Chapter 2

Review of Frequency-Domain Analysis

In this chapter we present a brief review of signal and system analysis in the frequency domain. This material is assumed to be known from previous study, so we shall keep the details to a minimum and skip almost all proofs. We consider both continuous-time and discrete-time signals. We pay attention to periodic signals, and present both complex (conventional) and real (cosine and sine) Fourier series. We include a brief summary of stationary random signals, since background on random signals is necessary for certain sections in this book.

2.1 Continuous-Time Signals and Systems

Let $x(t)$ be a continuous-time signal whose values can be real or complex. The Fourier transform of the signal is[1]

$$X^{\mathrm{F}}(\omega) = \{\mathcal{F}x\}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \quad \omega \in \mathbb{R}, \tag{2.1}$$

subject to the existence of the right side in a well-defined sense. A sufficient condition for the existence of the Fourier transform (2.1) is

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty. \tag{2.2}$$

Subject to further conditions, the inverse Fourier transform exists and is given by

$$x(t) = \{\mathcal{F}^{-1}X^{\mathrm{F}}\}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^{\mathrm{F}}(\omega)e^{j\omega t}d\omega, \quad t \in \mathbb{R}, \tag{2.3}$$

for every point t at which $x(t)$ is continuous. The following three conditions together, known as *Dirichlet conditions*, are sufficient for (2.3) to hold at every continuity point of $x(t)$:

1. $x(t)$ satisfies (2.2).
2. $x(t)$ is continuous, except for discontinuity points whose number on any finite interval is finite; the limits at both sides of each discontinuity point exist.
3. $x(t)$ has a finite number of minima and maxima on any finite interval.

The variable ω is called the *angular frequency*,[2] and is measured in radians per second (rad/s). The variable $f = \omega/2\pi$ is called the *frequency*, and is measured in hertz (Hz).

The main properties of the Fourier transform are as follows.

1. **Linearity**

$$z(t) = ax(t) + by(t) \iff Z^{\mathrm{F}}(\omega) = aX^{\mathrm{F}}(\omega) + bY^{\mathrm{F}}(\omega), \quad a, b \in \mathbb{C}. \tag{2.4}$$

2. **Time shift**

$$y(t) = x(t - \tau) \iff Y^{\mathrm{F}}(\omega) = e^{-j\omega\tau}X^{\mathrm{F}}(\omega), \quad \tau \in \mathbb{R}. \tag{2.5}$$

3. **Frequency shift (modulation)**

$$y(t) = e^{j\lambda t}x(t) \iff Y^{\mathrm{F}}(\omega) = X^{\mathrm{F}}(\omega - \lambda), \quad \lambda \in \mathbb{R}. \tag{2.6}$$

4. **Time and frequency scale**

$$y(t) = x(at) \iff Y^{\mathrm{F}}(\omega) = \frac{1}{|a|}X^{\mathrm{F}}\left(\frac{\omega}{a}\right), \quad a \in \mathbb{R},\ a \neq 0. \tag{2.7}$$

5. **Time-domain convolution**

$$z(t) = \{x * y\}(t) \iff Z^{\mathrm{F}}(\omega) = X^{\mathrm{F}}(\omega)Y^{\mathrm{F}}(\omega). \tag{2.8}$$

6. **Time-domain multiplication**

$$z(t) = x(t)y(t) \iff Z^{\mathrm{F}}(\omega) = \frac{1}{2\pi}\{X^{\mathrm{F}} * Y^{\mathrm{F}}\}(\omega). \tag{2.9}$$

7. **Parseval's theorem**

$$\int_{-\infty}^{\infty} x(t)\bar{y}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^{\mathrm{F}}(\omega)\bar{Y}^{\mathrm{F}}(\omega)d\omega, \tag{2.10}$$

(where the bar denotes complex conjugation) and its special case

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X^{\mathrm{F}}(\omega)|^2 d\omega. \tag{2.11}$$

8. **Conjugation**

$$y(t) = \bar{x}(t) \iff Y^{\mathrm{F}}(\omega) = \bar{X}^{\mathrm{F}}(-\omega). \tag{2.12}$$

9. **Symmetry** If $x(t)$ is real valued then

$$X^{\mathrm{F}}(-\omega) = \bar{X}^{\mathrm{F}}(\omega), \tag{2.13a}$$
$$\Re\{X^{\mathrm{F}}(-\omega)\} = \Re\{X^{\mathrm{F}}(\omega)\}, \tag{2.13b}$$
$$\Im\{X^{\mathrm{F}}(-\omega)\} = -\Im\{X^{\mathrm{F}}(\omega)\}, \tag{2.13c}$$
$$|X^{\mathrm{F}}(-\omega)| = |X^{\mathrm{F}}(\omega)|, \tag{2.13d}$$
$$\measuredangle X^{\mathrm{F}}(-\omega) = -\measuredangle X^{\mathrm{F}}(\omega). \tag{2.13e}$$

(where $\Re$ denotes the real part, $\Im$ denotes the imaginary part, and $\measuredangle$ denotes the angle of the corresponding complex number).

10. **Realness**

$$x(t) = \bar{x}(-t) \iff X^{\mathrm{F}}(\omega) \text{ is real.} \tag{2.14}$$

A continuous-time signal of great importance is the *impulse* (or *delta*) *function* $\delta(t)$, formally defined by the *sifting property*[3]

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \tag{2.15}$$

for any function $f(t)$ that is continuous at $t = 0$.

A *dynamic system* (or simply a *system*) is an object that accepts signals, operates on them, and yields other signals. The eventual interest of the engineer is in physical systems: electrical, mechanical, thermal, physiological, and so on. However, here we regard a system as a mathematical operator. In particular, a continuous-time, *single-input, single-output* (SISO) system is an operator that assigns to a given input signal $x(t)$ a unique output signal $y(t)$. A SISO system is thus characterized by the family of signals $x(t)$ it is permitted to accept (the input family), and by the mathematical relationship between signals in the input family and their corresponding outputs $y(t)$ (the output family). The input family almost never contains all possible continuous-time signals. For example, consider a system whose output signal is the time derivative of the input signal (such a system is called a differentiator). The input family of this system consists of all differentiable signals, and only such signals.

When representing a physical system by a mathematical one, we must remember that the representation is only approximate in general. For example, consider a parallel connection of a resistor R and a capacitor C, fed from a current source $i(t)$. The common mathematical description of such a system is by a differential equation relating the voltage across the capacitor $v(t)$ to the input current:

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} = i(t). \tag{2.16}$$

However, this relationship is only approximate. It neglects effects such as nonlinearity of the resistor, leakage in the capacitor, temperature induced variations of the resistance and the capacitance, and energy dissipation resulting from electromagnetic radiation. Approximations of this kind are made in all areas of science and engineering; they are not to be avoided, only used with care.

Of special importance to us here (and to system theory in general) is the class of linear systems. A SISO system is said to be *linear* if it satisfies the following two properties:

1. **Additivity:** The response to a sum of two input signals is the sum of the responses to the individual signals. If $y_i(t)$ is the response to $x_i(t)$, $i = 1, 2$, then the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. **Homogeneity:** The response to a signal multiplied by a scalar is the response to the given signal, multiplied by the same scalar. If $y(t)$ is the response to $x(t)$, then the response to $ax(t)$ is $ay(t)$ for all a.

Another important property that a system may possess is time invariance. A system is said to be *time invariant* if shifting the input signal in time by a fixed amount causes the same shift in time of the output signal, but no other change. If $y(t)$ is the response to $x(t)$, then the response to $x(t - t_0)$ is $y(t - t_0)$ for every fixed t_0 (positive or negative).

The resistor-capacitor system described by (2.16) is linear, provided the capacitor has zero charge in the absence of input current. This follows from linearity of the differential equation. The system is time invariant as well; however, if the resistance R or the capacitance C vary in time, the system is not time invariant.

A system that is both linear and time invariant is called *linear time invariant*, or LTI. All systems treated in this book are linear time invariant.

The Dirac delta function $\delta(t)$ may or may not be in the input family of a given LTI system. If it is, we denote by $h(t)$ the response to $\delta(t)$, and call it the *impulse response* of the system.[4] For example, the impulse response of the resistor-capacitor circuit described by (2.16) is

$$h(t) = \begin{cases} e^{-t/RC}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

If the impulse response of an LTI system exists, it completely characterizes the input-output relationship of the system. Given an input signal $x(t)$, the (so-called *zero-state*) output signal $y(t)$ is the convolution[5]

$$y(t) = \{h * x\}(t). \tag{2.17}$$

The signal is in the input family of the system if and only if the right side of (2.17) exists.

The *frequency response* of an LTI system is the Fourier transform $H^{\mathrm{F}}(\omega)$ of the impulse response. The frequency-domain counterpart of (2.17) is

$$Y^{\mathrm{F}}(\omega) = H^{\mathrm{F}}(\omega)X^{\mathrm{F}}(\omega). \tag{2.18}$$

Not every LTI system possessing an impulse response necessarily has frequency response, since $H^{\mathrm{F}}(\omega)$ may not exist. For example, the frequency response of a system whose impulse response is $h(t) = e^t$ is not defined.

2.2 Specific Signals and Their Transforms

In this section we list a few continuous-time signals that will be used repeatedly in this book, along with their Fourier transforms.

2.2.1 The Delta Function and the DC Function

We have already introduced the Dirac delta function $\delta(t)$ in (2.15). The Fourier transform of the delta function is

$$\{\mathcal{F}\delta\}(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1. \tag{2.19}$$

The function whose value is 1 for all t is called the DC function, and will be denoted by $1(t)$ (DC stands for *direct current*, a traditional electrical engineering term). By an argument dual to (2.19), the Fourier transform of the DC function is

$$\{\mathcal{F}1\}(\omega) = 2\pi\delta(\omega). \tag{2.20}$$

Figure 2.1 illustrates the delta and the DC functions.

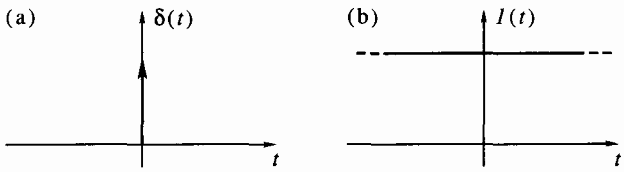

Figure 2.1 The Dirac delta function (a) and the DC function (b).

2.2.2 Complex Exponentials and Sinusoids

The *complex exponential function* is

$$x(t) = e^{j\omega_0 t}. \tag{2.21}$$

The parameter ω_0 is the *angular frequency* of the complex exponential; it is a real number, either positive or negative. In the special case $\omega_0 = 0$ we get the DC function.

The Fourier transform of the complex exponential function is obtained from (2.20) and the modulation property (2.6) as

$$X^{\mathrm{F}}(\omega) = 2\pi\delta(\omega - \omega_0). \tag{2.22}$$

A *sinusoidal* function has the general form

$$x(t) = A\cos(\omega_0 t + \phi_0). \tag{2.23}$$

The parameters A, ω_0, and ϕ are, respectively, the amplitude, angular frequency, and initial phase. The amplitude and the angular frequency are real and positive. The initial phase is assumed to be in the range $[-\pi, \pi)$ for uniqueness. In the special case $A = 1$, $\phi = 0$ we get the cosine function. In the special case $A = 1$, $\phi = -0.5\pi$ we get the sine function.

From (2.22) and the two Euler formulas

$$\cos(\omega_0 t) = 0.5(e^{j\omega_0 t} + e^{-j\omega_0 t}), \quad \sin(\omega_0 t) = 0.5j(e^{-j\omega_0 t} - e^{j\omega_0 t}), \tag{2.24}$$

we get the Fourier transforms of the cosine and sine functions:

$$\{\mathcal{F}\cos\}(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)], \tag{2.25a}$$

$$\{\mathcal{F}\sin\}(\omega) = j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]. \tag{2.25b}$$

Using the trigonometric formula

$$\cos(\omega_0 t + \phi_0) = \cos\phi_0\cos(\omega_0 t) - \sin\phi_0\sin(\omega_0 t),$$

we get the following for the general sinusoidal function $x(t)$ in (2.23):

$$\begin{aligned} X^{\mathrm{F}}(\omega) &= A\pi(\cos\phi_0 + j\sin\phi_0)\delta(\omega - \omega_0) + A\pi(\cos\phi_0 - j\sin\phi_0)\delta(\omega + \omega_0) \\ &= A\pi e^{j\phi_0}\delta(\omega - \omega_0) + A\pi e^{-j\phi_0}\delta(\omega + \omega_0). \end{aligned} \tag{2.26}$$

Suppose we feed the sinusoidal signal $x(t)$ defined in (2.23) to the input of an LTI system whose frequency response is $H^{\mathrm{F}}(\omega)$. Then, according to (2.18) and (2.26), the Fourier transform of the output signal will be

$$Y^{\mathrm{F}}(\omega) = H^{\mathrm{F}}(\omega_0)\pi e^{j\phi_0}\delta(\omega - \omega_0) + H^{\mathrm{F}}(-\omega_0)\pi e^{-j\phi_0}\delta(\omega + \omega_0). \tag{2.27}$$

Let us write the frequency response as

$$H^{\mathrm{F}}(\omega) = |H^{\mathrm{F}}(\omega)|e^{j\psi(\omega)}, \tag{2.28}$$

where $\psi(\omega)$ is the phase response of the system. Assuming that the system's impulse response is real, we have

$$|H^{\mathrm{F}}(\omega)| = |H^{\mathrm{F}}(-\omega)|, \quad \psi(\omega) = -\psi(-\omega). \tag{2.29}$$

Therefore,

$$Y^{\mathrm{F}}(\omega) = |H^{\mathrm{F}}(\omega_0)|[\pi e^{j[\phi_0+\psi(\omega_0)]}\delta(\omega - \omega_0) + \pi e^{-j[\phi_0+\psi(\omega_0)]}\delta(\omega + \omega_0)]. \tag{2.30}$$

It follows that

$$y(t) = |H^{\mathrm{F}}(\omega_0)|\cos[\omega_0 t + \phi_0 + \psi(\omega_0)]. \tag{2.31}$$

This result is a fundamental relationship between sinusoidal signals and LTI systems; it can be expressed as follows:

> When a sinusoidal signal at frequency ω_0 is fed to a real LTI system, the output is a sinusoidal signal at the same frequency. The ratio of magnitudes of the output and input signals is the magnitude response of the system at ω_0, and the difference in phases is the phase response of the system at ω_0.

Because of this property, sinusoidal signals are said to be *eigenfunctions* of LTI systems. For comparison, recall that multiplication of a matrix by an eigenvector of the same matrix yields the eigenvector again, multiplied by the corresponding eigenvalue. Thus eigenfunctions of an LTI system fulfill a role similar to that of eigenvectors of a matrix, and the system's frequency response fulfills a role similar to that of eigenvalues.

2.2.3 The rect and the sinc

The *rectangular* function, or *rect* (also known as *box*, *boxcar*, or *pulse*), is defined by

$$\operatorname{rect}(t) = \begin{cases} 1, & |t| < 0.5, \\ 0.5, & |t| = 0.5, \\ 0, & |t| > 0.5. \end{cases} \tag{2.32}$$

The *sinc* function is defined by

$$\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases} \tag{2.33}$$

The Fourier transform of the rect function is

$$\{\mathcal{F}\operatorname{rect}\}(\omega) = \int_{-0.5}^{0.5} e^{-j\omega t}dt = -\frac{1}{j\omega}(e^{-j0.5\omega} - e^{j0.5\omega}) = \operatorname{sinc}\left(\frac{\omega}{2\pi}\right). \tag{2.34}$$

By a dual argument, the Fourier transform of the sinc function is

$$\{\mathcal{F}\operatorname{sinc}\}(\omega) = \operatorname{rect}\left(\frac{\omega}{2\pi}\right). \tag{2.35}$$

Figure 2.2 illustrates the rect and the sinc functions.

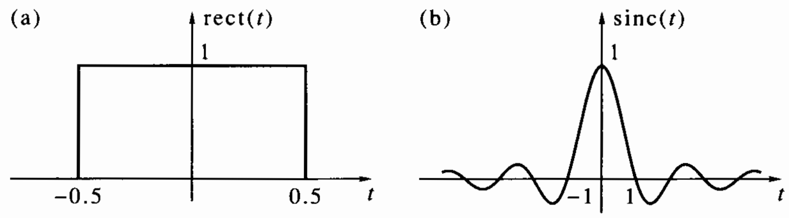

Figure 2.2 The rect function (a) and the sinc function (b).

2.2.4 The Gaussian Function

The Gaussian function is defined by

$$g(t) = e^{-0.5t^2}, \quad t \in \mathbb{R}. \tag{2.36}$$

The Fourier transform of the Gaussian function is derived as follows:

$$G^{\mathrm{F}}(\omega) = \int_{-\infty}^{\infty} e^{-0.5t^2}e^{-j\omega t}dt = e^{-0.5\omega^2}\int_{-\infty}^{\infty} e^{-0.5(t+j\omega)^2}dt = \sqrt{2\pi}e^{-0.5\omega^2}. \tag{2.37}$$

As we see, the Fourier transform of a Gaussian function is a Gaussian function of ω (up to a scale factor). The pair $g(t)$ and $G^{\mathrm{F}}(\omega)$ are shown in Figure 2.3. The Gaussian is a rare example of a signal and its Fourier transform being both real and positive valued for all t and ω.

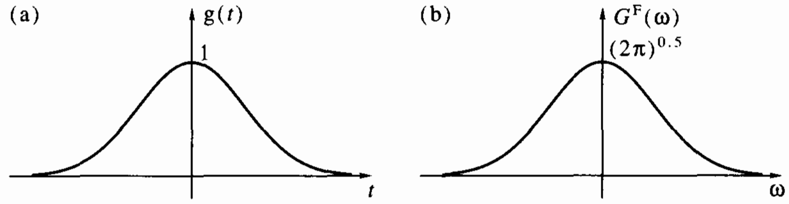

Figure 2.3 The Gaussian function (a) and its Fourier transform (b).

2.3 Continuous-Time Periodic Signals

A continuous-time signal $x(t)$ is said to be periodic if there exists $T > 0$ such that

$$x(t) = x(t+T), \quad \text{for all } t \in \mathbb{R}. \tag{2.38}$$

The smallest T for which (2.38) holds is called the *period.* A periodic signal satisfying certain smoothness conditions admits a Fourier series representation:

$$x(t) = \{S^{-1}X^{\mathrm{S}}\}(t) = \sum_{k=-\infty}^{\infty} X^{\mathrm{S}}[k] \exp\left(\frac{j2\pi kt}{T}\right), \tag{2.39}$$

where the sequence of coefficients $\{X^{\mathrm{S}}[k]\}$ is given by

$$X^{\mathrm{S}}[k] = \{Sx\}[k] = \frac{1}{T}\int_0^T x(t)\exp\left(-\frac{j2\pi kt}{T}\right)dt, \quad k \in \mathbb{Z}. \tag{2.40}$$

Note that (2.39) applies equally well to $x(t)$ and its restriction on any single period, say on $[0, T]$, or on $[-0.5T, 0.5T]$. Therefore, we shall use Fourier series interchangeably for signals defined on intervals and for their periodic extensions.

Parseval's theorem for Fourier series is

$$\sum_{k=-\infty}^{\infty} |X^{\mathrm{S}}[k]|^2 = \frac{1}{T}\int_0^T |x(t)|^2 dt. \tag{2.41}$$

When $x(t)$ is a real signal, its Fourier coefficients satisfy $\bar{X}^{\mathrm{S}}[k] = X^{\mathrm{S}}[-k]$. Then we get from (2.39)

$$x(t) = X^{\mathrm{S}}[0] + \sum_{k=1}^{\infty} 2|X^{\mathrm{S}}[k]|\cos\left(\frac{2\pi kt}{T} + \phi[k]\right), \quad \phi[k] = \measuredangle X^{\mathrm{S}}[k]. \tag{2.42}$$

The individual terms $2|X^{\mathrm{S}}[k]|\cos(2\pi kt/T + \phi[k])$ are called the *harmonics* of $x(t)$.

Periodic signals do not satisfy the condition (2.2), so they do not possess a Fourier transform in the usual sense. However, they can be formally represented as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^{\mathrm{F}}(\omega)e^{j\omega t}d\omega, \tag{2.43}$$

where

$$X^{\mathrm{F}}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} X^{\mathrm{S}}[k]\delta\left(\omega - \frac{2\pi k}{T}\right). \tag{2.44}$$

The right side of (2.44) is the accepted definition of the Fourier transform of the periodic signal (2.39). As we see, the Fourier transform of a periodic signal is supported on a discrete set of frequencies—the integer multiples of $2\pi/T$. It is common to refer to these frequencies as *spectral lines.* More generally, the Fourier transform of any sum (finite or convergent infinite) of complex exponential signals consists of spectral lines at the frequencies of the terms of the sum.

The following relation between periodic signals and linear systems is of special interest:

Theorem 2.1 When the input signal of an LTI system is periodic, the output (if exists) is also a periodic signal.

Proof Let $x(t)$ be periodic with period T, and let $h(t)$ be the impulse response of the LTI system. Then the output $y(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{2.45}$$

Therefore,

$$y(t+T) = \int_{-\infty}^{\infty} h(\tau)x(t+T-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = y(t), \tag{2.46}$$

so $y(t)$ is periodic. □

2.4 The Impulse Train

The *impulse train* $p_T(t)$ is defined by

$$p_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT), \tag{2.47}$$

where T is a positive constant, called the *period*. Figure 2.4 depicts the impulse train.

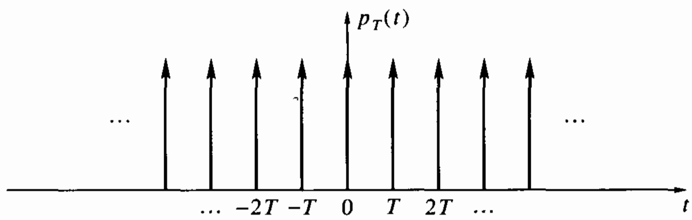

Figure 2.4 The impulse train $p_T(t)$.

An alternative expression for the impulse train is given by the *Poisson sum formula*:

Theorem 2.2

$$p_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \exp\left(\frac{j2\pi kt}{T}\right). \tag{2.48}$$

An informal proof Since $p_T(t)$ is periodic, it admits the Fourier series

$$p_T(t) = \sum_{k=-\infty}^{\infty} P_T^{\mathrm{S}}[k]\exp\left(\frac{j2\pi kt}{T}\right), \tag{2.49}$$

where

$$P_T^{\mathrm{S}}[k] = \frac{1}{T}\int_{-T/2}^{T/2} p_T(t)\exp\left(-\frac{j2\pi kt}{T}\right)dt = \frac{1}{T}. \tag{2.50}$$

Substitution of (2.50) in (2.49) yields (2.48). This proof is informal, since we have not proved the existence of the Fourier series (2.49) in this case. A formal proof can be found in the mathematical literature. □

A dual form of the Poisson formula is

$$\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right) = \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\omega T}. \tag{2.51}$$

The dual form is obtained by substituting

$$t \leftarrow \omega, \quad T \leftarrow \frac{2\pi}{T}$$

in (2.48). In both (2.48) and (2.51), n (or k) can be replaced by $-n$ ($-k$, respectively), since the summations are from $-\infty$ to ∞.

Using the Poisson formula, we can compute the Fourier transform of $p_T(t)$ as follows (this is again an informal derivation):

$$\begin{aligned} P_T^{\mathrm{F}}(\omega) &= \int_{-\infty}^{\infty} p_T(t) e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} e^{-j\omega nT} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right). \end{aligned} \tag{2.52}$$

The last equality in (2.52) follows from (2.51). We have thus obtained the following interesting result: The Fourier transform of an impulse train in t is an impulse train in ω, up to a scale factor. If the period of the given function is T, the period of the transform is $2\pi/T$.

2.5 Real Fourier Series*

Consider again the Fourier series pair (2.39), (2.40). As we know, the Fourier coefficients $X^{\mathrm{S}}[k]$ are complex even when the signal $x(t)$ is real. The Fourier series of a real signal can also be written in the real form (2.42), which is equivalent to

$$x(t) = X^{\mathrm{S}}[0] + 2\sum_{k=1}^{\infty} \left[\Re\{X^{\mathrm{S}}[k]\} \cos\left(\frac{2\pi kt}{T}\right) - \Im\{X^{\mathrm{S}}[k]\} \sin\left(\frac{2\pi kt}{T}\right)\right]. \tag{2.53}$$

This representation requires two sets of basis functions—sines and cosines. There exist, however, modified forms of Fourier series for which the coefficients of a real signal are real and the basis functions are either sines or cosines (but not both simultaneously). The derivation of the modified series relies on the following idea: Instead of regarding $x(t)$ as a signal supported on the interval $[0, T]$, we regard it as half of another signal, supported on the interval $[-T, T]$. The extended signal can be arbitrary on $[-T, 0)$, but two natural choices suggest themselves:

1. Force the extended signal to be a real even function of t, that is, define

$$x_{\mathrm{e}}(t) = \begin{cases} x(t), & 0 \le t \le T, \\ x(-t), & -T \le t < 0. \end{cases} \tag{2.54}$$

   Note that $x_{\mathrm{e}}(t)$ has no jump at $t = 0$. Moreover, if we extend $x_{\mathrm{e}}(t)$ periodically with period $2T$, there will be no jumps at $t = \pm T$. However, there is a possible discontinuity of the derivative of $x_{\mathrm{e}}(t)$ [assuming that $x(t)$ is differentiable] at $t = 0$, and at $t = \pm T$ if $x_{\mathrm{e}}(t)$ is extended periodically.

2. Force the extended signal to be an imaginary odd function of t, that is, define

$$x_{\mathrm{o}}(t) = \begin{cases} jx(t), & 0 \le t \le T, \\ -jx(-t), & -T \le t < 0. \end{cases} \tag{2.55}$$

   In this case there is a possible discontinuity at $t = 0$, and at $t = \pm T$ if $x_{\mathrm{o}}(t)$ is extended periodically.

Let us write the Fourier series pair for $x_e(t)$. We have from (2.40),

$$\begin{aligned} X_e^S[k] &= \frac{1}{2T}\int_{-T}^{T} x_e(t)\exp\left(-\frac{j2\pi kt}{2T}\right)dt \\ &= \frac{1}{2T}\int_{-T}^{0} x(-t)\exp\left(-\frac{j\pi kt}{T}\right)dt + \frac{1}{2T}\int_{0}^{T} x(t)\exp\left(-\frac{j\pi kt}{T}\right)dt \\ &= \frac{1}{T}\int_{0}^{T} x(t)\cos\left(\frac{\pi kt}{T}\right)dt. \end{aligned} \tag{2.56}$$

The coefficients $X_e^S[k]$ are real and symmetric. Furthermore, $x(t)$, which is equal to $x_e(t)$ on $[0, T]$, is given by (2.39) as

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X_e^S[k]\exp\left(j\frac{2\pi kt}{2T}\right) \\ &= X_e^S[0] + \sum_{k=-\infty}^{-1} X_e^S[k]\exp\left(\frac{j\pi kt}{T}\right) + \sum_{k=1}^{\infty} X_e^S[k]\exp\left(\frac{j\pi kt}{T}\right) \\ &= X_e^S[0] + 2\sum_{k=1}^{\infty} X_e^S[k]\cos\left(\frac{\pi kt}{T}\right). \end{aligned} \tag{2.57}$$

The representation (2.57) is known as the *cosine Fourier series*. Both the coefficients and the basis functions of this series are real, and only cosine basis functions are used. The cosine series is especially useful when the derivative of $x(t)$ is zero at $t = 0$ and $t = T$, because then the derivative of $x_e(t)$ has no jumps at these points.

Next let us write the Fourier series pair for $x_o(t)$. We have from (2.40),

$$\begin{aligned} X_o^S[k] &= \frac{1}{2T}\int_{-T}^{T} x_o(t)\exp\left(-\frac{j2\pi kt}{2T}\right)dt \\ &= \frac{1}{2jT}\int_{-T}^{0} x(-t)\exp\left(-\frac{j\pi kt}{T}\right)dt - \frac{1}{2jT}\int_{0}^{T} x(t)\exp\left(-\frac{j\pi kt}{T}\right)dt \\ &= \frac{1}{T}\int_{0}^{T} x(t)\sin\left(\frac{\pi kt}{T}\right)dt. \end{aligned} \tag{2.58}$$

The coefficients $X_o^S[k]$ are real and antisymmetric. In particular, $X_o^S[0] = 0$. The signal $x(t)$, which is equal to $x_o(t)/j$ on $[0, T]$, is given by (2.39) as

$$\begin{aligned} x(t) &= \frac{1}{j}\sum_{k=-\infty}^{\infty} X_o^S[k]\exp\left(j\frac{2\pi kt}{2T}\right) \\ &= \frac{1}{j}\sum_{k=-\infty}^{-1} X_o^S[k]\exp\left(\frac{j\pi kt}{T}\right) + \frac{1}{j}\sum_{k=1}^{\infty} X_o^S[k]\exp\left(\frac{j\pi kt}{T}\right) \\ &= 2\sum_{k=1}^{\infty} X_o^S[k]\sin\left(\frac{\pi kt}{T}\right). \end{aligned} \tag{2.59}$$

The representation (2.59) is known as the *sine Fourier series*. As in the case of the cosine Fourier series, both the coefficients and the basis functions of the sine series are real. This series is especially useful in cases where $x(t)$ is zero at $t = 0$ and $t = T$, because then $x_o(t)$ has no jumps at these points.

Fourier cosine and sine series are useful for solving partial differential equations with boundary conditions. Such equations are common in physics. When physical laws impose a zero value of the function at the boundaries, the sine series is useful. When they impose a zero value of the derivative at the boundaries, the cosine series is useful.

2.6 Continuous-Time Random Signals*

A random signal cannot be described by a unique, well-defined mathematical formula. Instead, it can be described by probabilistic laws. In this book we shall use random signals only occasionally, so detailed knowledge of them is not required. We do assume, however, familiarity with the notions of probability, random variables, expectation, variance and covariance. We give here the basic definitions pertaining to random signals and a few of their properties. We shall limit ourselves to real-valued random signals.

A continuous-time *random signal* (or *random process*) is a signal $x(t)$ whose value at each time point is a random variable. Random signals appear often in real life. Examples include:

1. The noise heard from a radio receiver that is not tuned to an operating channel.
2. The noise heard from a helicopter rotor.
3. Electrical signals recorded from a human brain through electrodes put in contact with the skull (these are called *electroencephalograms*, or EEGs).
4. Mechanical vibrations sensed in a vehicle moving on a rough terrain.
5. Angular motion of a boat in the sea caused by waves and wind.

Common to all these examples is the irregular appearance of the signal—see Figure 2.5.

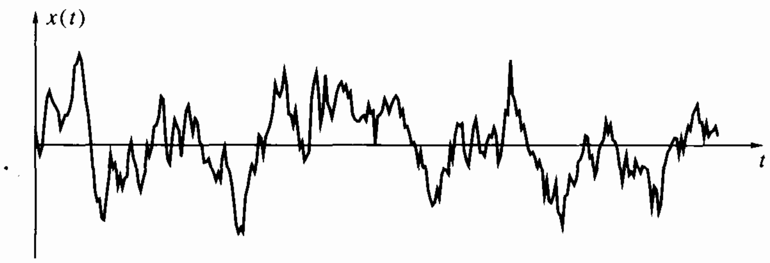

Figure 2.5 A continuous-time random signal.

A signal of interest may be accompanied by an undesirable random signal, which interferes with the signal of interest and limits its usefulness. For example, the typical hiss of audiocassettes limits the usefulness of such cassettes in playing high-fidelity music. In such cases, the undesirable random signal is usually called *noise*. Occasionally "noise" is understood as a synonym for a random signal, but more often it is used only when the random signal is considered harmful or undesirable.

2.6.1 Mean, Variance, and Covariance

The value of a random variable x is governed by its probability distribution. This distribution has, in general, a *mean*, which we denote by

$$\mu_x = E(x)$$

[where $E(\cdot)$ denotes expectation[6]], and a *variance*, which we denote by

$$\gamma_x = E(x - \mu_x)^2.$$

The square root of the variance, $\gamma_x^{1/2}$, is called the *standard deviation* of the random variable and is denoted by σ_x. The mean and the standard deviation are measured in the same physical units as the random variable itself. For example, if the random variable is a voltage across the terminals of a battery, the mean and the standard deviation are measured in volts, whereas the variance is measured in volts squared.

Two random variables x and y governed by a joint probability distribution have a *covariance*, defined by

$$\gamma_{x,y} = E[(x - \mu_x)(y - \mu_y)].$$

The covariance can be positive, negative, or zero; it obeys the *Cauchy-Schwarz* inequality[7]

$$|\gamma_{x,y}| \le \sigma_x \sigma_y. \tag{2.60}$$

Two random variables are said to be *uncorrelated* if their covariance is zero. If x and y are the same random variable, their covariance is equal to the variance of x.

A random signal $x(t)$ has mean and variance at every time point. The mean and the variance depend on t in general, so we denote them as functions of time, $\mu_x(t)$ and $\gamma_x(t)$, respectively. Thus, the mean and the variance of a random signal are

$$\mu_x(t) = E[x(t)], \quad \gamma_x(t) = E[x(t) - \mu_x(t)]^2. \tag{2.61}$$

The covariance of a random signal at two different time points t_1, t_2 is denoted by

$$\gamma_x(t_1, t_2) = E\{[x(t_1) - \mu_x(t_1)][x(t_2) - \mu_x(t_2)]\}. \tag{2.62}$$

Note that, whereas $\mu_x(t)$ and $\gamma_x(t)$ are functions of a single variable (the time t), the covariance is a function of two time variables.

2.6.2 Wide-Sense Stationary Signals

A random signal $x(t)$ is called *wide-sense stationary* (WSS) if it satisfies the following two properties:[8]

1. The mean $\mu_x(t)$ is the same at all time points, that is,

$$\mu_x(t) = \mu_x = \text{const}. \tag{2.63}$$

2. The covariance $\gamma_x(t_1, t_2)$ depends only on the difference between t_1 and t_2, that is,

$$\gamma_x(t_1, t_2) = \kappa_x(t_1 - t_2). \tag{2.64}$$

For a WSS signal, we denote the difference $t_1 - t_2$ by τ, and call it the *lag variable* of the function $\kappa_x(\tau)$. The function $\kappa_x(\tau)$ is called the *covariance function* of $x(t)$. Thus, the covariance function of a WSS random signal is

$$\kappa_x(\tau) = E\{[x(t + \tau) - \mu_x][x(t) - \mu_x]\}. \tag{2.65}$$

The right side of (2.65) is independent of t by definition of wide-sense stationarity. Note that (2.65) can also be expressed as (see Problem 2.37)

$$\kappa_x(\tau) = E[x(t + \tau)x(t)] - \mu_x^2. \tag{2.66}$$

The main properties of the covariance function are as follows:

1. $\kappa_x(0)$ is the variance of $x(t)$, that is,

$$\kappa_x(0) = E[x(t) - \mu_x]^2 = \gamma_x. \tag{2.67}$$

2. $\kappa_x(\tau)$ is a symmetric function of τ, since

$$\begin{aligned}\kappa_x(\tau) &= E\{[x(t + \tau) - \mu_x][x(t) - \mu_x]\} \\ &= E\{[x(t) - \mu_x][x(t + \tau) - \mu_x]\} = \kappa_x(-\tau).\end{aligned} \tag{2.68}$$

3. By the Cauchy-Schwarz inequality we have

$$|\kappa_x(\tau)| \le \sigma_x\sigma_x = \gamma_x, \quad \text{for all } \tau. \tag{2.69}$$

The random signal shown in Figure 2.5 is wide-sense stationary. As we see, a WSS signal looks more or less the same at different time intervals. Although its detailed form varies, its overall (or macroscopic) shape does not. An example of a random signal that is not stationary is a seismic wave during an earthquake. Figure 2.6 depicts such a wave. As we see, the amplitude of the wave shortly before the beginning of the earthquake is small. At the start of the earthquake the amplitude grows suddenly, sustains its amplitude for a certain time, then decays. Another example of a nonstationary signal is human speech. Although whether a speech signal is essentially random can be argued, it definitely has certain random features. Speech is not stationary, since different phonemes have different characteristic waveforms. Therefore, as the spoken sound moves from phoneme to phoneme (for example, from "f" to "i" to "sh" in "fish"), the macroscopic shape of the signal varies.

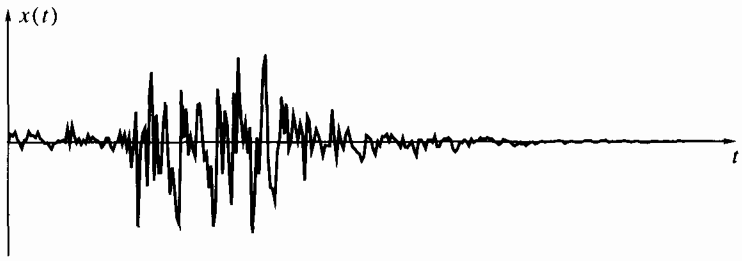

Figure 2.6 A seismic wave during an earthquake.

2.6.3 The Power Spectral Density

The Fourier transform of a WSS random signal requires a special definition, because (2.1) does not exist as a standard integral when $x(t)$ is a WSS random signal. However, the restriction of such a signal to a finite interval, say $[-0.5T, 0.5T]$, does possess a standard Fourier transform. The Fourier transform of a finite segment of a random signal appears random as well. For example, Figure 2.7 shows the magnitude of the Fourier transform of a finite interval of the random signal shown in Figure 2.5. As we see, this figure is difficult to interpret and its usefulness is limited.

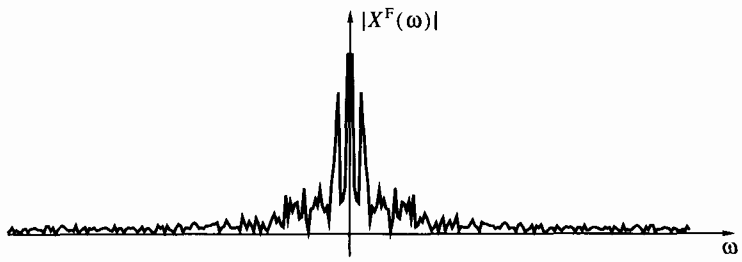

Figure 2.7 Magnitude of the Fourier transform of a finite segment of the signal in Figure 2.5.

A more meaningful way of representing random signals in the frequency domain is by their power spectra. The *power spectral density* (PSD), or *power spectrum*, of a WSS signal is defined as the Fourier transform of its covariance function,

$$K_x^{\mathrm{F}}(\omega) = \int_{-\infty}^{\infty} \kappa_x(\tau) e^{-j\omega\tau} d\tau, \tag{2.70}$$

provided the right side exists. A sufficient condition for existence of the PSD is

$$\int_{-\infty}^{\infty} |\kappa_x(\tau)| d\tau < \infty. \tag{2.71}$$

The inverse relationship is

$$\kappa_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_x^{\mathrm{F}}(\omega) e^{j\omega\tau} d\omega. \tag{2.72}$$

The PSD is not a random entity, since $\kappa_x(\tau)$ is not a random function. For example, the PSD of the random signal shown in Figure 2.5 is shown in Figure 2.8. Comparing this figure with Figure 2.7, we see that it looks smooth and well behaved.

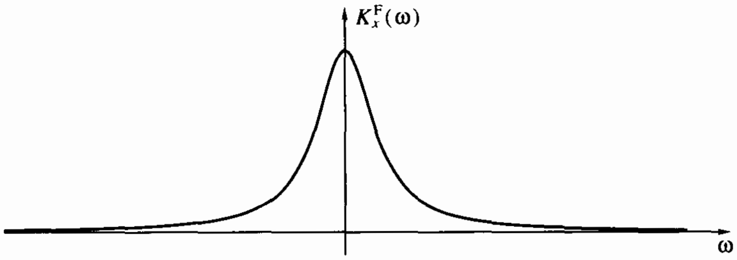

Figure 2.8 Power spectral density of the signal in Figure 2.5.

The PSD is expressed in physical units of power per hertz. For example, if $x(t)$ is measured in volts, then $K_x^{\mathrm{F}}(\omega)$ is measured in V^2/Hz. The value of $K_x^{\mathrm{F}}(\omega)$ can be interpreted as the average power density of the signal at frequency ω; stated in different words, the total average power of the signal in the frequency range $[\omega - 0.5d\omega, \omega + 0.5d\omega]$ is approximately $(2\pi)^{-1}K_x^{\mathrm{F}}(\omega)d\omega$ when $d\omega$ is small. This shows why the units of $K_x^{\mathrm{F}}(\omega)$ are power per hertz: It is because $(2\pi)^{-1}d\omega = df$, and df is measured in hertz.

The main properties of the power spectral density of a WSS signal are as follows:

1. It is real and symmetric in ω, because $\kappa_x(\tau)$ is real and symmetric in τ.
2. It is nonnegative, that is,

$$K_x^{\mathrm{F}}(\omega) \geq 0, \quad \text{for all } \omega. \tag{2.73}$$

   This property will follow from Theorem 2.3; see page 26.
3. The integral of the PSD over all frequencies is proportional to the variance of the signal, that is,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} K_x^{\mathrm{F}}(\omega) d\omega = \kappa_x(0) = \gamma_x. \tag{2.74}$$

   This follows from (2.72) when substituting $\tau = 0$.

Example 2.1 Here are a few examples of covariance functions and their PSDs.

1. A random signal $v(t)$ whose power spectral density is constant at all frequencies is called *white noise*. Thus, for white noise we have

$$K_v^{\mathrm{F}}(\omega) = N_0 1(\omega). \tag{2.75}$$

Correspondingly,

$$\kappa_v(\tau) = N_0 \delta(\tau). \tag{2.76}$$

As we see from (2.76) [or from (2.75) and (2.74)], white noise has infinite variance, so it exists only as a mathematical object, rather than as a physical signal. Nevertheless, it is a useful approximation of certain physical signals, as well as an important mathematical tool in the theory of random signals. The name *white* is borrowed from optics: The spectrum of white light is approximately constant at all frequencies within the range of visible light. Figure 2.9 illustrates (rather crudely) white noise.

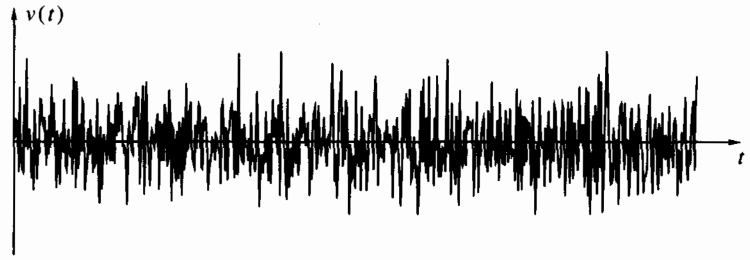

Figure 2.9 White noise.

2. A random signal $v(t)$ whose power spectral density is constant at all frequencies on a finite frequency interval and zero elsewhere is called *band-limited white noise.* Thus, for band-limited white noise we have

$$K_v^{\mathrm{F}}(\omega) = N_0 \operatorname{rect}\left(\frac{\omega}{2\omega_{\mathrm{m}}}\right) \tag{2.77}$$

for some constant ω_{m}. Correspondingly,

$$\kappa_v(\tau) = \frac{N_0 \omega_{\mathrm{m}}}{\pi} \operatorname{sinc}\left(\frac{\omega_{\mathrm{m}} \tau}{\pi}\right). \tag{2.78}$$

Band-limited white noise has a finite variance, equal to $N_0 \omega_{\mathrm{m}} / \pi$.

3. A WSS signal $x(t)$ whose covariance function is

$$\kappa_x(\tau) = N_0 e^{-\omega_0 |\tau|} \tag{2.79}$$

for some positive constant ω_0 is said to be *exponentially correlated.* The parameter $1/\omega_0$ is called the *correlation time constant* of the signal. The PSD of (2.79) is computed as follows:

$$\begin{aligned} K_x^{\mathrm{F}}(\omega) &= N_0 \int_{-\infty}^{0} e^{(\omega_0 - j\omega)\tau} d\tau + N_0 \int_{0}^{\infty} e^{-(\omega_0 + j\omega)\tau} d\tau \\ &= \frac{N_0}{\omega_0 - j\omega} + \frac{N_0}{\omega_0 + j\omega} = \frac{2N_0 \omega_0}{\omega^2 + \omega_0^2}. \end{aligned} \tag{2.80}$$

The random signal shown in Figure 2.5 is exponentially correlated and the PSD shown in Figure 2.8 corresponds to (2.80). □

An important relationship exists between the PSD of a WSS random signal and the Fourier transform of a finite segment of the signal. This relationship is known as the *Wiener-Khintchine theorem,* and is given as follows:

Theorem 2.3 (Wiener-Khintchine) Let $x(t)$ be a WSS random signal whose mean is zero and whose covariance function $\kappa_x(\tau)$ satisfies (2.71). Let

$$y(t) = \begin{cases} x(t), & -0.5T \le t \le 0.5T, \\ 0, & \text{otherwise,} \end{cases} \tag{2.81}$$

that is, $y(t)$ is the restriction of $x(t)$ to the interval $[-0.5T, 0.5T]$. Then

$$\lim_{T\to\infty} \frac{1}{T} E(|Y^{\mathrm{F}}(\omega)|^2) = K_x^{\mathrm{F}}(\omega). \tag{2.82}$$

Partial proof We have

$$\begin{aligned} \frac{1}{T}E(|Y^{\mathrm{F}}(\omega)|^2) &= \frac{1}{T}E\left\{\left[\int_{-0.5T}^{0.5T} x(t)e^{-j\omega t}dt\right]\left[\int_{-0.5T}^{0.5T} x(s)e^{j\omega s}ds\right]\right\} \\ &= \frac{1}{T}\int_{-0.5T}^{0.5T}\int_{-0.5T}^{0.5T} E[x(t)x(s)]e^{-j\omega(t-s)}dtds = \frac{1}{T}\int_{-0.5T}^{0.5T}\int_{-0.5T}^{0.5T} \kappa_x(t-s)e^{-j\omega(t-s)}dtds \\ &= \frac{1}{T}\int_{-T}^{T} \kappa_x(u)e^{-j\omega u}\left[\int_0^{T-|u|} 1\cdot dv\right]du = \int_{-T}^{T}\left(1-\frac{|u|}{T}\right)\kappa_x(u)e^{-j\omega u}du. \end{aligned} \tag{2.83}$$

The part of the proof that is skipped, since it is beyond the mathematical level of this book, is the equality of the following two limits:

$$\int_{-\infty}^{\infty} |a(t)|dt < \infty \implies \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)a(t)dt = \lim_{T\to\infty}\int_{-T}^{T} a(t)dt = \int_{-\infty}^{\infty} a(t)dt. \tag{2.84}$$

Taking the limit of (2.83) and using (2.84) finally gives

$$\lim_{T\to\infty}\frac{1}{T}E(|Y^{\mathrm{F}}(\omega)|^2) = \int_{-\infty}^{\infty}\kappa_x(u)e^{-j\omega u}du = K_x^{\mathrm{F}}(\omega). \tag{2.85}$$

Corollary The PSD $K_x^{\mathrm{F}}(\omega)$ is a nonnegative function of ω, since it is the limit of nonnegative functions. □

2.6.4 WSS Signals and LTI Systems

Consider an LTI system whose impulse response $h(t)$ satisfies

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty. \tag{2.86}$$

When a WSS signal is given at the input of such a system, the output signal is also WSS. This is formally stated and proved in the following theorem.

Theorem 2.4 Let $x(t)$ be a WSS signal with mean μ_x and covariance function $\kappa_x(\tau)$. Let

$$y(t) = \{h * x\}(t),$$

where $h(t)$ satisfies (2.86); then $y(t)$ is WSS. The mean and covariance function of $y(t)$ are given by

$$\mu_y = \mu_x\int_{-\infty}^{\infty} h(s)ds, \tag{2.87a}$$

$$\kappa_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(u)\kappa_x(\tau - s + u)dsdu. \tag{2.87b}$$

The PSD of $y(t)$ is given by

$$K_y^{\mathrm{F}}(\omega) = K_x^{\mathrm{F}}(\omega)|H^{\mathrm{F}}(\omega)|^2. \tag{2.88}$$

Proof

1.
$$\mu_y = E[y(t)] = E\left\{\int_{-\infty}^{\infty} h(s)x(t-s)ds\right\} = \int_{-\infty}^{\infty} h(s)E[x(t-s)]ds$$
$$= \int_{-\infty}^{\infty} h(s)\mu_x ds = \mu_x \int_{-\infty}^{\infty} h(s)ds. \tag{2.89}$$

2.
$$\kappa_y(\tau) = E\left\{\left[\int_{-\infty}^{\infty} h(s)x(t+\tau-s)ds\right]\left[\int_{-\infty}^{\infty} h(u)x(t-u)du\right]\right\} - \mu_y^2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(u)E[x(t+\tau-s)x(t-u)]dsdu - \mu_y^2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(u)[\kappa_x(\tau-s+u)+\mu_x^2]dsdu - \mu_y^2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(u)\kappa_x(\tau-s+u)dsdu. \tag{2.90}$$

Since both μ_y and κ_y are found to be independent of t, the output signal $y(t)$ is indeed WSS.

3. Using (2.87b) and the definition of PSD, we can write

$$K_y^{\mathrm{F}}(\omega) = \int_{-\infty}^{\infty} \kappa_y(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(u)\kappa_x(\tau-s+u)e^{-j\omega\tau}dsdud\tau$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(u)e^{-j\omega(s-u)}K_x^{\mathrm{F}}(\omega)dsdu$$
$$= \left[\int_{-\infty}^{\infty} h(s)e^{-j\omega s}ds\right]\left[\int_{-\infty}^{\infty} h(u)e^{j\omega u}du\right]K_x^{\mathrm{F}}(\omega)$$
$$= H^{\mathrm{F}}(\omega)\bar{H}^{\mathrm{F}}(\omega)K_x^{\mathrm{F}}(\omega) = K_x^{\mathrm{F}}(\omega)|H^{\mathrm{F}}(\omega)|^2. \tag{2.91}$$

In passing from the first to the second line, we integrated over τ and used the time-shift property of the Fourier transform (2.5). □

Formula (2.88) is a major result in random signal theory. It gives a new interpretation to the frequency response of an LTI system. In Section 2.2.2 we saw that an LTI system operates on a sinusoidal signal to multiply its magnitude by the magnitude response of the system and shift its phase by the phase response. Now we see that the square-magnitude response $|H^{\mathrm{F}}(\omega)|^2$ multiplies the power spectral density of a WSS random signal at each frequency.

An important special case of (2.88) is applicable when the input signal is white noise. In this case we get

$$K_y^{\mathrm{F}}(\omega) = N_0|H^{\mathrm{F}}(\omega)|^2. \tag{2.92}$$

Therefore, the PSD of the response to white noise is proportional to the square magnitude of the frequency response of the system.

2.7 Discrete-Time Signals and Systems

Let $x[n]$ be a discrete-time signal whose values can be real or complex. The formal definition of the Fourier transform of the signal is[9]

$$X^{\mathrm{f}}(\theta) = \{\mathcal{F}x\}(\theta) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}, \quad \theta \in \mathbb{R}, \tag{2.93}$$

subject to the existence of the right side in a well-defined sense. A sufficient condition for the existence of the right side is

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty. \tag{2.94}$$

The inverse Fourier transform is then given by

$$x[n] = \{\mathcal{F}^{-1}X^{\mathrm{f}}\}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{\mathrm{f}}(\theta)e^{j\theta n}d\theta, \quad n \in \mathbb{Z}. \tag{2.95}$$

We remark that the Fourier transform of $x[n]$ can also be defined if (2.94) is violated, but the weaker condition

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \tag{2.96}$$

holds. In this case, the sum (2.93) exists in the sense that[10]

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left|X^{\mathrm{f}}(\theta) - \sum_{n=-N}^{N} x[n]e^{-j\theta n}\right|^2 d\theta = 0. \tag{2.97}$$

The variable θ is called the *angular frequency* and is measured in radians per sample.

The main properties of the Fourier transform of a sequence are as follows.

1. **Linearity**

$$z[n] = ax[n] + by[n] \iff Z^{\mathrm{f}}(\theta) = aX^{\mathrm{f}}(\theta) + bY^{\mathrm{f}}(\theta), \quad a, b \in \mathbb{C}. \tag{2.98}$$

2. **Periodicity**

$$X^{\mathrm{f}}(\theta) = X^{\mathrm{f}}(\theta + 2\pi k), \quad \theta \in \mathbb{R},\ k \in \mathbb{Z}. \tag{2.99}$$

3. **Time shift**

$$y[n] = x[n-m] \iff Y^{\mathrm{f}}(\theta) = e^{-j\theta m}X^{\mathrm{f}}(\theta), \quad m \in \mathbb{Z}. \tag{2.100}$$

4. **Frequency shift (modulation)**

$$y[n] = e^{j\lambda n}x[n] \iff Y^{\mathrm{f}}(\theta) = X^{\mathrm{f}}(\theta - \lambda), \quad \lambda \in \mathbb{R}. \tag{2.101}$$

5. **Time-domain convolution**

$$z[n] = \{x * y\}[n] \iff Z^{\mathrm{f}}(\theta) = X^{\mathrm{f}}(\theta)Y^{\mathrm{f}}(\theta). \tag{2.102}$$

6. **Time-domain multiplication**

$$z[n] = x[n]y[n] \iff Z^{\mathrm{f}}(\theta) = \frac{1}{2\pi}\{X^{\mathrm{f}} * Y^{\mathrm{f}}\}(\theta). \tag{2.103}$$

7. **Parseval's theorem**

$$\sum_{n=-\infty}^{\infty} x[n]\bar{y}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{\mathrm{f}}(\theta)\bar{Y}^{\mathrm{f}}(\theta)d\theta, \tag{2.104}$$

and its special case

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X^{\mathrm{f}}(\theta)|^2 d\theta. \tag{2.105}$$

8. **Conjugation**

$$y[n] = \bar{x}[n] \iff Y^{\mathrm{f}}(\theta) = \bar{X}^{\mathrm{f}}(-\theta). \tag{2.106}$$

9. **Symmetry** if $x[n]$ is real valued then

$$X^{\mathrm{f}}(-\theta) = \bar{X}^{\mathrm{f}}(\theta), \tag{2.107a}$$

$$\Re\{X^{\mathrm{f}}(-\theta)\} = \Re\{X^{\mathrm{f}}(\theta)\}, \tag{2.107b}$$

$$\Im\{X^{\mathrm{f}}(-\theta)\} = -\Im\{X^{\mathrm{f}}(\theta)\}, \tag{2.107c}$$

$$|X^{\mathrm{f}}(-\theta)| = |X^{\mathrm{f}}(\theta)|, \tag{2.107d}$$

$$\measuredangle X^{\mathrm{f}}(-\theta) = -\measuredangle X^{\mathrm{f}}(\theta). \tag{2.107e}$$

10. **Realness**

$$x[n] = \bar{x}[-n] \iff X^{\mathrm{f}}(\theta) \text{ is real.} \tag{2.108}$$

A discrete-time signal of great importance is the *unit sample*, or *unit impulse*, denoted by $\delta[n]$ and defined by

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.109}$$

Note that we distinguish between the continuous-time delta and the discrete-time delta by the typography of the argument (in parentheses for the former, in brackets for the latter).

A *discrete-time SISO system* is a system whose input and output signals, $x[n]$ and $y[n]$, are in discrete time. The notions of linearity and time invariance apply to discrete-time systems in the same way as to continuous-time systems. The response of a discrete-time LTI system to the unit-sample signal $\delta[n]$ is called the *unit-sample response*, or the *impulse response* of the system, and is denoted by $h[n]$. The response of the system to an input signal $x[n]$ is given by the discrete-time convolution

$$y[n] = \{h * x\}[n], \tag{2.110}$$

provided the right side exists. The *frequency response* of the system is $H^{\mathrm{f}}(\theta)$, the Fourier transform of the impulse response (if the Fourier transform exists). The frequency-domain relationship between input and output is

$$Y^{\mathrm{f}}(\theta) = H^{\mathrm{f}}(\theta)X^{\mathrm{f}}(\theta). \tag{2.111}$$

2.8 Discrete-Time Periodic Signals

A discrete-time signal $x[n]$ is called *periodic* if there exists an integer $N > 0$ such that

$$x[n] = x[n+N], \quad \text{for all } n \in \mathbb{Z}. \tag{2.112}$$

The smallest N for which (2.112) holds is called the *period.*

Unlike continuous-time signals, a discrete-time sinusoidal signal

$$x[n] = \cos(\theta_0 n + \phi_0) \tag{2.113}$$

is not necessarily periodic. It is periodic if and only if $\theta_0/2\pi$ is a rational number (Problem 2.35). The formal Fourier transform of this signal (regardless of whether it is periodic) is

$$X^{\mathrm{f}}(\theta) = \pi e^{j\phi_0}\delta(\theta - \theta_0) + \pi e^{-j\phi_0}\delta(\theta + \theta_0). \tag{2.114}$$

The Fourier series of a discrete-time periodic signal is called the *discrete Fourier transform* (DFT), and we shall discuss it in detail in Chapter 4.

As in the case of continuous-time signals, we have the following relation between discrete-time periodic signals and linear systems:

Theorem 2.5 When the input signal of a discrete-time LTI system is periodic, the output is also a periodic signal.

The easy proof is left as an exercise to the reader; see Problem 2.33. □

2.9 Discrete-Time Random Signals*

Most of the basic definitions and properties of continuous-time random signals carry over to discrete-time random signals. Therefore, we shall briefly summarize the concepts and results pertaining to discrete-time random signals, rather than repeating the material of Section 2.6 in full detail.

A discrete-time random signal is a sequence $x[n]$ whose value at each time point is a random variable. The mean and the variance of a discrete-time random signal are

$$\mu_x[n] = E(x[n]), \quad \gamma_x[n] = E(x[n] - \mu_x[n])^2. \tag{2.115}$$

The covariance at two different time points n_1, n_2 is denoted by

$$\gamma_x[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1])(x[n_2] - \mu_x[n_2])\}. \tag{2.116}$$

Note that the covariance is a function of two integer variables.

A discrete-time random signal $x[n]$ is called *wide-sense stationary* (WSS) if it satisfies the following two properties:

1. The mean $\mu_x[n]$ is the same at all time points, that is,

$$\mu_x[n] = \mu_x = \text{const}. \tag{2.117}$$

2. The covariance $\gamma_x[n_1, n_2]$ depends only on the difference between n_1 and n_2, that is,

$$\gamma_x[n_1, n_2] = \kappa_x[n_1 - n_2]. \tag{2.118}$$

For a WSS signal, we denote the difference $n_1 - n_2$ by m and call it the *lag variable* of the sequence $\kappa_x[m]$. The sequence $\kappa_x[m]$ is called the *covariance sequence* of $x[n]$. Thus, the covariance sequence of a WSS random signal is

$$\kappa_x[m] = E\{(x[n+m] - \mu_x)(x[n] - \mu_x)\} = E(x[n+m]x[n]) - \mu_x^2. \tag{2.119}$$

The main properties of the covariance sequence are as follows:

1. $\kappa_x[0]$ is the variance of $x[n]$, that is,

$$\kappa_x[0] = E(x[n] - \mu_x)^2 = \gamma_x. \tag{2.120}$$

2. $\kappa_x[m]$ is a symmetric function of m.
3.

$$|\kappa_x[m]| \le \gamma_x, \quad \text{for all } m. \tag{2.121}$$

The *power spectral density* (PSD) of a WSS signal is the Fourier transform of its covariance sequence:

$$K_x^{\mathrm{f}}(\theta) = \sum_{m=-\infty}^{\infty} \kappa_x[m]e^{-j\theta m}, \tag{2.122}$$

provided the right side exists. A sufficient condition for existence of the PSD is

$$\sum_{m=-\infty}^{\infty} |\kappa_x[m]| < \infty. \tag{2.123}$$

The inverse relationship is

$$\kappa_x[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_x^f(\theta)e^{j\theta m}d\theta. \tag{2.124}$$

The main properties of the power spectral density of a WSS signal are as follows:

1. It is real and symmetric in θ.
2. It is nonnegative, that is,

$$K_x^f(\theta) \geq 0, \quad \text{for all } \theta. \tag{2.125}$$

3. The integral of the PSD over $[-\pi, \pi]$ is proportional to the variance of the signal, that is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_x^f(\theta)d\theta = \kappa_x[0] = \gamma_x. \tag{2.126}$$

Example 2.2 Here are two examples of covariance sequences of discrete-time signals and their PSDs.

1. A discrete-time random signal whose power spectral density is constant at all frequencies is called *discrete-time white noise.* Thus, for a discrete-time white noise $v[n]$ we have

$$K_v^f(\theta) = \gamma_v, \quad \kappa_v[m] = \gamma_v\delta[m]. \tag{2.127}$$

Unlike continuous-time white noise, there is nothing nonphysical in a discrete-time white noise. Its variance is finite and the covariance of any two different samples is zero. The MATLAB command `randn(1,N)` can be used to generate N samples of discrete-time, zero mean, unit variance white noise.

2. A WSS signal $x[n]$ whose covariance sequence is

$$\kappa_x[m] = \gamma_x e^{-\theta_0|m|} \tag{2.128}$$

for some positive constant θ_0 is said to be *exponentially correlated.* The PSD of (2.128) is computed as follows:

$$\begin{aligned} K_x^f(\theta) &= \gamma_x \sum_{m=-\infty}^{-1} e^{(\theta_0 - j\theta)m} + \gamma_x \sum_{m=0}^{\infty} e^{-(\theta_0+j\theta)m} = \frac{\gamma_x e^{-\theta_0+j\theta}}{1-e^{-\theta_0+j\theta}} + \frac{\gamma_x}{1-e^{-\theta_0-j\theta}} \\ &= \frac{\gamma_x(1-e^{-2\theta_0})}{1-2e^{-\theta_0}\cos\theta + e^{-2\theta_0}}. \end{aligned} \tag{2.129}$$

□

The Wiener-Khintchine theorem for discrete-time signals is:

Theorem 2.6 Let $x[n]$ be a WSS discrete-time random signal whose mean is zero and whose covariance sequence $\kappa_x[m]$ satisfies (2.123). Let N be odd and define

$$y[n] = \begin{cases} x[n], & -0.5(N-1) \leq n \leq 0.5(N-1), \\ 0, & \text{otherwise,} \end{cases} \tag{2.130}$$

that is, $y[n]$ is the restriction of $x[n]$ to the interval $[-0.5(N-1), 0.5(N-1)]$. Then

$$\lim_{N\to\infty} \frac{1}{N} E(|Y^f(\theta)|^2) = K_x^f(\theta). \tag{2.131}$$

The proof is similar to that of Theorem 2.3 and will be skipped. □

Consider a discrete-time LTI system whose impulse response $h[n]$ satisfies

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{2.132}$$

Then,

Theorem 2.7 Let $x[n]$ be a discrete-time WSS signal with mean μ_x and covariance sequence $\kappa_x[m]$. Let

$$y[n] = \{h * x\}[n],$$

where $h[n]$ satisfies (2.132). Then $y[n]$ is WSS. The mean and the covariance sequence of $y[n]$ are given by

$$\mu_y = \mu_x \sum_{n=-\infty}^{\infty} h[n], \tag{2.133a}$$

$$\kappa_y[m] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[n]h[k]\kappa_x[m-n+k]. \tag{2.133b}$$

The PSD of $y[n]$ is given by

$$K_y^{\mathrm{f}}(\theta) = K_x^{\mathrm{f}}(\theta)|H^{\mathrm{f}}(\theta)|^2. \tag{2.134}$$

The proof is similar to that of Theorem 2.4 and will be skipped. □

An important special case of (2.134) is applicable when the input signal is white noise $v[n]$. In this case we get

$$K_y^{\mathrm{f}}(\theta) = \gamma_v|H^{\mathrm{f}}(\theta)|^2. \tag{2.135}$$

As we see, the PSD of the response to white noise is proportional to the square magnitude of the frequency response of the system. Using (2.126), we get from (2.135)

$$\gamma_y = \frac{\gamma_v}{2\pi}\int_{-\pi}^{\pi} |H^{\mathrm{f}}(\theta)|^2 d\theta. \tag{2.136}$$

This relationship provides a convenient means of computing the variance of the response of an LTI system to white noise input. The quantity

$$\mathrm{NG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H^{\mathrm{f}}(\theta)|^2 d\theta \tag{2.137}$$

is called the *noise gain* of the system. The noise gain of an LTI system is a measure of the ratio between the output and input variances when the input is white noise.

2.10 Summary and Complements

2.10.1 Summary

In this chapter we reviewed frequency-domain analysis and its relationships to linear system theory. The fundamental operation is the Fourier transform of a continuous-time signal, defined in (2.1), and the inverse transform, given in (2.3). The Fourier transform is a mathematical operation that (1) detects sinusoidal components in a signal and enables the computation of their amplitudes and phases and (2) provides the amplitude and phase density of nonperiodic signals as a function of the frequency. Among the properties of the Fourier transform, the most important is perhaps the convolution property (2.8). The reason is that the response of a linear time-invariant (LTI) system to an arbitrary input is the convolution between the input signal and the impulse response of the system (2.17). From this it follows that the Fourier transform of the output signal is the Fourier transform of the input, multiplied by the frequency response of the system (2.18).

We introduced a few common signals and their Fourier transforms; in particular: the delta function $\delta(t)$, the DC function $1(t)$, the complex exponential, sinusoidal signals, the rectangular function $\mathrm{rect}(t)$, the sinc function $\mathrm{sinc}(t)$, and the Gaussian function. Complex exponentials and sinusoids are eigenfunctions of LTI systems: They undergo change of amplitude and phase, but their functional form is preserved when passed through an LTI system.

Continuous-time periodic signals were introduced next. Such signals admit a Fourier series expansion (2.39). A periodic signal of particular interest is the impulse train $p_T(t)$. The Fourier transform of an impulse train is an impulse train in the frequency domain (2.52). The impulse train satisfies the Poisson sum formula (2.48). A continuous-time signal on a finite interval can be represented by a Fourier series (2.39). A continuous-time, real-valued signal on a finite interval can also be represented by either a cosine Fourier series (2.57) or a sine Fourier series (2.59).

We reviewed continuous-time random signals, in particular wide-sense stationary (WSS) signals. A WSS signal is characterized by a constant mean and a covariance function depending only on the lag variable. The Fourier transform of the covariance function is called the power spectral density (PSD) of the signal. The PSD of a real-valued WSS signal is real, symmetric, and nonnegative. Two examples of WSS signals are white noise and band-limited white noise. The PSD of the former is constant for all frequencies, whereas that of the latter is constant and nonzero on a finite frequency interval. The PSD of a WSS signal satisfies the Wiener–Khintchine theorem 2.3.

When a WSS signal passes through an LTI system, the output is WSS as well. The PSD of the output is the product of the PSD of the input and the square magnitude of the frequency response of the system (2.88). In particular, when the input signal is white noise, the PSD of the output signal is proportional to the square magnitude of the frequency response (2.92).

Frequency-domain analysis of discrete-time signals parallels that of continuous-time signals in many respects. The Fourier transform of a discrete-time signal is defined in (2.93) and its inverse is given in (2.95). The Fourier transform of a discrete-time signal is periodic, with period 2π. Discrete-time periodic signals and discrete-time random signals are defined in a manner similar to the corresponding continuous-time signals and share similar properties.

The material in this chapter is covered in many books. For general signals and system theory, see Oppenheim and Willsky [1983], Gabel and Roberts [1987], Kwakernaak and Sivan [1991], or Haykin and Van Veen [1997]. For random signals and their relation to linear systems, see Papoulis [1991] or Gardner [1986].

2.10.2 Complements

1. [p. 11] The definition of the Fourier transform is not completely standard. Variations of the definition (2.1) include:

   (a) Normalization:

$$X^{\mathrm{F}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{2.138}$$

   With this definition, the inverse transform (2.3) has a factor $1/\sqrt{2\pi}$ (instead of $1/2\pi$).

(b) Sign reversal of the frequency variable:

$$X^{\mathrm{F}}(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt. \tag{2.139}$$

In this case, the inverse transform has a negative sign in the exponent.

(c) Use of frequency (instead of angular frequency) as the transform variable:

$$X^{\mathrm{F}}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt. \tag{2.140}$$

In this case, the inverse transform is

$$x(t) = \int_{-\infty}^{\infty} X^{\mathrm{F}}(f)e^{j2\pi ft}df. \tag{2.141}$$

(d) Various combinations of the above.

2. [p. 11] Beginners sometimes find the notion of negative frequencies difficult to comprehend. Often a student would say: "We know that only positive frequencies exist physically!" A possible answer is: "Think of a rotating wheel; a wheel rotating clockwise is certainly different from one rotating counterclockwise. So, if you define the angular velocity of a clockwise-rotating wheel as positive, you must define that of a counterclockwise-rotating wheel as negative. The angular frequency of a signal fulfills the same role as the angular velocity of a rotating wheel, so it can have either polarity." For real-valued signals, the conjugate symmetry property of the Fourier transform (2.13a) disguises the existence of negative frequencies. However, for complex signals, positive and negative frequencies are fundamentally distinct. For example, the complex signal $e^{j\omega_0 t}$ with $\omega_0 > 0$ is different from a corresponding signal with $\omega_0 < 0$.

   Complex signals are similarly difficult for beginners to comprehend, since such signals are not commonly encountered as physical entities. Rather, they usually serve as convenient mathematical representations for real signals of certain types. An example familiar to electrical engineering students is the phasor representation of AC voltages and currents (e.g., in sinusoidal steady-state analysis of electrical circuits). A real voltage $v_{\mathrm{m}}\cos(\omega_0 t + \phi_0)$ is represented by the phasor $V = v_{\mathrm{m}}e^{j\phi_0}$. The phasor represents the real AC signal by a complex DC signal V.

3. [p. 12] The Dirac delta function is not a function in the usual sense, since it is infinite at $t = 0$. There is a mathematical framework that treats the delta function (and its derivatives) in a rigorous manner. This framework, called the *theory of distributions*, is beyond the scope of this book; see Kwakernaak and Sivan [1991, Supplement C] for a relatively elementary treatment of this subject. We shall continue to use the delta function but not in rigor, as common in engineering books.

4. [p. 13] A common misconception is that every LTI system has an impulse response. The following example shows that this is not true. Let $x(t)$ be in the input family if and only if (1) $x(t)$ is continuous, except at a countable number of points t; (2) the discontinuity at each such point is a finite jump, that is, the limits at both sides of the discontinuity exist; (3) the sum of absolute values of all discontinuity jumps is finite. Let $y(t)$ be the sum of all jumps of $x(\tau)$ at discontinuity points $\tau < t$. This system is linear and time invariant, but it has no impulse response because $\delta(t)$ is not in the input family. Consequently, its response to $x(t)$ cannot be described by a convolution. This example is by Kailath [1980].

5. [p. 14] The term *zero state* is related to the assumption that the system has no memory, or that its response to the input $x(t) = 0,\ -\infty < t < \infty$ is identically

zero. For example, an electrical circuit containing capacitors is in zero state if its capacitors are completely discharged before the input signal is applied. If some capacitors are charged, the circuit may have nonzero output even in the absence of any input. In this case, the system is not strictly linear since a linear system must, by definition, give zero response to zero input.

6. [p. 21] If x is a random variable possessing a probability density function $p(x)$ and $g(x)$ is any function of x, the expectation of $g(x)$ is

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)p(x)dx, \tag{2.142}$$

provided the right side exists. If x is a vector random variable, the density $p(x)$ is multidimensional and then the integral should be understood as a multiple integral.

7. [p. 22] The Cauchy-Schwarz inequality applies to any pair of mathematical objects for which an *inner product* is defined. If we denote the inner product of two objects x and y by $\langle x, y\rangle$, then the Cauchy-Schwarz inequality is

$$|\langle x, y\rangle| \le \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}, \tag{2.143}$$

with equality if and only if x and y are equal up to a proportionality constant. For example, if x and y are vectors in the three-dimensional Euclidean space, then

$$\langle x, y\rangle = \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}\cos\alpha, \tag{2.144}$$

where α is the angle between the two vectors. In this case, the Cauchy-Schwarz inequality simply states that the cosine of any angle is not larger than 1 in magnitude. A few other objects to which the Cauchy-Schwarz inequality applies are:

(a) Vectors in the complex N-dimensional Euclidean space; for those

$$\left|\sum_{n=1}^{N} x_n\bar{y}_n\right| \le \left[\sum_{n=1}^{N} |x_n|^2\right]^{1/2} \left[\sum_{n=1}^{N} |y_n|^2\right]^{1/2}. \tag{2.145}$$

(b) Complex sequences having finite energy; for those

$$\left|\sum_{n=-\infty}^{\infty} x_n\bar{y}_n\right| \le \left[\sum_{n=-\infty}^{\infty} |x_n|^2\right]^{1/2} \left[\sum_{n=-\infty}^{\infty} |y_n|^2\right]^{1/2}. \tag{2.146}$$

(c) Complex-valued functions, square integrable on a certain domain; for those

$$\left|\int x(t)\bar{y}(t)dt\right| \le \left[\int |x(t)|^2 dt\right]^{1/2} \left[\int |y(t)|^2 dt\right]^{1/2}. \tag{2.147}$$

(d) Random variables having finite second moments; for those

$$E(xy) \le [E(x^2)]^{1/2}[E(y^2)]^{1/2}. \tag{2.148}$$

8. [p. 22] Wide-sense stationarity should be distinguished from *strict-sense stationarity*, which is a much stronger property; see Papoulis [1991]. Strict-sense stationarity is outside the scope of our discussion.

9. [p. 27] Many books refer to the Fourier transform of a sequence as the *discrete-time Fourier transform*. We avoid this terminology for the following reasons: (1) to avoid confusion with the discrete Fourier transform, which we shall study in Chapter 4; (2) because the transform itself is not in discrete time, but in the continuous variable θ; (3) because it is redundant—the transformed object being a sequence uniquely defines the transform.

10. [p. 28] The integral in (2.97) is *not* a standard (Riemann) integral, but a *Lebesgue* integral. The discussion of such integrals is outside the scope of this book; see Rudin [1964].

2.11 Problems

2.1 Compute the Fourier transform of the continuous-time signal

$$x(t) = \begin{cases} C, & a \le t \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where a, b, C are real constants. Use the Fourier transform of the rect function, and the shift and scale properties (2.5), (2.7) of the Fourier transform. Then verify the result by a direct computation.

2.2 Compute the Fourier transform of the continuous-time signal

$$x(t) = \begin{cases} 1 - |t|, & -1 \le t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

in two different ways.

2.3 Compute the Fourier transforms of the continuous-time signals

$$x_1(t) = \begin{cases} e^{-\alpha t}, & t \ge 0, \\ 0, & t < 0, \end{cases} \qquad x_2(t) = \begin{cases} te^{-\alpha t}, & t \ge 0, \\ 0, & t < 0, \end{cases}$$

where $\alpha < 0$.

2.4 Compute the Fourier transform of the continuous-time signal

$$x(t) = \text{rect}(t)\cos(\omega_0 t),$$

where ω_0 is a real positive constant.

2.5 Does the signal $x(t) = \cosh(t)$ have a Fourier transform? If so, compute it. If not, explain the reason (cosh denotes a hyperbolic cosine).

2.6 Repeat Problem 2.5 for the signal $x(t) = \cosh(t)\text{rect}(t)$.

2.7 Let $g(t)$ be the Gaussian function, defined in (2.36), and let $h(t) = \{g * g\}(t)$. Compute $h(t)$, using the Fourier transform of $g(t)$.

2.8 The signal $x(t)$ is passed through the LTI system whose impulse response is $h(t)$, where

$$x(t) = \text{sinc}^2\left(\frac{\omega_0 t}{\pi}\right), \quad h(t) = \text{sinc}\left(\frac{\omega_0 t}{\pi}\right).$$

Compute the output $y(t)$ of the system.

2.9 Let $X^{\text{F}}(\omega)$ be as shown in Figure 2.10. Compute the inverse Fourier transform $x(t)$. Hint: You can use your solution to Problem 2.8.

2.10 We saw in (2.36), (2.37) that the inverse Fourier transform of $G^{\text{F}}(\omega) = e^{-0.5\omega^2}$ is a positive function. Is the same true for

$$X^{\text{F}}(\omega) = e^{-0.5(\omega-\omega_0)^2} + e^{-0.5(\omega+\omega_0)^2},$$

where ω_0 is a constant real number?

2.11 Let $x(t)$ be a continuous-time signal possessing a derivative $dx(t)/dt$ and a Fourier transform $X^{\text{F}}(\omega)$.

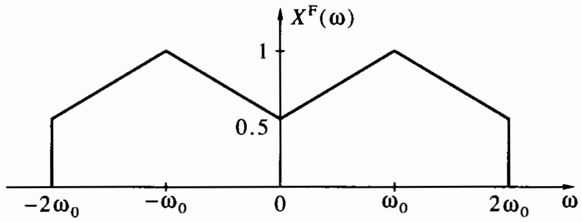

Figure 2.10 Pertaining to Problem 2.9.

(a) Prove the derivative property of the Fourier transform

$$y(t) = \frac{dx(t)}{dt} \implies Y^F(\omega) = j\omega X^F(\omega). \tag{2.149}$$

(b) Use part a and Parseval's theorem to express the quantity

$$\int_{-\infty}^{\infty} \left| \frac{dx(t)}{dt} \right|^2 dt$$

in terms of $X^F(\omega)$.

2.12 In this problem we derive the Fourier transforms of the sign function and the unit-step function.

(a) Find the Fourier transform of the signal

$$x(t) = \begin{cases} e^{-\alpha t}, & t > 0, \\ 0, & t = 0, \\ -e^{\alpha t}, & t < 0, \end{cases} \qquad \text{where} \quad \alpha > 0.$$

(b) The *sign function* $\text{sign}(t)$ is defined by

$$\text{sign}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

Find the Fourier transform of the sign function.

(c) The *unit-step function* $v(t)$ is defined by

$$v(t) = \begin{cases} 1, & t > 0, \\ 0.5, & t = 0, \\ 0, & t < 0. \end{cases}$$

Find the Fourier transform of the unit-step function. Hint: Express $v(t)$ in terms of $\text{sign}(t)$ and another signal.

2.13 Find the inverse Fourier transform of

$$H^F(\omega) = -j\,\text{sign}(\omega). \tag{2.150}$$

Hint: Use the solution to Problem 2.12. The LTI system whose frequency response is $H^F(\omega)$ is called a *Hilbert transformer.* The signal

$$y(t) = \{x * h\}(t)$$

is called the *Hilbert transform* of $x(t)$.

2.14 Let

$$y(t) = x(t) + j\{x * h\}(t),$$

where $h(t)$ is the impulse response of $H^{\mathrm{F}}(\omega)$ given in (2.150). Find $Y^{\mathrm{F}}(\omega)$ in terms of $X^{\mathrm{F}}(\omega)$. The signal $y(t)$ is called the *analytic signal* of $x(t)$.

2.15 This problem discusses the spectra of certain modulated signals.

(a) Let

$$y(t) = x(t)\cos(\omega_0 t), \quad \omega_0 > 0.$$

Express $Y^{\mathrm{F}}(\omega)$ in terms of $X^{\mathrm{F}}(\omega)$.

(b) Repeat part a for

$$y(t) = x(t)\sin(\omega_0 t), \quad \omega_0 > 0.$$

The operation in either part a or part b is called *double-side-band modulation* (DSB).

(c) Let

$$y(t) = x_1(t)\cos(\omega_0 t) - x_2(t)\sin(\omega_0 t), \quad \omega_0 > 0.$$

Express $Y^{\mathrm{F}}(\omega)$ in terms of $X_1^{\mathrm{F}}(\omega)$ and $X_2^{\mathrm{F}}(\omega)$.
This operation is called *quadrature amplitude modulation* (QAM); it is widely used in digital communication.

2.16 Consider the modulation operations described in Problem 2.15. Assume that the signal $x(t)$ is such that $X^{\mathrm{F}}(\omega) = 0$ for $|\omega| > \omega_{\mathrm{m}}$, where ω_{m} is fixed [for part c assume the same for both $x_1(t)$ and $x_2(t)$]. Assume further that $\omega_0 > \omega_{\mathrm{m}}$. For each of the three modulation types, let $z(t)$ be the corresponding analytic signal of $y(t)$, as defined in Problem 2.14. Find $Z^{\mathrm{F}}(\omega)$ in all three cases.

2.17 Let $x(t)$ be a real signal. Prove that:

(a)

$$x(t) = x(-t) \implies X^{\mathrm{F}}(\omega) \text{ is real,} \tag{2.151}$$

(b)

$$x(t) = -x(-t) \implies X^{\mathrm{F}}(\omega) \text{ is imaginary.} \tag{2.152}$$

2.18 For a real signal $x(t)$ we define

$$x_{\mathrm{e}}(t) = 0.5x(t) + 0.5x(-t), \tag{2.153a}$$

$$x_{\mathrm{o}}(t) = 0.5x(t) - 0.5x(-t). \tag{2.153b}$$

The signals $x_{\mathrm{e}}(t)$, $x_{\mathrm{o}}(t)$ are called the *even* and *odd* parts of $x(t)$, respectively. Prove that

$$X_{\mathrm{e}}^{\mathrm{F}}(\omega) = \Re\{X^{\mathrm{F}}(\omega)\}, \quad X_{\mathrm{o}}^{\mathrm{F}}(\omega) = j\Im\{X^{\mathrm{F}}(\omega)\}. \tag{2.154}$$

2.19 Let $x(t)$ be a real signal satisfying

$$x(t) = 0, \quad t < 0,$$

and $x(t)$ has no delta component at $t = 0$.

(a) Show that

$$x_{\mathrm{o}}(t) = \mathrm{sign}(t)x_{\mathrm{e}}(t).$$

(b) Show that $\Im\{X^{\mathrm{F}}(\omega)\}$ is the Hilbert transform of $\Re\{X^{\mathrm{F}}(\omega)\}$, as defined in Problem 2.13.

2.20 For a signal $x(t)$, let

$$P_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X^{\mathrm{F}}(\omega)|^2 d\omega.$$

The *centroid* of a signal $x(t)$ and that of its Fourier transform are defined as

$$C_x = P_x^{-1}\int_{-\infty}^{\infty} t|x(t)|^2 dt, \tag{2.155a}$$

$$C_X = P_x^{-1}\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega|X^{\mathrm{F}}(\omega)|^2 d\omega. \tag{2.155b}$$

The *effective width* of a signal $x(t)$ and that of its Fourier transform are defined as

$$W_x = \left[P_x^{-1}\int_{-\infty}^{\infty} (t - C_x)^2|x(t)|^2 dt\right]^{1/2}, \tag{2.156a}$$

$$W_X = \left[P_x^{-1}\frac{1}{2\pi}\int_{-\infty}^{\infty} (\omega - C_X)^2|X^{\mathrm{F}}(\omega)|^2 d\omega\right]^{1/2}. \tag{2.156b}$$

(a) Compute the centroid and effective width of the signal

$$x(t) = e^{-t}, \quad t \geq 0.$$

(b) Compute the effective width of the signal $\operatorname{rect}(t)$.

(c) Explain why, for any real signal $x(t)$, the centroid of $X^{\mathrm{F}}(\omega)$ is zero.

2.21 Let $x(t)$ be a signal whose Fourier transform $X^{\mathrm{F}}(\omega)$ exists. Prove that

$$\sum_{n=-\infty}^{\infty} x(t - nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{2\pi k}{T}\right)\exp\left(j\frac{2\pi k t}{T}\right). \tag{2.157}$$

Hint: Derive the Fourier series representation of the left side of (2.157).

2.22 Let

$$x(t) = \begin{cases} e^{-\alpha t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \qquad y(t) = \sum_{n=-\infty}^{\infty} x(t - nT).$$

Find the Fourier series coefficients of the periodic signal $y(t)$.

2.23 Define a *pulse train* signal by

$$r_{T,\Delta}(t) = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \operatorname{rect}\left(\frac{t - nT}{\Delta}\right), \quad \text{where} \quad 0 < \Delta < T. \tag{2.158}$$

Compute the Fourier transform of $r_{T,\Delta}(t)$.

2.24 Define an *alternating impulse train*

$$q_T(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - nT),$$

see Figure 2.11. Compute $Q_T^{\mathrm{F}}(\omega)$, the Fourier transform of $q_T(t)$.

2.25 The impulse train $p_T(t)$ is passed through an LTI filter whose impulse response is

$$h(t) = \operatorname{sinc}\left(\frac{0.5t}{T}\right)\cos\left(\frac{\pi t}{T}\right).$$

Find the output $y(t)$ of the filter. Hint: Solve in the frequency domain.

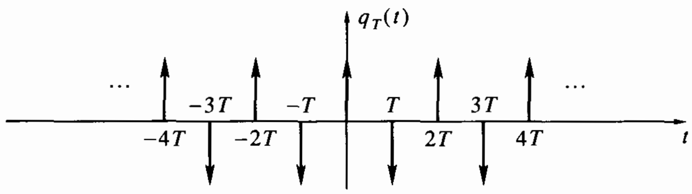

Figure 2.11 Pertaining to Problem 2.24.

2.26 The form of the Poisson sum formula given in (2.48) is

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \exp\left(j\frac{2\pi k t}{T}\right). \tag{2.159}$$

In mathematical texts, the formula is usually stated as

$$\sum_{n=-\infty}^{\infty} x(nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{2\pi k}{T}\right), \tag{2.160}$$

provided $x(t)$ is continuous at the points nT and $X^{\mathrm{F}}(\omega)$ is continuous at the points $2\pi k/T$. Show that (2.159) implies (2.160).

2.27 Compute the Fourier transform of the discrete-time signal

$$x[n] = \begin{cases} C, & n_1 \le n \le n_2, \\ 0, & \text{otherwise,} \end{cases}$$

where n_1, n_2 are integer constants, and C is a real constant.

2.28 The Fourier transform of a discrete-time signal $x[n]$ is

$$X^{\mathrm{f}}(\theta) = \cos\theta + \sin(2\theta).$$

Compute $x[n]$.

2.29 Let $x[n]$ be the signal

$$x[n] = \begin{cases} \{1, -1, -2, 4, -2, -1, 1\}, & -3 \le n \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the following quantities without finding $X^{\mathrm{f}}(\theta)$ first.

(a) $X^{\mathrm{f}}(0)$.

(b) $\measuredangle X^{\mathrm{f}}(\theta)$.

(c) $\int_{-\pi}^{\pi} X^{\mathrm{f}}(\theta)d\theta$.

(d) $X^{\mathrm{f}}(\pi)$.

(e) $\int_{-\pi}^{\pi} |X^{\mathrm{f}}(\theta)|^2 d\theta$.

(f) $\frac{dX^{\mathrm{f}}(\theta)}{d\theta}\big|_{\theta=0}$.

2.30 Can the function

$$X^{\mathrm{f}}(\theta) = \cos(0.5\theta), \quad \theta \in \mathbb{R}$$

be the Fourier transform of a discrete-time signal $x[n]$? If so, find $x[n]$. If not, explain the reason.

2.31 This problem discusses the Fourier transform of a sequence modulated by sign alternations.

(a) If $X^{\mathrm{f}}(\theta)$ is the Fourier transform of $x[n]$, what is the Fourier transform of $(-1)^n x[n]$?

(b) Express $\sum_{n=-\infty}^{\infty}(-1)^n x[n]$ in terms of the Fourier transform of $x[n]$.

2.32 We are given two linear discrete-time systems. The response of the first to a unit impulse $\delta[n-k]$ is $h_1[n] = \sin[3(n-k)]$ and that of the second is $h_2[n] = \sin[3(n+k)]$. Is either system time invariant?

2.33 Prove Theorem 2.5.

2.34 Let $x[n]$ be a discrete-time signal with Fourier transform $X^{\mathrm{f}}(\theta)$, and let

$$Y^{\mathrm{f}}(\theta) = X^{\mathrm{f}}(\theta) + X^{\mathrm{f}}(\theta - \pi).$$

Prove that $Y^{\mathrm{f}}(\theta)$ depends only on the even-indexed signal values $x[2m]$, and is independent of the odd-indexed values $x[2m+1]$.

2.35 Prove that

$$x[n] = \cos(\theta_0 n + \phi_0)$$

is periodic if and only if $\theta_0 = 2\pi p/q$, where p and q are positive integers. Find the period N in case the condition is satisfied.

2.36 The *correlation* (or *cross-correlation*) of two continuous-time signals is defined as

$$z(t) = \{x \star y\}(t) = \int_{-\infty}^{\infty} x(t+\tau)\bar{y}(\tau)d\tau. \tag{2.161}$$

Similarly, the correlation of two discrete-time signals is

$$z[n] = \{x \star y\}[n] = \sum_{m=-\infty}^{\infty} x[n+m]\bar{y}[m]. \tag{2.162}$$

Express the correlation operation in the frequency domain, for both continuous-time and discrete-time signals.

2.37* Prove that (2.65) implies (2.66).

2.38* Suppose we have two discrete-time LTI systems connected in series, so the frequency response of the series connection is

$$H^{\mathrm{f}}(\theta) = H_1^{\mathrm{f}}(\theta)H_2^{\mathrm{f}}(\theta).$$

Recall the definition of noise gain (2.137). Let NG_1, NG_2, NG be the noise gains of the corresponding frequency response functions.

(a) Show that, in general,

$$\mathrm{NG} \neq \mathrm{NG}_1\mathrm{NG}_2.$$

(b) If $H_1^{\mathrm{f}}(\theta) = C$, where C is a real positive constant, what can you say about NG in this special case?

2.39* This problem discusses the effect of an LTI system on white noise in the covariance domain.

(a) White noise $x(t)$, whose covariance function $\kappa_x(\tau)$ is as in (2.76), is given at the input of an LTI system with impulse response $h(t)$. Use (2.87) to compute $\kappa_y(\tau)$, the covariance function of the output, in this special case.

(b) Repeat part a for the discrete-time case. The white noise at the input has covariance sequence $\kappa_x[m]$, as given in (2.127), and the LTI system has impulse response $h[n]$. Find $\kappa_y[m]$.

2.40* A discrete-time white noise $x[n]$ with zero mean and variance γ_x is fed to an ideal low-pass filter whose frequency response is

$$H^{\mathrm{f}}(\theta) = \mathrm{rect}\left(\frac{\theta}{2\theta_{\mathrm{c}}}\right), \quad 0 < \theta_{\mathrm{c}} < \pi.$$

Let $y[n]$ be the response of the filter to $x[n]$. Compute the covariance sequence $\kappa_y[m]$ of $y[n]$.

2.41* Show that the Cauchy-Schwarz inequality for random variables (2.148) follows from the inequality for functions (2.147). Hint: Use (2.142) for the vector random variable consisting of $\{x, y\}$.

2.42* Let $x(t)$ be a continuous-time signal on $[-0.5T, 0.5T]$. The signal has both a Fourier series (2.39) and a Fourier transform (2.1).

(a) Express the Fourier coefficients $X^{\mathrm{S}}[k]$ in terms of the Fourier transform $X^{\mathrm{F}}(\omega)$.

(b) Express the Fourier transform $X^{\mathrm{F}}(\omega)$ in terms of the Fourier coefficients $X^{\mathrm{S}}[k]$.

2.43* This problem illustrates the phenomenon of *harmonic distortions* caused by nonlinear operations on sinusoidal signals, in this case by hard limiting. Let $x(t)$ be the periodic signal

$$x(t) = \begin{cases} \cos(2\pi t), & -A \le \cos(2\pi t) \le A, \\ -A, & \cos(2\pi t) < -A, \\ A, & \cos(2\pi t) > A, \end{cases} \qquad \text{where} \quad 0 < A < 1.$$

(a) Compute the Fourier series coefficients of $x(t)$ as a function of A. Hint: Compute separately for even k, for $k = \pm 1$, and for all other odd k.

(b) Compute the numerical values of the ratios $\{X^{\mathrm{S}}[k]/X^{\mathrm{S}}[1],\ 2 \le k \le 10\}$ for $A = 0.9, 0.5, 0.1, 0.01$.

(c) State your conclusions from these results.

2.44* Let

$$x(t) = \cos(\omega_{\mathrm{m}} t), \quad y(t) = [1 + a x(t)] \sin(\omega_{\mathrm{c}} t),$$

where $0 < a < 1$ and $\omega_{\mathrm{m}} \ll \omega_{\mathrm{c}}$. The operation of constructing $y(t)$ from $x(t)$ is called *amplitude modulation* (AM). In this problem we assume, for simplicity, that $\omega_{\mathrm{c}} = M\omega_{\mathrm{m}}$, where M is an integer much larger than 1.

(a) Show that $y(t)$ is periodic with period $2\pi/\omega_{\mathrm{m}}$.

(b) Find the Fourier series coefficients of $y(t)$.

(c) Define

$$z(t) = \begin{cases} y(t), & y(t) \ge 0, \\ 0, & y(t) < 0. \end{cases}$$

The signal $z(t)$ is called *half-wave rectification* of $y(t)$. Since $a < 1$, we have

$$z(t) = \begin{cases} y(t), & \sin(\omega_c t) \geq 0, \\ 0, & \sin(\omega_c t) < 0. \end{cases}$$

Show that $z(t)$ is periodic with period $2\pi/\omega_m$.

(d) Find the Fourier series coefficients $Z^S[k]$ of $z(t)$ in the range

$$-(M+1) \leq k \leq M+1.$$

(e) Suggest a way to extract the signal $x(t)$ from $z(t)$.

2.45* In this problem we seek to prove the *uncertainty principle*:

> The product of the effective width of a signal and the effective width of its Fourier transform is not smaller than 0.5.

We use the definitions of centroids C_x, C_X and effective widths W_x, W_X, as given in Problem 2.20. We assume, for simplicity, that the signal is real and that the centroids of both the signal and its transforms are zero. However, the proof can be extended to signals not thus restricted.

(a) Prove that, if

$$\lim_{t\to\infty} t|x(t)|^2 = \lim_{t\to-\infty} t|x(t)|^2 = 0,$$

then

$$-\int_{-\infty}^{\infty} t\frac{d|x(t)|^2}{dt}dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = P_x.$$

Hint: Use integration by parts.

(b) Use Problem 2.11 to express W_X in terms of $dx(t)/dt$.

(c) Use the Cauchy-Schwarz inequality (2.147) to find a lower bound on $W_x W_X$.

(d) Finally, use part b to show that

$$W_x W_X \geq 0.5. \tag{2.163}$$

The uncertainty principle is of fundamental importance in Fourier theory. It shows that a signal cannot be arbitrarily narrow in both time and frequency. The celebrated *Heisenberg uncertainty principle* in quantum mechanics is a consequence of (2.163) as well.

2.46* Let $x[n]$ be a discrete-time signal that is identically zero for $n \geq N$ and $n < 0$, for some integer N. Prove that the set

$$\{\theta: -\pi \leq \theta < \pi,\ X^f(\theta) = 0\}$$

contains a finite number of points at most. Hint: A polynomial equation has only a finite number of solutions.

2.47* Consider the claim that for two sequences $x[n]$, $y[n]$, $\{x * y\} = 0$ implies that at least one of the two sequences is identically zero. Is this claim true? If so, prove it; if not, give a counterexample.

2.48* Repeat Problem 2.47 if the two sequences $x[n]$, $y[n]$ have finite durations.

2.49* This problem illustrates an application of Fourier transform for evaluation of infinite sums.

(a) Evaluate the infinite sum

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

Hint: Find a sequence $x[n]$ and its Fourier transform $X^{\mathrm{f}}(\theta)$ such that Parseval's theorem (2.105) will yield the required result.

(b) Use part a to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$