# An Alternative to the Hamming Code in the Class of SEC–DED Codes in Semiconductor Memory

Alexander A. Davydov and Leonid M. Tombak

**Abstract—The Π code constructed by Panchenko is studied. The Π code to be an alternative to the Hamming code in the class of single-error correcting and double-error detecting codes (SEC–DED codes) is also considered. The Π code has a smaller number of words of weight 4 and provides a larger probability of the triple-independent-error detection than the shortened Hamming code with the same parameters. In this work shortening algorithms for the Π code are proposed, and parity check matrices of the [39,32], [72,64], [137,128] shortened Π codes are constructed. The codes obtained can detect byte errors of length 4.**

## I. Introduction

Every word of a semiconductor memory is usually encoded by an error correcting code [20]. Errors appearing in the memory are classified to be either independent errors or byte errors [1]–[8], [13]–[18]. In this correspondence the following strategies for memory protection [1]–[10], [13]–[18], [21, p. 177–182] are considered.

1) A linear code of length $n$, minimum distance $d=4$ and redundancy $r=\lceil \log_2 n \rceil+1$ is used. All single errors are corrected and all double errors and some triple errors are detected.
2) In addition to Strategy 1 the same code detects all byte errors of length 4.

Let $\Delta_3$ be the ratio of the number of triple independent errors that may be detected by a code to the total number of triple errors. Denote by $A_4$ the number of words of weight 4 in a code. For Strategies 1 and 2, the memory reliability substantially depends on the value of $\Delta_3$. It is known that $\Delta_3 = 1-4A_4/\binom{n}{3}$ [13]. Hence, it is useful to decrease the value of $A_4$.

It should be noted that the maximum number of 1's in rows of a parity check matrix and regular structure of the matrix are also important for memory protection systems [4], [6], [7], [13]–[18], [21].

Manuscript received May 10, 1990; revised January 16, 1991. This work was presented in part at the II All Union Conference on Actual Problems of Informatics and Computers (INFORMATICS-87), Erevan, U.S.S.R., October 1987, and at the VIII Conference of International COMPCONTROL Committee on Computer Applications in Production Management and Engineering (COMPCONTROL-87), Moscow, U.S.S.R., October 1987.

The authors are with the Institute for Problems of Cybernetics of Academy of Sciences of the U.S.S.R., Vavilov Street 37, 117312 Moscow, U.S.S.R.

IEEE Log Number 9143294.

The Hamming code is the most well known of codes with $d=4$. The problem of $A_4$ minimization for the shortened Hamming code was considered in [3], [4], [9], [11], [13], [14], and [21]. (Throughout this correspondence the word "shortened" may be omitted in a code name.) Let $a_4^H(n,r)$ be the minimum of $A_4$ over all $[n, n-r]$ Hamming codes. In [9] evaluations of $a_4^H(n,r)$ were obtained.

There are linear codes with $d=4$ that are not equivalent to the Hamming code [10], [12], [19]. Let $a_4^L(n,r)$ be the minimum of $A_4$ over all linear $[n,n-r]$ codes with $d=4$. In [11], [19] evaluations of $a_4^L(n,r)$ were obtained. In [19] Panchenko constructed the Π code that is not equivalent to the Hamming code and has $A_4 < a_4^H(n,r)$.

Let $N_r = 17\cdot 2^{r-6}$. In [12] it is proved that there exist only three nonequivalent quasiperfect binary linear codes with $d=4$, $n>N_r$: the Hamming code with $n=2^{r-1}$, the Π code with $n=5\cdot 2^{r-4}$, and the Ω code with $n=9\cdot 2^{r-5}$. Any binary linear code with $d=4$, $n>N_r$, is a shortening of one of these codes.

Let $B_k=[b_k,\cdots,b_k]$ be a matrix consisting of equal columns $b_k$, where $b_k$ is the binary representation of $k$. Let $D=2^{r-4}$, $M=2^{r-5}$,

$$G=\begin{bmatrix}10001\\01001\\00101\\00011\end{bmatrix},\qquad Q=\begin{bmatrix}00000 & 1111\\10001 & 0000\\01001 & 1001\\00101 & 0101\\00011 & 0011\end{bmatrix}. \tag{1}$$

The parity check matrix $P_r$ of the nonshortened $[n,n-r]$ Π code with $n=5\cdot 2^{r-4}$, $r\geq 5$, has the following form:

$$P_r=\left[\begin{array}{c|c|c|c|c} B_0 & B_1 & B_2 & \cdots & B_{D-1}\\ \hline G & G & G & \cdots & G\end{array}\right], \tag{2}$$

where $B_k$ is a $(r-4)\times 5$ matrix.

For example, the parity check matrix of the [40,33] Π code is

$$P_7=\left[\begin{array}{c|c|c|c|c|c|c|c} 00000 & 00000 & 00000 & 00000 & 11111 & 11111 & 11111 & 11111\\ 00000 & 00000 & 11111 & 11111 & 00000 & 00000 & 11111 & 11111\\ 00000 & 11111 & 00000 & 11111 & 00000 & 11111 & 00000 & 11111\\ \hline 10001 & 10001 & 10001 & 10001 & 10001 & 10001 & 10001 & 10001\\ 01001 & 01001 & 01001 & 01001 & 01001 & 01001 & 01001 & 01001\\ 00101 & 00101 & 00101 & 00101 & 00101 & 00101 & 00101 & 00101\\ 00011 & 00011 & 00011 & 00011 & 00011 & 00011 & 00011 & 00011 \end{array}\right]. \tag{3}$$

The parity check matrix $Q_r$ [12] of the nonshortened $[n,n-r]$ Ω code with $n=9\cdot 2^{r-5}$ has the following form:

$$Q_r=\left[\begin{array}{c|c|c|c|c} B_0 & B_1 & B_2 & \cdots & B_{M-1}\\ \hline Q & Q & Q & \cdots & Q\end{array}\right], \tag{4}$$

where $B_k$ is a $(r-5)\times 9$ matrix.

Codes with $n=2^{r-2}+r$, $r\leq 9$, used in a memory, have $n>N_r$.

In Section II we construct two shortening algorithms for the Π code. We consider the following ranges of code length $n$:

$$\max\{5\cdot 2^{r-4}-8, 9\cdot 2^{r-5}-1, 17\cdot 2^{r-6}+1\}\leq n\leq 5\cdot 2^{r-4}. \tag{5}$$

$$\max\{5\cdot 2^{r-4}-25, 17\cdot 2^{r-6}+1\}\leq n\leq 5\cdot 2^{r-4}. \tag{6}$$

The range (5) includes [39,32] and [72,64] codes. In the range (5) the first algorithm gives the global minimum of $A_4$, i.e., $A_4=a_4^L(n,r)$. The range (6) includes [137,128] codes. In the

0018-9448/91/0500-0897$01.00 

TABLE I
VALUES OF $A_4$ FOR THE $\Pi$ CODES SHORTENED BY ALGORITHM 1 AND LOWER BOUNDS OF $A_4$ FOR THE SHORTENED HAMMING CODES

| $i$ | $n$ $(r=7)$ | $\Pi$ Code $A_4=$ | Hamming Code $A_4\geq$ | $n$ $(r=8)$ | $\Pi$ Code $A_4=$ | Hamming Code $A_4\geq$ |
|---|---|---|---|---|---|---|
| 0 | 40 | 1190 | 1480 | 80 | 10300 | 12578 |
| 1 | 39 | 1071 | 1332 | 79 | 9785 | 11944 |
| 2 | 38 | 959 | 1191 | 78 | 9285 | 11335 |
| 3 | 37 | 854 | 1063 | 77 | 8800 | 10748 |
| 4 | 36 | 756 | 945 | 76 | 8330 | 10185 |
| 5 | 35 | 665 | 838 | 75 | 7875 | 9644 |
| 6 | | | | 74 | 7455 | 9124 |
| 7 | | | | 73 | 7048 | 8626 |
| 8 | | | | 72 | 6654 | 8157 |

range (6) the second algorithm provides smaller values of $A_4$ in comparison with the Hamming code and the $\Omega$ code. However, this algorithm does not give the best shortened $\Pi$ code. In Section II we construct the parity check matrices of the [39,32], [72,64], and [137,128] shortened $\Pi$ codes for strategy I.

In Section III we construct the parity check matrices of the [72,64] and [137,128] shortened $\Pi$ codes for the strategy II.

Structures of the obtained matrices are regular. Therefore, these matrices are suitable for VLSI implementation.

All obtained parity check matrices of the $\Pi$ code have a larger value of $\Delta_3$ than corresponding matrices of the Hamming code and the $\Omega$ code. Therefore, the $\Pi$ code for strategies I, II provides the best reliability of the memory in the class of linear codes with $n=2^{r-2}+r$.

Some results of this work are introduced (without proofs) in [10].

## II. THE SHORTENED $\Pi$ CODES WITH THE BEST DETECTING CAPABILITY IN THE CLASS OF BINARY LINEAR CODE WITH $d=4$

In order to count $A_4$ we represent a nonzero column $s_t$, which does not belong to the parity check matrix of an $[n,n-r]$ code ("external" column), as the sum of two columns $h_{t,j}$ and $h_{t,j+1}$, which belong to the matrix [14], [19]. Denote by $m(t)$ the number of various representations of the column $s_t$. Then the following relations hold:

$$s_t=h_{t,1}+h_{t,2}=h_{t,3}+h_{t,4}=\cdots=h_{t,2m(t)-1}+h_{t,2m(t)}, \tag{7}$$

where

$$t=\overline{1,\cdots,2^r-1-n},\qquad m(t)\in\overline{\{0,\cdots,\lfloor n/2\rfloor\}}.$$

$$A_4=\frac{1}{3}\sum_{t=1}^{2^r-1-n}\binom{m(t)}{2}=\frac{1}{3}\sum_{j=2}^{\lfloor n/2\rfloor}\binom{j}{2}F_j, \tag{8}$$

where $F_j$ is the number of external columns $s_t$, for which $m(t)=j$.

The main idea of proposed algorithms is to decrease $F_{\lfloor n/2\rfloor}$.

*Algorithm 1:* We shorten the matrix $P_r$ by $i$ columns, $i\leq 8$. We delete columns of $P_r$ in the following order:

$$\begin{bmatrix} b_\gamma \\ \hline g_{15}\end{bmatrix},\begin{bmatrix} b_\gamma \\ \hline g_{8}\end{bmatrix},\begin{bmatrix} b_\gamma \\ \hline g_{4}\end{bmatrix},\begin{bmatrix} b_\gamma \\ \hline g_{2}\end{bmatrix},\begin{bmatrix} b_\gamma \\ \hline g_{1}\end{bmatrix},\begin{bmatrix} b_\delta \\ \hline g_{15}\end{bmatrix},\begin{bmatrix} b_\nu \\ \hline g_{8}\end{bmatrix},\begin{bmatrix} b_{\mathscr{K}} \\ \hline g_{4}\end{bmatrix}, \tag{9}$$

where $g_v$ is a column of matrix $G$, corresponding to the binary representation of $v$, and columns $b_\gamma, b_\delta, b_\nu, b_{\mathscr{K}}$ are distinct.

Throughout this correspondence the expression $j=\{a,b\}$, $F_j=\{c,d\}$ means that $F_a=c$, $F_b=d$. Let $E=5\cdot 2^{r-5}$.

*Theorem 1:* For $r\times n$ matrices with $n=5\cdot 2^{r-4}-i$ obtained by Algorithm 1 the nonzero values $F_j$ can be represented as follows:

$$\begin{aligned}
&i=0: j=\{D,E\}, && F_j=\{10D,D-1\}.\\
&i=1: j=\{D-1,D,E-1\}, && F_j=\{4D,6D,D-1\}.\\
&i=2: j=\{D-2,D-1,D,E-2\}, && F_j=\{D-1,6D+1,3D,D-1\}.\\
&i=3: j=\{D-2,D-1,D,E-3\}, && F_j=\{3D-3,6D+3,D,D-1\}.\\
&i=4: j=\{D-2,D-1,E-4\}, && F_j=\{6D-6,4D+6,D-1\}.\\
&i=5: j=\{D-2,D-1,E-5\}, && F_j=\{10D-10,10,D-1\}.\\
&i=6: j=\{D-3,D-2,D-1,E-6,E-5\}, && F_j=\{4D-8,6D+2,6,D-2,1\}.\\
&i=7: j=\{D-4,D-3,D-2,D-1,E-7,E-6\}, && F_j=\{D-4,6D-8,3D+9,3,D-3,2\}.\\
&i=8: j=\{D-4,D-3,D-2,D-1,E-8,E-7\}, && F_j=\{3D-12,6D,D+11,1,D-4,3\}.
\end{aligned} \tag{10}$$

*Proof:* See the Appendix. □

Denote by $\Delta_3^L(n,r)$ and $\Delta_3^H(n,r)$ the maximum of $\Delta_3$ over all linear $[n,n-r]$ codes with $d=4$ and over all $[n,n-r]$ Hamming codes respectively.

*Theorem 2:* In the range (5) the $[n,n-r]$ $\Pi$ code obtained by Algorithm 1 has the minimum number of words of weight 4 and the maximum probability of triple-independent-error detection over all linear $[n,n-r]$ codes with $d=4$, i.e., this $\Pi$ code has $A_4=a_4^L(n,r)$ and $\Delta_3=\Delta_3^L(n,r)$.

*Proof:* See the Appendix. □

Using the relations (8), (10) and results of [9] we obtain Table I.

In order to shorten the matrices $P_7$ and $P_8$ we use Algorithm 1. Let $r=7$, $i=1$, and $\gamma=7$. Then the parity check matrix $H_{39}$ of the [39,32] $\Pi$ code is the matrix $P_7$ with the last column omitted. Take $r=8$, $i=8$, $\gamma=15$, $\delta=14$, $\nu=13$, and $\mathscr{H}=12$. The parity check matrix $H_{72}$ of the [72, 64] $\Pi$ code has the following form:

$$H_{72}=\left[\begin{array}{c|c|c|c|c|c|c|c}
00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000\\
00000 & 00000 & 00000 & 00000 & 11111 & 11111 & 11111 & 11111\\
00000 & 00000 & 11111 & 11111 & 00000 & 00000 & 11111 & 11111\\
00000 & 11111 & 00000 & 11111 & 00000 & 11111 & 00000 & 11111\\
\hline
10001 & 10001 & 10001 & 10001 & 10001 & 10001 & 10001 & 10001\\
01001 & 01001 & 01001 & 01001 & 01001 & 01001 & 01001 & 01001\\
00101 & 00101 & 00101 & 00101 & 00101 & 00101 & 00101 & 00101\\
00011 & 00011 & 00011 & 00011 & 00011 & 00011 & 00011 & 00011
\end{array}\right.$$

$$\left.\begin{array}{|c|c|c|c|c|c|c}
11111 & 11111 & 11111 & 11111 & 11\;11 & 111\;1 & 1111\\
00000 & 00000 & 00000 & 00000 & 11\;11 & 111\;1 & 1111\\
00000 & 00000 & 11111 & 11111 & 00\;00 & 000\;0 & 1111\\
00000 & 11111 & 00000 & 11111 & 00\;00 & 111\;1 & 0000\\
\hline
10001 & 10001 & 10001 & 10001 & 10\;01 & 100\;1 & 1000\\
01001 & 01001 & 01001 & 01001 & 01\;01 & 010\;1 & 0100\\
00101 & 00101 & 00101 & 00101 & 00\;01 & 001\;1 & 0010\\
00011 & 00011 & 00011 & 00011 & 00\;11 & 000\;1 & 0001
\end{array}\right]. \tag{11}$$

For $H_{39}$:

$$A_4=a_4^L(39,7)=1071, \qquad \Delta_3=\Delta_3^L(39,7)=0.5312.$$

For the Hamming code [9]:

$$a_4^H(39,7)\ge 1332, \qquad \Delta_3^H(39,7)\le 0.4170.$$

For $H_{72}$:

$$A_4=a_4^L(72,8)=6654, \qquad \Delta_3=\Delta_3^L(72,8)=0.5537.$$

For the Hamming code [9]:

$$a_4^H(72,8)\ge 8157, \qquad \Delta_3^H(72,8)\le 0.4529.$$

Denote by $\Gamma$ the maximum number of 1's in rows of a parity check matrix. Matrices $H_{39}$ and $H_{72}$ have $\Gamma=19$ and $\Gamma=34$, respectively. The parity check matrices of the [39, 32] and [72, 64] Hamming codes constructed in [13], [14] have $\Gamma=15$ and $\Gamma=27$, $A_4=8392$, $\Delta_3=0.4361$.

*Algorithm 2:* We shorten matrix $P_r$ by $i$ columns, where $i\le 25$. We delete submatrices $\left[\begin{smallmatrix}B_u\\ G\end{smallmatrix}\right]$, where $u=k_\nu$, $\nu=1,\cdots,$ $q=\lfloor i/5\rfloor$, any three and four columns of the set $\{b_{k_1},b_{k_2},\cdots,b_{k_q}\}$ are linearly independent. If $i\ne 5q$, then one submatrix is deleted incompletely.

For $r\times n$ matrices with $n=5\cdot 2^{r-4}-5q$ obtained by Algorithm 2 the nonzero values of $F_j$ can be represented as follows:

$$\begin{aligned} j&=\{D-2q, D-2q+2, D-q, E-5q, E-5q+5\},\\ F_j&=\{10D-10-5q(q-1), 5q(q-1), 10,\\ &\qquad D-1-q(q-1)/2, q(q-1)/2\}. \end{aligned} \tag{12}$$

This relation can be proved similar to Theorem 1.

*Theorem 3:* In the range (6) the $[n,n-r]$ $\Pi$ code obtained by Algorithm 2 has a smaller value of $A_4$ than any $[n,n-r]$ codes with $d=4$ obtained by shortening of linear codes nonequivalent to the $\Pi$ code.

*Proof:* See the Appendix. □

Let $r=9$, $i=23$, $q=5$, $k_1=31$, $k_2=30$, $k_3=29$, $k_4=27$, and $k_5=23$. The parity check matrix $H_{137}$ of the [137, 128] code obtained from $P_9$ by Algorithm 2 has the following form:

$$H_{137}=\left[\begin{array}{c|c|c|c|c|c|c|c|c} B_0 & B_1 & \cdots & B_{22} & \hat{B}_{23} & B_{24} & B_{25} & B_{26} & B_{28}\\ \hline G & G & \cdots & G & \hat{G} & G & G & G & G \end{array}\right], \tag{13}$$

where $\hat{B}_{23}=[b_{23}b_{23}]$ is $5\times 2$ matrix; $\hat{G}=[g_1g_2]$ is $4\times 2$ matrix.

For $H_{137}$: $A_4=45488$, $\Delta_3=0.5660$, $\Gamma=62$. (Note that we construct the parity check matrix of the [137, 128] $\Pi$ code with $A_4=45443$, but this matrix does not have a regular structure).

For the Hamming code:

$$a_4^H(137,9)\ge 15182,\ \Delta_3^H(137,9)\le 0.4735,\ \Gamma\ge 54,$$

[4], [9]. [13]. In [4] the parity check matrix of the [137, 128] Hamming code with $A_4=56252$, $\Delta_3=0.4733$, $\Gamma=55$ is described.

Algorithm 2 gives matrices with more regular structure than Algorithm 1. For example, the parity check matrix $H_{72}^*$ of the [72, 64] $\Pi$ code obtained by Algorithm 2 is more regular than $H_{72}$. For $H_{72}^*$: $A_4=6657$, $\Gamma=35$.

## III. Parity Check Matrices of the $\Pi$ Codes Detecting Byte Errors of Length 4

The parity check matrix $H_{72}^4$ of the [72, 64] $\Pi$ code detecting byte errors of length 4 has the following form:

$$H_{72}^4=\left[\begin{array}{c|c|c|c|c|c|c|c|c}
0000 & 0000 & 0000 & & 1111 & 1111 & 1111 & 0001 & 0001\\
0000 & 0000 & 0000 & \cdots & 1111 & 1111 & 1111 & 0010 & 0110\\
0000 & 0000 & 1111 & & 0000 & 1111 & 1111 & 0100 & 1011\\
0000 & 1111 & 0000 & & 1111 & 0000 & 1111 & 1000 & 1100\\
\hline
1000 & 1000 & 1000 & & 1000 & 1000 & 1000 & 1111 & 1111\\
0100 & 0100 & 0100 & \cdots & 0100 & 0100 & 0100 & 1111 & 1111\\
0010 & 0010 & 0010 & & 0010 & 0010 & 0010 & 1111 & 1111\\
0001 & 0001 & 0001 & & 0001 & 0001 & 0001 & 1111 & 1111
\end{array}\right]. \tag{14}$$

The syndromes of byte errors of length 4 are not identical with any column of the matrix $H_{72}^4$. For $H_{72}^4$: $A_4=7221$, $\Delta_3=0.5156$, $\Gamma=36$.

Corresponding matrices for the Hamming code have $A_4=8408$, $\Delta_3=0.4363$, $\Gamma=27$ [4] and $A_4=8200$, $\Delta_3=0.45$, $\Gamma=31$ [15].

The parity check matrix $H_{137}^4$ of the [137,128] Π code detecting the byte errors of length 4 has the following form:

$$H_{137}^4=\left[\begin{array}{c|c} 0\cdots 0 & 1\cdots 1 \\ \hline H_{72}^4 & H_{65}^4 \end{array}\right], \tag{15}$$

where $H_{65}^4$ is the matrix $H_{72}^4$ with the last 7 columns omitted. For $H_{137}^4$: $A_4=54885$, $\Delta_3=0.4763$, $\Gamma=68$. For the Hamming code the corresponding matrix $H^{(10)}$ in [4] has $A_4=57339$, $\Delta_3=0.4529$, $\Gamma=65$.

## Conclusion

The Π code proposed by Panchenko belongs to the class of SEC–DED codes. In this correspondence we construct the parity check matrices of the [39,32], [72,64], and [137,128] shortened Π codes. The obtained matrices provide a smaller number of words of weight 4 and a larger probability of triple-independent-error detection as compared with the Hamming codes.

The constructed [39,32] and [72,64] shortened Π codes have the minimum number of words of weight 4 in the class of all linear codes with the same parameters.

On the other hand, the parity check matrices of the Π code have more 1's in rows than corresponding matrices of the Hamming code.

The Π code is a reasonable alternative to the Hamming code in the class of SEC–DED codes.

## Acknowledgment

The authors thank the referees for their helpful suggestions.

## Appendix

*Notations and Definitions.*

The set of numbers $F_j$ (see (8)) is called an *F spectrum*. If external columns are partitioned into noncross groups, then $F_j^{(v)}$ denotes the number of columns belonging to $v$th group, for which $m(t)=j$. The set of numbers $F_j^{(v)}$ is called a *partial F spectrum*. Obviously,

$$F_j=\sum_v F_j^{(v)}. \tag{A.1}$$

Denote by $G_i=[g_i\ g_i \cdots g_i]$ a matrix consisting of equal columns $g_i$, where $g_i$ is the binary 4-bit representation of $i$. Let $B^*=[b_0 b_1 \cdots b_{D-1}]$ be a $(r-4)\times 2^{r-4}$ matrix consisting of all distinct columns $b_j$ of length $r-4$, where $b_j$ is the binary representation of $j$. Denote by $B^*(d_1,\cdots,d_k)$ the matrix $B^*$ with columns $b_{d_1},\cdots,b_{d_k}$ omitted. Let

$$A_0=\left[\frac{B^*(0)}{G_0}\right],\qquad A_i=\left[\frac{B^*}{G_i}\right],$$

$$A_i(d_1,\cdots,d_k)=\left[\frac{B^*(d_1,\cdots,d_k)}{G_i}\right],\qquad i=1,\cdots,15.$$

The matrix $P_r$ can be represented as follows:

$$P_r=[A_1A_2A_4A_8A_{15}].$$

Let $Y=\{1,2,4,8,15\}$, $\quad U=\{3,5,6,7,9,10,11,12,13,14\}$.

*Definition:* Let

$$a=\left[\frac{b_{i_a}}{g_k}\right]\in A_k,\ k\in\{0,\cdots,15\}.$$

Then $b_{i_a}$ is called the locator of column $a$, and $g_k$ is called the indicator of column $a$.

*Proof of Theorem 1:* We consider the case $i=8$. For $i\neq 8$ the proof is analogous.

For $i=8$ the Π code shortened by Algorithm 1 has the following parity check matrix:

$$H_{5D-8}=[A_1(\gamma)A_2(\gamma)A_4(\gamma,\mathscr{H})A_8(\gamma,\nu)A_{15}(\gamma,\delta)].$$

The deleted columns of $P_r$ are external for a shortened matrix, but these columns cannot be represented in the form (7). Hence, the $F$ spectrum does not depend on the deleted columns. All sums of the form $g_i+g_j$ for $i,j\in Y$, $i\neq j$, are distinct:

$$\begin{aligned}&g_1+g_2=g_3;\ g_4+g_8=g_{12},\ g_4+g_{15}=g_{11},\ g_8+g_{15}=g_7;\\ &g_1+g_4=g_5,\ g_1+g_8=g_9,\ g_1+g_{15}=g_{14},\\ &g_2+g_4=g_6,\ g_2+g_8=g_{10},\ g_2+g_{15}=g_{13}.\end{aligned} \tag{A.2}$$

For any external column $a\in A_k$, $k\neq 0$, in every sum of the relation (7) one summand belongs to a matrix $A_i$ and the other summand belongs to a matrix $A_s$, where $i,s\in Y$, $g_i+g_s=g_k$. For $a\in A_0$ both summands belong to a matrix $A_i$, $i\in Y$. For the matrix $H_{5D-8}$ we partition external columns into 4 groups: 1) the matrix $A_0$, 2) the matrix $A_3$, 3) the matrices $A_5,A_6,A_9,A_{10},A_{13},A_{14}$, 4) the matrices $A_7,A_{11},A_{12}$. Partial $F$ spectrums of external columns belonging to different matrices of one group are equal (see (A.2)). The partial $F$ spectrums of the groups of columns are as follows:

$$\begin{aligned}&j=\{E-8,E-7\},\ F_j^{(1)}=\{D-4,3\};\\ &j=\{D-2,D-1\},\ F_j^{(2)}=\{D-1,1\};\\ &j=\{D-4,D-3\},\ F_j^{(3)}=\{3D-12,12\};\\ &j=\{D-3,D-2\},\ F_j^{(4)}=\{6D-12,12\}.\end{aligned} \tag{A.3}$$

For example, we calculate $F_j^{(3)}$. Columns $b_\gamma,b_\nu,b_{\mathscr{H}}$ are distinct, so columns of $A_{12}$ with locators $b_\gamma+b_\gamma=b_0, b_\gamma+b_\nu, b_\gamma+b_{\mathscr{H}}, b_\nu+b_{\mathscr{H}}$ can be obtained by $D-3$ ways as a sum of columns belonging to $A_4(\gamma,\mathscr{H})$ and $A_8(\gamma,\nu)$. Everyone of the other $D-4$ columns of $A_{12}$ is obtained in $D-4$ ways. We may consider $A_{11}$ and $A_7$ in the same way. Therefore, in the third group of external columns $m(t)=D-3$ for $3\cdot 4=12$ columns and $m(t)=D-4$ for $3(D-4)=3D-12$ columns.

So it is important that columns $b_\gamma,b_\delta,b_\nu,b_{\mathscr{H}}$ are distinct but the $F$ spectrum does not depend on concrete values of $\gamma,\delta,\nu,\mathscr{H}$.

The proof of necessity can be obtained from (A.1) and (A.3). □

*Proof of Theorem 2:* According to [12], in the range (5) the following linear binary codes with $d=4$ exist: the Π code and its shortenings, the shortened Hamming codes, the Ω code and its shortenings.

We show that the shortened Π code obtained by Algorithm 1 is the best code over all shortened Π codes. For $n-1$ external columns the number of representations $m(t)$ is reduced by one if the code of length $n$ is shortened by one symbol. According to (8), in order to minimize the value of $A_4$ in a shortened code we should reduce the maximal values of $m(t)$. For the nonshortened $\Pi$ code external columns can be partitioned into 2 groups: 1) the matrix $A_0$, 2) the matrices $A_k$, $k \in U$. The partial $F$ spectrums of these groups of columns are as follows:

$$j = \{E\},\ F_j^{(1)} = \{D-1\};\ \mathscr{H} = \{D\},\ F_{\mathscr{H}}^{(2)} = \{10D\}. \quad \text{(A.4)}$$

Therefore, in the range (5) for any shortening of the $\Pi$ code by $i \le 8$ symbols we have $j > \mathscr{H}$.

Denote by $\Pi^f$ a shortened $\Pi$ code. For the code $\Pi^f$ we introduce the following notations. Let $X^f$ be the set of deleted columns of $P_r$ and let $X_i^f$ be the set of deleted columns with the indicator $g_i$, $i \in Y$. Obviously,

$$X^f = \bigcup_{i \in Y} X_i^f.$$

Let the column

$$\left[\frac{b_{k_i}}{g_i}\right]$$

be a representative of the set $X_i^f$. Denote by $X_*^f$ a set containing one representative of every $X_i^f \neq \varnothing$, $i \in Y$. Let $|X|$ be the cardinality of a set $X$. Evidently, $X_*^f \subseteq X^f$, $|X_*^f| \le 5$. Let the set of numbers $F_k^{(v,f)}$ be the partial $F$ spectrum of columns belonging to the matrix $A_v$, $v \in U$. Denote by $F^{(v,f)}$ this partial $F$ spectrum.

*Lemma 1:* For any $X^f, X_*^f$ there exists on equivalent the matrix $P_r$ for which all columns of $X_*^f$ have equal locators.

*Proof:* We give the algorithm of this equivalent transformation. Assume that $X_i^f \neq \varnothing$, $i \in Y$, i.e., $|X_*^f| = 5$. Let $u(j) = 2^{j-1}$.

1) For $j = 2,3,4$ we sum the $(r+1-j)$th row of $P_r$ with the rows in which $b_{k_{u(j)}}$ differs from $b_{k_1}$. As a result four columns of $X_*^f$ have locator $b_{k_1}$ and the fifth one has a locator $b_{k_{15}}^*$.
2) We add the sum of the 4 lower rows of $P_r$ to the rows in which $b_{k_1}$ differs from $b_{k_{15}}^*$. Now all columns of $X_*^f$ have locator $b_{k_{15}}^*$. □

Let

$$X_{\max}^f = \max_{i \in Y} |X_i^f|,\ X_{\min}^f = \min_{i \in Y} |X_i^f|.$$

A set $X^f$ is called nonoptimal if $X_{\max}^f - X_{\min}^f \ge 2$. Denote by $A_4(\Pi^f)$ the number of words of weight 4 in $\Pi^f$.

*Lemma 2:* Let $\Pi^1$ be a code such that the set $X^1$ is nonoptimal and $|X^1| \le 8$. Then there is a code $\Pi^2$ with $|X^2| = |X^1|$, $A_4(\Pi^2) < A_4(\Pi^1)$.

*Proof:* Let $X_{\min}^1 = |X_1^1|$, $X_{\max}^1 = |X_2^1|$.

Since $X^1$ is a nonoptimal set and $|X^1| \le 8$, we have $X_{\min}^1 = |X_1^1| \le 1$. So we should consider two cases: $X_{\min}^1 = 0$ and $X_{\min}^1 = 1$. We consider the first case. (The second case can be studied in much the same way.)

For $i = 2,4,8,15$ we assume that $|X_i^1| > 0$. Let

$$X_i^1 = \left\{\frac{b_{k_i}}{g_i}, \cdots, \frac{b_{r_i}}{g_i}\right\}, \qquad i \in \{2,4,8,15\}.$$

By Lemma 1, $b_{k_2} = b_{k_4} = b_{k_8} = b_{k_{15}}$. Let $\Pi^2$ be the code with

$$X^2 = X^1 \cup \left\{\frac{b_{r_2}}{g_1}\right\} \Big\backslash \left\{\frac{b_{r_2}}{g_2}\right\}. \quad \text{(A.5)}$$

From (A.5), it follows that

$$|X^1| = |X^2|,\ X_i^2 = X_i^1, \qquad i = 4,8,15,$$

$$X_2^2 = X_2^1 \Big\backslash \left\{\frac{b_{r_2}}{g_2}\right\}, \qquad X_1^2 = X_1^1 \cup \left\{\frac{b_{r_2}}{g_1}\right\} = \left\{\frac{b_{r_2}}{g_1}\right\}.$$

Constructions of codes $\Pi^1, \Pi^2$ result in

$$F^{(7,1)} = F^{(7,2)},\ F^{(11,1)} = F^{(11,2)},\ F^{(12,1)} = F^{(12,2)}. \quad \text{(A.6)}$$

According to (8), (A.1), it holds that

$$A_4(\Pi^f) = \frac{1}{3} \sum_{v \in U} \sum_{k=2}^{[n/2]} \binom{k}{2} F_k^{(v,f)}. \quad \text{(A.7)}$$

Let

$$\Phi_v \triangleq \sum_{k=2}^{[n/2]} \binom{k}{2} \left(F_k^{(v,1)} - F_k^{(v,2)}\right).$$

From (A.6), it follows that $\Phi_v = 0$ for $v = 7,11,12$. Now we can compare the partial $F^{(0,f)}$ spectrums. For this comparison we use the code $\Pi^3$ with

$$X^3 = X^1 \Big\backslash \left\{\frac{b_{r_2}}{g_2}\right\} = X^2 \Big\backslash \left\{\frac{b_{r_2}}{g_1}\right\}.$$

For the code $\Pi^1$, we see there are (resp. $\Pi^2$) $2^{r-4} - |X_1^2|$ (resp. $2^{r-4} - 1$) columns of $A_0$ for which the value of $m(t)$ is decreased by one with respect to the code $\Pi^3$.

Let $C = |X_2^1| - 1$, $N_j = 2^{r-4} - |X_j^1|$. For a code $\Pi^f$ denote by $k(v,f,u)$ the number of representations in (7) of the $u$th column belonging to $A_v$. From the constructions of codes $\Pi^1, \Pi^2$ it follows that $|k(v,1,u) - k(v,2,u)| \le 1$. Consequently,

$$\begin{aligned}\Phi_0 &= \sum_{u=1}^{C} \left( \binom{k(0,1,u)}{2} - \binom{k(0,1,u)-1}{2} \right) \\ &= \sum_{u=1}^{C} (k(0,1,u)-1). \end{aligned} \quad \text{(A.8)}$$

Reasoning along similar lines, we obtained (A.9)–(A.12).

$$\Phi_3 = \sum_{u=1}^{C} (k(3,1,u)-1). \quad \text{(A.9)}$$

$$\Phi_5 + \Phi_6 = \sum_{u=1}^{N_4} (k(5,1,u) - k(6,1,u)) > 0. \quad \text{(A.10)}$$

The last formula follows from relations $|X_2^1| > |X_1^1| + 1 = |X_1^2|$, $|X_4^1| = |X_4^2|$. From these statements we can obtain that $k(5,1,u) > k(6,1,u)$ for any $u$. In addition, the columns

$$\left[\frac{b_{r_2}}{g_2}\right], \qquad \left[\frac{b_{r_2}}{g_1}\right]$$

have equal locators. In much the same way we have

$$\Phi_9 = \Phi_{10} = \sum_{u=1}^{N_8} (k(9,1,u) - k(10,1,u)) > 0, \quad (A.11)$$

$$\Phi_{13} + \Phi_{14} = \sum_{u=1}^{N_{15}} (k(14,1,u) - k(13,1,u)) > 0. \quad (A.12)$$

Let $\pi > 0$. From (A.7)–(A.12), it follows that

$$A_4(\Pi^1) - A_4(\Pi^2) = \sum_{u=1}^{|X_2^1|} k(0,1,u) - \sum_{m=1}^{|X_2^1|} k(3,1,m) + \pi. \quad (A.13)$$

From (A.4), it follows that $k(0,1,u) > k(3,1,m)$ for any $m, u \in \{1,\cdots,2^{r-4}\}$. Consequently, $A_4(\Pi^1) > A_4(\Pi^2)$. □

According to Lemmas 1 and 2, the code obtained by Algorithm 1 is optimal for any $i \le 6$. If $i = 7$, we should consider two nonequivalent shortened $\Pi$ codes: 1) the code obtained by Algorithm 1; 2) the code obtained by deleting the following columns:

$$Z \triangleq \left\{ \frac{b_\gamma}{g_{15}}, \frac{b_\gamma}{g_8}, \frac{b_\gamma}{g_4}, \frac{b_\gamma}{g_2}, \frac{b_\gamma}{g_1}, \frac{b_\delta}{g_{15}}, \frac{b_\delta}{g_8} \right\}. \quad (A.14)$$

If $i = 8$, we should consider three nonequivalent shortened $\Pi$ codes: 1) the code obtained by Algorithm 1; 2) and 3) the codes obtained by deleting of the following columns:

$$\left\{ Z, \frac{b_\nu}{g_4} \right\} \text{ and } \left\{ Z, \frac{b_\delta}{g_4} \right\} \text{ respectively.} \quad (A.15)$$

We obtain the $F$ spectrums in the cases (A.14), (A.15) and conclude that the $\Pi$ code obtained by Algorithm 1 has a smaller value of $A_4$ than other $\Pi$ codes.

According to [19], in the range (5) for any length $n$ there exists a shortened $[n, n-r]$ $\Pi$ code that has a smaller value of $A_4$ in comparison with any shortened $[n, n-r]$ Hamming code. Hence, the $\Pi$ code shortened by Algorithm 1 has a smaller value of $A_4$ than any shortened Hamming code with the same parameters.

The range (5) contains the $\Omega$ code only for $r = 6,7,8$. In the range (5) we obtained the values of $A_4$ for the $\Omega$ code using a computer. These values are larger than the $\Pi$ codes shortened by Algorithm 1. For example, the [72, 64] $\Omega$ code has $A_4 = 7742$ (compare with Table I). This finishes the proof of Theorem 2. □

*Proof of Theorem 3:* According to [12], we consider the Hamming code, the $\Pi$ code, and the $\Omega$ code.

The range (6) includes the $\Omega$ code only for $r = 6,\cdots,9$. We obtained the value of $A_4$ for the $\Omega$ code with $r = 6,7,8$, by hand and for $r = 9$, $n \ge 141$ by computer. Algorithm 2 gives smaller values. The $F$ spectrum of the $[144, 144-9]$ $\Omega$ code is as follows: $j = \{72, 48, 16\}$, $F_j = \{15, 112, 240\}$. Hence, any shortened [137, 128] $\Omega$ code cannot be better than the [137, 128] $\Omega$ code with $F$ spectrum of the form $j = \{65, 41, 16, 15\}$, $F_j = \{15, 112, 149, 91\}$ and with $A_4 = 50159$. But according to (12), the [140, 131] $\Pi$ code shortened by Algorithm 2 with $i = 20$ has $A_4 = 49670$.

The Hamming code is a code with even weights. According to [9] and [11, formula (3)], it follows that for shortened Hamming codes $A_4 > \binom{n}{2}\left(\binom{n}{2}/2^{r-1} - 1\right)/6$. It can be verified that in the range (6) for the shortened Hamming codes the values of $A_4$ are larger than for the $\Pi$ code shortened by Algorithm 2. □

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