes negative values when  $w$  is negative. Thus, it cannot be the power spectral density of a wide-sense stationary process.
- c. The function  $S_{XX}(w) = \frac{8}{w^2 + 16}$  is an even function and is also a non-negative function. Thus, it can be the power spectral density of a wide-sense stationary process.
- d. The function  $S_{XX}(w) = \frac{5w^2}{1 + 3w^2 + 4w^4}$  is an even function that is also a non-negative function. Thus, it can be the power spectral density of a wide-sense stationary process.

- e. The function  $S_{XX}(w) = \frac{5w}{1 + 3w^2 + 4w^4}$  is not an even function and takes negative values when  $w$  is negative. Thus, it cannot be the power spectral density of a wide-sense stationary process.

8.34 A bandlimited white noise has the power spectral density defined by

$$S_{NN}(w) = \begin{cases} 0.01 & 400\pi \leq |w| \leq 500\pi \\ 0 & \text{otherwise} \end{cases}$$

The power spectral density can be sketched as shown below.



The mean-square value of the process is given by

$$\begin{aligned} E[N^2(t)] &= R_{NN}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{NN}(w) dw = \frac{1}{2\pi} \left\{ \int_{-500\pi}^{-400\pi} 0.01 dw + \int_{400\pi}^{500\pi} 0.01 dw \right\} \\ &= \frac{0.01}{2\pi} \{ [w]_{-500\pi}^{-400\pi} + [w]_{400\pi}^{500\pi} \} = \frac{0.01}{2\pi} \{ 200\pi \} \\ &= 1 \end{aligned}$$

8.35 The autocorrelation function of a wide-sense stationary noise process  $N(t)$  is given by

$$R_{NN}(\tau) = Ae^{-4|\tau|}$$

where  $A$  is a constant. The power spectral density can be determined by noting from Table 8.1 that  $e^{-a|\tau|} \leftrightarrow \frac{2a}{a^2 + w^2}$ . Thus, since  $a = 4$ , we have that

$$S_{NN}(w) = A \left\{ \frac{2(4)}{w^2 + 4^2} \right\} = \frac{8A}{w^2 + 16}$$

8.36 The processes  $X(t)$  and  $Y(t)$  are defined as follows:

$$\begin{aligned} X(t) &= A \cos(w_0 t) + B \sin(w_0 t) \\ Y(t) &= B \cos(w_0 t) - A \sin(w_0 t) \end{aligned}$$

where  $w_0$  is a constant, and  $A$  and  $B$  zero-mean and uncorrelated random variables with variances  $\sigma_A^2 = \sigma_B^2 = \sigma^2$ . The cross power spectral density of  $X(t)$  and  $Y(t)$ ,  $S_{XY}(w)$ , can be obtained as follows:

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = E[\{A \cos(w_0 t) + B \sin(w_0 t)\} \{B \cos(w_0 t + w_0 \tau) - A \sin(w_0 t + w_0 \tau)\}] \\ &= E[B^2] \sin(w_0 t) \cos(w_0 t + w_0 \tau) - E[A^2] \cos(w_0 t) \sin(w_0 t + w_0 \tau) \\ &= \sigma^2 \{ \sin(w_0 t) \cos(w_0 t + w_0 \tau) - \cos(w_0 t) \sin(w_0 t + w_0 \tau) \} = -\sigma^2 \sin(w_0 \tau) \\ S_{XY}(w) &= j\sigma^2 \pi \{ \delta(w - w_0) - \delta(w + w_0) \} \end{aligned}$$

8.37  $X(t)$  and  $Y(t)$  are both zero-mean and wide-sense stationary processes and the random process  $Z(t) = X(t) + Y(t)$ . The power spectral density of  $Z(t)$  can be obtained as follows:

$$\begin{aligned} R_{ZZ}(t, t + \tau) &= E[Z(t)Z(t + \tau)] = E[\{X(t) + Y(t)\} \{X(t + \tau) + Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau) + X(t)Y(t + \tau) + Y(t)X(t + \tau) + Y(t)Y(t + \tau)] \\ &= E[X(t)X(t + \tau)] + E[X(t)Y(t + \tau)] + E[Y(t)X(t + \tau)] + E[Y(t)Y(t + \tau)] \\ &= R_{XX}(\tau) + E[X(t)Y(t + \tau)] + E[Y(t)X(t + \tau)] + R_{YY}(\tau) \end{aligned}$$

a. If  $X(t)$  and  $Y(t)$  are jointly wide-sense stationary, then we obtain

$$\begin{aligned} R_{ZZ}(t, t + \tau) &= R_{XX}(\tau) + E[X(t)Y(t + \tau)] + E[Y(t)X(t + \tau)] + R_{YY}(\tau) \\ &= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau) = R_{ZZ}(\tau) \\ S_{ZZ}(w) &= \int_{-\infty}^{\infty} R_{ZZ}(\tau) e^{-jw\tau} d\tau = S_{XX}(w) + S_{XY}(w) + S_{YX}(w) + S_{YY}(w) \end{aligned}$$

- b. If  $X(t)$  and  $Y(t)$  are orthogonal, then  $E[X(t)Y(t+\tau)] = E[Y(t)X(t+\tau)] = 0$ , and we obtain

$$\begin{aligned} R_{ZZ}(t, t+\tau) &= R_{XX}(\tau) + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] + R_{YY}(\tau) \\ &= R_{XX}(\tau) + R_{YY}(\tau) = R_{ZZ}(\tau) \\ S_{ZZ}(w) &= \int_{-\infty}^{\infty} R_{ZZ}(\tau) e^{-jw\tau} d\tau = S_{XX}(w) + S_{YY}(w) \end{aligned}$$

8.38  $X(t)$  and  $Y(t)$  are jointly stationary random processes that have the crosscorrelation function

$$R_{XY}(\tau) = 2e^{-2\tau} \quad \tau \geq 0$$

- a. The cross power spectral density  $S_{XY}(w)$  is given by

$$S_{XY}(w) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-jw\tau} d\tau = \int_0^{\infty} 2e^{-2\tau} e^{-jw\tau} d\tau = \int_0^{\infty} 2e^{-(2+jw)\tau} d\tau = 2 \left[ -\frac{e^{-(2+jw)\tau}}{2+jw} \right]_0^{\infty} = \frac{2}{2+jw}$$

- b. The cross power spectral density  $S_{YX}(w)$  is given by

$$\begin{aligned} S_{YX}(w) &= \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-jw\tau} d\tau = \int_{-\infty}^{\infty} R_{XY}(-\tau) e^{-jw\tau} d\tau = \int_{-\infty}^{\infty} R_{XY}(u) e^{jwu} du \\ &= S_{XY}(-w) = S_{XY}^*(w) \\ &= \frac{2}{2-jw} \end{aligned}$$

8.39 Two jointly stationary random processes  $X(t)$  and  $Y(t)$  have the cross power spectral density given by

$$S_{XY}(w) = \frac{1}{-w^2 + j4w + 4} = \frac{1}{(2+jw)^2}$$

From Table 8.1, we find that the corresponding crosscorrelation function is

$$R_{XY}(\tau) = \tau e^{-2\tau} \quad \tau \geq 0$$

8.40  $X(t)$  and  $Y(t)$  are zero-mean independent wide-sense stationary random processes with the following power spectral densities:

$$S_{XX}(w) = \frac{4}{w^2 + 4}$$

$$S_{YY}(w) = \frac{w^2}{w^2 + 4}$$

$W(t)$  is defined as follows:  $W(t) = X(t) + Y(t)$ .

a. The power spectral density of  $W(t)$  can be obtained as follows:

$$\begin{aligned} R_{WW}(t, t + \tau) &= E[W(t)W(t + \tau)] = E[\{X(t) + Y(t)\}\{X(t + \tau) + Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau)] + E[X(t)]E[Y(t + \tau)] + E[Y(t)]E[X(t + \tau)] + E[Y(t)Y(t + \tau)] \\ &= R_{XX}(\tau) + R_{YY}(\tau) = R_{WW}(\tau) \end{aligned}$$

$$S_{WW}(w) = S_{XX}(w) + S_{YY}(w) = \frac{4}{w^2 + 4} + \frac{w^2}{w^2 + 4} = \frac{4 + w^2}{w^2 + 4} = 1$$

b. The cross power spectral density  $S_{XW}(w)$  can be obtained as follows:

$$\begin{aligned} R_{XW}(t, t + \tau) &= E[X(t)W(t + \tau)] = E[X(t)\{X(t + \tau) + Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau)] + E[X(t)]E[Y(t + \tau)] = R_{XX}(\tau) \end{aligned}$$

$$S_{XW}(w) = S_{XX}(w) = \frac{4}{w^2 + 4}$$

c. The cross power spectral density  $S_{YW}(w)$  is given by

$$\begin{aligned} R_{YW}(t, t + \tau) &= E[Y(t)W(t + \tau)] = E[Y(t)\{X(t + \tau) + Y(t + \tau)\}] \\ &= E[Y(t)]E[X(t + \tau)] + E[Y(t)Y(t + \tau)] = R_{YY}(\tau) \end{aligned}$$

$$S_{YW}(w) = S_{YY}(w) = \frac{w^2}{w^2 + 4}$$



8.41  $X(t)$  and  $Y(t)$  are zero-mean independent wide-sense stationary random processes with the following power spectral densities:

$$S_{XX}(w) = \frac{4}{w^2 + 4}$$

$$S_{YY}(w) = \frac{w^2}{w^2 + 4}$$

$V(t)$  and  $W(t)$  are defined as follows:

$$V(t) = X(t) + Y(t)$$

$$W(t) = X(t) - Y(t)$$

The cross power spectral density  $S_{VW}(w)$  can be obtained as follows:

$$\begin{aligned} R_{VW}(t, t + \tau) &= E[V(t)W(t + \tau)] = E[\{X(t) + Y(t)\}\{X(t + \tau) - Y(t + \tau)\}] \\ &= E[X(t)X(t + \tau)] - E[X(t)]E[Y(t + \tau)] + E[Y(t)]E[X(t + \tau)] - E[Y(t)Y(t + \tau)] \\ &= R_{XX}(\tau) - R_{YY}(\tau) = R_{VW}(\tau) \\ S_{VW}(w) &= S_{XX}(w) - S_{YY}(w) = \frac{4 - w^2}{w^2 + 4} \end{aligned}$$

8.42  $X(t)$ ,  $-\infty < t < \infty$ , is a zero-mean wide-sense stationary random process with the following power spectral density:

$$S_{XX}(w) = \frac{2}{1 + w^2} \quad -\infty < w < \infty$$

The random process  $Y(t)$  is defined by

$$Y(t) = \sum_{k=0}^2 X(t+k) = X(t) + X(t+1) + X(t+2)$$

a. The mean of  $Y(t)$  is given by

$$E[Y(t)] = E[X(t)] + E[X(t+1)] + E[X(t+2)] = 0$$

- b. To find the variance of  $Y(t)$ , we note that because the mean of  $Y(t)$  is zero, the variance is equal to the second moment. That is,  $\sigma_{Y(t)}^2 = E[Y^2(t)]$ . Thus, we need to find the autocorrelation function that will enable us to find the second moment, as follows:

$$\begin{aligned} R_{YY}(t, t+\tau) &= E[Y(t)Y(t+\tau)] = E[\{X(t) + X(t+1) + X(t+2)\} \{X(t+\tau) + X(t+\tau+1) + X(t+\tau+2)\}] \\ &= 3R_{XX}(\tau) + 2R_{XX}(\tau+1) + R_{XX}(\tau+2) + 2R_{XX}(\tau-1) + R_{XX}(\tau-2) = R_{YY}(\tau) \\ E[Y^2(t)] &= R_{YY}(0) = 3R_{XX}(0) + 2R_{XX}(1) + R_{XX}(2) + 2R_{XX}(-1) + R_{XX}(-2) \\ &= 3R_{XX}(0) + 4R_{XX}(1) + 2R_{XX}(2) \end{aligned}$$

where the last equality follows from the fact that for a wide-sense stationary process, the autocorrelation function is an even function; therefore,  $R_{XX}(-\tau) = R_{XX}(\tau)$ . Now,

since  $\frac{2a}{a^2 + w^2} \leftrightarrow e^{-a|\tau|}$ , we have that

$$\begin{aligned} S_{XX}(w) &= \frac{2}{1 + w^2} \Rightarrow R_{XX}(\tau) = e^{-|\tau|} \\ \sigma_{Y(t)}^2 &= E[Y^2(t)] = 3R_{XX}(0) + 4R_{XX}(1) + 2R_{XX}(2) = 3 + 4e^{-1} + 2e^{-2} = 4.7422 \end{aligned}$$

8.43  $X(t)$  and  $Y(t)$  are wide-sense stationary processes and  $Z(t) = X(t) + Y(t)$ .

- a. The autocorrelation function of  $Z(t)$  is given by

$$\begin{aligned} R_{ZZ}(t, t+\tau) &= E[Z(t)Z(t+\tau)] = E[\{X(t) + Y(t)\} \{X(t+\tau) + Y(t+\tau)\}] \\ &= E[X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau)] \\ &= E[X(t)X(t+\tau)] + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau)] \\ &= R_{XX}(\tau) + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau) + R_{YY}(\tau) \end{aligned}$$

- b. If  $X(t)$  and  $Y(t)$  are jointly wide-sense stationary, then we have that

$$\begin{aligned} R_{ZZ}(t, t + \tau) &= R_{XX}(\tau) + R_{XY}(t, t + \tau) + R_{YX}(t, t + \tau) + R_{YY}(\tau) \\ &= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau) \end{aligned}$$

- c. If  $X(t)$  and  $Y(t)$  are jointly wide-sense stationary, then the power spectral density of  $Z(t)$  is given by

$$S_{ZZ}(w) = \int_{-\infty}^{\infty} R_{ZZ}(\tau) e^{-jw\tau} d\tau = S_{XX}(w) + S_{XY}(w) + S_{YX}(w) + S_{YY}(w)$$

- d. If  $X(t)$  and  $Y(t)$  are uncorrelated, then  $R_{ZZ}(t, t + \tau) = R_{XX}(\tau) + R_{YY}(\tau) + 2\mu_X\mu_Y = R_{ZZ}(\tau)$ , and the power spectral density of  $Z(t)$  is given by

$$S_{ZZ}(w) = S_{XX}(w) + S_{YY}(w) + 4\mu_X\mu_Y\delta(w)$$

- e. If  $X(t)$  and  $Y(t)$  are orthogonal, then the power spectral density of  $Z(t)$  is given by

$$S_{ZZ}(w) = \int_{-\infty}^{\infty} R_{ZZ}(\tau) e^{-jw\tau} d\tau = S_{XX}(w) + S_{YY}(w)$$

## Section 8.8: Discrete-time Random Processes

- 8.44 A random sequence  $X[n]$  has the autocorrelation function  $R_{XX}[m] = a^m$ ,  $m = 0, 1, 2, \dots$ , where  $|a| < 1$ . Its power spectral density is given by

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}[m] e^{-j\Omega m} = \sum_{m=0}^{\infty} a^m e^{-j\Omega m} = \sum_{m=0}^{\infty} [a e^{-j\Omega}]^m = \frac{1}{1 - a e^{-j\Omega}}$$

- 8.45 A wide-sense stationary continuous-time process  $X(t)$  has the autocorrelation function given by

$$R_{X_c X_c}(\tau) = e^{-2|\tau|} \cos(w_0 \tau)$$

From Table 8.1, the power spectral density of  $X(t)$  is given by

$$S_{X_c X_c}(w) = \frac{2}{4 + (w - w_0)^2} + \frac{2}{4 + (w + w_0)^2}$$

$X(t)$  is sampled with a sampling period 10 seconds to produce the discrete-time process  $X[n]$ . Thus, the power spectral density of  $X[n]$  is given by

$$S_{XX}(\Omega) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} S_{X_c X_c}\left(\frac{\Omega - 2\pi m}{T_s}\right) = \frac{1}{10} \sum_{m=-\infty}^{\infty} \left\{ \frac{2}{4 + \left(\frac{\Omega - 2\pi m}{10} - w_0\right)^2} + \frac{2}{4 + \left(\frac{\Omega - 2\pi m}{10} + w_0\right)^2} \right\}$$

8.46 Periodic samples of the autocorrelation function of white noise  $N(t)$  with period  $T$  are defined by

$$R_{NN}(kT) = \begin{cases} \sigma_N^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

The power spectral density of the process is given by

$$S_{NN}(\Omega) = \sum_{k=-\infty}^{\infty} R_{NN}[k] e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} R_{NN}(kT) e^{-j\Omega k} = \sigma_N^2 e^{-j\Omega 0} = \sigma_N^2$$

8.47 The autocorrelation function  $R_{XX}[k]$  of  $X[n]$  is given by

$$R_{XX}[k] = \begin{cases} \sigma_X^2 & k = 0 \\ \frac{4\sigma_X^2}{k^2 \pi^2} & k = \text{odd} \\ 0 & k = \text{even} \end{cases}$$

The power spectral density  $S_{XX}(\Omega)$  of the process is given by

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$$\begin{aligned}
S_{XX}(\Omega) &= \sum_{k=-\infty}^{\infty} R_{XX}[k]e^{-j\Omega k} = \sigma_X^2 + \frac{4\sigma_X^2}{\pi^2} \left\{ e^{-j\Omega} + e^{j\Omega} + \frac{e^{-3j\Omega} + e^{3j\Omega}}{9} + \frac{e^{-5j\Omega} + e^{5j\Omega}}{25} + \dots \right\} \\
&= \sigma_X^2 + \frac{8\sigma_X^2}{\pi^2} \left\{ \frac{e^{-j\Omega} + e^{j\Omega}}{2} + \frac{1}{9} \left[ \frac{e^{-3j\Omega} + e^{3j\Omega}}{2} \right] + \frac{1}{25} \left[ \frac{e^{-5j\Omega} + e^{5j\Omega}}{2} \right] + \dots \right\} \\
&= \sigma_X^2 + \frac{8\sigma_X^2}{\pi^2} \left\{ \cos(\Omega) + \frac{1}{9} \cos(3\Omega) + \frac{1}{25} \cos(5\Omega) + \frac{1}{49} \cos(7\Omega) + \dots \right\}
\end{aligned}$$



## Section 9.2: Linear Systems with Deterministic Input

9.1 Given the “sawtooth” function  $x(t)$  defined in the interval  $[-T, T]$ :

$$x(t) = \begin{cases} 1 + \frac{t}{T} & -T \leq t \leq 0 \\ 1 - \frac{t}{T} & 0 \leq t \leq T \end{cases}$$

The Fourier transform of  $x(t)$  is given by

$$\begin{aligned} X(w) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-T}^0 \left(1 + \frac{t}{T}\right) e^{-j\omega t} dt + \int_0^T \left(1 - \frac{t}{T}\right) e^{-j\omega t} dt \\ &= \left[ -\frac{e^{-j\omega t}}{j\omega} \right]_{-T}^0 + \frac{1}{T} \int_{-T}^0 t e^{-j\omega t} dt + \left[ -\frac{e^{-j\omega t}}{j\omega} \right]_0^T - \frac{1}{T} \int_0^T t e^{-j\omega t} dt \\ &= \frac{1}{j\omega} (e^{j\omega T} - 1) + \frac{1}{j\omega} (1 - e^{-j\omega T}) + \frac{1}{T} \int_{-T}^0 t e^{-j\omega t} dt - \frac{1}{T} \int_0^T t e^{-j\omega t} dt \\ &= \frac{2}{w} \left\{ \frac{e^{j\omega T} - e^{-j\omega T}}{2j} \right\} + \frac{1}{T} \int_{-T}^0 t e^{-j\omega t} dt - \frac{1}{T} \int_0^T t e^{-j\omega t} dt = \frac{2}{w} \sin(\omega T) + \frac{1}{T} \int_{-T}^0 t e^{-j\omega t} dt - \frac{1}{T} \int_0^T t e^{-j\omega t} dt \end{aligned}$$

Let  $u = t \Rightarrow du = dt$ , and let  $dv = e^{-j\omega t} dt \Rightarrow v = -e^{-j\omega t}/j\omega$ . Thus, we have that

$$\begin{aligned} \int_{-T}^0 t e^{-j\omega t} dt &= \left[ -\frac{t e^{-j\omega t}}{j\omega} \right]_{-T}^0 + \frac{1}{j\omega} \int_{-T}^0 e^{-j\omega t} dt = -\frac{T e^{j\omega T}}{j\omega} + \frac{1}{j\omega} \left[ -\frac{e^{-j\omega t}}{j\omega} \right]_{-T}^0 = -\frac{T e^{j\omega T}}{j\omega} + \frac{1}{\omega^2} [1 - e^{j\omega T}] \\ \int_0^T t e^{-j\omega t} dt &= \left[ -\frac{t e^{-j\omega t}}{j\omega} \right]_0^T + \frac{1}{j\omega} \int_0^T e^{-j\omega t} dt = -\frac{T e^{-j\omega T}}{j\omega} + \frac{1}{j\omega} \left[ -\frac{e^{-j\omega t}}{j\omega} \right]_0^T = -\frac{T e^{-j\omega T}}{j\omega} + \frac{1}{\omega^2} [e^{-j\omega T} - 1] \end{aligned}$$

Thus,

$$\begin{aligned}
X(w) &= \frac{2}{w} \sin(wT) + \frac{1}{T} \int_{-T}^0 t e^{-jw t} dt - \frac{1}{T} \int_0^T t e^{-jw t} dt \\
&= \frac{2}{w} \sin(wT) + \frac{1}{T} \left\{ \frac{1}{w^2} (1 - e^{jwT}) - \frac{T e^{jwT}}{jw} + \frac{T e^{-jwT}}{jw} - \frac{1}{w^2} (e^{-jwT} - 1) \right\} \\
&= \frac{2}{w} \sin(wT) + \frac{2}{w^2 T} - \frac{2}{w^2 T} \left( \frac{e^{jwT} + e^{-jwT}}{2} \right) - \frac{2}{w} \left( \frac{e^{jwT} - e^{-jwT}}{2j} \right) \\
&= \frac{2}{w} \sin(wT) + \frac{2}{w^2 T} - \frac{2}{w^2 T} \cos(wT) - \frac{2}{w} \sin(wT) \\
&= \frac{2}{w^2 T} \{1 - \cos(wT)\}
\end{aligned}$$

9.2 Given that

$$y(t) = \frac{d}{dt} x(t)$$

The Fourier transform of  $y(t)$  can be obtained as follows:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jw t} dw \\
y(t) &= \frac{d}{dt} x(t) = \frac{1}{2\pi} \frac{d}{dt} \int_{-\infty}^{\infty} X(w) e^{jw t} dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) \frac{d}{dt} e^{jw t} dw \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} jw X(w) e^{jw t} dw \\
Y(w) &= jw X(w)
\end{aligned}$$

That is,  $y(t) = \frac{d}{dt} x(t) \leftrightarrow jw X(w)$ .

9.3 Given that



$$y(t) = e^{jw_0 t} x(t)$$

where  $w_0 > 0$  is a constant, the Fourier transform of  $y(t)$  is given by

$$\begin{aligned} Y(w) &= \int_{-\infty}^{\infty} y(t) e^{-jw t} dt = \int_{-\infty}^{\infty} e^{jw_0 t} x(t) e^{-jw t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(w - w_0) t} dt \\ &= X(w - w_0) \end{aligned}$$

9.4 Given that

$$y(t) = x(t - t_0)$$

where  $t_0 > 0$  is a constant, the Fourier transform of  $y(t)$  is given by

$$Y(w) = \int_{-\infty}^{\infty} y(t) e^{-jw t} dt = \int_{-\infty}^{\infty} x(t - t_0) e^{-jw t} dt$$

Let  $u = t - t_0 \Rightarrow du = dt$ . Thus, we obtain

$$\begin{aligned} Y(w) &= \int_{-\infty}^{\infty} x(t - t_0) e^{-jw t} dt = \int_{-\infty}^{\infty} x(u) e^{-jw(u + t_0)} du = e^{-jw t_0} \int_{-\infty}^{\infty} x(u) e^{-jw u} du \\ &= e^{-jw t_0} X(w) \end{aligned}$$

9.5 Given that

$$y(t) = x(at)$$

where  $a > 0$  is a constant, the Fourier transform of  $y(t)$  is given by

$$Y(w) = \int_{-\infty}^{\infty} y(t)e^{-jw t} dt = \int_{-\infty}^{\infty} x(at)e^{-jw t} dt$$

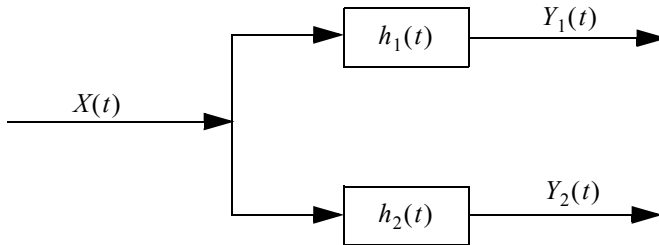
Let  $u = at \Rightarrow dt = \frac{du}{a}$ . Thus,

$$Y(w) = \int_{-\infty}^{\infty} x(at)e^{-jw t} dt = \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(u)e^{-j\left(\frac{w}{a}\right)u} du & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(u)e^{-j\left(\frac{w}{a}\right)u} du & a < 0 \end{cases}$$

$$= \frac{1}{|a|} X\left(\frac{w}{a}\right)$$

### Section 9.3: Linear Systems with Continuous Random Input

9.6 A stationary zero-mean random signal  $X(t)$  is the input to two filters, as shown below.



The power spectral density of  $X(t)$  is  $S_{XX}(w) = N_0/2$ , and the filter impulse responses are given by

$$h_1(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_2(t) = \begin{cases} 2e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The system responses of the filters are given by

$$H_1(w) = \int_{-\infty}^{\infty} h_1(t)e^{-jw t} dt = \int_0^1 e^{-jw t} dt = \left[ -\frac{e^{-jw t}}{jw} \right]_0^1 = \frac{1}{jw} \{1 - e^{-jw}\}$$

$$H_2(w) = \int_{-\infty}^{\infty} h_2(t)e^{-jw t} dt = 2 \int_0^{\infty} e^{-t} e^{-jw t} dt = 2 \int_0^{\infty} e^{-(1+jw)t} dt = \frac{2}{1+jw} [-e^{-(1+jw)t}]_0^{\infty} = \frac{2}{1+jw}$$

Thus,  $Y_i(w) = H_i(w)S_{XX}(w) = \frac{1}{2}N_0 H_i(w)$ ,  $i = 1, 2$ .

1. The mean  $E[Y_i(t)]$  of the output signal  $Y_i(t)$ , for  $i = 1, 2$  is given by

$$E[Y_i(t)] = \mu_X(t) * h_i(t)$$

Thus, if  $X(t)$  is a zero-mean process, then  $Y_i(t)$  is also a zero-mean output process.

The second moment  $E[Y_i^2(t)]$  can be obtained as follows:

$$E[Y_i^2(t)] = R_{Y_i Y_i}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{Y_i Y_i}(w) dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_i(w)|^2 S_{XX}(w) dw = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} |H_i(w)|^2 dw$$

Thus, for  $E[Y_1^2(t)]$  we obtain

$$\begin{aligned} E[Y_1^2(t)] &= \frac{N_0}{4\pi} \int_{-\infty}^{\infty} |H_1(w)|^2 dw = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} \frac{2}{w^2} \{1 - \cos(w)\} dw = \frac{N_0}{\pi} \int_{-\infty}^{\infty} \left[ \frac{\sin(w/2)}{w/2} \right]^2 dw \\ &= \frac{2N_0}{\pi} \int_0^{\infty} \left[ \frac{\sin(w/2)}{w/2} \right]^2 dw \end{aligned}$$

where the last equality follows from the fact that the integrand is an even function. Let  $u = w/2 \Rightarrow dw = 2du$ . Thus, we obtain

$$E[Y_1^2(t)] = \frac{2N_0}{\pi} \int_0^\infty \left[ \frac{\sin(w/2)}{w/2} \right]^2 dw = \frac{4N_0}{\pi} \int_0^\infty \left[ \frac{\sin(u)}{u} \right]^2 du = \frac{4N_0}{\pi} \left( \frac{\pi}{2} \right) = 2N_0$$

Similarly, for  $E[Y_2^2(t)]$  we obtain

$$E[Y_2^2(t)] = \frac{N_0}{4\pi} \int_{-\infty}^\infty |H_2(w)|^2 dw = \frac{N_0}{4\pi} \int_{-\infty}^\infty \frac{4}{1+w^2} dw = \frac{N_0}{\pi} \int_{-\infty}^\infty \frac{1}{1+w^2} dw$$

Let  $w = \tan(\theta) \Rightarrow dw = [\sec(\theta)]^2 d\theta$ . Now, when  $w = -\infty$ ,  $\theta = -\pi/2$ . Similarly when  $w = \infty$ ,  $\theta = \pi/2$ . Thus, we obtain

$$\begin{aligned} E[Y_2^2(t)] &= \frac{N_0}{\pi} \int_{-\infty}^\infty \frac{1}{1+w^2} dw = \frac{N_0}{\pi} \int_{-\pi/2}^{\pi/2} \frac{[\sec(\theta)]^2 d\theta}{1 + [\tan(\theta)]^2} = \frac{N_0}{\pi} \int_{-\pi/2}^{\pi/2} \frac{[\sec(\theta)]^2 d\theta}{[\sec(\theta)]^2} = \frac{N_0}{\pi} \int_{-\pi/2}^{\pi/2} d\theta \\ &= \frac{N_0}{\pi} [\theta]_{-\pi/2}^{\pi/2} = \frac{N_0}{\pi} (\pi) = N_0 \end{aligned}$$

2. Since  $X(t)$  is a noise function,  $R_{XX}(\tau) = (N_0/2)\delta(\tau)$  and the crosscorrelation function  $R_{Y_1 Y_2}(t, t + \tau)$  is given by

$$\begin{aligned}
R_{Y_1 Y_2}(t, t + \tau) &= E[Y_1(t)Y_2(t + \tau)] = E\left[\int_{-\infty}^{\infty} h_1(u)X(t - u)du \int_{-\infty}^{\infty} h_2(v)X(t + \tau - v)dv\right] \\
&= \int_0^1 \int_0^{\infty} h_1(u)h_2(v)E[X(t - u)X(t + \tau - v)]dvdu \\
&= \int_0^1 \int_0^{\infty} 2e^{-v}R_{XX}(\tau + u - v)dvdu = \int_0^1 \int_0^{\infty} 2e^{-v}(N_0/2)\delta(\tau + u - v)dvdu \\
&= N_0 \int_0^1 e^{-(u + \tau)}du = N_0 e^{-\tau}[-e^{-u}]_0^1 = N_0 e^{-\tau}[1 - e^{-1}] \\
&= 0.632 N_0 e^{-\tau}
\end{aligned}$$

9.7  $X(t)$  is a wide-sense stationary process with autocorrelation function  $R_{XX}(\tau) = e^{-4|\tau|}$  and is the input to a linear system whose impulse response is  $h(t) = 2e^{-7t}$ ,  $t \geq 0$ . The output process is  $Y(t)$ . The power spectral density of  $X(t)$  and the system response of the system are given by

$$\begin{aligned}
S_{XX}(w) &= \frac{8}{w^2 + 16} \\
H(w) &= \frac{2}{7 + jw}
\end{aligned}$$

1. The power spectral density of  $Y(t)$  is given by

$$S_{YY}(w) = |H(w)|^2 S_{XX}(w) = \left(\frac{2}{7 + jw}\right)\left(\frac{2}{7 - jw}\right)\left(\frac{8}{w^2 + 16}\right) = \frac{32}{(w^2 + 49)(w^2 + 16)}$$

2. The cross-spectral power density  $S_{XY}(w)$  is given by

$$S_{XY}(w) = H(w)S_{XX}(w) = \left(\frac{2}{7 + jw}\right)\left(\frac{8}{w^2 + 16}\right) = \frac{16}{(7 + jw)(w^2 + 16)}$$

3. The crosscorrelation function  $R_{XY}(\tau)$  can be obtained as follows:

$$S_{XY}(w) = \frac{16}{(7+jw)(w^2+16)} = \frac{16}{(7+jw)(4+jw)(4-jw)} \equiv \frac{A}{7+jw} + \frac{B}{4+jw} + \frac{C}{4-jw}$$

$$A = (7+jw)S_{XY}(w)|_{jw=-7} = \frac{16}{(-3)(11)} = -\frac{16}{33}$$

$$B = (4+jw)S_{XY}(w)|_{jw=-4} = \frac{16}{(3)(8)} = \frac{2}{3}$$

$$C = (4-jw)S_{XY}(w)|_{jw=4} = \frac{16}{(11)(8)} = \frac{2}{11}$$

Thus, we have that

$$S_{XY}(w) = \frac{2/3}{4+jw} - \frac{16/33}{7+jw} + \frac{2/11}{4-jw}$$

$$R_{XY}(\tau) = \begin{cases} \frac{2}{3}e^{-4\tau} - \frac{16}{33}e^{-7\tau} & \tau \geq 0 \\ \frac{2}{11}e^{4\tau} & \tau < 0 \end{cases}$$

9.8 A linear system has a transfer function given by

$$H(w) = \frac{w}{w^2 + 15w + 50}$$

- a. The power spectral density of the output process when the input function is a stationary random process  $X(t)$  with an autocorrelation function  $R_{XX}(\tau) = 10e^{-|\tau|}$  can be obtained as follows:

$$S_{XX}(w) = \frac{20}{1+w^2}$$

$$S_{YY}(w) = |H(w)|^2 S_{XX}(w) = \frac{20w^2}{(1+w^2)(w^2+15w+50)^2}$$

- b. The power spectral density of the output process when the input function is a white noise that has a mean-square value of  $1.2 \text{ V}^2/\text{Hz}$  can be obtained as follows:

$$\begin{aligned} R_{XX}(\tau) &= 1.2\delta(\tau) \\ S_{XX}(w) &= 1.2 \\ S_{YY}(w) &= |H(w)|^2 S_{XX}(w) = \frac{1.2w^2}{(w^2 + 15w + 50)^2} \end{aligned}$$

- 9.9 A linear system has the impulse response  $h(t) = e^{-at}$ , where  $t \geq 0$  and  $a > 0$ . The power transfer function of the system,  $|H(w)|^2$ , can be obtained as follows:

$$\begin{aligned} H(w) &= \frac{1}{a + jw} \\ |H(w)|^2 &= \left( \frac{1}{a + jw} \right) \left( \frac{1}{a - jw} \right) = \frac{1}{a^2 + w^2} \end{aligned}$$

- 9.10 The white noise with power spectral density  $N_0/2$  is the input to a system with the impulse response  $h(t) = e^{-at}$ , where  $t \geq 0$  and  $a > 0$ . The power spectral density of the output process can be obtained as follows:

$$\begin{aligned} S_{XX}(w) &= \frac{N_0}{2} \\ H(w) &= \frac{1}{a + jw} \\ S_{YY}(w) &= |H(w)|^2 S_{XX}(w) = \frac{N_0}{2(a^2 + w^2)} \end{aligned}$$

- 9.11 The power transfer function of a system is given by

$$|H(w)|^2 = \frac{64}{[16 + w^2]^2} = \left( \frac{8}{16 + w^2} \right)^2 = H(w)H^*(w) \Rightarrow H(w) = \frac{8}{16 + w^2}$$

From Table 8.1 we have that

$$H(w) = \frac{8}{16 + w^2} = \frac{2(4)}{4^2 + w^2} \leftrightarrow e^{-4|t|}$$

Thus, the impulse function  $h(t)$  of the system is given by

$$h(t) = e^{-4|t|}$$

9.12 A wide-sense stationary process  $X(t)$  has the autocorrelation function given by

$$R_{XX}(\tau) = \cos(w_0\tau) \Rightarrow S_{XX}(w) = \pi\{\delta(w - w_0) + \delta(w + w_0)\}$$

The process is input to a system with the power transfer function

$$|H(w)|^2 = \frac{64}{[16 + w^2]^2}$$

a. The power spectral density of the output process is given by

$$S_{YY}(w) = |H(w)|^2 S_{XX}(w) = \frac{64\pi}{[16 + w_0^2]^2} + \frac{64\pi}{[16 + w_0^2]^2} = \frac{128\pi}{[16 + w_0^2]^2}$$

b. Given that  $Y(t)$  is the output process, the cross-power spectral density  $S_{XY}(w)$  is obtained by noting that the system response  $H(w)$  is given by  $H(w) = 8/(16 + w^2)$ . Thus, we have that

$$S_{YY}(w) = H(w)S_{XX}(w) = \frac{8\pi}{16 + w_0^2} + \frac{8\pi}{16 + w_0^2} = \frac{16\pi}{16 + w_0^2}$$

9.13 A causal system is used to generate an output process  $Y(t)$  with the power spectral density

$$S_{YY}(w) = \frac{2a}{a^2 + w^2}$$

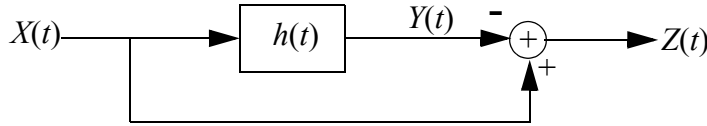


Since

$$S_{YY}(w) = \frac{2a}{a^2 + w^2} = |H(w)|^2 S_{XX}(w) = |1|^2 \frac{2a}{a^2 + w^2} \Rightarrow H(w) = 1$$

we conclude that the impulse response  $h(t)$  of the system is  $h(t) = \delta(t)$ .

9.14  $X(t)$  is a wide-sense stationary process that is the input to a linear system with impulse response  $h(t)$ , and  $Y(t)$  is the output process. Another process  $Z(t)$  that is obtained as follows:  $Z(t) = X(t) - Y(t)$ , as shown below.



a. The autocorrelation function  $R_{ZZ}(\tau)$  is given by

$$\begin{aligned} R_{ZZ}(\tau) &= E[Z(t)Z(t+\tau)] = E[\{X(t) - Y(t)\}\{X(t+\tau) - Y(t+\tau)\}] \\ &= E[X(t)(X(t+\tau))] - E[X(t)Y(t+\tau)] - E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau)] \\ &= R_{XX}(\tau) - R_{XY}(\tau) - R_{YX}(\tau) + R_{YY}(\tau) \\ &= R_{XX}(\tau) - R_{XX}(\tau)*h(\tau) - R_{XX}(\tau)*h(-\tau) + R_{XX}(\tau)*h(-\tau)*h(\tau) \end{aligned}$$

b. The power spectral density  $S_{ZZ}(w)$  is given by

$$\begin{aligned} S_{ZZ}(w) &= S_{XX}(w) - H(w)S_{XX}(w) - H^*(w)S_{XX}(w) + |H(w)|^2 S_{XX}(w) \\ &= \{1 - H(w) - H^*(w) + |H(w)|^2\} S_{XX}(w) \end{aligned}$$

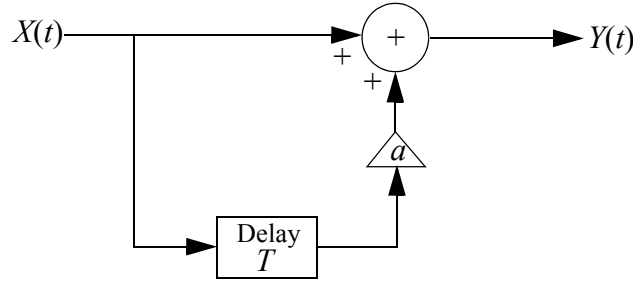
c. The crosscorrelation function  $R_{XZ}(\tau)$  is given by

$$\begin{aligned} R_{XZ}(\tau) &= E[X(t)Z(t+\tau)] = E[X(t)\{X(t+\tau) - Y(t+\tau)\}] = E[X(t)(X(t+\tau))] - E[X(t)Y(t+\tau)] \\ &= R_{XX}(\tau) - R_{XY}(\tau) = R_{XX}(\tau) - R_{XX}(\tau)*h(\tau) \end{aligned}$$

d. The crosspower spectral density  $S_{XZ}(w)$  is given by

$$S_{XZ}(w) = S_{XX}(w) - H(w)S_{XX}(w) = \{1 - H(w)\}S_{XX}(w)$$

9.15 In the system shown below an output process  $Y(t)$  is the sum of an input process  $X(t)$  and a delayed version of  $X(t)$  that is scaled (or multiplied) by a factor  $a$ .



- a. The equation that governs the system is given by  $Y(t) = X(t) + aX(t - T)$ .
- b. The crosscorrelation function  $R_{XY}(\tau)$  is given by

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t + \tau)] = E[X(t)\{X(t + \tau) + aX(t + \tau - T)\}] \\ &= E[X(t)(X(t + \tau))] + E[aX(t)X(t + \tau - T)] = E[X(t)(X(t + \tau))] + aE[X(t)X(t + \tau - T)] \\ &= R_{XX}(\tau) + aR_{XX}(\tau - T) \end{aligned}$$

- c. The crosspower spectral density  $S_{XY}(w)$  is given by

$$S_{XY}(w) = S_{XX}(w) + aS_{XX}(w)e^{-jwT} = \{1 + ae^{-jwT}\}S_{XX}(w)$$

- d. From the result above, the transfer function  $H(w)$  of the system is given by

$$H(w) = \frac{S_{XY}(w)}{S_{XX}(w)} = 1 + ae^{-jwT}$$

- e. The power spectral density of  $Y(t)$  is given by

$$\begin{aligned}
S_{YY}(w) &= |H(w)|^2 S_{XX}(w) = H(w)H^*(w)S_{XX}(w) = \{1 + ae^{-jwT}\} \{1 + ae^{jwT}\} S_{XX}(w) \\
&= \{1 + ae^{jwT} + ae^{-jwT} + a^2\} S_{XX}(w) = \left\{1 + 2a \left[ \frac{e^{jwT} + e^{-jwT}}{2} \right] + a^2\right\} S_{XX}(w) \\
&= \{1 + 2\cos(wT) + a^2\} S_{XX}(w)
\end{aligned}$$

9.16  $X(t)$  and  $Y(t)$  are two jointly wide-sense stationary processes, and  $Z(t) = X(t) + Y(t)$  is the input to a linear system with impulse response  $h(t)$ .

a. The autocorrelation function of  $Z(t)$  is given by

$$\begin{aligned}
R_{ZZ}(\tau) &= E[Z(t)Z(t+\tau)] = E[\{X(t) + Y(t)\} \{X(t+\tau) + Y(t+\tau)\}] \\
&= E[X(t)(X(t+\tau))] + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau)] \\
&= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)
\end{aligned}$$

b. The power spectral density of  $Z(t)$  is given by

$$\begin{aligned}
S_{ZZ}(w) &= S_{XX}(w) + S_{XY}(w) + S_{YX}(w) + S_{YY}(w) \\
&= S_{XX}(w) + S_{XY}(w) + S_{XY}^*(w) + S_{YY}(w)
\end{aligned}$$

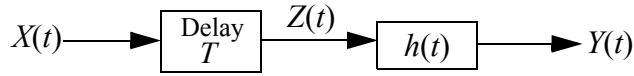
c. The crosspower spectral density  $S_{ZV}(w)$  of the input process  $Z(t)$  and the output process  $V(t)$  is given by

$$S_{ZV}(w) = H(w)S_{ZZ}(w) = H(w)\{S_{XX}(w) + S_{XY}(w) + S_{YX}(w) + S_{YY}(w)\}$$

d. The power spectral density of the output process  $V(t)$  is given by

$$S_{VV}(w) = |H(w)|^2 S_{ZZ}(w) = |H(w)|^2 \{S_{XX}(w) + S_{XY}(w) + S_{YX}(w) + S_{YY}(w)\}$$

9.17  $X(t)$  is a wide-sense stationary process and  $Z(t) = X(t-d)$ , where  $d$  is a constant delay.  $Z(t)$  is the input to a linear system with impulse response  $h(t)$ , as shown below.



- a. The autocorrelation function of  $Z(t)$  is given by

$$R_{ZZ}(\tau) = E[Z(t)Z(t+\tau)] = E[X(t-d)(X(t+\tau-d))] = R_{XX}(\tau)$$

- b. The power spectral density  $S_{ZZ}(w)$  is  $S_{ZZ}(w) = S_{XX}(w)$ .  
 c. The crosscorrelation function  $R_{ZX}(\tau)$  is given by

$$R_{ZX}(\tau) = E[Z(t)X(t+\tau)] = E[X(t-d)X(t+\tau)] = R_{XX}(\tau+d)$$

- d. The crosspower spectral density  $S_{ZX}(w)$  is  $S_{ZX}(w) = S_{XX}(w)e^{jwd}$   
 e. The power spectral density  $S_{YY}(w)$  of the output process  $Y(t)$  is given by

$$S_{YY}(w) = |H(w)|^2 S_{ZZ}(w) = |H(w)|^2 S_{XX}(w)$$

9.18  $X(t)$  is a zero-mean wide-sense stationary white noise process with average power  $N_0/2$  that is the input to a linear system with the transfer function

$$H(w) = \frac{1}{a+jw}$$

where  $a > 0$ . Thus, we have that

$$R_{XX}(\tau) = \frac{N_0}{2} \delta(\tau) \Rightarrow S_{XX}(w) = \frac{N_0}{2}$$

- a. Since from Table 8.1  $e^{-at} \leftrightarrow \frac{1}{a+jw}$  for  $a > 0, t \geq 0$ , the impulse response of the system is given by

$$h(t) = e^{-at} \quad a > 0, t \geq 0$$

- b. The crosspower spectral density  $S_{XY}(w)$  of the input process and the output process  $Y(t)$  is given by

$$S_{XY}(w) = H(w)S_{XX}(w) = \frac{N_0}{2(a+jw)}$$

- c. The crosscorrelation function  $R_{XY}(\tau)$  is the inverse Fourier transform of  $S_{XY}(w)$ , which is

$$R_{XY}(\tau) = \frac{N_0}{2}e^{-a\tau} \quad \tau \geq 0$$

- d. The crosscorrelation function  $R_{YX}(\tau)$  is given by  $R_{YX}(\tau) = R_{XY}(-\tau) = \frac{N_0}{2}e^{a\tau}$ ,  $\tau < 0$ .
- e. The crosspower spectral density  $S_{YX}(w)$  is given by

$$S_{YX}(w) = H^*(w)S_{XX}(w) = \frac{N_0}{2(a-jw)}$$

- f. The power spectral density  $S_{YY}(w)$  of the output process is given by

$$S_{YY}(w) = |H(w)|^2 S_{XX}(w) = \frac{N_0}{2} \left( \frac{1}{a^2 + w^2} \right)$$

## Section 9.4: Linear Systems with Discrete Random Input

9.19 A linear system has an impulse response given by

$$h[n] = \begin{cases} e^{-an} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

where  $a > 0$  is a constant. The transfer function of the system is given by

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} e^{-an}e^{-j\Omega n} = \sum_{n=0}^{\infty} e^{-(a+j\Omega)n} = \frac{1}{1 - e^{-(a+j\Omega)}}$$

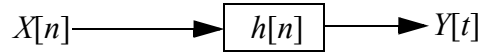
9.20 A linear system has an impulse response given by

$$h[n] = \begin{cases} e^{-an} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

where  $a > 0$  is a constant. Assume that the autocorrelation function of the input sequence to this system is defined by

$$R_{XX}[n] = b^n \quad 0 < b < 1, n \geq 0$$

Thus, the system can be represented as shown below.



The power spectral density of the output process can be obtained as follows:

$$H(\Omega) = \frac{1}{1 - e^{-(a+j\Omega)}}$$

$$S_{XX}(\Omega) = \sum_{n=0}^{\infty} b^n e^{-j\Omega n} = \sum_{n=0}^{\infty} [be^{-j\Omega}]^n = \frac{1}{1 - be^{-j\Omega}}$$

$$\begin{aligned} S_{YY}(\Omega) &= |H(\Omega)|^2 S_{XX}(\Omega) = H(\Omega)H^*(\Omega)S_{XX}(\Omega) = \left\{ \frac{1}{1 - e^{-(a+j\Omega)}} \right\} \left\{ \frac{1}{1 - e^{-(a-j\Omega)}} \right\} \left\{ \frac{1}{1 - be^{-j\Omega}} \right\} \\ &= \frac{1}{1 - 2e^{-a} \cos(\Omega) + e^{-2a}} \left\{ \frac{1}{1 - be^{-j\Omega}} \right\} \end{aligned}$$

9.21 The autocorrelation function of a discrete-time random sequence  $X[n]$  is given by

$$R_{XX}[m] = e^{-b|m|}$$

where  $b > 0$  is a constant. The power spectral density of the sequence can be obtained as follows:

$$R_{XX}[m] = \begin{cases} e^{-bm} & m \geq 0 \\ e^{bm} & m < 0 \end{cases}$$

$$\begin{aligned} S_{XX}(\Omega) &= \sum_{m=-\infty}^{\infty} R_{XX}[m] e^{-j\Omega m} = \sum_{m=-\infty}^{-1} e^{bm} e^{-j\Omega m} + \sum_{m=0}^{\infty} e^{-bm} e^{-j\Omega m} = \sum_{m=1}^{\infty} e^{-bm} e^{j\Omega m} + \sum_{n=0}^{\infty} e^{-bm} e^{-j\Omega m} \\ &= 1 + \sum_{m=1}^{\infty} e^{-bm} \{e^{j\Omega m} + e^{-j\Omega m}\} = 1 + 2 \sum_{m=1}^{\infty} e^{-bm} \left\{ \frac{e^{j\Omega m} + e^{-j\Omega m}}{2} \right\} \\ &= 1 + 2 \sum_{m=1}^{\infty} e^{-bm} \cos(m\Omega) \end{aligned}$$

9.22 A linear system has an impulse response given by

$$h[n] = \begin{cases} e^{-an} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

where  $a > 0$  is a constant. The autocorrelation function of the input discrete-time random sequence  $X[n]$  is given by

$$R_{XX}[m] = e^{-b|m|}$$

From earlier results, the power spectral density of the output process can be obtained as follows:

$$\begin{aligned}
 H(\Omega) &= \frac{1}{1 - e^{-(a+j\Omega)}} \\
 S_{XX}(\Omega) &= 1 + 2 \sum_{m=1}^{\infty} e^{-bm} \cos(m\Omega) \\
 S_{YY}(\Omega) &= |H(\Omega)|^2 S_{XX}(\Omega) = H(\Omega) H^*(\Omega) S_{XX}(\Omega) \\
 &= \left\{ \frac{1}{1 - e^{-(a+j\Omega)}} \right\} \left\{ \frac{1}{1 - e^{-(a-j\Omega)}} \right\} \left\{ 1 + 2 \sum_{m=1}^{\infty} e^{-bm} \cos(m\Omega) \right\} \\
 &= \frac{1}{1 - 2e^{-a} \cos(\Omega) + e^{-2a}} \left\{ 1 + 2 \sum_{m=1}^{\infty} e^{-bm} \cos(m\Omega) \right\}
 \end{aligned}$$

9.23 A wide-sense stationary continuous-time process  $X_c(t)$  has the autocorrelation function given by

$$R_{X_c X_c}(\tau) = e^{-4|\tau|}$$

Thus, the power spectral density of  $X_c(t)$  is given by

$$S_{X_c X_c}(w) = \frac{8}{16 + w^2}$$

If  $X_c(t)$  is sampled with a sampling period 10 seconds to produce the discrete-time process  $X[n]$ , the power spectral density of  $X[n]$  is given by

$$S_{XX}(\Omega) = \frac{1}{T} \sum_{k=1}^{\infty} S_{X_c X_c} \left( \frac{\Omega - 2\pi k}{T} \right) = \frac{1}{10} \sum_{k=1}^{\infty} \frac{8}{16 + \left( \frac{\Omega - 2\pi k}{10} \right)^2}$$



9.24 A wide-sense stationary continuous-time process  $X_c(t)$  has the autocorrelation function given by

$$R_{X_c X_c}(\tau) = e^{-4|\tau|}$$

$X_c(t)$  is sampled with a sampling period 10 seconds to produce the discrete-time sequence  $X[n]$ . The sequence is then input to a system with the impulse response

$$h[n] = \begin{cases} e^{-an} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

From earlier results, we have that

$$S_{X_c X_c}(w) = \frac{8}{16 + w^2}$$

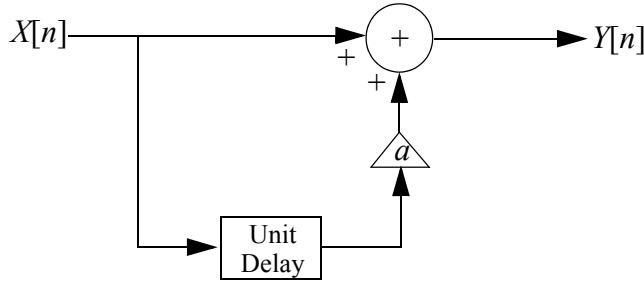
$$S_{XX}(\Omega) = \frac{1}{10} \sum_{k=-\infty}^{\infty} \frac{8}{16 + \left(\frac{\Omega - 2\pi k}{10}\right)^2}$$

$$H(\Omega) = \frac{1}{1 - e^{-(a + j\Omega)}}$$

$$S_{YY}(\Omega) = |H(\Omega)|^2 S_{XX}(\Omega) = H(\Omega)H^*(\Omega)S_{XX}(\Omega) = \left\{ \frac{1}{1 - 2e^{-a}\cos(\Omega) + e^{-2a}} \right\} S_{XX}(\Omega)$$

$$= \frac{1}{10} \left\{ \frac{1}{1 - 2e^{-a}\cos(\Omega) + e^{-2a}} \right\} \sum_{k=-\infty}^{\infty} \frac{8}{16 + \left(\frac{\Omega - 2\pi k}{10}\right)^2}$$

9.25 In the system shown below an output sequence  $Y[n]$  is the sum of an input sequence  $X[n]$  and a version of  $X[n]$  that has been delayed by one unit and scaled (or multiplied) by a factor  $a$ .



- a. The equation that governs the system is  $Y[n] = X[n] + aX[n-1]$
- b. The crosscorrelation function  $R_{XY}[m]$  is

$$\begin{aligned} R_{XY}[m] &= E[X[n]Y[n+m]] = E[X[n]\{X[n+m] + aX[n+m-1]\}] \\ &= E[X[n]X[n+m]] + aE[X[n]X[n+m-1]] \\ &= R_{XX}[m] + aR_{XX}[m-1] \end{aligned}$$

- c. The crosspower spectral density  $S_{XY}(\Omega)$  is given by

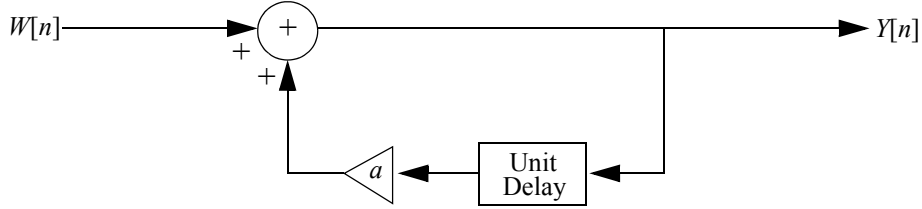
$$S_{XY}(\Omega) = S_{XX}(\Omega) + aS_{XX}(\Omega)e^{-j\Omega} = [1 + ae^{-j\Omega}]S_{XX}(\Omega)$$

- d. The transfer function  $H(\Omega)$  of the system is given by

$$H(\Omega) = \frac{S_{XY}(\Omega)}{S_{XX}(\Omega)} = 1 + ae^{-j\Omega}$$

## Section 9.5: Autoregressive Moving Average Processes

9.26 In the figure below, we are given that  $|a| < 1$ , and the random process  $W[n]$  is a sequence of independent and identically distributed random variables with zero mean and standard deviation  $\beta$ . It is assumed also that the random process  $Y[n]$  has zero mean.



- a. This is an example of a first-order autoregressive process, and the equation that governs the system is  $Y[n] = W[n] + aY[n-1]$
- b. The general structure of the output process  $Y[n]$  can be obtained as follows:

$$Y[0] = W[0]$$

$$Y[1] = aY[0] + W[1] = aW[0] + W[1]$$

$$Y[2] = aY[1] + W[2] = a\{aW[0] + W[1]\} + W[2] = a^2W[0] + aW[0] + W[2]$$

$$Y[n] = \sum_{k=0}^n a^k W[n-k]$$

Thus, the autocorrelation function of  $Y[n]$  is given by

$$\begin{aligned} R_{YY}[n, n+m] &= E[Y[n]Y[n+m]] = E\left[\sum_{k=0}^n a^k W[n-k] \sum_{j=0}^n a^j W[n+m-j]\right] \\ &= \sum_{k=0}^n \sum_{j=0}^n a^k a^j E[W[n-k]W[n+m-j]] \end{aligned}$$

Since the  $W[n]$  are independent and identically distributed with  $E[W[n]] = 0$  and  $E[W^2[n]] = \beta^2$ , we have that  $E[W[n-k]W[n+m-j]] = 0$  except when  $n-k = n+m-j$ ; that is, when  $j = m+k$ . Thus, the autocorrelation function becomes

$$R_{YY}[n, n+m] = \sum_{k=0}^n \beta^2 a^k a^{m+k} = \beta^2 a^m \sum_{k=0}^n a^{2k} = \frac{\beta^2 a^{|m|} \{1 - a^{2(n+1)}\}}{1 - a^2}$$

Since  $R_{YY}[n, n+m]$  is not independent of  $n$ ,  $Y[n]$  is not a wide-sense stationary process.

- c. Since we established that  $Y[n]$  is not a wide-sense stationary process, we are not required to obtain the power transfer function.
- d. The crosscorrelation function  $R_{WY}[n, n+m]$  is given by

$$\begin{aligned} R_{WY}[n, n+m] &= E[W[n]Y[n+m]] = E[W[n]\{W[n+m] + aY[n+m-1]\}] \\ &= E[W[n]W[n+m]] + aE[W[n]Y[n+m-1]] \\ &= R_{WW}[n, n+m] + aR_{WY}[n, n+m-1] \end{aligned}$$

- e. The autocorrelation function  $R_{WW}[n, n+m]$  of the input process is given by

$$R_{WW}[n, n+m] = \beta^2 \delta[m] = R_{WW}[m].$$

9.27 For an MA(2) process, if we assume that  $W[n]$  is a zero-mean process with variance  $\sigma_W^2$ , we have that

$$\begin{aligned} Y[n] &= \beta_0 W[n] + \beta_1 W[n-1] + \beta_2 W[n-2] \\ E[Y[n]] &= E[\beta_0 W[n] + \beta_1 W[n-1] + \beta_2 W[n-2]] = \beta_0 E[W[n]] + \beta_1 E[W[n-1]] + \beta_2 E[W[n-2]] \\ &= 0 \\ \sigma_Y[n] &= E[Y^2[n]] = E[(\beta_0 W[n] + \beta_1 W[n-1] + \beta_2 W[n-2])(\beta_0 W[n] + \beta_1 W[n-1] + \beta_2 W[n-2])] \\ &= \sigma_W^2 \{\beta_0^2 + \beta_1^2 + \beta_2^2\} \\ R_{YY}[n, n+m] &= E[Y[n]Y[n+m]] \\ &= E[\{\beta_0 W[n] + \beta_1 W[n-1] + \beta_2 W[n-2]\} \{\beta_0 W[n+m] + \beta_1 W[n+m-1] + \beta_2 W[n+m-2]\}] \\ &= \beta_0^2 r_{00} + \beta_0 \beta_1 r_{01} + \beta_0 \beta_2 r_{02} + \beta_0 \beta_1 r_{10} + \beta_1^2 r_{11} + \beta_1 \beta_2 r_{12} + \beta_0 \beta_2 r_{20} + \beta_1 \beta_2 r_{21} + \beta_2^2 r_{22} \\ &= \beta_0^2 r_{00} + \beta_1^2 r_{11} + \beta_2^2 r_{22} + \beta_0 \beta_1 \{r_{01} + r_{10}\} + \beta_0 \beta_2 \{r_{02} + r_{20}\} + \beta_1 \beta_2 \{r_{12} + r_{21}\} \end{aligned}$$

where

$$\begin{aligned}
r_{00} &= E[W[n]W[n+m]] = R_{WW}[n, n+m] \\
r_{01} &= E[W[n]W[n+m-1]] = R_{WW}[n, n+m-1] \\
r_{02} &= E[W[n]W[n+m-2]] = R_{WW}[n, n+m-2] \\
r_{10} &= E[W[n-1]W[n+m]] = R_{WW}[n-1, n+m] \\
r_{11} &= E[W[n-1]W[n+m-1]] = R_{WW}[n-1, n+m-1] \\
r_{12} &= E[W[n-1]W[n+m-2]] = R_{WW}[n-1, n+m-2] \\
r_{20} &= E[W[n-2]W[n+m]] = R_{WW}[n-2, n+m] \\
r_{21} &= E[W[n-2]W[n+m-1]] = R_{WW}[n-2, n+m-1] \\
r_{22} &= E[W[n-2]W[n+m-2]] = R_{WW}[n-2, n+m-2]
\end{aligned}$$

Thus, we obtain

$$R_{YY}[n, n+m] = \begin{cases} \{\beta_0^2 + \beta_1^2 + \beta_2^2\} \sigma_W^2 & m = 0 \\ \{\beta_0\beta_1 + \beta_1\beta_2\} \sigma_W^2 & m = \pm 1 \\ \beta_0\beta_2 \sigma_W^2 & m = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

## 9.28 For the MA(2) process

$$Y[n] = W[n] + 0.7W[n-1] - 0.2W[n-2]$$

we use the results of Problem 9.27 with  $\beta_0 = 1, \beta_1 = 0.7, \beta_2 = -0.2$  to obtain the result

$$R_{YY}[n, n+m] = \begin{cases} 1.53\sigma_W^2 & m = 0 \\ 0.66\sigma_W^2 & m = \pm 1 \\ -0.2\sigma_W^2 & m = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

9.29 The autocorrelation function of the output process of the following AR(2) process

$$Y[n] = 0.7Y[n-1] - 0.2Y[n-2] + W[n]$$

can be obtained as follows:

$$\begin{aligned} R_{YY}[n, n+m] &= E[Y[n]Y[n+m]] \\ &= E[\{0.7Y[n-1] - 0.2Y[n-2] + W[n]\} \{0.7Y[n+m-1] - 0.2Y[n+m-2] + W[n+m]\}] \\ &= 0.49R_{YY}[n-1, n+m-1] + 0.04R_{YY}[n-2, n+m-2] + R_{WW}[m] + A + B + C + D + F + G \end{aligned}$$

where

$$\begin{aligned} A &= -(0.7)(0.2)E[Y[n-1]Y[n+m-2]] = -0.14R_{YY}[n-1, n+m-2] \\ B &= 0.7E[Y[n-1]W[n+m]] = 0.7R_{YW}[n-1, n+m] \\ C &= -(0.2)(0.7)E[Y[n-2]Y[n+m-1]] = -0.14R_{YY}[n-2, n+m-1] \\ D &= -0.2E[Y[n-2]W[n+m]] = -0.2R_{YW}[n-2, n+m] \\ F &= 0.7E[W[n]Y[n+m-1]] = 0.7R_{WY}[n, n+m-1] \\ G &= -0.2E[W[n]Y[n+m-2]] = -0.2R_{WY}[n, n+m-2] \end{aligned}$$

9.30 Given the following ARMA(1, 1) process where  $|\alpha| < 1$ ,  $|\beta| < 1$  and  $Y[n] = 0$  for  $n < 0$ :

$$Y[n] = \alpha Y[n-1] + W[n] + \beta W[n-1]$$

We assume that  $W[n]$  is a zero-mean white random process with variance  $E[W[n]W[k]] = \sigma_W^2 \delta[n-k]$  and  $W[n] = 0$  for  $n < 0$ .

- a. A general expression for the  $Y[n]$  in terms of only  $W[n]$  and its delayed versions can be obtained as follows:

---


$$\begin{aligned}
Y[n] &= \alpha Y[n-1] + W[n] + \beta W[n-1] = \alpha \{ \alpha Y[n-2] + W[n-1] + \beta W[n-2] \} + W[n] + \beta W[n-1] \\
&= \alpha^2 Y[n-2] + \alpha \beta W[n-2] + (\alpha + \beta) W[n-1] + W[n] \\
&= \alpha^2 \{ \alpha Y[n-3] + W[n-2] + \beta W[n-3] \} + \alpha \beta W[n-2] + (\alpha + \beta) W[n-1] + W[n] \\
&= \alpha^3 Y[n-3] + \alpha^2 \beta W[n-3] + \alpha(\alpha + \beta) W[n-2] + (\alpha + \beta) W[n-1] + W[n] \\
&= \alpha^3 \{ \alpha Y[n-4] + W[n-3] + \beta W[n-2] \} + \alpha^2 \beta W[n-3] + \alpha(\alpha + \beta) W[n-2] + (\alpha + \beta) W[n-1] + W[n] \\
&= W[n] + (\alpha + \beta) W[n-1] + \alpha(\alpha + \beta) W[n-2] + \alpha^2(\alpha + \beta) W[n-3] + \alpha^4 Y[n-4] \\
&\dots \\
&= W[n] + (\alpha + \beta) \sum_{k=1}^n \alpha^{k-1} W[n-k]
\end{aligned}$$

6. Using the above results, the autocorrelation function of the ARMA(1,1) process is given by

$$\begin{aligned}
R_{YY}[n, n+m] &= E[Y[n]Y[n+m]] \\
&= E \left[ \left\{ W[n] + (\alpha + \beta) \sum_{k=1}^n \alpha^{k-1} W[n-k] \right\} \left\{ W[n+m] + (\alpha + \beta) \sum_{j=1}^{n+m} \alpha^{j-1} W[n+m-j] \right\} \right] \\
&= S_{11} + S_{12} + S_{21} + S_{22}
\end{aligned}$$

where

$$S_{11} = E[W[n]W[n+m]] = R_{WW}[n, n+m]$$

$$S_{12} = (\alpha + \beta) \sum_{j=1}^{n+m} \alpha^{j-1} E[W[n]W[n+m-j]] = (\alpha + \beta) \sum_{j=1}^{n+m} \alpha^{j-1} R_{WW}[n, n+m-j]$$

$$S_{21} = (\alpha + \beta) \sum_{k=1}^n \alpha^{k-1} E[W[n-k]W[n+m]] = (\alpha + \beta) \sum_{k=1}^n \alpha^{k-1} R_{WW}[n-k, n+m]$$

$$S_{22} = (\alpha + \beta)^2 \sum_{k=1}^n \sum_{j=1}^{n+m} \alpha^{k-1} \alpha^{j-1} E[W[n-k]W[n+m-j]] = (\alpha + \beta)^2 \sum_{k=1}^n \sum_{j=1}^{n+m} \alpha^{k-1} \alpha^{j-1} R_{WW}[n-k, n+m-j]$$

Since  $E[W[n]W[k]] = R_{WW}[n, k] = \sigma_W^2 \delta[n-k]$ , we have that

$$S_{11} = R_{WW}[n, n+m] = \sigma_W^2 \delta[m]$$

$$S_{12} = (\alpha + \beta) \sum_{j=1}^{n+m} \alpha^{j-1} R_{WW}[n, n+m-j] = \sigma_W^2 (\alpha + \beta) \alpha^{m-1}$$

$$S_{21} = (\alpha + \beta) \sum_{k=1}^n \alpha^{k-1} R_{WW}[n-k, n+m] = \sigma_W^2 (\alpha + \beta) \alpha^{-m-1}$$

$$\begin{aligned} S_{22} &= (\alpha + \beta)^2 \sum_{k=1}^n \sum_{j=1}^{n+m} \alpha^{k-1} \alpha^{j-1} R_{WW}[n-k, n+m-j] = \sigma_W^2 (\alpha + \beta)^2 \sum_{k=1}^n \alpha^{k-1} \alpha^{m+k-1} \\ &= \sigma_W^2 (\alpha + \beta)^2 \alpha^{m-2} \sum_{k=1}^n \alpha^{2k} = \sigma_W^2 (\alpha + \beta)^2 \alpha^{m-2} \left\{ \frac{1}{1-\alpha^2} - 1 \right\} = \frac{\sigma_W^2 (\alpha + \beta)^2 \alpha^m}{1-\alpha^2} \end{aligned}$$

Thus, the autocorrelation function is given by

$$R_{YY}[n, n+m] = \sigma_W^2 \left\{ \delta[m] + \frac{\alpha + \beta}{\alpha} (\alpha^m + \alpha^{-m}) + \frac{(\alpha + \beta)^2 \alpha^m}{1 - \alpha^2} \right\}$$



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9.31 The expression for the MA(5) process is

$$Y[n] = \beta_0 W[n] + \beta_1 W[n-1] + \beta_2 W[n-2] + \beta_3 W[n-3] + \beta_4 W[n-4] + \beta_5 W[n-5]$$

9.32 The expression for the AR(5) process is

$$Y[n] = a_1 Y[n-1] + a_2 Y[n-2] + a_3 Y[n-3] + a_4 Y[n-4] + a_5 Y[n-5] + \beta_0 W[n]$$

9.33 The expression for the ARMA(4,3) process is

$$Y[n] = a_1 Y[n-1] + a_2 Y[n-2] + a_3 Y[n-3] + a_4 Y[n-4] + \beta_0 W[n] + \beta_1 W[n-1] + \beta_2 W[n-2] + \beta_3 W[n-3]$$



## Section 10.2: Bernoulli Process

- 10.1  $Y[n] = 3X[n] + 1$ , where  $X[n]$  is a Bernoulli process with a success probability  $p$ . Thus, the mean and variance of  $X[n]$  are given by

$$\begin{aligned} E[X[n]] &= p \\ \sigma_{X[n]}^2 &= p(1-p) \end{aligned}$$

Therefore, the mean and variance of  $Y[n]$  are given by

$$\begin{aligned} E[Y[n]] &= E[3X[n] + 1] = 3E[X[n]] + 1 = 3p + 1 \\ \sigma_{Y[n]}^2 &= \text{Var}(3X[n] + 1) = 9\sigma_{X[n]}^2 = 9p(1-p) \end{aligned}$$

- 10.2 Let the random variable  $K(7)$  denote the number of nondefective components among the 7 components. Then the PMF of  $K(7)$  has the binomial distribution with  $p = 0.8$ , as follows:

$$p_{K(7)}(k) = \binom{7}{k} (0.8)^k (0.2)^{7-k} \quad k = 0, 1, \dots, 7$$

Thus, the probability of selecting three nondefective components is

$$p_{K(7)}(3) = \binom{7}{3} (0.8)^3 (0.2)^4 = \frac{7!}{3!4!} (0.8)^3 (0.2)^4 = 0.0287$$

- 10.3 Let the random variable  $N(15)$  denote the number of survivors of the disease. Then  $N(15)$  has the binomial distribution with  $p = 0.3$  and PMF

$$p_{N(15)}(n) = \binom{15}{n} (0.3)^n (0.7)^{15-n} \quad n = 0, 1, \dots, 15$$

- a. The probability that at least 10 survive is given by

$$\begin{aligned}
 P[N \geq 10] &= \sum_{n=10}^{15} p_N(n) \\
 &= \binom{15}{10} (0.3)^{10} (0.7)^5 + \binom{15}{11} (0.3)^{11} (0.7)^4 + \binom{15}{12} (0.3)^{12} (0.7)^3 + \binom{15}{13} (0.3)^{13} (0.7)^2 + \\
 &\quad \binom{15}{14} (0.3)^{14} (0.7) + (0.3)^{15} \\
 &= 0.00365
 \end{aligned}$$

- b. The probability that the number of survivors is at least 3 and at most 8 is given by

$$\begin{aligned}
 P[3 \leq N \leq 8] &= \sum_{n=3}^8 p_N(n) \\
 &= \binom{15}{3} (0.3)^3 (0.7)^{12} + \binom{15}{4} (0.3)^4 (0.7)^{11} + \binom{15}{5} (0.3)^5 (0.7)^{10} + \binom{15}{6} (0.3)^6 (0.7)^9 + \\
 &\quad \binom{15}{7} (0.3)^7 (0.7)^8 + \binom{15}{8} (0.3)^8 (0.7)^7 \\
 &= 0.8579
 \end{aligned}$$

- c. The probability that exactly 6 survive is given by

$$P[N = 6] = \binom{15}{6} (0.3)^6 (0.7)^9 = 0.1472$$

10.4 Let  $X_k$  be a random variable that denotes the number of trials up to and including the trial that results in the  $k$ th success. Then  $X_k$  is an  $k$ th-order Pascal random variable whose PMF is given by

$$p_{X_k}(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k} \quad k = 1, 2, \dots; n = k, k+1, \dots$$

where  $p = 0.8$ .

- a. The probability that the first success occurs on the fifth trial is given by

$$P[X_1 = 5] = \binom{5-1}{1-1} p^1 (1-p)^4 = p(1-p)^4 = 0.8(0.2)^4 = 0.00128$$

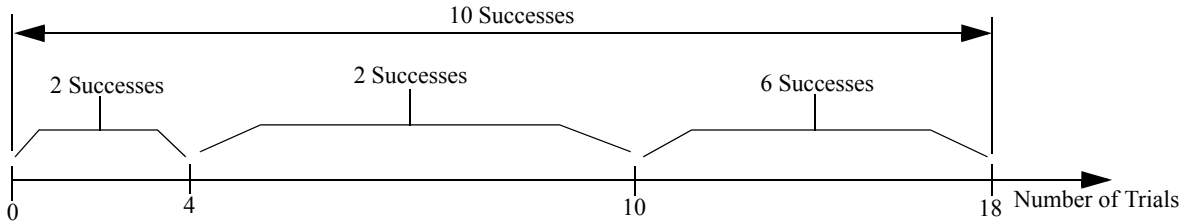
- b. The probability that the third success occurs on the eighth trial is given by

$$P[X_3 = 8] = \binom{8-1}{3-1} p^3 (1-p)^5 = \binom{7}{2} p^3 (1-p)^5 = 21 p^3 (1-p)^5 = 21(0.8)^3 (0.2)^5 = 0.00344$$

- c. The probability that there are two successes by the fourth trial, there are four successes by the tenth trial and there are ten successes by the eighteenth trial can be obtained by partitioning the timeline as follows:

1. There are 2 successes in the first 4 trials
2. There are 2 successes in the next 6 trials
3. There are 6 successes in the next 8 trials

These intervals are illustrated in the following diagram.



Since these intervals are nonoverlapping, the events occurring within them are independent. Thus, the probability,  $Q$ , of the event is given by

$$\begin{aligned} Q &= \left\{ \binom{4}{2} p^2 (1-p)^2 \right\} \left\{ \binom{6}{2} p^2 (1-p)^4 \right\} \left\{ \binom{8}{6} p^6 (1-p)^2 \right\} = \binom{4}{2} \binom{6}{2} \binom{8}{6} p^{10} (1-p)^8 \\ &= 6(15)(28) p^{10} (1-p)^8 = 2520(0.8)^{10} (0.2)^8 = 0.00069 \end{aligned}$$

10.5 Let the random variable  $N$  denote the number of guests that come for the dinner. Then  $N$  has the binomial distribution with the PMF

$$p_N(n) = \binom{12}{n} p^n (1-p)^{12-n} = \binom{12}{n} (0.4)^n (0.6)^{12-n} \quad n = 0, 1, \dots, 12$$

Let  $X$  denote the event that she has a sit-down dinner. Thus,  $\bar{X}$  denotes the event that she has a buffet-style dinner.

a. The probability that she has a sit-down dinner is given by

$$\begin{aligned} P[X] &= \sum_{n=0}^6 p_N(n) = \sum_{n=0}^6 \binom{12}{n} (0.4)^n (0.6)^{12-n} \\ &= (0.6)^{12} + 12(0.4)(0.6)^{11} + 66(0.4)^2(0.6)^{10} + 220(0.4)^3(0.6)^9 + 495(0.4)^4(0.6)^8 + \\ &\quad 792(0.4)^5(0.6)^7 + 924(0.4)^6(0.6)^6 \\ &= 0.8418 \end{aligned}$$

b. The probability that she has a buffet-style dinner is given by

$$P[\bar{X}] = 1 - P[X] = 0.1582$$

c. The probability that there are at most three guests is given by

$$\begin{aligned} P[N \leq 3] &= \sum_{n=0}^3 p_N(n) = \sum_{n=0}^3 \binom{12}{n} (0.4)^n (0.6)^{12-n} \\ &= (0.6)^{12} + 12(0.4)(0.6)^{11} + 66(0.4)^2(0.6)^{10} + 220(0.4)^3(0.6)^9 \\ &= 0.2253 \end{aligned}$$

10.6 Let  $X_k$  be a random variable that denotes the number of trials up to and including the trial that results in the  $k$ th success. Then  $X_k$  is an  $k$ th-order Pascal random variable whose PMF is given by

$$p_{X_k}(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k} = \binom{n-1}{k-1} (0.4)^k (0.6)^{n-k} \quad k = 1, 2, \dots; n = k, k+1, \dots$$

- a. The probability that the house where they make their first sale is the fifth house they visit is given by

$$P[X_1 = 5] = p_{X_1}(5) = \binom{5-1}{1-1} (0.4)^1 (0.6)^4 = (0.4)(0.6)^4 = 0.05184$$

- b. Let the random variable  $X(10)$  denote the number of sets of cookie packs they sell given that they visited 10 houses on a particular day. Then  $X(10)$  is a binomially distributed random variable with the PMF

$$p_{X(10)}(x) = \binom{10}{x} (0.4)^x (0.6)^{10-x} \quad x = 0, 1, \dots, 10$$

Thus, the probability that they sold exactly 6 sets of cookie packs is given by

$$P[X(10) = 6] = p_{X(10)}(6) = \binom{10}{6} (0.4)^6 (0.6)^4 = 210(0.4)^6 (0.6)^4 = 0.1115$$

- c. The probability that on a particular day the third set of cookie packs is sold at the seventh house that the girls visit is given by

$$P[X_3 = 7] = p_{X_3}(7) = \binom{7-1}{3-1} (0.4)^3 (0.6)^4 = \binom{6}{2} (0.4)^3 (0.6)^4 = 15(0.4)^3 (0.6)^4 = 0.1244$$

### Section 10.3: Random Walk

- 10.7 Since there are 11 balls and the balls are drawn with replacement, the probability of success (i.e., drawing a green ball) in each game is  $p = 6/11$ . With  $k = 50$  and  $N = 100$ , the probability that Jack will go bankrupt (i.e., ruined) is given by

$$r_{50} = \frac{\left(\frac{1-p}{p}\right)^k - \left(\frac{1-p}{p}\right)^N}{1 - \left(\frac{1-p}{p}\right)^N} = \frac{\left(\frac{5}{6}\right)^{50} - \left(\frac{5}{6}\right)^{100}}{1 - \left(\frac{5}{6}\right)^{100}} = 0.00011$$

Thus, the probability that he will not go bankrupt is  $1 - r_{50} = 0.9999$ .

10.8 When in state  $i \neq 0, N$ , he plays a game. If he wins, he moves to state  $i + 1$ ; otherwise, he moves to state  $i - 1$ . Let  $D_i$  be a random variable that denotes the duration of a game in which a player starts in state  $i$ . Thus,  $E[D_i] = d_i$ , where  $d_0 = 0$  and  $d_N = 0$ . Let  $W$  denote the event that he wins a game and  $L$  the event that he loses a game.

a. Let  $p$  denote the probability that he wins a game. Then  $d_i$  is given by

$$\begin{aligned} d_i &= E[D_i|W]P[W] + E[D_i|L]P[L] = p[1 + d_{i+1}] + (1-p)[1 + d_{i-1}] \\ &= 1 + pd_{i+1} + (1-p)d_{i-1} \end{aligned}$$

Since  $p = 1/2$ , we have that

$$d_i = \begin{cases} 0 & i = 0, N \\ 1 + \frac{d_{i+1} + d_{i-1}}{2} & i = 1, 2, \dots, N-1 \end{cases}$$

b. From the above relationship, we have that

$$d_i = 1 + \frac{d_{i+1} + d_{i-1}}{2} \Rightarrow d_{i+1} = 2d_i - d_{i-1} - 2$$

Thus,

$$\begin{aligned} d_2 &= 2d_1 - d_0 - 2 = 2(d_1 - 1) \\ d_3 &= 2d_2 - d_1 - 2 = 2\{2(d_1 - 1)\} - d_1 - 2 = 3(d_1 - 2) \\ d_4 &= 2d_3 - d_2 - 2 = 2\{3(d_1 - 2)\} - 2(d_1 - 1) - 2 = 4(d_1 - 3) \\ d_5 &= 2d_4 - d_3 - 2 = 2\{4(d_1 - 3)\} - 3(d_1 - 2) - 2 = 5(d_1 - 4) \end{aligned}$$

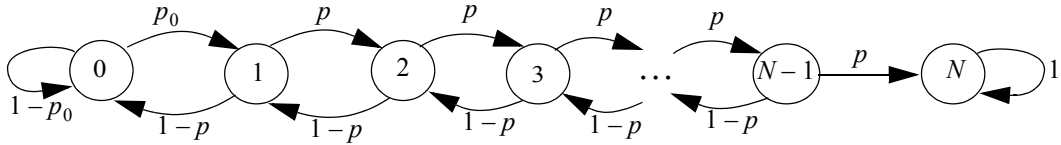


From these results we see that in general, we have that  $d_i = i(d_1 - i + 1)$ . Since  $d_N = 0 = N(d_1 - N + 1) \Rightarrow d_1 = N - 1$ , we obtain

$$d_i = i(d_1 - i + 1) = i(N - 1 - i + 1) = (N - i)i \quad i = 1, 2, \dots, N - 1$$

10.9 Given a random walk with *reflecting barrier* at zero such that when state 0 is reached the process moves to state 1 with probability  $p_0$  or stays at state 0 with probability  $1 - p_0$ . Let state  $i$  denote the state in which player A has a total of  $\$i$ .

a. The state transition diagram of the process is as follows:



b. The probability of player B being ruined when the process is currently in state  $i$ ,  $r_i$ , can be obtained from the following relationship:

$$r_i = \begin{cases} pr_{i+1} + (1-p)r_{i-1} & i = 1, 2, \dots, N-1 \\ p_0r_1 + (1-p_0)r_0 & i = 0 \\ 0 & i = N \end{cases}$$

10.10 The total available amount is  $N = 9 + 6 = 15$ .

a. Since Ben started with  $\$9$  and the probability that he wins a game is  $p = 0.6$ , the probability that he is ruined is given by

$$r_9 = \frac{\left(\frac{1-p}{p}\right)^9 - \left(\frac{1-p}{p}\right)^{15}}{1 - \left(\frac{1-p}{p}\right)^{15}} = \frac{\left(\frac{0.4}{0.6}\right)^9 - \left(\frac{0.4}{0.6}\right)^{15}}{1 - \left(\frac{0.4}{0.6}\right)^{15}} = \frac{(2/3)^9 - (2/3)^{15}}{1 - (2/3)^{15}} = 0.02378$$

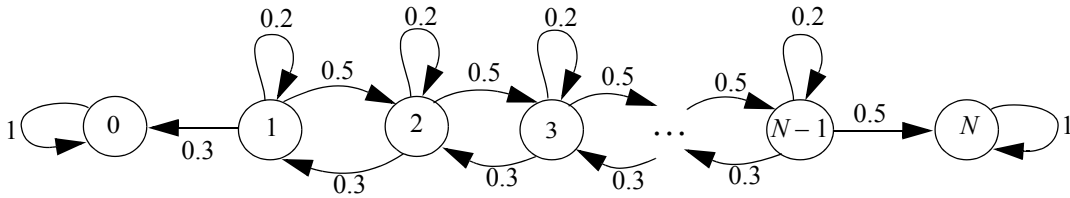
b. Since Jerry started with  $\$6$  and the probability that he wins a game is  $q = 0.4$ , the probability that he is ruined is given by

$$r_6 = \frac{\left(\frac{1-q}{q}\right)^6 - \left(\frac{1-q}{q}\right)^{15}}{1 - \left(\frac{1-q}{q}\right)^{15}} = \frac{\left(\frac{0.6}{0.4}\right)^6 - \left(\frac{0.6}{0.4}\right)^{15}}{1 - \left(\frac{0.6}{0.4}\right)^{15}} = \frac{(3/2)^6 - (3/2)^{15}}{1 - (3/2)^{15}} = 0.97622$$

$$= 1 - r_9$$

10.11 Let  $k$  denote the state in which Ben has a total of  $\$k$  left. The total amount is  $N = 15$ .

- a. The state transition diagram of the process is given by



- b. If  $r_k$  denotes the probability that Ben is ruined, given that the process is currently in state  $k$ , the expression for  $r_k$  in the first game when the process is in state  $k$  is given by

$$r_k = 0.5r_{k+1} + 0.3r_{k-1} + 0.2r_k \Rightarrow 0.8r_k = 0.5r_{k+1} + 0.3r_{k-1} \Rightarrow r_k = 1.25\{0.5r_{k+1} + 0.3r_{k-1}\}$$

## Section 10.4: Gaussian Process

10.12  $X(t)$  is a wide-sense stationary Gaussian process with the autocorrelation function

$$R_{XX}(\tau) = 4 + e^{-|\tau|}$$

The expected value of  $X(t)$  is given by

$$E[X(t)] = \pm \sqrt{\lim_{|\tau| \rightarrow \infty} R_{XX}(\tau)} = \pm \sqrt{4} = \pm 2$$

Let  $X_1 = X(0), X_2 = X(1), X_3 = X(3), X_4 = X(6)$ . Then

$$C_{ij} = \text{Cov}(X_i, X_j) = R_{XX}(i, j) - \mu_X(i)\mu_X(j) = R_{XX}(j - i) - 4 = e^{-|j-i|}$$

Thus, the covariance matrix for the random variables  $X(0), X(1), X(3), X(6)$  is given by

$$C_{XX} = \begin{bmatrix} 1 & e^{-1} & e^{-3} & e^{-6} \\ e^{-1} & 1 & e^{-2} & e^{-5} \\ e^{-3} & e^{-2} & 1 & e^{-3} \\ e^{-6} & e^{-5} & e^{-3} & 1 \end{bmatrix}$$

10.13  $X(t)$  has an autocorrelation function

$$R_{XX}(\tau) = \frac{4 \sin(\pi \tau)}{\pi \tau}$$

Its expected value is given by

$$E[X(t)] = \pm \sqrt{\lim_{|\tau| \rightarrow \infty} R_{XX}(\tau)} = 0$$

Let  $X_1 = X(t), X_2 = X(t+1), X_3 = X(t+2), X_4 = X(t+3)$ . Then

$$C_{ij} = \text{Cov}(X_i, X_j) = R_{XX}(i, j) - \mu_X(i)\mu_X(j) = R_{XX}(j - i)$$

Thus, the covariance matrix for the random variables  $X(t), X(t+1), X(t+2),$  and  $X(t+3)$  is given by

$$C_{XX} = \begin{bmatrix} 4 & \frac{4 \sin(\pi)}{\pi} & \frac{4 \sin(2\pi)}{2\pi} & \frac{4 \sin(3\pi)}{3\pi} \\ \frac{4 \sin(-\pi)}{-\pi} & 4 & \frac{4 \sin(\pi)}{\pi} & \frac{4 \sin(2\pi)}{2\pi} \\ \frac{4 \sin(-2\pi)}{-2\pi} & \frac{4 \sin(-\pi)}{-\pi} & 4 & \frac{4 \sin(\pi)}{\pi} \\ \frac{4 \sin(-3\pi)}{-3\pi} & \frac{4 \sin(-2\pi)}{-2\pi} & \frac{4 \sin(-\pi)}{-\pi} & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

10.14  $X(t)$  is a Gaussian random process with a mean  $E[X(t)] = 0$  and autocorrelation function  $R_{XX}(\tau) = e^{-|\tau|}$ . The random variable  $A$  is defined as follows:

$$A = \int_0^1 X(t) dt$$

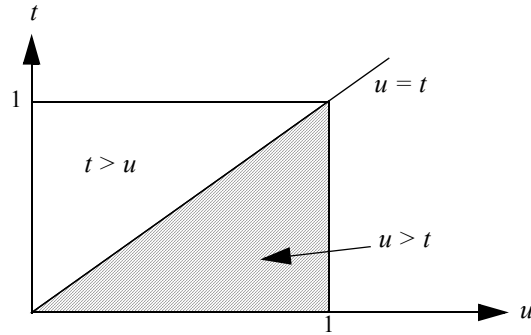
Then

a.  $E[A] = E\left[\int_0^1 X(t) dt\right] = \int_0^1 E[X(t)] dt = 0$

b. Since  $E[A] = 0$ ,  $\sigma_A^2 = E[A^2]$ . Thus,

$$\sigma_A^2 = E[A^2] = E\left[\int_0^1 X(t) dt \int_0^1 X(u) du\right] = \int_0^1 \int_0^1 E[X(t)X(u)] dt du = \int_0^1 \int_0^1 R_{XX}(u-t) dt du = \int_0^1 \int_0^1 e^{-|u-t|} dt du$$

Consider the following figure:



Since

$$e^{-|u-t|} = \begin{cases} e^{-(u-t)} & u \geq t \\ e^{-(t-u)} & t > u \end{cases}$$

we have that

$$\begin{aligned} \sigma_A^2 &= \int_0^1 \int_0^1 e^{-|u-t|} dt du = \int_{u=0}^1 \int_{(t=0)}^u e^{-(u-t)} dt du + \int_{t=0}^1 \int_{u=0}^t e^{-(t-u)} du dt \\ &= 2 \int_{u=0}^1 \int_{(t=0)}^u e^{-(u-t)} dt du = 2 \int_{u=0}^1 e^{-u} [e^u - 1] du = 2 \int_{u=0}^1 [1 - e^{-u}] du \\ &= 2[u + e^{-u}]_0^1 = 2(1 + e^{-1} - 1) = 2e^{-1} \\ &= 0.7357 \end{aligned}$$

10.15  $X(t)$  is a Gaussian random process with a mean  $E[X(t)] = 0$  and autocorrelation function  $R_{XX}(\tau) = e^{-|\tau|}$ . The random variable  $A$  is defined as follows:

$$A = \int_0^B X(t) dt$$

where  $B$  is a uniformly distributed random variable with values between 1 and 5 and is independent of the random process  $X(t)$ . Then

a. The mean of  $A$  is given by

$$\begin{aligned} E[A] &= \int_{b=1}^5 E[A|B=b]f_B(b)db = \int_{b=1}^5 E\left[\int_{t=0}^b X(t)dt\right]f_B(b)db = \int_{b=1}^5 \int_{t=0}^b E[X(t)]f_B(b)db \\ &= 0 \end{aligned}$$

b. Since the mean of  $A$  is zero, the variance of  $A$ ,  $\sigma_A^2 = E[A^2]$ ; that is,

$$\begin{aligned} \sigma_A^2 &= E[A^2] = \int_{b=1}^5 E[A^2|B=b]f_B(b)db = \int_{b=1}^5 E\left[\int_{t=0}^b \int_{u=0}^b X(t)X(u)dtdu\right]f_B(b)db \\ &= \int_{b=1}^5 \int_{t=0}^b \int_{u=0}^b E[X(t)X(u)]f_B(b)dtdu db = \int_{b=1}^5 \int_{t=0}^b \int_{u=0}^b R_{XX}(t, u)f_B(b)dtdu db \\ &= \int_{b=1}^5 \int_{t=0}^b \int_{u=0}^b e^{-|u-t|}f_B(b)dtdu db = 2 \int_{b=1}^5 \int_{t=0}^b \int_{u=0}^t e^{-(t-u)}f_B(b)dtdu db \\ &= 2 \int_{b=1}^5 (b + e^{-b} - 1)f_B(b)db = \frac{2}{4} \int_{b=1}^5 (b + e^{-b} - 1)db = \frac{1}{2} \left[ \frac{b^2}{2} - e^{-b} - b \right]_1^5 = \frac{1}{2} [8 + e^{-1} - e^{-5}] \\ &= 4.1806 \end{aligned}$$

## Section 10.5: Poisson Process

10.16 Since buses arrive according to a Poisson process with an average rate of 5 buses per hour, the times  $X$  between bus arrivals are exponentially distributed with the PDF

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

Now,  $\lambda = 5$  buses/hour or  $\lambda = 5/60 = 1/12$  buses/minute. Since Chris just missed the last bus, the time until the next bus arrives is the random variable  $X$ . Therefore, the probability that he waits more than 20 minutes before boarding a bus is given by

$$P[X > 20] = 1 - P[X \leq 20] = e^{-20\lambda} = e^{-5/3} = 0.18887$$

10.17 Since cars arrive according to a Poisson process at an average rate of 12 cars per hour, the PMF of  $N$ , the number of cars that arrive within an interval of  $t$  minutes, is given by

$$p_N(n, t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad n = 0, 1, 2, \dots$$

where  $\lambda = 12/60 = 1/5$  cars/minute. Thus, the probability that one or more cars will be waiting when the attendant comes back from a 2-minute break is given by

$$P[N \geq 1, t = 2] = 1 - P[N = 0, t = 2] = 1 - p_N(0, 2) = 1 - e^{-2/5} = 0.3297$$

10.18 Since cars arrive according to a Poisson process at an average rate of 50 cars per hour, the PMF of  $N$ , the number of cars that arrive over an interval of length  $t$ , is given by

$$p_N(n, t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad n = 0, 1, 2, \dots$$

where  $\lambda = 50/60 = 5/6$  cars/minute. Let  $W$  denote the event that a waiting line occurs. Then, the probability that a waiting line will occur at the station is given by

$$\begin{aligned} P[W] &= \sum_{n=2}^{\infty} p_N(n, 1) = 1 - p_N(0, 1) - p_N(1, 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda} = 1 - (1 + \lambda)e^{-\lambda} \\ &= 1 - \frac{11}{6}e^{-5/6} = 0.2032 \end{aligned}$$

10.19 Let  $\lambda$  denote the average arrival rate of cars per minute and  $K$  the number of cars that arrive over an interval of  $t$  minutes. Then, the PMF of  $K$  is given by

$$p_K(k, t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad k = 0, 1, 2, \dots$$

- a. Given that the probability that 3 cars will arrive at a parking lot in a 5-minute interval is 0.14, we have that

$$p_K(3, 5) = \frac{(5\lambda)^3 e^{-5\lambda}}{3!} = \frac{125\lambda^3 e^{-5\lambda}}{6} = 0.14 \Rightarrow \lambda^3 e^{-5\lambda} = \frac{6(0.14)}{125} = 0.00672$$

Solving the above equation numerically, we obtain  $\lambda = 1$ .

- b. The probability that no more than 2 cars arrive in a 10-minute interval is given by

$$\begin{aligned} P[K \leq 2, t = 10] &= p_K(0, 10) + p_K(1, 10) + p_K(2, 10) = e^{-10\lambda} \{1 + 10\lambda + 50\lambda^2\} \\ &= e^{-10} \{1 + 10 + 50\} = 61e^{-10} = 0.00277 \end{aligned}$$

10.20 Let  $N$  denote the number of telephone calls that arrive at the switching center during an interval of length  $t$  seconds. The PMF of  $N$  is given by

$$p_N(n, t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad n = 0, 1, 2, \dots$$

where  $\lambda = 75/60 = 1.25$  calls/second. The probability that more than 3 calls arrive within a 5-second period is given by

$$\begin{aligned} P[N > 3, t = 5] &= 1 - P[N \leq 2, t = 5] = 1 - \{p_N(0, 5) + p_N(1, 5) + p_N(2, 5)\} \\ &= 1 - e^{-6.25} \{1 + 6.25 + 19.53125\} = 1 - 26.78125e^{-6.25} = 0.9483 \end{aligned}$$

10.21 Let  $M$  denote the number of claims paid in an  $n$ -week period. Then the PMF and expected value of  $M$  are given by



$$p_M(m, n) = \frac{(\lambda n)^m e^{-\lambda n}}{m!}$$

$$E[M] = \lambda n$$

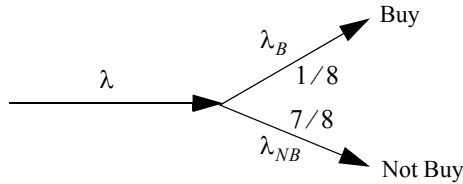
where  $\lambda = 5$ . Let  $X$  denote the amount paid on a policy. Since  $X$  is uniformly distributed between \$2,000.00 and \$10,000.00, its mean is given by

$$E[X] = \frac{2,000 + 10,000}{2} = 6,000$$

Thus, the expected total amount of money in dollars,  $E[T]$ , that the company pays out in a 4-week period is given by

$$E[T] = E[M]E[X] = (5)(4)(6,000) = 120,000$$

10.22 This is an example of subdivision of a Poisson process, which is illustrated in the figure below.



If  $\lambda$  is the arrival rate of customers and  $\lambda_B$  denotes the arrival rate of customers who buy books at the bookstore, then we know that

$$\lambda_B = \frac{\lambda}{8} = \frac{10}{8} = 1.25$$

Let  $K$  denote the number of books that the bookstore sells in one hour. Then we know that  $K$  is a Poisson random variable with the PMF

$$p_K(k) = \frac{\lambda_B^k e^{-\lambda_B}}{k!} = \frac{(1.25)^k e^{-1.25}}{k!} \quad k = 0, 1, 2, \dots$$

- a. The probability that the bookstore sells no book during a particular hour is given by

$$P[K = 0] = p_K(0) = e^{-1.25} = 0.2865$$

- b. Let  $X$  denote the time between book sales. Then  $X$  is an exponentially distributed random variable with the PDF

$$f_X(x) = \lambda_B e^{-\lambda_B x} = 1.25 e^{-1.25x} \quad x \geq 0$$

10.23 Let  $Y$  denote the life of a bulb. Since  $Y$  is an exponentially distributed random variable with rate  $\lambda$  (or mean  $1/\lambda = 200$ ), the failure (or burnout) rate when  $k$  bulbs are still operational is  $k\lambda$ .

- a. Since the lifetimes are exponentially distributed, when Joe comes back the lifetimes of the bulbs start from scratch because of the forgetfulness property of the exponential distribution. Thus, the 6 bulbs will operate as a “superbulb” whose failure rate is  $6\lambda$ . Since the time until the superbulb fails is also exponentially distributed, the expected time until the next bulb failure occurs is  $\frac{1}{6\lambda} = \frac{1}{6} \left( \frac{1}{\lambda} \right) = \frac{200}{6} = 33.33$  hours
- b. By the time Joe went for the break, 4 bulbs had failed. Thus, given that all 6 bulbs were still working by the time he came back, the time between the 4th failure and the next failure, which is the 5th failure, is the duration of the interval (or gap) entered by random incidence. Therefore, the expected length of time from the instant the 4th bulb failed until the instant the 5th bulb failed is given by

$$\frac{2}{6\lambda} = \frac{1}{3} \left( \frac{1}{\lambda} \right) = \frac{200}{3} = 66.67$$

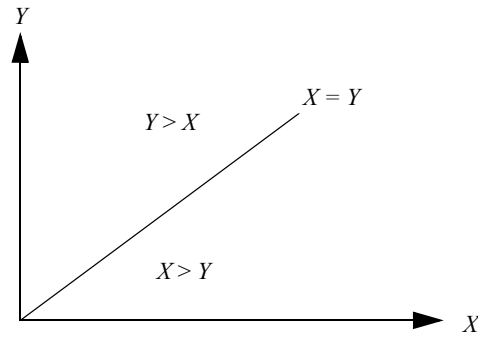
10.24 Let  $X$  denote the time to serve a customer. Then the PDF of  $X$  is given by

$$f_X(x) = \lambda e^{-\lambda x} = 0.25e^{-0.25x} \quad x \geq 0$$

The time  $Y$  to serve customers  $B$  and  $C$  is the second-order Erlang random variable whose PDF is given by

$$f_Y(y) = \lambda^2 y e^{-\lambda y} = 0.0625 y e^{-0.25y} \quad y \geq 0$$

The probability that customer  $A$  is still in the bank after customers  $B$  and  $C$  leave is simply the probability that  $X$  is greater than  $Y$ , which can be obtained as follows:



$$P[X > Y] = \int_{x=0}^{\infty} \int_{y=0}^x f_{XY}(x, y) dy dx = \int_{x=0}^{\infty} \int_{y=0}^x f_X(x) f_Y(y) dy dx = \lambda^3 \int_{x=0}^{\infty} e^{-\lambda x} \int_{y=0}^x y e^{-\lambda y} dy dx$$

Let  $u = y \Rightarrow du = dy$ , and let  $dv = e^{-\lambda y} dy \Rightarrow v = -e^{-\lambda y} / \lambda$ . Thus,

$$\begin{aligned}
 P[X > Y] &= \lambda^3 \int_{x=0}^{\infty} e^{-\lambda x} \int_{y=0}^x y e^{-\lambda y} dy dx = \lambda^3 \int_{x=0}^{\infty} e^{-\lambda x} \left\{ [-y e^{-\lambda y} / \lambda]_0^x + \frac{1}{\lambda} \int_{y=0}^x e^{-\lambda y} dy \right\} dx \\
 &= \lambda^3 \int_{x=0}^{\infty} e^{-\lambda x} \left\{ [-x e^{-\lambda x} / \lambda] + \frac{1}{\lambda} \left[ -\frac{e^{-\lambda y}}{\lambda} \right]_0^x \right\} dx = \lambda^3 \int_{x=0}^{\infty} \left\{ \frac{e^{-\lambda x}}{\lambda^2} - \frac{e^{-2\lambda x}}{\lambda^2} - \frac{x e^{-2\lambda x}}{\lambda} \right\} dx \\
 &= \lambda^3 \left\{ \left[ -\frac{e^{-\lambda x}}{\lambda^3} \right]_0^{\infty} + \left[ \frac{e^{-2\lambda x}}{2\lambda^3} \right]_0^{\infty} - \frac{1}{2\lambda^2} \int_{y=0}^x 2\lambda x e^{-2\lambda x} dx \right\} = \lambda^3 \left\{ \frac{1}{\lambda^3} - \frac{1}{2\lambda^3} - \frac{1}{2\lambda^2} \left( \frac{1}{2\lambda} \right) \right\} \\
 &= 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}
 \end{aligned}$$

Note that another way to solve the problem is to use the forgetfulness property of the exponential distribution as follows. Let  $X_A$  denote the time to serve  $A$ ,  $X_B$  the time to serve  $B$ , and  $X_C$  the time to serve  $C$ . Let the mean time to serve customer  $A$  be  $1/\lambda_A$ , the mean time to serve customer  $B$  be  $1/\lambda_B$ , and the mean time to serve customer  $C$  be  $1/\lambda_C$ , where  $\lambda_A = \lambda_B = \lambda_C = 1/4$ . The probability that  $B$  leaves before  $A$  is given by

$$P[X_A > X_B] = \frac{\lambda_B}{\lambda_A + \lambda_B}$$

Because of the forgetfulness property of the exponential distribution, after  $B$  leaves,  $A$ 's service starts from scratch. Thus, the probability that  $C$  leaves before  $A$  is given by

$$P[X_A > X_C] = \frac{\lambda_C}{\lambda_A + \lambda_C}$$

Thus, the probability that customer  $A$  is still in the bank after the other two customers leave is given by

$$\begin{aligned}
 P[X_B + X_C < X_A] &= P[X_A > X_B] P[X_A > X_C | X_A > X_B] = P[X_A > X_B] P[X_A > X_C] \\
 &= \left\{ \frac{\lambda_B}{\lambda_A + \lambda_B} \right\} \left\{ \frac{\lambda_C}{\lambda_A + \lambda_C} \right\} = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{4}
 \end{aligned}$$

10.25 Since the times between component failures are exponentially distributed, the number  $N$  of failures within an interval of length  $t$  is a Poisson random variable with rate  $\lambda$ , where  $1/\lambda = 4 \times 60 = 240$  seconds or  $\lambda = 1/240$ . Thus, the PMF of  $N$  is given by

$$p_N(n, t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad n = 0, 1, 2, \dots$$

Therefore, the probability that at least one component failure occurs within a 30-minute period is given by

$$P[N \geq 1] = 1 - P[N = 0] = 1 - e^{-30\lambda} = 1 - e^{-30/240} = 1 - e^{-0.125} = 0.1175$$

10.26 Let  $T$  denote the interval between student arrival times at the professor's office. Then the PDF of  $T$  is given by

$$f_T(t) = \lambda e^{-\lambda t} \quad t \geq 0$$

where  $\lambda = 4$  students/hour. Let  $X$  denote the time that elapses from the instant one session ends until the time the next session begins.

- a. Given that a tutorial has just ended and there are no students currently waiting for the professor, the mean time until another tutorial can start in hours is given by the mean time until 3 students arrive, which is the following:

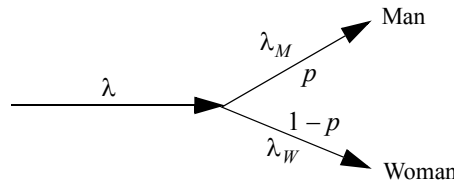
$$E[X] = E[T_1 + T_2 + T_3] = 3E[T] = \frac{3}{\lambda} = \frac{3}{4}$$

That is, the mean time between the two sessions is  $3/4$  hours or 45 minutes.

- b. Given that one student was waiting when the tutorial ended, the probability that the next tutorial does not start within the first 2 hours is the probability that the time until the second of two other students arrives is greater than 2 hours measured from the time the last session ended, which is the probability that a second-order random variable  $X_2$  with parameter  $\lambda$  is greater than 2 hours. That is,

$$P[X_2 > 2] = 1 - P[X_2 \leq 1] = \sum_{k=0}^1 \frac{(2\lambda)^k e^{-2\lambda}}{k!} = e^{-2\lambda} + 2\lambda e^{-2\lambda} = (1 + 2\lambda)e^{-2\lambda} = 9e^{-8} = 0.0030$$

10.27 This is an example of subdivision of a Poisson process. If  $\lambda_M$  is the arrival rate of male customers and  $\lambda_W$  is the arrival rate of female customers, we can represent the process as shown below.



Let  $N_M$  denote the number of men who arrive in an interval of length 2 hours, and let  $N_W$  denote the number of women who arrive in an interval of length 2 hours. Since both  $N_M$  and  $N_W$  are Poisson random variables with rates  $\lambda_M = p\lambda$  and  $\lambda_W = (1-p)\lambda$ , respectively, where  $\lambda = 6$ , we have that

$$E[N_M] = 2\lambda_M = 2p(6) = 12p = 8 \Rightarrow p = \frac{8}{12} = \frac{2}{3}$$

Thus, the average number of women who arrived over the same period is given by

$$E[N_W] = 2\lambda_W = 2(1-p)(6) = 12(1-p) = 12\left(\frac{1}{3}\right) = 4$$

10.28 Let  $X$  denote the time until a bulb from set  $A$  fails and  $Y$  the time until a bulb from set  $B$  fails. Then the PDFs and expected values of  $X$  and  $Y$  are given by

$$f_X(x) = \lambda_A e^{-\lambda_A x}, x \geq 0$$

$$E[X] = \frac{1}{\lambda_A} = 200$$

$$f_Y(y) = \lambda_B e^{-\lambda_B y}, y \geq 0$$

$$E[Y] = \frac{1}{\lambda_B} = 400$$

Let  $p_A$  denote the probability that a bulb from set  $A$  fails before a bulb from set  $B$ . Then we have that

$$p_A = \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{(1/200)}{(1/200) + (1/400)} = \frac{2}{3}$$

Thus, the probability  $p_B$  that a bulb from set  $B$  fails before a bulb from set  $A$  is given by

$$p_B = 1 - p_A = \frac{1}{3}$$

- a. Let  $K$  denote the number of set  $B$  bulbs that fail out of the 8 bulbs. Then  $K$  has a binomial distribution whose PMF is given by

$$p_K(k) = \binom{8}{k} p_B^k (1 - p_B)^{8-k} = \binom{8}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{8-k} \quad k = 0, 1, \dots, 8$$

Thus, the probability that exactly 5 of those 8 bulbs are from set  $B$  is given by

$$P[K = 5] = p_K(5) = \binom{8}{5} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^3 = 0.0683$$

- b. Since the two-bulb arrangement constitutes a competing Poisson process, the composite failure rate is  $\lambda = \lambda_A + \lambda_B$ . The time  $V$  until a bulb fails is exponentially distributed with the PDF and CDF

$$f_V(v) = \lambda e^{-\lambda v}, v \geq 0$$

$$F_V(v) = 1 - e^{-\lambda v}$$

Thus, the probability that no bulb will fail in the first 100 hours is given by

$$P[V > 100] = 1 - F_V(100) = e^{-100\lambda} = e^{-100\left(\frac{1}{200} + \frac{1}{400}\right)} = e^{-3/4} = 0.4724$$

- c. The mean time between two consecutive bulbs failures is given by

$$E[V] = \frac{1}{\lambda} = \frac{1}{\frac{1}{200} + \frac{1}{400}} = \frac{400}{3} = 133.33$$

10.29 Let  $X$  be a random variable that denotes the times between plane arrivals. Since the number of planes arriving within any time interval is a Poisson random variable with a mean rate of  $\lambda = 2$  planes/hour, the PDF of  $X$  is given by

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

where  $E[X] = 1/2$  hours or 30 minutes. Given that Vanessa arrived at the airport and had to wait to catch the next flight.

- Due to the forgetfulness property of the exponential distribution, the mean time between the instant Vanessa arrived at the airport until the time the next plane arrived is the same as  $E[X] = 30$  minutes.
- The time  $T$  between the arrival time of the last plane that took off from the Manchester airport before Vanessa arrived and the arrival time of the plane that she boarded is the gap Vanessa entered by random incidence. Thus,  $E[T] = 2E[X] = 1$  hour.

10.30 We are given three lightbulbs that have independent and identically distributed lifetimes  $T$  with PDF  $f_T(t) = \lambda e^{-\lambda t}$ ,  $t \geq 0$ . Bob has a pet that requires the light in his apartment to be always on, which prompts Bob to keep three lightbulbs on with the hope that at least one bulb will be operational when he is not at the apartment.



- 
- a. Probabilistically speaking, given that Bob is about to leave the apartment and all three bulbs are working fine, Bob gains nothing by replacing all three bulbs with new ones before he leaves because the time until any one of the 3 bulbs fails is statistically identical to the time to failure of a new bulb. This is the result of the forgetfulness property of the exponential distribution.
- b. The 3 bulbs behave as a single system with a failure rate  $\lambda_X = 3\lambda$ . Thus, the time  $X$  until the first bulb fails is exponentially distributed with the PDF

$$f_X(x) = \lambda_X e^{-\lambda_X x} = 3\lambda e^{-3\lambda x} \quad x \geq 0$$

- c. Given that Bob is going away for an indefinite period of time and all three bulbs are working fine before he leaves, the random variable  $Y$ , which denotes the time until the third bulb failure after he leaves, can be obtained as follows. Let  $X_1$  denote the time that elapses from the instant Bob leaves until the first bulb fails,  $X_2$  the time between the first bulb failure and the second bulb failure, and  $X_3$  the time between the second bulb failure and the third bulb failure. Then,  $X_1$  is exponentially distributed with parameter  $3\lambda$ ,  $X_2$  is exponentially distributed with parameter  $2\lambda$ , and  $X_3$  is exponentially distributed with parameter  $\lambda$ . That is, the PDFs of  $X_1$ ,  $X_2$ , and  $X_3$  are given, respectively, by

$$f_{X_1}(x) = 3\lambda e^{-3\lambda x} \quad x \geq 0$$

$$f_{X_2}(x) = 2\lambda e^{-2\lambda x} \quad x \geq 0$$

$$f_{X_3}(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

Thus, we have that  $Y = X_1 + X_2 + X_3$ . Because of the forgetfulness property of the underlying exponential distribution, the random variables  $X_1$ ,  $X_2$ , and  $X_3$  are independent. Therefore, the PDF of  $Y$  is the convolution of the PDFs of the three random variables. That is,

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * f_{X_3}(y)$$

- d. The expected value of  $Y$  is

$$E[Y] = E[X_1] + E[X_2] + E[X_3] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{11}{6\lambda}$$

10.31 Let  $X$  denote the lifetime of the 60-watt bulb and  $Y$  the lifetime of the 100-watt bulb. Then the PDFs of  $X$  and  $Y$  are given by

$$f_X(x) = \lambda e^{-\lambda x}$$

$$f_Y(y) = \mu e^{-\mu y}$$

$$E[X] = \frac{1}{\lambda} = 60 \Rightarrow \lambda = \frac{1}{60}$$

$$E[Y] = \frac{1}{\mu} = 100 \Rightarrow \mu = \frac{1}{100}$$

- a. The probability that the 60-watt bulb fails before the 100-watt bulb is given by

$$P[X < Y] = \frac{\lambda}{\lambda + \mu} = \frac{(1/60)}{(1/60) + (1/100)} = \frac{5}{8}$$

- b. The time until the first of the two bulbs fails is  $T = \min(X, Y)$ . Thus, the mean value of  $T$  is

$$E[T] = \frac{1}{\lambda + \mu} = \frac{1}{(1/60) + (1/100)} = \frac{600}{16} = \frac{75}{2} = 37.5$$

- c. Due to the forgetfulness property of the exponential distribution, given that the 60-watt bulb has not failed after 300 hours, the probability that it will last at least another 100 hours is given by

$$P[X \geq 100] = e^{-100\lambda} = e^{-100/60} = e^{-5/3} = 0.18887$$

10.32 The lifetime  $X$  of each motor has the PDF  $f_X(x) = \lambda e^{-\lambda x}$ ,  $x \geq 0$ ,  $\lambda > 0$ , and the lifetimes of the motors are independent. If the machine can operate properly when at least 3 of the 5 motors are functioning, then it fails when the 3rd motor fails.

This is an example of a combination of independent Poisson processes. Thus, initially the 5 motors probabilistically operate as one unit with failure rate  $\lambda_5 = 5\lambda$ . Then, after the first failure, the 4 remaining motors operate as a unit with rate  $\lambda_4 = 4\lambda$  due to the forgetfulness property of the exponential distribution, and so on until only one motor is left and the rate is  $\lambda_1 = \lambda$ . Thus, if the random variable  $Y$  is the time until the machine fails, then  $E[Y]$  is given by

$$E[Y] = \frac{1}{\lambda_5} + \frac{1}{\lambda_4} + \frac{1}{\lambda_3} = \frac{1}{5\lambda} + \frac{1}{4\lambda} + \frac{1}{3\lambda} = \frac{47}{60\lambda}$$

10.33 Let  $X$  denote the time until a PC fails. Then the PDF of  $X$  is given by

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

where  $E[X] = 1/\lambda = 50 \Rightarrow \lambda = 1/50$ . Similarly, let  $Y$  denote the time to repair a PC after it fails. Then the PDF of  $Y$  is given by

$$f_Y(y) = \mu e^{-\mu y} \quad y \geq 0$$

where  $E[Y] = 1/\mu = 3 \Rightarrow \mu = 1/3$ . Given that Alice has two identical personal computers and she uses one PC at a time and the other is a backup that is used when one fails. The probability that she is idle because neither PC is operational is the probability that the time to repair a failed PC is greater than the time until the other PC fails. Thus, if  $A$  is the event that Alice is idle, we have that

$$P[A] = P[X < Y] = \frac{\lambda}{\lambda + \mu} = \frac{(1/50)}{(1/50) + (1/3)} = \frac{3}{53} = 0.0566$$

10.34 Let the random variable  $X$  denote the times between arrivals of cars from the northbound section of the intersection. Then the PDF of  $X$  is given by

$$f_X(x) = \lambda_N e^{-\lambda_N x} \quad x \geq 0$$

Similarly, let the random variable  $Y$  denote the times between arrivals of cars from the eastbound section. Then the PDF of  $Y$  is given by

$$f_Y(y) = \lambda_E e^{-\lambda_E y} \quad y \geq 0$$

- a. Given that there is currently no car at the intersection, the probability that a northbound car arrives before an eastbound car is given by the probability that  $X$  is smaller than  $Y$ , which is

$$P[X < Y] = \frac{\lambda_N}{\lambda_N + \lambda_E}$$

- b. Given that there is currently no car at the intersection, the event that the fourth northbound car arrives before the second eastbound car can occur as follows:
1. The first 4 arrivals are northbound cars. The probability of this event is the probability that there are 4 successes in 4 Bernoulli trials, where the probability of success is  $p = \lambda_N / (\lambda_N + \lambda_E)$ . Thus, the event is defined by a binomial random variable with 4 successes and no failure.
  2. There are 3 successes in the first 4 Bernoulli trials and the 5th trial results in a success. Thus, this event is defined by the 4th-order Pascal random variable in which the 4th success occurs in the 5th trial.

Since these two events are mutually exclusive, the probability  $q$  that the fourth northbound car arrives before the second eastbound car is given by

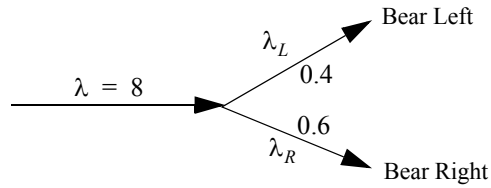
$$\begin{aligned} q &= \binom{4}{4} p^4 (1-p)^0 + \binom{5-1}{4-1} p^4 (1-p)^1 = p^4 + \binom{4}{3} p^4 (1-p) = 4p^4 (1-p) + p^4 \\ &= p^4 \{4(1-p) + 1\} = \left( \frac{\lambda_N}{\lambda_N + \lambda_E} \right)^4 \left\{ 4 \left( \frac{\lambda_E}{\lambda_N + \lambda_E} \right) + 1 \right\} \end{aligned}$$

10.35 This is an example of subdivision of a Poisson process. Let  $\lambda_R$  denote the arrival rate of cars that bear right and let  $\lambda_L$  denote the arrival rate of cars that bear left. Now,

$$\lambda_R = 0.6\lambda = 0.6 \times 8 = 4.8$$

$$\lambda_L = 0.4\lambda = 0.4 \times 8 = 3.2$$

The process is illustrated in the following figure.



- a. Let  $R$  denote the number of cars that bear right in an interval of length  $t$ . Since  $R$  is a Poisson random variable, its PMF is given by

$$p_R(r, t) = \frac{(\lambda_R t)^r e^{-\lambda_R t}}{r!} = \frac{(4.8t)^r e^{-4.8t}}{r!} \quad r = 0, 1, \dots$$

The probability that at least four cars bear right at the fork in 3 minutes is given by

$$\begin{aligned} P[R \geq 4, t = 3] &= 1 - P[R < 4, t = 3] = 1 - \{p_R(0, 3) + p_R(1, 3) + p_R(2, 3) + p_R(3, 3)\} \\ &= 1 - e^{-14.4} \left\{ 1 + 14.4 + \frac{14.4^2}{2} + \frac{14.4^3}{6} \right\} = 0.9996 \end{aligned}$$

- b. Since  $R$  and  $L$  are independent Poisson random variables, the probability that 2 cars bear left at the fork in 3 minutes, given that 3 cars bear right at the fork in 3 minutes, is simply the probability that 2 cars bear left in 3 minutes, which is given by

$$P[L = 2, t = 3 | R = 3, t = 3] = P[L = 2, t = 3] = \frac{(3\lambda_L)^2 e^{-3\lambda_L}}{2!} = \frac{(9.6)^2 e^{-9.6}}{2} = 0.00312$$

- c. Given that 10 cars arrive at the fork in three minutes, the probability that 4 of the cars bear right at the fork is given by the binomial distribution

$$P[(R = 4, t = 3), (L = 6, t = 3)] = \binom{10}{4} \left( \frac{\lambda_R}{\lambda_R + \lambda_L} \right)^4 \left( \frac{\lambda_L}{\lambda_R + \lambda_L} \right)^6 = \binom{10}{4} \left( \frac{4.8}{8} \right)^4 \left( \frac{3.2}{8} \right)^6 = \binom{10}{4} (0.6)^4 (0.4)^6 \\ = 0.1115$$

## Section 10.7: Discrete-Time Markov Chains

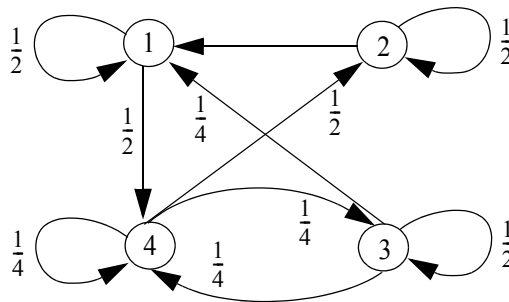
10.36 The missing elements denoted by  $x$  in the following transition probability matrix are

$$P = \begin{bmatrix} x & 1/3 & 1/3 & 1/3 \\ 1/10 & x & 1/5 & 2/5 \\ x & x & x & 1 \\ 3/5 & 2/5 & x & x \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/10 & 3/10 & 1/5 & 2/5 \\ 0 & 0 & 0 & 1 \\ 3/5 & 2/5 & 0 & 0 \end{bmatrix}$$

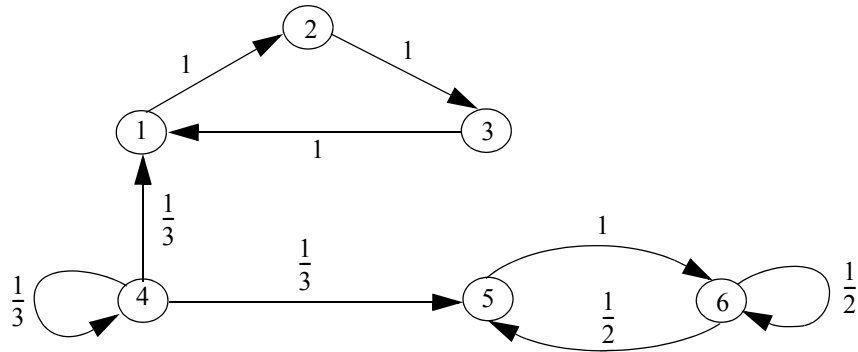
10.37 We are given the Markov chain with the following transition probability matrix

$$P = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \end{bmatrix}$$

The state transition diagram is as follows:



10.38 We are given a Markov chain with the following state-transition diagram.



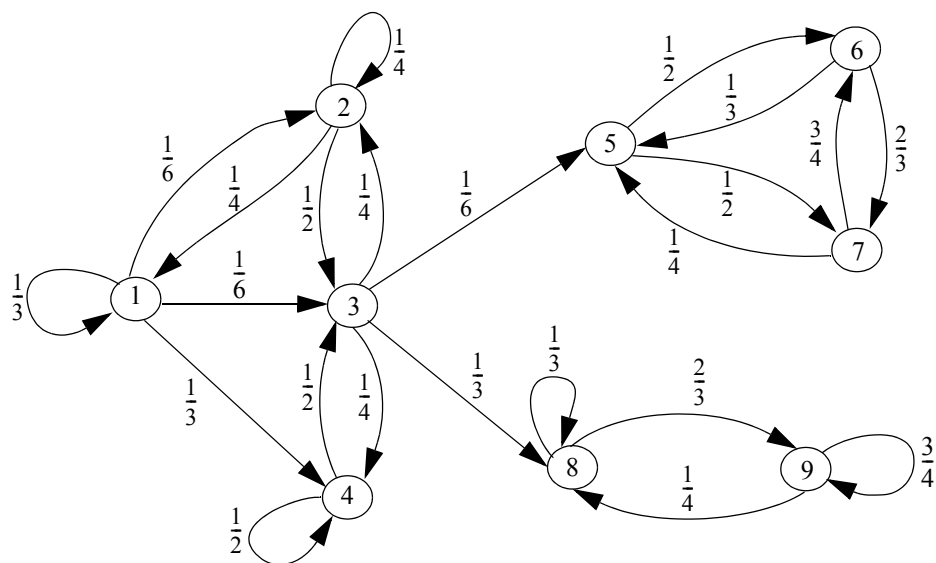
a. The transition probability matrix is given by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

b. Recurrent states:  $\{1, 2, 3, 5, 6\}$

c. The only transient state is  $\{4\}$

10.39 We are given the Markov chain with the following state-transition diagram.



- a. Transient states:  $\{1, 2, 3, 4\}$   
 Recurrent states:  $\{5, 6, 7, 8, 9\}$   
 Periodic states: None
- b. There are 2 chains of recurrent states, which are
  1. Chain 1:  $\{5, 6, 7\}$
  2. Chain 2:  $\{8, 9\}$
- c. The transition probability matrix of the process is given by



$$P = \begin{bmatrix} 1/3 & 1/6 & 1/6 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 1/6 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 3/4 \end{bmatrix}$$

- d. Given that the process starts in state 1, let  $A$  denote the event that the process leaves the transient states  $\{1, 2, 3, 4\}$ . Given event  $A$ , the probability that the process enters the chain  $\{8, 9\}$  is given by

$$P[1 \rightarrow 8|A] = \frac{1/3}{1/3 + 1/6} = \frac{2}{3}$$

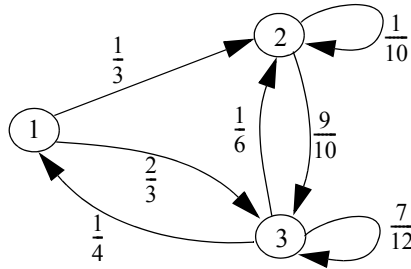
After entering the chain  $\{8, 9\}$  the limiting probability that it is in state 8 can be obtained as follows. Given that the process is in chain  $\{8, 9\}$ , let  $\pi_k$  denote the limiting-state probability that the process is in state  $k$ ,  $k = 8, 9$ .

$$\begin{aligned} \pi_8 &= \frac{1}{3}\pi_8 + \frac{1}{4}\pi_9 \Rightarrow \frac{2}{3}\pi_8 = \frac{1}{4}\pi_9 \Rightarrow \pi_9 = \frac{8}{3}\pi_8 \\ 1 &= \pi_8 + \pi_9 = \pi_8\left(1 + \frac{8}{3}\right) \Rightarrow \frac{11}{3}\pi_8 = 1 \Rightarrow \pi_8 = \frac{3}{11} \end{aligned}$$

Thus, given that the process starts in state 1, the probability that it is in state 8 after an infinitely large number of transitions is the probability that it enters the chain  $\{8, 9\}$  multiplied by the limiting state probability of its being in state 8 once it enters that chain. That is, this probability exists and is equal to

$$\frac{2}{3} \times \frac{3}{11} = \frac{2}{11}$$

10.40 For the following three-state Markov chain



we have that

- a. Transient states: None  
 Recurrent states:  $\{1, 2, 3\}$   
 Periodic states: None  
 Chain of recurrent states: 1 chain:  $\{1, 2, 3\}$
- b. Since the process is an irreducible and aperiodic Markov chain, the limiting-state probabilities exist and can be obtained as follows. Let  $\pi_k$  denote the limiting-state probability that the process is in state  $k$ ,  $k = 1, 2, 3$ .

$$\pi_1 = \frac{1}{4}\pi_3 \Rightarrow \pi_3 = 4\pi_1$$

$$\pi_2 = \frac{1}{3}\pi_1 + \frac{1}{10}\pi_2 + \frac{1}{6}\pi_3 \Rightarrow \frac{9}{10}\pi_2 = \frac{1}{3}\pi_1 + \frac{4}{6}\pi_1 = \pi_1 \Rightarrow \pi_2 = \frac{10}{9}\pi_1$$

$$1 = \pi_1 + \pi_2 + \pi_3 = \pi_1 + \frac{10}{9}\pi_1 + 4\pi_1 = \frac{55\pi_1}{9} \Rightarrow \pi_1 = \frac{9}{55}$$

$$\pi_1 = \frac{9}{55}$$

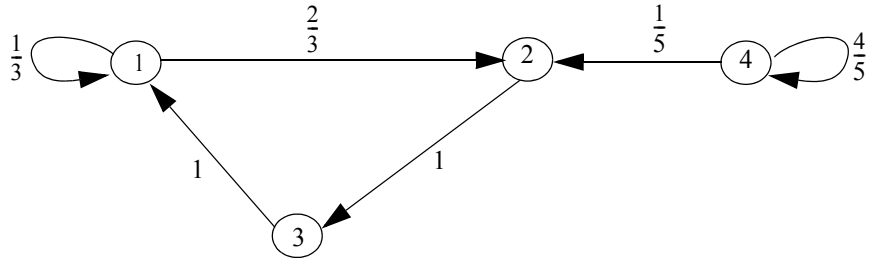
$$\pi_2 = \frac{10}{9} \times \frac{9}{55} = \frac{2}{11}$$

$$\pi_3 = 4\left(\frac{9}{55}\right) = \frac{36}{55}$$

- c. Given that the process is currently in state 1, the probability  $P[A]$  that it will be in state 3 at least once during the next two transitions is given by

$$\begin{aligned}
P[A] &= P[(1 \rightarrow 3 \rightarrow 2) \cup (1 \rightarrow 3 \rightarrow 3) \cup (1 \rightarrow 2 \rightarrow 3)] \\
&= P[1 \rightarrow 3 \rightarrow 2] + P[1 \rightarrow 3 \rightarrow 3] + P[1 \rightarrow 2 \rightarrow 3] \\
&= \left(\frac{2}{3}\right)\left(\frac{1}{6}\right) + \left(\frac{2}{3}\right)\left(\frac{7}{12}\right) + \left(\frac{1}{3}\right)\left(\frac{9}{10}\right) = \frac{1}{9} + \frac{7}{18} + \frac{3}{10} = \frac{4}{5}
\end{aligned}$$

10.41 For the following Markov chain



- Transient states:  $\{4\}$
- Periodic states: None
- State 3 is a recurrent state that belongs to the only chain of recurrent states, which is  $\{1, 2, 3\}$ . Therefore, it has a limiting-state probability, which can be determined as follows. Let  $\pi_k$  denote the limiting-state probability that the process is in state  $k$ ,  $k = 1, 2, 3, 4$ .

$$\pi_1 = \frac{1}{3}\pi_1 + \pi_3 \Rightarrow \pi_3 = \frac{2}{3}\pi_1$$

$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{5}\pi_4$$

$$\pi_3 = \pi_2$$

$$\pi_4 = \frac{4}{5}\pi_4 \Rightarrow \pi_4 = 0$$

$$1 = \pi_1 + \pi_2 + \pi_3 + \pi_4 = \pi_1 + \frac{2}{3}\pi_1 + \frac{2}{3}\pi_1 = \frac{7}{3}\pi_1 \Rightarrow \pi_1 = \frac{3}{7}$$

$$\pi_3 = \pi_2 = \left(\frac{2}{3}\right)\pi_1 = \frac{2}{3} \times \frac{3}{7} = \frac{2}{7}$$

- d. Assuming that the process begins in state 4, let  $X$  denote the number of trials up to and including the trial in which the process enters state 2 for the first time. Then  $X$  is a geometrically distributed random variable with success probability  $p = 1/5$  and PMF

$$p_X(x) = p(1-p)^{x-1} = \frac{1}{5}\left(\frac{4}{5}\right)^{x-1} \quad x = 1, 2, \dots$$

When the process leaves state 2, it takes exactly 2 trials to enter state 1. Given that it has just entered state 1, let  $Y$  denote the number of trials up to and including that in which it enters state 2. Then  $Y$  is a geometrically distributed random variable with success probability  $q = 2/3$  and PMF

$$p_Y(y) = q(1-q)^{y-1} = \frac{2}{3}\left(\frac{1}{3}\right)^{y-1} \quad y = 1, 2, \dots$$

Thus,  $K$ , the number of trials up to and including the trial in which the process enters state 2 for the *second* time, is given by  $K = X + 2 + Y$ . Since  $X$  and  $Y$  are independent random variables, we have that the z-transform of  $K$  is given by

$$\begin{aligned} G_K(z) &= E[z^K] = E[z^{X+2+Y}] = E[z^2]E[z^X]E[z^Y] = z^2 G_X(z)G_Y(z) \\ &= z^2 \left\{ \frac{z(1/5)}{1 - \frac{4}{5}z} \right\} \left\{ \frac{z(2/3)}{1 - \frac{1}{3}z} \right\} = \frac{2z^4}{(5-4z)(3-z)} \end{aligned}$$

#### 10.42 The transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

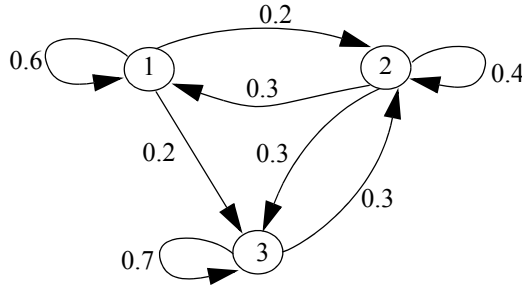
is a doubly stochastic matrix. Thus, the limiting state probabilities are

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

10.43 Consider the following transition probability matrix:

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$

a. The state-transition diagram is given by



b. Given that the process is currently in state 1, the probability that it will be in state 2 at the end of the third transition,  $p_{12}(3)$ , can be obtained as follows:

$$\begin{aligned}
 p_{12}(3) &= P[1 \rightarrow 1 \rightarrow 1 \rightarrow 2] + P[1 \rightarrow 1 \rightarrow 2 \rightarrow 2] + P[1 \rightarrow 1 \rightarrow 3 \rightarrow 2] + P[1 \rightarrow 3 \rightarrow 2 \rightarrow 2] + \\
 &\quad P[1 \rightarrow 3 \rightarrow 3 \rightarrow 2] + P[1 \rightarrow 2 \rightarrow 3 \rightarrow 2] + P[1 \rightarrow 2 \rightarrow 2 \rightarrow 2] + P[1 \rightarrow 2 \rightarrow 1 \rightarrow 2] \\
 &= (0.6)^2(0.2) + (0.6)(0.2)(0.4) + (0.6)(0.2)(0.3) + (0.2)(0.7)(0.3) + (0.2)(0.3)(0.4) + \\
 &\quad (0.2)(0.3)^2 + (0.2)(0.4)^2 + (0.2)^2(0.3) \\
 &= 0.284
 \end{aligned}$$

Another way to obtain  $p_{12}(3)$  is that it is the entry on row 1 and column 2 of the matrix  $P^3$ , which is given by

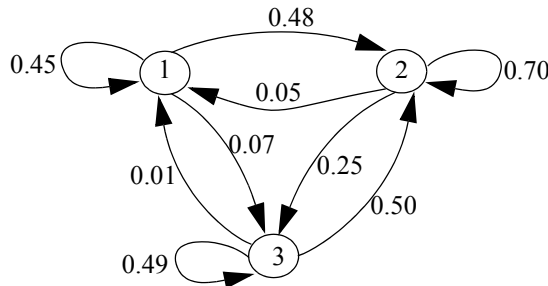
$$P^3 = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}^3 = \begin{bmatrix} 0.330 & 0.284 & 0.224 \\ 0.273 & 0.301 & 0.426 \\ 0.153 & 0.324 & 0.523 \end{bmatrix}$$

- c. Given that the process is currently in state 1, the probability  $f_{13}(4)$  that the first time it enters state 3 is the fourth transition is given by

$$\begin{aligned} f_{13}(4) &= P[1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 3] + P[1 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 3] + P[1 \rightarrow 1 \rightarrow 2 \rightarrow 2 \rightarrow 3] + \\ &\quad P[1 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 3] + P[1 \rightarrow 2 \rightarrow 1 \rightarrow 1 \rightarrow 3] + P[1 \rightarrow 2 \rightarrow 2 \rightarrow 1 \rightarrow 3] + \\ &\quad P[1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3] + P[1 \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow 3] \\ &= (0.6)^3(0.2) + (0.6)^2(0.2)(0.3) + (0.6)(0.2)(0.4)(0.3) + (0.6)(0.2)(0.3)(0.2) + \\ &\quad (0.2)(0.3)(0.6)(0.2) + (0.2)(0.4)(0.3)(0.2) + (0.2)(0.3)(0.2)(0.3) + (0.2)(0.4)^2(0.3) \\ &= 0.1116 \end{aligned}$$

10.44 The process operates as follows. Given that a person is raised in state 1, he will enter state 1 with probability 0.45, state 2 with probability 0.48, and state 3 with probability 0.07. Given that a person is in state 2, he will enter state 1 with probability 0.05, state 2 with probability 0.70, and state 3 with probability 0.25. Finally, given that a person is in state 3, he will enter state 1 with probability 0.01, state 2 with probability 0.50, and state 3 with probability 0.49.

- a. The state-transition diagram of the process is given by the following:



- 
- b. The transition probability matrix of the process is given by

$$P = \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}$$

- c. The limiting-state probabilities can be obtained as follows. Let  $\pi_k$  denote the limiting-state probability that the process is in state  $k$ ,  $k = 1, 2, 3$ .

$$\pi_1 = 0.45\pi_1 + 0.05\pi_2 + 0.01\pi_3 \Rightarrow 0.55\pi_1 = 0.05\pi_2 + 0.01\pi_3$$

$$\pi_2 = 0.48\pi_1 + 0.70\pi_2 + 0.50\pi_3 \Rightarrow 48\pi_1 = 0.30\pi_2 - 50\pi_3$$

$$1 = \pi_1 + \pi_2 + \pi_3$$

The solution to the above system of equations is

$$\pi_1 = 0.057$$

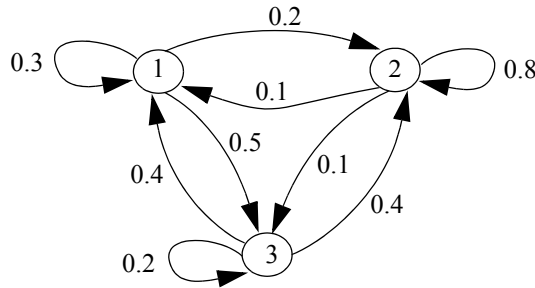
$$\pi_2 = 0.555$$

$$\pi_3 = 0.388$$

This result can be interpreted as follows to the layperson. On the long run, 5.7% of the population will be in the upper class, 55.5% of the population will be in the middle class, and 38.8% of the population will be in the lower class.

- 10.45 The model is equivalent to the following. Given that the process is in state 1, it will enter state 1 with probability 0.3, state 2 with probability 0.2, and state 3 with probability 0.5. Similarly, given that the process is in state 2, it will enter state 1 with probability 0.1, state 2 with probability 0.8, and state 3 with probability 0.1. Finally, given that the process is in state 3, it will enter state 1 with probability 0.4, state 2 with probability 0.4, and state 3 with probability 0.2.

- a. The state-transition diagram for the process is as follows:



- b. The transition probability matrix for the process is the following:

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}$$

- c. The limiting-state probabilities can be obtained as follows. Let  $\pi_k$  denote the limiting-state probability that the process is in state  $k$ ,  $k = 1, 2, 3$ .

$$\pi_1 = 0.3\pi_1 + 0.1\pi_2 + 0.4\pi_3 \Rightarrow \pi_1 = \frac{0.1\pi_2 + 0.4\pi_3}{0.7} = \frac{\pi_2 + 4\pi_3}{7}$$

$$\pi_2 = 0.2\pi_1 + 0.8\pi_2 + 0.4\pi_3 \Rightarrow \pi_1 = \frac{0.2\pi_2 - 0.4\pi_3}{0.2} = \pi_2 - 2\pi_3$$

$$1 = \pi_1 + \pi_2 + \pi_3$$

The solution to the above system of equations is

$$\pi_1 = 0.2$$

$$\pi_2 = 0.6$$

$$\pi_3 = 0.2$$

- d. Given that the taxi driver is currently in town 2 and is waiting to pick up his first customer for the day, the probability that the first time he picks up a passenger to town 2 is when he picks up his third passenger for the day is  $f_{22}(3)$ , which is given by



$$\begin{aligned}
f_{22}(3) &= P[2 \rightarrow 1 \rightarrow 1 \rightarrow 2] + P[2 \rightarrow 1 \rightarrow 3 \rightarrow 2] + P[2 \rightarrow 3 \rightarrow 3 \rightarrow 2] + P[2 \rightarrow 3 \rightarrow 1 \rightarrow 2] \\
&= (0.1)(0.3)(0.2) + (0.1)(0.5)(0.4) + (0.1)(0.2)(0.4) + (0.1)(0.4)(0.2) \\
&= 0.042
\end{aligned}$$

- e. Given that he is currently in town 2, the probability that his third passenger from now will be going to town 1 is  $p_{21}(3)$ , which is given by

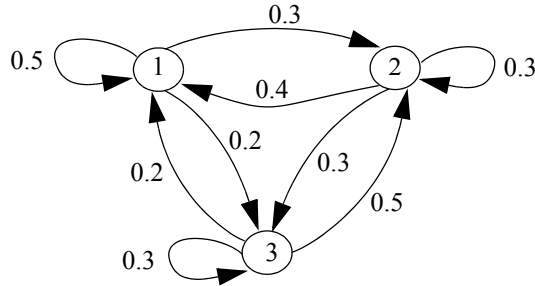
$$\begin{aligned}
p_{21}(3) &= P[2 \rightarrow 2 \rightarrow 2 \rightarrow 1] + P[2 \rightarrow 2 \rightarrow 1 \rightarrow 1] + P[2 \rightarrow 2 \rightarrow 3 \rightarrow 1] + P[2 \rightarrow 1 \rightarrow 3 \rightarrow 1] + \\
&\quad P[2 \rightarrow 1 \rightarrow 1 \rightarrow 1] + P[2 \rightarrow 3 \rightarrow 2 \rightarrow 1] + P[2 \rightarrow 3 \rightarrow 3 \rightarrow 1] + P[2 \rightarrow 3 \rightarrow 1 \rightarrow 1] + \\
&\quad P[2 \rightarrow 1 \rightarrow 2 \rightarrow 1] \\
&= (0.8)^2(0.1) + (0.8)(0.1)(0.3) + (0.8)(0.1)(0.4) + (0.1)(0.5)(0.4) + (0.1)(0.3)^2 + \\
&\quad (0.1)(0.4)(0.1) + (0.1)(0.2)(0.4) + (0.1)(0.4)(0.3) + (0.1)(0.2)(0.1) \\
&= 0.175
\end{aligned}$$

Note that  $p_{21}(3)$  can also be obtained from the entry in the first column of the second row of the matrix  $P^3$  as follows:

$$P^3 = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}^3 = \begin{bmatrix} 0.243 & 0.506 & 0.251 \\ 0.175 & 0.650 & 0.175 \\ 0.232 & 0.544 & 0.224 \end{bmatrix}$$

- 10.46 New England fall weather can be classified as sunny (state 1), cloudy (state 2), or rainy (state 3). The transition probabilities are as follows: Given that it is sunny on any given day, then on the following day it will be sunny again with probability 0.5, cloudy with probability 0.3, and rainy with probability 0.2. Given that it is cloudy on any given day, then on the following day it will be sunny with probability 0.4, cloudy again with probability 0.3, and rainy with probability 0.3. Finally, given that it is rainy on any given day, then on the following day it will be sunny with probability 0.2, cloudy with probability 0.5, and rainy again with probability 0.3.

- a. Thus, the state-transition diagram of New England fall weather is given by



- b. The transition probability matrix of New England fall weather is given by

$$P = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

- c. Given that it is sunny today (i.e., in state 1), the probability that it will be sunny four days from now is  $p_{11}(4)$ , which is obtained from the entry in the first row and first column of the matrix  $P^4$ , where

$$P^4 = \begin{bmatrix} 0.3873 & 0.3518 & 0.2609 \\ 0.3862 & 0.3524 & 0.2614 \\ 0.3852 & 0.3528 & 0.2620 \end{bmatrix}$$

Thus, the required probability is  $p_{11}(4) = 0.3873$ .

- d. The limiting-state probabilities of the weather can be obtained as follows. Let  $\pi_k$  denote the limiting-state probability that the process is in state  $k$ ,  $k = 1, 2, 3$ .

$$\pi_1 = 0.5\pi_1 + 0.4\pi_2 + 0.2\pi_3 \Rightarrow \pi_1 = 0.8\pi_2 + 0.2\pi_3$$

$$\pi_2 = 0.3\pi_1 + 0.3\pi_2 + 0.5\pi_3 \Rightarrow \pi_1 = \frac{7}{3}\pi_2 - \frac{5}{3}\pi_3$$

$$1 = \pi_1 + \pi_2 + \pi_3$$

From the above system of equations we obtain the solution

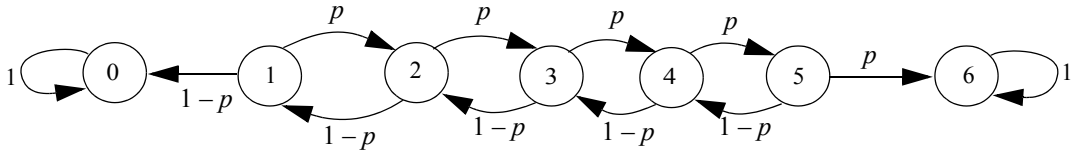
$$\pi_1 = \frac{34}{88} = 0.3863$$

$$\pi_2 = \frac{31}{88} = 0.3523$$

$$\pi_3 = \frac{23}{88} = 0.2614$$

10.47 Let state  $k$  denote the event that the student currently has a total of  $\$k$ ,  $k = 0, 1, \dots, 6$ .

a. The state-transition diagram of the process is given by



b. We know that the ruin probability  $r_k$  for a player that starts with  $\$k$  is given by

$$r_k = \begin{cases} \frac{[(1-p)/p]^k - [(1-p)/p]^N}{1 - [(1-p)/p]^N} & p \neq \frac{1}{2} \\ \frac{N-k}{N} & p = \frac{1}{2} \end{cases}$$

Thus, when  $N = 6$  and  $k = 3$  we obtain

$$r_3 = \begin{cases} \frac{[(1-p)/p]^3 - [(1-p)/p]^6}{1 - [(1-p)/p]^6} & p \neq \frac{1}{2} \\ \frac{1}{2} & p = \frac{1}{2} \end{cases}$$

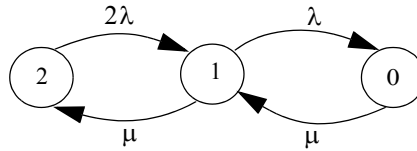
c. The probability that he stops after he has doubled his original amount is the probability that he is not ruined, which is given by

$$r_6 = 1 - r_3$$

## Section 10.8: Continuous-Time Markov Chains

10.48 Let state  $k$  denote the number of operational PCs.

- a. Since each PC fails independently of the other, the failure rate when both PCs are operational is  $2\lambda$ . Thus, the state-transition-rate diagram of the process is given by



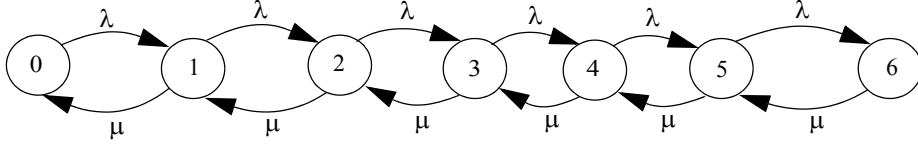
- b. Let  $p_k$  denote the limiting state probability that the process is in state  $k$ ,  $k = 0, 1, 2$ . Then the fraction of time that both machines are down,  $p_0$ , can be found by using local balance equations as follows:

$$\begin{aligned}
 2\lambda p_2 &= \mu p_1 \Rightarrow p_2 = \frac{\mu}{2\lambda} p_1 = \frac{1}{2\rho} p_1 \\
 \lambda p_1 &= \mu p_0 \Rightarrow p_1 = \frac{\mu}{\lambda} p_0 = \frac{1}{\rho} p_0 \Rightarrow p_2 = \frac{1}{2\rho^2} p_0 \\
 1 &= p_0 + p_1 + p_2 = p_0 \left\{ 1 + \frac{1}{\rho} + \frac{1}{2\rho^2} \right\} \\
 p_0 &= \frac{1}{1 + \frac{1}{\rho} + \frac{1}{2\rho^2}} = \frac{2\rho^2}{1 + 2\rho + 2\rho^2}
 \end{aligned}$$

where  $\rho = \lambda/\mu$ .

10.49 Let the state  $k$  denote the number of chairs that are occupied by customers, including the chair that the customer who is currently receiving a haircut is sitting on. Thus,  $k$  has the values  $k = 0, 1, \dots, 6$ .

- a. The state-transition-rate diagram of the process is given by



- b. Let  $p_k$  denote the limiting-state probability that the process is in state  $k$ , and let the parameter  $\rho = \lambda/\mu$ . Then from local balance we obtain the following results:

$$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0 = \rho p_0$$

$$\lambda p_1 = \mu p_2 \Rightarrow p_2 = \frac{\lambda}{\mu} p_1 = \rho^2 p_0$$

In general it can be shown that  $p_k = \rho^k p_0$ ,  $k = 0, 1, \dots, 6$ . Assuming that  $\rho < 1$ , then from total probability we have that

$$\sum_{k=0}^6 p_k = p_0 \sum_{k=0}^6 \rho^k = \frac{p_0(1-\rho^7)}{1-\rho} = 1 \Rightarrow p_0 = \frac{1-\rho}{1-\rho^7}$$

Thus, the probability that there are three waiting customers in the shop is the probability that the process is in state  $k = 4$ , which is given by

$$p_4 = \rho^4 p_0 = \frac{\rho^4(1-\rho)}{1-\rho^7}$$

- c. The probability that an arriving customer leaves without receiving a haircut is the probability that there is no available chair when the customer arrives, which is the probability that the process is in state  $k = 6$ , which is given by

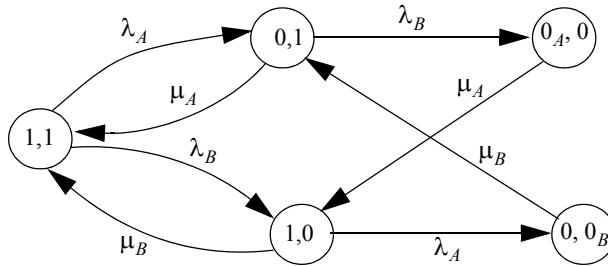
$$p_6 = \rho^6 p_0 = \frac{\rho^6(1-\rho)}{1-\rho^7}$$

- d. The probability that an arriving customer does not have to wait is the probability that the customer found the place empty, which is the probability that the process is in state  $k = 0$  and is given by

$$p_0 = \frac{1 - \rho}{1 - \rho^7}$$

10.50 Let the state of the process be denoted by the pair  $(a, b)$ , where  $a = 1$  if machine A is up and  $a = 0$  otherwise, and  $b = 1$  if machine B is up and  $b = 0$  otherwise. Also, let the state  $(0_A, 0)$  be the state in which both machines are down but machine A failed first and was being repaired when machine B failed. Similarly, let the state  $(0, 0_B)$  be the state in which both machines are down but machine B failed first and was being repaired when machine A failed.

- a. The state-transition-rate diagram of the process is given by



- b. Let  $p_k$  denote the limiting-state probability that the process is in state  $k$ . If we define  $\rho_A = \lambda_A/\mu_A$  and  $\rho_B = \lambda_B/\mu_B$ , then from local balance we have that

$$\begin{aligned}
p_{1,1}\lambda_A &= \mu_A p_{0,1} \Rightarrow p_{0,1} = \frac{\lambda_A}{\mu_A} p_{1,1} = \rho_A p_{1,1} \\
p_{1,1}\lambda_B &= \mu_B p_{1,0} \Rightarrow p_{1,0} = \frac{\lambda_B}{\mu_B} p_{1,1} = \rho_B p_{1,1} \\
p_{1,0}\lambda_A &= \mu_B p_{0,0_B} \Rightarrow p_{0,0_B} = \frac{\lambda_A}{\mu_B} p_{1,0} = \frac{\lambda_A}{\mu_B} \rho_B p_{1,1} \\
p_{0,1}\lambda_B &= \mu_A p_{0_A,0} \Rightarrow p_{0_A,0} = \frac{\lambda_B}{\mu_A} p_{0,1} = \frac{\lambda_B}{\mu_A} \rho_A p_{1,1}
\end{aligned}$$

From total probability we obtain

$$\begin{aligned}
1 &= p_{1,1} + p_{1,0} + p_{0,1} + p_{0_A,0} + p_{0,0_B} = p_{1,1} \left\{ 1 + \rho_A + \rho_B + \frac{\rho_A \lambda_B}{\mu_A} + \frac{\rho_B \lambda_A}{\mu_B} \right\} \\
p_{1,1} &= \frac{1}{1 + \rho_A + \rho_B + \frac{\rho_A \lambda_B}{\mu_A} + \frac{\rho_B \lambda_A}{\mu_B}} = \frac{\mu_A \mu_B}{1 + \mu_B \lambda_A + \mu_A \lambda_B + \rho_A \lambda_B \mu_B + \rho_B \lambda_A \mu_A}
\end{aligned}$$

Thus, the probability that both PCs are down is given by

$$\begin{aligned}
p_{0_A,0} + p_{0,0_B} &= p_{1,1} \left\{ \frac{\lambda_B}{\mu_A} \rho_A + \frac{\lambda_A}{\mu_B} \rho_B \right\} = p_{1,1} \left\{ \frac{\lambda_A \lambda_B}{\mu_A^2} + \frac{\lambda_A \lambda_B}{\mu_B^2} \right\} = p_{1,1} \left\{ \frac{\lambda_A \lambda_B \{\mu_A^2 + \mu_B^2\}}{\mu_A^2 \mu_B^2} \right\} \\
&= \frac{\lambda_A \lambda_B \{\mu_A^2 + \mu_B^2\}}{\mu_A \mu_B \{1 + \mu_B \lambda_A + \mu_A \lambda_B + \rho_A \lambda_B \mu_B + \rho_B \lambda_A \mu_A\}}
\end{aligned}$$

- c. The probability that PC A is the first to fail given that both PCs have failed is the probability that the process is in state  $(0, 0_A)$  given that both machines have failed and is given by

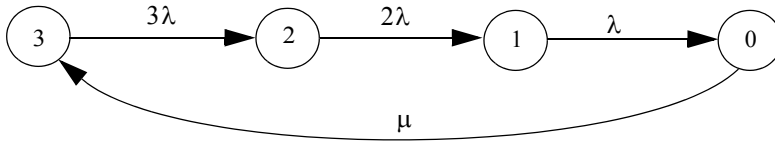
$$\frac{p_{0_A,0}}{p_{0_A,0} + p_{0,0_B}} = \frac{\frac{\lambda_B}{\mu_A} \rho_A p_{1,1}}{\frac{\lambda_B}{\mu_A} \rho_A p_{1,1} + \frac{\lambda_A}{\mu_B} \rho_B p_{1,1}} = \frac{\frac{\lambda_A \lambda_B}{\mu_A^2}}{\frac{\lambda_A \lambda_B}{\mu_A^2} + \frac{\lambda_A \lambda_B}{\mu_B^2}} = \frac{\mu_B^2}{\mu_A^2 + \mu_B^2}$$

- d. The probability that both PCs are up is the probability that the process is in state  $(1, 1)$  and is given by

$$p_{1,1} = \frac{\mu_A \mu_B}{1 + \mu_B \lambda_A + \mu_A \lambda_B + \rho_A \lambda_B \mu_B + \rho_B \lambda_A \mu_A}$$

10.51 Let state  $k$  denote the number of lightbulbs that have not failed.

- a. The state-transition-rate diagram of the process is given by



- b. Let  $p_k$  denote the limiting-state probability that the process is in state  $k$ . Then from global balance we obtain

$$3\lambda p_3 = \mu p_0 \Rightarrow p_0 = \frac{3\lambda}{\mu} p_3$$

$$3\lambda p_3 = 2\lambda p_2 \Rightarrow p_2 = \frac{3}{2} p_3$$

$$2\lambda p_2 = \lambda p_1 \Rightarrow p_1 = 2p_2 = 3p_3$$

$$1 = p_3 + p_2 + p_1 + p_0 = p_3 \left\{ 1 + 1.5 + 3 + \frac{3\lambda}{\mu} \right\} = p_3 \{ 5.5 + 3\rho \}$$

$$p_3 = \frac{1}{5.5 + 3\rho}$$

where  $\rho = \lambda/\mu$ . Thus, the probability that only one lightbulb is working is

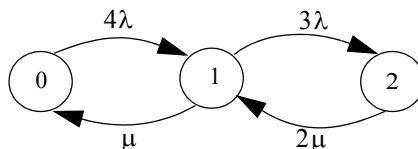
$$p_1 = 3p_3 = \frac{3}{5.5 + 3\rho}$$

- c. The probability that all three lightbulbs are working is  $p_3 = 1/(5.5 + 3\rho)$ .



10.52 Let  $k$  denote the state in which  $k$  lines are busy,  $k = 0, 1, 2$ .

a. The state-transition-rate diagram of the process is given by



b. The fraction of time that the switchboard is blocked is the limiting-state probability that the process is in state 2, which can be obtained as follows. Let  $p_k$  denote the limiting-state probability that the process is in state  $k$ . If we define  $\rho = \lambda/\mu$ , then from local balance we have that

$$4\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{4\lambda}{\mu} p_0 = 4\rho p_0$$

$$3\lambda p_1 = 2\mu p_2 \Rightarrow p_2 = \frac{3\lambda}{2\mu} p_1 = 6\rho^2 p_0$$

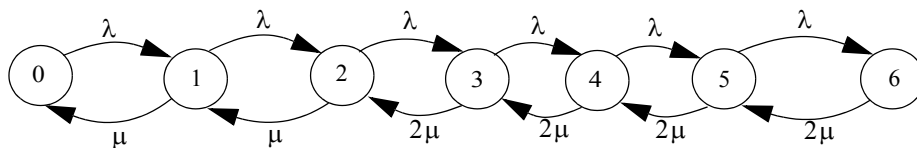
$$1 = p_0 + p_1 + p_2 = p_0 \{1 + 4\rho + 6\rho^2\} \Rightarrow p_0 = \frac{1}{1 + 4\rho + 6\rho^2}$$

Thus, the fraction of time that the switchboard is blocked is

$$p_2 = 6\rho^2 p_0 = \frac{6\rho^2}{1 + 4\rho + 6\rho^2}$$

10.53 Let  $k$  denote the number of customers at the service facility, where  $k = 0, 1, \dots, 6$ .

a. The state-transition-rate diagram of the process is given by



b. Let  $p_k$  denote the limiting-state probability that the process is in state  $k$ . If we define  $\rho = \lambda/\mu$ , then from local balance we obtain

$$\begin{aligned}\lambda p_0 &= \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0 = \rho p_0 \\ \lambda p_1 &= \mu p_2 \Rightarrow p_2 = \frac{\lambda}{\mu} p_1 = \rho^2 p_0 \\ \lambda p_2 &= 2\mu p_3 \Rightarrow p_3 = \frac{\lambda}{2\mu} p_2 = \frac{\rho^3}{2} p_0 \\ \lambda p_3 &= 2\mu p_4 \Rightarrow p_4 = \frac{\lambda}{2\mu} p_3 = \frac{\rho^4}{4} p_0 \\ \lambda p_4 &= 2\mu p_5 \Rightarrow p_5 = \frac{\lambda}{2\mu} p_4 = \frac{\rho^5}{8} p_0 \\ \lambda p_5 &= 2\mu p_6 \Rightarrow p_6 = \frac{\lambda}{2\mu} p_5 = \frac{\rho^6}{16} p_0\end{aligned}$$

From total probability we have that

$$\begin{aligned}1 &= \sum_{k=0}^6 p_k = p_0 \left\{ 1 + \rho + \rho^2 + \frac{\rho^3}{2} + \frac{\rho^4}{4} + \frac{\rho^5}{8} + \frac{\rho^6}{16} \right\} \\ p_0 &= \frac{1}{1 + \rho + \rho^2 + \frac{\rho^3}{2} + \frac{\rho^4}{4} + \frac{\rho^5}{8} + \frac{\rho^6}{16}} = \frac{16}{16(1 + \rho + \rho^2) + 8\rho^3 + 4\rho^4 + 2\rho^5 + \rho^6}\end{aligned}$$

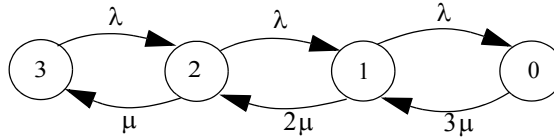
Thus, the probability  $q$  that both attendants are busy attending to customers is 1 minus the probability that at least one attendant is idle, which is given by

$$q = 1 - \{p_0 + p_1 + p_2\} = 1 - \frac{1 + \rho + \rho^2}{1 + \rho + \rho^2 + \frac{\rho^3}{2} + \frac{\rho^4}{4} + \frac{\rho^5}{8} + \frac{\rho^6}{16}} = \frac{16(1 + \rho + \rho^2)}{16(1 + \rho + \rho^2) + 8\rho^3 + 4\rho^4 + 2\rho^5 + \rho^6}$$

- c. The probability that neither attendant is busy is given by  $p_0$ , which is given above.

10.54 Let  $k$  denote the number of taxis waiting at the station, where  $k = 0, 1, 2, 3$ .

- a. The state-transition-rate diagram of the process is given as follows:



- b. Let  $p_k$  denote the limiting-state probability that the process is in state  $k$ . If we define  $\rho = \lambda/\mu$ , then from local balance we obtain

$$\lambda p_3 = \mu p_2 \Rightarrow p_2 = \frac{\lambda}{\mu} p_3 = \rho p_3$$

$$\lambda p_2 = 2\mu p_1 \Rightarrow p_1 = \frac{\lambda}{2\mu} p_2 = \frac{\rho^2}{2} p_3$$

$$\lambda p_1 = 3\mu p_0 \Rightarrow p_0 = \frac{\lambda}{3\mu} p_1 = \frac{\rho^3}{6} p_3$$

From total probability we have that

$$1 = p_3 + p_2 + p_1 + p_0 = p_3 \left\{ 1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{6} \right\}$$

$$p_3 = \frac{1}{1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{6}} = \frac{6}{6 + 6\rho + 3\rho^2 + \rho^3}$$

Thus, the probability that an arriving customer sees exactly one taxi at the station is the limiting-state probability that the process is in state 1, which is given by

$$p_1 = \frac{\rho^2}{2} p_3 = \frac{(\rho^2/2)6}{6 + 6\rho + 3\rho^2 + \rho^3} = \frac{3\rho^2}{6 + 6\rho + 3\rho^2 + \rho^3}$$

- c. The probability that an arriving customer goes to another taxicab company is the probability that the process is in state 0, which is given by

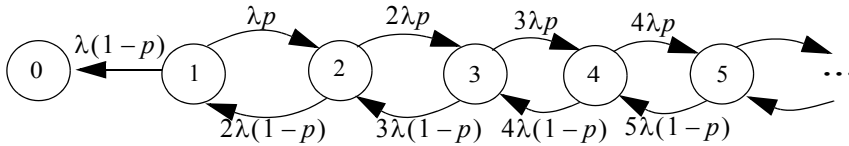
$$p_0 = \frac{\rho^3}{6} p_3 = \frac{(\rho^3/6)6}{6 + 6\rho + 3\rho^2 + \rho^3} = \frac{\rho^3}{6 + 6\rho + 3\rho^2 + \rho^3}$$

10.55 Let  $X(t) = k$  denote the state at time  $t$ .

- a. When  $k = 1$ , the particle splits with rate  $\lambda p$  and disappears with rate  $\lambda(1-p)$ . When  $k = n$ , there are  $n$  particles, each of which is acting independently. Therefore, the birth and death rates of the process are given, respectively, by

$$\begin{aligned} b_k &= k\lambda p & k &= 1, 2, \dots \\ d_k &= k\lambda(1-p) & k &= 1, 2, \dots \end{aligned}$$

- b. The state-transition-rate diagram of the process is as follows:



## Section 11.2: Sampling Theory

11.1 A sample size of 5 results in the sample values 9, 7, 1, 4, and 6.

- a. The sample mean is  $\bar{X} = \frac{9+7+1+4+6}{5} = \frac{27}{5} = 5.4$
- b. The sample variance is given by

$$\begin{aligned} S^2 &= \frac{1}{5} \sum_{k=1}^5 (X_k - 5.4)^2 = \frac{1}{5} \{ (9 - 5.4)^2 + (7 - 5.4)^2 + (1 - 5.4)^2 + (4 - 5.4)^2 + (6 - 5.4)^2 \} \\ &= \frac{1}{5} \{ (3.6)^2 + (1.6)^2 + (-4.4)^2 + (-1.4)^2 + (0.6)^2 \} = \frac{1}{5} \{ 12.96 + 2.56 + 19.36 + 1.96 + 0.36 \} \\ &= \frac{37.20}{5} = 7.44 \end{aligned}$$

- c. The unbiased estimate of the sample variance is given by

$$\hat{S}^2 = \frac{n}{n-1} S^2 = \left( \frac{5}{4} \right) (7.44) = 9.3$$

11.2 Given that true mean and true variance are

$$\begin{aligned} \sigma_X^2 &= 144 \\ \mu_X &= 70 \end{aligned}$$

it is desired to estimate the mean by sampling a subset of the scores, without replacement.

- a. The standard deviation of the sample mean when only 10 scores are used can be obtained as follows:

$$\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{10} \left( \frac{50-10}{50-1} \right) = \frac{144}{10} \left( \frac{40}{49} \right) = \frac{576}{49} \Rightarrow \sigma_{\bar{X}} = \sqrt{\frac{576}{49}} = \frac{24}{7} = 3.428$$

- b. Let  $n$  be the sample size required for the standard deviation of the sample mean to be 1% of the true mean. Then we have that

$$\sigma_{\bar{X}} = \sqrt{\frac{144}{n} \left( \frac{50-n}{49} \right)} = 70 \left( \frac{1}{100} \right) = 0.7 \Rightarrow \frac{144}{n} \left( \frac{50-n}{49} \right) = 0.49 \Rightarrow 144(50-n) = 0.49(49n) = 24.01n$$

$$n = \frac{144 \times 50}{144 + 24.01} = \frac{7200}{168.01} = 42.85 \approx 43$$

- 11.3 The true mean and true variance are given, respectively, by  $\mu_X = 24$ ,  $\sigma_X^2 = 324$ . If a random sample of size 81 is taken from a population, the sample standard deviation mean and sample mean are given by

$$\sigma_{\bar{X}} = \sqrt{\frac{\sigma_X^2}{n}} = \sqrt{\frac{324}{81}} = \sqrt{4} = 2$$

$$\bar{X} = \mu_X = 24$$

Using the central limit theorem, the probability that the sample mean lies between 23.9 and 24.2 is given by

$$\begin{aligned} P[23.9 < \bar{X} < 24.2] &= F_{\bar{X}}(24.2) - F_{\bar{X}}(23.9) = \Phi\left(\frac{24.2 - \bar{X}}{2}\right) - \Phi\left(\frac{23.9 - \bar{X}}{2}\right) \\ &= \Phi\left(\frac{24.2 - 24}{2}\right) - \Phi\left(\frac{23.9 - 24}{2}\right) = \Phi(0.1) - \Phi(-0.05) = \Phi(0.1) - \{1 - \Phi(0.05)\} \\ &= \Phi(0.1) + \Phi(0.05) - 1 = 0.5398 + 0.5199 - 1 = 0.0597 \end{aligned}$$

- 11.4 A random number generator produces three-digit random numbers that are uniformly distributed between 0.000 and 0.999. Thus, the true mean and true variance are

$$\mu_X = \frac{0.000 + 0.999}{2} = 0.4995$$

$$\sigma_X^2 = \frac{(0.999 - 0.000)^2}{12} = \frac{0.999^2}{12}$$

- a. If the generator produces the sequence of numbers 0.276, 0.123, 0.072, 0.324, 0.815, 0.312, 0.432, 0.283, 0.717, the sample mean is given by

$$\bar{X} = \frac{1}{9}\{0.276 + 0.123 + 0.072 + 0.324 + 0.815 + 0.312 + 0.432 + 0.283 + 0.717\} = 0.3727$$

- b. When we have a sample size of  $n$ , the variance of the sample mean of numbers produced by the random number generator is given by

$$\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n} = \frac{0.999^2}{12n} = \frac{0.0832}{n}$$

- c. Let  $n$  be the sample size required to obtain a sample mean whose standard deviation is no greater than 0.01. Then we have that

$$\begin{aligned}\sigma_{\bar{X}} &= \sqrt{\frac{0.999^2}{12n}} \leq 0.01 \Rightarrow \frac{0.999^2}{12n} \leq 0.0001 \\ n &\geq \frac{0.999^2}{12(0.0001)} = 831.67 \approx 832\end{aligned}$$

11.5 The PDF of the Student's  $t$  distribution is given by

$$f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

where  $\nu = n - 1$  is the number of degrees of freedom and  $n$  is the sample size. When  $t = 2$ , we obtain

$$f_T(2) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{4}{\nu}\right)^{-(\nu+1)/2}$$

a. At 6 degrees of freedom we obtain

$$\begin{aligned} f_T(2)|_{v=6} &= \frac{\Gamma\left(\frac{6+1}{2}\right)}{\sqrt{6\pi}\Gamma\left(\frac{6}{2}\right)} \left(1 + \frac{4}{6}\right)^{-(6+1)/2} = \frac{\Gamma(3.5)}{\sqrt{6\pi}\Gamma(3)} \left(\frac{5}{3}\right)^{-3.5} = \frac{(2.5)(1.5)(0.5)\sqrt{\pi}}{2!\sqrt{\pi}\sqrt{6}} \left(\frac{5}{3}\right)^{-3.5} \\ &= \frac{(2.5)(1.5)(0.5)}{2\sqrt{6}} \left(\frac{5}{3}\right)^{-3.5} = 0.0640 \end{aligned}$$

b. At 12 degrees of freedom we obtain

$$\begin{aligned} f_T(2)|_{v=12} &= \frac{\Gamma\left(\frac{12+1}{2}\right)}{\sqrt{12\pi}\Gamma\left(\frac{12}{2}\right)} \left(1 + \frac{4}{12}\right)^{-(12+1)/2} = \frac{\Gamma(6.5)}{\sqrt{12\pi}\Gamma(6)} \left(\frac{4}{3}\right)^{-6.5} \\ &= \frac{(5.5)(4.5)(3.5)(2.5)(1.5)(0.5)\sqrt{\pi}}{5!\sqrt{\pi}\sqrt{12}} \left(\frac{4}{3}\right)^{-6.5} \\ &= \frac{(5.5)(4.5)(3.5)(2.5)(1.5)(0.5)}{120\sqrt{12}} \left(\frac{4}{3}\right)^{-6.5} = 0.0602 \end{aligned}$$

### Section 11.3: Estimation Theory

11.6 Given that  $\bar{X} = 120$ ,  $\sigma = 10$ , when the sample size is  $n$ , the confidence limits for the 90% confidence level are

$$\bar{X} \pm \frac{k_{90}\sigma}{\sqrt{n}} = 120 \pm \frac{1.64(10)}{\sqrt{n}} = 120 \pm \frac{16.4}{\sqrt{n}}$$

where the value  $k_{90} = 1.64$  was obtained from Table 11.1.

a. When  $n = 100$ , we obtain the limits as

$$120 \pm \frac{16.4}{\sqrt{100}} = 120 \pm \frac{16.4}{10} = 120 \pm 1.64$$



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- b. When  $n = 25$ , we obtain the limits as

$$120 \pm \frac{16.4}{\sqrt{25}} = 120 \pm \frac{16.4}{5} = 120 \pm 3.28$$

- 11.7 We are given a population size of  $N = 200$  and a sample size of  $n = 50$ . Since the population size is not very large compared to the sample size, the confidence limits are given by

$$\bar{X} \pm k \sigma_{\bar{X}} = \bar{X} \pm k \frac{\sigma_X}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} = 75 \pm k \frac{10}{\sqrt{50}} \sqrt{\frac{200-50}{200-1}} = 75 \pm k \frac{10}{\sqrt{50}} \sqrt{\frac{150}{199}} = 75 \pm 10k \sqrt{\frac{3}{199}} = 75 \pm 1.23k$$

- a. From Table 11.1, the value for  $k$  at the 95% confidence level is  $k = 1.96$ . Thus, the confidence limits are  $75 \pm 1.23(1.96) = 75 \pm 2.41$ .
- b. At the 99% confidence level, the value for  $k$  is  $k = 2.58$ . Thus, the confidence limits are  $75 \pm 1.23(2.58) = 75 \pm 3.17$ .
- c. To obtain the value of  $k$  that gives the confidence limits of  $75 \pm 1$ , we solve the equation

$$75 \pm 1.23k = 75 \pm 1 \Rightarrow k = 1/1.23 = 0.81$$

The area under the standard normal curve from 0 to 0.81 is  $0.7910 - 0.5 = 0.2910$ . Thus, the required degree of confidence is the area  $2(0.2910) = 0.5820$ , which means that the confidence level is 58%.

- 11.8 Given that  $\mu$  is the true mean, the true standard deviation is 24, and the number of students is  $n = 36$ , the probability that the estimate differs from the true mean by 3.6 marks is given by

$$\begin{aligned}
P[|\bar{X} - \mu| \leq 3.6] &= P[\mu - 3.6 \leq \bar{X} \leq \mu + 3.6] = F_{\bar{X}}(\mu + 3.6) - F_{\bar{X}}(\mu - 3.6) \\
&= \Phi\left(\frac{\mu + 3.6 - \mu}{\sigma/\sqrt{36}}\right) - \Phi\left(\frac{\mu - 3.6 - \mu}{\sigma/\sqrt{36}}\right) = \Phi\left(\frac{3.6}{24/6}\right) - \Phi\left(\frac{-3.6}{24/6}\right) = \Phi(0.9) - \Phi(-0.9) \\
&= \Phi(0.9) - \{1 - \Phi(0.9)\} = 2\Phi(0.9) - 1 = 2(0.8159) - 1 \\
&= 0.6318
\end{aligned}$$

11.9 From Table 11.1, the values of  $k$  corresponding to the 90% and 99.9% confidence levels are  $k = 1.64$  and  $k = 3.29$ , respectively. If we denote the sample sizes for the 90% and 99.9% confidence levels by  $m$  and  $n$ , respectively, then

$$\begin{aligned}
P\left[\mu_X - 1.64\sqrt{\frac{\sigma_X^2}{m}} \leq \bar{X} \leq \mu_X + 1.64\sqrt{\frac{\sigma_X^2}{m}}\right] &= 0.9 \\
P\left[\mu_X - 3.29\sqrt{\frac{\sigma_X^2}{n}} \leq \bar{X} \leq \mu_X + 3.29\sqrt{\frac{\sigma_X^2}{n}}\right] &= 0.999
\end{aligned}$$

If the confidence limits are to be the same for both cases, we have that

$$1.64\sqrt{\frac{\sigma_X^2}{m}} = 3.29\sqrt{\frac{\sigma_X^2}{n}} \Rightarrow \frac{1.64^2 \sigma_X^2}{m} = \frac{3.29^2 \sigma_X^2}{n} \Rightarrow \frac{n}{m} = \frac{3.29^2}{1.64^2} = 4.024$$

Thus, we require a fourfold increase in sample size.

11.10 If we consider selecting a red ball as success, then the success probability is  $p = 0.7$ . Since each selection is a Bernoulli trial, the variance is given by

$$\sigma^2 = p(1-p) = (0.7)(0.3) = 0.21$$

If  $n = 60$ , the 95% confidence limits for the actual proportion of red balls in the box are given by

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$$\bar{X} \pm k_{95} \frac{\sigma}{\sqrt{n}} = \bar{X} \pm 1.96 \sqrt{\frac{0.21}{60}} = \bar{X} \pm 0.116$$

11.11 Let  $K$  denote the number of red balls among the 20 balls drawn. If  $p$  denotes the probability of drawing a red ball, then the PMF of  $K$  is

$$p_K(k) = \binom{20}{k} p^k (1-p)^{20-k}$$

The likelihood function is given by

$$L(p; k) = \binom{20}{k} p^k (1-p)^{20-k}$$

The value of  $p$  that maximizes this function can be obtained as follows:

$$\begin{aligned} \log L(p; k) &= \log \binom{20}{k} + k \log p + (20-k) \log(1-p) \\ \frac{\partial}{\partial p} \log L(p; k) &= \frac{k}{p} - \frac{20-k}{1-p} = 0 \Rightarrow k(1-p) = p(20-k) \Rightarrow k = 20p \\ \hat{p} &= \frac{k}{20} \end{aligned}$$

If  $k = 12$ , we obtain

$$\hat{p} = \frac{k}{20} = \frac{12}{20} = 0.6$$

11.12  $X$  denotes the number of balls drawn until a green ball appears, and  $p$  denotes the fraction of green balls in the box. The PMF of  $X$  is given by

$$p_X(x) = p(1-p)^{x-1}$$

If the operation is repeated  $n$  times to obtain the sample  $X_1, X_2, \dots, X_n$ , then the likelihood function of the sample is given by

$$\begin{aligned} L(p, x_1, x_2, \dots, x_n) &= [p(1-p)^{x_1-1}][p(1-p)^{x_2-1}] \dots [p(1-p)^{x_n-1}] \\ &= p^n (1-p)^{x_1+x_2+\dots+x_n-n} = p^n (1-p)^{y-n} \end{aligned}$$

where  $y = x_1 + x_2 + \dots + x_n$ . The value of  $p$  that maximizes the function can be obtained as follows:

$$\begin{aligned} \log L(p) &= n \log p + (y-n) \log(1-p) \\ \frac{\partial}{\partial p} \log L(p) &= \frac{n}{p} - \frac{y-n}{1-p} = 0 \Rightarrow n(1-p) = p(y-n) \\ \hat{p} &= \frac{n}{y} = \frac{n}{x_1 + x_2 + \dots + x_n} \end{aligned}$$

11.13 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} 2 & 0 \leq y \leq x; 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The marginal PDF of  $X$  and its significant statistics are given by

$$\begin{aligned} f_X(x) &= \int_0^x f_{XY}(x, y) dy = \int_0^x 2 dy = 2x, \quad 0 \leq x \leq 1 \\ E[X] &= \int_0^1 2x^2 dx = \left. \frac{2x^3}{3} \right|_0^1 = \frac{2}{3} \\ E[X^2] &= \int_0^1 2x^3 dx = \left. \frac{2x^4}{4} \right|_0^1 = \frac{1}{2} \\ \sigma_X^2 &= E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \end{aligned}$$

Similarly, the marginal PDF  $Y$  and its significant statistics are given by

$$f_Y(y) = \int_{x=y}^1 f_{XY}(x, y) dx = \int_y^1 2 dx = 2(1-y), \quad 0 \leq y \leq 1$$

$$E[Y] = \int_0^1 2y(1-y) dy = \left[ y^2 - \frac{2y^3}{3} \right]_0^1 = \frac{1}{3}$$

$$E[Y^2] = \int_0^1 2y^2(1-y) dy = \left[ \frac{2y^3}{3} - \frac{2y^4}{4} \right]_0^1 = \frac{1}{6}$$

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$E[XY] = \int_{x=0}^1 \int_{y=0}^x 2xy dy dx = \int_{x=0}^1 x[y^2]_0^x dx = \int_{x=0}^1 x^3 dx = \left[ \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

$$\sigma_{XY} = E[XY] - E[X]E[Y] = \frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$$

- a. Let  $\hat{Y} = aX + b$  denote the best linear estimate of  $Y$  in terms of  $X$ . The values of  $a$  and  $b$  that give the minimum mean squared error are known to be as follows:

$$a^* = \frac{\sigma_{XY}}{\sigma_X^2} = \frac{1/36}{1/18} = \frac{1}{2}$$

$$b^* = E[Y] - \frac{\sigma_{XY}E[X]}{\sigma_X^2} = \frac{1}{3} - \frac{1/36(2)}{1/18} = \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 0$$

- b. The minimum mean squared error corresponding to the best linear estimate is given by

$$e_{mms} = \sigma_Y^2 - \frac{(\sigma_{XY})^2}{\sigma_X^2} = \frac{1}{18} - \frac{(1/36)^2}{1/18} = \frac{1}{18} - \frac{1}{72} = \frac{1}{24}$$

- c. The best nonlinear estimate of  $Y$  in terms of  $X$  is given by

$$\hat{Y} = g(X) = E[Y|X=x]$$

Now, the conditional PDF of  $Y$  given  $X$  is

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}, \quad 0 \leq y \leq 1$$
$$E[Y|X] = g(X) = \int_0^1 y f_{Y|X}(y|x) dy = \int_0^1 \frac{y}{x} dy = \left[ \frac{y^2}{2x} \right]_0^1 = \frac{1}{2x}$$

11.14 The joint PDF of  $X$  and  $Y$  is given by

$$f_{XY}(x, y) = \begin{cases} \frac{2}{3}(x + 2y) & 0 < x < 1; 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The marginal PDF of  $X$  and its significant statistics are as follows:

$$f_X(x) = \int_0^1 f_{XY}(x, y) dy = \frac{2}{3} \int_0^1 (x + 2y) dy = \frac{2}{3} [xy + y^2]_0^1 = \frac{2}{3}(1 + x), \quad 0 < x < 1$$
$$E[X] = \int_0^1 x f_X(x) dx = \frac{2}{3} \int_0^1 x(1 + x) dx = \frac{2}{3} \left[ \frac{x^2}{2} + \frac{x^3}{3} \right]_0^1 = \frac{5}{9}$$
$$E[X^2] = \int_0^1 x^2 f_X(x) dx = \frac{2}{3} \int_0^1 x^2(1 + x) dx = \frac{2}{3} \left[ \frac{x^3}{3} + \frac{x^4}{4} \right]_0^1 = \frac{7}{18}$$
$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{7}{18} - \frac{25}{81} = \frac{13}{162}$$

Similarly, the marginal PDF of  $Y$  and its significant statistics are as follows:

$$f_Y(y) = \int_0^1 f_{XY}(x, y) dx = \frac{2}{3} \int_0^1 (x + 2y) dx = \frac{2}{3} \left[ \frac{x^2}{2} + 2yx \right]_0^1 = \frac{1}{3}(1 + 4y), 0 < y < 1$$

$$E[Y] = \int_0^1 y f_Y(y) dy = \frac{1}{3} \int_0^1 y(1 + 4y) dy = \frac{1}{3} \left[ \frac{y^2}{2} + \frac{4y^3}{3} \right]_0^1 = \frac{11}{18}$$

$$E[Y^2] = \int_0^1 y^2 f_Y(y) dy = \frac{1}{3} \int_0^1 y^2(1 + 4y) dy = \frac{1}{3} \left[ \frac{y^3}{3} + y^4 \right]_0^1 = \frac{4}{9}$$

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = \frac{4}{9} - \frac{121}{324} = \frac{23}{324}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{x + 2y}{1 + x}$$

$$E[Y|X] = \int_0^1 y f_{Y|X}(y|x) dy = \int_0^1 \frac{y(x + 2y)}{1 + x} dy = \frac{1}{1 + x} \left[ \frac{xy^2}{2} + \frac{2y^3}{3} \right]_0^1 = \frac{3x + 4}{6(1 + x)}$$

$$\begin{aligned} E[XY] &= \int_{x=0}^1 \int_{y=0}^1 xy f_{XY}(x, y) dy dx = \frac{2}{3} \int_{x=0}^1 \int_{y=0}^1 xy(x + 2y) dy dx \\ &= \frac{2}{3} \int_{x=0}^1 x \left[ \frac{xy^2}{2} + \frac{2y^3}{3} \right]_0^1 dx = \frac{2}{3} \int_{x=0}^1 \left[ \frac{x^2}{2} + \frac{2x}{3} \right] dx = \frac{2}{3} \left[ \frac{x^3}{6} + \frac{x^2}{3} \right]_0^1 = \frac{1}{3} \end{aligned}$$

$$\sigma_{XY} = E[XY] - E[X]E[Y] = \frac{1}{3} - \left( \frac{5}{9} \right) \left( \frac{11}{18} \right) = -\frac{1}{162}$$

- a. A linear estimate of  $Y$  in terms of  $X$  is given by  $\hat{Y} = aX + b$ . The values of  $a$  and  $b$  that give the minimum mean squared error of the estimate are

$$a^* = \frac{\sigma_{XY}}{\sigma_X^2} = \frac{-1/162}{13/162} = -\frac{1}{13}$$

$$b^* = E[Y] - \frac{\sigma_{XY}E[X]}{\sigma_X^2} = E[Y] - a^*E[X] = \frac{11}{18} + \left( \frac{1}{13} \right) \left( \frac{5}{9} \right) = \frac{17}{26}$$

- b. The minimum mean squared error corresponding to the linear estimate is given by

$$e_{mms} = \sigma_Y^2 - \frac{(\sigma_{XY})^2}{\sigma_X^2} = \frac{23}{324} - \frac{(-1/162)^2}{13/162} = \frac{23}{324} - \frac{1}{(13)(162)} = \frac{206}{2106}$$

c. The best nonlinear estimate of  $Y$  in terms of  $X$  is given by

$$\hat{Y} = g(X) = E[Y|X = x] = \frac{3x + 4}{6(1 + x)}$$

## Section 11.4: Hypothesis Testing

11.15 The population proportion of success (or population mean) is  $p = 0.6$ . Since the experiment is essentially a Bernoulli trial, the population variance is  $\sigma^2 = p(1 - p) = 0.24$ . The sample proportion of success (or sample mean) is

$$\bar{p} = \frac{15}{36} = 0.42$$

Since the sample proportion is less than population proportion, the null and alternate hypotheses can be set up as follows:

$$H_0: p = 0.60$$

$$H_1: p < 0.60$$

Thus, we have a left-tail test whose  $z$ -score is

$$z = \frac{\bar{p} - p}{\sigma / \sqrt{n}} = \frac{0.42 - 0.60}{\sqrt{0.24/36}} = \frac{-6(0.18)}{\sqrt{0.24}} = -2.204$$

a. At the 0.05 level of significance, the critical  $z$ -score for the left-tail test is  $z_c = -1.645$ . That is, we reject any null hypothesis that lies in the region  $z_c \leq -1.645$ . Since the score  $z = -2.204$  lies in this region, we reject  $H_0$  and accept  $H_1$ .



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- b. At the 0.01 level of significance, the critical  $z$ -score for the left-tail test is  $z_c = -2.33$ . Since the score  $z = -2.204$  lies in the acceptance region, which is  $z > 2.33$ , we accept  $H_0$  and reject  $H_1$ .

11.16 The population mean is  $p = 0.95$ , and the corresponding population variance is  $\sigma^2 = p(1-p) = (0.95)(0.05) = 0.0475$ . The sample mean is

$$\bar{p} = \frac{200 - 18}{200} = 0.91$$

Since the sample mean is less than the population mean, the null and alternate hypotheses can be set up as follows:

$$H_0: p = 0.95$$

$$H_1: p < 0.95$$

Thus, we have a left-tail test whose  $z$ -score is

$$z = \frac{\bar{p} - p}{\sigma / \sqrt{n}} = \frac{0.91 - 0.95}{\sqrt{0.0475 / 200}} = -0.04 \sqrt{\frac{200}{0.0475}} = -2.595$$

- a. At the 0.05 level of significance, the critical  $z$ -score for the left-tail test is  $z_c = -1.645$ . That is, we reject any null hypothesis that lies in the region  $z_c \leq -1.645$ . Since the score  $z = -2.595$  lies in this region, we reject  $H_0$  and accept  $H_1$ .
- b. At the 0.01 level of significance, the critical  $z$ -score for the left-tail test is  $z_c = -2.33$ . Since the score  $z = -2.595$  lies in the rejection region, we still reject  $H_0$  and accept  $H_1$ .

11.17 The population mean is  $\mu = 500$  and a standard deviation  $\sigma = 75$ . The sample mean is  $\bar{X} = 510$  with  $n = 100$  observations. Since the sample mean is greater than the population mean, we can set up the null and alternate hypotheses as follows:

$$H_0: \mu = 500$$

$$H_1: \mu > 500$$

Thus, we have a right-tail test whose  $z$ -score is

$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \frac{510 - 500}{75 / \sqrt{100}} = \frac{100}{75} = 1.33$$

For a right-tail test the critical  $z$ -score at the 95% level of confidence is  $z_c = 1.645$ . That is, we reject any null hypothesis that lies in the region  $z_c \geq 1.645$ . Since  $z = 1.33$  lies in the acceptance region, we accept  $H_0$  and reject  $H_1$ . This means that there is no statistical difference between the sample mean and the population at the 95% level of confidence.

- 11.18 The population mean is  $\mu = 20$  and the standard deviation is  $\sigma = 5$ , but the sample mean is  $\bar{X} = 18$  with  $n = 36$  observations. Since the sample mean is less than the population mean, we can set up the null and althernate hypotheses as follows:

$$H_0: \mu = 20$$

$$H_1: \mu < 20$$

This is a left-tail test whose  $z$ -score is given by

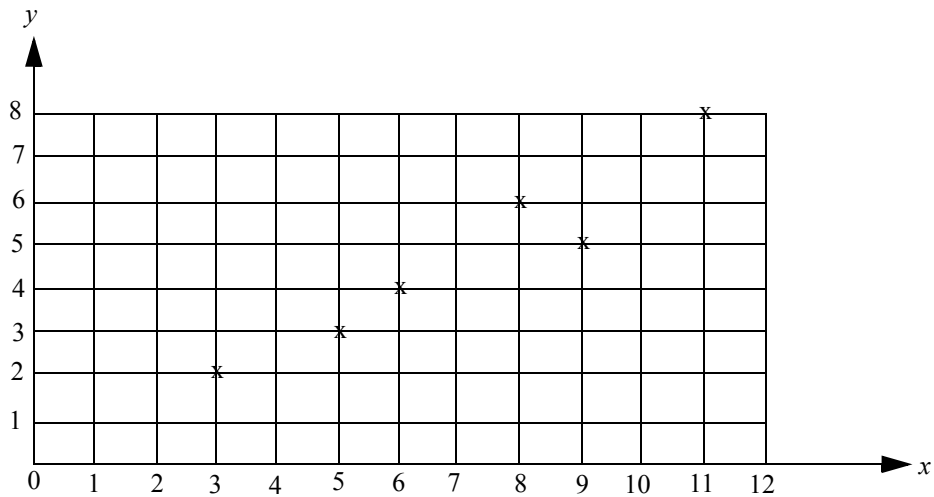
$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = \frac{18 - 20}{5 / \sqrt{36}} = \frac{-2(6)}{5} = -2.4$$

For a left-tail test the critical  $z$ -score at the 95% level of confidence is  $z_c = -1.645$ . That is, we reject any null hypothesis that lies in the region  $z_c \leq -1.645$ . Since  $z = -2.4$  lies in this region, we reject  $H_0$  and accept  $H_1$ .

### Section 11.5: Curve Fitting and Linear Regression

- 11.19 Given the recorded  $(x, y)$  pairs  $(3, 2)$ ,  $(5, 3)$ ,  $(6, 4)$ ,  $(8, 6)$ ,  $(9, 5)$  and  $(11, 8)$ .

- a. The scatter diagram for these data is as shown below.



- b. To find the linear regression line  $y = a + bx$  of  $y$  on  $x$  that best fits these data we proceed as follows:

$x$	$y$	$x^2$	$xy$	$y^2$
3	2	9	6	4
5	3	25	15	9
6	4	36	24	16
8	6	64	48	36
9	5	81	45	25
11	8	121	88	64
$\sum x = 42$	$\sum y = 28$	$\sum x^2 = 336$	$\sum xy = 226$	$\sum y^2 = 154$

The values of  $a$  and  $b$  that make the line best fit the above data are given by

$$b^* = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2} = \frac{6(226) - 42(28)}{6(336) - (42)^2} = \frac{1356 - 1176}{2016 - 1764} = 0.714$$

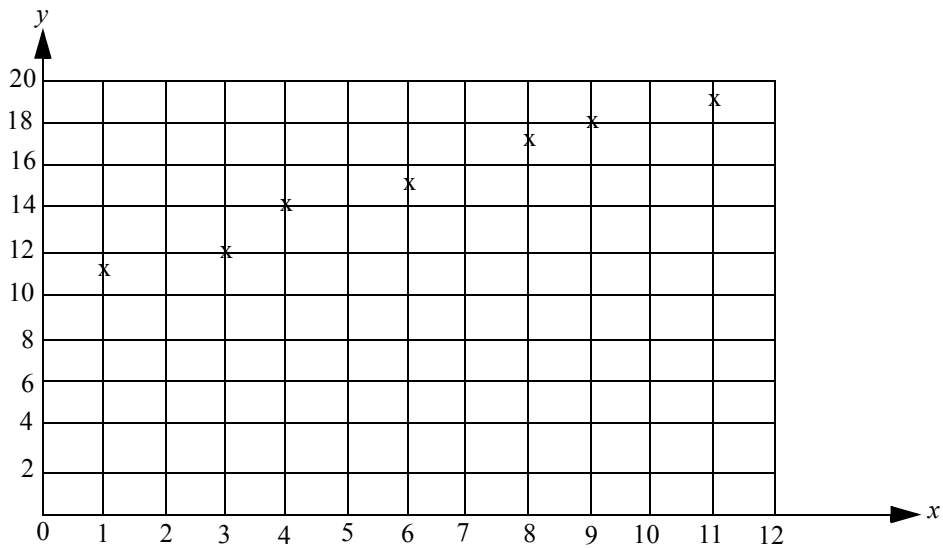
$$a^* = \frac{\sum_{i=1}^n y_i - b \sum_{i=1}^n x_i}{n} = \frac{28 - 0.714(42)}{6} = \frac{28 - 29.938}{6} = -0.33$$

Thus, the best line is  $y = -0.33 + 0.714x$ .

c. When  $x = 15$ , we obtain the estimate  $y = -0.33 + 0.714(15) = 10.38$ .

11.20 Given the recorded  $(x, y)$  pairs  $(1, 11)$ ,  $(3, 12)$ ,  $(4, 14)$ ,  $(6, 15)$ ,  $(8, 17)$ ,  $(9, 18)$ , and  $(11, 19)$ .

a. The scatter diagram for these data is as shown:



- b. To find the linear regression line  $y = a + bx$  of  $y$  on  $x$  that best fits these data, we proceed as follows:

$x$	$y$	$x^2$	$xy$	$y^2$
1	11	1	11	121
3	12	9	36	144
4	14	16	56	196
6	15	36	90	225
8	17	64	136	289
9	18	81	162	324
11	19	121	209	361
$\sum x = 42$	$\sum y = 106$	$\sum x^2 = 328$	$\sum xy = 700$	$\sum y^2 = 1660$

The values of  $a$  and  $b$  that make the line best fit the above data are given by

$$b^* = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2} = \frac{7(700) - 42(106)}{7(328) - (42)^2} = \frac{4900 - 4452}{2296 - 1764} = 0.842$$

$$a^* = \frac{\sum_{i=1}^n y_i - b \sum_{i=1}^n x_i}{n} = \frac{106 - 0.842(42)}{7} = \frac{70.63}{7} = 10.09$$

Thus, the best line is  $y = 10.09 + 0.842x$ .

c. When  $x = 20$ , we estimate  $y$  to be  $y = 10.09 + 0.842(20) = 26.93$ .

11.21 The ages  $x$  and systolic blood pressures  $y$  of 12 people are shown in the following table:

Age ( $x$ )	56	42	72	36	63	47	55	49	38	42	68	60
Blood Pressure ( $y$ )	147	125	160	118	149	128	150	145	115	140	152	155

a. The least-squares regression line  $y = a + bx$  of  $y$  on  $x$  can be obtained as follows:

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$x$	$y$	$x^2$	$xy$	$y^2$
56	147	3136	8323	21609
42	125	1764	5250	15625
72	160	5184	11520	25600
36	118	1296	4248	13924
63	149	3969	9387	22201
47	128	2209	6016	16384
55	150	3025	8250	22500
49	145	2401	7105	21025
38	115	1444	4370	13225
42	140	1764	5880	19600
68	152	4624	10336	23104
60	155	3600	9300	24025
$\sum x = 628$	$\sum y = 1684$	$\sum x^2 = 34416$	$\sum xy = 89985$	$\sum y^2 = 238822$

The values of  $a$  and  $b$  that make the line best fit the above data are given by

$$b^* = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2} = \frac{12(89985) - 628(1684)}{12(34416) - (628)^2} = \frac{1079820 - 1057552}{412992 - 394384} = 1.2$$

$$a^* = \frac{\sum_{i=1}^n y_i - b \sum_{i=1}^n x_i}{n} = \frac{1684 - 1.2(628)}{12} = 77.71$$

Thus, the best line is  $y = 77.71 + 1.2x$ .

- b. The estimate of the blood pressure of a person whose age is 45 years is given by  
 $y = 77.71 + 1.2(45) = 131.71$ .

11.22 The given table is as follows:

Couple	1	2	3	4	5	6	7	8	9	10	11	12
Planned Number of Children ( $x$ )	3	3	0	2	2	3	0	3	2	1	3	2
Actual Number of Children ( $y$ )	4	3	0	4	4	3	0	4	3	1	3	1

- a. The least-squares regression line  $y = a + bx$  of  $y$  on  $x$  can be obtained as follows:



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$x$	$y$	$x^2$	$xy$	$y^2$
3	4	9	12	16
3	3	9	9	9
0	0	0	0	0
2	4	4	8	16
2	4	4	8	16
3	3	9	9	9
0	0	0	0	0
3	4	9	12	16
2	3	4	6	9
1	1	1	1	1
3	3	9	9	9
2	1	4	2	1
$\sum x = 24$	$\sum y = 30$	$\sum x^2 = 62$	$\sum xy = 76$	$\sum y^2 = 102$

The values of  $a$  and  $b$  that make the line best fit the above data are given by

$$b^* = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2} = \frac{12(76) - 24(30)}{12(62) - (24)^2} = \frac{912 - 720}{744 - 576} = 1.143$$

$$a^* = \frac{\sum_{i=1}^n y_i - b \sum_{i=1}^n x_i}{n} = \frac{30 - 1.143(24)}{12} = 0.214$$

Thus, the best line is  $y = 0.214 + 1.143x$ .

- b. The estimate for the number of children that a couple who had planned to have 5 children actually had is given by  $y = 0.214 + 5(1.143) = 5.929 = 6$ .