Chapter 7

Review of z-Transforms and Difference Equations

The z-transform fulfills, for discrete-time signals, the same need that the Laplace transform fulfills for continuous-time signals: It enables us to replace operations on signals by operations on complex functions. Like the Laplace transform, the z-transform is mainly a tool of convenience, rather than necessity. Frequency-domain analysis, as we have seen, can be dealt with both theoretically and practically without the z-transform. However, for certain operations the convenience of using the z-transform (or the Laplace transform in the continuous-time case) outweighs the burden of having to learn yet another tool. This is especially true when dealing with linear filtering and linear systems in general. Applications of the z-transform in linear system analysis include:

1. Time-domain interpretation of LTI systems responses.
2. Stability testing.
3. Block-diagram manipulation of systems consisting of subsystems connected in cascade, parallel, and feedback.
4. Decomposition of systems into simple building blocks.
5. Analysis of systems and signals that do not possess Fourier transforms (e.g., unstable LTI systems).

In this chapter we give the necessary background on the z-transform and its relation to linear systems.[1] We pay special attention to *rational systems*, that is, systems that can be described by difference equations. We shall use the bilateral z-transform almost exclusively. This is in contrast with continuous-time systems, where the unilateral Laplace transform is usually emphasized. The unilateral z-transform will be dealt with briefly, in connection to the solution of difference equations with initial conditions.

Proper understanding of the material in this chapter requires certain knowledge of complex function theory, in particular analytic functions and their basic properties. If you are not familiar with this material, you can still use the main results given here, especially the ones concerning rational systems, and take their derivations as a matter of belief.

7.1 The z-Transform

The *bilateral* (or *two-sided*) *z-transform* of a sequence $x[n]$ is the complex-valued function[2]

$$\{\mathcal{Z}x\}(z) = X^{z}(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \tag{7.1}$$

where z belongs to a subset of the complex numbers $\mathbb{C}$, called the *region of convergence* (ROC) of the transform. By definition, the region of convergence is the set of all z such that the right side of (7.1) is absolutely convergent, that is,[3]

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty. \tag{7.2}$$

To determine the shape of the region of convergence, let us write the transform variable z in a polar form, that is, as $z = re^{j\theta}$, where r and θ are real, $r \geq 0$ and $-\pi \leq \theta < \pi$. We then get

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]| \cdot |re^{j\theta}|^{-n} = \sum_{n=-\infty}^{\infty} |x[n]|r^{-n}. \tag{7.3}$$

Define[4]

$$R_1 = \inf\left\{r \geq 0 : \sum_{n=-\infty}^{\infty} |x[n]|r^{-n} < \infty\right\}, \tag{7.4a}$$

$$R_2 = \sup\left\{r \geq 0 : \sum_{n=-\infty}^{\infty} |x[n]|r^{-n} < \infty\right\}. \tag{7.4b}$$

Then we get that the region of convergence includes the annulus

$$R_1 < |z| < R_2, \tag{7.5}$$

provided this annulus is nonempty.

The numbers R_1 and R_2 depend on the sequence $x[n]$, as seen from (7.4).[5] The region of convergence may include the circle $|z| = R_1$ or the circle $|z| = R_2$ or both, but there is no general criterion for testing these possibilities.[6] In any case, we call the set (7.5) the *interior* of the region and the union of the two circles the *boundary* of the region. Figure 7.1 illustrates the region of convergence of the z-transform.

Recall that a function $f(z)$ is *analytic* in a domain in the complex plane if it is differentiable at every point of the domain. A fundamental property of the z-transform, which we quote without proof, is

Theorem 7.1 The z-transform is an analytic function of z in the interior of its region of convergence. □

The sequence $x[n]$ can be computed from its z-transform as given by the following theorem:

Theorem 7.2 (inverse z-transform)

$$x[n] = \frac{1}{2\pi j}\oint X^{z}(z)z^{n-1}dz, \tag{7.6}$$

where the complex integration is over any simple contour in the interior of the region of convergence of $X^{z}(z)$ that circles the point $z = 0$ exactly once in the counterclockwise direction.

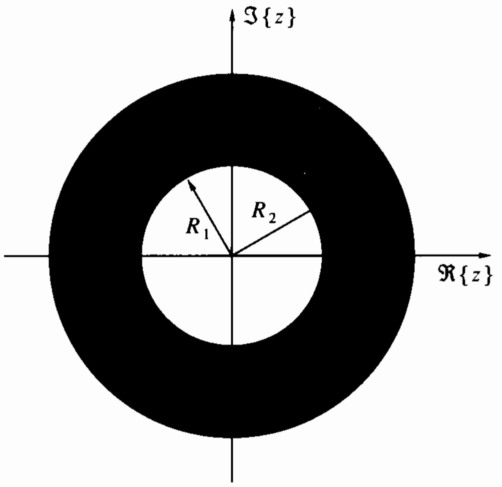

Figure 7.1 The region of convergence of the z-transform.

Proof The proof relies on the following result from complex function theory:

$$\frac{1}{2\pi j}\oint z^k dz = \delta[k+1] = \begin{cases} 1, & k = -1, \\ 0, & k \in \mathbb{Z},\ k \neq -1, \end{cases} \tag{7.7}$$

where the integral is over any simple contour that circles the point $z = 0$ exactly once in the counterclockwise direction. We get from (7.6) and (7.7)

$$\begin{aligned} \frac{1}{2\pi j}\oint X^z(z) z^{n-1} dz &= \frac{1}{2\pi j}\oint \left[\sum_{k=-\infty}^{\infty} x[k] z^{-k}\right] z^{n-1} dz \\ &= \sum_{k=-\infty}^{\infty} x[k]\left[\frac{1}{2\pi j}\oint z^{n-k-1} dz\right] \\ &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]. \end{aligned} \tag{7.8}$$

For the substitution made in the derivation to be valid, the contour must lie in the interior of the region of convergence of $X^z(z)$. □

Table 7.1 gives the z-transforms of a few common sequences. In this table, $v[n]$ denotes the discrete-time, unit-step function. Methods of deriving the transforms in this table are discussed later in this chapter.

Example 7.1 The following special cases illustrate various points related to the definition of the z-transform and its region of convergence.

1. The z-transform of $x[n] = \delta[n]$ is obviously $X^z(z) = 1$. Here the z-transform is defined on the entire complex plane.
2. Let

$$x[n] = \begin{cases} a^n, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{7.9}$$

$x[n]$	$X^z(z)$	ROC
$\delta[n]$	1	all z
$v[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$a^n v[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n v[-(n+1)]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n v[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$n^2 a^n v[n]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| > \|a\|$
$a^n \cos(\theta_0 n)\, v[n]$	$\frac{1-a\cos\theta_0\, z^{-1}}{1-2a\cos\theta_0\, z^{-1}+a^2 z^{-2}}$	$\|z\| > \|a\|$
$a^n \sin(\theta_0 n)\, v[n]$	$\frac{a\sin\theta_0\, z^{-1}}{1-2a\cos\theta_0\, z^{-1}+a^2 z^{-2}}$	$\|z\| > \|a\|$

Table 7.1 Common sequences and their z-transforms.

Then

$$X^z(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}, \quad |z| > |a|. \tag{7.10}$$

In this example $R_1 = |a|$ and $R_2 = \infty$.

3. Let

$$x[n] = \begin{cases} -a^n, & n < 0, \\ 0, & n \ge 0. \end{cases} \tag{7.11}$$

Then

$$\begin{aligned} X^z(z) &= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{n=1}^{\infty} (za^{-1})^n \\ &= -\frac{za^{-1}}{1-za^{-1}} = \frac{1}{1-az^{-1}}, \quad |z| < |a|. \end{aligned} \tag{7.12}$$

In this example $R_1 = 0$ and $R_2 = |a|$. The z-transforms of (7.9) and (7.11) have the same functional form, but disjoint regions of convergence. As we see, the region of convergence is an indispensable part of the transform: Without knowledge of this region, we cannot tell whether the function $1/(1 - az^{-1})$ comes from the sequence (7.9) or from (7.11).

4. Let

$$x[n] = a^{|n|}, \quad -\infty < n < \infty. \tag{7.13}$$

Then

$$\begin{aligned} X^z(z) &= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n} = \frac{az}{1-az} + \frac{1}{1-az^{-1}} \\ &= \frac{1-a^2}{(1-az)(1-az^{-1})}, \quad |a| < |z| < |a|^{-1}. \end{aligned} \tag{7.14}$$

Here $R_1 = |a|$ and $R_2 = |a|^{-1}$ provided $|a| < 1$; otherwise the region of convergence is empty. This example shows that there exist sequences that have no z-transforms.

5. Let

$$x[n] = \frac{1}{|n| + 1}, \quad -\infty < n < \infty. \tag{7.15}$$

We have

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=-\infty}^{\infty} \frac{1}{|n| + 1} = \infty,$$

so (7.2) does not converge for $|z| = 1$. This implies lack of convergence for $|z| < 1$, since then

$$\sum_{n=0}^{\infty} |x[n]z^{-n}| > \sum_{n=0}^{\infty} |x[n]| = \infty,$$

as well as lack of convergence for $|z| > 1$, since then

$$\sum_{n=-\infty}^{0} |x[n]z^{-n}| > \sum_{n=-\infty}^{0} |x[n]| = \infty.$$

In summary, (7.2) does not converge for any z, so $x[n]$ does not have a z-transform. Note the difference between this example and the preceding one: There the sequence had exponential growth for both positive and negative n when $|a| > 1$. Here the sequence decays to zero for both positive and negative n, but it does not decay fast enough.

6. Let

$$x[n] = \frac{1}{n^2 + 1}, \quad -\infty < n < \infty. \tag{7.16}$$

Then (see Problem 2.49),

$$\sum_{n=-\infty}^{\infty} |x[n]| = \frac{\pi^2}{3} - 1 < \infty, \tag{7.17}$$

so the z-transform exists on the circle $|z| = 1$. However, the sum (7.2) diverges for all $|z| \neq 1$ in this case. The interior of the region of convergence of $X^z(z)$ is therefore empty, and $X^z(z)$ is not analytic anywhere.

7. Let $x[n]$ be a finite-duration sequence; that is, $x[n] = 0$ for $n < n_1$ and for $n > n_2$. Then

$$X^z(z) = \sum_{n=n_1}^{n_2} x[n]z^{-n}. \tag{7.18}$$

The sum is finite, so it converges for all $0 < |z| < \infty$; therefore $R_1 = 0$ and $R_2 = \infty$.

8. A sequence $x[n]$ satisfying $x[n] = 0$ for all $n < 0$ is called a *causal* sequence. For such a sequence,

$$X^z(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{7.19}$$

Let R_1 be as defined in (7.4a). Then

$$|z| > R_1 \implies \sum_{n=0}^{\infty} |x[n]z^{-n}| = \sum_{n=0}^{\infty} |x[n]| \cdot |z|^{-n} < \infty, \tag{7.20}$$

so z is in the region of convergence for all $|z| > R_1$, meaning that $R_2 = \infty$.

9. A sequence $x[n]$ satisfying $x[n] = 0$ for all $n > 0$ is called an *anticausal* sequence. For such a sequence,

$$X^z(z) = \sum_{n=-\infty}^{0} x[n]z^{-n}. \tag{7.21}$$

Let R_2 be as defined in (7.4b). Then

$$|z| < R_2 \implies \sum_{n=-\infty}^{0} |x[n]z^{-n}| = \sum_{n=-\infty}^{0} |x[n]| \cdot |z|^{-n} < \infty, \tag{7.22}$$

so z is in the region of convergence for all $|z| < R_2$, meaning that $R_1 = 0$.

10. Suppose we are given the z-transform

$$X^z(z) = \frac{z(z+1.2)}{(z-0.4)(z-2)}. \tag{7.23}$$

What is the corresponding sequence $x[n]$? Since the region of convergence is not given, there is no unique answer. Let us, however, explore the various possibilities. At the points $z = 0.4$ and $z = 2$ the function $X^z(z)$ is singular. Since the z-transform is an analytic function in the interior of its region of convergence, the circles $|z| = 0.4$ and $|z| = 2$ cannot be in the interior of the region of convergence. Nevertheless, they can belong to the boundary of the region. Now, since the region of convergence is always of the form (7.5), there are only three possibilities:

$$\begin{aligned} 0 <&|z| < 0.4 \quad &\text{(case I)},\\ 0.4 <&|z| < 2 \quad &\text{(case II)},\\ 2 <&|z| < \infty \quad &\text{(case III)}. \end{aligned}$$

This is illustrated in Figure 7.2.

To find $x[n]$ in the three cases, we express $X^z(z)$ in the form

$$X^z(z) = \frac{2z}{z-2} - \frac{z}{z-0.4} = \frac{2}{1-2z^{-1}} - \frac{1}{1-0.4z^{-1}}. \tag{7.24}$$

Now parts 2, 3, 8, and 9 of Example 7.1 lead us to conclude that

$$x[n] = \begin{cases} \begin{cases} -2\times 2^n + 0.4^n, & n < 0, \\ 0, & n \ge 0, \end{cases} & \text{(case I)}, \\ \begin{cases} -2\times 2^n, & n < 0, \\ -0.4^n, & n \ge 0, \end{cases} & \text{(case II)}, \\ \begin{cases} 2\times 2^n - 0.4^n, & n \ge 0, \\ 0, & n < 0, \end{cases} & \text{(case III)}. \end{cases} \tag{7.25}$$

□

7.2 Properties of the z-Transform

In this section we list the main properties of the z-transforms. Most of these properties are shared by Fourier transforms of sequences, as listed in Section 2.7. However, in the case of the z-transform we must also specify the region of convergence at which the property holds. This was not a problem in the case of the Fourier transform, since the range of argument θ of the Fourier transform is always the entire real line.

The sequences $x[n]$, $y[n]$, $w[n]$ mentioned next have z-transforms $X^z(z)$, $Y^z(z)$, $W^z(z)$ and radii of convergence (R_{x1}, R_{x2}), (R_{y1}, R_{y2}), (R_{w1}, R_{w2}), respectively. In each case, the region of convergence may include either boundary or both.

1. **Linearity**

$$w[n] = ax[n] + by[n] \implies W^z(z) = aX^z(z) + bY^z(z), \quad a, b \in \mathbb{C}, \tag{7.26}$$

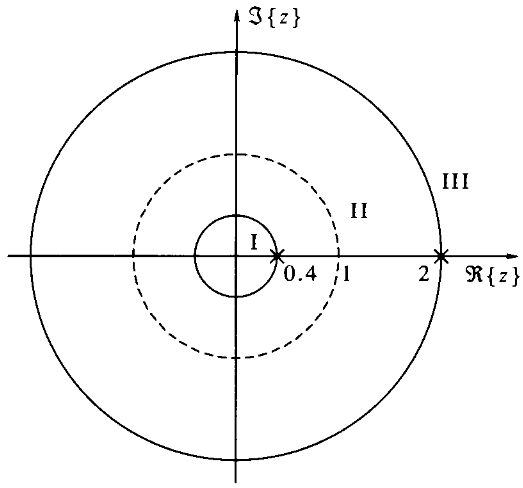

Figure 7.2 Possible regions of convergence in Example 7.1, part 10.

where

$$R_{w1} \leq \max\{R_{x1}, R_{y1}\}, \quad R_{w2} \geq \min\{R_{x2}, R_{y2}\}.$$

The proof of (7.26) is easy and will be skipped.

2. **Time shift**

$$w[n] = x[n-m] \iff W^z(z) = z^{-m}X^z(z), \quad m \in \mathbb{Z}. \tag{7.27}$$

The region of convergence of $W^z(z)$ is identical to that of $X^z(z)$ with one exception: If $x[n]$ is anticausal then $X^z(z)$ exists on $z = 0$, but $W^z(z)$ may not exist on $z = 0$ if m is positive.

Proof

$$W^z(z) = \sum_{n=-\infty}^{\infty} x[n-m]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{-(n+m)} = z^{-m}X^z(z). \tag{7.28}$$

3. **Multiplication by a geometric series**

$$w[n] = a^n x[n] \iff W^z(z) = X^z\left(\frac{z}{a}\right), \quad a \in \mathbb{C},\ a \neq 0, \tag{7.29}$$

where

$$R_{w1} = |a|R_{x1}, \quad R_{w2} = |a|R_{x2}.$$

Proof

$$W^z(z) = \sum_{n=-\infty}^{\infty} a^n x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X^z\left(\frac{z}{a}\right). \tag{7.30}$$

4. **Time reversal**

$$w[n] = x[-n] \iff W^z(z) = X^z(z^{-1}), \tag{7.31}$$

where

$$R_{w1} = (R_{x2})^{-1}, \quad R_{w2} = (R_{x1})^{-1}.$$

Proof

$$W^z(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{n} = X^z(z^{-1}). \tag{7.32}$$

5. **Differentiation**

$$w[n] = nx[n] \iff W^z(z) = -z\frac{dX^z(z)}{dz}, \tag{7.33}$$

$$R_{w1} = R_{x1}, \quad R_{w2} = R_{x2}.$$

Proof

$$-z\frac{dX^z(z)}{dz} = -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n]z^{-n} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-(n+1)}$$

$$= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = W^z(z). \tag{7.34}$$

6. **Time-domain convolution**

$$w[n] = \{x * y\}[n] \iff W^z(z) = X^z(z)Y^z(z), \tag{7.35}$$

where

$$R_{w1} \le \max\{R_{x1}, R_{y1}\}, \quad R_{w2} \ge \min\{R_{x2}, R_{y2}\}.$$

Proof

$$W^z(z) = \sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} x[m]y[n-m]\right]z^{-(n-m)-m}$$

$$= \sum_{m=-\infty}^{\infty} x[m]z^{-m}\sum_{k=-\infty}^{\infty} y[k]z^{-k} = X^z(z)Y^z(z). \tag{7.36}$$

7. **Time-domain multiplication**

$$w[n] = x[n]y[n] \iff W^z(z) = \frac{1}{2\pi j}\oint X^z(u)Y^z\left(\frac{z}{u}\right)u^{-1}du, \tag{7.37}$$

where

$$R_{w1} \le R_{x1}R_{y1}, \quad R_{w2} \ge R_{x2}R_{y2}.$$

Proof Using (7.7), we get

$$\frac{1}{2\pi j}\oint X^z(u)Y^z\left(\frac{z}{u}\right)u^{-1}du$$

$$= \frac{1}{2\pi j}\oint\left[\sum_{n=-\infty}^{\infty} x[n]u^{-n}\right]\left[\sum_{k=-\infty}^{\infty} y[k](z/u)^{-k}\right]u^{-1}du$$

$$= \sum_{n=-\infty}^{\infty} x[n]\sum_{k=-\infty}^{\infty} y[k]z^{-k}\left[\frac{1}{2\pi j}\oint u^{k-n-1}du\right]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\sum_{k=-\infty}^{\infty} y[k]z^{-k}\delta[k-n] = \sum_{n=-\infty}^{\infty} x[n]y[n]z^{-n} = w[n]. \tag{7.38}$$

The contour of integration must be such that

$$R_{x1} < |u| < R_{x2} \text{ and } R_{y1} < |z/u| < R_{y2}.$$

Such a contour is guaranteed to exist if

$$R_{x1}R_{y1} < |z| < R_{x2}R_{y2},$$

hence the region of convergence of $W^z(z)$.

7.3 Transfer Functions

Recall that a discrete-time LTI system is completely characterized, as far as its input-output relationships are concerned, by its impulse response sequence $h[n]$. The z-transform $H^z(z)$ of the impulse response, if exists, is called the *transfer function* of the system. Not every LTI system possesses a transfer function, since not every sequence has a z-transform. However, systems of practical interest usually have transfer functions, which is to say that $H^z(z)$ exists on a subset of the complex plane.

Of particular interest are LTI systems that are stable in the sense of having the bounded-input, bounded-output (BIBO) property. An LTI system is BIBO-stable if its response to any bounded input signal is a bounded signal. Let $x[n]$ be any input to the system and $y[n]$ the corresponding output. Then the system is BIBO-stable if

$$\begin{aligned}&|x[n]| \le B_1 \text{ for all } n \in \mathbb{Z} \implies\\ &\text{there exists } B_2 \text{ such that } |y[n]| \le B_2 \text{ for all } n \in \mathbb{Z}.\end{aligned} \tag{7.39}$$

The BIBO-stability of an LTI system can be determined from the impulse response of the system, as stated in the following theorem.

Theorem 7.3 A discrete-time LTI system is BIBO-stable if and only if its impulse response satisfies

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{7.40}$$

Proof Suppose the impulse response of the system satisfies (7.40) and let $x[n]$ be bounded by B_1. Then,

$$\begin{aligned}|y[n]| &= \left|\sum_{m=-\infty}^{\infty} h[m]x[n-m]\right| \le \sum_{m=-\infty}^{\infty} |h[m]|\,|x[n-m]|\\ &\le B_1 \sum_{m=-\infty}^{\infty} |h[m]| < \infty.\end{aligned} \tag{7.41}$$

Therefore, by taking $B_2 = B_1 \sum_{m=-\infty}^{\infty} |h[m]|$, we deduce that the system is BIBO-stable. This proves sufficiency of the condition (7.40). To prove necessity, assume that (7.40) does not hold; that is, the left side is infinite. Consider the input sequence

$$x[n] = \begin{cases} \dfrac{h[-n]}{|h[-n]|}, & h[-n] \ne 0,\\ 0, & h[-n] = 0.\end{cases} \tag{7.42}$$

This sequence is bounded in magnitude by 1. The output at time $n = 0$ is

$$y[0] = \sum_{m=-\infty}^{\infty} h[m]x[-m] = \sum_{m=-\infty}^{\infty} |h[m]| = \infty, \tag{7.43}$$

so the output is not bounded. This proves that (7.40) is necessary for BIBO-stability.□

Corollary An LTI system is BIBO-stable if and only if the region of convergence of its transfer function includes the circle $|z| = 1$.
Proof Substitution $|z| = 1$ makes the left sides of (7.40) and (7.2) equal (with the obvious change of $x[n]$ to $h[n]$). □

The circle $|z| = 1$ is called the *unit circle*. It is therefore common to say that an LTI system is stable if and only if the region of convergence of its transfer function includes the unit circle.

Another important property that an LTI system may possess is causality. A system is said to be *causal* if, for any time n, the response $y[n]$ depends only on past and present inputs $\{x[m],\ -\infty < m \le n\}$, but not on future inputs $\{x[m],\ m > n\}$.

Theorem 7.4 An LTI system is causal if and only if its impulse response $h[n]$ is a causal sequence.

Proof Decompose the system's output as follows:

$$y[n] = \sum_{m=-\infty}^{-1} h[m]x[n-m] + \sum_{m=0}^{\infty} h[m]x[n-m]. \tag{7.44}$$

The system is causal if and only if the first term is identically zero for an arbitrary input signal; but this is possible if and only if $h[m] = 0$ for all $m < 0$. □

As we saw in Example 7.1, part 8, the region of convergence of the transfer function of a causal system is of the form $R_1 < |z| < \infty$. It may also include the boundary $|z| = R_1$, but there is no general criterion for testing this.

Referring to Example 7.1, part 10, and assuming that $x[n]$ is the impulse response of an LTI system, we see that:

1. The system is not stable and not causal in case I.
2. The system is stable, but not causal, in case II.
3. The system is causal, but not stable, in case III.

The region of convergence of the transfer function $H^z(z)$ of an LTI system that is both causal and stable must include the unit circle and all points outside it. In particular, $H^z(z)$ may possess no singularities in that region. An alternative statement of this property is:

All the singularities of the transfer function of a stable and causal LTI system must be inside the unit circle.

7.4 Systems Described by Difference Equations

7.4.1 Difference Equations

A differential equation relates a function $y(t)$ and a few of its derivatives to another function $x(t)$ and a few of its derivatives. Similarly, a *difference equation* relates a sequence $y[n]$ and a few of its time shifts to another sequence $x[n]$ and a few of its time shifts.

Linear differential equations with constant coefficients are of special importance in the theory of continuous-time LTI systems. Similarly, linear difference equations with constant coefficients are of special importance in the theory of discrete-time LTI systems. A linear difference equation with constant coefficients is, by definition, the following relationship between two sequences $x[n]$ and $y[n]$:

$$\begin{aligned} y[n] &= -a_1 y[n-1] - \cdots - a_p y[n-p] + b_0 x[n] + b_1 x[n-1] + \cdots + b_q x[n-q] \\ &= -\sum_{i=1}^{p} a_i y[n-i] + \sum_{i=0}^{q} b_i x[n-i]. \end{aligned} \tag{7.45}$$

The numbers p and q are called, respectively, the *denominator* and *numerator orders* of the difference equation. The constants $\{a_1, a_2, \ldots, a_p\}$ are called the *denominator* *coefficients*, and the constants $\{b_0, b_1, \ldots, b_q\}$ are called the *numerator coefficients* (the reason for these names will soon become clear). We assume that $a_p \neq 0$ and $b_q \neq 0$, because otherwise we can delete zero-valued a coefficients or b coefficients and reduce p or q accordingly.

The difference equation (7.45) represents a causal relationship between the input sequence $x[n]$ and the output sequence $y[n]$. In other words, $y[n]$ depends on its own past values and on past and present (but not future) values of $x[n]$. A system whose input-output relationship obeys a difference equation such as (7.45) is linear and time invariant; see Problem 7.11.

The transfer function of a system described by a difference equation can be computed as follows. Take the z-transform of (7.45) and use the linearity and time-shift properties of the z-transform to obtain

$$Y^z(z) = -\sum_{i=1}^{p} a_i z^{-i} Y^z(z) + \sum_{i=0}^{q} b_i z^{-i} X^z(z) = -Y^z(z)\sum_{i=1}^{p} a_i z^{-i} + X^z(z)\sum_{i=0}^{q} b_i z^{-i}, \tag{7.46}$$

hence

$$H^z(z) = \frac{Y^z(z)}{X^z(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}} = \frac{b(z)}{a(z)}. \tag{7.47}$$

A transfer function that has the form (7.47) is called *rational*, since it is a ratio of two polynomials in the variable z^{-1}, namely $a(z)$ and $b(z)$. The polynomials are called the *numerator* and *denominator polynomials* of the transfer function, for obvious reasons.

It is occasionally useful to express (7.47) in positive powers of z. To this end, let b_{q-r} be the first nonzero numerator coefficient. Thus, for example, $r = q$ if $b_0 \neq 0$, etc. Then we can write from (7.47)

$$H^z(z) = \begin{cases} \dfrac{z^{p-q}(b_{q-r}z^r + \cdots + b_q)}{z^p + a_1 z^{p-1} + \cdots + a_p}, & p \geq q, \\[2ex] \dfrac{b_{q-r}z^r + \cdots + b_q}{z^{q-p}(z^p + a_1 z^{p-1} + \cdots + a_p)}, & p < q. \end{cases} \tag{7.48}$$

Now the numerator and the denominator on the right side are polynomials in z. If $p \geq q$, the denominator and numerator degrees are p and $p - q + r$, respectively. If $p < q$, the denominator and numerator degrees are q and r, respectively. In any case, since $r \leq q$, the numerator degree is not larger than the denominator degree.

A rational transfer function in positive powers of the variable (s in continuous time, z in discrete time) whose numerator degree is not larger than its denominator degree is called *proper*. The transfer function in (7.48) is proper because the system it represents is causal.[7] If the denominator and numerator degrees are equal, the transfer function is said to be *exactly proper*. This happens when $r = q$, that is, when $b_0 \neq 0$. The transfer function is exactly proper if $y[n]$ depends on the present value of $x[n]$, not only on its past values. If $b_0 = 0$, then $y[n]$ depends only on past values of $x[n]$. In this case the numerator degree of (7.48) will be strictly less than the denominator degree. Such a transfer function is called *strictly proper*.

7.4.2 Poles and Zeros

According to the fundamental theorem of algebra, a polynomial $P(z)$ of degree n has exactly n roots, including multiplicities. Thus, the polynomial can be written as a product of first-order factors

$$P(z) = \prod_{i=1}^{n}(z - \lambda_i). \tag{7.49}$$

The roots λ_i can be either real or complex. However, if the coefficients of the polynomial are real, complex roots come in conjugate pairs; in other words, if λ is a root of multiplicity m, then $\bar{\lambda}$ is also a root of multiplicity m.

Let n_{d}, n_{n} denote, respectively, the degrees of the denominator and numerator polynomials in (7.48), in either of the two cases. Let $\{\alpha_i,\ 1 \le i \le n_{\mathrm{d}}\}$ be the roots of the denominator polynomial in (7.48) and $\{\beta_i,\ 1 \le i \le n_{\mathrm{n}}\}$ the roots of the numerator polynomial. Then we can write (7.48) as

$$H^z(z) = b_{q-r} z^{p-q} \frac{\prod_{i=1}^{r}(z-\beta_i)}{\prod_{i=1}^{p}(z-\alpha_i)} = b_{q-r} \frac{\prod_{i=1}^{n_{\mathrm{n}}}(z-\beta_i)}{\prod_{i=1}^{n_{\mathrm{d}}}(z-\alpha_i)}. \tag{7.50}$$

The numbers α_i are called the *poles* of the system, and the numbers β_i are called the *zeros*. These numbers can be real or complex; however, complex poles or zeros come in conjugate pairs if the coefficients of the difference equation (7.45) are real. Of the poles, exactly p are nonzero, because of the condition $a_p \neq 0$. There are $q-p$ additional poles at $z = 0$ if $q > p$. Of the zeros, exactly r are nonzero, because of the condition $b_q \neq 0$. There are $p - q$ additional zeros at $z = 0$ if $p > q$.

Two polynomials are said to be *coprime* if no polynomial of degree 1 or more is a factor of both. Thus, two polynomials are coprime if and only if they have no root in common. If the numerator and denominator polynomials of a rational transfer function are coprime, the corresponding system is said to be *minimal*. The reason for this name is that the system cannot be described by a difference equation having smaller p and q. If the two polynomials do have a common factor, the system is said to be *nonminimal*. A nonminimal system can be brought to a minimal form by canceling the greatest common factor of the numerator and denominator polynomials. In dealing with rational transfer functions, we assume that the function is given in a minimal form unless stated otherwise.

7.4.3 Partial Fraction Decomposition

Another useful form of a rational transfer function is as a *partial fraction decomposition*. Partial fraction decomposition can be derived from either (7.47) (with negative powers of z) or (7.48) (with positive powers of z). It will turn out to be more useful to work with negative powers of z. Consider first the case where $q \ge p$. Then we can write (7.47) in the form

$$H^z(z) = c_0 + \cdots + c_{q-p} z^{-(q-p)} + \frac{d_0 + \cdots + d_{p-1} z^{-(p-1)}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}} = c(z) + \frac{d(z)}{a(z)}. \tag{7.51}$$

The coefficients of the new polynomials can be obtained by equating coefficients of powers of z^{-1} in the equality

$$c(z)a(z) + d(z) = b(z). \tag{7.52}$$

We have $q + 1$ equations in $q + 1$ unknowns: the numbers $\{c_0, \dots, c_{q-p}, d_0, \dots, d_{p-1}\}$. These equations can be written explicitly as follows:

$$\sum_{i=0}^{\min\{q-p,k\}} c_i a_{k-i} + d_k = b_k, \quad 0 \le k \le p-1, \tag{7.53a}$$

$$\sum_{i=k-p}^{\min\{q-p,k\}} c_i a_{k-i} = b_k, \quad p \le k \le q. \tag{7.53b}$$

It can be shown that this set of equations is always nonsingular, so there is always a unique solution.

The case $q < p$ is a special case of (7.51), with $c(z) = 0$ and $d(z) = b(z)$. Thus, by computing the partial fraction decomposition of $d(z)/a(z)$ we shall cover all cases. Assume, further, that all the poles of the system are *simple*; that is, no two of them are equal. Then we have

$$\frac{d(z)}{a(z)} = \frac{d_0 + d_1 z^{-1} + \cdots + d_{p-1} z^{-(p-1)}}{\prod_{i=1}^{p}(1 - \alpha_i z^{-1})} = \sum_{i=1}^{p} \frac{A_i}{1 - \alpha_i z^{-1}}. \tag{7.54}$$

The coefficient A_k can be found by multiplying (7.54) by $(1 - \alpha_k z^{-1})$ and then substituting $z = \alpha_k$. This gives

$$A_k = \frac{d_0 + d_1 \alpha_k^{-1} + \cdots + d_{p-1} \alpha_k^{-(p-1)}}{\prod_{i \neq k}(1 - \alpha_i \alpha_k^{-1})} = \frac{d_0 \alpha_k^{p-1} + d_1 \alpha_k^{p-2} + \cdots + d_{p-1}}{\prod_{i \neq k}(\alpha_k - \alpha_i)}. \tag{7.55}$$

Since the α_i are all distinct, the denominator is nonzero. Therefore, A_k is well defined.

In summary, the partial fraction decomposition of $H^z(z)$ is given by

$$H^z(z) = c_0 + \cdots + c_{q-p} z^{-(q-p)} + \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k z^{-1}}, \tag{7.56}$$

with A_k as in (7.55). We remark that (7.56) can be generalized to transfer functions of systems with multiple poles. However, such systems are rare in digital signal processing applications, so this generalization is of little use to us and we shall not discuss it here.

The procedure `tf2pf` in Program 7.1 implements the partial fraction decomposition (7.56) in MATLAB. The first part of the program computes $c(z)$ by solving the linear equations (7.53). The matrix `temp` is the coefficient matrix of the linear equations. It is built using the MATLAB function `toeplitz`. A *Toeplitz* matrix is a matrix whose elements are equal along the diagonals. Such a matrix is uniquely defined by its first column and first row. You are encouraged to learn more about this function by typing `help toeplitz`. The second part of the program computes the α_k and A_k. The MATLAB function `residue` can also be used for partial fraction decomposition. However, `residue` was programmed to deal with polynomials expressed in positive powers of the argument, so it is more suitable for transfer functions in the s domain. It can be adapted for use with negative powers, but this requires care.

The procedure `pf2tf` in Program 7.2 implements the inverse operation, that is, the conversion of partial fraction decomposition to a transfer function. It iteratively brings the partial fractions under a common denominator, using convolution to perform multiplication of polynomials. It takes the real part of the results at the end, since it implicitly assumes that the transfer function is real.

7.4.4 Stability of Rational Transfer Functions

A causal LTI system is stable if and only if it possesses no singularities in the domain $|z| \geq 1$. For a rational system, this is equivalent to saying that all poles are inside the unit circle. Testing the stability of a rational LTI system by explicitly computing its poles may be inconvenient, however, if the order p of the denominator polynomial is high. There exist several tests for deciding whether a polynomial $a(z)$ is stable, that is, has all its roots inside the unit circle, without explicitly finding the roots. The best known are the *Jury test* and the *Schur-Cohn test*. Here we describe the latter.

Let there be given a monic pth-order polynomial in powers of z^{-1}; *monic* means that the coefficient of z^0 is 1. Denote the polynomial as

$$a_p(z) = 1 + a_{p,1} z^{-1} + \cdots + a_{p,p} z^{-p} \tag{7.57}$$

(the reason for labeling the coefficients with two indices will become clear presently). Since $a_{p,p}$ is, up to a sign, the product of all roots of $a_p(z)$, the condition $|a_{p,p}| < 1$ is necessary for stability of $a_p(z)$. This condition is not sufficient, however. For example, the polynomial $1 + 4z^{-1} + 0.5z^{-2}$ is not stable. The Schur-Cohn test begins by testing whether $|a_{p,p}| < 1$. If this condition is violated, the polynomial is not stable. If it holds, the polynomial is reduced to a degree $p - 1$ by computing

$$a_{p-1}(z) = (1 - a_{p,p}^2)^{-1}[a_p(z) - a_{p,p}z^{-p}a_p(z^{-1})]. \tag{7.58}$$

The coefficients of the reduced-degree polynomial are

$$a_{p-1,k} = (1 - a_{p,p}^2)^{-1}[a_{p,k} - a_{p,p}a_{p,p-k}], \quad 0 \le k \le p - 1. \tag{7.59}$$

The polynomial $a_{p-1}(z)$ is monic since $a_{p-1,0}$, as given by (7.59), is equal to 1. The Schur-Cohn theorem asserts that, given that $|a_{p,p}| < 1$, the polynomial $a_p(z)$ is stable if and only if $a_{p-1}(z)$ is stable. Since $a_{p-1}(z)$ has degree lower than that of $a_p(z)$, we can repeat the test recursively. At the mth step we are given a polynomial

$$a_{p-m}(z) = 1 + a_{p-m,1}z^{-1} + \cdots + a_{p-m,p-m}z^{-(p-m)}. \tag{7.60}$$

We first test if $|a_{p-m,p-m}| < 1$. If not, we conclude that the initial polynomial $a_p(z)$ is not stable and terminate the recursion. If yes, we perform the degree-reduction step

$$a_{p-m-1}(z) = (1 - a_{p-m,p-m}^2)^{-1}[a_{p-m}(z) - a_{p-m,p-m}z^{-(p-m)}a_{p-m}(z^{-1})], \tag{7.61}$$

or, equivalently,

$$a_{p-m-1,k} = (1 - a_{p-m,p-m}^2)^{-1}[a_{p-m,k} - a_{p-m,p-m}a_{p-m,p-m-k}], \ 0 \le k \le p - m - 1. \tag{7.62}$$

We then decrease m by 1 and repeat. If the procedure does not terminate prior to $m = p$, we conclude that the initial polynomial $a_p(z)$ is stable.

A proof of the Schur-Cohn test is beyond the scope of this book. The procedure `sctest` in Program 7.3 implements the test. In this procedure, the conversion of the polynomial to monic (by dividing by the leading coefficient) is performed first, thus enabling the procedure to handle nonmonic polynomials. Next the magnitude of the last coefficient is tested. If it is not smaller than 1, the procedure terminates with a negative result. Otherwise the reduced-degree polynomial is computed and the program calls itself recursively. There is no explicit multiplication by $(1 - a_{p-m,p-m}^2)^{-1}$, since the conversion to monic at the beginning takes care of this. If the degree of the input polynomial is zero, the program exits with positive result.

Example 7.2 Consider the second-order polynomial

$$a_2(z) = 1 + a_1z^{-1} + a_2z^{-2}.$$

The first step in the Schur-Cohn test gives the necessary condition for stability

$$|a_2| < 1.$$

The reduced-degree polynomial is

$$a_1(z) = 1 + \frac{a_1(1 - a_2)}{1 - a_2^2}z^{-1} = 1 + \frac{a_1}{1 + a_2}z^{-1}.$$

The second step in the test gives the additional condition

$$-1 < \frac{a_1}{1 + a_2} < 1.$$

In summary, $a_2(z)$ is stable if and only if

$$-1 < a_2 < 1, \quad 1 + a_1 + a_2 > 0, \quad 1 - a_1 + a_2 > 0. \tag{7.63}$$

Figure 7.3 shows the so-called triangle of stability of the coefficients of a second-order polynomial. □

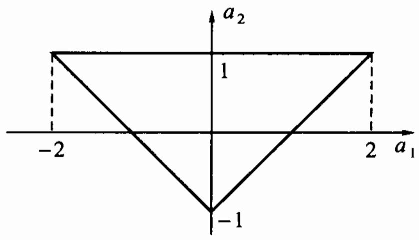

Figure 7.3 The triangle of stability of a second-order polynomial; points inside the triangle correspond to a stable polynomial.

7.4.5 The Noise Gain of Rational Transfer Functions*

In Section 2.9 we introduced the noise gain of an LTI system. The noise gain measures the ratio between the output and input variances of the system when the input is white noise. Here we show how to compute the noise gain of a system whose transfer function is rational and stable.

By Parseval's theorem,

$$\mathrm{NG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H^{\mathrm{f}}(\theta)|^2 d\theta = \sum_{n=-\infty}^{\infty} |h[n]|^2. \tag{7.64}$$

Therefore, in the special case of a transfer function whose denominator polynomial is $a(z) = 1$, we get

$$\mathrm{NG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |b(e^{j\theta})|^2 d\theta = \sum_{k=0}^{q} b_k^2. \tag{7.65}$$

In this case, the noise gain can be easily computed from the numerator coefficients.

When $a(z) \neq 1$, the sum in (7.64) is infinite, so it cannot be computed exactly in a direct manner. An indirect procedure can be used for computing the noise gain exactly, as shown in the following derivation. Let

$$G^{\mathrm{f}}(\theta) = |H^{\mathrm{f}}(\theta)|^2 = \sum_{n=-\infty}^{\infty} g[n]e^{-j\theta n}. \tag{7.66}$$

We have from (2.137) that the noise gain is

$$\mathrm{NG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H^{\mathrm{f}}(\theta)|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} G^{\mathrm{f}}(\theta) d\theta = g[0]. \tag{7.67}$$

However, (7.66) also implies that

$$G^{z}(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} = H^{z}(z)H^{z}(z^{-1}) = \frac{b(z)b(z^{-1})}{a(z)a(z^{-1})}. \tag{7.68}$$

We assume for now that the numerator and denominator polynomials have equal degrees, that is, $p = q$. To find the constant coefficient in the series decomposition of (7.68), it is convenient to express the right side in the form

$$\frac{b(z)b(z^{-1})}{a(z)a(z^{-1})} = \frac{c(z)}{a(z)} + \frac{c(z^{-1})}{a(z^{-1})}, \tag{7.69}$$

where $c(z)$ is a polynomial of degree p in powers of z^{-1}. Such a polynomial exists if and only if

$$a(z)c(z^{-1}) + a(z^{-1})c(z) = b(z)b(z^{-1}). \tag{7.70}$$

This equation can be written explicitly in terms of the coefficients of the three polynomials as

$$\begin{aligned}(1 + \cdots + a_p z^{-p})(c_0 + \cdots + c_p z^p) + (1 + \cdots + a_p z^p)(c_0 + \cdots + c_p z^{-p})\\ = (b_0 + b_1 z^{-1} + \cdots + b_p z^{-p})(b_0 + b_1 z + \cdots + b_p z^p).\end{aligned} \tag{7.71}$$

Equation (7.71) holds if and only if the coefficients of z^i are equal on both sides for all $-p \le i \le p$. Because of the symmetry in this equation, however, we need look only at $0 \le i \le p$. We then get that (7.71) is equivalent to the set of linear equations

$$\left\{\begin{bmatrix} 1 & a_1 & \dots & a_p \\ 0 & 1 & \dots & a_{p-1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} + \begin{bmatrix} 1 & a_1 & \dots & a_p \\ a_1 & a_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ a_p & 0 & \dots & 0 \end{bmatrix}\right\} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & \dots & b_p \\ 0 & b_0 & \dots & b_{p-1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & b_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \end{bmatrix}. \tag{7.72}$$

The coefficient matrix in (7.72) is nonsingular if $a(z)$ is a stable polynomial (we shall not prove this here). Therefore, (7.72) has a unique solution for the coefficients c_k. This implies that the decomposition (7.69) exists and is unique.

Now that we have established the existence and uniqueness of (7.69), let us denote by $f[n]$ the impulse response of $c(z)/a(z)$, so

$$\frac{c(z)}{a(z)} = \sum_{n=0}^{\infty} f[n]z^{-n}. \tag{7.73}$$

Then we have by symmetry,

$$\frac{c(z^{-1})}{a(z^{-1})} = \sum_{n=0}^{\infty} f[n]z^{n}. \tag{7.74}$$

Therefore we can write, from (7.68), (7.69),

$$\sum_{n=-\infty}^{\infty} g[n]z^{-n} = \sum_{n=0}^{\infty} f[n](z^{-n} + z^n). \tag{7.75}$$

In particular,

$$g[0] = \mathrm{NG} = 2f[0]. \tag{7.76}$$

From (7.74), however, when substituting $z = 0$, we see that $f[0] = c_0$. Therefore,

$$\mathrm{NG} = 2c_0. \tag{7.77}$$

In summary, the noise gain can be computed by solving the linear equations (7.72); then $2c_0$ is the desired result.

The procedure `nsgain` in Program 7.4 implements the noise-gain computation in MATLAB. The case $p = 0$ is handled first, using (7.65). The case $p \ne q$ (but $p \ne 0$) is handled by extending one of the polynomials to degree $\max\{p, q\}$ by adding zero-valued coefficients. The coefficient matrix of the linear equations is built by calling the functions `toeplitz` and `hankel`. We have already commented on Toeplitz matrices when discussing partial fraction decomposition. A *Hankel* matrix is a matrix whose elements are equal along the antidiagonals. Such a matrix is uniquely defined by its first column and last row. You are encouraged to learn more about this function by typing `help hankel`.

7.5 Inversion of the z-Transform

The general formula for the inverse z-transform is (7.6). However, this formula is seldom used for actual computation of the inverse z-transform. In this section we describe common methods for inverse z-transform calculation.

If the region of convergence includes the unit circle, we can use the unit circle as the contour of integration. Substitution of $z = e^{j\theta}$ and $dz = je^{j\theta}d\theta$ in (7.6) gives

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{z}(e^{j\theta})e^{j\theta n}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{f}(\theta)e^{j\theta n}d\theta, \tag{7.78}$$

so the inverse z-transform coincides with an inverse Fourier transform. If the region of convergence does not include the unit circle, we can take a circle with radius r (greater or smaller than 1, depending on the region of convergence) and substitute $z = re^{j\theta}$ and $dz = jre^{j\theta}d\theta$ to get

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{z}(re^{j\theta})r^{n}e^{j\theta n}d\theta = r^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} X^{z}(re^{j\theta})e^{j\theta n}d\theta. \tag{7.79}$$

Therefore, the inverse z-transform can be computed by evaluating the inverse Fourier transform of $X^{z}(re^{j\theta})$ and multiplying the result by the sequence r^n.

Sometimes $X^{z}(z)$ can be expanded as a known power series (e.g., a series available in standard tables). The following example illustrates such a case.

Example 7.3 Let

$$X^{z}(z) = -\log_{e}(1 - az^{-1}) = az^{-1} + \frac{1}{2}a^{2}z^{-2} + \frac{1}{3}a^{3}z^{-3} + \cdots, \quad |z| > |a|. \tag{7.80}$$

This series is given in most standard series tables. Therefore,

$$x[n] = \begin{cases} \frac{1}{n}a^{n}, & n \geq 1, \\ 0, & n < 1. \end{cases} \tag{7.81}$$

□

A general and powerful tool for inverse z-transform computation is the Cauchy residue theorem. If a function $f(z)$ is analytic in a domain in the complex plane except at a single point z_0 in the domain, the *residue* of $f(z)$ at z_0 is

$$\text{res}\{f(z) \text{ at } z_0\} = \frac{1}{2\pi j}\oint f(z)dz, \tag{7.82}$$

where the contour of integration lies in the domain of analyticity of $f(z)$ and circles z_0 once in the counterclockwise direction. If $f(z)$ has a finite number of singularities inside the contour, say $\{z_k,\ 1 \leq k \leq K\}$, and is analytic otherwise on the contour and inside it, then

$$\frac{1}{2\pi j}\oint f(z)dz = \sum_{k=1}^{K} \text{res}\{f(z) \text{ at } z_k\}. \tag{7.83}$$

Equation (7.83) is the Cauchy residue theorem.

If $f(z)$ is singular at z_0 and there exists an integer m such that

$$g(z) \triangleq f(z)(z - z_0)^{m}$$

is analytic at z_0, then z_0 is said to be a *pole* of order m. A singularity point that is not an mth-order pole for any m is called an *essential singularity*. The residue of $f(z)$ at an mth-order pole z_0 is given by

$$\text{res}\{f(z) \text{ at } z_0\} = \frac{1}{(m-1)!}\left.\frac{d^{m-1}g(z)}{dz^{m-1}}\right|_{z=z_0}. \tag{7.84}$$

The Cauchy residue theorem can be used for inverse z-transform computation as follows. Assume that $X^z(z)$ has only a finite number of poles and no essential singularities in the region $0 \le |z| \le R_1$ (where R_1 is the inner radius of the region of convergence). Then, according to (7.83),

$$x[n] = \frac{1}{2\pi j}\oint X^z(z)z^{n-1}dz = \sum_{k=1}^{K_n} \text{res}\{X^z(z)z^{n-1} \text{ at } z_{k,n}\}, \tag{7.85}$$

where $\{z_{k,n},\ 1 \le k \le K_n\}$ are the residues of $X^z(z)z^{n-1}$ in the region $0 \le |z| \le R_1$. The poles of $X^z(z)z^{n-1}$ in this region are the poles of $X^z(z)$, and possibly a pole at $z = 0$. The residues can be computed from (7.84).

Example 7.4 Let

$$X^z(z) = \frac{z}{(z-a)^2}, \quad |z| > |a|. \tag{7.86}$$

Then

$$X^z(z)z^{n-1} = \frac{z^n}{(z-a)^2}, \quad |z| > |a|. \tag{7.87}$$

For $n \ge 0$, the only singular point is $z = a$, which is a second-order pole. At this point we have $g(z) = z^n$, so

$$\left.\frac{dg(z)}{dz}\right|_{z=a} = \left. nz^{n-1}\right|_{z=a} = na^{n-1}. \tag{7.88}$$

For $n < 0$, the two singular points are $z = a$ (a second-order pole, as before) and $z = 0$, which is a pole of order $m = -n$. At the point $z = 0$ we have $g(z) = (z-a)^{-2}$, so

$$\begin{aligned}\left.\frac{1}{(m-1)!}\frac{d^{m-1}g(z)}{dz^{m-1}}\right|_{z=0} &= \left.\frac{m!}{(m-1)!}(a-z)^{-(m+1)}\right|_{z=0} \\ &= ma^{-(m+1)} = -na^{n-1}.\end{aligned} \tag{7.89}$$

The residue at $z = a$ is na^{n-1}, as before. Therefore, the sum of the two residues is zero. In summary,

$$x[n] = \begin{cases} na^{n-1}, & n \ge 0, \\ 0, & n < 0. \end{cases} \tag{7.90}$$

□

Our main interest here is the case of $X^z(z)$ rational and, in particular, $X^z(z)$ is the transform of a *causal* sequence. If all the poles of $X^z(z)$ are distinct, it can be written as a partial fraction decomposition [cf. (7.56)]

$$X^z(z) = \sum_{i=0}^{q-p} c_i z^{-i} + \sum_{k=1}^{p} \frac{A_k}{1-\alpha_k z^{-1}}. \tag{7.91}$$

Therefore we have, by the linearity and time-shift properties, and by Example 7.1, part 2,

$$x[n] = \sum_{i=0}^{q-p} c_i\delta[n-i] + \sum_{k=1}^{p} A_k\alpha_k^n, \quad n \ge 0. \tag{7.92}$$

The poles α_k can be real or complex. If the coefficients of the rational function $X^z(z)$ are real, complex poles must appear in conjugate pairs. Assume, for example, that α_1 and α_2 are conjugates of each other, say

$$\alpha_1 = \rho e^{j\phi}, \quad \alpha_2 = \rho e^{-j\phi}.$$

Then A_1 and A_2 must also be conjugate of each other, say

$$A_1 = a + jb, \quad A_2 = a - jb.$$

Then

$$\begin{aligned} A_1\alpha_1^n + A_2\alpha_2^n &= (a + jb)\rho^n e^{jn\phi} + (a - jb)\rho^n e^{-jn\phi} \\ &= \rho^n(a + jb)[\cos(n\phi) + j\sin(n\phi)] + \rho^n(a - jb)[\cos(n\phi) - j\sin(n\phi)] \\ &= 2\rho^n[a\cos(n\phi) - b\sin(n\phi)]. \end{aligned} \tag{7.93}$$

This conversion can be performed for all complex pairs in (7.92), thus bringing it to a purely real form.

The procedure `invz` in Program 7.5 implements the computation of the inverse z-transform of a rational causal transfer function. Inverse z-transform computation by partial fraction decomposition can be generalized to the case of repeated poles, as well as to the case of rational noncausal transfer functions. These generalizations are not discussed here.

When learning the z-transform, you should get acquainted with the characteristic shapes of sequences associated with different pole locations. The shape is mainly determined by (1) whether the pole is real or complex, and (2) whether it is inside, on, or outside the unit circle. Figure 7.4 illustrates the sequences associated with real poles for six different cases: $\alpha > 1$, $\alpha = 1$, $0 < \alpha < 1$, $-1 < \alpha < 0$, $\alpha = -1$, and $\alpha < -1$. As we see, the sequences are divergent when $|\alpha| > 1$, convergent when $|\alpha| < 1$, and have constant amplitude when $|\alpha| = 1$. When $\alpha < 0$, the sequence is oscillatory. This is in contrast with continuous-time signals, which are never oscillatory when the corresponding pole is real.

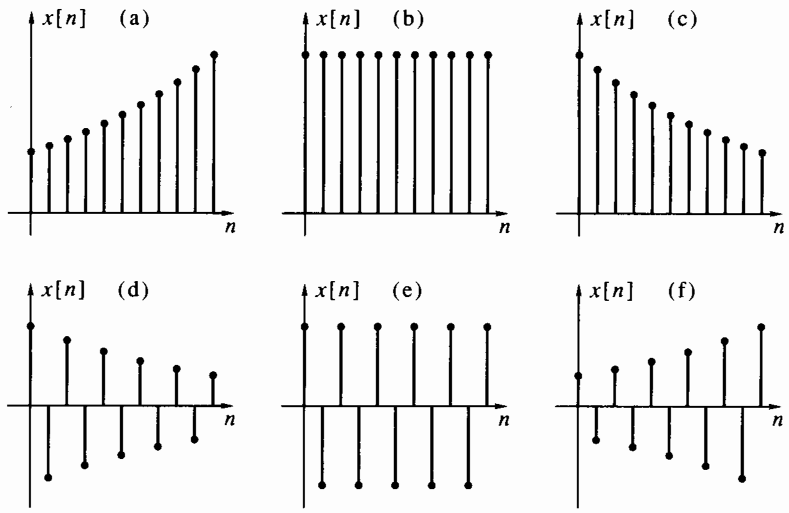

Figure 7.4 The sequences associated with a single real pole at $z = \alpha$ in the z-transform: (a) $\alpha > 1$, (b) $\alpha = 1$, (c) $0 < \alpha < 1$, (d) $-1 < \alpha < 0$, (e) $\alpha = -1$, (f) $\alpha < -1$.

Figure 7.5 illustrates the sequences associated with complex poles for six different cases: $|\alpha| > 1$ and $\measuredangle\alpha < 0.5\pi$, $|\alpha| = 1$ and $\measuredangle\alpha < 0.5\pi$, $|\alpha| < 1$ and $\measuredangle\alpha < 0.5\pi$, $|\alpha| < 1$ and $\measuredangle\alpha > 0.5\pi$, $|\alpha| = 1$ and $\measuredangle\alpha > 0.5\pi$, and finally $|\alpha| > 1$ and $\measuredangle\alpha > 0.5\pi$. The convergence-divergence behavior is the same as for real poles. However, in the case $\alpha = 1$, the constant amplitude property is not evident, because a discrete-time sinusoidal signal is not necessarily periodic. The sequences are always oscillatory and in general appear quite irregular.

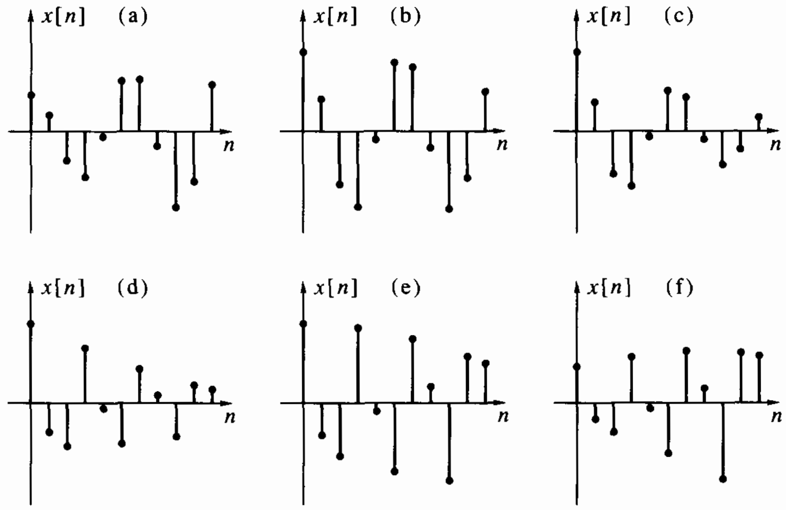

Figure 7.5 The sequences associated with a pair of complex poles at $z = \alpha, z = \bar{\alpha}$ in the z-transform: (a) $|\alpha| > 1$, $\measuredangle\alpha < 0.5\pi$, (b) $|\alpha| = 1$, $\measuredangle\alpha < 0.5\pi$, (c) $|\alpha| < 1$, $\measuredangle\alpha < 0.5\pi$, (d) $|\alpha| < 1$, $\measuredangle\alpha > 0.5\pi$, (e) $|\alpha| = 1$, $\measuredangle\alpha > 0.5\pi$, (f) $|\alpha| > 1$, $\measuredangle\alpha > 0.5\pi$.

7.6 Frequency Responses of Rational Transfer Functions

The frequency response of a BIBO-stable system is obtained from the transfer function by the substitution $z = e^{j\theta}$. For rational transfer functions, it is especially convenient to use the pole-zero factorization (7.50) of the transfer function. We have in this case

$$H^{\mathrm{f}}(\theta) = b_{q-r}e^{j\theta(p-q)}\frac{\prod_{i=1}^{r}(e^{j\theta} - \beta_i)}{\prod_{i=1}^{p}(e^{j\theta} - \alpha_i)}. \tag{7.94}$$

Since the magnitude of a product of complex numbers is the product of the magnitudes of the factors, we get from (7.94)

$$|H^{\mathrm{f}}(\theta)| = b_{q-r}\frac{\prod_{i=1}^{r}|e^{j\theta} - \beta_i|}{\prod_{i=1}^{p}|e^{j\theta} - \alpha_i|}. \tag{7.95}$$

Similarly, since the phase of a product of complex numbers is the sum of the phases of the factors, we get

$$\measuredangle H^{\mathrm{f}}(\theta) = \theta(p-q) + \sum_{i=1}^{r}\measuredangle(e^{j\theta} - \beta_i) - \sum_{i=1}^{p}\measuredangle(e^{j\theta} - \alpha_i). \tag{7.96}$$

Figure 7.6 illustrates how to determine the magnitude and phase of the frequency response from the pole-zero plot of the transfer function. Each frequency θ is represented by a point on the unit circle. The transfer function represented by this figure is

$$H^{\mathrm{z}}(z) = \frac{1 + 0.2z^{-1}}{[1 - 0.867z^{-1}][1 - (0.067 + j0.867)z^{-1}][1 - (0.067 - j0.867)z^{-1}]}.$$

The complex numbers $(e^{j\theta} - \beta_i)$ are represented by vectors pointing from the zero locations β_i to the point on the unit circle. Similarly, the complex numbers $(e^{j\theta} - \alpha_i)$

are represented by vectors pointing from the pole locations α_i to the point on the unit circle. To compute the magnitude response at a specific θ, we must form the product of magnitudes of all vectors pointing from the zeros, then form the product of magnitudes of all vectors pointing from the poles, divide the former by the latter, and finally multiply by the constant factor b_{q-r}. To compute the phase response at a specific θ, we must form the sum of angles of all vectors pointing from the zeros, then form the sum of angles of all vectors pointing from the poles, subtract the latter from the former, and finally add the linear-phase term $\theta(p-q)$. For the transfer function represented by Figure 7.6 we get the frequency response shown in Figure 7.7.

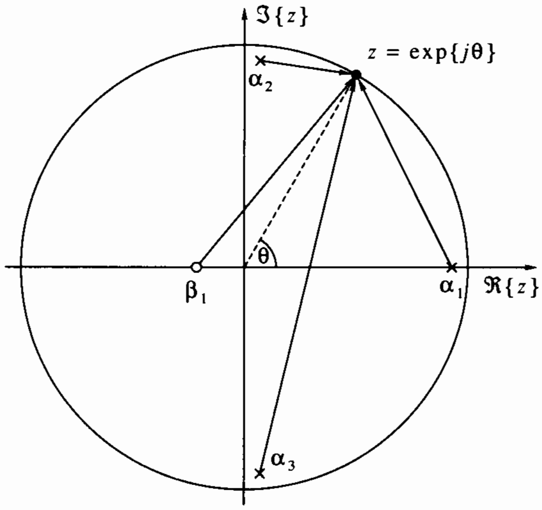

Figure 7.6 Using the pole-zero plot to obtain the frequency response.

The graphical procedure leads to the following observations:

1. A real pole near $z = 1$ results in a high DC gain, whereas a real pole near $z = -1$ results in a high gain at $\theta = \pi$.
2. Complex poles near the unit circle result in a high gain near the frequencies corresponding to the phase angles of the poles.
3. A real zero near $z = 1$ results in a low DC gain, whereas a real zero near $z = -1$ results in a low gain at $\theta = \pi$.
4. Complex zeros near the unit circle result in a low gain near the frequencies corresponding to the phase angles of the poles.

Examination of the pole-zero pattern of the transfer function thus permits rapid, although coarse, estimation of the general nature of the frequency response.

A MATLAB implementation of frequency response computation of a rational system does not require a pole-zero factorization, but can be performed directly on the polynomials of the transfer function. The procedure `frqresp` in Program 7.6 illustrates this computation. The program has three modes of operation. In the first mode, only the polynomial coefficients and the desired number of frequency points K are given. The program then selects K equally spaced points on the interval $[0, \pi]$ and performs the computation by dividing the zero-padded FFTs of the numerator and the denominator. In the second mode, the program is given a number of points and a frequency interval. The program then selects K equally spaced points on the given interval and

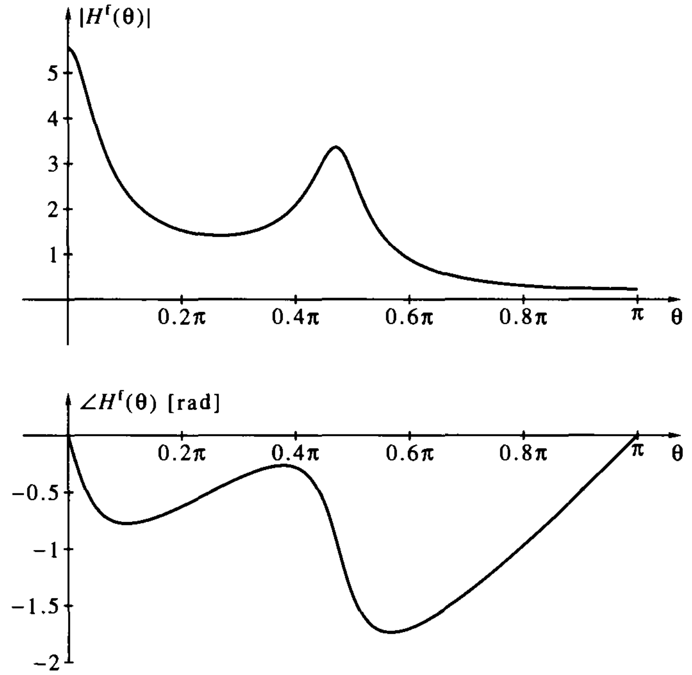

Figure 7.7 The frequency response corresponding to the pole-zero plot in Figure 7.6.

performs the computation by dividing the chirp Fourier transforms of the numerator and the denominator. In the third mode, the program is given a vector of individual frequencies, not necessarily equally spaced, and performs the computation point by point. We remark that the MATLAB function `freqz`, located in the Signal Processing Toolbox, performs a similar computation.

7.7 The Unilateral z-Transform

The *unilateral*, or *one-sided*, z-transform of a sequence $x[n]$ is defined as

$$X_+^z(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{7.97}$$

The definition applies to general sequences, whether causal or not. Equality of $X^z(z)$ and $X_+^z(z)$ holds if and only if the sequence $x[n]$ is causal.

The region of convergence of the unilateral z-transform includes the open annulus

$$R_1 < |z| < \infty, \tag{7.98}$$

where

$$R_1 = \inf\left\{r \geq 0 : \sum_{n=0}^{\infty} |x[n]|r^{-n} < \infty\right\}. \tag{7.99}$$

In addition, it may or may not include the boundary $|z| = R_1$.

The inverse unilateral z-transform formula is identical to that of the bilateral transform, (7.6). However, it yields only the values of $x[n]$ for $n \geq 0$. The unilateral z-transform provides no information on values of $x[n]$ for $n < 0$.

The unilateral z-transform shares certain, but not all properties of the bilateral z-transform. Of special interest to us here are its two shift properties, given in the following theorem.

Theorem 7.5 Let

$$y[n] = x[n+m], \quad m > 0, \tag{7.100a}$$

$$u[n] = x[n-m], \quad m > 0. \tag{7.100b}$$

Then,

$$Y_+^z(z) = z^m X_+^z(z) - \sum_{i=0}^{m-1} x[i] z^{m-i}, \tag{7.101a}$$

$$U_+^z(z) = z^{-m} X_+^z(z) + \sum_{i=-m}^{-1} x[i] z^{-(m+i)}. \tag{7.101b}$$

Proof of (7.101a)

$$\begin{aligned} Y_+^z(z) &= \sum_{n=0}^{\infty} x[n+m] z^{-n} = z^m \sum_{n=0}^{\infty} x[n+m] z^{-(n+m)} = z^m \sum_{i=m}^{\infty} x[i] z^{-i} \\ &= z^m \left[\sum_{i=0}^{\infty} x[i] z^{-i} - \sum_{i=0}^{m-1} x[i] z^{-i} \right] = z^m X_+^z(z) - \sum_{i=0}^{m-1} x[i] z^{m-i}. \end{aligned} \tag{7.102}$$

Proof of (7.101b)

$$\begin{aligned} U_+^z(z) &= \sum_{n=0}^{\infty} x[n-m] z^{-n} = z^{-m} \sum_{n=0}^{\infty} x[n-m] z^{-(n-m)} = z^{-m} \sum_{i=-m}^{\infty} x[i] z^{-i} \\ &= z^{-m} \left[\sum_{i=-m}^{-1} x[i] z^{-i} + \sum_{i=0}^{\infty} x[i] z^{-i} \right] = z^{-m} X_+^z(z) + \sum_{i=-m}^{-1} x[i] z^{-(m+i)}. \end{aligned} \tag{7.103}$$

□

Two other properties of the unilateral z-transform are given by the following theorems.

Theorem 7.6 (initial value theorem) The initial value $x[0]$ of the sequence $x[n]$ can be obtained from its unilateral z-transform by

$$x[0] = \lim_{z \to \infty} X_+^z(z). \tag{7.104}$$

Proof Take the limits of both sides of (7.97) as $z \to \infty$ and observe that the only term on the right side that does not go to zero is $x[0]$. □

Theorem 7.7 (final value theorem) Assume that $X_+^z(z)$ is such that $(z-1)X_+^z(z)$ exists on the unit circle. Then

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X_+^z(z). \tag{7.105}$$

Proof Observe that if $(z-1)X_+^z(z)$ exists on the unit circle, then either $X_+^z(z)$ exists on the unit circle, or $X_+^z(z)$ has a simple pole at $z = 1$, which is canceled by the factor $(z-1)$. In the former case the sequence $\{x[n],\ n \geq 0\}$ is stable, so necessarily $\lim_{n \to \infty} x[n] = 0$. Indeed, in this case $\lim_{z \to 1}(z-1)X_+^z(z) = 0$. In the latter case $X_+^z(z)$ admits the additive decomposition

$$X_+^z(z) = \frac{A}{z-1} + B(z), \tag{7.106}$$

where $B(z)$ exists on the unit circle, so it is the transform of a stable sequence. We get the constant A from (7.106), multiplying by $(z-1)$ and taking the limit $z \to 1$:

$$A = \lim_{z \to 1}(z-1)X_+^z(z). \tag{7.107}$$

On the other hand, the inverse z-transform of (7.106) gives

$$x[n] = A + b[n], \quad n \geq 1. \tag{7.108}$$

Taking the limit as $n \to \infty$ and using the stability of $b[n]$ gives

$$A = \lim_{n \to \infty} x[n]. \tag{7.109}$$

□

An important application of the final value theorem is for computing the final (or steady-state) value of the response of a stable and causal LTI system to a unit-step input. The z-transform of a unit step is $1/(1-z^{-1})$, so the steady-state response to a unit step of a system whose transfer function is $H^z(z)$ is

$$\lim_{n \to \infty} y[n] = \lim_{z \to 1} \frac{(z-1)H^z(z)}{(1-z^{-1})} = \lim_{z \to 1} zH^z(z) = H^z(1). \tag{7.110}$$

The quantity $H^z(1)$ is called the *DC gain* of the system. Although $H^z(1)$ may exist for any $H^z(z)$, it is meaningful only for a stable and causal transfer function.

The main use of the unilateral z-transform is for solving difference equations with initial conditions. Consider the general difference equation

$$y[n] = -\sum_{i=1}^{p} a_i y[n-i] + \sum_{i=0}^{q} b_i x[n-i]. \tag{7.111}$$

and assume that we are given the initial conditions

$$\{y[-p], y[-p+1], \ldots, y[-1]\}.$$

Assume also that the input sequence $x[n]$ is causal, so $x[n] = 0$ for $n < 0$. Because the difference equation is linear, we can express its solution as

$$y[n] = y_{\text{zir}}[n] + y_{\text{zsr}}[n], \tag{7.112}$$

where

1. $y_{\text{zir}}[n]$ is the *zero-input response*, or the *natural response*, defined as the solution of (7.111) when the initial conditions are as given above, while the input $x[n]$ is identically zero.
2. $y_{\text{zsr}}[n]$ is the *zero-state response*, or the *forced response*, defined as the solution of (7.111) when the initial conditions are zero and the input $x[n]$ is present.

Computation of $y_{\text{zsr}}[n]$ requires only the techniques we studied in Section 7.5, since

$$Y_{\text{zsr}}^z(z) = \frac{b(z)X^z(z)}{a(z)}. \tag{7.113}$$

If $X^z(z)$ is rational, we can invert the right side of (7.113) using partial fraction decomposition. If $X^z(z)$ is not rational but its numerical values are known, we can compute $y_{\text{zsr}}[n]$ numerically using, for example, the MATLAB function `filter`.

Computation of $y_{\text{zir}}[n]$ involves solving the homogeneous equation

$$y[n] = -\sum_{i=1}^{p} a_i y[n-i], \quad n \geq 0, \tag{7.114}$$

subject to the initial conditions

$$\{y[-p], y[-p+1], \ldots, y[-1]\}.$$

The solution can be obtained using the unilateral z-transform, as follows. We transform (7.114) and use the shift property (7.101) to get

$$Y_+^z(z) = -\sum_{i=1}^{p} a_i \left[z^{-i} Y_+^z(z) + \sum_{l=-i}^{-1} y[l] z^{-(i+l)} \right]. \tag{7.115}$$

Therefore,

$$Y_+^z(z) = -\frac{\sum_{i=1}^{p} a_i \sum_{l=-i}^{-1} y[l] z^{-(i+l)}}{1 + \sum_{i=1}^{p} a_i z^{-i}}. \tag{7.116}$$

This can also be written, after rearrangement of the numerator, as

$$Y_+^z(z) = -\frac{\sum_{i=0}^{p-1} \left[\sum_{l=i-p}^{-1} y[l] a_{i-l} \right] z^{-i}}{a(z)}. \tag{7.117}$$

Finally, $y_{\mathrm{zir}}[n]$ is obtained by computing the inverse z-transform of (7.117).

The procedure `numzir` in Program 7.7 implements the computation of the numerator of (7.117). The zero-input response can then be computed by calling `invz`, or the partial fraction decomposition of (7.117) be computed by calling `tf2pf`, as needed.

7.8 Summary and Complements

7.8.1 Summary

In this chapter we introduced the z-transform of a discrete-time signal (7.1) and discussed its use to discrete-time linear system theory. The z-transform is a complex function of a complex variable. It is defined on a domain in the complex plane called the region of convergence. The ROC usually has the form of an annulus. The inverse z-transform is given by the complex integration formula (7.6).

The z-transform of the impulse response of a discrete-time LTI system is called the transfer function of the system. Of particular interest are systems that are stable in the BIBO sense. For such systems, the ROC of the transfer function includes the unit circle. If the system is also causal, the ROC includes the unit circle and all that is outside it. Therefore, all the singularities of the transfer function of a stable and causal system must be inside the unit circle.

Of special importance are causal LTI systems whose transfer functions are rational functions of z. Such a system can be described by a difference equation (7.45). Its transfer function is characterized by the numerator and denominator polynomials. The roots of these two polynomials are called zeros and poles, respectively. For a stable system, all the poles are inside the unit circle. The stability of a rational transfer function can be tested without explicitly finding the poles, but by means of the Schur-Cohn test, which requires only simple rational operations. A rational transfer function whose poles are simple (i.e., of multiplicity one) can be expressed by partial fraction decomposition (7.56).

Several methods exist for computation of inverse z-transforms. Contour integration is the most general, but usually the least convenient method. The Cauchy residue theorem and power series expansions are convenient in certain cases. Partial fraction decomposition is the preferred method for computing inverse z-transforms of rational functions.

The frequency response of an LTI system can be computed from its transfer function by substituting $z = e^{j\theta}$. This is especially convenient when the transfer function is given in a factored form. Regions of low magnitude response (in the vicinity of zeros) and high magnitude response (in the vicinity of poles) can be determined by visual examination of the pole-zero map of the system.

The unilateral z-transform (7.97) is also useful, mainly for solving difference equations with initial conditions. The solution of such an equation is conveniently expressed as a sum of two terms: the zero-input response and the zero-state response. The former is best obtained by the unilateral z-transform, whereas the latter can be done using the bilateral z-transform.

7.8.2 Complements

1. [p. 205] The earliest references on sampled-data systems, which paved the way to the z-transforms, are by MacColl [1945] and Hurewicz [1947]. The z-transform was developed independently by a number of people in the late 1940s and early 1950s. The definitive reference in the western literature is by Ragazzini and Zadeh [1952] and that in the eastern literature is Tsypkin [1949, 1950]. Barker [1952] and Linvill [1951] have proposed definitions similar to the z-transform. Jury [1954] has invented the modified z-transform, which we shall mention later in this book (Problem 10.35).
2. [p. 206] In complex function theory, the z-transform is a special case of a Laurent series: It is the Laurent series of $X^z(z)$ around the point $z_0 = 0$. The inverse z-transform formula is the inversion formula of Laurent series; see, for example, Churchill and Brown [1984].
3. [p. 206] The region of convergence of the z-transform is defined as the set of all complex numbers z such that the sum of the absolute values of the series converges, as seen in (7.2). Why did we require absolute convergence when the z-transform is defined as a sum of complex numbers, as seen in (7.1)? The reason is that the value of an infinite sum such as (7.1) can potentially depend on the order in which the elements of the series are added. In general, a series $\sum_{n=1}^{\infty} a_n$ may converge (that is, yield a finite result) even when $\sum_{n=1}^{\infty} |a_n| = \infty$. However, in such a case the value of $\sum_{n=1}^{\infty} a_n$ will vary if the order of terms is changed. Such a series is said to be *conditionally convergent*. On the other hand, if the sum of absolute values is finite, the sum will be independent of the order of summation. Such a series is said to be *absolutely convergent*. In the z-transform we are summing a two-sided sequence, so we do not want the sum to depend on the order of terms. The requirement of absolute convergence (7.2) eliminates the problem and guarantees that (7.1) be unambiguous.
4. [p. 206] Remember that (1) the *infimum* of a nonempty set of real numbers $\mathbb{S}$, denoted by $\inf\{\mathbb{S}\}$, is the largest number having the property of being smaller than all members of $\mathbb{S}$; (2) the *supremum* of a nonempty set of real numbers $\mathbb{S}$, denoted by $\sup\{\mathbb{S}\}$, is the smallest number having the property of being larger than all members of $\mathbb{S}$. The infimum is also called *greatest lower bound*; the supremum is also called *least upper bound*. Every nonempty set of real numbers has a unique infimum and a unique supremum. If the set is bounded from above, its supremum is finite; otherwise it is defined as ∞. If the set is bounded from below, its infimum is finite; otherwise it is defined as $-\infty$.
5. [p. 206] The *Cauchy-Hadamard theorem* expresses the radii of convergence R_1 and R_2 explicitly in terms of the sequence values:

$$R_1 = \limsup_{n\to\infty} |x[n]|^{1/n}, \quad R_2 = \liminf_{n\to\infty} |x[-n]|^{-1/n}. \tag{7.118}$$

6. [p. 206] The *extended complex plane* is obtained by adding a single point $z = \infty$ to the conventional complex plane $\mathbb{C}$. The point $z = \infty$ has modulus (magnitude) larger than that of any other complex number; its argument (phase) is undefined. By comparison, the point $z = 0$ has modulus smaller than that of any other complex number and undefined argument. The region of convergence may be extended to include the point $z = \infty$ if and only if the sequence is causal; see the discussion in Example 7.1, part 8.
7. [p. 215] In continuous-time systems, properness is related to realizability, not causality. For example, the continuous-time transfer function $H^L(s) = s$ is not proper. It represents pure differentiation, which is a causal, but not a physically realizable operation.

7.9 MATLAB Programs

Program 7.1 Partial fraction decomposition of a rational transfer function.

```
function [c,A,alpha] = tf2pf(b,a);
% Synopsis: [c,A,alpha] = tf2pf(b,a).
% Partial fraction decomposition of b(z)/a(z). The polynomials are in
% negative powers of z. The poles are assumed distinct.
% Input parameters:
% a, b: the input polynomials
% Output parameters:
% c: the free polynomial; empty if deg(b) < deg(a)
% A: the vector of gains of the partial fractions
% alpha: the vector of poles.

% Compute c(z) and d(z)
p = length(a)-1; q = length(b)-1;
a = (1/a(1))*reshape(a,1,p+1);
b = (1/a(1))*reshape(b,1,q+1);
if (q >= p), % case of nonempty c(z)
   temp = toeplitz([a,zeros(1,q-p)]',[a(1),zeros(1,q-p)]);
   temp = [temp,[eye(p); zeros(q-p+1,p)]];
   temp = temp\b';
   c = temp(1:q-p+1)'; d = temp(q-p+2:q+1)';
else
   c = []; d = [b,zeros(1,p-q-1)];
end

% Compute A and alpha
alpha = cplxpair(roots(a)).'; A = zeros(1,p);
for k = 1:p,
   temp = prod(alpha(k)-alpha(find(1:p ~= k)));
   if (temp == 0), error('Repeated roots in TF2PF');
   else, A(k) = polyval(d,alpha(k))/temp; end
end
```

Program 7.2 Conversion of partial fraction decomposition to a rational transfer function.

```
function [b,a] = pf2tf(c,A,alpha);
% Synopsis: [b,a] = pf2tf(c,A,alpha).
% Conversion of partial fraction decomposition to the form b(z)/a(z).
% The polynomials are in negative powers of z.
% Input parameters:
% c: the free polynomial; empty if deg(b) < deg(a)
% A: the vector of gains of the partial fractions
% alpha: the vector of poles.
% Output parameters:
% a, b: the output polynomials

p = length(alpha); d = A(1); a = [1,-alpha(1)];
for k = 2:p,
   d = conv(d,[1,-alpha(k)]) + A(k)*a;
   a = conv(a,[1,-alpha(k)]);
end
if (length(c) > 0),
   b = conv(c,a) + [d,zeros(1,length(c))];
else, b = d; end
a = real(a); b = real(b);
```

Program 7.3 The Schur-Cohn stability test.

```
function s = sctest(a);
% Synopsis: s = sctest(a).
% Schur-Cohn stability test.
% Input:
% a: coefficients of polynomial to be tested.
% Output:
% s: 1 if stable, 0 if unstable.

n = length(a);
if (n == 1), s = 1; % a zero-order polynomial is stable
else,
   a = reshape((1/a(1))*a,1,n); % make the polynomial monic
   if (abs(a(n)) >= 1), s = 0; % unstable
   else,
      s = sctest(a(1:n-1)-a(n)*fliplr(a(2:n))); % recursion
   end
end
```

Program 7.4 Computation of the noise gain of a rational transfer function.

```
function ng = nsgain(b,a);
% Synopsis: ng = nsgain(b,a).
% Computes the noise gain of a rational system b(z)/a(z).
% Input parameters:
% b, a: the numerator and denominator coefficients.
% Output:
% ng: the noise gain

p = length(a)-1; q = length(b)-1; n = max(p,q);
if (p == 0), ng = sum(b.^2); return, end
a = [reshape(a,1,p+1),zeros(1,n-p)];
b = [reshape(b,1,q+1),zeros(1,n-q)];
mat = toeplitz([1; zeros(n,1)],a) + ...
      hankel(a',[a(n+1),zeros(1,n)]);
vec = toeplitz([b(1); zeros(n,1)],b)*b';
vec = mat\vec; ng = 2*vec(1);
```

Program 7.5 The inverse z-transform of a rational transfer function.

```
function x = invz(b,a,N);
% Synopsis: x = invz(b,a,N).
% Computes first N terms of the inverse z-transform
% of the rational transfer function b(z)/a(z).
% The poles are assumed distinct.
% Input parameters:
% b, a: numerator and denominator input polynomials
% N: number of points to be computed
% Output:
% x: the inverse sequence.

[c,A,alpha] = tf2pf(b,a);
x = zeros(1,N);
x(1:length(c)) = c;
for k = 1:length(A),
   x = x+A(k)*(alpha(k)).^(0:N-1);
end
x = real(x);
```

Program 7.6 Frequency response of a rational transfer function.

```
function H = frqresp(b,a,K,theta);
% Synopsis: H = frqresp(b,a,K,theta).
% Frequency response of b(z)/a(z) on a given frequency interval.
% Input parameters:
% b, a: numerator and denominator polynomials
% K: the number of frequency response points to compute
% theta: if absent, the K points are uniformly spaced on [0, pi];
%        if present and theta is a 1-by-2 vector, its entries are
%        taken as the end points of the interval on which K evenly
%        spaced points are placed; if the size of theta is different
%        from 2, it is assumed to be a vector of frequencies for which
%        the frequency response is to be computed, and K is ignored.
% Output:
% H: the frequency response vector.

if (nargin == 3),
   H = fft(b,2*K-2)./fft(a,2*K-2); H = H(1:K);
elseif (length(theta) == 2),
   t0 = theta(1); dt = (theta(2)-theta(1))/(K-1);
   H = chirpf(b,t0,dt,K)./chirpf(a,t0,dt,K);
else
   H = zeros(1,length(theta));
   for i = 1:length(theta),
      H(i) = sum(b.*exp(-j*theta(i)*(0:length(b)-1)))/ ...
             sum(a.*exp(-j*theta(i)*(0:length(a)-1)));
   end
end
```

Program 7.7 Computation of the numerator of (7.117).

```
function b = numzir(a,yinit);
% Synopsis: b = numzir(a,yinit).
% Compute the numerator polynomial for finding the zero-input
% response of the homogeneous equation a(z)y(z) = 0.
% Input parameters:
% a: the coefficient polynomial of the homogeneous equation
% yinit: the vector of y[-1], y[-2], ..., y[-p].
% Output:
% b: the numerator.

p = length(a) - 1; a = fliplr(reshape(-a(2:p+1),1,p));
b = conv(reshape(yinit,1,p),a); b = fliplr(b(1:p));
```

7.10 Problems

7.1 Derive the z-transforms of

$$x_1[n] = \begin{cases} \cos(\theta_0 n), & n \ge 0, \\ 0, & n < 0, \end{cases} \qquad x_2[n] = \begin{cases} \sin(\theta_0 n), & n \ge 0, \\ 0, & n < 0, \end{cases}$$

$$x_3[n] = \begin{cases} n\cos(\theta_0 n), & n \ge 0, \\ 0, & n < 0, \end{cases} \qquad x_4[n] = \begin{cases} n\sin(\theta_0 n), & n \ge 0, \\ 0, & n < 0. \end{cases}$$

Specify the region of convergence of the transforms.

7.2 Find the z-transform of

$$x[n] = \begin{cases} n, & 1 \le n \le N, \\ 2N - n, & N+1 \le n \le 2N-1, \\ 0, & \text{otherwise.} \end{cases}$$

7.3 Use the differentiation property (7.33) to find the z-transform of the sequences

$$x_1[n] = \begin{cases} na^n, & n \ge 0, \\ 0, & n < 0, \end{cases} \qquad x_2[n] = \begin{cases} n^2 a^n, & n \ge 0, \\ 0, & n < 0. \end{cases}$$

Specify the region of convergence of the transforms.

7.4 Show that the z-transform of the sequence

$$x[n] = \begin{cases} \dfrac{(n+1)(n+2)\cdots(n+m)a^n}{m!}, & n \ge 0, \\ 0, & n < 0, \end{cases}$$

(where $m = 1, 2, \ldots$) is

$$X^z(z) = \frac{1}{(1 - az^{-1})^{m+1}}.$$

Hint: Use induction.

7.5 Can the function $X^z(z) = \bar{z}$ be the z-transform of any sequence $x[n]$? If so, what is $x[n]$? If not, why not?

7.6 Let $x[n]$ be a finite-duration signal on $0 \le n \le N-1$.

(a) Express $X^d[k]$, the DFT of the signal, as a function of $X^z(z)$.

(b) Express $X^z(z)$ as a function of $X^d[k]$.

7.7 Let $x[n]$ be a symmetric sequence, that is, $x[n] = x[-n]$ for all n.

(a) Prove that $x[n]$ possesses a z-transform in a region in the complex plane if and only if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$.

(b) Let

$$X^z(z) = \frac{z^{-1}}{(1 - 0.5z^{-1})(1 - 2z^{-1})}.$$

What is the region of convergence of $X^z(z)$ if $x[n]$ is symmetric and what is $x[n]$?

7.8 Give an example of a causal sequence that does not have a z-transform.

7.9 Let $X^z(z)$ be the z-transform of $x[n]$. What is the z-transform of $n^2 x[n]$?

7.10 Let $x[n]$, $y[n]$ be causal sequences. Prove that

$$\frac{1}{2\pi j}\oint X^z(z)Y^z(z)z^{-1}dz = \left[\frac{1}{2\pi j}\oint X^z(z)z^{-1}dz\right]\cdot\left[\frac{1}{2\pi j}\oint Y^z(z)z^{-1}dz\right].$$

7.11 Prove that a system obeying a difference equation such as (7.45) is linear and time invariant. Assume that $y[n]$ is identically zero when $x[n]$ is identically zero.

7.12 We are given a system whose transfer function is

$$H^z(z) = (1 - Az^{-3})\sum_{k=0}^{2}\frac{b_k}{1 - W_3^k z^{-1}}, \quad \text{where } W_3 = e^{j2\pi/3}.$$

(a) For what value(s) of A will the system be BIBO-stable?

(b) For A found in part a:

i. Compute the impulse response of the system $h[n]$.

ii. Compute $H^f(\theta)$ for $\theta = 0, 2\pi/3, 4\pi/3$.

iii. Express the relation between $\{b_0, b_1, b_2\}$ and $\{h[0], h[1], h[2]\}$ in matrix-vector form. Interpret this equation.

7.13 The signal $y(t)$ obeys a differential equation

$$\frac{dy(t)}{dt} = -\alpha y(t) + x(t),$$

where $\alpha > 0$. The signal is sampled at interval T. Prove that the sampled signal $y[n]$ obeys a difference equation

$$y[n] = ay[n-1] + u[n].$$

Express a and $u[n]$ in terms of α, T, and $x[n]$.

7.14 An LTI system has the transfer function

$$H^z(z) = \sum_{k=2}^{10}\left(\frac{1}{1 - kz^{-1}} + \frac{1}{1 - k^{-1}z^{-1}}\right).$$

Find the region of convergence and the impulse response of the system if it is known that the system is (a) causal (but not necessarily stable) and (b) stable (but not necessarily causal).

7.15 A causal LTI system has four zeros at $z = \pm 0.5 \pm j0.5$ and two poles at $z = \pm j0.5$. The system has no other poles or zeros, except possibly at $z = 0$.

(a) What is the minimum number of poles or zeros that the system must have at $z = 0$?

(b) Draw an approximate magnitude response plot of the system.

(c) Define

$$g[n] = (-1)^n h[n],$$

where $h[n]$ is the impulse response of the given system. Find the poles and zeros of $G^z(z)$.

7.16 A stable LTI system has the transfer function

$$H^z(z) = \frac{3(1 - z^{-1})}{(1 - 0.5z^{-1})(1 - 2z^{-1})}.$$

Find the impulse response of the system.

7.17 The *step response* of an LTI system is its response to the unit-step signal $v[n]$.

(a) Explain how to compute the step response of an LTI system obeying a difference equation. Assume that all the poles of the system are simple, and none of them is at $z = 1$.

(b) Compute the step response of the system

$$H^z(z) = \frac{2 + 2.7z^{-1} - 0.36z^{-2}}{1 + 0.5z^{-1} - 0.36z^{-2}}.$$

7.18 We are given a causal LTI system whose transfer function is

$$H^z(z) = \frac{1 + z^{-1} + z^{-2}}{(1 - 0.8z^{-1})(1 + 0.5z^{-1})}.$$

The impulse response $h[n]$ of the system is used for generating a new impulse response $h_1[n]$ according to

$$h_1[n] = \gamma^n h[n],$$

where γ is constant.

(a) Write the transfer function $H_1^z(z)$ and the corresponding difference equation.

(b) Is $H_1^z(z)$ stable if $|\gamma| < 1$? If $|\gamma| > 1$?

(c) Generalize to an arbitrary causal, rational, and stable LTI system $H^z(z)$ of orders p, q.

7.19 Find the difference equation of the LTI system whose impulse response is

$$h[n] = \begin{cases} a^{n-1}\cos(\theta_0 n), & n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

7.20 We are given a causal LTI system obeying the difference equation

$$y[n] = 2y[n-1] - y[n-2] + x[n-1].$$

The input $x[n]$ is generated from the output of the system and from an external input $v[n]$ according to

$$x[n] = K(v[n] - 0.5v[n-1] - y[n] + 0.5y[n-1]),$$

where K is constant. Find all values of K for which the transfer function from $v[n]$ to $y[n]$ is stable.

7.21 Bob, Nick, and Dave are given a causal and stable LTI system and are told that its transfer function is

$$H^z(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}.$$

They are asked to find the parameters b_0, b_1, a_1. Bob feeds a unit impulse to the system and reports that $h[0] = 1$. Nick feeds a unit step to the system and reports that the DC gain is 4. Dave feeds the signal $\cos(\pi n/3)$ to the system and reports that the amplitude of the sinusoidal signal at the output is 2. Based on this information, what are the values of b_0, b_1, a_1?

7.22 The response of a causal LTI system to the input signal.

$$x[n] = 0.5^n, \quad n \geq 0$$

is known to be

$$y[n] = 0.25^n, \quad n \geq 0.$$

Find the impulse response of the system.

7.23 A causal LTI system is described by the difference equation

$$y[n] = -0.5y[n-1] + x[n] + x[n-1].$$

The input to the system is

$$x[n] = \begin{cases} 1, & n \text{ even}, \quad n \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

What is the output of the system?

7.24 This problem discusses the computation of the coefficients of a difference equation from a finite set of impulse response values.

(a) Show that the impulse response of the system described by the difference equation (7.45) satisfies

$$h[n] = -a_1 h[n-1] - a_2 h[n-2] - \ldots - a_p h[n-p], \quad \text{for all } n > q. \tag{7.119}$$

(b) Deduce from (7.119) that the denominator coefficients can be obtained from the impulse response coefficients by solving the linear equations

$$\begin{bmatrix} h[q] & h[q-1] & \ldots & h[q-p+1] \\ h[q+1] & h[q] & \ldots & h[q-p+2] \\ \vdots & \vdots & \ldots & \vdots \\ h[q+p-1] & h[q+p-2] & \ldots & h[q] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = - \begin{bmatrix} h[q+1] \\ h[q+2] \\ \vdots \\ h[q+p] \end{bmatrix}. \tag{7.120}$$

It can be shown that the coefficient matrix of this set of equations is nonsingular (hence has a unique solution) if and only if the polynomials $a(z)$, $b(z)$ are coprime.

(c) Show that the numerator coefficients can be obtained from the formula

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_q \end{bmatrix} = \begin{bmatrix} h[0] & 0 & \ldots & 0 \\ h[1] & h[0] & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ h[q] & h[q-1] & \ldots & h[q-p] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix}. \tag{7.121}$$

7.25 Let $G^z(z)$ and $H^z(z)$ be the causal LTI systems

$$G^z(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}, \quad H^z(z) = \frac{1 - 0.25z^{-1}}{1 - 0.5z^{-1}}.$$

Find a_1, a_2 such that $\{g[0], g[1], g[2]\}$ will be equal to $\{h[0], h[1], h[2]\}$, respectively. For a_1, a_2 you have found, is $g[3]$ equal to $h[3]$ as well? Hint: Use Problem 7.24.

7.26 Write a MATLAB program that implements the Schur–Cohn test without recursive calls.

7.27 Find the inverse z-transforms of the functions

$$X_1^z(z) = \exp(z^{-1}), \quad X_2^z(z) = \cos(z^{-1}), \quad X_3^z(z) = \sin(z^{-1}),$$

where the ROC is $z \neq 0$ in all cases.

7.28 Find the causal sequence $x[n]$ whose Fourier transform is

$$X^f(\theta) = \frac{1}{e^{j\theta} + \frac{1}{6} - \frac{1}{6}e^{-j\theta}}.$$

7.29 An LTI system has the transfer function

$$H^{z}(z) = \frac{a_p + a_{p-1}z^{-1} + \cdots + a_1 z^{-(p-1)} + z^{-p}}{1 + a_1 z^{-1} + \cdots + a_{p-1}z^{-(p-1)} + a_p z^{-p}}.$$

Find $|H^{f}(\theta)|$, the magnitude response of the system, as a function of θ and the coefficients $a_1, \ldots, a_p$. Interpret the result.

7.30 A causal LTI has the z-transform

$$H^{z}(z) = \frac{z^p - (1-\varepsilon)}{z^p + (1-\varepsilon)},$$

where p is odd and $0 < \varepsilon \ll 1$.

(a) Compute the poles and the zeros of the system. Is the system stable?

(b) Draw the pole-zero map of the system for $p = 5$.

(c) Draw an approximate magnitude response plot of the system for $p = 5$ and $\varepsilon = 0.1$.

7.31 The program `frqresp` may be sensitive to roundoff errors if the order of the $H^{z}(z)$ is large. Write a MATLAB program `frqrspzp` that computes the frequency response $H^{f}(\theta)$ from the pole-zero factorization (7.50). As we shall see in Chapter 10, digital filters are often designed to give the poles and zeros directly, so the program you are asked to write here will be useful later.

7.32 Let $x[n]$ be a periodic signal with period N, and $y[n]$ be one period of $x[n]$, that is

$$y[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Express $X_{+}^{z}(z)$ in terms of $Y_{+}^{z}(z)$ and state its region of convergence (note that these are unilateral z-transforms).

(b) Find $X_{+}^{z}(z)$ in the special case where $N = 6$ and

$$y[n] = \begin{cases} 1, & 0 \le n \le 2, \\ -1, & 3 \le n \le 5. \end{cases}$$

7.33 A signal $x[n]$ is fed to two LTI systems, yielding outputs $y_1[n]$, $y_2[n]$, respectively. The LTI systems obey the difference equations

$$y_1[n] = 0.5y_1[n-1] + x[n], \quad y_2[n] = 0.5y_2[n-1] - 2x[n-1].$$

The signal $w[n]$ is generated according to

$$w[n] = y_1[n] + y_2[n]$$

and fed to a third LTI system obeying the difference equation

$$y_3[n] = 2.5y_3[n-1] - y_3[n-2] + w[n].$$

(a) Find the response of $y_3[n]$ to a unit impulse at the input $x[n]$.

(b) Is the transfer function from $x[n]$ to $y_3[n]$ BIBO-stable?

(c) Find the DC gain of the system.

(d) Assume that $y_1[n]$ and $y_2[n]$ are zero for $n < 0$, but $y_3[-1] = y_3[-2] = 1$. Find the zero-input response of $y_3[n]$.

(e) Reconcile the result in part d with your answer to part b.

7.34* This problem explores the existence of a z-transform of the sinc function.

(a) Does the sequence $\operatorname{sinc}(0.5n)$ possess a z-transform? If so, find it and specify the region of convergence. If not, explain why.

(b) Generalize part a to $\operatorname{sinc}(rn)$, where r is a rational number in the range $(0, 1)$.

The result of part b holds for any $r \in (0, 1)$, whether rational or not. However, the proof of the irrational case is much more difficult.

7.35* Repeat Problem 7.34 for the sequence $\operatorname{sinc}^2(0.5n)$, then for $\operatorname{sinc}^2(rn)$.

7.36* This problem discusses the z-transform of a subsequence of a given sequence.

(a) The causal signal $x[n]$ has z-transform

$$X^z(z) = \frac{3}{1 - 0.4z^{-1} - 0.32z^{-2}}.$$

Define

$$y[n] = x[2n], \quad n \geq 0.$$

Find $Y^z(z)$, the z-transform of $y[n]$.

(b) Repeat part a for

$$X^z(z) = \frac{2(1 - 0.5z^{-1})}{1 - z^{-1} + 0.5z^{-2}}.$$

(c) Let $X^z(z)$ be a causal rational function and let $y[n]$ be related to $x[n]$ as in part a. Assume that the poles of $X^z(z)$ are simple. Is $Y^z(z)$ a rational function?

(d) Repeat part c if

$$y[n] = x[Mn], \quad n \geq 0,$$

where M is a fixed positive integer.

(e) If $y[n] = x[2n]$ and

$$Y^z(z) = \frac{1}{1 - 0.25z^{-1}},$$

what is $x[n]$?

7.37* Compute the noise gain of the filter

$$H^z(z) = \frac{b_0}{1 + a_1 z^{-1}}$$

as a function of b_0 and a_1.