Chapter 10

Infinite Impulse Response Filters

As we said in Chapter 8, the most common design method for digital IIR filters is based on designing an analog IIR filter and then transforming it to an equivalent digital filter. Accordingly, this chapter includes two main topics: analog IIR filter design and analog-to-digital transformations of IIR filters.

Well-developed design methods exist for analog low-pass filters. We therefore discuss such filters first. The main classes of analog low-pass filters are (1) Butterworth filters; (2) Chebyshev filters, of which there are two kinds; and (3) elliptic filters. These filters differ in the nature of their magnitude responses, as well as in their respective complexity of design and implementation. Familiarity with all classes helps one to choose the most suitable filter class for a specific application.

The design of analog filters other than low pass is based on frequency transformations. Frequency transformations enable obtaining a desired high-pass, band-pass, or band-stop filter from a prototype low-pass filter of the same class. They are discussed after the sections on low-pass filter classes.

The next topic in this chapter is the transformation of a given analog IIR filter to a similar digital filter, which could be implemented by digital techniques. Similarity is required in both magnitude and phase responses of the filters. Since the frequency response of an analog filter is defined for $-\infty < \omega < \infty$, whereas that of a digital filter is restricted to $-\pi \le \theta < \pi$ (beyond which it is periodic), the two cannot be made identical. We shall therefore be concerned with similarity over a limited frequency range, usually the low frequencies.

Of the many transformation methods discussed in the literature, we shall restrict ourselves to three: the impulse invariant method, the backward difference method, and the bilinear transform. The first two are of limited applicability, but their study is pedagogically useful. The third is the best and most commonly used method of analog-to-digital filter transformation, so this is the one we emphasize.

IIR filter design usually concentrates on the magnitude response and regards the phase response as secondary. The next topic in this chapter explores the effect of phase distortions of digital IIR filters. We show that phase distortions due to variable group delay may be significant, even when the pass-band ripple of the filter is low.

The final topic discussed in this chapter is that of analog systems interfaced to a digital environment, also called sampled-data systems. This topic is marginal to digital signal processing but has great importance in related fields (such as digital control), and its underlying mathematics is well suited to the material in this chapter.

10.1 Analog Filter Basics

Analog filters are specified in a manner similar to digital filters; the main difference is that frequencies are specified in the ω domain (in rad/s), rather than in the θ domain. Thus, a low-pass analog filter is specified in terms of its pass-band edge frequency ω_p, stop-band edge frequency ω_s, pass-band ripple δ_p, and stop-band attenuation δ_s. For analog filters, the pass-band magnitude response is usually required to be in the range $[1-\delta_p, 1]$. This is mainly a matter of convenience, since the filter's gain can be easily adjusted to make the pass-band ripple symmetrical with respect to 1 (see Problem 10.4). Also recall the definitions

$$A_p = -20\log_{10}(1-\delta_p) \approx 8.6859\delta_p, \tag{10.1a}$$

$$A_s = -20\log_{10}\delta_s. \tag{10.1b}$$

It will be convenient to introduce the following auxiliary parameters, which are functions of the basic specification parameters:

$$d = \left[\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}\right]^{1/2} = \left(\frac{10^{0.1A_p}-1}{10^{0.1A_s}-1}\right)^{1/2}, \tag{10.2}$$

$$k = \frac{\omega_p}{\omega_s}. \tag{10.3}$$

The parameter d is called the *discrimination factor*; the parameter k is called the *selectivity factor*. As we shall see later, these are the primary design parameters for analog filters.

Two other parameters of analog filters are the following:

1. The -3 dB frequency (also called the *cutoff frequency*) ω_{3db}, defined as the frequency at which the magnitude response of the filter is $1/\sqrt{2}$ of its nominal value at the pass band.
2. The asymptotic attenuation at high frequencies. This parameter is determined by the difference between the denominator and numerator degrees. It follows from (8.5) that

$$H^F(\omega) \approx b_0(j\omega)^{q-p} \tag{10.4}$$

for large enough ω (e.g., larger than about 5 times the largest absolute value of all poles and zeros). Then from (10.4) we can write

$$20\log_{10}|H^F(\omega)| \approx 20\log_{10}|b_0| - 20(p-q)\log_{10}\omega. \tag{10.5}$$

It is common to express this by saying that the asymptotic attenuation is $20(p-q)$ dB/decade.

The -3 dB frequency is occasionally useful for digital filters as well. The asymptotic attenuation is not defined for a digital filter, because the digital frequency θ is of interest only in the range $[-\pi, \pi]$.

Classical analog filters are constructed from a desired square-magnitude response function. This function is usually of the form (except for Chebyshev filter of the second kind, to be studied in Section 10.3.2)

$$|H^F(\omega)|^2 = \frac{1}{1+\Lambda\left(\frac{\omega}{\omega_0}\right)}, \tag{10.6}$$

where $\Lambda(\cdot)$ is either a polynomial or a rational function, called the *attenuation function*, and ω_0 is a reference frequency. The attenuation function is nonnegative and is designed to be small in the pass band and large in the stop band. If $\Lambda(\cdot)$ is monotone, so is $|H^{\mathrm{F}}(\omega)|^2$. If $\Lambda(\cdot)$ is oscillatory in one of the bands (or both), $|H^{\mathrm{F}}(\omega)|^2$ exhibits ripple in that band. By constraining $\Lambda(\cdot)$ to be polynomial or rational, we impose on $H^{\mathrm{L}}(s)$ to be rational, thus guaranteeing that it be realizable. The order of the polynomial, or the rational function, determines the order of $H^{\mathrm{L}}(s)$. In addition, the attenuation function may depend on other parameters, used for tuning the behavior of $|H^{\mathrm{F}}(\omega)|^2$.

The square-magnitude function $|H^{\mathrm{F}}(\omega)|^2$ determines the product $H^{\mathrm{L}}(s)H^{\mathrm{L}}(-s)$, but it does not uniquely determine the transfer function $H^{\mathrm{L}}(s)$. The set of poles of $H^{\mathrm{L}}(s)H^{\mathrm{L}}(-s)$ consists of the poles of $H^{\mathrm{L}}(s)$ and their mirror images with respect to the imaginary axis in the s plane. The requirement that $H^{\mathrm{L}}(s)$ be stable determines this transfer function uniquely: $H^{\mathrm{L}}(s)$ must include all poles of $H^{\mathrm{L}}(s)H^{\mathrm{L}}(-s)$ on the left half of the s plane and only those.

The analog filters that we shall study in the following sections are all constructed in a similar manner, differing only in the choice of attenuation functions. Because of the different attenuation functions, the filters differ in their response characteristics.

10.2 Butterworth Filters

A low-pass Butterworth filter is defined in terms of its square-magnitude frequency response

$$|H^{\mathrm{F}}(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2N}}, \tag{10.7}$$

where N is an integer (which will soon be seen to be the filter's order), and ω_0 is a parameter. Figure 10.1 illustrates the square magnitude of the frequency response for $N = 3$.

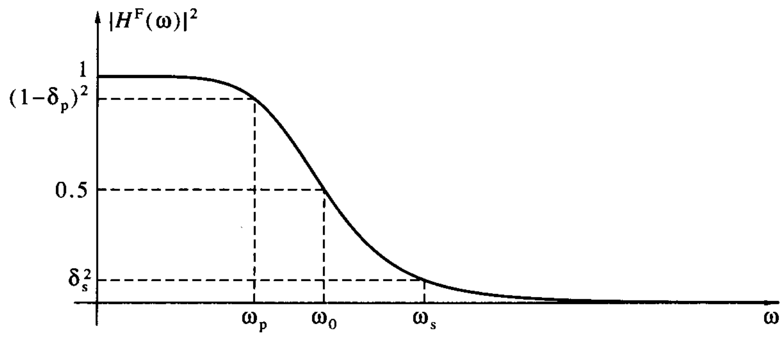

Figure 10.1 The square magnitude of the frequency response of a low-pass Butterworth filter, $N = 3$.

The salient properties of a low-pass Butterworth filter are as follows:

1. The magnitude response is a monotonically decreasing function of ω.
2. The maximum gain occurs at $\omega = 0$ and is $|H^{\mathrm{F}}(0)| = 1$.
3. We have $|H^{\mathrm{F}}(\omega_0)| = \sqrt{0.5}$, so the -3 dB point is ω_0 rad/s.
4. The asymptotic attenuation at high frequencies is $20N$ dB/decade.
5. The square magnitude of the frequency response satisfies

$$\left.\frac{\partial^k |H^{\mathrm{F}}(\omega)|^2}{\partial \omega^k}\right|_{\omega=0} = 0, \quad 1 \le k \le 2N - 1. \tag{10.8}$$

Thus, the magnitude response is nearly constant at low frequencies (the constant being 1). Because of this property, low-pass Butterworth filters are said to be *maximally flat at DC.*

The transfer function of Butterworth filter can be obtained by the following calculation. By substituting $s = j\omega$, or $\omega = s/j$, we get from (10.7)

$$H^{\mathrm{L}}(s)H^{\mathrm{L}}(-s) = \frac{1}{1 + \left(\frac{s}{j\omega_0}\right)^{2N}} = \frac{1}{1 + (-1)^N \left(\frac{s}{\omega_0}\right)^{2N}}. \tag{10.9}$$

The right side of (10.9) has $2N$ poles, which are the $2N$ complex roots of $(-1)^{(N+1)}$, multiplied by ω_0. The poles are given by the explicit expression

$$s_k = \omega_0 \exp\left[j\frac{(N + 1 + 2k)\pi}{2N}\right], \quad 0 \le k \le 2N - 1. \tag{10.10}$$

Of those, the poles corresponding to $0 \le k \le N - 1$ are on the left side of the complex plane. These must correspond to $H^{\mathrm{L}}(s)$, since we require $H^{\mathrm{L}}(s)$ to be stable. The remaining roots are the poles of $H^{\mathrm{L}}(-s)$. Figure 10.2 illustrates the pole locations of Butterworth filters of odd and even orders.

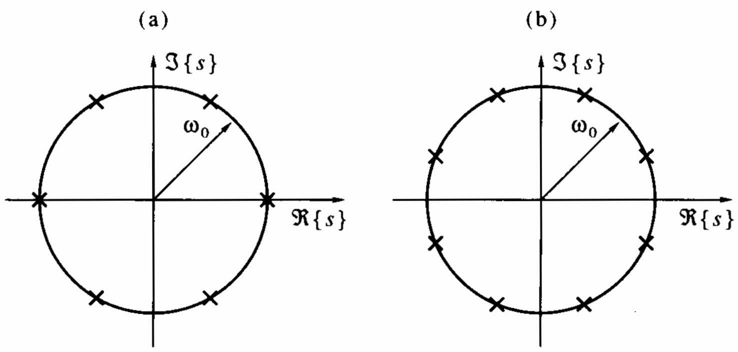

Figure 10.2 The poles of a low-pass Butterworth filter: (a) $N = 3$; (b) $N = 4$. The N poles of $H^{\mathrm{L}}(s)$ are on the left side of the s plane and those of $H^{\mathrm{L}}(-s)$ are on the right side.

A Butterworth filter has no zeros, but its constant coefficient in the numerator has to be adjusted so that $H^F(0) = 1$. In summary, Butterworth filter of order N is given by

$$H^{\mathrm{L}}(s) = \prod_{k=0}^{N-1} \frac{-s_k}{s - s_k}, \tag{10.11}$$

where

$$\begin{aligned} s_k &= \omega_0 \cos\left[\frac{\pi}{2} + \frac{(2k+1)\pi}{2N}\right] + j\omega_0 \sin\left[\frac{\pi}{2} + \frac{(2k+1)\pi}{2N}\right] \\ &= -\omega_0 \sin\left[\frac{(2k+1)\pi}{2N}\right] + j\omega_0 \cos\left[\frac{(2k+1)\pi}{2N}\right], \quad 0 \le k \le N - 1. \end{aligned} \tag{10.12}$$

Table 10.1 gives the coefficients of the polynomials $a(s)$ in the transfer function

$$H^{\mathrm{L}}(s) = \frac{1}{a(s)} = \frac{1}{s^N + a_1 s^{N-1} + \cdots + a_{N-1}s + 1} \tag{10.13}$$

of an Nth-order LP Butterworth filter having $\omega_0 = 1$, for $2 \le N \le 6$. Such a filter is said to be *normalized.*

N	a_1	a_2	a_3	a_4	a_5
2	1.4142				
3	2.0000	2.0000			
4	2.6131	3.4142	2.6131		
5	3.2361	5.2361	5.2361	3.2361	
6	3.8637	7.4641	9.1416	7.4641	3.8637

Table 10.1 Coefficients of the denominator polynomials of normalized low-pass Butterworth filters of orders 2 through 6.

Designing a Butterworth filter to meet given specifications proceeds as follows. First we compute the discrimination factor d and the selectivity factor k from the given values of δ_p, δ_s, ω_p, and ω_s, using the formulas (10.2) and (10.3). Then we observe from (10.7) and Figure 10.1 that

$$\frac{1}{1+\left(\frac{\omega_p}{\omega_0}\right)^{2N}} \geq (1-\delta_p)^2, \quad \frac{1}{1+\left(\frac{\omega_s}{\omega_0}\right)^{2N}} \leq \delta_s^2. \tag{10.14}$$

This gives

$$\left(\frac{\omega_p}{\omega_0}\right)^{2N} \leq (1-\delta_p)^{-2}-1, \quad \left(\frac{\omega_s}{\omega_0}\right)^{2N} \geq \delta_s^{-2}-1. \tag{10.15}$$

Dividing these two expressions gives

$$\left(\frac{\omega_s}{\omega_p}\right)^{2N} \geq \frac{\delta_s^{-2}-1}{(1-\delta_p)^{-2}-1}. \tag{10.16}$$

Using the definitions of d (10.2) and k (10.3) gives

$$\left(\frac{1}{k}\right)^{2N} \geq \left(\frac{1}{d}\right)^2. \tag{10.17}$$

Finally, taking the logarithms and dividing again gives

$$N \geq \frac{\log_e(1/d)}{\log_e(1/k)}. \tag{10.18}$$

The right side of (10.18) will not be an integer in general, so in practice we take N as the smallest integer larger than $\log_e(1/d)/\log_e(1/k)$. The frequency ω_0 can be chosen anywhere in the interval (Problem 10.3)

$$\omega_p\left[(1-\delta_p)^{-2}-1\right]^{-1/2N} \leq \omega_0 \leq \omega_s\left[\delta_s^{-2}-1\right]^{-1/2N}. \tag{10.19}$$

In summary, the design procedure for a low-pass Butterworth filter is as follows:

Low-pass Butterworth filter design procedure

1. Compute d and k as a function of δ_p, δ_s, ω_p, and ω_s, using (10.2) and (10.3).
2. Compute N, using (10.18), and round upward to the nearest integer.
3. Choose ω_0, using (10.19).
4. Compute the poles s_k, using (10.12).
5. Compute the transfer function $H^L(s)$, using (10.11).

Example 10.1 Design a low-pass Butterworth filter to meet the specifications:

$$\delta_{\mathrm{p}} = 0.001, \quad \delta_{\mathrm{s}} = 0.001, \quad \omega_{\mathrm{p}} = 1\,\mathrm{rad/s}, \quad \omega_{\mathrm{s}} = 2\,\mathrm{rad/s}.$$

The selectivity and discrimination factors are

$$d = 4.4755 \times 10^{-5}, \quad k = 0.5.$$

We get

$$N \geq 14.45 \implies N = 15.$$

Therefore,

$$1.2301\,\mathrm{rad/s} \leq \omega_0 \leq 1.2619\,\mathrm{rad/s}.$$

The poles of the resulting filter (with $\omega_0 = 1.2301$) are

$$\begin{aligned} s_{0,14} &= -0.1286 \pm j1.2234, & s_{1,13} &= -0.3801 \pm j1.1699, \\ s_{2,12} &= -0.6150 \pm j1.0653, & s_{3,11} &= -0.8231 \pm j0.9141, \\ s_{4,10} &= -0.9952 \pm j0.7230, & s_{5,9} &= -1.1238 \pm j0.5003, \\ s_{6,8} &= -1.2032 \pm j0.2558, & s_7 &= -1.2301. \end{aligned}$$

The magnitude response of the filter is shown in Figure 10.3. Part a shows the response in the frequency range 0 to 2 rad/s; part b shows the pass band at an expanded scale. The filter meets the pass-band specification exactly, but its transition band is slightly narrower than the specification. □

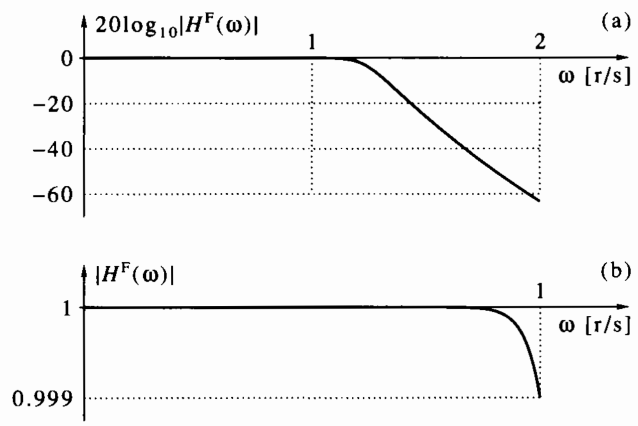

Figure 10.3 Magnitude response of the Butterworth filter in Example 10.1: (a) full range; (b) pass-band details.

10.3 Chebyshev Filters

Chebyshev has noted that $\cos(N\alpha)$ is a polynomial of degree N in $\cos\alpha$. Furthermore, $\cosh(N\alpha)$ is *the same* polynomial (of degree N) in $\cosh\alpha$. Correspondingly, Chebyshev polynomial of degree N is defined as[1]

$$T_N(x) = \begin{cases} \cos(N \arccos x), & |x| \leq 1, \\ \cosh(N \operatorname{arccosh} x), & |x| > 1. \end{cases} \tag{10.20}$$

Chebyshev polynomials can be constructed by the following recursive formula:

Theorem 10.1

$$T_N(x) = 2xT_{N-1}(x) - T_{N-2}(x), \quad T_0(x) = 1, \ T_1(x) = x. \tag{10.21}$$

Proof We have the following trigonometric identities:

$$\cos(N\alpha) = \cos[(N-1)\alpha]\cos\alpha - \sin[(N-1)\alpha]\sin\alpha, \tag{10.22a}$$

$$\cos[(N-2)\alpha] = \cos[(N-1)\alpha]\cos\alpha + \sin[(N-1)\alpha]\sin\alpha. \tag{10.22b}$$

Adding these two identities gives

$$\cos(N\alpha) + \cos[(N-2)\alpha] = 2\cos[(N-1)\alpha]\cos\alpha, \tag{10.23}$$

which is identical to (10.21). □

The main properties of Chebyshev polynomials are as follows:

1. For $|x| \le 1$, $|T_N(x)| \le 1$, and it oscillates between -1 and $+1$ a number of times proportional to N. This is obvious from the definition (10.20).
2. For $|x| > 1$, $|T_N(x)| > 1$, and it is monotonically increasing in $|x|$. This is also obvious from the definition (10.20).
3. Chebyshev polynomials of odd orders are odd functions of x (i.e., they contain only odd powers of x) and Chebyshev polynomials of even orders are even functions of x (i.e., they contain only even powers of x). The proof is left as an exercise (Problem 10.7).
4. $T_N(0) = \pm 1$ for even N, and $T_N(0) = 0$ for odd N. The proof is again left as an exercise (Problem 10.8).
5. $|T_N(\pm 1)| = 1$ for all N. This follows from the definition (10.20).

Figure 10.4 shows the graphs of Chebyshev polynomials of orders 1 through 4.

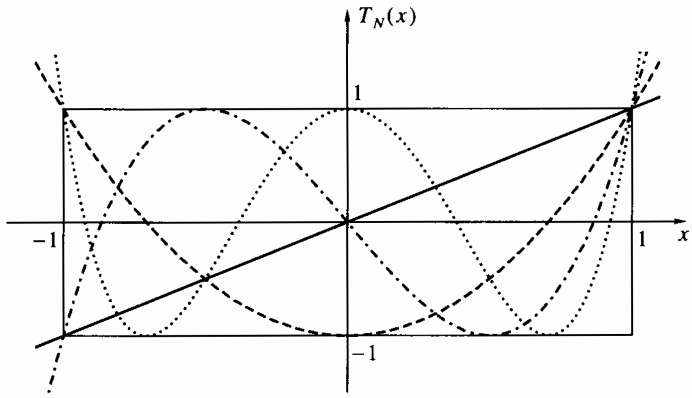

Figure 10.4 Chebyshev polynomials. Solid line: $N = 1$, dashed line: $N = 2$, dot-dashed line: $N = 3$, dotted line: $N = 4$.

The main property of Chebyshev polynomials—being oscillatory in the range $|x| \le 1$ and monotone outside it—is used for constructing filters that are equiripple in either the pass band or the stop band. Thus, the magnitude response oscillates between the permitted minimum and maximum values in the band a number of times depending on the filter's order. The equiripple property provides sharper transition between the pass band and the stop band, compared with that obtained when the magnitude response is monotone. As a result, the order of a Chebyshev filter needed to achieve given specifications is usually smaller than that of a Butterworth filter.

There are two kinds of Chebyshev filter: The first is equiripple in the pass band and monotonically decreasing in the stop band, whereas the second is monotonically decreasing in the pass band and equiripple in the stop band. The second kind is also called *inverse Chebyshev.*

10.3.1 Chebyshev Filter of the First Kind

Chebyshev filter of the first kind, or Chebyshev-I for short, is defined by the square-magnitude frequency response

$$|H^{\mathrm{F}}(\omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\omega}{\omega_0}\right)}, \tag{10.24}$$

where N is an integer (which will soon be seen to be the filter's order), and ω_0 and ε are parameters. Figure 10.5 illustrates the square magnitudes of the frequency responses for $N = 2$ and $N = 3$.

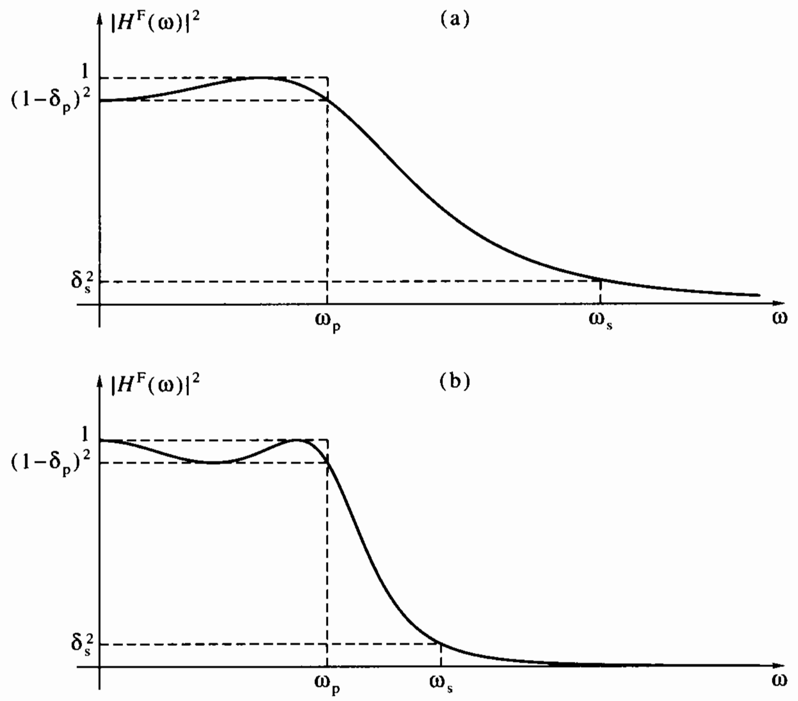

Figure 10.5 The square magnitude of the frequency response of a low-pass Chebyshev filter of the first kind: (a) $N = 2$; (b) $N = 3$.

The salient properties of a low-pass Chebyshev-I filter are as follows:

1. For $0 \le \omega \le \omega_0$ we have, by the properties of Chebyshev polynomials,

$$\frac{1}{1+\varepsilon^2} \le |H^{\mathrm{F}}(\omega)|^2 \le 1. \tag{10.25}$$

2. From property 4 of Chebyshev polynomials we get that

$$|H^{\mathrm{F}}(0)|^2 = \begin{cases} 1/(1+\varepsilon^2), & N \text{ even}, \\ 1, & N \text{ odd}. \end{cases} \tag{10.26}$$

3. For $\omega > \omega_0$, the response is monotonically decreasing, because of the monotone behavior of $T_N(x)$ for $|x| > 1$. Furthermore, since $T_N(x)$ is an Nth-order polynomial, the asymptotic attenuation of the filter is $20N$ dB/decade, same as that of Butterworth filter of the same order.

The poles of an Nth-order low-pass Chebyshev-I filter are given by the formula

$$\begin{aligned} s_k = &-\omega_0 \sinh\left(\frac{1}{N}\operatorname{arcsinh}\frac{1}{\varepsilon}\right)\sin\left[\frac{(2k+1)\pi}{2N}\right] \\ &+ j\omega_0 \cosh\left(\frac{1}{N}\operatorname{arcsinh}\frac{1}{\varepsilon}\right)\cos\left[\frac{(2k+1)\pi}{2N}\right], \quad 0 \le k \le N-1. \end{aligned} \tag{10.27}$$

The poles are located on an ellipse whose principal radii are $\sinh(N^{-1}\operatorname{arcsinh}\varepsilon^{-1})$ (horizontal) and $\cosh(N^{-1}\operatorname{arcsinh}\varepsilon^{-1})$ (vertical). The proof of (10.27) is not given here. Figure 10.6 illustrates the pole locations of a Chebyshev-I filter for odd and even N. The following identities can be used for computing the inverse hyperbolic functions:

$$\operatorname{arcsinh}(x) = \log_e(x + \sqrt{x^2+1}), \quad \operatorname{arccosh}(x) = \log_e(x + \sqrt{x^2-1}). \tag{10.28}$$

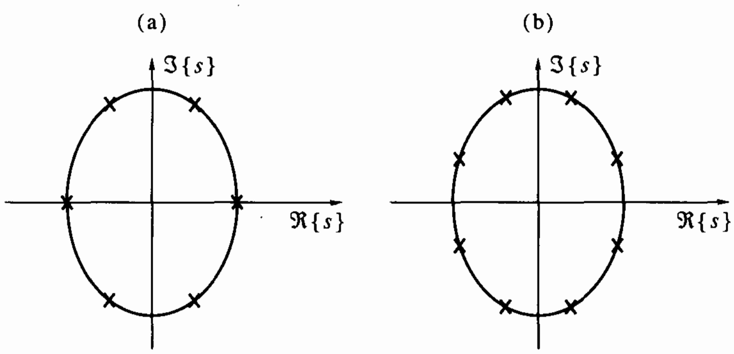

Figure 10.6 The poles of a low-pass Chebyshev filter of the first kind: (a) $N = 3$; (b) $N = 4$. The N poles of $H^{\mathrm{L}}(s)$ are on the left side of the s plane and those of $H^{\mathrm{L}}(-s)$ are on the right side.

A low-pass Chebyshev filter of the first kind has no zeros. Thus, the formula for the transfer function $H^{\mathrm{L}}(s)$ is

$$H^{\mathrm{L}}(s) = H_0 \prod_{k=0}^{N-1} \frac{-s_k}{s - s_k}, \tag{10.29}$$

where

$$H_0 = \begin{cases} (1+\varepsilon^2)^{-1/2}, & N \text{ even}, \\ 1, & N \text{ odd}. \end{cases} \tag{10.30}$$

Designing a Chebyshev-I filter to meet given specifications proceeds as follows. First, we compute the discrimination factor d and the selectivity factor k from the given values of δ_{p}, δ_{s}, ω_{p}, and ω_{s}, using (10.2) and (10.3). Then we observe from (10.24) that

$$\frac{1}{1 + \varepsilon^2 T_N^2\left(\frac{\omega_{\mathrm{p}}}{\omega_0}\right)} \ge (1 - \delta_{\mathrm{p}})^2, \tag{10.31}$$

$$\frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\omega_s}{\omega_0}\right)} \le \delta_s^2. \tag{10.32}$$

Equation (10.31) can be satisfied with equality by choosing

$$\omega_0 = \omega_p, \quad \varepsilon = [(1-\delta_p)^{-2} - 1]^{1/2}. \tag{10.33}$$

Then (10.32) can be satisfied by choosing

$$T_N(1/k) = \cosh[N \operatorname{arccosh}(1/k)] \ge \frac{1}{d}. \tag{10.34}$$

The inequality (10.34) leads to

$$N \ge \frac{\operatorname{arccosh}(1/d)}{\operatorname{arccosh}(1/k)}. \tag{10.35}$$

In practice, we take N as the smallest integer larger than the right side of (10.35). We note that the right side of (10.35) is always smaller than the right side of (10.18); see Problem 10.10. Thus, the order of a Chebyshev filter meeting given specifications will be smaller than that of Butterworth filter meeting the same specifications, or equal to it at most. In summary, the design procedure for a low-pass Chebyshev-I filter is as follows:

Low-pass Chebyshev-I filter design procedure

1. Compute d and k as a function of δ_p, δ_s, ω_p, and ω_s, using (10.2) and (10.3).
2. Compute N, using (10.35), and round upward to the nearest integer.
3. Compute ω_0 and ε, using (10.33).
4. Compute the poles s_k, using (10.27).
5. Compute H_0, using (10.30).
6. Compute the transfer function $H^L(s)$, using (10.29).

Example 10.2 Design a low-pass Chebyshev-I filter to meet the same specifications as in Example 10.1, therefore having the same selectivity and discrimination factors. We get

$$\varepsilon = 0.04475, \quad \omega_0 = 1\,\text{rad/s}, \quad N \ge 8.13 \implies N = 9.$$

The poles of the resulting filter are

$$\begin{aligned} s_{0,8} &= -0.0755 \pm j1.0739, \quad s_{1,7} = -0.2175 \pm j0.9444, \\ s_{2,6} &= -0.3332 \pm j0.7009, \quad s_{3,5} = -0.4087 \pm j0.3730, \\ s_4 &= -0.4349. \end{aligned}$$

The magnitude response of the filter is shown in Figure 10.7. Part a shows the response in the frequency range 0 to 2 rad/s; part b shows the pass band at an expanded scale. The filter meets the pass-band specification exactly, but its transition band is slightly narrower than the specification. □

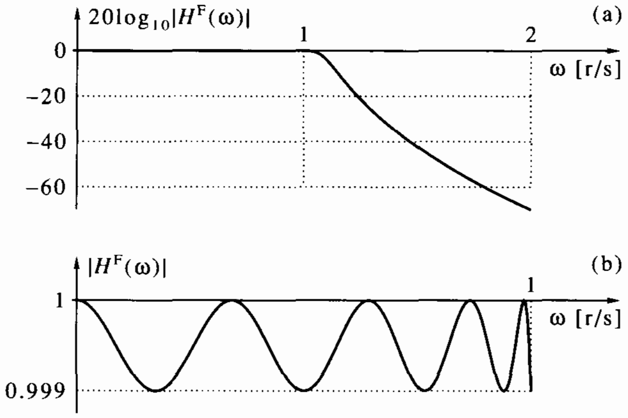

Figure 10.7 Magnitude response of the Chebyshev filter in Example 10.2: (a) full range; (b) pass-band details.

10.3.2 Chebyshev Filter of the Second Kind

Chebyshev filter of the second kind, or Chebyshev-II for short, is defined by the square-magnitude frequency response

$$|H^F(\omega)|^2 = 1 - \frac{1}{1 + \varepsilon^2 T_N^2\left(\frac{\omega_0}{\omega}\right)} = \frac{\varepsilon^2 T_N^2\left(\frac{\omega_0}{\omega}\right)}{1 + \varepsilon^2 T_N^2\left(\frac{\omega_0}{\omega}\right)}, \tag{10.36}$$

where N is the filter's order, and ω_0 and ε are parameters. Figure 10.8 illustrates the square magnitudes of the frequency responses for $N = 2$ and $N = 3$.

The salient properties of a low-pass Chebyshev-II filter are as follows:

1. For $\omega \geq \omega_0$ we have, by the properties of Chebyshev polynomials,
$$0 \leq |H^F(\omega)|^2 \leq \frac{\varepsilon^2}{1+\varepsilon^2}. \tag{10.37}$$
2. $|H^F(0)|^2 = 1$ for all N, ω_0, and $\varepsilon > 0$.
3. For $0 \leq \omega < \omega_0$, the response is monotonically decreasing, because of the monotone behavior of $T_N(x)$ for $|x| > 1$.
4. In the limit as ω tends to infinity we have
$$\lim_{\omega\to\infty} |H^F(\omega)|^2 = \frac{\varepsilon^2 T_N^2(0)}{1 + \varepsilon^2 T_N^2(0)} = \begin{cases} \dfrac{\varepsilon^2}{1+\varepsilon^2}, & N \text{ even}, \\ 0, & N \text{ odd}. \end{cases} \tag{10.38}$$
5. Since, for even N, $|H^F(\omega)|^2$ approaches a nonzero constant as ω tends to infinity, the asymptotic attenuation is 0 dB/decade for even N. For odd N we can find the asymptotic attenuation as follows. Near $x = 0$, Chebyshev polynomial of an odd order is approximately given by $T_N(x) \approx K_N x$, where K_N is constant. Therefore we have for large ω,
$$|H^F(\omega)|^2 \approx \frac{\varepsilon^2 K_N^2 \omega_0^2}{\omega^2}. \tag{10.39}$$
Thus, for odd N, the asymptotic attenuation is 20 dB/decade.

The poles of the transfer function $H^L(s)$ of a low-pass Chebyshev-II filter are inversely proportional to those of a Chebyshev-I filter of the same order. Denoting the

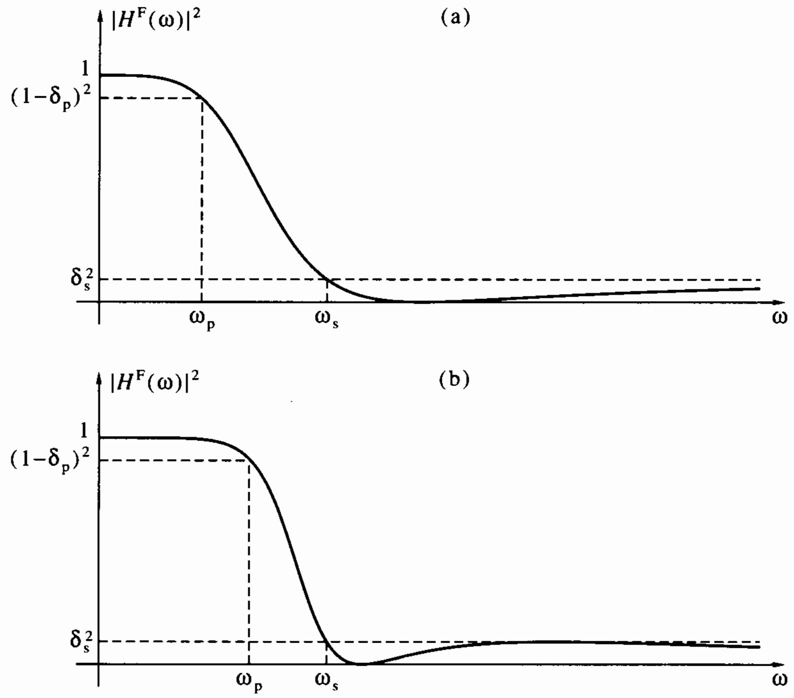

Figure 10.8 The square magnitude of the frequency response of a low-pass Chebyshev filter of the second kind: (a) $N = 2$; (b) $N = 3$.

poles by $\{v_k,\ 0 \le k \le N-1\}$, we have

$$v_k = \frac{\omega_0^2}{s_k}, \quad 0 \le k \le N-1, \tag{10.40}$$

where the s_k are given by (10.27). The poles of a Chebyshev filter of the second kind are *not* located on an ellipse.

Contrary to filters of the first kind, Chebyshev filters of the second kind have zeros. As is clear from the definition of $|H^F(\omega)|^2$, the zeros are on the imaginary axis, at the frequencies for which $T_N(\omega_0/\omega) = 0$. The zeros of $T_N(x)$ are easily seen from the definition to be at $\{\cos[(2k+1)\pi/2N],\ 0 \le k \le N-1\}$. Therefore, the zeros of $H^L(s)$ are at

$$u_k = \frac{j\omega_0}{\cos\left[\dfrac{(2k+1)\pi}{2N}\right]}, \quad 0 \le k \le N-1. \tag{10.41}$$

When N is even, there are N finite zeros. When N is odd, there are only $N-1$ zeros, since $k = (N-1)/2$ gives a zero at infinity.

In summary, the transfer function $H^L(s)$ is given by

$$H^L(s) = \prod_{k=0}^{N-1} \frac{v_k(s-u_k)}{u_k(s-v_k)}, \tag{10.42}$$

where $(s-u_k)/u_k$ is replaced by -1 if $u_k = \infty$.

Designing a Chebyshev filter of the second kind to meet given specifications proceeds as follows. First we compute the discrimination factor d and the selectivity factor k from the given values of δ_p, δ_s, ω_p, and ω_s, using (10.2) and (10.3). Then we observe from (10.36) that

$$\frac{\varepsilon^2 T_N^2\left(\frac{\omega_0}{\omega_p}\right)}{1+\varepsilon^2 T_N^2\left(\frac{\omega_0}{\omega_p}\right)} \geq (1-\delta_p)^2, \qquad (10.43)$$

$$\frac{\varepsilon^2 T_N^2\left(\frac{\omega_0}{\omega_s}\right)}{1+\varepsilon^2 T_N^2\left(\frac{\omega_0}{\omega_s}\right)} \leq \delta_s^2. \qquad (10.44)$$

Equation (10.44) can be satisfied with equality by choosing

$$\omega_0 = \omega_s, \quad \varepsilon = (\delta_s^{-2} - 1)^{-1/2}. \qquad (10.45)$$

Then we can satisfy (10.43) by choosing

$$T_N(1/k) = \cosh[N \operatorname{arccosh}(1/k)] \geq \frac{1}{d}, \qquad (10.46)$$

hence

$$N \geq \frac{\operatorname{arccosh}(1/d)}{\operatorname{arccosh}(1/k)}. \qquad (10.47)$$

In practice, we take N as the smallest integer larger than the right side of (10.47). For a given set of specifications, both kinds of Chebyshev filter will have the same order. Therefore, the choice between the two kinds depends only on the desired ripple characteristics (i.e., whether ripple is permitted in the pass band or in the stop band). In summary, the design procedure for a low-pass Chebyshev-II filter is as follows:

Low-pass Chebyshev-II filter design procedure

1. Compute d and k as a function of δ_p, δ_s, ω_p, and ω_s, using (10.2) and (10.3).
2. Compute N, using (10.47), and round upward to the nearest integer.
3. Compute ω_0 and ε, using (10.45).
4. Compute the s_k, using (10.27).
5. Compute the poles v_k, using (10.40).
6. Compute the zeros u_k, using (10.41).
7. Compute the transfer function $H^L(s)$, using (10.42), where $(s-u_k)/u_k$ is replaced by -1 if $u_k = \infty$.

Example 10.3 Design a low-pass Chebyshev-II filter to meet the same specifications as in Example 10.1, therefore having the same selectivity and discrimination factors. We get

$$\varepsilon = 0.001, \quad \omega_0 = 2\,\text{rad/s}, \quad N \geq 8.13 \implies N = 9.$$

The poles of the resulting filter are

$$v_{0,8} = -0.1762 \pm j1.4520, \quad v_{1,7} = -0.5750 \pm j1.4770,$$
$$v_{2,6} = -1.1069 \pm j1.3496, \quad v_{3,5} = -1.7533 \pm j0.9273,$$
$$v_4 = -2.1084.$$

The zeros of the filter are

$$u_{0,8} = \pm j2.0308, \quad u_{1,7} = \pm j2.3094, \quad u_{2,6} = \pm j3.1114, \quad u_{3,5} = \pm j5.8476.$$

The magnitude response of the filter is shown in Figure 10.9. Part a shows the response in the frequency range 0 to 4 rad/s; part b shows the pass band at an expanded scale. The filter meets the stop-band specification exactly, but its transition band is narrower than the specification, so its tolerance in the pass band is much narrower than the specification. □

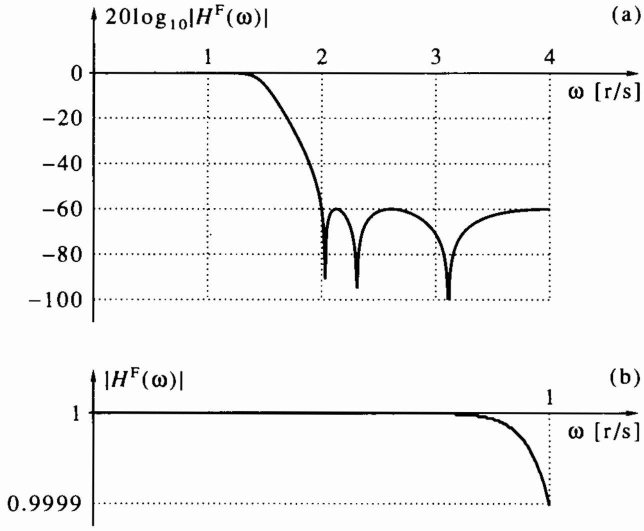

Figure 10.9 Magnitude response of the Chebyshev filter in Example 10.3: (a) full range; (b) pass-band details.

10.4 Elliptic Filters*

Elliptic filters are equiripple in both the pass band and the stop band. They achieve the minimal possible order for given specifications. A low-pass elliptic filter is defined by the square-magnitude response

$$|H^{\mathrm{F}}(\omega)|^2 = \frac{1}{1 + \varepsilon^2 R_N^2\left(\frac{\omega}{\omega_0}\right)}, \tag{10.48}$$

where $R_N(x)$ is a *Chebyshev rational function* of degree N. The main properties of this function are as follows:

1. It is an even function of x for even N, and an odd function of x for odd N (similar to Chebyshev polynomials).
2. In the range $-1 \le x \le 1$, the function oscillates between -1 and 1 and all its zeros are in this range. Therefore, $|H^{\mathrm{F}}(\omega)|^2$ oscillates between 1 and $1/(1+\varepsilon^2)$ for $0 \le |\omega| \le \omega_0$.
3. In the range $1 < |x| < \infty$, $|R_N(x)|$ oscillates between $1/d$ and ∞, where d is a design parameter (later identified with the discrimination factor d). As a result, $|H^{\mathrm{F}}(\omega)|^2$ oscillates between 0 and $1/(1+\varepsilon^2/d^2)$ in the range $\omega_0 < |\omega| < \infty$.

Figure 10.10 shows the graphs of typical Chebyshev rational functions of orders 2 through 5, in the range $-1 \le x \le 1$.

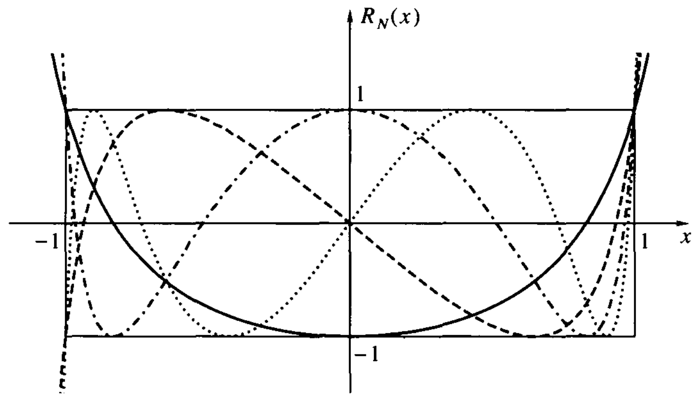

Figure 10.10 Chebyshev rational functions. Solid line: $N = 2$, dashed line: $N = 3$, dot-dashed line: $N = 4$, dotted line: $N = 5$.

Figure 10.11 illustrates the square magnitudes of the frequency responses of a low-pass elliptic filter for $N = 2$ and $N = 3$. Comparing this with Figure 10.5, we see that an elliptic filter indeed has a narrower transition band than a Chebyshev filter having the same order and tolerances.

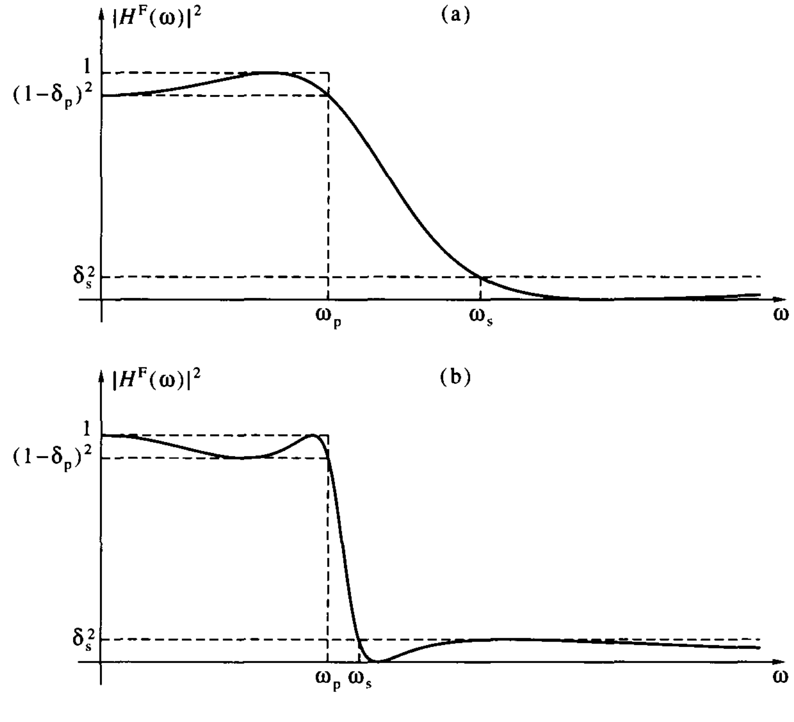

Figure 10.11 The square magnitude of the frequency response of a low-pass elliptic filter: (a) $N = 2$; (b) $N = 3$.

Before we characterize the Chebyshev rational functions, we state a few results on elliptic integrals and Jacobi elliptic functions. We give here the necessary notations and definitions of elliptic integrals and functions, but omit the proofs. A detailed discussion of this topic is given in Antoniou [1993].

The *elliptic integral of the first kind* is defined as

$$u(\phi, m) = \int_0^{\phi} (1 - m \sin^2 x)^{-1/2} dx, \tag{10.49}$$

where $0 < m < 1$. When we substitute $\phi = 0.5\pi$, we get the *complete elliptic integral of the first kind*

$$K(m) = u(0.5\pi, m) = \int_0^{0.5\pi} (1 - m \sin^2 x)^{-1/2} dx. \tag{10.50}$$

The complete elliptic integral of the first kind is computed in MATLAB by the function `ellipke`. It is a monotone increasing function of m, tending to 0.5π as m approaches 0, and tending to ∞ as m approaches 1.

The Jacobi elliptic sine function is obtained from the elliptic integral of the first kind by the following two operations:

1. Invert the function $u(\phi, m)$; that is, express ϕ as a function of u, with m as a parameter, say $\phi(u, m)$.
2. The Jacobi elliptic sine function is then

$$\mathrm{sn}(u, m) = \sin[\phi(u, m)]. \tag{10.51}$$

The Jacobi elliptic sine function is a periodic function, with period $4K(m)$. It is computed in MATLAB by the function `ellipj` (which also computes two other Jacobi elliptic functions, of no interest to us here).

A Chebyshev rational function has the form

$$R_N(x) = Cx^{(N \bmod 2)} \prod_{l=1}^{\lfloor N/2 \rfloor} \frac{x^2 - z_l^2}{x^2 - p_l^2}, \quad \text{where} \quad C = \prod_{l=1}^{\lfloor N/2 \rfloor} \frac{1 - z_l^2}{1 - p_l^2}. \tag{10.52}$$

In the present case, the parameters N, $\{z_l, p_l, \ 0 \le l \le N/2\}$ are derived from the selectivity factor k and the discrimination factor d of the filter to be designed according to the following formulas:

$$N = \frac{K(k^2)K(1 - d^2)}{K(1 - k^2)K(d^2)}, \tag{10.53}$$

$$z_l = \begin{cases} \mathrm{sn}[2lK(k^2)/N], & N \text{ odd}, \\ \mathrm{sn}[(2l - 1)K(k^2)/N], & N \text{ even}, \end{cases} \qquad p_l = (kz_l)^{-1}. \tag{10.54}$$

After we have computed the poles and the zeros of $R_N(x)$, we can determine $H^{\mathrm{L}}(s)$ from (10.48). To this end, we substitute $\omega = -js$ and $\omega_0 = 1$. We get

$$R_N(-js) = C(-js)^{(N \bmod 2)} \prod_{l=1}^{\lfloor N/2 \rfloor} \frac{s^2 + z_l^2}{s^2 + p_l^2}. \tag{10.55}$$

Substitution of (10.55) in (10.48) gives

$$H^{\mathrm{L}}(s)H^{\mathrm{L}}(-s) = \frac{\prod_{l=1}^{\lfloor N/2 \rfloor}(s^2 + p_l^2)^2}{\prod_{l=1}^{\lfloor N/2 \rfloor}(s^2 + p_l^2)^2 + (C\varepsilon)^2(-s^2)^{(N \bmod 2)} \prod_{l=1}^{\lfloor N/2 \rfloor}(s^2 + z_l^2)^2}. \tag{10.56}$$

Therefore, the zeros of $H^{\mathrm{L}}(s)$ are $\{jp_l, -jp_l, \ 1 \le l \le \lfloor N/2 \rfloor\}$ and the poles are the subset of N roots of the denominator on the right side of (10.56) whose real part is negative.

The following steps summarize the design procedure of a low-pass elliptic filter having $\omega_{\mathrm{p}} = 1$ and meeting given specifications:

Low-pass elliptic filter design procedure

1. Compute the discrimination factor d and the selectivity factor k, using (10.2) and (10.3).
2. Compute
$$\varepsilon = [(1-\delta_{\rm p})^{-2} - 1]^{1/2}. \tag{10.57}$$
3. Compute the order N from (10.53). Usually N will be noninteger. Therefore, we have to round N upward to the nearest integer, then search for a new k (which will be larger than the given k) such that (10.53) is satisfied exactly with the integer N. This guarantees that the pass-band specifications will be met exactly, whereas the stop-band specifications will be exceeded. The search for new k is performed numerically, as explained in Section 10.5.
4. Compute z_l, p_l, and C as given by (10.54) and (10.52).
5. Compute the zeros of $H^{\rm L}(s)$ as
$$u_{2l-1} = jp_l,\ u_{2l} = -jp_l, \quad 1 \le l \le \lfloor N/2 \rfloor. \tag{10.58}$$
6. Compute the poles of $H^{\rm L}(s)$ as follows:
   (a) Solve the polynomial equation
$$\prod_{l=1}^{\lfloor N/2 \rfloor} (s^2 + p_l^2)^2 + (\varepsilon C)^2(-s^2)^{(N \bmod 2)} \prod_{l=1}^{\lfloor N/2 \rfloor} (s^2 + z_l^2)^2 = 0. \tag{10.59}$$
   (b) Take the poles v_l of $H^{\rm L}(s)$ as the subset of roots with negative real parts (there are always exactly N such roots).
7. Let
$$H_0 = \begin{cases} (1+\varepsilon^2)^{-1/2}\dfrac{\prod_{l=1}^{N}(-v_l)}{\prod_{l=1}^{N}(-u_l)}, & N \text{ even}, \\ \dfrac{\prod_{l=1}^{N}(-v_l)}{\prod_{l=1}^{N-1}(-u_l)}, & N \text{ odd}. \end{cases} \tag{10.60}$$
8. Finally, the transfer function of the elliptic filter is given by
$$H^{\rm L}(s) = H_0 \frac{\prod_{l=1}^{2\lfloor N/2 \rfloor}(s-u_l)}{\prod_{l=1}^{N}(s-v_l)}. \tag{10.61}$$

Example 10.4 Design a low-pass elliptic filter to meet the same specifications as in Example 10.1, therefore having the same selectivity and discrimination factors. We get
$$\varepsilon = 0.04475, \quad \omega_0 = 1\,\text{rad/s}, \quad k = 0.5486, \quad N = 6.$$
The poles of the resulting filter are
$$v_{1,2} = -0.1259 \pm j0.8386, \quad v_{3,4} = -0.4995 \pm j0.8914,$$
$$v_{5,6} = -1.2454 \pm j0.5955.$$
The zeros of the filter are
$$u_{1,2} = \pm j1.0353, \quad u_{3,4} = \pm j1.4142, \quad u_{5,6} = \pm j3.8637.$$
The magnitude response of the filter is shown in Figure 10.12. Part a shows the response in the frequency range 0 to 4 rad/s; part b shows the pass band at an expanded scale. The filter meets the pass-band specification exactly, but its transition band is slightly narrower than the specification. □

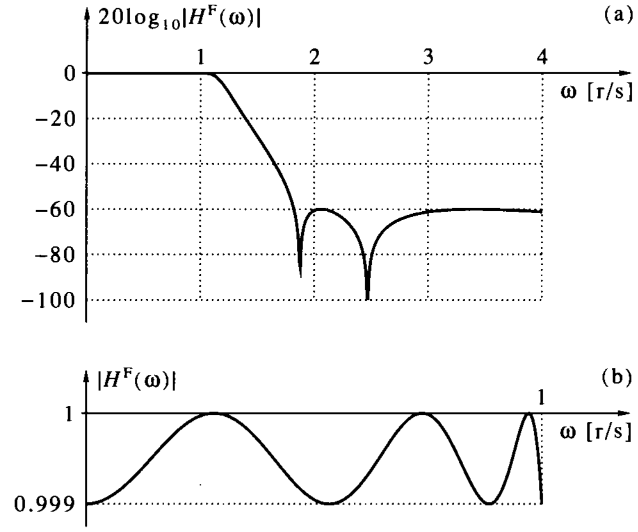

Figure 10.12 Magnitude response of the elliptic filter in Example 10.4: (a) full range; (b) pass-band details.

10.5 MATLAB Programs for Analog Low-Pass Filters

Table 10.2 summarizes the main properties and design equations of the four analog filters described in the preceding sections.

Filter	Butterworth	Chebyshev-I	Chebyshev-II	Elliptic
Pass band	monotone	equiripple	monotone	equiripple
Stop band	monotone	monotone	equiripple	equiripple
Order	(10.18)	(10.35)	(10.47)	(10.53)
ω_0	(10.19)	(10.33)	(10.45)	1
ε	N/A	(10.33)	(10.45)	(10.57)
K	N/A	N/A	N/A	(10.50)
Poles	(10.12)	(10.27)	(10.40)	(10.59)
Zeros	none	none	(10.41)	(10.58)
$H^L(s)$	(10.11)	(10.29)	(10.42)	(10.60), (10.61)

Table 10.2 Summary of properties and design equations of analog low-pass filter.

For a given set of specification parameters (band-edge frequencies and tolerances), the elliptic filter always has the smallest order of the four filter classes. Therefore, the elliptic filter is usually the preferred class for general IIR filtering applications. The other three classes are appropriate when monotone magnitude response is required in certain bands. If monotone response is required in the pass band, Chebyshev-II filter is appropriate; if monotone response is required in the stop band, Chebyshev-I filter is appropriate; if monotone response is required for all frequencies, Butterworth filter is appropriate.

The procedure `analoglp` in Program 10.1 computes the numerator and denominator polynomials of the four low-pass filter classes, as well as the poles, zeros, and constant gain. The implementation is straightforward. First, the poles and the constant gain are computed, depending on the filter class. In the case of Chebyshev-II filter, the zeros are computed as well. In the case of elliptic filter, a call is made to `elliplp`; see Program 10.2. This program implements the design of a low-pass elliptic filter as described in Section 10.4. Finally, the poles, zeros, and constant gain are expanded to form the two polynomials.

The procedure `lpspec` in Program 10.3 computes the parameters of a low-pass filter of one of the four classes according to given specifications. The inputs to the program are the four parameters ω_p, ω_s, δ_p, δ_s. The program provides the filter's order N, the frequency ω_0, the parameter ε, and, for elliptic filters, the parameter $m = k^2$. In the case of elliptic filter, a call is made to `ellord`, shown in Program 10.4. This program first computes the order N, using formula (10.53). Since the right side of (10.53) is not an integer in general, N is taken as the nearest larger integer. Then a search is performed over m (recall that $m = k^2$) such that (10.53) is satisfied exactly. The search is performed as follows:

1. First, m is increased in a geometric series, by 1.1 each time, until the right side of (10.53) becomes larger than N. The original m and m thus found bracket the value of m to be computed.
2. The method of *false position* is then used for finding the exact m within the brackets. In this method, a straight line is passed between the two end points and m is found such that the straight line has ordinate equal to N. With this m, the right side of (10.53) is computed again. If it is less than N, the lower end point of the bracketing interval is moved to the new m. If it is greater than N, the upper end point of the bracketing interval is moved to the new m. This iteration is terminated when the right side of (10.53) is equal to N to within 10^{-6}. The iteration is guaranteed to converge, since the right side of (10.53) is a monotone increasing function of m.

10.6 Frequency Transformations

The design of analog filters other than low pass is usually done by designing a low-pass filter of the desired class first (Butterworth, Chebyshev, or elliptic), and then transforming the resulting filter to get the desired frequency response: high pass, band pass, or band stop. Transformations of this kind are called *frequency transformations*. We first define frequency transformations in general, and then discuss special cases.

Let $f(\cdot)$ be a given rational function and let s be the Laplace transform variable. Define a transformed complex variable $\tilde{s}$ by

$$s = f(\tilde{s}).$$

This definition is *implicit*: It expresses s in terms of $\tilde{s}$, but it should be understood as defining $\tilde{s}$ in terms of s. Each of the two complex variables has a frequency variable associated with it, that is,

$$s = j\omega, \quad \tilde{s} = j\tilde{\omega}.$$

The corresponding frequency transformation is therefore

$$\omega = -jf(j\tilde{\omega}).$$

Let $H^{\mathrm{L}}(s)$ denote the transfer function of a low-pass filter. Define the transformed filter $\tilde{H}^{\mathrm{L}}(\tilde{s})$ by

$$\tilde{H}^{\mathrm{L}}(\tilde{s}) = H^{\mathrm{L}}(s)\Big|_{s=f(\tilde{s})}. \tag{10.62}$$

The equivalent frequency-domain transformation is

$$\tilde{H}^{\mathrm{F}}(\tilde{\omega}) = H^{\mathrm{F}}(\omega)\Big|_{\omega=-jf(\tilde{\omega})}. \tag{10.63}$$

Equation (10.62) should be interpreted in the following manner: Each occurrence of the variable s in the formula for $H^{\mathrm{L}}(s)$ is replaced by the formula $f(\tilde{s})$. This yields a new function in the variable $\tilde{s}$, which is denoted by $\tilde{H}^{\mathrm{L}}(\tilde{s})$. Equation (10.63) should be interpreted in a similar manner.

Example 10.5 Let

$$H^{\mathrm{L}}(s) = \frac{1}{s+3}, \quad s = \frac{2}{\tilde{s}}.$$

Then

$$\tilde{H}^{\mathrm{L}}(\tilde{s}) = \frac{1}{(2/\tilde{s})+3} = \frac{\tilde{s}}{3\tilde{s}+2}.$$

As we see, $H^{\mathrm{L}}(s)$ is a low-pass filter, and $\tilde{H}^{\mathrm{L}}(\tilde{s})$ is a high-pass filter. □

Since we have restricted $f(\cdot)$ to be a rational function, and since $H^{\mathrm{L}}(s)$ is rational, it follows that $\tilde{H}^{\mathrm{L}}(\tilde{s})$ is a rational function of $\tilde{s}$. Moreover, $f(\cdot)$ must preserve the stability of the filter. Thus, the left half s plane should be transformed to the left half $\tilde{s}$ plane, the right half s plane should be transformed to the right half $\tilde{s}$ plane, and the imaginary axis $j\omega$ should be transformed to the imaginary axis $j\tilde{\omega}$. The behavior of $\tilde{H}^{\mathrm{F}}(\tilde{\omega})$ must be according to the desired filter type in the transformed domain: high pass, band pass, or band stop. Finally, the function $f(\cdot)$ should have the minimal possible order, to minimize the complexity of the transformed filter.

The general procedure for designing an analog filter by a frequency transformation is as follows. Let us call the domain of s and ω the *design domain*, and the domain of $\tilde{s}$ and $\tilde{\omega}$ the *target domain*. It is convenient to think of the low-pass filter as normalized, that is, to think of s and ω as dimensionless numbers. The target-domain quantities $\tilde{s}$ and $\tilde{\omega}$ are measured, as usual, in radians per second. The specifications are assumed to be given in the target domain. The first step is then to convert them to specifications of a low-pass filter in the design domain, that is, to $\delta_{\mathrm{p}}, \delta_{\mathrm{s}}, \omega_{\mathrm{p}}, \omega_{\mathrm{s}}$. This conversion depends on the filter type. The next step is to design the low-pass filter $H^{\mathrm{L}}(s)$ according to these specifications, using one of the methods studied in the preceding sections. The final step is to transform the design to the target domain, using (10.62). The specific procedures for the different transformation types are described next.

10.6.1 Low-Pass to Low-Pass Transformation

Low-pass to low-pass frequency transformation is not strictly needed, because the design methods of low-pass filters we have presented are sufficiently general; however, we present it for completeness. The low-pass to low-pass transformation is

$$s = \frac{\tilde{s}}{\omega_{\mathrm{c}}}, \quad \omega = \frac{\tilde{\omega}}{\omega_{\mathrm{c}}}, \tag{10.64}$$

where ω_{c} is a positive parameter. This transformation stretches (or contracts) the frequency axis by a constant factor. For example, if $H^{\mathrm{L}}(s)$ is a low-pass filter with band-edge frequencies ω_{p} and ω_{s}, then $\tilde{H}^{\mathrm{L}}(\tilde{s}) = H^{\mathrm{L}}(\tilde{s}/\omega_{\mathrm{c}})$ is a low-pass filter with band-edge frequencies $\omega_{\mathrm{c}}\omega_{\mathrm{p}}$ and $\omega_{\mathrm{c}}\omega_{\mathrm{s}}$. The choice of ω_{c} is arbitrary; however, the following are convenient choices for the different filter classes:

1. For a Butterworth filter choose $\omega_{\mathrm{c}} = \tilde{\omega}_0$.

2. For a Chebyshev-I filter choose $\omega_c = \widetilde{\omega}_p$.
3. For a Chebyshev-II filter choose $\omega_c = \widetilde{\omega}_s$.
4. For an elliptic filter choose $\omega_c = \sqrt{\widetilde{\omega}_p \widetilde{\omega}_s}$.

All these choices cause the parameter ω_0 of the low-pass filter to be equal to 1.

Low-pass to low-pass transformation can be carried out in terms of either the numerator and denominator polynomials or the pole-zero factorization of the low-pass filter. Let

$$H^{L}(s) = \frac{\sum_{k=0}^{q} b_k s^{q-k}}{\sum_{k=0}^{p} a_k s^{p-k}} = \frac{b_0 \prod_{k=1}^{q}(s - u_k)}{\prod_{k=1}^{p}(s - v_k)}, \quad a_0 = 1. \tag{10.65}$$

Then the transformed filter is given by

$$\widetilde{H}^{L}(\widetilde{s}) = \frac{\omega_c^{p-q} \sum_{k=0}^{q} b_k \omega_c^{k} \widetilde{s}^{q-k}}{\sum_{k=0}^{p} a_k \omega_c^{k} \widetilde{s}^{p-k}} = \frac{b_0 \omega_c^{p-q} \prod_{k=1}^{q}(\widetilde{s} - u_k \omega_c)}{\prod_{k=1}^{p}(\widetilde{s} - v_k \omega_c)}. \tag{10.66}$$

10.6.2 Low-Pass to High-Pass Transformation

The transformation

$$s = \frac{\omega_c}{\widetilde{s}}, \quad \omega = -\frac{\omega_c}{\widetilde{\omega}}, \tag{10.67}$$

where ω_c is a positive parameter, is low pass to high pass. This is illustrated in Figure 10.13, which shows how a low-pass Butterworth filter (in the ω domain) is transformed to a high-pass Butterworth filter (in the $\widetilde{\omega}$ domain). For convenience, we have omitted the sign reversal of the frequencies. The sign reversal is immaterial, since the magnitude response is a symmetric function of the frequency.

Suppose we wish to design a high-pass filter with band edge frequencies $\widetilde{\omega}_p$ and $\widetilde{\omega}_s$. We do it by designing a low-pass filter $H^{L}(s)$ with $\omega_p = \omega_c/\widetilde{\omega}_p$ and $\omega_s = \omega_c/\widetilde{\omega}_s$. The tolerance parameters δ_p and δ_s used for the low-pass design are the same as the desired tolerances of the high-pass filter. Then $\widetilde{H}^{L}(\widetilde{s}) = H^{L}(\omega_c/\widetilde{s})$ will be a high-pass filter with the given band-edge frequencies and tolerance parameters. The convenient choices for Butterworth, Chebyshev, and elliptic filters are the same as for the low-pass to low-pass transformation.

The low-pass to high-pass transformation is a first-order rational function, so the order of $\widetilde{H}^{L}(\widetilde{s})$ is the same as that of $H^{L}(s)$. To show that it preserves stability, let $\widetilde{s} = \widetilde{\sigma} + j\widetilde{\omega}$ be the decomposition of $\widetilde{s}$ into real and imaginary parts. Then the corresponding decomposition of s is

$$s = \sigma + j\omega = \frac{\omega_c \widetilde{\sigma}}{\widetilde{\sigma}^2 + \widetilde{\omega}^2} - j\frac{\omega_c \widetilde{\omega}}{\widetilde{\sigma}^2 + \widetilde{\omega}^2}. \tag{10.68}$$

We find that σ and $\widetilde{\sigma}$ always have the same sign, which implies the stability property.

Low-pass to high-pass transformation can be carried out in terms of either the numerator and denominator polynomials or the pole-zero factorization of the low-pass filter. Let the filter $H^{L}(s)$ be as in (10.65). Then the transformed filter is given by

$$\widetilde{H}^{L}(\widetilde{s}) = \frac{\widetilde{s}^{p-q} \sum_{k=0}^{q} b_k \omega_c^{q-k} \widetilde{s}^{k}}{\sum_{k=0}^{p} a_k \omega_c^{p-k} \widetilde{s}^{k}} = \frac{b_0 \prod_{k=1}^{q}(-u_k)}{\prod_{k=1}^{p}(-v_k)} \cdot \frac{\widetilde{s}^{p-q} \prod_{k=1}^{q}(\widetilde{s} - \omega_c u_k^{-1})}{\prod_{k=1}^{p}(\widetilde{s} - \omega_c v_k^{-1})}. \tag{10.69}$$

As we see, the number of zeros of a high-pass filter is always equal to the number of poles. If $H^{L}(s)$ is Butterworth or Chebyshev-I, it has no zeros. Therefore, in this case, $\widetilde{H}^{L}(\widetilde{s})$ will have p zeros at $\widetilde{s} = 0$. If $H^{L}(s)$ is Chebyshev-II or elliptic, it has $2\lfloor p/2 \rfloor$

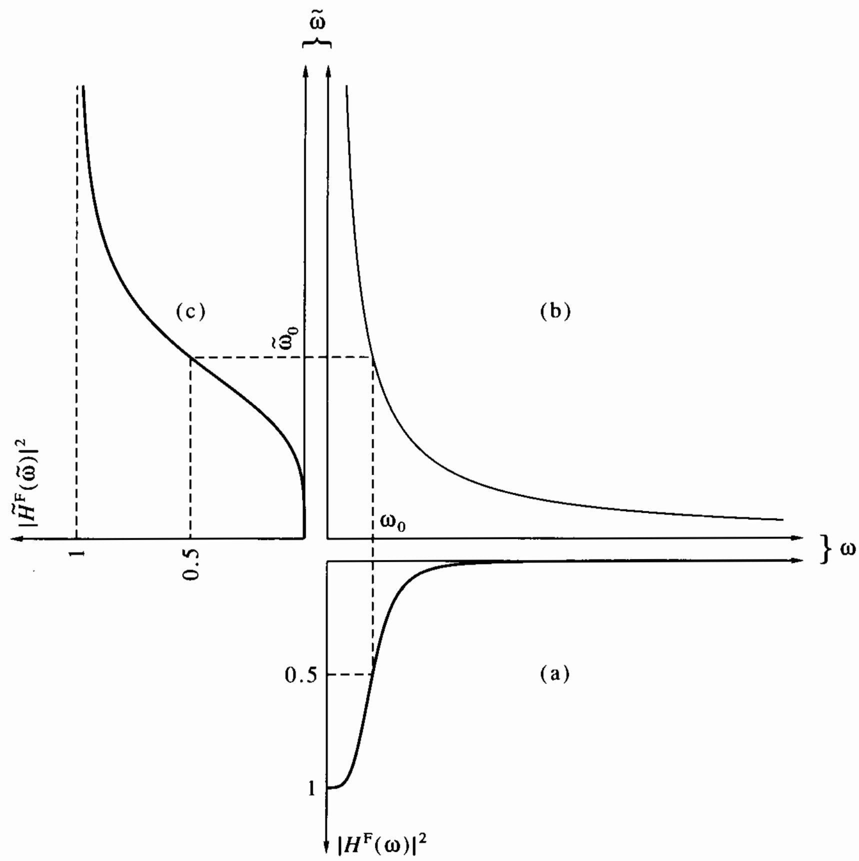

Figure 10.13 Low-pass to high-pass transformation of an analog filter: (a) frequency response of the low-pass filter (amplitude axis pointing downward); (b) $\tilde{\omega}$ as a function of ω; (c) frequency response of the high-pass filter (amplitude axis pointing to left).

zeros on the imaginary axis. These will be transformed to zeros on the imaginary axis according to (10.69). Also, there will be an additional zero at $\tilde{s} = 0$ if p is odd, and no additional zero if p is even.

Formula (10.69) is convenient for computer implementation of the transformation, as we shall see in Section 10.6.5. In summary, the design procedure for a high-pass analog filter is as follows:

High-pass analog filter design procedure

1. Choose ω_c as an arbitrary positive parameter, for example, $\omega_c = 1$.
2. Given the high-pass filter specifications $\tilde{\omega}_p$, $\tilde{\omega}_s$, $\tilde{\delta}_p$, $\tilde{\delta}_s$, let

$$\delta_p = \tilde{\delta}_p, \quad \delta_s = \tilde{\delta}_s, \quad \omega_p = \frac{\omega_c}{\tilde{\omega}_p}, \quad \omega_s = \frac{\omega_c}{\tilde{\omega}_s}. \tag{10.70}$$

3. Design a low-pass analog filter $H^L(s)$ to meet the specifications ω_p, ω_s, δ_p, δ_s.
4. Obtain the analog high-pass filter $\tilde{H}^L(\tilde{s})$, using (10.69).

Example 10.6 We wish to design a high-pass filter according to the specification

$$\widetilde{\omega}_s = 0.5, \quad \widetilde{\omega}_p = 5, \quad \delta_s = 0.01, \quad \delta_p = 0.01.$$

We get, with $\omega_c = 1$,

$$\omega_p = 0.2, \quad \omega_s = 2.$$

Therefore,

$$d = 0.01432, \quad k = 0.1.$$

For Butterworth filter we get

$$N = 3,\ \omega_0 = 0.3829,\ H^L(s) = \frac{0.05614}{s^3 + 0.7658s^2 + 0.2932s + 0.05614}.$$

The low-pass to high-pass transformation gives

$$\widetilde{H}^L(\widetilde{s}) = \frac{\widetilde{s}^3}{\widetilde{s}^3 + 5.2231\widetilde{s}^2 + 13.6405\widetilde{s} + 17.8115}.$$

For Chebyshev-I filter we get

$$N = 3,\ \omega_0 = 0.2,\ \varepsilon = 0.1425,\ H^L(s) = \frac{0.01404}{s^3 + 0.4005s^2 + 0.1102s + 0.01404}.$$

The low-pass to high-pass transformation gives

$$\widetilde{H}^L(\widetilde{s}) = \frac{\widetilde{s}^3}{\widetilde{s}^3 + 7.8507\widetilde{s}^2 + 28.5325\widetilde{s} + 71.2461}.$$

For Chebyshev-II filter we get

$$N = 3,\ \omega_0 = 2,\ \varepsilon = 0.01,\ H^L(s) = \frac{0.06s^2 + 0.32}{s^3 + 1.3492s^2 + 0.9084s + 0.32}.$$

The low-pass to high-pass transformation gives

$$\widetilde{H}^L(\widetilde{s}) = \frac{\widetilde{s}^3 + 0.1875\widetilde{s}}{\widetilde{s}^3 + 2.8385\widetilde{s}^2 + 4.2160\widetilde{s} + 3.1248}.$$

For an elliptic filter we get

$$N = 3,\ \omega_0 = 0.2,\ \varepsilon = 0.1425,\ m = 0.0773,$$

$$H^L(s) = \frac{0.02116s^2 + 0.01446}{s^3 + 0.3958s^2 + 0.1084s + 0.01446}.$$

The low-pass to high-pass transformation gives

$$\widetilde{H}^L(\widetilde{s}) = \frac{\widetilde{s}^3 + 1.4631\widetilde{s}}{\widetilde{s}^3 + 7.4970\widetilde{s}^2 + 27.3713\widetilde{s} + 69.1456}.$$

□

10.6.3 Low-Pass to Band-Pass Transformation

The transformation

$$s = \frac{\widetilde{s}^2 + \omega_l\omega_h}{\widetilde{s}(\omega_h - \omega_l)}, \quad \omega = \frac{\widetilde{\omega}^2 - \omega_l\omega_h}{\widetilde{\omega}(\omega_h - \omega_l)}, \tag{10.71}$$

where ω_h and ω_l are positive parameters satisfying $\omega_h > \omega_l$, is low pass to band pass, as illustrated in Figure 10.14, which shows how a low-pass Butterworth filter (in the ω domain) is transformed to a band-pass Butterworth filter (in the $\widetilde{\omega}$ domain). Since (10.71) is a second-order rational function, to each ω there correspond two values of $\widetilde{\omega}$. Therefore, a total of four values of $\widetilde{\omega}$ correspond to $\pm\omega$. Figure 10.14 shows the two positive values of $\widetilde{\omega}$ corresponding to each positive ω. Note the asymmetric form of the band-pass filter with respect to the center frequency $\sqrt{\omega_l\omega_h}$.

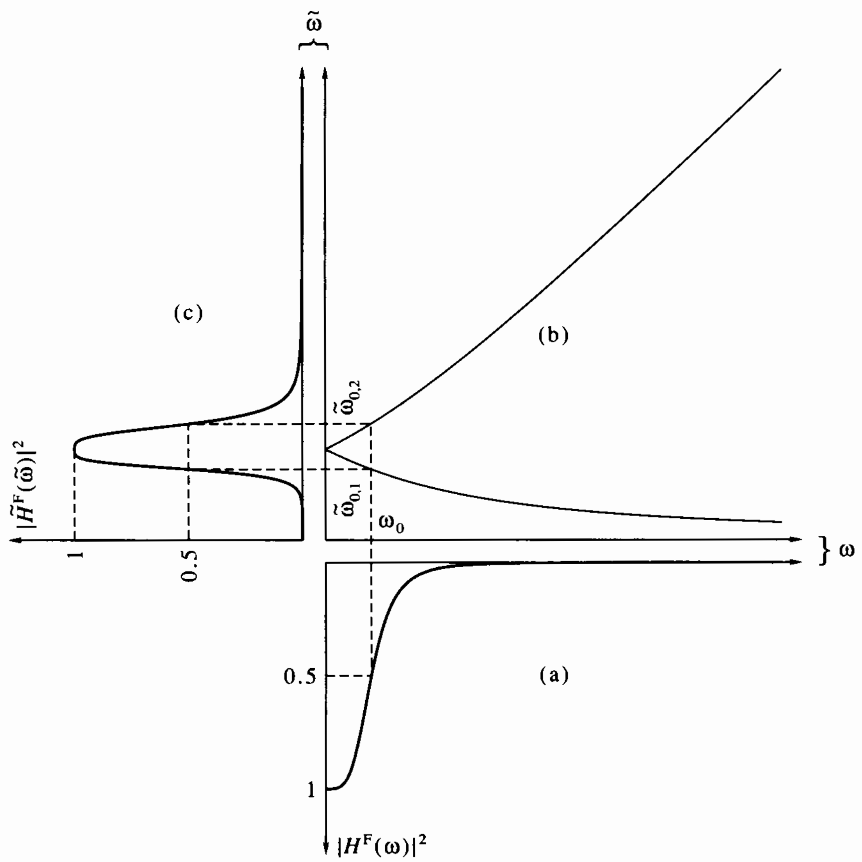

Figure 10.14 Low-pass to band-pass transformation of an analog filter: (a) frequency response of the low-pass filter (amplitude axis pointing downward); (b) $\tilde{\omega}$ as a function of ω; (c) frequency response of the band-pass filter (amplitude axis pointing to left).

Suppose we wish to design a band-pass filter with tolerance parameters $\tilde{\delta}_{\mathrm{p}}$, $\tilde{\delta}_{\mathrm{s},1}$, $\tilde{\delta}_{\mathrm{s},2}$, and band-edge frequencies $\tilde{\omega}_{\mathrm{s},1}$, $\tilde{\omega}_{\mathrm{p},1}$, $\tilde{\omega}_{\mathrm{p},2}$, $\tilde{\omega}_{\mathrm{s},2}$. In general, the specifications of the two transition bands and the two stop bands are independent, therefore possibly different. On the other hand, the low-pass filter has only one transition band and one stop band. Therefore, the low-pass design must be such that all specification parameters of the band-pass filter are met or surpassed. This implies that the tolerance parameters of the low-pass filter are to be chosen as

$$\delta_{\mathrm{p}} = \tilde{\delta}_{\mathrm{p}}, \quad \delta_{\mathrm{s}} = \min\{\tilde{\delta}_{\mathrm{s},1}, \tilde{\delta}_{\mathrm{s},2}\}. \tag{10.72}$$

The transformation parameters ω_{l}, ω_{h} should be chosen in one of two ways:

1. Such that both $\tilde{\omega}_{\mathrm{p},1}$ and $\tilde{\omega}_{\mathrm{p},2}$ will be transformed to the same ω_{p}, for example, to $\omega_{\mathrm{p}} = 1$. The choice

$$\omega_{\mathrm{l}} = \tilde{\omega}_{\mathrm{p},1}, \quad \omega_{\mathrm{h}} = \tilde{\omega}_{\mathrm{p},2} \tag{10.73}$$

accomplishes this, since

$$\frac{\tilde{\omega}_{\mathrm{p},1}^2 - \tilde{\omega}_{\mathrm{p},1}\tilde{\omega}_{\mathrm{p},2}}{\tilde{\omega}_{\mathrm{p},1}(\tilde{\omega}_{\mathrm{p},2} - \tilde{\omega}_{\mathrm{p},1})} = -1, \quad \frac{\tilde{\omega}_{\mathrm{p},2}^2 - \tilde{\omega}_{\mathrm{p},1}\tilde{\omega}_{\mathrm{p},2}}{\tilde{\omega}_{\mathrm{p},2}(\tilde{\omega}_{\mathrm{p},2} - \tilde{\omega}_{\mathrm{p},1})} = 1, \tag{10.74}$$

and the frequencies 1 and -1 are equivalent. With this choice, the stop-band frequencies are transformed to

$$\omega_{s,1} = \frac{\widetilde{\omega}_{s,1}^2 - \widetilde{\omega}_{p,1}\widetilde{\omega}_{p,2}}{\widetilde{\omega}_{s,1}(\widetilde{\omega}_{p,2} - \widetilde{\omega}_{p,1})}, \quad \omega_{s,2} = \frac{\widetilde{\omega}_{s,2}^2 - \widetilde{\omega}_{p,1}\widetilde{\omega}_{p,2}}{\widetilde{\omega}_{s,2}(\widetilde{\omega}_{p,2} - \widetilde{\omega}_{p,1})}. \tag{10.75}$$

A smaller value of ω_s corresponds to a narrower transition band. Therefore, the stop-band frequency of the low-pass filter must be chosen as

$$\omega_s = \min\{|\omega_{s,1}|, |\omega_{s,2}|\}, \tag{10.76}$$

to satisfy both transition-band requirements.

2. Such that both $\widetilde{\omega}_{s,1}$ and $\widetilde{\omega}_{s,2}$ will be transformed to the same ω_s, for example, to $\omega_s = 1$. The choice

$$\omega_l = \widetilde{\omega}_{s,1}, \quad \omega_h = \widetilde{\omega}_{s,2} \tag{10.77}$$

accomplishes this, since

$$\frac{\widetilde{\omega}_{s,1}^2 - \widetilde{\omega}_{s,1}\widetilde{\omega}_{s,2}}{\widetilde{\omega}_{s,1}(\widetilde{\omega}_{s,2} - \widetilde{\omega}_{s,1})} = -1, \quad \frac{\widetilde{\omega}_{s,2}^2 - \widetilde{\omega}_{s,1}\widetilde{\omega}_{s,2}}{\widetilde{\omega}_{s,2}(\widetilde{\omega}_{s,2} - \widetilde{\omega}_{s,1})} = 1, \tag{10.78}$$

and the frequencies ± 1 are equivalent. With this choice, the pass-band frequencies are transformed to

$$\omega_{p,1} = \frac{\widetilde{\omega}_{p,1}^2 - \widetilde{\omega}_{s,1}\widetilde{\omega}_{s,2}}{\widetilde{\omega}_{p,1}(\widetilde{\omega}_{s,2} - \widetilde{\omega}_{s,1})}, \quad \omega_{p,2} = \frac{\widetilde{\omega}_{p,2}^2 - \widetilde{\omega}_{s,1}\widetilde{\omega}_{s,2}}{\widetilde{\omega}_{p,2}(\widetilde{\omega}_{s,2} - \widetilde{\omega}_{s,1})}. \tag{10.79}$$

A larger value of ω_p corresponds to a narrower transition band. Therefore, the pass-band frequency of the low-pass filter must be chosen as

$$\omega_p = \max\{|\omega_{p,1}|, |\omega_{p,2}|\}, \tag{10.80}$$

to satisfy both transition-band requirements.

The low-pass to band-pass transformation is a second-order rational function, therefore the order of $\widetilde{H}^L(\widetilde{s})$ is twice that of $H^L(s)$. To show that it preserves stability, let $\widetilde{s} = \widetilde{\sigma} + j\widetilde{\omega}$ be the decomposition of $\widetilde{s}$ into real and imaginary parts. Then the corresponding decomposition of s is

$$s = \sigma + j\omega = \frac{\widetilde{\sigma}}{\omega_h - \omega_l}\left(1 + \frac{\omega_l\omega_h}{\widetilde{\sigma}^2 + \widetilde{\omega}^2}\right) + j\frac{\widetilde{\omega}}{\omega_h - \omega_l}\left(1 - \frac{\omega_l\omega_h}{\widetilde{\sigma}^2 + \widetilde{\omega}^2}\right). \tag{10.81}$$

We get that σ and $\widetilde{\sigma}$ always have the same sign, which implies the stability property.

The low-pass to band-pass transformation is not convenient to express when $H^L(s)$ is given as a ratio of two polynomials, but it is easy when $H^L(s)$ is given in a factored form. For a single zero u_k we have

$$s - u_k = \frac{\widetilde{s}^2 + \omega_l\omega_h}{\widetilde{s}(\omega_h - \omega_l)} - u_k = \frac{\widetilde{s}^2 - u_k(\omega_h - \omega_l)\widetilde{s} + \omega_l\omega_h}{\widetilde{s}(\omega_h - \omega_l)}, \tag{10.82}$$

and similarly for v_k. Let the pole-zero factorization of $H^L(s)$ be as in (10.65). Then we get, by substituting (10.82) for all poles and zeros,

$$\widetilde{H}^L(\widetilde{s}) = \frac{b_0(\omega_h - \omega_l)^{p-q}\widetilde{s}^{p-q}\prod_{k=1}^{q}[\widetilde{s}^2 - u_k(\omega_h - \omega_l)\widetilde{s} + \omega_l\omega_h]}{\prod_{k=1}^{p}[\widetilde{s}^2 - v_k(\omega_h - \omega_l)\widetilde{s} + \omega_l\omega_h]}. \tag{10.83}$$

Now (10.83) can be expanded to a ratio of two polynomials, or factored further to first-order complex factors, as needed.

The number of zeros of a band-pass filter depends on the class of the low-pass filter from which it is derived, and whether its order is even or odd. In any case, the zeros are all on the imaginary axis. See Problem 10.16 for further discussion of this point. In summary, the design procedure for a band-pass analog filter is as follows:

Band-pass analog filter design procedure

1. Given the band-pass filter specifications $\tilde{\omega}_{p,1}$, $\tilde{\omega}_{s,1}$, $\tilde{\omega}_{p,2}$, $\tilde{\omega}_{s,2}$, $\tilde{\delta}_p$, $\tilde{\delta}_{s,1}$, $\tilde{\delta}_{s,2}$, choose δ_p, δ_s according to (10.72), and ω_l, ω_h according to (10.73).
2. Let $\omega_p = 1$, and compute ω_s according to (10.75), (10.76).
3. Design a low-pass analog filter $H^L(s)$ to meet the specifications ω_p, ω_s, δ_p, δ_s, and find its poles, zeros, and constant gain.
4. Obtain the analog band-pass filter $\tilde{H}^L(\tilde{s})$, using (10.83).

Example 10.7 We wish to design a band-pass filter according to the specification

$$\tilde{\omega}_{s,1} = 0.2, \quad \tilde{\omega}_{p,1} = 0.5, \quad \tilde{\omega}_{p,2} = 2, \quad \tilde{\omega}_{s,2} = 6, \quad \tilde{\delta}_{s,1} = \tilde{\delta}_{s,2} = 0.1, \quad \delta_p = 0.1.$$

We get, with $\omega_l = \tilde{\omega}_{p,1}$, $\omega_h = \tilde{\omega}_{p,2}$,

$$\omega_p = 1, \quad \omega_s = \min\{3.2, 3.8889\} = 3.2.$$

Therefore,

$$d = 0.048677, \quad k = 0.3125.$$

For Butterworth filter we get

$$N = 3, \quad \omega_0 = 1.2734, \quad H^L(s) = \frac{2.0647}{s^3 + 2.5467s^2 + 3.2429s + 2.0647}.$$

The low-pass to band-pass transformation gives

$$\tilde{H}^L(\tilde{s}) = \frac{6.9685\tilde{s}^3}{\tilde{s}^6 + 3.8201\tilde{s}^5 + 10.2966\tilde{s}^4 + 14.6087\tilde{s}^3 + 10.2966\tilde{s}^2 + 3.8201\tilde{s} + 1}.$$

For Chebyshev-I filter we get

$$N = 3, \ \omega_0 = 1, \ \varepsilon = 0.4843, \ H^L(s) = \frac{0.5162}{s^3 + 1.0213s^2 + 1.2716s + 0.5162}.$$

The low-pass to band-pass transformation gives

$$\tilde{H}^L(\tilde{s}) = \frac{1.7421\tilde{s}^3}{\tilde{s}^6 + 1.5320\tilde{s}^5 + 5.8610\tilde{s}^4 + 4.8062\tilde{s}^3 + 5.8610\tilde{s}^2 + 1.5320\tilde{s} + 1}.$$

For Chebyshev-II filter we get

$$N = 3, \ \omega_0 = 3.2, \ \varepsilon = 0.1005, \ H^L(s) = \frac{0.9648s^2 + 13.1732}{s^3 + 4.4972s^2 + 9.6471s + 13.1732}.$$

The low-pass to band-pass transformation gives

$$\tilde{H}^L(\tilde{s}) = \frac{1.4472\tilde{s}^5 + 47.3542\tilde{s}^3 + 1.4472\tilde{s}}{\tilde{s}^6 + 6.7458\tilde{s}^5 + 24.7059\tilde{s}^4 + 57.9513\tilde{s}^3 + 24.7059\tilde{s}^2 + 6.7458\tilde{s} + 1}.$$

For an elliptic filter we get

$$N = 2, \ \omega_0 = 1, \ \varepsilon = 0.4843, \ m = 0.1770, \ H^L(s) = \frac{0.1s^2 + 1.0772}{s^2 + 1.0678s + 1.1969}.$$

The low-pass to band-pass transformation gives

$$\tilde{H}^L(\tilde{s}) = \frac{0.1\tilde{s}^4 + 2.6237\tilde{s}^2 + 0.1}{\tilde{s}^4 + 1.6017\tilde{s}^3 + 4.6930\tilde{s}^2 + 1.6017\tilde{s} + 1}.$$

□

10.6.4 Low-Pass to Band-Stop Transformation

The transformation

$$s = \frac{\tilde{s}(\omega_{\mathrm{h}} - \omega_{\mathrm{l}})}{\tilde{s}^2 + \omega_l \omega_h}, \quad \omega = \frac{\tilde{\omega}(\omega_{\mathrm{h}} - \omega_{\mathrm{l}})}{\omega_{\mathrm{l}}\omega_{\mathrm{h}} - \tilde{\omega}^2}, \tag{10.84}$$

where ω_{h} and ω_{l} are positive parameters satisfying $\omega_{\mathrm{h}} > \omega_{\mathrm{l}}$, is low pass to band stop. This is illustrated in Figure 10.15, which shows how a low-pass Butterworth filter (in the ω domain) is transformed to a band-stop Butterworth filter (in the $\tilde{\omega}$ domain). As in the case of low-pass to band-pass transformation, to each ω there correspond two values of $\tilde{\omega}$. Therefore, a total of four values of $\tilde{\omega}$ correspond to $\pm\omega$. Figure 10.15 shows the two positive values of $\tilde{\omega}$ corresponding to each positive ω. Note the asymmetric form of the band-stop filter with respect to the center frequency $\sqrt{\omega_{\mathrm{l}}\omega_{\mathrm{h}}}$.

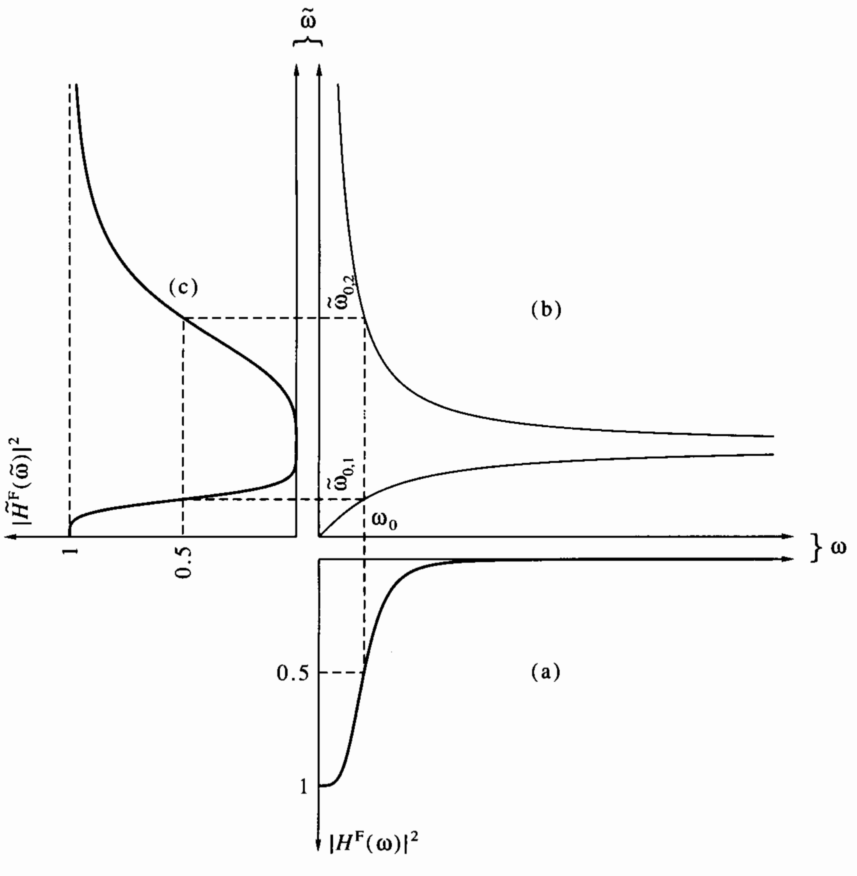

Figure 10.15 Low-pass to band-stop transformation of an analog filter: (a) frequency response of the low-pass filter (amplitude axis pointing downward); (b) $\tilde{\omega}$ as a function of ω; (c) frequency response of the band-stop filter (amplitude axis pointing to left).

Suppose we wish to design a band-stop filter with tolerance parameters $\tilde{\delta}_{\mathrm{p},1}$, $\tilde{\delta}_{\mathrm{p},2}$, $\tilde{\delta}_{\mathrm{s}}$, and band-edge frequencies $\tilde{\omega}_{\mathrm{p},1}$, $\tilde{\omega}_{\mathrm{s},1}$, $\tilde{\omega}_{\mathrm{s},2}$, $\tilde{\omega}_{\mathrm{p},2}$. As in the case of band-pass filter, the low-pass design must be such that *all* specification parameters of the band-stop filter are met or surpassed. This implies that the tolerance parameters of the low-pass filter are to be chosen as

$$\delta_p = \min\{\tilde{\delta}_{p,1}, \tilde{\delta}_{p,2}\}, \quad \delta_s = \tilde{\delta}_s. \tag{10.85}$$

The transformation parameters ω_l, ω_h should be chosen in one of two ways:

1. Such that both $\tilde{\omega}_{p,1}$ and $\tilde{\omega}_{p,2}$ will be transformed to $\omega_p = 1$. The choice

$$\omega_l = \tilde{\omega}_{p,1}, \quad \omega_h = \tilde{\omega}_{p,2} \tag{10.86}$$

accomplishes this. With this choice, the stop-band frequencies are transformed to

$$\omega_{s,1} = \frac{\tilde{\omega}_{s,1}(\tilde{\omega}_{p,2} - \tilde{\omega}_{p,1})}{\tilde{\omega}_{p,1}\tilde{\omega}_{p,2} - \tilde{\omega}_{s,1}^2}, \quad \omega_{s,2} = \frac{\tilde{\omega}_{s,2}(\tilde{\omega}_{p,2} - \tilde{\omega}_{p,1})}{\tilde{\omega}_{p,1}\tilde{\omega}_{p,2} - \tilde{\omega}_{s,2}^2}. \tag{10.87}$$

The stop-band frequency of the low-pass filter must be chosen as

$$\omega_s = \min\{|\omega_{s,1}|, |\omega_{s,2}|\}, \tag{10.88}$$

to satisfy both transition-band requirements.

2. Such that both $\tilde{\omega}_{s,1}$ and $\tilde{\omega}_{s,2}$ will be transformed to $\omega_s = 1$. The choice

$$\omega_l = \tilde{\omega}_{s,1}, \quad \omega_h = \tilde{\omega}_{s,2} \tag{10.89}$$

accomplishes this. With this choice, the pass-band frequencies are transformed to

$$\omega_{p,1} = \frac{\tilde{\omega}_{p,1}(\tilde{\omega}_{s,2} - \tilde{\omega}_{s,1})}{\tilde{\omega}_{s,1}\tilde{\omega}_{s,2} - \tilde{\omega}_{p,1}^2}, \quad \omega_{p,2} = \frac{\tilde{\omega}_{p,2}(\tilde{\omega}_{s,2} - \tilde{\omega}_{s,1})}{\tilde{\omega}_{s,1}\tilde{\omega}_{s,2} - \tilde{\omega}_{p,2}^2}. \tag{10.90}$$

The pass-band frequency of the low-pass filter must be chosen as

$$\omega_p = \max\{|\omega_{p,1}|, |\omega_{p,2}|\}, \tag{10.91}$$

to satisfy both transition-band requirements.

The low-pass to band-stop transformation can be performed by transforming from low-pass to high-pass first, using (10.67), followed by a low-pass to band-pass transformation (10.71). Since each of these transformations preserves stability, so does the low-pass to band-stop transformation.

Let us write the low-pass to band-stop transformation for $H^L(s)$ in a factored form, as we did for the low-pass to band-pass transformation. For a single zero u_k we have

$$s - u_k = \frac{\tilde{s}(\omega_h - \omega_l)}{\tilde{s}^2 + \omega_l\omega_h} - u_k = \frac{(-u_k)[\tilde{s}^2 - u_k^{-1}(\omega_h - \omega_l)\tilde{s} + \omega_l\omega_h]}{\tilde{s}^2 + \omega_l\omega_h}, \tag{10.92}$$

and similarly for v_k. Let the pole-zero factorization of $H^L(s)$ be as in (10.65). Then, substituting (10.92) for all poles and zeros, we can write

$$\tilde{H}^L(\tilde{s}) = \frac{b_0 \prod_{k=1}^{q}(-u_k)}{\prod_{k=1}^{p}(-v_k)} \cdot \frac{(\tilde{s}^2 + \omega_l\omega_h)^{p-q} \prod_{k=1}^{q}[\tilde{s}^2 - u_k^{-1}(\omega_h - \omega_l)\tilde{s} + \omega_l\omega_h]}{\prod_{k=1}^{p}[\tilde{s}^2 - v_k^{-1}(\omega_h - \omega_l)\tilde{s} + \omega_l\omega_h]}. \tag{10.93}$$

Now (10.93) can be expanded to a ratio of two polynomials, or factored further to first-order complex factors, as needed.

The number of zeros of a band-stop filter is always $2p$, and they are all on the imaginary axis. See Problem 10.17 for further discussion of this point. The low-pass to band-stop transformation can also be carried out by using the low-pass to high-pass transformation (10.69) and the low-pass to band-pass transformation (10.83) in succession. In summary, the design procedure for a band-stop analog filter is as follows:

Band-stop analog filter design procedure

1. Given the band-stop filter specifications $\tilde{\omega}_{p,1}$, $\tilde{\omega}_{s,1}$, $\tilde{\omega}_{p,2}$, $\tilde{\omega}_{s,2}$, $\tilde{\delta}_p$, $\tilde{\delta}_{p,1}$, $\tilde{\delta}_{p,2}$, choose δ_s, δ_s according to (10.85), and ω_l, ω_h according to (10.86).
2. Let $\omega_p = 1$, and compute ω_s according to (10.87), (10.88).
3. Design a low-pass analog filter $H^L(s)$ to meet the specifications ω_p, ω_s, δ_p, δ_s, and find its poles, zeros, and constant gain.
4. Obtain the analog band-pass filter $\tilde{H}^L(\tilde{s})$, using (10.93).

10.6.5 MATLAB Implementation of Frequency Transformations

The procedure `analogtr` in Program 10.5 implements the frequency transformation formulas. It uses the pole-zero factorization of the low-pass filter as input and provides both the pole-zero factorization and the expanded polynomials of the transformed filter. It uses (10.66) for low-pass to low-pass transformation, (10.69) for low-pass to high-pass transformation, and (10.83) for low-pass to band-pass transformation. For low-pass to band-stop transformation it calls itself recursively twice, first performing low-pass to high-pass transformation with $\omega_c = 1$, and then performing low-pass to band-pass transformation.

10.7 Impulse Invariant Transformation

We now turn our attention to the problem of transforming an analog IIR filter into the digital domain. We assume that we are given a real, causal, stable, rational filter $H^L(s)$, designed to meet given specifications. We wish to obtain a digital IIR filter $H^z(z)$ from $H^L(s)$. The main requirements for $H^z(z)$ are as follows:

1. It should be real, causal, stable, and rational.
2. Its order should not be greater than that of $H^L(s)$. This feature is not absolutely necessary but is highly desired, since we do not wish to increase the complexity of the digital filter beyond necessity.
3. Its frequency response should be close to that of the analog filter in the frequency range of interest (the relationship between analog and digital frequencies being $\theta = \omega T$).

In addition, the transformation method should be simple, convenient to implement, and applicable to all classes and types of analog filters.

The first method we present is called the *impulse invariant method.* Let $h(t)$ be the impulse response of an analog filter, and $h[n]$ the impulse response of the desired digital filter. The impulse invariant method defines $h[n]$ as proportional to the samples of $h(t)$; in other words,

$$h[n] = Th(nT). \tag{10.94}$$

The following procedure is used to obtain the digital filter's transfer function:

1. Compute the inverse Laplace transform of $H^L(s)$ to get $h(t)$.
2. Sample the impulse response at interval T and multiply by T to obtain $h[n]$.
3. Compute the z-transform of the sequence $h[n]$.

This sequence of operations is denoted by the operator

$$H^{z}(z) = \{Z_T H^{L}\}(z). \tag{10.95}$$

The name *impulse invariant* should be obvious from the definition. The constant scale factor T can be explained as follows. The DC gains of the analog and digital filters are given, respectively, by

$$H^{L}(0) = \int_0^{\infty} h(t)dt, \quad H^{z}(1) = \sum_{n=0}^{\infty} h[n] = \sum_{n=0}^{\infty} Th(nT). \tag{10.96}$$

The sum, including the factor T, can be regarded as a zero-order (Euler) approximation of the integral.[2] Thus, the factor T causes the two filters to have approximately equal DC gains.

Example 10.8 Consider the first-order analog filter $H^{L}(s) = \alpha/(s+\alpha)$. The corresponding impulse response is

$$h(t) = \alpha e^{-\alpha t} v(t),$$

so

$$h[n] = Th(nT) = \alpha T e^{-n\alpha T} v[n].$$

Therefore,

$$H^{z}(z) = \frac{\alpha T}{1 - e^{-\alpha T} z^{-1}}.$$

As we see, a first-order analog filter is transformed to a first-order digital filter, and stability is preserved, since $\alpha > 0$ implies $0 < e^{-\alpha T} < 1$. We shall soon see that these properties hold for general filters (not necessarily first order). The DC gain of $H^{z}(z)$ is

$$H^{z}(1) = \frac{\alpha T}{1 - e^{-\alpha T}}.$$

Since $e^{-\alpha T} \approx 1 - \alpha T$ if $\alpha T \ll 1$, we get that $H^{z}(1) \approx 1$ in this case. □

In the next example, the impulse invariant method fails.

Example 10.9 Consider the first-order analog high-pass filter

$$H^{L}(s) = \frac{s}{s+\alpha} = 1 - \frac{\alpha}{s+\alpha}.$$

The corresponding impulse response is

$$h(t) = \delta(t) - \alpha e^{-\alpha t} v(t).$$

Now the presence of the delta term in $h(t)$ prevents sampling of the impulse response, so $h[n]$ is not defined. The problem is fundamental, not just technical. Suppose, for example, that we try to substitute $T^{-1}\delta[n]$ for the sampling of $\delta(t)$. This gives

$$H^{z}(z) = 1 - \frac{\alpha T}{1 - e^{-\alpha T} z^{-1}} = \frac{1 - \alpha T - e^{-\alpha T} z^{-1}}{1 - e^{-\alpha T} z^{-1}}.$$

The resulting digital filter will be high pass if $\alpha T \ll 1$, but not necessarily so otherwise. For example, if $\alpha T = 1$, then $H^{z}(z)$ becomes a low-pass filter with negative gain and an additional unit delay. □

The conclusion from the preceding example is that the impulse invariant transformation is not suitable for analog filters whose transfer function is *exactly proper* (i.e., their numerator and denominator polynomials are of the same degree), since the corresponding impulse response contains a term $\delta(t)$. High-pass and band-stop filters have exactly proper transfer functions, so the impulse invariant method cannot be used for such filters. On the other hand, low-pass and band-pass filters usually have *strictly proper* transfer functions (i.e., the numerator degree is strictly less than the denominator degree). Such functions have no delta function terms in their impulse response, so the impulse invariant method can be applied to them (see, however, Problem 10.31 in this regard).

Any rational strictly proper transfer function $H^{\mathrm{L}}(s)$ can be decomposed to partial fractions. If the poles of $H^{\mathrm{L}}(s)$ are simple, the decomposition has the form

$$H^{\mathrm{L}}(s) = \frac{b_0 s^q + b_1 s^{q-1} + \cdots + b_q}{s^p + a_1 s^{p-1} + \cdots + a_p} = \sum_{k=1}^{p} \frac{C_k}{s - \lambda_k}, \tag{10.97}$$

where p is the filter's order, $\{\lambda_k\}$ are the poles, and $\{C_k\}$ are the residues. Correspondingly,

$$h(t) = \sum_{k=1}^{p} C_k e^{\lambda_k t}, \quad h[n] = \sum_{k=1}^{p} C_k T e^{n\lambda_k T}, \tag{10.98}$$

so

$$H^{\mathrm{z}}(z) = \sum_{k=1}^{p} \frac{C_k T}{1 - e^{\lambda_k T} z^{-1}}. \tag{10.99}$$

The transformed digital filter $H^{\mathrm{z}}(z)$ has the following properties:

1. Its order is the same as that of the analog filter, because the common denominator on the right side has degree p.
2. Its poles are mapped according to

   $$\lambda_k \longrightarrow e^{\lambda_k T}, \quad 1 \le k \le p.$$

   We have

   $$\Re\{\lambda_k\} < 0 \Longrightarrow |e^{\lambda_k T}| < 1, \quad 1 \le k \le p.$$

   Consequently, a stable analog filter $H^{\mathrm{L}}(s)$ is transformed to a stable digital filter $H^{\mathrm{z}}(z)$.
3. The zeros of $H^{\mathrm{z}}(z)$ do not bear a simple relationship to those of $H^{\mathrm{L}}(s)$. When the right side of (10.99) is brought to a common denominator, the numerator will be a polynomial of degree $p-1$ in z^{-1} in general, regardless of the degree q of the numerator of $H^{\mathrm{L}}(s)$. The roots of this polynomial will not depend on $\{\lambda_k, C_k, T\}$ in an obvious manner.

These properties hold also in case the analog filter has multiple poles, but the corresponding formulas are omitted.

The procedure `impinv` in Program 10.6 implements the impulse invariant method. It first decomposes the analog transfer function into partial fractions, using the MATLAB function `residue`. It then transforms the poles and the gains as given by (10.99), and finally calls `pf2tf` (described in Section 7.4) to bring the partial fractions under a common denominator. The program issues an error message if the numerator degree of the analog filter is not smaller than the denominator degree.

The frequency response $H^{\mathrm{f}}(\theta)$ of the digital filter is related to that of the analog filter $H^{\mathrm{F}}(\omega)$ by the sampling theorem,

$$H^{\mathrm{f}}(\theta) = \sum_{k=-\infty}^{\infty} H^{\mathrm{F}}\left(\frac{\theta - 2\pi k}{T}\right) \tag{10.100}$$

(note the cancellation of $1/T$ in front of the sum, because of the inclusion of the factor T in our definition of the impulse invariant method). Since, for a rational analog filter,

$H^F(\omega)$ is never band limited, the frequency response of the digital filter *is always aliased.* If $H^L(s)$ is low pass or band pass, aliasing can be made negligible by choosing the sampling frequency $1/T$ high enough such that the fraction of energy in the range $|\omega| > \pi/T$ will be negligible. If $H^L(s)$ is high pass or band stop, the impulse invariant method cannot be used at all. We concluded this before based on the properties of the impulse response, and now we conclude it again based on the frequency response: The frequency response of these two filter classes does not decay to zero, so the right side of (10.100) does not converge.

When a digital filter needs to be designed according to given specifications, the impulse invariant method becomes problematic even in the case of low-pass or band-pass filters. Consider, for example, the design of a low-pass filter with tolerance parameters δ_p, δ_s and band-edge frequencies θ_p, θ_s. As a first attempt, we may use the given tolerance parameters for the analog filter and choose the band-edge frequencies as $\omega_p = \theta_p/T$, $\omega_s = \theta_s/T$. When transforming the analog filter designed with these parameters, using the impulse invariance method, we will usually discover that the digital filter does not meet the specifications. This happens because aliasing causes the replicas $H^F((\theta - 2\pi k)/T)$ to add up and spoil both pass-band and stop-band tolerances. A possible solution is to design the analog filter with narrower tolerances, but this requires experimentation, so the design procedure ceases to be straightforward.

Example 10.10 Recall the low-pass Chebyshev filter of the second kind designed in Example 10.3. That filter has order $N = 9$, therefore it has 9 poles and 8 zeros, making its transfer function strictly proper. As shown in Figure 10.9, its frequency response meets the stop-band specification and exceeds the pass-band specification. We transform this filter to a digital filter, using the impulse invariant method with $T = 1$. The poles of the digital filter thus obtained are

$$\alpha_{1,2} = 0.0993 \pm j0.8325, \quad \alpha_{3,4} = 0.0695 \pm j0.5584,$$
$$\alpha_{5,6} = 0.0725 \pm j0.3225, \quad \alpha_{7,8} = 0.1039 \pm j0.1386,$$
$$\alpha_9 = 0.1214.$$

The zeros of the filter are

$$\beta_{1,2} = 0.3817 \pm j2.6660, \quad \beta_{3,4} = -0.2993 \pm j0.9055,$$
$$\beta_{5,6} = -0.4315 \pm j0.4880, \quad \beta_7 = -0.2590, \quad \beta_8 = -0.0672.$$

Figure 10.16 shows the resulting frequency response. As we see, the digital filter meets neither the stop-band nor the pass-band specifications. This failure is due to the analog filter's asymptotic attenuation of only 20 dB/decade, which is not enough to prevent aliasing from spoiling the frequency response of the digital filter. □

In summary, the impulse invariant method has the advantages of preserving the order and stability of the analog filter. On the other hand, there is a distortion of the shape of the frequency response that is due to aliasing. Reducing the aliasing effect requires high sampling rates, thus limiting the usefulness of the method. Besides, the method is not applicable to all filter types. As a result of these drawbacks, this method is not in common use.

10.8 The Backward Difference Method*

As we recall, the analog-domain variable s represents differentiation. Therefore, we can try to replace s by an approximate differentiation operator in the digital domain.

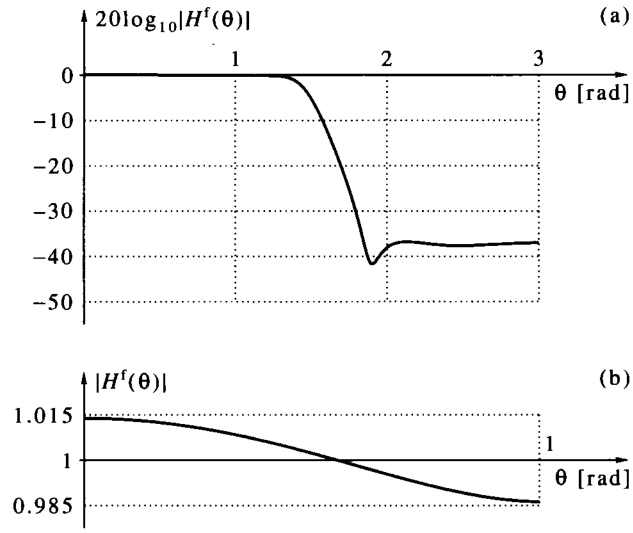

Figure 10.16 Frequency response of the Chebyshev filter in Example 10.10: (a) full range; (b) pass-band details.

A simple approximation to differentiation is given by

$$y(t) = \frac{dx(t)}{dt} \implies y(nT) \approx \frac{x(nT) - x(nT - T)}{T},$$

so

$$Y^z(z) \approx \frac{1 - z^{-1}}{T} X^z(z).$$

This approximation suggests the s-to-z transformation

$$s \longleftarrow \frac{1 - z^{-1}}{T}. \tag{10.101}$$

The operator on the right side of (10.101) is called the *backward difference operator*, since it represents an approximate differentiation, using past and present values of the function to be differentiated. The transfer function $H^z(z)$ is then obtained as

$$H^z(z) = H^L(s)\Big|_{s = \frac{1 - z^{-1}}{T}}. \tag{10.102}$$

The operation (10.102) is called *backward difference transformation.*

Example 10.11 Consider again the low-pass filter $H^L(s) = \alpha/(s + \alpha)$. We get

$$H^z(z) = \frac{\alpha T}{1 - z^{-1} + \alpha T} = \frac{\alpha T}{1 + \alpha T} \cdot \frac{1}{1 - (1 + \alpha T)^{-1} z^{-1}}. \tag{10.103}$$

As we see, the digital filter is also of first order and is stable. □

To develop a general formula for the transformed digital filter, let the filter $H^L(s)$ be given in a factored form

$$H^L(s) = \frac{b_0 \prod_{k=1}^{q} (s - u_k)}{\prod_{k=1}^{p} (s - v_k)}. \tag{10.104}$$

A single zero u_k is transformed as follows:

$$s - u_k = \frac{1}{T}(1 - z^{-1} - u_k T) = \frac{1 - u_k T}{T}\left(1 - \frac{1}{1 - u_k T} z^{-1}\right), \tag{10.105}$$

and similarly for a single pole. Therefore, the transformed filter has the factored form

$$H^{z}(z) = \frac{b_0 T^{p-q} \prod_{k=1}^{q}(1-u_k T)}{\prod_{k=1}^{p}(1-v_k T)} \cdot \frac{\prod_{k=1}^{q}[1-(1-u_k T)^{-1}z^{-1}]}{\prod_{k=1}^{p}[1-(1-v_k T)^{-1}z^{-1}]}. \tag{10.106}$$

It follows from (10.106) that the order of $H^{z}(z)$ is equal to that of $H^{\mathrm{L}}(s)$. We now show that the transformation preserves the stability of $H^{\mathrm{L}}(s)$. We have

$$z = \frac{1}{1-sT} = \frac{1}{1-\sigma T - j\omega T}, \tag{10.107}$$

so

$$|z| = \frac{1}{[(1-\sigma T)^2 + (\omega T)^2]^{1/2}}. \tag{10.108}$$

We get that $\sigma < 0$ implies $|z| < 1$. If $H^{\mathrm{L}}(s)$ is stable, all its poles have negative real parts, so the corresponding poles of $H^{z}(z)$ will have moduli less than 1. This proves that the backward difference transformation preserves stability.

Considering (10.107) further, we see that

$$\sigma = 0 \implies z = \frac{1}{1-j\omega T} \implies z - \frac{1}{2} = \frac{1+j\omega T}{2(1-j\omega T)} \implies \left|z - \frac{1}{2}\right| = \frac{1}{2}. \tag{10.109}$$

This shows that the imaginary axis in the s domain is mapped to the circle of radius 0.5 centered at $z = 0.5$ in the z domain. It is *not* mapped to the circle $|z| = 1$ and, consequently, the backward difference method is not a ω-to-θ transformation. We can therefore expect that the frequency response $H^{\mathrm{f}}(\theta)$ will be considerably distorted with respect to $H^{\mathrm{F}}(\omega)$. The left half s plane is mapped to the disk $|z - 0.5| \le 0.5$ in the z plane, which completely lies in the right half z plane; see Figure 10.17. An analog high-pass filter cannot be mapped to a digital high-pass filter because the poles of the digital filter cannot lie in the correct region, which is the left half of the z plane in this case. In summary, the backward difference method is crude and consequently rarely used.

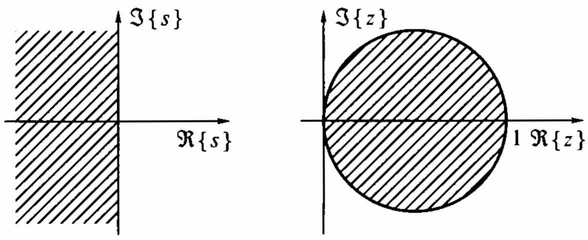

Figure 10.17 The backward difference transform from the s plane to the z plane: The imaginary axis maps to the circle, the left half plane maps to the disk.

10.9 The Bilinear Transform

10.9.1 Definition and Properties of the Bilinear Transform

The bilinear transform can be regarded as a correction of the backward difference method. It is defined by the substitution

$$s \longleftarrow \frac{2}{T} \cdot \frac{z-1}{z+1}. \tag{10.110}$$

Given an analog filter $H^{\mathrm{L}}(s)$, the transfer function of the corresponding digital filter is

$$H^{\mathrm{z}}(z) = H^{\mathrm{L}}(s)\Big|_{s=\frac{2(z-1)}{T(z+1)}}. \tag{10.111}$$

The s-to-z transformation (10.110) can be interpreted as an approximation of continuous-time integration by discrete-time trapezoidal integration; see Problem 10.32.

Example 10.12 Consider the two filters

$$H_1^{\mathrm{L}}(s) = \frac{\alpha}{s+\alpha}, \quad H_2^{\mathrm{L}}(s) = \frac{s}{s+\alpha}.$$

Recall that the former is low pass and the latter is high pass. We get from (10.111)

$$H_1^{\mathrm{z}}(z) = \frac{\alpha}{\frac{2(z-1)}{T(z+1)}+\alpha} = \frac{0.5\alpha T}{1+0.5\alpha T}\cdot\frac{1+z^{-1}}{1-\frac{1-0.5\alpha T}{1+0.5\alpha T}z^{-1}}, \tag{10.112a}$$

$$H_2^{\mathrm{z}}(z) = \frac{\frac{2(z-1)}{T(z+1)}}{\frac{2(z-1)}{T(z+1)}+\alpha} = \frac{1}{1+0.5\alpha T}\cdot\frac{1-z^{-1}}{1-\frac{1-0.5\alpha T}{1+0.5\alpha T}z^{-1}}. \tag{10.112b}$$

As we see, the two digital filters are first order, and they have a stable pole at $z = (1-0.5\alpha T)/(1+0.5\alpha T)$. The filter $H_1^{\mathrm{z}}(z)$ is low pass (it has a zero at $z=-1$), and $H_2^{\mathrm{z}}(z)$ is high pass (it has a zero at $z = 1$). □

In general, let the filter $H^{\mathrm{L}}(s)$ be given in a factored form

$$H^{\mathrm{L}}(s) = \frac{b_0\prod_{k=1}^{q}(s-u_k)}{\prod_{k=1}^{p}(s-v_k)}, \tag{10.113}$$

A single zero u_k is transformed as follows:

$$\begin{aligned} s-u_k &= \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}} - 0.5Tu_k\right] = \frac{2[(1-0.5Tu_k)-(1+0.5Tu_k)z^{-1}]}{T(1+z^{-1})} \\ &= \frac{2(1-0.5Tu_k)}{T(1+z^{-1})}\left[1-\frac{1+0.5Tu_k}{1-0.5Tu_k}z^{-1}\right], \end{aligned} \tag{10.114}$$

and similarly for a single pole. Therefore, the transformed filter has the factored form

$$\begin{aligned} H^{\mathrm{z}}(z) = {}& \frac{b_0(0.5T)^{p-q}\prod_{k=1}^{q}(1-0.5Tu_k)}{\prod_{k=1}^{p}(1-0.5Tv_k)} \\ & \cdot\frac{(1+z^{-1})^{p-q}\prod_{k=1}^{q}\left[1-\frac{1+0.5Tu_k}{1-0.5Tu_k}z^{-1}\right]}{\prod_{k=1}^{p}\left[1-\frac{1+0.5Tv_k}{1-0.5Tv_k}z^{-1}\right]}. \end{aligned} \tag{10.115}$$

The procedure `bilin` in Program 10.7 implements the bilinear transformation, using the factored form (10.115). Its operation is similar to that of Program 10.5.

It is evident from (10.115) that the bilinear transform preserves the number of poles p; hence it preserves the order of the filter. The number of zeros increases from q to p when $p > q$; in this case, the additional $p-q$ zeros are at $z=-1$. Contrary to the backward difference method, the left half s plane is now mapped to the entire unit disc, rather than to a part of it. Moreover, the imaginary axis is mapped to the unit circle, see Figure 10.18. Therefore, (10.110) is a true frequency-to-frequency transformation. This is shown by the following derivation:

$$z = \frac{1+0.5sT}{1-0.5sT} = \frac{1+0.5\sigma T + j0.5\omega T}{1-0.5\sigma T - j0.5\omega T}, \tag{10.116}$$

so

$$|z| = \left[\frac{(1+0.5\sigma T)^2 + (0.5\omega T)^2}{(1-0.5\sigma T)^2 + (0.5\omega T)^2}\right]^{1/2}. \tag{10.117}$$

We see that $\sigma < 0$ if and only if $|z| < 1$, and $\sigma = 0$ if and only if $|z| = 1$. Therefore, the bilinear transform preserves stability of the transformed filter.

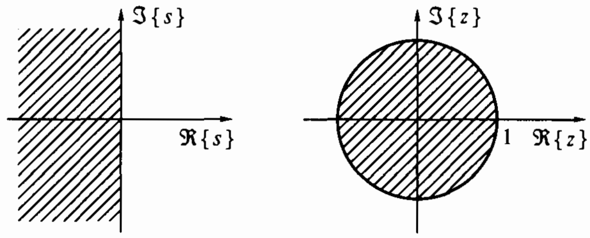

Figure 10.18 The bilinear transform from the s plane to the z plane: The imaginary axis maps to the circle, the left half plane maps to the disk.

For $\sigma = 0$ (i.e., $s = j\omega$), we get

$$z = \frac{1 + j0.5\omega T}{1 - j0.5\omega T} = e^{j\theta}, \tag{10.118}$$

where

$$\theta = 2\arctan(0.5\omega T). \tag{10.119}$$

The digital-domain frequency θ is therefore *warped* with respect to the analog frequency ω, the warping function being $2\arctan(0.5\omega T)$. At low frequencies, $\theta \approx \omega T$, which is the familiar linear relationship. The analog frequencies $\omega = \pm\infty$ are mapped to $\theta = \pm\pi$. The frequency mapping *is not aliased*; that is, the relationship between ω and θ is one-to-one. Consequently, there are no limitations on the use of the bilinear transform; it is adequate for *all* filter types.

Figure 10.19 illustrates the frequency warping introduced by the bilinear transform. The filter in this figure is a second-order, high-pass elliptic filter.

To overcome the frequency warping introduced by the bilinear transform, it is common to *prewarp* the specifications of the analog filter, so that after warping they will be located at the desired frequencies. For example, suppose we wish to design a low-pass filter with band-edge frequencies θ_p, θ_s. We transform these frequencies to corresponding analog-domain band-edge frequencies, using the inverse of (10.119), that is,

$$\omega_p = \frac{2}{T}\tan\left(\frac{\theta_p}{2}\right), \quad \omega_s = \frac{2}{T}\tan\left(\frac{\theta_s}{2}\right). \tag{10.120}$$

We then design the analog filter, using the band-edge frequencies thus obtained. After the analog filter has been transformed using the bilinear transform, the resulting digital filter will have its band-edge frequencies in the right places.

Since prewarping is performed in the beginning of the design procedure, and bilinear transformation is performed in the end, the value of T used is immaterial, as long as it is the same in both. Taking T as the sampling interval enables physical interpretation of the analog frequencies, but taking $T = 1$ or $T = 2$ may be more convenient for hand computations.

In summary, the design procedure for a digital IIR filter using the bilinear transform is as follows:

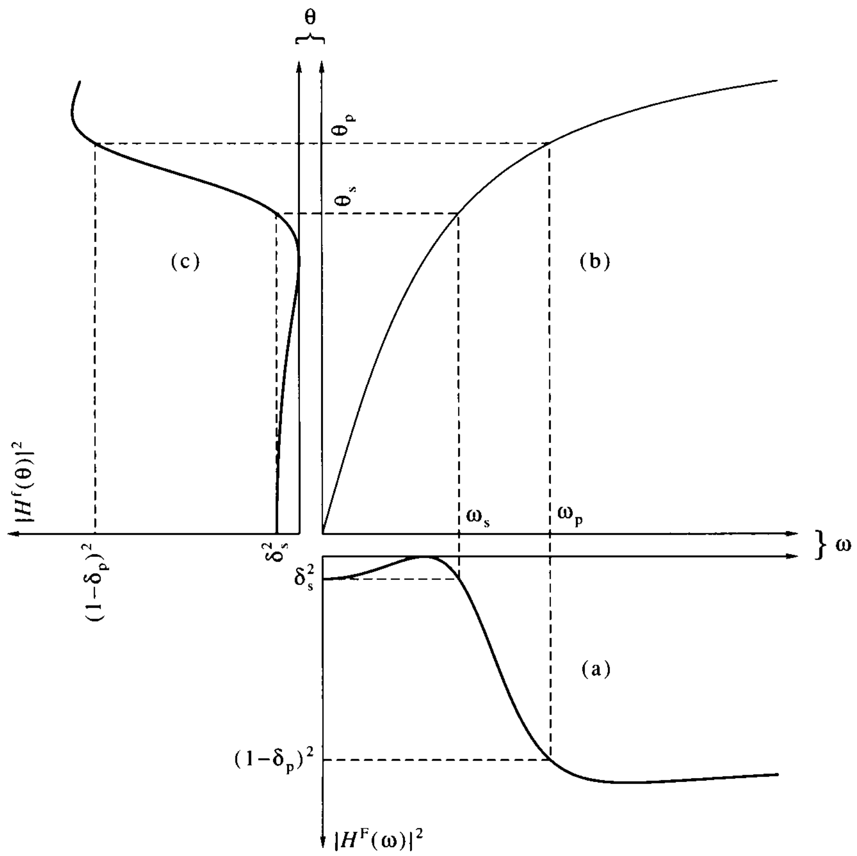

Figure 10.19 The frequency warping of the bilinear transform: (a) frequency response of the analog filter (amplitude axis pointing downward); (b) θ as a function of ω; (c) frequency response of the digital filter (amplitude axis pointing to left).

Digital IIR filter design procedure

1. Convert each specified band-edge frequency of the digital filter to a corresponding band-edge frequency of an analog filter, using (10.120). Leave the pass-band tolerance δ_p and the stop-band tolerance δ_s unchanged.
2. Design an analog filter $H^L(s)$ of the desired type, according to the transformed specifications.
3. Transform $H^L(s)$ to a digital filter $H^z(z)$, using (10.115).

Example 10.13 We wish to design a fourth-order, band-pass digital Butterworth filter with -3 dB frequencies 2000 and 3000 Hz; the sampling frequency is 8000 Hz. The standard low-pass Butterworth filter of order 2 is

$$H_0^L(s) = \frac{1}{s^2 + 1.4142s + 1}.$$

The prewarped -3 dB frequencies are

$$\omega_l = 16{,}000 \tan(0.25\pi) = 16{,}000 \text{ rad/s},$$

$$\omega_{\rm h} = 16,000\tan(0.375\pi) = 38,627 \text{ rad/s}.$$

It is convenient in this case, since the frequency range of interest is in kilohertz, to specify angular frequencies in multiples of 10^4 rad/s, and time in multiples of 10^{-4} second. The low-pass to band-pass transformation is then

$$s = \frac{\tilde{s}^2 + \omega_{\rm l}\omega_{\rm h}}{(\omega_{\rm h} - \omega_{\rm l})\tilde{s}} = \frac{\tilde{s}^2 + 6.1803}{2.2627\tilde{s}}.$$

Therefore, the band-pass Butterworth filter is

$$\tilde{H}^{\rm L}(\tilde{s}) = \frac{1}{\left(\frac{\tilde{s}^2 + 6.1803}{2.2627\tilde{s}}\right)^2 + 1.4142\left(\frac{\tilde{s}^2 + 6.1803}{2.2627\tilde{s}}\right) + 1}$$
$$= \frac{5.1198\tilde{s}^2}{\tilde{s}^4 + 3.2\tilde{s}^3 + 17.48\tilde{s}^2 + 19.777\tilde{s} + 38.196}.$$

The corresponding digital filter is given by

$$H^z(z) = \frac{13.106\left(\frac{z-1}{z+1}\right)^2}{\left(\frac{z-1}{z+1}\right)^4 + 13.107\left(\frac{z-1}{z+1}\right)^3 + 44.749\left(\frac{z-1}{z+1}\right)^2 + 31.643\left(\frac{z-1}{z+1}\right) + 38.196}$$
$$= \frac{0.0976(1 - 2z^{-2} + z^{-4})}{1 + 1.2189z^{-1} + 1.3333z^{-2} + 0.6667z^{-3} + 0.3333z^{-4}}.$$

□

10.9.2 MATLAB Implementation of IIR Filter Design

The procedure `iirdes` in Program 10.8 combines the programs mentioned in Section 10.5 and the program for the bilinear transform to a complete digital IIR filter design program. The program accepts the desired filter class (Butterworth, Chebyshev-I, Chebyshev-II, or elliptic), the desired frequency response type (low pass, high pass, band pass, or band stop), the band-edge frequencies, the pass-band ripple, and the stop-band attenuation. The program first prewarps the digital frequencies, using sampling interval $T = 1$ (this choice is arbitrary). It then transforms the specifications to the specifications of the prototype low-pass filter. Next, the order N and the parameters ω_0, ε are computed from the specifications. The low-pass filter is designed next, transformed to the appropriate analog band, then to digital, using the bilinear transform (again with $T = 1$). The program provides both the polynomials and the pole-zero factorization of the z-domain transfer function.

10.9.3 IIR Filter Design Examples

We now illustrate IIR filter design based on the bilinear transform by several examples. We use the specification examples given in Section 8.2 and present design results that meet these specifications. We show the magnitude responses of the filters, but do not list their coefficients. You can easily obtain the coefficients, as well as the poles and zeros, with the program `iirdes`.

Example 10.14 Consider the low-pass filter whose specifications were given in Example 8.1. Butterworth, Chebyshev-I, Chebyshev-II, and elliptic filters that meet these specifications have orders $N = 27, 9, 9, 5$, respectively. Figure 10.20 shows the magnitude responses of these filters. □

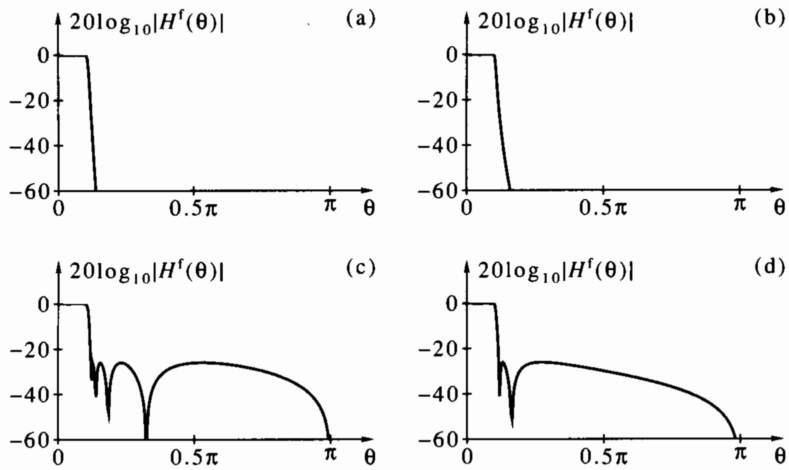

Figure 10.20 Magnitude responses of the filters in Example 10.14: (a) Butterworth; (b) Chebyshev-I; (c) Chebyshev-II; (d) elliptic.

Example 10.15 Consider the high-pass filter whose specifications were given in Example 8.2. Butterworth, Chebyshev-I, Chebyshev-II, and elliptic filters that meet these specifications have orders $N = 15, 8, 8, 5$, respectively. Figure 10.21 shows the magnitude responses of these filters. □

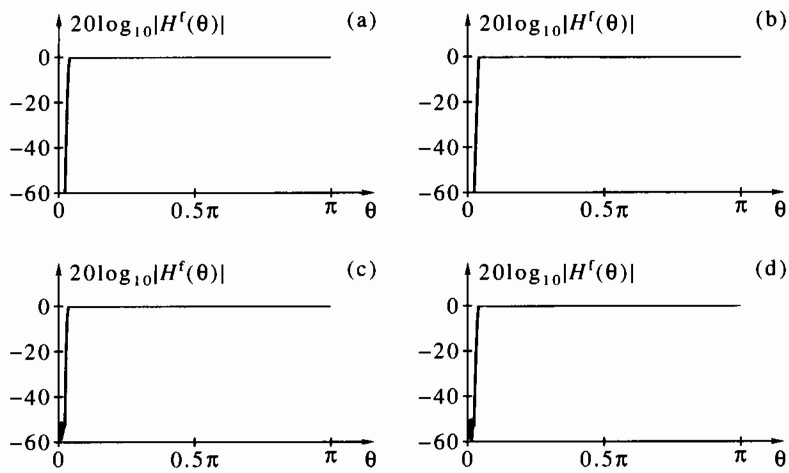

Figure 10.21 Magnitude responses of the filters in Example 10.15: (a) Butterworth; (b) Chebyshev-I; (c) Chebyshev-II; (d) elliptic.

Example 10.16 Consider the five filters whose specifications were given in Example 8.3. Here we design the band-pass filter for the second channel. Butterworth, Chebyshev-I, Chebyshev-II, and elliptic filters that meet these specifications have orders $N = 48, 22, 22, 14$, respectively. Figure 10.22 shows the magnitude responses of these filters. □

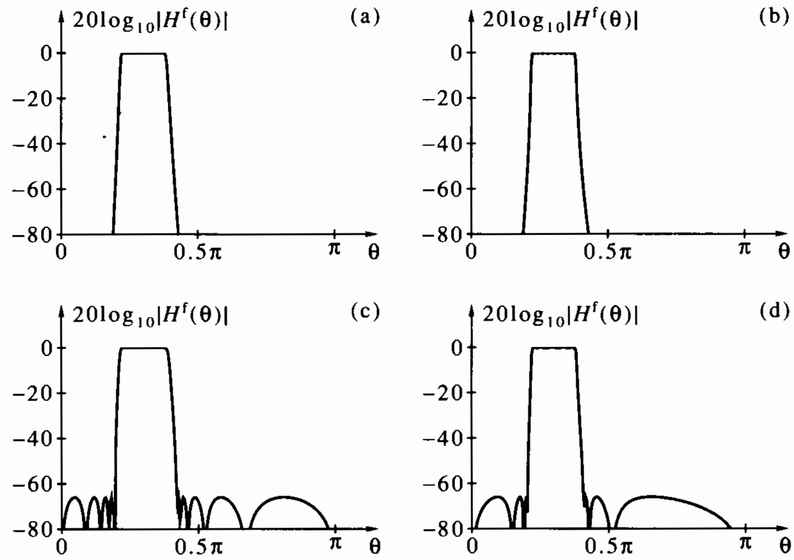

Figure 10.22 Magnitude responses of the filters in Example 10.16: (a) Butterworth; (b) Chebyshev-I; (c) Chebyshev-II; (d) elliptic.

Example 10.17 Consider the band-stop filter whose specifications were given in Example 8.4. Butterworth, Chebyshev-I, Chebyshev-II, and elliptic filters that meet these specifications have orders $N = 16, 10, 10, 8$, respectively. Figure 10.23 shows the magnitude responses of these filters. □

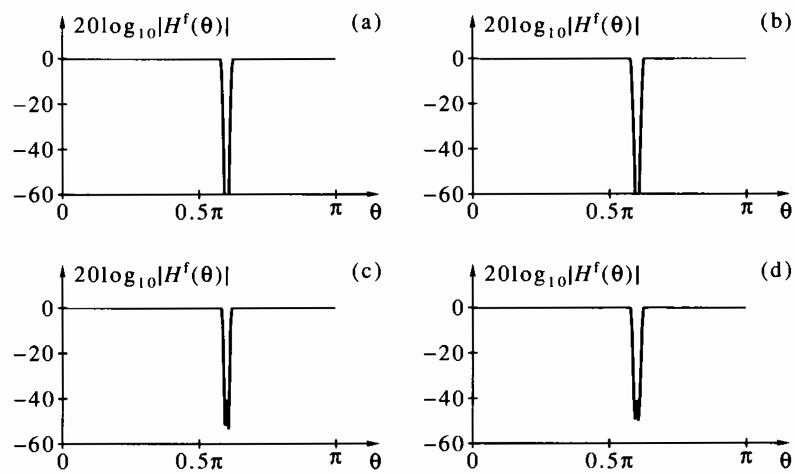

Figure 10.23 Magnitude responses of the filters in Example 10.17: (a) Butterworth; (b) Chebyshev-I; (c) Chebyshev-II; (d) elliptic.

In conclusion, the bilinear transform facilitates design of IIR filters meeting prescribed specifications without trial and error. Elliptic filters have the smallest order. Therefore, an elliptic filter should usually be the preferred choice, unless the application requires monotone response in the pass band, or the stop band, or both.

10.10 The Phase Response of Digital IIR Filters

Digital IIR filters do not have linear phase. That is, their group delay is not constant for all frequencies. Even if we restrict ourselves to the pass band, we still find that the group delay is not constant there. The group delay of a low-pass IIR filter typically increases monotonically in the pass band, reaches a maximum in the transition band, and then decreases monotonically. As a result, signals in the pass band that are fed to such a filter are usually distorted at the output, even if the pass-band ripple of the filter is very small. Distortion occurs, since each frequency component is delayed by a different amount, so the relative phases of the frequency components at the output are different from those at the input. The following example illustrates this phenomenon.

Example 10.18 Consider the following digital low-pass filter specifications:

$$\theta_p = 0.1\pi, \; \theta_s = 0.2\pi, \; \delta_p = \delta_s = 0.001.$$

We design four filters meeting these specifications, using the program `iirdes`. The filters are Butterworth, Chebyshev-I, Chebyshev-II, and elliptic. We then compute the group delay of the four filters in the frequency range $[0, 0.2\pi]$ (which includes the pass band and the transition band), using the program `grpdly`. The results are shown in Figure 10.24. As we see, the Chebyshev filter of the second kind (represented by the dot-dashed line) has the best group delay response of the four: Its value is the smallest at all pass-band frequencies and it is the most nearly constant in the pass band.

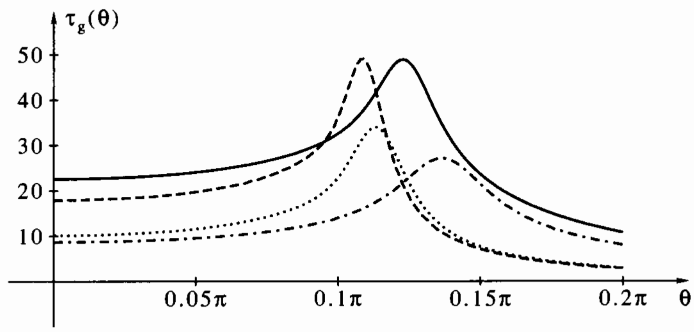

Figure 10.24 The group delay of the four filters in Example 10.18. Solid line: Butterworth, dashed line: Chebyshev-I, dot-dashed line: Chebyshev-II, dotted line: elliptic.

To illustrate the distortion caused by the nonlinear phase, we input the signal

$$x[n] = \sum_{m=1}^{4} \frac{1}{2m-1} \sin[0.0125\pi(2m-1)n]$$

to the four filters. This signal consists of the first four odd harmonics of a square wave whose fundamental period is 160 samples. Its highest frequency is 0.0875π, so it is completely in the pass band of the four filters. The signal is shown in Figure 10.25. Note that we do not depict it as vertical sticks, but as a continuous line, for better visual appearance. The continuous line can be regarded as the continuous-time signal corresponding to $x[n]$.

Let us denote the signal at the output of any of the four filters by $y[n]$. For meaningful comparison of $x[n]$ and $y[n]$, we must advance $y[n]$ by the group delay of the corresponding filter at the fundamental signal frequency, that is, by $\tau_g(0.0125\pi)$. Then, if the filter had a constant group delay and unit gain in the pass band, we would

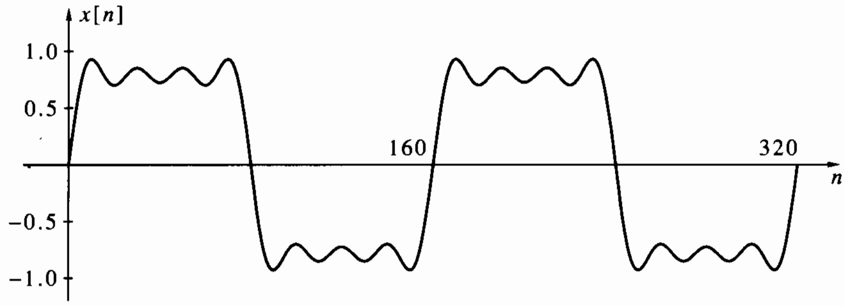

Figure 10.25 Input test signal in Example 10.18.

expect $y[n]$ to be equal to $x[n]$. In reality, we get the responses shown in Figure 10.26. The solid line in each plot shows the input $x[n]$ and the dotted line shows the advanced $y[n]$. The group delays are rounded to integer values. They are 23, 18, 9, and 10 for the four filters, respectively. Since the differences between $y[n]$ and $x[n]$ are not clearly visible in Figure 10.26, we also show, in Figure 10.27, plots of the differences $y[n] - x[n]$. If the filter had a constant group delay, we would expect the error to be bounded by 0.001, which is the value of the pass-band ripple. As we see, the actual differences are larger by two orders of magnitude, due to the distortion caused by the nonlinear phase. The Chebyshev filter of the second kind is the best of the four, in agreement with what we saw in Figure 10.24.

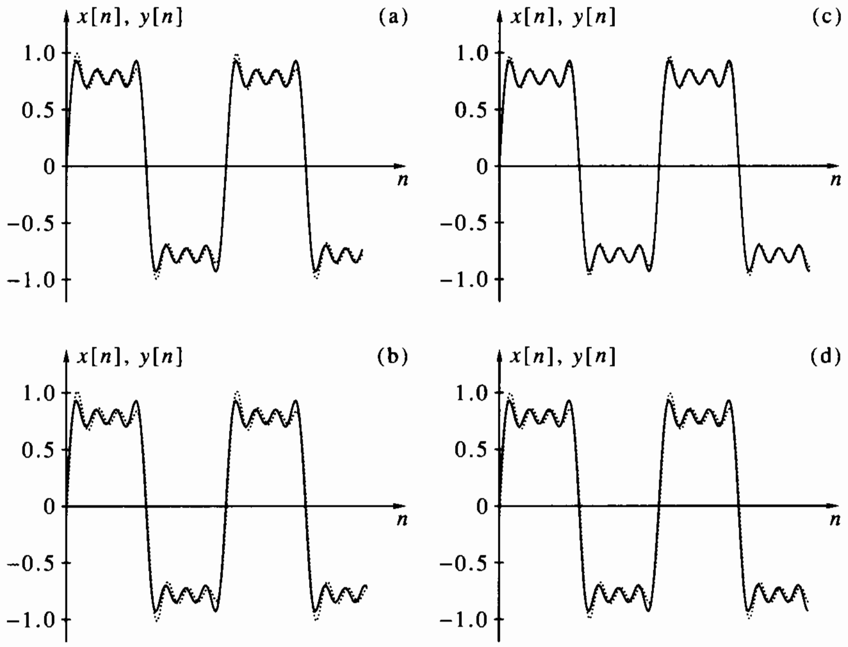

Figure 10.26 Input and output test signals in Example 10.18: (a) Butterworth; (b) Chebyshev-I; (c) Chebyshev-II; (d) elliptic. Solid line: input $x[n]$, dotted line: output $y[n]$.

In summary, nonlinear phase is a major drawback of IIR filters. However, to put this drawback in proper perspective, we reiterate that analog filters do not have linear

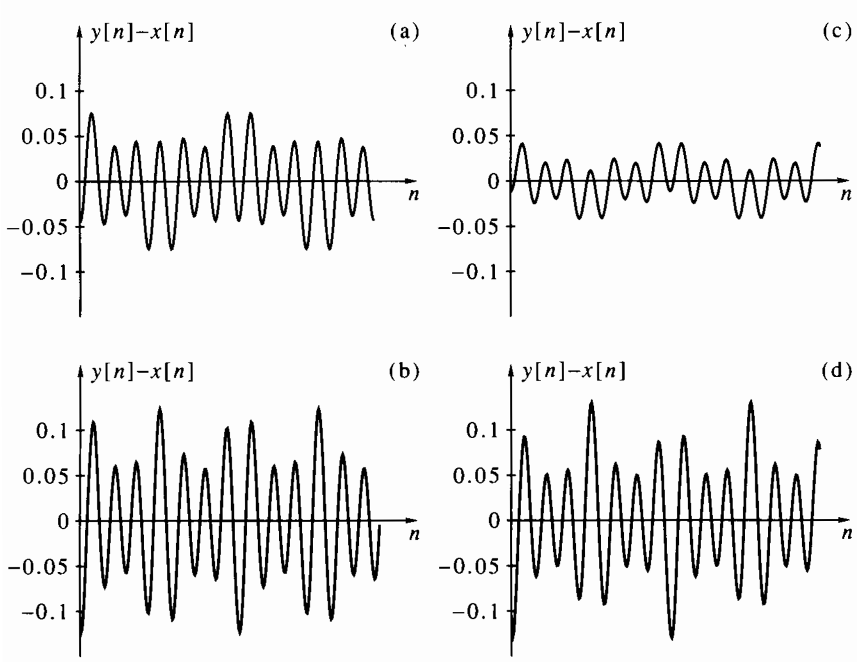

Figure 10.27 The difference between the input and output test signals in Example 10.18: (a) Butterworth; (b) Chebyshev-I; (c) Chebyshev-II; (d) elliptic.

phase either. Despite this, analog filter technology dominated electrical engineering until as recently as the early 1980s. □

As we saw in Chapter 9, digital FIR filters can have linear phase in the pass band, thereby eliminating phase distortions completely. This is illustrated in the next example.

Example 10.19 Let us repeat Example 10.18, using an FIR filter that meets the same specifications, namely

$$\theta_p = 0.1\pi, \quad \theta_s = 0.2\pi, \quad \delta_p = \delta_s = 0.001.$$

The design is carried out using the program `firkais`. The resulting filter has order $N = 76$. We feed the signal shown in Figure 10.25 at the input of the FIR filter. Figure 10.28 shows the difference between the output signal, advanced by the group delay (which is 38 in this case), and the input signal. As we see, the difference signal is well below the pass-band tolerance and is smaller than the difference signal in Example 10.18 by three orders of magnitude. The lesson from this example is that FIR filters indeed offer a great advantage over IIR filters in eliminating distortions due to nonlinear phase response. □

10.11 Sampled-Data Systems*

In this chapter we discussed the design of digital IIR filters. We conclude the chapter by showing how an analog system can be made to work in a digital environment. This topic is marginal to digital signal processing, but it is of great importance in digital

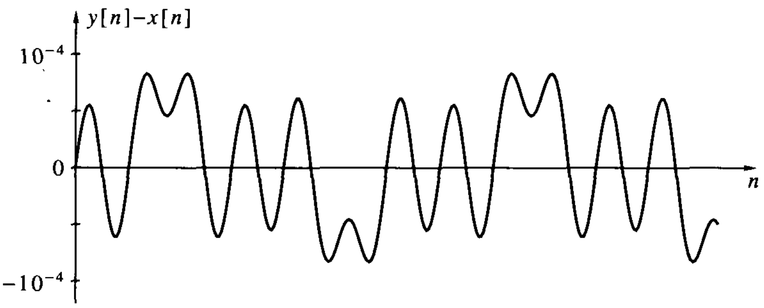

Figure 10.28 The difference between the output and the input signals in Example 10.19.

control applications. The reason is that control systems are often analog by nature and cannot be replaced by discrete-time systems. They include elements such as motors, gears, and power amplifiers. If we wish to interact with such a system digitally, we must interface to it in a proper manner and understand the effect of the interface on the system's behavior.

A *sampled-data system* is a continuous-time system whose input and output are discrete-time signals. To feed a discrete-time signal to a continuous-time system, it is necessary to convert it first to a continuous-time signal. This is usually accomplished by a zero-order hold at the input of the system. Similarly, to get a discrete-time signal from a continuous-time system, it is necessary to sample its output. Figure 10.29 shows the general structure of a sampled-data system. Such an arrangement is also known as a *hybrid system*.

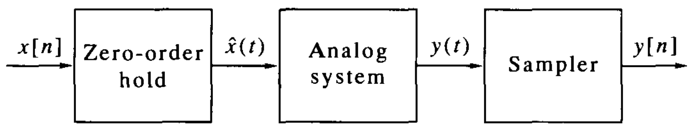

Figure 10.29 A sampled-data system.

We wish to derive the relationship between the discrete-time input signal $x[n]$ and the discrete-time output signal $y[n]$, under the condition that the continuous-time system is linear time invariant. We shall show that in this case, the two signals are related by discrete-time convolution. Thus, the sampled-data system is linear time invariant as well. We denote the impulse response of the continuous-time system by $g(t)$ and its transfer function by $G^{\mathrm{L}}(s)$.

We have at the output of the ZOH,

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x[m] h_{\mathrm{zoh}}(t - mT). \tag{10.121}$$

Recall that the transfer function of a ZOH is

$$H_{\mathrm{zoh}}^{\mathrm{L}}(s) = \frac{1 - e^{-sT}}{s} \tag{10.122}$$

[this is obtained from (3.34) upon replacement of $j\omega$ by s]. The Laplace transform of

$\hat{x}(t)$ is therefore

$$\hat{X}^{\mathrm{L}}(s) = \sum_{m=-\infty}^{\infty} x[m]H_{\mathrm{zoh}}^{\mathrm{L}}(s)e^{-msT} = \sum_{m=-\infty}^{\infty} x[m]\frac{1-e^{-sT}}{s}e^{-msT}$$
$$= \sum_{m=-\infty}^{\infty} x[m]\frac{e^{-msT}-e^{-(m+1)sT}}{s}. \tag{10.123}$$

The Laplace transform of $y(t)$, the output of the continuous-time system, is

$$Y^{\mathrm{L}}(s) = G^{\mathrm{L}}(s)\hat{X}^{\mathrm{L}}(s) = \sum_{m=-\infty}^{\infty} x[m]\frac{G^{\mathrm{L}}(s)(e^{-msT}-e^{-(m+1)sT})}{s}$$
$$= \sum_{m=-\infty}^{\infty} x[m]F^{\mathrm{L}}(s)(e^{-msT}-e^{-(m+1)sT}), \tag{10.124}$$

where we define

$$F^{\mathrm{L}}(s) = \frac{G^{\mathrm{L}}(s)}{s}. \tag{10.125}$$

Let $f(t)$ be the inverse Laplace transform of $F^{\mathrm{L}}(s)$. Then the inverse Laplace transform of (10.124) is

$$y(t) = \sum_{m=-\infty}^{\infty} x[m]\{f(t-mT)-f(t-mT-T)\}. \tag{10.126}$$

The sampled sequence $y[n]$ is given by

$$y[n] = y(nT) = \sum_{m=-\infty}^{\infty} x[m]\{f(nT-mT)-f(nT-mT-T)\}$$
$$= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \{x*h\}[n], \tag{10.127}$$

where we define

$$h[n] = f(nT) - f(nT-T). \tag{10.128}$$

In conclusion, the input-output relationship of the block diagram in Figure 10.29 is a discrete-time convolution. Therefore, the equivalent discrete-time system is linear time invariant. Its transfer function $H^{z}(z)$ is computed by the following procedure:

1. Compute the inverse Laplace transform of $F^{\mathrm{L}}(s)$ given in (10.124), and denote the result by $f(t)$. This is mathematically expressed by
$$f(t) = \left\{\mathcal{L}^{-1}\left[\frac{G^{\mathrm{L}}(s)}{s}\right]\right\}(t).$$
2. Sample $f(t)$ to obtain $f(nT)$.
3. Subtract from $f(nT)$ its unit-delayed version $f(nT-T)$ to get $h[n]$.
4. Compute the z-transform of $h[n]$ to obtain $H^{z}(z)$.

Note that we can also compute the z-transform of $f(nT)$ first, say $F^{z}(z)$, and then

$$H^{z}(z) = (1-z^{-1})F^{z}(z).$$

Example 10.20 Let the continuous-time system have a transfer function

$$G^{\mathrm{L}}(s) = \frac{C}{s(s+\alpha)}.$$

This system consists of an integrator and a real pole; it is a common model for a DC motor, for example. We have

$$\frac{G^{\mathrm{L}}(s)}{s} = \frac{C}{s^2(s+\alpha)} = \frac{C}{\alpha s^2} - \frac{C}{\alpha^2 s} + \frac{C}{\alpha^2(s+\alpha)},$$

so

$$f(t) = \frac{Ct}{\alpha} - \frac{C}{\alpha^2} + \frac{Ce^{-\alpha t}}{\alpha^2}, \quad f(nT) = \frac{CTn}{\alpha} - \frac{C}{\alpha^2} + \frac{Ce^{-\alpha Tn}}{\alpha^2}.$$

The z-transform of this sequence is (cf. Table 7.1)

$$F^z(z) = \frac{CTz^{-1}}{\alpha(1 - z^{-1})^2} - \frac{C}{\alpha^2(1 - z^{-1})} + \frac{C}{\alpha^2(1 - e^{-\alpha T}z^{-1})}.$$

Finally,

$$\begin{aligned} H^z(z) = (1 - z^{-1})F^z(z) &= \frac{CTz^{-1}}{\alpha(1 - z^{-1})} - \frac{C}{\alpha^2} + \frac{C(1 - z^{-1})}{\alpha^2(1 - e^{-\alpha T}z^{-1})} \\ &= \frac{C[(\alpha T + e^{-\alpha T})z^{-1} - e^{-\alpha T}(1 + \alpha T)z^{-2}]}{\alpha^2(1 - z^{-1})(1 - e^{-\alpha T}z^{-1})}. \end{aligned}$$

As we see, the equivalent discrete-time system inherits the integrator and the first-order pole from the continuous-time system, but it also has a unit delay and a zero, which the continuous-time system does not. This result—the inheritance of poles and the appearance of zeros in the discrete-time transfer function—is typical of sampled-data systems. □

10.12 Summary and Complements

10.12.1 Summary

This chapter was devoted to the design of digital infinite impulse response (IIR) filters, in particular, to design by means of analog filters. The classical analog filters have different ripple characteristics: Butterworth is monotone at all frequencies, Chebyshev-I is monotone in the stop band and equiripple in the pass band, Chebyshev-II is monotone in the pass band and equiripple in the stop band, and an elliptic filter is equiripple in all bands. Design formulas were given for these filter classes. Among the four classes, elliptic filters have the smaller order for a given set of specifications, whereas Butterworth filters have the largest order.

When an analog filter other than low pass needs to be designed, a common procedure is to design a prototype low-pass filter of the desired class, and then to transform the low-pass filter by a rational frequency transformation. Standard transformations were given for high-pass, band-pass, and band-stop filters.

After an analog filter has been designed, it must be transformed to the digital domain. The preferred method for this purpose is the bilinear transform. The bilinear transform preserves the order and stability of the analog filter. It is suitable for filters of all classes and types, and is straightforward to compute. The frequency response of the digital filter is related to that of the analog filter from which it was derived through the frequency-warping formula (10.119). At low frequencies the frequency warping is small, but at frequencies close to π it is significant. Prewarping of the discrete-time band-edge frequencies prior to the analog design guarantees that the digital filter obtained as a result of the design will meet the specifications.

Other methods for analog-to-digital filter transformation are the impulse invariant and the backward difference methods; both are inferior to the bilinear transform.

We reiterate the main advantages and disadvantages of IIR filters:

1. Advantages:
   (a) Straightforward design of standard IIR filters, thanks to the existence of well-established analog filter design techniques and simple transformation procedures.

   (b) Low complexity of implementation when compared to FIR filters, especially in the case of elliptic filters.
   (c) Relatively short delays, since practical IIR filters are usually minimum phase.
2. Disadvantages:
   (a) IIR filters do not have linear phase.
   (b) IIR filters are much less flexible than FIR filters in achieving nonstandard frequency responses.
   (c) Design techniques other than those based on analog filters are not readily available, and are complex to develop and implement.
   (d) Although theoretically stable, IIR filters may become unstable when their coefficients are truncated to a finite word length. Therefore, stability must be carefully verified; see Section 11.5 for further discussion.

10.12.2 Complements

1. [p. 333] The notation $T_N(x)$ for the Chebyshev polynomials is derived from the name *Tschebyscheff*, the German spelling of Chebyshev.
2. [p. 357] Euler approximation is based on the observation that the area under the function $h(t)$ between the abscissas nT and $(n+1)T$ is approximately equal to the area of the rectangle having base T and height $h(nT)$, that is,

$$\int_{nT}^{(n+1)T} h(t)dt \approx Th(nT).$$

   By dividing the entire range $[0, \infty)$ to intervals of length T each and using this approximation for each interval, we obtain (10.96).

10.13 MATLAB Programs

Program 10.1 Design of analog low-pass filters.

```
function [b,a,v,u,C] = analoglp(typ,N,w0,epsilon,m);
% Synopsis: [b,a,v,u,C] = analoglp(typ,N,w0,epsilon,m).
% Butterworth, Chebyshev-I or Chebyshev-II low-pass filter.
% Input parameters:
% typ: filter class: 'but', 'ch1', 'ch2', or 'ell'
% N: the filter order
% w0: the frequency parameter
% epsilon: the tolerance parameter; not needed for Butterworth
% m: parameter needed for elliptic filters.
% Output parameters:
% b, a: numerator and denominator polynomials
% v, u, C: poles, zeros, and constant gain.

if (typ == 'ell'),
   [v,u,C] = elliplp(N,w0,epsilon,m);
   a = 1; for i = 1:N, a = conv(a,[1, -v(i)]); end
   b = C; for i = 1:length(u), b = conv(b,[1, -u(i)]); end
   a = real(a); b = real(b); C = real(C); return
end
k = (0.5*pi/N)*(1:2:2*N-1); s = -sin(k); c = cos(k);
if (typ == 'but'), v  = w0*(s+j*c);
elseif (typ(1:2) == 'ch'),
   f = 1/epsilon; f = log(f+sqrt(1+f^2))/N;
   v = w0*(sinh(f)*s+j*cosh(f)*c);
end
if (typ == 'ch2'),
   v = (w0^2)./v;
   if (rem(N,2) == 0), u = j*w0./c;
   else, u = j*w0./[c(1:(N-1)/2),c((N+3)/2:N)]; end
end
a = 1; for k = 1:N, a = conv(a,[1, -v(k)]); end
if (typ == 'but' | typ == 'ch1'),
   C = prod(-v); b = C; u = [];
elseif (typ == 'ch2'),
   C = prod(-v)/prod(-u); b = C;
   for k = 1:length(u), b = conv(b,[1, -u(k)]); end;
end
if (typ == 'ch1' & rem(N,2) == 0),
   f = (1/sqrt(1+epsilon^2)); b = f*b; C = f*C; end
a = real(a); b = real(b); C = real(C);
```

Program 10.2 Design of a low-pass elliptic filter.

```
function [v,u,C] = elliplp(N,w0,epsilon,m);
% Synopsis: [v,u,C] = elliplp(N,w0,epsilon,m).
% Designs a low-pass elliptic filter.
% Input parameters:
% N: the order
% w0: the pass-band edge
% epsilon, m: filter parameters.
% Output parameters:
% v, u, C: poles, zeros, and constant gain of the filter.

flag = rem(N,2); K = ellipke(m);
if (~flag), lmax = N/2; l = (1:lmax)-0.5;
else, lmax = (N-1)/2; l = 1:lmax; end
zl = ellipj((2*K/N)*l,m); pl = 1 ./(sqrt(m)*zl);
f = prod((1-pl.^2)./(1-zl.^2));
u = w0*reshape([j*pl; -j*pl],1,2*lmax);
a = 1;
for l = 1:lmax,
   for i = 1:2, a = conv(a,[1,0,pl(l)^2]); end
end
b = 1;
for l = 1:lmax,
   for i = 1:2, b = conv(b,[1,0,zl(l)^2]); end
end
b = (f*epsilon)^2*b;
if (flag), b = -[b,0,0]; a = [0,0,a]; end
v = roots(a+b).'; v = w0*v(find(real(v) < 0));
C = prod(-v)./prod(-u);
if (~flag), C = C/sqrt(1+epsilon^2); end, C = real(C);
```

Program 10.3 The parameters of an analog low-pass filter as a function of the specification parameters.

```
function [N,w0,epsilon,m] = lpspec(typ,wp,ws,deltap,deltas);
% Synopsis: [N,w0,epsilon,k,q] = lpspec(typ,wp,ws,deltap,deltas).
% Butterworth, Chebyshev-I or Chebyshev-II low-pass filter
% parameter computation from given specifications.
% Input parameters:
% typ: the filter class:
%      'but' for Butterworth
%      'ch1' for Chebyshev-I
%      'ch2' for Chebyshev-II
%      'ell' for elliptic
% wp, ws: band-edge frequencies
% deltap, deltas: pass-band and stop-band tolerances.
% Output parameters:
% N: the filter order
% w0: the frequency parameter
% epsilon: the tolerance parameter; not supplied for Butterworth
% m: parameter supplied in case of elliptic filter.

d = sqrt(((1-deltap)^(-2)-1)/(deltas^(-2)-1)); di = 1/d;
k = wp/ws; ki =1/k;
if (typ == 'but'),
   N = ceil(log(di)/log(ki));
   w0 = wp*((1-deltap)^(-2)-1)^(-0.5/N);
   nargout = 2;
elseif (typ(1:2) == 'ch'),
   N = ceil(log(di+sqrt(di^2-1))/log(ki+sqrt(ki^2-1)));
   nargout = 3;
   if (typ(3) == '1'),
      w0 = wp; epsilon = sqrt((1-deltap)^(-2)-1);
   elseif (typ(3) == '2'),
      w0 = ws; epsilon = 1/sqrt(deltas^(-2)-1);
   end
elseif (typ == 'ell'),
   w0 = wp; epsilon = sqrt((1-deltap)^(-2)-1);
   [N,m]  = ellord(k,d);
   nargout = 4;
end
```

Program 10.4 Computation of the order of an elliptic low-pass filter.

```
function [N,m] = ellord(k,d);
% Synopsis: [N,m] = ellord(k,d).
% Finds the order and the parameter m of an elliptic filter.
% Input parameters:
% k, d: the selectivity and discrimination factors.
% Output parameters:
% N: the order
% m: the parameter for the Jacobi elliptic function.

m0 = k^2; C = ellipke(1-d^2)/ellipke(d^2);
N0  = C*ellipke(m0)/ellipke(1-m0);
if (abs(N0-round(N0))) <= 1.0e-6,
   N = round(N0); m = m0; return; end
N = ceil(N0); m = 1.1*m0;
while (C*ellipke(m)/ellipke(1-m) < N), m = 1.1*m; end
N1 = C*ellipke(m)/ellipke(1-m);
while (abs(N1-N) >= 1.0e-6),
   a = (N1-N0)/(m-m0);
   mnew = m0+(N-N0)/a;
   Nnew = C*ellipke(mnew)/ellipke(1-mnew);
   if (Nnew < N), m0 = mnew; N0 = Nnew;
   else, m = mnew; N1 = Nnew; end
end
```

Program 10.5 Frequency transformations of analog filters.

```
function [b,a,vout,uout,Cout] = analogtr(typ,vin,uin,Cin,w);
% Synopsis: [b,a,vout,uout,Cout] = analogtr(typ,vin,uin,Cin,w).
% Performs frequency transformations of analog low-pass filters.
% Input parameters:
% typ: the transformation type:
%      'l' for low-pass to low-pass
%      'h' for low-pass to high-pass
%      'p' for low-pass to band-pass
%      's' for low-pass to band-stop
% vi, ui, Cin: the poles, zeros, and constant gain of the low-pass
% w: equal to omega_c for 'l' or 'h'; a 1 by 2 matrix of
%    [omega_l, omega_h] for 'p' or 's'.
% Output parameters:
% b, a: the output polynomials
% vout, uout, Cout: the output poles, zeros, and constant gain.

p = length(vin); q = length(uin);
if (typ == 'l'),
   uout = w*uin; vout = w*vin; Cout = w^(p-q)*Cin;
elseif (typ == 'h'),
   uout = [w./uin,zeros(1,p-q)]; vout = w./vin;
   Cout = prod(-uin)*Cin/prod(-vin);
elseif (typ == 'p'),
   wl = w(1); wh = w(2); uout = []; vout = [];
   for k = 1:q,
      uout = [uout,roots([1,-uin(k)*(wh-wl),wl*wh]).']; end
   uout = [uout,zeros(1,p-q)];
   for k = 1:p,
      vout = [vout,roots([1,-vin(k)*(wh-wl),wl*wh]).']; end
   Cout = (wh-wl)^(p-q)*Cin;
elseif (typ == 's'),
   [t1,t2,t3,t4,t5] = analogtr('h',vin,uin,Cin,1);
   [t1,t2,vout,uout,Cout] = analogtr('p',t3,t4,t5,w);
end
a = 1; b = 1;
for k = 1:length(vout), a = conv(a,[1,-vout(k)]); end
for k = 1:length(uout), b= conv(b,[1,-uout(k)]); end
a = real(a); b = real(Cout*b); Cout = real(Cout);
```

Program 10.6 Impulse invariant transformation of an analog filter.

```
function [bout,aout] = impinv(bin,ain,T);
% Synopsis: [bout,aout] = impinv(bin,ain,T).
% Computes the impulse invariant transformation of an analog filter.
% Input parameters:
% bin, ain: the numerator and denominator polynomials of the
%           analog filter
% T: the sampling interval
% Output parameters:
% bout, aout: the numerator and denominator polynomials of the
%             digital filter.

if (length(bin) >= length(ain)),
   error('Analog filter in IMPINV is not strictly proper'); end
[r,p,k] = residue(bin,ain);
[bout,aout] = pf2tf([],T*r,exp(T*p));
```

Program 10.7 Bilinear transformation of an analog filter.

```
function [b,a,vout,uout,Cout] = bilin(vin,uin,Cin,T);
% Synopsis: [b,a,vout,uout,Cout] = bilin(vin,uin,Cin,T).
% Computes the bilinear transform of an analog filter.
% Input parameters:
% vi, ui, Cin: the poles, zeros, and constant gain of the
%              analog filter
% T: the sampling interval.
% Output parameters:
% b, a: the output polynomials
% vout, uout, Cout: the output poles, zeros, and constant gain.

p = length(vin); q = length(uin);
Cout = Cin*(0.5*T)^(p-q)*prod(1-0.5*T*uin)/prod(1-0.5*T*vin);
uout = [(1+0.5*T*uin)./(1-0.5*T*uin),-ones(1,p-q)];
vout = (1+0.5*T*vin)./(1-0.5*T*vin);
a = 1; b = 1;
for k = 1:length(vout), a = conv(a,[1,-vout(k)]); end
for k = 1:length(uout), b= conv(b,[1,-uout(k)]); end
a = real(a); b = real(Cout*b); Cout = real(Cout);
```

Program 10.8 Digital IIR filter design.

```
function [b,a,v,u,C] = iirdes(typ,band,theta,deltap,deltas);
% Synopsis: [b,a,v,u,C] = iirdes(typ,band,theta,deltap,deltas).
% Designs a digital IIR filter to meet given specifications.
% Input parameters:
% typ: the filter class: 'but', 'ch1', 'ch2', or 'ell'
% band: 'l' for LP, 'h' for HP, 'p' for BP, 's' for BS
% theta: an array of band-edge frequencies, in increasing
%        order; must have 2 frequencies if 'l' or 'h',
%        4 if 'p' or 's'
% deltap: pass-band ripple/s (possibly 2 for 's')
% deltas: stop-band ripple/s (possibly 2 for 'p')
% Output parameters:
% b, a: the output polynomials
% v, u, C: the output poles, zeros, and constant gain.

% Prewarp frequencies (with T = 1)
omega = 2*tan(0.5*theta);
% Transform specifications
if (band == 'l'), wp = omega(1); ws = omega(2);
elseif (band == 'h'), wp = 1/omega(2); ws = 1/omega(1);
elseif (band == 'p'),
   wl = omega(2); wh = omega(3); wp = 1;
   ws = min(abs((omega([1,4]).^2-wl*wh) ...
        ./((wh-wl)*omega([1,4]))));
elseif (band == 's'),
   wl = omega(2); wh = omega(3); ws = 1;
   wp = 1/min(abs((omega([1,4]).^2-wl*wh) ...
        ./((wh-wl)*omega([1,4]))));
end
% Get low-pass filter parameters
[N,w0,epsilon,m] = lpspec(typ,wp,ws,min(deltap),min(deltas));
% Design low-pass filter
[b,a,v1,u1,C1] = analoglp(typ,N,w0,epsilon,m);
% Transform to the required band
ww = 1; if (band == 'p' | band == 's'), ww = [wl,wh]; end
[b,a,v2,u2,C2] = analogtr(band,v1,u1,C1,ww);
% Perform bilinear transformation
[b,a,v,u,C] = bilin(v2,u2,C2,1);
```

10.14 Problems

10.1 Explain how the approximation (10.4) is derived.

10.2 This problem examines certain properties of the discrimination factor.

(a) Is the discrimination factor d defined in (10.2) typically greater than 1 or less than 1? Explain.

(b) Derive an approximation for d under the assumption that both δ_p and δ_s are much smaller than 1.

10.3 Derive (10.19). Explain the meaning of equality at the lower end of the range, and the meaning of equality at the higher end of the range.

10.4 An analog filter is required to have pass-band ripple $1 \pm \delta'_\mathrm{p}$ and stop-band attenuation δ'_s. Show how to choose δ_p, δ_s for an analog filter $H^\mathrm{L}(s)$, and a gain C such that the filter $CH^\mathrm{L}(s)$ will meet the requirements.

10.5 We are given the analog filter

$$H^\mathrm{L}(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}.$$

For this filter, we are given that

$$A_\mathrm{p} = 0.5\,\mathrm{dB}, \quad A_\mathrm{s} = 20\,\mathrm{dB}.$$

Find the discrimination factor d and the selectivity factor k of this filter to four decimal places.

10.6 Let $H_1^\mathrm{L}(s)$ and $H_2^\mathrm{L}(s)$ be normalized Butterworth filters of orders 2 and 3, respectively. Let $H^\mathrm{L}(s)$ be their cascade connection, that is, $H^\mathrm{L}(s) = H_1^\mathrm{L}(s)H_2^\mathrm{L}(s)$.

(a) Show that $H^\mathrm{L}(s)$ is *not* a Butterworth filter.

(b) Show that, even though $H^\mathrm{L}(s)$ is not a Butterworth filter, it shares certain properties of a fifth order normalized Butterworth filter; specify which properties.

10.7 Prove that $T_N(x)$ is a symmetric function of x when N is even, and an antisymmetric function of x when N is odd.

10.8 Use (10.21) and show that $T_N(0) = \pm 1$ for even N, and $T_N(0) = 0$ for odd N.

10.9 Let $H^\mathrm{L}(s)$ be a Chebyshev-I low-pass filter, with

$$N = 3, \quad \omega_\mathrm{p} = 3, \quad \omega_\mathrm{s} = 10, \quad \delta_\mathrm{s} = 0.02.$$

The filter meets the specifications *exactly* in both the pass band and the stop band. Compute $H^\mathrm{L}(s)$ and express it as a ratio of two polynomials, with coefficients accurate to 4 decimal digits.

10.10 Consider the function

$$y(x) = \frac{\log_e(x + \sqrt{x^2 - 1})}{\log_e x}.$$

(a) Prove that $y(x)$ tends to infinity as x approaches 1 from above; prove that it tends to 1 as x approaches ∞.

(b) Use MATLAB for plotting this function in the range $1.01 \le x \le 10$. What is the nature of this function?

(c) Use your conclusion from part b and show that, for given analog low-pass filter specifications, the order of a Chebyshev filter will always be smaller than or equal to the order of a Butterworth filter.

10.11 A second-order, low-pass analog Chebyshev-II filter is known to satisfy $H^F(1) = 0$. The filter known to have been designed with $\varepsilon = 0.1$. Compute $H^L(s)$ and express it as a ratio of two polynomials with real coefficients.

10.12 Joan, a junior engineer, is assigned the following task. She is given an analog low-pass filter and told that it is Chebyshev, but she does not know whether it is of the first or second kind. She is told that the filter was designed using the procedures described in this chapter. Joan is requested to find the transfer function of the filter, with coefficient values to four decimal digits.

Joan performs a series of tests, and finds that:

- $|H^F(0)|^2 = 1$.
- $|H^F(10)|^2 = 0.2$.
- $H^F(10.8239) = H^F(25.1313) = 0$.
- $H^F(\omega)$ is not identically zero for any frequency in the range $0 \le \omega \le 1000$, except for the two aforementioned frequencies.

Based on this information, help Joan to find the transfer function to the required accuracy.

10.13 We wish to sample a musical signal in order to store it on a digital tape. The sampling frequency is to be 48 kHz. It is required to design an analog low-pass antialiasing for this purpose. The specifications of the filter are

$$\omega_p = (2\pi \cdot 19)10^3, \quad \omega_s = (2\pi \cdot 24)10^3, \quad \delta_p = 0.05, \quad \delta_s = 10^{-4}.$$

As a result of the nature of music signals (whose energy typically decreases as the frequency increases), the pass-band magnitude response is required to be monotone. For each eligible filter among the ones we have studied in this chapter, compute the relevant design parameters (N, ε, ω_0, ...).

10.14 It is required to design an analog band-pass filter whose frequency response has no ripple in the pass band. The specifications are

$$\tilde{\omega}_{s,1} = 20, \quad \tilde{\omega}_{p,1} = 50, \quad \tilde{\omega}_{p,2} = 20{,}000, \quad \tilde{\omega}_{s,2} = 45{,}000.$$

$$\tilde{A}_p = 3\,\text{dB}, \quad \tilde{A}_{s,1} = \tilde{A}_{s,2} = 25\,\text{dB}.$$

What is the minimum order of a filter that meets these specifications?

10.15 Sharon and Irwin were asked to design an analog band-pass filter according to the specifications

$$\tilde{\delta}_p = 0.02, \quad \tilde{\delta}_{s,1} = \tilde{\delta}_{s,2} = 0.005,$$

$$\tilde{\omega}_{s,1} = 5, \quad \tilde{\omega}_{p,1} = 10, \quad \tilde{\omega}_{p,2} = 40, \quad \tilde{\omega}_{s,2} = 160.$$

The filter must be ripple free in all bands.

Sharon designed a band-pass Butterworth filter, as was studied in this chapter. Irwin decided to construct the band-pass filter by a cascade connection of a high-pass

Butterworth filter $H_1^{\mathrm{L}}(s)$ and a low-pass Butterworth filter $H_2^{\mathrm{L}}(s)$. He used band-edge frequencies $\widetilde{\omega}_{\mathrm{s},1}, \widetilde{\omega}_{\mathrm{p},1}$ for the high-pass filter and band-edge frequencies $\widetilde{\omega}_{\mathrm{p},2}, \widetilde{\omega}_{\mathrm{s},2}$ for the low-pass filter. He used the same δ_{p} for both filters, and the same δ_{s}, choosing these two parameters to satisfy the tolerances $\widetilde{\delta}_{\mathrm{p}}, \widetilde{\delta}_{\mathrm{s},1}, \widetilde{\delta}_{\mathrm{s},2}$ in the overall filter.

Whose design leads to a band-pass filter of lower order?

10.16 Find the number of zeros of a band-pass filter $\widetilde{H}^{\mathrm{L}}(\widetilde{s})$, derived from a low-pass filter $H^{\mathrm{L}}(s)$ according to (10.83). Treat the four filter classes separately. Show that, in all cases, the zeros of the band-pass filter are on the imaginary axis.

10.17 Find the number of zeros of a band-stop filter $\widetilde{H}^{\mathrm{L}}(\widetilde{s})$, derived from a low-pass filter $H^{\mathrm{L}}(s)$ according to (10.93). Treat the four filter classes separately. Show that, in all cases, the zeros of the band-stop filter are on the imaginary axis.

10.18 Explain why the impulse invariant transform is not applicable to Chebyshev-II and elliptic filters of even orders (even when they are low pass or band pass).

10.19 The digital filter

$$H^{z}(z) = \frac{0.008502z^{-1} + 0.007272z^{-2}}{1 - 2.5060z^{-1} + 2.1470z^{-2} - 0.6252z^{-3}}$$

is known to have been obtained from a Chebyshev-I analog low-pass filter by an impulse invariant transformation with $T = 0.1$. Find the parameters ε and ω_0 of the analog filter [use MATLAB for finding the poles of $H^{z}(z)$].

10.20 A Chebyshev-I filter of order $N = 3$ and $\omega_0 = 1$ is known to have a pole at $s = -1$ rad/s.

(a) Find the other two poles of the filter and its parameter ε.

(b) The filter is transformed to the z domain using a bilinear transform with $T = 2$. Compute the transfer function of the digital filter $H^{z}(z)$.

10.21 A first-order analog filter $H^{\mathrm{L}}(s)$ has a zero at $s = -2$, a pole at $s = -2/3$, and its DC gain is $H^{\mathrm{L}}(0) = 1$. Bilinear transformation of $H^{\mathrm{L}}(s)$ yields the digital filter $H^{z}(z) = K/(1 - \alpha z^{-1})$. Find K, α, and the sampling interval T.

10.22 We are given the digital filter

$$H^{z}(z) = \frac{2\theta_0^2(1+z)^2}{(z-1)^2 + 2\theta_0(z^2-1) + 2\theta_0^2(z+1)^2},$$

where θ_0 is a given positive parameter. We are also given that $H^{z}(z)$ was obtained from an analog filter $H^{\mathrm{L}}(s)$ by a bilinear transform with $T = 2$.

(a) What type of filter is $H^{z}(z)$? Low pass, high pass, band pass, or band stop? Give reasons.

(b) Compute the transfer function, as well as the poles and zeros, of the analog filter $H^{\mathrm{L}}(s)$. Is this filter Butterworth, Chebyshev-I, Chebyshev-II, or elliptic?

(c) At what frequency $\omega_{3\mathrm{db}}$ will the analog filter have an attenuation of 3 dB?

(d) At what frequency $\theta_{3\mathrm{db}}$ will the digital filter have an attenuation of 3 dB?

10.23 We are given a digital filter $H^{z}(z)$ having two zeros at $z = -1$, a pole at $z = j\alpha$, and a pole at $z = -j\alpha$, where α is real, $0.6 < \alpha < 1$. The filter was obtained from an analog filter $H^{\mathrm{L}}(s)$ using the bilinear transform.

(a) Draw an approximate plot of $H^{\mathrm{f}}(\theta)$ in the range $0 \le \theta \le \pi$.

(b) Compute $H^{\mathrm{L}}(s)$ and express it as a ratio of two polynomials, with α and T as parameters.

(c) If $\alpha = 1/\sqrt{2}$ and $T = 1$, is $H^{\mathrm{L}}(s)$ Butterworth, Chebyshev-I, Chebyshev-II, or elliptic?

10.24 Figure 10.30 shows the pole-zero maps of three digital filters. For each of the three, state if it could have been obtained from Butterworth, Chebyshev-I, or Chebyshev-II analog filter, using a bilinear transform with $T = 1$. If so, write the corresponding $H^{\mathrm{L}}(s)$ and state its kind. If not, explain why.

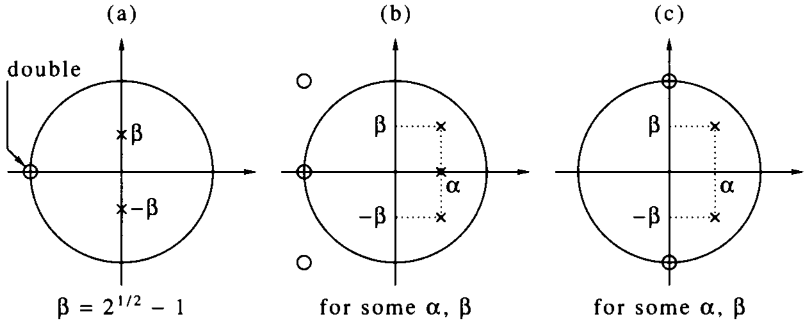

Figure 10.30 Pertaining to Problem 10.24; the circles have radii 1; ∘s indicate zeros; ×s indicate poles.

10.25 A digital IIR filter is to meet the following requirements:

- Its denominator degree p and numerator degree q should be equal.
- It must have infinite attenuation at frequency $\theta = \pi/3$.
- Its poles must be equal to those of a normalized Butterworth filter transformed to the digital domain by a bilinear transform with $T = \sqrt{2}$.
- Its DC gain must be equal to 1.
- It must have minimal order.

Find the transfer function of the filter.

10.26 As we recall, the transformation $s = \omega_0/\tilde{s}$ converts a normalized analog low-pass filter to an unnormalized high-pass filter. Let us transform s to z and $\tilde{s}$ to $\tilde{z}$, using the bilinear transform in both cases, with T as a parameter.

(a) Express $\tilde{z}$ as a function of z.

(b) Prove that the unit circle in the z plane is transformed to the unit circle in the $\tilde{z}$ plane, and find the frequency variable $\tilde{\theta}$ as a function of θ. It is convenient to define an auxiliary parameter depending on ω_0 and T.

(c) Show that the transformation from z to $\tilde{z}$ is low pass to high pass.

(d) Suppose that we are given a low-pass filter $H^{z}(z)$ whose pass-band cutoff frequency is θ_{p}, and we wish to obtain a high-pass filter in the $\tilde{z}$ domain whose pass-band cutoff frequency is $\tilde{\theta}_{\mathrm{p}}$. Find ω_0 in the transformation that will achieve this.

10.27 We are given a low-pass digital Butterworth filter $H^z(z)$ whose -3 dB frequency is θ_0. We transform this filter using the substitution

$$\tilde{H}^z(\tilde{z}) = H^z(z)\Big|_{z^{-1}=\frac{\tilde{z}^{-1}-\alpha}{1-\alpha\tilde{z}^{-1}}},$$

where $-1 < \alpha < 1$.

(a) Show that $\tilde{H}^z(\tilde{z})$ is also a low-pass filter.

(b) We want the -3 dB frequency of $\tilde{H}^z(\tilde{z})$ to be $\tilde{\theta}_0$. Find α that will accomplish this, as a function of θ_0 and $\tilde{\theta}_0$.

10.28* It is required to design a multiband digital IIR filter according to the following specifications:

- Gain 1 with $\delta_p = 0.01$ in the frequency band 0–0.2π.
- Gain 0 with $\delta_s = 0.05$ in the frequency band 0.25π–0.45π.
- Gain 1 with $\delta_p = 0.01$ in the frequency band 0.5π–0.7π.
- Gain 0 with $\delta_s = 0.05$ in the frequency band 0.75π–π.

It is proposed to build the filter as a parallel connection of two filters, $H_1^z(z)$ and $H_2^z(z)$, where the first is low pass and the second is band pass.

Design the system. Note: This problem does not have a unique solution. Any solution meeting the specifications is acceptable. However, the lower the complexity of the system, the better.

10.29* The Nth-order *Bessel polynomial* $B_N(s)$ is defined by

$$B_N(s) = \sum_{k=0}^{N} b_{N,k}s^k, \quad \text{where} \quad b_{N,k} = \frac{(2N-k)!}{2^{N-k}k!(N-k)!}.$$

The Nth-order *Bessel filter* is defined by

$$H^L(s) = \frac{1}{B_N(s)}.$$

(a) Show that Bessel polynomials satisfy the recursion

$$B_N(s) = (2N-1)B_{N-1}(s) + s^2 B_{N-2}(s), \quad B_0(s) = 1,\ B_1(s) = s + 1.$$

(b) Write a MATLAB program that computes the coefficients of the Nth-order Bessel polynomial.

(c) Compute and plot the magnitude and phase responses of a Bessel filter of orders 2 through 8. Observe in particular the phase response at low frequency. State your conclusions about the characteristics of Bessel filters.

10.30* Let $H^L(s)$ be a given analog filter. Define $g(t)$ as the response of the filter to a unit-step input, that is,

$$g(t) = \int_0^t h(\tau)d\tau.$$

Let $g[n] = g(nT)$ be the point sampling of $g(t)$. Define $H^z(z)$ as the digital filter whose response to a discrete-time, unit-step input is $g[n]$, that is,

$$g[n] = \sum_{m=0}^{n} h[m].$$

The filter $H^z(z)$ is called the *step invariant transform* of $H^L(s)$.

(a) Derive a formula and a method of computation for the step invariant transform.

(b) Compare this method with the impulse invariant transform from all points of view we have discussed.

10.31* Define the following s-to-z transformation

$$H^{\mathrm{z}}(z) = H^{\mathrm{L}}(s)\Big|_{s=\frac{z-1}{T}}. \tag{10.129}$$

The operation (10.102) is called *forward difference transformation*, in analogy to (10.102). Show that this transformation does not preserve stability; that is, a stable analog filter $H^{\mathrm{L}}(s)$ may be transformed to an unstable digital filter $H^{\mathrm{z}}(z)$.

10.32* The purpose of this problem is to show that the bilinear transform can be interpreted as approximation of continuous-time integration by discrete-time *trapezoidal integration.*

Let $y(t)$ be the integral of $x(t)$, that is,

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \iff Y^{\mathrm{L}}(s) = \frac{X^{\mathrm{L}}(s)}{s}.$$

Perform a bilinear transform on the s-domain description of the integral and obtain a z-domain transfer function relating $X^{\mathrm{z}}(z)$ to $Y^{\mathrm{z}}(z)$. Then transform back to time domain. Finally, interpret the resulting difference equation as approximate integration and explain the name *trapezoidal integration.* Hint: The integral relationship between $y(t)$ and $x(t)$ implies

$$y(nT) = y(nT - T) + \int_{nT-T}^{nT} x(\tau)d\tau.$$

10.33* This problem introduces the *Goertzel algorithm* [Goertzel, 1958] for computing the Fourier transform of a finite-duration signal at a single frequency point θ_0. The relation of this problem to IIR filters will become clear soon.

(a) Show that the operation

$$X^{\mathrm{f}}(\theta_0) = \sum_{n=0}^{N-1} x[n]e^{-j\theta_0 n}$$

can be expressed as

$$X^{\mathrm{f}}(\theta_0) = e^{-j\theta_0(N-1)}\{x * h\}[N-1], \tag{10.130}$$

where $h[n]$ is the impulse response of a causal IIR filter, given by

$$h[n] = e^{j\theta_0 n}, \quad n \geq 0.$$

(b) Show that the filter $h[n]$ has the transfer function

$$H^{\mathrm{z}}(z) = \frac{1}{1 - e^{j\theta_0}z^{-1}}.$$

Hence show that, if $x[n]$ is a general complex sequence, (10.130) can be performed in about $4N$ real multiplications and $4N$ real additions. Note that this is no more efficient than direct computation of $X^{\mathrm{f}}(\theta_0)$.

(c) Show that $H^{\mathrm{z}}(z)$ can be written as

$$H^{\mathrm{z}}(z) = (1 - e^{-j\theta_0}z^{-1})H_1^{\mathrm{z}}(z),$$

where

$$H_1^{\mathrm{z}}(z) = \frac{1}{1 - 2\cos\theta_0 z^{-1} + z^{-2}}.$$

(d) Show that (10.130) can be carried out by the following steps:

$$y[n] = \{x * h_1\}[n], \quad 0 \le n \le N-1, \tag{10.131a}$$

$$X^{\mathrm{f}}(\theta_0) = e^{-j\theta_0(N-1)}(y[N-1] - e^{-j\theta_0}y[N-2]). \tag{10.131b}$$

(e) Count the total number of real operations in (10.131). Compare it with the number of operations in direct computation of $X^{\mathrm{f}}(\theta_0)$.

10.34* Consider a sampled-data system such as the one shown in Figure 10.29, except that instead of the ZOH there is a reconstructor whose impulse response is

$$h(t) = \begin{cases} 1 + \frac{t}{T}, & 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the transfer function $H^{\mathrm{z}}(z)$ from $x[n]$ to $y[n]$.

(b) Compute $H^{\mathrm{z}}(z)$ for the special case $G^{\mathrm{L}}(s) = 1/s$.

10.35* Consider the sampled-data system described in Section 10.11. Suppose that the input signal $x[n]$ is delayed by Δ before being fed to the ZOH. In other words, the response of the ZOH to $x[n]$ is $h_{\mathrm{zoh}}(t - nT - \Delta)$. Assume that $\Delta < T$. Assume also that $y[n]$ is obtained by sampling synchronously with $x[n]$ *before the delay*, that is, $y[n] = y(nT)$ as before. Show that the equivalent discrete-time system relating $y[n]$ to $x[n]$ has the impulse response

$$h[n] = u(nT) - u(nT - T),$$

where

$$u(t) = \left\{\mathcal{L}^{-1}\left[\frac{G(s)e^{-s\Delta}}{s}\right]\right\}(t).$$

In control applications it is common to define

$$m = 1 - \frac{\Delta}{T}$$

and express $h[n]$ as a function of the parameter m. The function $H^{\mathrm{z}}(z)$ is then called the *modified z-transform* of $G^{\mathrm{L}}(s)$ [Jury, 1954]. Modified z-transforms are used when the input to the ZOH is delayed with respect to the system's clock, for example, because of finite computation time.