Chapter 4

The Discrete Fourier Transform

A discrete-time signal $x[n]$ can be recovered unambiguously from its Fourier transform $X^{\mathrm{f}}(\theta)$ through the inverse transform formula (2.95). This, however, requires knowledge of $X^{\mathrm{f}}(\theta)$ for all $\theta \in [-\pi, \pi]$. Knowledge of $X^{\mathrm{f}}(\theta)$ at a finite subset of frequencies is not sufficient, since the sequence $x[n]$ has an infinite number of terms in general. If, however, the signal has finite duration, say $\{x[n],\ 0 \le n \le N-1\}$, then knowledge of $X^{\mathrm{f}}(\theta)$ at N frequency points *may* be sufficient for recovering the signal, provided these frequencies are chosen properly. In other words, we may be able to sample the Fourier transform at N points and compute the signal from these samples. An intuitive justification of this claim can be given as follows: The Fourier transform is a linear operation. Therefore, the values of $X^{\mathrm{f}}(\theta)$ at N values of θ, say $\{\theta[k],\ 0 \le k \le N-1\}$, provide N linear equations at N unknowns: the signal values $\{x[n],\ 0 \le n \le N-1\}$. We know from linear algebra that such a system of equations has a unique solution if the coefficient matrix is nonsingular. Therefore, if the frequencies are chosen so as to satisfy this condition, the signal values can be computed unambiguously.

The sampled Fourier transform of a finite-duration, discrete-time signal is known as the *discrete Fourier transform* (DFT). The DFT contains a finite number of samples, equal to the number of samples N in the given signal. The DFT is perhaps the most important tool of digital signal processing. We devote most of this chapter to a detailed study of this transform, and to the closely related concept of circular convolution. In the next chapter we shall study fast computational algorithms for the DFT, known as fast Fourier transform algorithms. Then, in Chapter 6, we shall examine the use of the DFT for practical problems.

The DFT is but one of a large family of transforms for discrete-time, finite-duration signals. Common to most of these transforms is their interpretation as frequency domain descriptions of the given signal. The magnitude of the transform of a sinusoidal signal should be relatively large at the frequency of the sinusoid and relatively small at other frequencies. If the transform is linear, then the transform of a sum of sinusoids is the sum of the transforms of the individual sinusoids, so it should have relatively large magnitudes at the corresponding frequencies. Therefore, a standard way of understanding and interpreting a transform is to examine its action on a sinusoidal signal.

Among the many relatives of the DFT, the discrete cosine transform (DCT) has gained importance in recent years, because of its use in coding and compression of images. In recognition of its importance, we devote a section in this chapter to the

DCT. The discrete sine transform (DST), which is closely related to the DCT, is also discussed briefly.

4.1 Definition of the DFT and Its Inverse

Let the discrete-time signal $x[n]$ have finite duration, say in the range $0 \le n \le N-1$. The Fourier transform of this signal is

$$X^{\mathrm{f}}(\theta) = \sum_{n=0}^{N-1} x[n]e^{-j\theta n}. \tag{4.1}$$

Let us sample the frequency axis using a total of N equally spaced samples in the range $[0, 2\pi)$, so the sampling interval is $2\pi/N$; in other words, we use the frequencies

$$\theta[k] = \frac{2\pi k}{N}, \quad 0 \le k \le N-1. \tag{4.2}$$

The result is, by definition, the discrete Fourier transform. Mathematically,

$$X^{\mathrm{d}}[k] = \{\mathcal{D}x\}[k] = \sum_{n=0}^{N-1} x[n]\exp\left(-\frac{j2\pi kn}{N}\right), \quad 0 \le k \le N-1. \tag{4.3}$$

Figure 4.1 illustrates the Fourier transform of a discrete-time signal and its DFT samples.

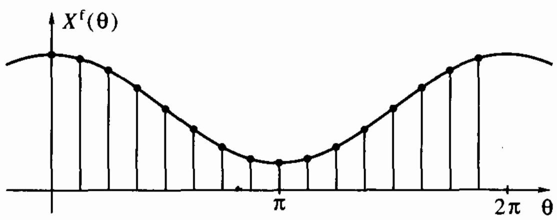

Figure 4.1 The relationship of the DFT to the Fourier transform. Solid line: Fourier transform; Circles: DFT samples (shown for $N = 16$).

In working with the DFT, it is common to use the notation

$$W_N = \exp\left(\frac{j2\pi}{N}\right). \tag{4.4}$$

Also, it is common to denote the DFT operation for a length-N signal by $\mathrm{DFT}_N\{x[n]\}$. With these notations we can rewrite (4.3) as

$$X^{\mathrm{d}}[k] = \mathrm{DFT}_N\{x[n]\} = \sum_{n=0}^{N-1} x[n]W_N^{-kn}, \quad 0 \le k \le N-1. \tag{4.5}$$

The sequence of integer powers of W_N, $\{W_N^n, \ -\infty < n < \infty\}$ is ubiquitous in the DFT world. This sequence is periodic with period N, since

$$W_N^N = \exp\left(\frac{j2\pi N}{N}\right) = e^{j2\pi} = 1 = W_N^0.$$

Figure 4.2 illustrates this sequence in the range $0 \le n \le N-1$, for even and odd values of N. The magnitude of all the numbers in the sequence is 1, and the phases are equally spaced, starting at zero. The phase π appears if and only if N is even, and then it corresponds to $n = N/2$.

The formula given in the following lemma is a useful tool in deriving various DFT-related results.

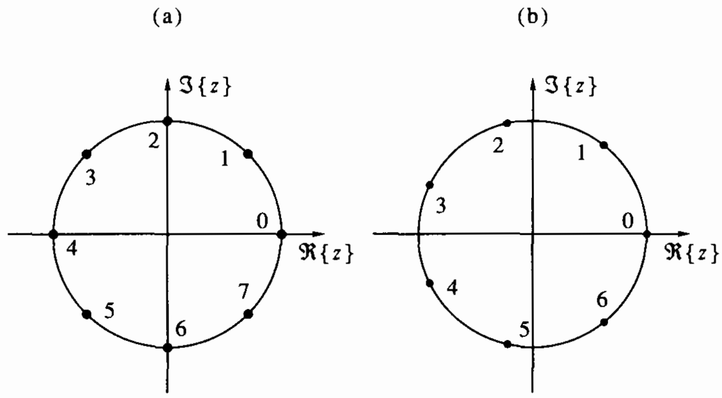

Figure 4.2 The sequence W_N^n in the complex plane: (a) even N; (b) odd N. The numbers indicate the values of n.

Theorem 4.1 (lemma)

$$\sum_{n=0}^{N-1} W_N^{kn} = N\delta[k \bmod N] = \begin{cases} 0, & (k \bmod N) \neq 0, \\ N, & (k \bmod N) = 0. \end{cases} \tag{4.6}$$

Proof The left side of (4.6) is a geometric series with parameter

$$q = W_N^k = \exp\left(\frac{j2\pi k}{N}\right).$$

When k is a multiple of N, say $k = mN$, we have $q = 1$ because

$$e^{j2\pi m} = 1, \quad \text{for all } m \in \mathbb{Z}.$$

Therefore, in this case, the sum evaluates to N. When k is not a multiple of N we have, by the summation formula for a geometric series,

$$\sum_{n=0}^{N-1} W_N^{kn} = \frac{W_N^{Nk} - 1}{W_N^k - 1} = \frac{1 - 1}{W_N^k - 1} = 0. \tag{4.7}$$

In summary,

$$\sum_{n=0}^{N-1} W_N^{kn} = N\delta[k \bmod N]. \tag{4.8}$$

□

Example 4.1 The DFT of

$$x[n] = \begin{cases} 1, & n = 0, \\ 0, & 1 \leq n \leq N-1, \end{cases} \tag{4.9}$$

is

$$X^{\mathrm{d}}[k] = 1, \quad 0 \leq k \leq N-1. \tag{4.10}$$

□

Example 4.2 The DFT of

$$x[n] = 1, \quad 0 \leq n \leq N-1, \tag{4.11}$$

is

$$X^{\mathrm{d}}[k] = \sum_{n=0}^{N-1} W_N^{-kn} = \begin{cases} N, & k = 0, \\ 0, & 1 \le k \le N-1. \end{cases} \tag{4.12}$$

□

Example 4.3 As we said at the beginning of this chapter, a good way of getting a feeling for a time-to-frequency transform is to examine its action on a sinusoidal signal. Since we are dealing with finite-duration signals, let us compute the DFT of a finite segment of a sinusoid:

$$x[n] = \cos(\theta_0 n + \phi_0), \quad 0 \le n \le N-1, \tag{4.13}$$

where $0 < \theta_0 < \pi$. Expressing the cosine in terms of two complex exponentials, we get

$$\begin{aligned} X^{\mathrm{d}}[k] &= \sum_{n=0}^{N-1} \cos(\theta_0 n + \phi_0) \exp\left(-\frac{j2\pi kn}{N}\right) \\ &= 0.5e^{j\phi_0} \sum_{n=0}^{N-1} \exp\left[jn\left(\theta_0 - \frac{2\pi k}{N}\right)\right] + 0.5e^{-j\phi_0} \sum_{n=0}^{N-1} \exp\left[-jn\left(\theta_0 + \frac{2\pi k}{N}\right)\right] \\ &= 0.5e^{j\phi_0} \frac{1 - e^{j\theta_0 N}}{1 - \exp\left[j\left(\theta_0 - \frac{2\pi k}{N}\right)\right]} + 0.5e^{-j\phi_0} \frac{1 - e^{-j\theta_0 N}}{1 - \exp\left[-j\left(\theta_0 + \frac{2\pi k}{N}\right)\right]}. \end{aligned} \tag{4.14}$$

In particular, consider the case of θ_0 an integer multiple of $2\pi/N$, say

$$\theta_0 = \frac{2\pi m}{N}.$$

Then we get

$$X^{\mathrm{d}}[k] = \begin{cases} 0.5Ne^{j\phi_0}, & k = m, \\ 0.5Ne^{-j\phi_0}, & k = N-m, \\ 0, & \text{otherwise}. \end{cases} \tag{4.15}$$

The magnitude of the DFT indeed peaks at $k = m$, as well as at $k = N - m$. The phase of the DFT at $k = m$ is ϕ_0, the phase of the given sinusoidal signal. The DFT at all other frequencies is zero in this case. The conclusion is that, at least for sinusoids at particular frequencies—the integer multiples of $2\pi/N$—the DFT yields the expected result. Figure 4.3(a) illustrates the magnitude DFT for $N = 64$, $\theta_0 = 2\pi \cdot 15/64$, and $\phi_0 = 0$.

When θ_0 is not an integer multiple of $2\pi/N$, the DFT expression is less transparent. However, we still see from (4.14) that the magnitude of the denominator of the first term is minimized for k nearest to $N\theta_0/2\pi$, and the magnitude of the denominator of the second term is minimized for k nearest to $N(1 - \theta_0/2\pi)$. The numerators vary in magnitude between 0 and 2, but they are not identically zero if θ_0 is not an integer multiple of $2\pi/N$. Therefore, the magnitude DFT peaks for $k \approx N\theta_0/2\pi$, as well as for $k \approx N(1 - \theta_0/2\pi)$. Figure 4.3(b) illustrates the magnitude DFT for $N = 64$, $\theta_0 = 2\pi \cdot 15.25/64$, and $\phi_0 = 0$. As we see, the peak is at $k = 15$, which is the integer nearest to 15.25. The DFT values at all other frequencies are not zero, since the frequency is not an integer multiple of $2\pi/N$. Figure 4.3(c) illustrates the magnitude DFT for $N = 64$, $\theta_0 = 2\pi \cdot 15.5/64$, and $\phi_0 = 0$. In this case, since the frequency is exactly in the middle between $k = 15$ and $k = 16$, the peaks at these two values of k are equal.

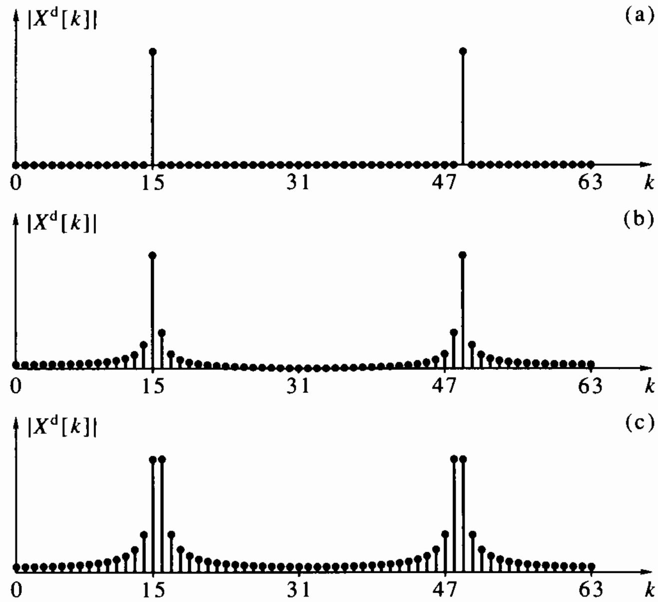

Figure 4.3 The magnitude DFT of a sinusoidal signal ($N = 64$): (a) $\theta_0 = 2\pi \cdot 15/64$; (b) $\theta_0 = 2\pi \cdot 15.25/64$; (c) $\theta_0 = 2\pi \cdot 15.5/64$.

In conclusion, the magnitude DFT of a sinusoidal signal indeed exhibits a peak at k approximately proportional to the frequency of the signal. If the frequency is an integer multiple of $2\pi/N$, we also get the phase of the sinusoid from the phase of the DFT at the peak point. □

The DFT values $\{X^d[k],\ 0 \le k \le N-1\}$ uniquely define the sequence $x[n]$ through the *inverse DFT formula* (IDFT):

Theorem 4.2 (inverse DFT)

$$x[n] = \{\mathcal{D}^{-1}X^d\}[n] = \mathrm{IDFT}_N\{X^d[k]\} = \frac{1}{N}\sum_{k=0}^{N-1} X^d[k]W_N^{kn}, \quad 0 \le n \le N-1. \tag{4.16}$$

Proof

$$\frac{1}{N}\sum_{k=0}^{N-1} X^d[k]W_N^{kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x[m]W_N^{-km}\right]W_N^{kn} = \frac{1}{N}\sum_{m=0}^{N-1} x[m]\left[\sum_{k=0}^{N-1} W_N^{(n-m)k}\right]$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]N\delta[(n-m) \bmod N] = x[n]. \tag{4.17}$$

□

The procedure `bfdft` in Program 4.1 illustrates possible MATLAB implementation of the DFT and its inverse. This is a brute-force implementation, obtained directly from the definition. The program computes the direct DFT if the variable `swtch` is 0, and the inverse DFT if it is nonzero. You are advised not to use this program for serious applications, since MATLAB offers a much more efficient implementation. Direct DFT in MATLAB is the function `fft` and inverse DFT is the function `ifft`. We shall learn about these implementations in Chapter 5.

Example 4.4 Let N be even, and

$$X^{\mathrm{d}}[k] = \begin{cases} 1, & k = 0, \\ -1, & k = 0.5N, \\ 0, & \text{otherwise}. \end{cases}$$

The inverse DFT is given by

$$x[n] = N^{-1}(W_N^0 - W_N^{nN/2}) = N^{-1}[1 - (-1)^n],$$

since $W_N^{N/2} = e^{j\pi} = -1$. □

A peculiarity of the DFT definition is the range of frequencies represented by the variable k. According to (4.2), the range $0 \le k \le \lfloor N/2 \rfloor$ corresponds to frequencies $0 \le \theta \le \pi$, whereas the range $\lceil N/2 \rceil \le k \le N-1$ corresponds to $\pi \le \theta < 2\pi$. The latter range is equivalent to $-\pi \le \theta < 0$. Therefore, the first $\lfloor N/2 \rfloor$ values of k correspond to positive frequencies and the remaining correspond to negative frequencies. The point $k = 0$ always corresponds to zero frequency and $k = N/2$ (for even N) corresponds to $\theta = \pm\pi$. When plotting $X^{\mathrm{d}}[k]$ as a function of k from 0 to $N-1$, we will see the positive frequencies on the left and the negative frequencies on the right, which is opposite to common custom. For example, low-pass frequency responses will appear as high pass, and vice versa. To avoid this confusion, one may wish to interchange the two halves of the DFT points before plotting them, that is, to rearrange the sequence (for even N) as

$$\{X^{\mathrm{d}}[N/2], X^{\mathrm{d}}[N/2+1], \ldots, X^{\mathrm{d}}[N-1], X^{\mathrm{d}}[0], \ldots, X^{\mathrm{d}}[N/2-1]\}.$$

Figure 4.4 illustrates the visual appearance of the DFT of a low-pass signal in the original order and after rearrangement.

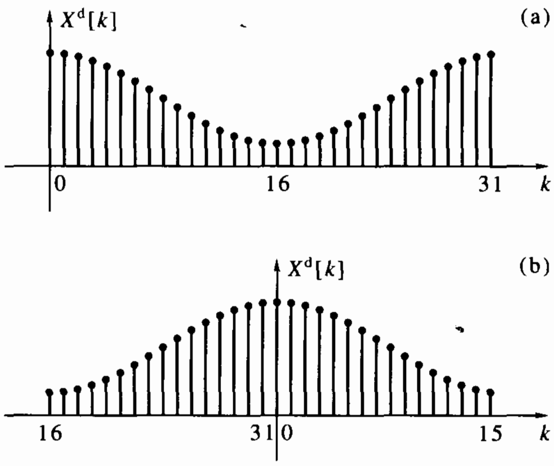

Figure 4.4 Rearrangement of the DFT: (a) index k in original order; (b) index k in a shifted order (shown for $N = 32$).

The following MATLAB expression interchanges the halves of a vector:

```
y = [x(ceil(length(x)/2)+1:length(x)), x(1:ceil(length(x)/2))];
```

the MATLAB function `fftshift` does the same operation.

The *frequency resolution* of the DFT is the spacing between two adjacent frequencies, that is, $\Delta\theta = 2\pi/N$. This number is also called the *frequency bin* of the DFT. The corresponding physical frequency resolution (or frequency bin), in hertz, is

$$\Delta f = 1/NT,$$

where T is the sampling interval. Note that the product NT is the total duration of the continuous-time signal from which the discrete-time signal was obtained by sampling. Therefore we get the following important relationship:

> The frequency resolution of the DFT is the inverse of the signal duration.[1]

Note, in particular, that the frequency resolution does not depend on the number of samples in the interval! This is perhaps counterintuitive and occasionally confusing, so it is worth bearing in mind.

So far we have defined the DFT for finite-duration signals. However, the definition remains unchanged if we consider discrete-time periodic signals instead, by using the DFT definition on *one period* of the signal. Thus, the DFT of a finite-duration signal $\{x[n],\ 0 \le n \le N-1\}$ is identical to the DFT of its periodic extension $\tilde{x}[n]$. The same holds for the transformed sequence $X^{\mathrm{d}}[k]$; instead of limiting k to the range $0 \le k \le N-1$, we can use its periodic extension for all $k \in \mathbb{Z}$. There is no need to change either of (4.5) or (4.16), since W_N^{kn} is periodic in both k and n (with period N in both cases). From now on we shall use the DFT and inverse DFT for the finite-duration sequences and their periodic extensions interchangeably.

The relationship

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k]W_N^{kn}, \quad n \in \mathbb{Z}, \tag{4.18}$$

can be viewed as a Fourier series representation of the periodic signal $\tilde{x}[n]$. Clearly, a discrete-time periodic signal has no more than a finite number of harmonics, equal to the period N. Thus, the IDFT formula (4.16) fulfills the role of Fourier series of discrete-time periodic signals, and (4.3) is the formula for the coefficients. Compare the two formulas with formulas (2.39) and (2.40), given in Section 2.3 for continuous-time periodic signals.

4.2 Matrix Interpretation of the DFT*

The DFT is a linear operation, acting on N-dimensional vectors (the sequences $\{x[n],\ 0 \le n \le N-1\}$), and producing N-dimensional vectors (the sequences $\{X^{\mathrm{d}}[k],\ 0 \le k \le N-1\}$). Therefore, it can be conveniently represented by an $N \times N$ matrix. Define

$$\boldsymbol{F}_N = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ W_N^0 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix},$$

$$\boldsymbol{x}_N = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}, \quad \boldsymbol{X}_N^{\mathrm{d}} = \begin{bmatrix} X^{\mathrm{d}}[0] \\ X^{\mathrm{d}}[1] \\ \vdots \\ X^{\mathrm{d}}[N-1] \end{bmatrix}. \tag{4.19}$$

Then we have from (4.3)

$$X_N^{\mathrm{d}} = \boldsymbol{F}_N \boldsymbol{x}_N. \tag{4.20}$$

The matrix $\boldsymbol{F}_N$ is called the *DFT matrix* of dimension N.

Example 4.5 The DFT matrices of dimensions 2, 3, 4 are as follows:

$$\boldsymbol{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{F}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5(1 + j\sqrt{3}) & -0.5(1 - j\sqrt{3}) \\ 1 & -0.5(1 - j\sqrt{3}) & -0.5(1 + j\sqrt{3}) \end{bmatrix}, \tag{4.21}$$

$$\boldsymbol{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}. \tag{4.22}$$

DFT matrices are convenient for hand computation of DFTs of short lengths. For example, let $x[n]$ be the signal $\{1, 3, 0, -2\}$. Then

$$X_4^{\mathrm{d}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - 5j \\ 0 \\ 1 + 5j \end{bmatrix}.$$

Therefore $X^{\mathrm{d}}[k] = \{2, 1 - 5j, 0, 1 + 5j\}$. □

The DFT matrix has the following properties:

1. The elements on the first row and the first column are 1, since $W_N^0 = 1$.
2. It is symmetric, since its (k, n)th element, W_N^{-kn}, is symmetric in k and n.
3. If $\bar{\boldsymbol{F}}_N$ is the complex conjugate of the DFT matrix, then

$$\boldsymbol{F}_N \bar{\boldsymbol{F}}_N' = N \boldsymbol{I}_N, \tag{4.23}$$

where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix. This follows from (4.6), since

$$(\boldsymbol{F}_N \bar{\boldsymbol{F}}_N')_{k,l} = \sum_{n=0}^{N-1} W_N^{-kn} W_N^{ln} = \sum_{n=0}^{N-1} W_N^{(l-k)n} = N\delta[(k - l) \bmod N]. \tag{4.24}$$

A complex $N \times N$ matrix Q satisfying $Q\bar{Q}' = \boldsymbol{I}_N$ is called a *unitary matrix.* If Q is real, it is called an *orthonormal matrix.* It follows from (4.23) that the matrix $N^{-1/2}\boldsymbol{F}_N$ is unitary. It is also symmetric (property 2), so it is a *symmetric unitary matrix.* The matrix $N^{-1/2}\boldsymbol{F}_N$ is called the *normalized DFT matrix.*

We have from (4.16)

$$\boldsymbol{x}_N = N^{-1} \bar{\boldsymbol{F}}_N X_N^{\mathrm{d}}, \tag{4.25}$$

so the inverse DFT operation can be described by the conjugate of the DFT matrix, with additional multiplication by N^{-1}.

The members of any set of N orthonormal vectors in an N-dimensional vector space form an orthonormal basis for the space. For example, the columns of the identity matrix $\boldsymbol{I}_N$ are orthonormal, so they form an orthonormal basis. This is called the *natural basis.* Let us denote these columns by $\{\boldsymbol{e}_{N,n},\ 0 \le n \le N - 1\}$. Then an arbitrary sequence $x[n]$, regarded as a vector $\boldsymbol{x}_N$, can be expressed as

$$\boldsymbol{x}_N = \sum_{n=0}^{N-1} x[n] \boldsymbol{e}_{N,n}. \tag{4.26}$$

The numbers $\{x[n],\ 0 \le n \le N-1\}$ are the coordinates of $\boldsymbol{x}_N$ in this basis. We emphasize that the $\boldsymbol{e}_{N,n}$ can be viewed as a basis for either a real N-dimensional space (if the $x[n]$ are restricted to real sequences) or a complex N-dimensional space (if the $x[n]$ are allowed to be complex sequences). Since the vectors $\boldsymbol{x}_N$ are discrete-time signals, we identify the natural coordinates of a vector as temporal sequences. In other words, the nth natural coordinate of a vector is its value at time n.

Since the columns of the matrix $N^{-1/2}\bar{\boldsymbol{F}}_N$ are orthonormal, we can regard them as basis vectors in a complex N-dimensional vector space. Let us denote these vectors by $\{N^{-1/2}\boldsymbol{f}_{N,k},\ 0 \le k \le N-1\}$. With this point of view in mind, we can express the inverse DFT relationship as

$$\boldsymbol{x}_N = \sum_{k=0}^{N-1} (N^{-1/2}X^{\mathrm{d}}[k])(N^{-1/2}\boldsymbol{f}_{N,k}). \tag{4.27}$$

Therefore, the numbers $\{N^{-1/2}X^{\mathrm{d}}[k],\ 0 \le k \le N-1\}$ are the coordinates of $\boldsymbol{x}_N$ in this basis. Since the basis is orthonormal, they are also the *projections* of the signal vector on the members of the basis.

We can now interpret Example 4.3 in geometrical terms, as follows. Since

$$\cos(\theta_0 n) = 0.5e^{j\theta_0 n} + 0.5e^{-j\theta_0 n},$$

it is a linear combination of two basis vectors when $\theta_0 = 2\pi m/N$. This explains why only two points in Figure 4.3(a) have nonzero ordinates. When $\theta_0 \ne 2\pi m/N$, both $e^{j\theta_0 n}$ and $e^{-j\theta_0 n}$ are close to basis vectors, but are not aligned with them. Therefore, $\cos(\theta_0 n)$ has nonzero projections on all basis vectors, as shown in Figure 4.3(b, c). Of those, the largest projections are on the basis vectors nearest to $\exp(j\theta_0 n)$ and $\exp(-j\theta_0 n)$, that is, on $\exp(j2\pi mn/N)$ and $\exp[j2\pi(N-m)n/N]$, where m is the integer nearest to $N\theta_0/2\pi$.

We summarize the preceding discussion as follows. The values of the DFT of a sequence $x[n]$ can be viewed as the coordinates of the sequence in a particular orthonormal basis (up to the constant factor $N^{-1/2}$). The vectors constituting this basis are the columns of the DFT matrix (again, up to a constant factor), therefore they are called the *DFT basis*. Figure 4.5 illustrates the DFT basis for $N = 8$. It is common to draw the basis vectors as staircase waveforms. The horizontal axis of the waveform is time, so the nth stair of the kth waveform indicates the nth coordinate of the kth basis vector in the natural basis. In this case we have 8 basis vectors, each having a real part and a complex part. For large N, the real-part waveforms will look more and more like continuous-time cosine functions at linearly increasing frequencies, whereas the imaginary-part waveforms will look like continuous-time sine functions.

4.3 Properties of the DFT

The discrete Fourier transform has properties similar to those of the usual Fourier transform. We now list the main properties of the DFT and their proofs.

1. **Linearity**

$$z[n] = ax[n] + by[n] \iff Z^{\mathrm{d}}[k] = aX^{\mathrm{d}}[k] + bY^{\mathrm{d}}[k], \quad a, b \in \mathbb{C}. \tag{4.28}$$

The proof follows immediately from (4.3).

2. **Periodicity**

$$X^{\mathrm{d}}[k] = X^{\mathrm{d}}[k+N]. \tag{4.29}$$

This is a direct result of the periodicity of W_N, as explained in Section 4.1.

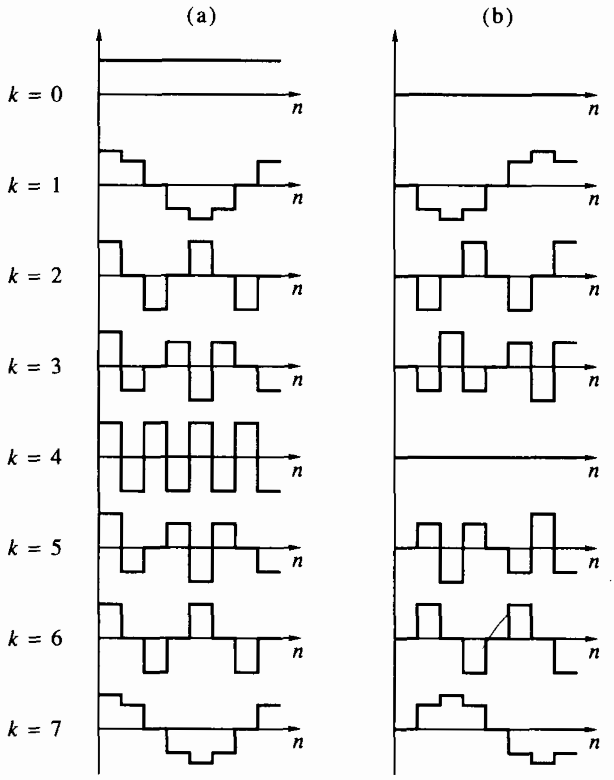

Figure 4.5 The DFT basis vectors for $N = 8$: (a) real part; (b) imaginary part.

3. **Circular shift**

$$y[n] = x[(n-m) \bmod N] \iff Y^{\mathrm{d}}[k] = W_N^{-km} X^{\mathrm{d}}[k], \quad m \in \mathbb{Z}. \tag{4.30}$$

Proof

$$\begin{aligned} Y^{\mathrm{d}}[k] &= \sum_{n=0}^{N-1} x[(n-m) \bmod N] W_N^{-kn} = \sum_{n=0}^{N-1} x[(n-m) \bmod N] W_N^{-k(n-m+m)} \\ &= W_N^{-km} \sum_{n=0}^{N-1} x[(n-m) \bmod N] W_N^{-k[(n-m) \bmod N]} = W_N^{-km} X^{\mathrm{d}}[k]. \end{aligned} \tag{4.31}$$

In passing from the first to the second line we used the property

$$W_N^{-k(l \bmod N)} = W_N^{-kl}, \quad \text{for all } k, l \in \mathbb{Z}.$$

4. **Frequency shift (modulation)**

$$y[n] = W_N^{mn} x[n] \iff Y^{\mathrm{d}}[k] = X^{\mathrm{d}}[(k-m) \bmod N], \quad m \in \mathbb{Z}. \tag{4.32}$$

Proof

$$\begin{aligned} Y^{\mathrm{d}}[k] &= \sum_{n=0}^{N-1} x[n] W_N^{mn} W_N^{-kn} = \sum_{n=0}^{N-1} x[n] W_N^{-n(k-m)} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{-n[(k-m) \bmod N]} = X^{\mathrm{d}}[(k-m) \bmod N]. \end{aligned} \tag{4.33}$$

5. **Parseval's theorem**

$$\sum_{n=0}^{N-1} x[n]\bar{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k]\bar{Y}^{\mathrm{d}}[k], \tag{4.34}$$

and its special case

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X^{\mathrm{d}}[k]|^2. \tag{4.35}$$

Proof Using the vector notations (4.19) for the sequence and its DFT, we can express (4.34) as

$$\bar{\boldsymbol{y}}'\boldsymbol{x} = N^{-1}(\bar{Y}^{\mathrm{d}})'X^{\mathrm{d}}. \tag{4.36}$$

Now, using the matrix representation of the DFT (4.20) and property (4.23) of the DFT matrix, we have

$$N^{-1}(\bar{Y}^{\mathrm{d}})'X^{\mathrm{d}} = N^{-1}\bar{\boldsymbol{y}}'\bar{\boldsymbol{F}}'_N\boldsymbol{F}_N\boldsymbol{x} = \bar{\boldsymbol{y}}'\boldsymbol{I}_N\boldsymbol{x} = \bar{\boldsymbol{y}}'\boldsymbol{x}, \tag{4.37}$$

which is the same as (4.36). The special case (4.35) is obtained by taking the sequence $y[n]$ equal to $x[n]$.

6. **Conjugation**

$$y[n] = \bar{x}[n] \iff Y^{\mathrm{d}}[k] = \bar{X}^{\mathrm{d}}[(N-k) \bmod N]. \tag{4.38}$$

Proof

$$Y^{\mathrm{d}}[k] = \sum_{n=0}^{N-1} \bar{x}[n]W_N^{-kn} = \overline{\left[\sum_{n=0}^{N-1} x[n]W_N^{kn}\right]}$$
$$= \overline{\left[\sum_{n=0}^{N-1} x[n]W_N^{-[(N-k) \bmod N]n}\right]} = \bar{X}^{\mathrm{d}}[(N-k) \bmod N]. \tag{4.39}$$

7. **Symmetry** If $x[n]$ is real valued then

$$X^{\mathrm{d}}[(N-k) \bmod N] = \bar{X}^{\mathrm{d}}[k], \tag{4.40a}$$
$$\Re\{X^{\mathrm{d}}[(N-k) \bmod N]\} = \Re\{X^{\mathrm{d}}[k]\}, \tag{4.40b}$$
$$\Im\{X^{\mathrm{d}}[(N-k) \bmod N]\} = -\Im\{X^{\mathrm{d}}[k]\}, \tag{4.40c}$$
$$|X^{\mathrm{d}}[(N-k) \bmod N]| = |X^{\mathrm{d}}[k]|, \tag{4.40d}$$
$$\measuredangle X^{\mathrm{d}}[(N-k) \bmod N] = -\measuredangle X^{\mathrm{d}}[k]. \tag{4.40e}$$

Proof Equality (4.40a) follows from

$$X^{\mathrm{d}}[(N-k) \bmod N] = \sum_{n=0}^{N-1} x[n]W_N^{-[(N-k) \bmod N]n} = \sum_{n=0}^{N-1} x[n]W_N^{kn} = \bar{X}^{\mathrm{d}}[k]. \tag{4.41}$$

The other four equalities follow from the first.

The conjugate symmetry property of real sequences requires clarification. Substitution of $k = 0$ gives $X^{\mathrm{d}}[0] = \bar{X}^{\mathrm{d}}[0]$. This is obvious, since

$$X^{\mathrm{d}}[0] = \sum_{n=0}^{N-1} x[n], \tag{4.42}$$

which is real. So, $k = 0$ is its own conjugate symmetric. Substitution of $k = 0.5N$ (if N is even) gives $X^{\mathrm{d}}[0.5N] = \bar{X}^{\mathrm{d}}[0.5N]$. This is again obvious, since

$$X^{\mathrm{d}}[0.5N] = \sum_{n=0}^{N-1} x[n]W_N^{-0.5Nn} = \sum_{n=0}^{N-1} (-1)^n x[n], \tag{4.43}$$

which is real. So, for even N, $k = 0.5N$ is its own conjugate symmetric. The other terms are not conjugate symmetric to themselves; for example, $k = 1$ is conjugate symmetric to $k = N - 1$, $k = 0.5N - 1$ is conjugate symmetric to $k = 0.5N + 1$, and so forth. If N is odd, $0.5N$ is not an integer, so the self-conjugate center point is missing. Figure 4.6 shows the symmetries in the odd and even cases.

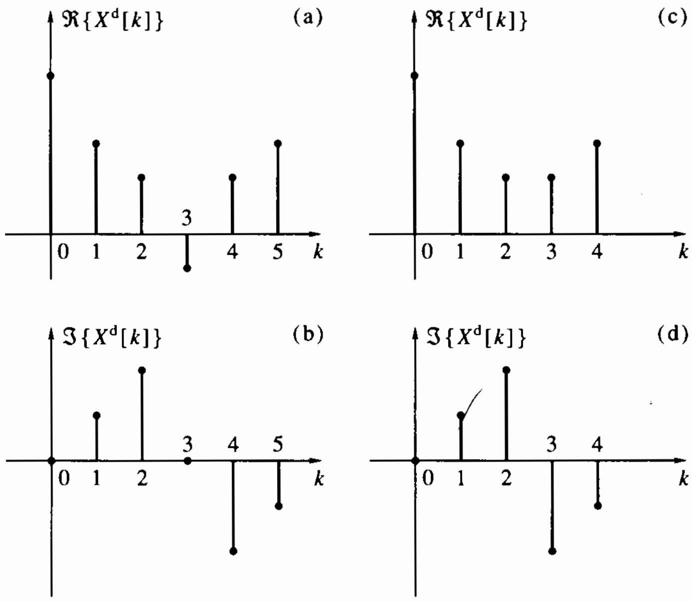

Figure 4.6 Symmetries in DFT of a real signal: (a) even N, real part; (b) even N, imaginary part; (c) odd N, real part; (d) odd N, imaginary part.

4.4 Zero Padding

The DFT of a length-N sequence is itself a length-N sequence, so it gives the frequency response of the signal at N points. Suppose we are interested in computing the frequency response at M equally spaced frequency points, where $M > N$. A simple device accomplishes this goal: We add $M - N$ zeros at the tail of the given sequence, thus forming a length-M sequence. The DFT of the new sequence has M frequency points. We prove that the values of the new DFT are indeed samples of the frequency response of the given signal at M equally spaced frequencies. Denote

$$x_a[n] = \begin{cases} x[n], & 0 \le n \le N - 1, \\ 0, & N \le n \le M - 1. \end{cases} \tag{4.44}$$

The operation of adding zeros to the tail of a sequence is called *zero padding*. The DFT of the zero-padded sequence $x_a[n]$ is given by

$$X_a^d[k] = \sum_{n=0}^{M-1} x_a[n] \exp\left(-\frac{j2\pi kn}{M}\right) = \sum_{n=0}^{N-1} x[n] \exp\left(-\frac{j2\pi kn}{M}\right) = X^f(\theta[k]), \tag{4.45}$$

where

$$\theta[k] = \frac{2\pi k}{M}, \quad 0 \le k \le M - 1. \tag{4.46}$$

As we see, $X_a^d[k]$ is indeed a sampling of $X^f(\theta)$ at M equally spaced frequency points in the range $[0, 2\pi)$.

The notation $\mathrm{DFT}_M\{x[n]\}$ is used for describing the zero-padded DFT operation. For example, if the length of $x[n]$ is 16, then $\mathrm{DFT}_{16}\{x[n]\}$ denotes the usual DFT, and $\mathrm{DFT}_{64}\{x[n]\}$ denotes the DFT of the sequence obtained by zero padding to length 64. The following MATLAB expression performs zero padding on a row vector:

```
y = [x, zeros(1,M-length(x)];
```

Figure 4.7 illustrates the zero-padding operation. Part a shows a signal of length $N = 8$, and part b shows the magnitude of its length-N DFT (note that we interchange the positive- and negative-index halves). Part c shows the signal obtained by zero padding to length $M = 32$, and part d shows the magnitude of the length-M DFT of the zero-padded signal.

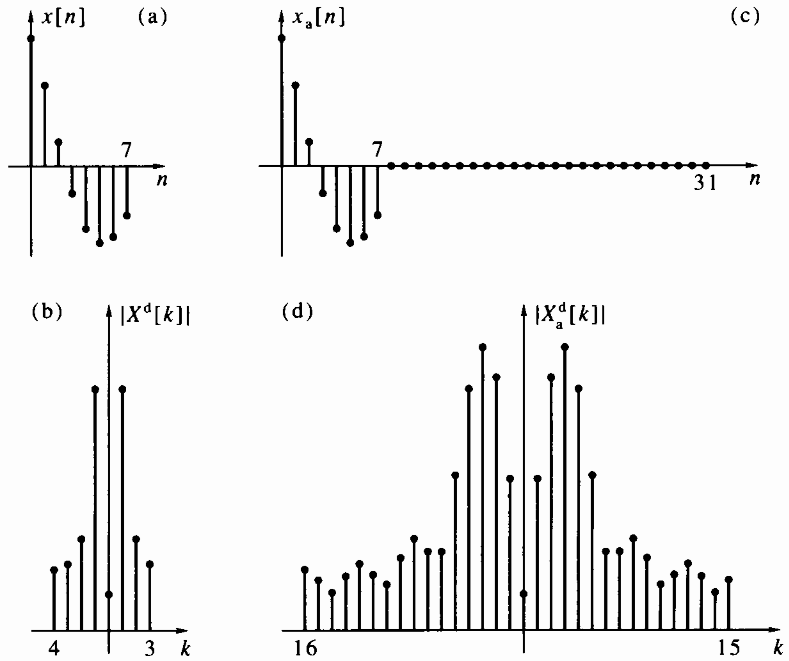

Figure 4.7 Increasing the DFT length by zero padding: (a) a signal of length 8; (b) the 8-point DFT of the signal (magnitude); (c) zero padding the signal to length 32; (d) the 32-point DFT of the zero-padded signal.

We can interpret the zero-padded DFT $X_a^d[l]$ as interpolation operation on $X^d[k]$. In particular, if M is an integer multiple of N, say $M = LN$, we have interpolation by a factor L. In this case, the points $X_a^d[kL]$ of the zero-padded DFT are identical to the corresponding points $X^d[k]$ of the conventional DFT. If M is not an integer multiple of N, most of the points $X^d[k]$ do not appear as points of $X_a^d[l]$ (Problem 4.21).

Zero padding is typically used for improving the visual continuity of plots of frequency responses. When plotting $X_a^d[k]$, we typically see more details than when plotting $X^d[k]$. However, the additional details do not represent additional information about the signal, since all the information is in the N given samples of $x[n]$. Indeed, computation of the inverse DFT of $X_a^d[k]$ gives the zero-padded sequence $x_a[n]$, which consists only of the $x[n]$ and zeros.

4.5 Zero Padding in the Frequency Domain*

The interpretation of zero padding given at the end of the preceding section raises an intriguing thought: If zero padding in the time domain provides interpolation in the frequency domain, then zero padding in the frequency domain must provide interpolation in the time domain. We should therefore be able to use zero padding of the DFT as a means of interpolating finite-duration, discrete-time signals. Implementation of this idea requires care, since we must preserve the symmetry properties of the zero-padded DFT. Let us consider, for simplicity, the case of odd N. The case of even N is left as an exercise (Problem 4.42). Also, we assume that M is an integer multiple of N, say $M = NL$, where L is the *interpolation factor.*

Assume we are given the DFT of a length-N sequence (where N is odd), and define the zero-padded DFT as

$$X_{\mathrm{i}}^{\mathrm{d}}[k] = \begin{cases} LX^{\mathrm{d}}[k], & 0 \le k \le \frac{N-1}{2}, \\ LX^{\mathrm{d}}[k-M+N], & M - \frac{N-1}{2} \le k \le M-1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.47}$$

for $M = LN$, $L > 0$. The new DFT has zero values at high frequencies; conjugate symmetry is preserved if possessed by $X^{\mathrm{d}}[k]$. Let us now define $x_{\mathrm{i}}[n]$ as the length-M IDFT of $X_{\mathrm{i}}^{\mathrm{d}}[k]$, that is,

$$x_{\mathrm{i}}[n] = \frac{1}{M}\sum_{k=0}^{M-1} X_{\mathrm{i}}^{\mathrm{d}}[k]W_M^{nk}, \quad 0 \le n \le M-1. \tag{4.48}$$

Then,

$$\begin{aligned} x_{\mathrm{i}}[n] &= \frac{1}{N}\sum_{k=0}^{(N-1)/2} X^{\mathrm{d}}[k]W_M^{nk} + \frac{1}{N}\sum_{k=M-(N-1)/2}^{M-1} X^{\mathrm{d}}[k-M+N]W_M^{nk} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\left[\sum_{k=0}^{(N-1)/2} W_M^{nk}W_N^{-mk} + W_M^{-nN}\sum_{k=(N+1)/2}^{N-1} W_M^{nk}W_N^{-mk}\right] \\ &= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\left[\frac{(W_M^{n}W_N^{-m})^{(N+1)/2}-1}{W_M^{n}W_N^{-m}-1} + \frac{1-(W_M^{n}W_N^{-m})^{-(N-1)/2}}{W_M^{n}W_N^{-m}-1}\right] \\ &= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\frac{\sin[\pi(n-mL)/L]}{\sin[\pi(n-mL)/M]}. \end{aligned} \tag{4.49}$$

The right side of (4.49) bears similarity to Shannon's reconstruction formula

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right). \tag{4.50}$$

This similarity is not coincidental. Shannon's formula describes the reconstruction of a continuous-time signal from its samples. Formula (4.49) describes the reconstruction of a discrete-time signal from a signal obtained after resampling a discrete-time signal at an L-times slower rate. In Shannon's formula, the impulse response of the interpolating filter is a sinc function. Here it is the function $\sin(\pi n/L)/\sin(\pi n/M)$, which can be thought of as a discrete-time (periodic) sinc. Note, in particular, that for values of n that are integer multiples of L, we get from (4.49)

$$x_{\mathrm{i}}[kL] = x[k],$$

so the interpolation becomes an identity at the time points of the original length-N signal.

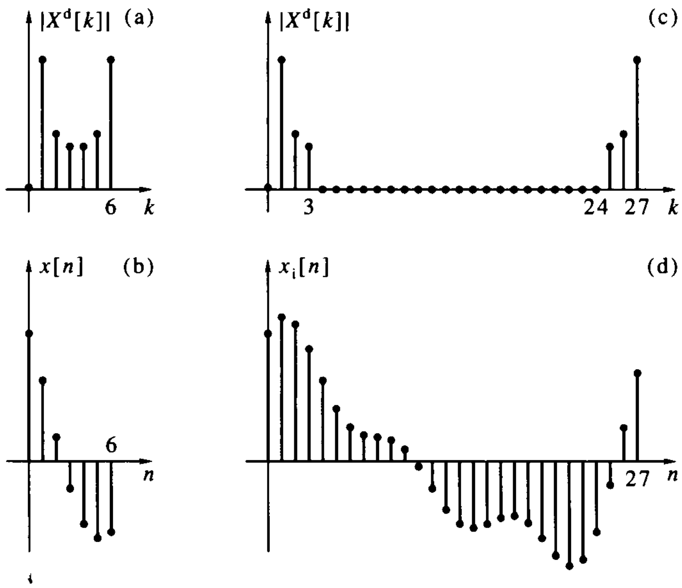

Figure 4.8 Interpolation by zero padding in the frequency domain: (a) DFT of a signal of length 7; (b) the time-domain signal; (c) zero padding the DFT to length 28; (d) the time-domain signal of the zero-padded DFT.

Figure 4.8 illustrates zero padding in the frequency domain. Part a shows the magnitude of the DFT of a signal of length $N = 7$, and part b shows the signal itself. Here we did not interchange the halves of the DFT, for better illustration of the operation (4.47). Part c shows the DFT after being zero padded to a length $M = 28$, and part d shows the inverse DFT of the zero-padded DFT. The result in part d exhibits two peculiarities:

1. The interpolated signal is rather wiggly. The wiggles, which are typical of this kind of interpolation, are introduced by the interpolating function $\sin(\pi n/L)/\sin(\pi n/M)$.
2. The last point of the original signal is $x[6]$, and it corresponds to the point $x_i[24]$ of the interpolated signal. The values $x_i[25]$ through $x_i[27]$ of the interpolated signal are actually interpolations involving the periodic extension of the signal. These values are not reliable, since they are heavily influenced by the initial values of $x[n]$ (note how $x[27]$ is pulled upward, closer to the value of $x[0]$).

Because of these phenomena, interpolation by zero padding in the frequency domain is not considered desirable and is not in common use (see, however, a continuation of this discussion in Problem 6.14).

4.6 Circular Convolution

You may have noticed that the convolution and multiplication properties were conspicuously missing from the list of properties of the DFT. This is because the DFT satisfies these properties only for a certain kind of convolution, which we now define.

Let $x[n]$ and $y[n]$ be two finite length sequences, of *equal* length N. We define their *circular convolution* as

$$z[n] = \{x \circledast y\}[n] = \sum_{m=0}^{N-1} x[m]y[(n-m) \bmod N], \quad 0 \le n \le N-1. \tag{4.51}$$

Other names for this operation are *cyclic convolution* and *periodic convolution.*

The circular convolution of two length-N sequences is itself a length-N sequence. It is convenient to think of a circular convolution as though the sequences are defined on points on a circle, rather than on a line. Take, for example, $N = 12$, and imagine the n coordinate as the hour on a watch, with 0 instead of 12. To perform the circular convolution, proceed as follows:

1. Spread the $x[n]$ clockwise, starting at the zero hour.
2. Spread the $y[n]$ counterclockwise, starting at the zero hour (i.e, the point $y[1]$ goes on the 11th hour, and so on).
3. To compute $z[n]$, rotate the sequence of $y[n]$s clockwise by n steps, then perform element-by-element multiplication of the two sequences, and sum.
4. Repeat for all n from 0 through 11.

Figure 4.9 illustrates this procedure using a 6-digit watch. In this figure we use $x[n] = y[n] = n$, so the numbers represent both indices and values of the two sequences. The value of n and the corresponding value of $z[n]$ is shown beneath each position of the watch.

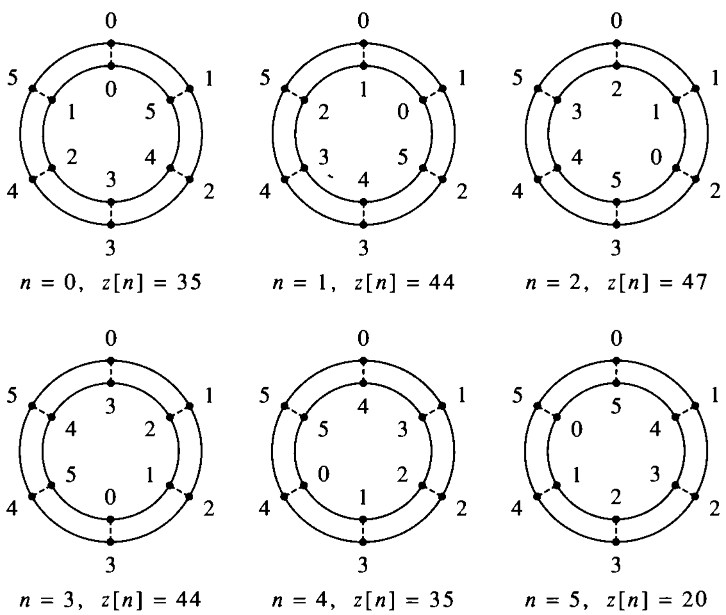

Figure 4.9 Illustration of circular convolution for $N = 6$. Outer circles: $x[m]$; inner circles: $y[n-m]$. The result of the convolution is the sum of products of sequence values near the ends of each dashed line.

Another useful way to look at circular convolution is to regard it as partial convolution of the periodic extensions of the two sequences. By "partial" we mean that summation is performed only over one period, not from $-\infty$ to ∞. Thus,

$$z[n] = \sum_{m=0}^{N-1} \tilde{x}[m]\tilde{y}[n-m]. \tag{4.52}$$

Does there exist a sequence y of length 4 such that

$$x \circledast y = w?$$

If so, find y; if not, prove that such y does not exist. Hint: Solve using DFT.

4.27 Given that $x = \{2, -1\}$, $w = x * y$, and $w = \{6, -1, 7, -4\}$, compute the sequence y using DFT.

4.28 Prove the commutativity and associativity of circular convolution (4.54).

4.29 We are given two sequences $x_1[n]$, $x_2[n]$ of length 75 each. It is known that $x[n] = 0$ for all $0 \le n \le 7$ and for all $60 \le n \le 74$, but is nonzero otherwise; $x_2[n]$ is nonzero in general. Let

$$y[n] = \{x_1 \circledast x_2\}[n], \quad z[n] = \{x_1 * x_2\}[n].$$

For what values of n will $y[n \bmod 75] = z[n]$ hold?

4.30 The sequences $x[n]$, $y[n]$, $z[n]$ have length 3 each, and it is known that

$$z[n] = \{x \circledast y\}[n].$$

We are given that

$$x[n] = \{1, 1, 0\}, \quad Z^{\mathrm{d}}[k] = \{2, -W_3^2, -W_3\}.$$

Find the sequence $y[n]$.

4.31 In Section 4.7 we showed how to perform linear convolution of sequences of lengths N_1, N_2 by zero padding to length $N_1 + N_2 - 1$. Suppose, instead, that we zero-pad the two sequences to length $M > N_1 + N_2 - 1$. How then is the circular convolution of the zero-padded sequences related to the linear convolution?

4.32 Suggest a procedure for performing linear convolution of *three* sequences of lengths N_1, N_2, N_3 by DFT.

4.33 The *circular correlation* (or *circular cross-correlation*) of two discrete-time signals, both of length N, is defined as (cf. Problem 2.36)

$$z[n] = \sum_{m=0}^{N-1} x[(m+n) \bmod N]\bar{y}[m]. \tag{4.114}$$

Express the circular correlation operation in the frequency domain.

4.34* Let

$$x(t) = t^3 + 1, \quad -\infty < t < \infty.$$

The signal is sampled at interval T, for $-N \le n \le N$, and shifted to the right by N, yielding a sequence of length $(2N+1)$. Find $X^{\mathrm{d}}[0]$ of this sequence.

4.35* Let $x(t)$ be the signal

$$x(t) = a_1 \cos(2\pi f t + \phi_1) + a_2 \cos(2\pi \cdot 1.25 f t + \phi_2).$$

The frequency f is unknown, but it is known to lie in the range

$$1000\,\text{Hz} < f < 1600\,\text{Hz}$$

(with strict inequalities). The parameters a_1, a_2, ϕ_1, ϕ_2 are real and unknown.

Theorem 4.4 (multiplication)

$$z[n] = x[n]y[n] \iff Z^{\mathrm{d}}[k] = \frac{1}{N}\{X^{\mathrm{d}} \circledast Y^{\mathrm{d}}\}[k]. \tag{4.58}$$

The proof follows from the previous one by duality: Start at right side and compute its IDFT using linearity and the modulation property (4.32). □

We restate the last two theorems, to emphasize their importance:

> The DFT of a circular convolution of two sequences is the product of the individual DFTs. The DFT of a product of two sequences is, up to a proportionality constant, the circular convolution of the individual DFTs.

Example 4.7 Consider again the sequences in Example 4.6. We have

$$X^{\mathrm{d}}[k] = \{9, 3 + j2, -7, 3 - j2\}, \quad Y^{\mathrm{d}}[k] = \{0, 3 - j7, -2, 3 + j7\}.$$

Therefore,

$$Z^{\mathrm{d}}[k] = \{0, 23 - j15, 14, 23 + j15\}.$$

Taking the inverse DFT of $Z^{\mathrm{d}}[k]$ gives

$$z[n] = \{15, 4, -8, 11\},$$

which is equal to the result in Example 4.6. □

The fact that it is circular, rather than conventional convolution that translates to multiplication in the frequency domain is a common source of confusion and mistakes. The following claim is often made by beginners: "Linear time-invariant filtering is equivalent to multiplication in the frequency domain. Therefore, we can perform linear filtering by computing the DFT of the input signal $X^{\mathrm{d}}[k]$, multiply by the DFT of the impulse response of the filter $H^{\mathrm{d}}[k]$ (which is presumably the frequency response of the filter), and compute the inverse DFT of the product." This claim is wrong, because the DFT is not the frequency response of either the signal or the filter. Rather, it is the sampling of the frequency response in the θ domain. Consequently, the inverse DFT of $H^{\mathrm{d}}[k]X^{\mathrm{d}}[k]$ is $\{x \circledast y\}[n]$, whereas the proper linear filtering operation is $\{x * y\}[n]$. Nevertheless, the idea behind the aforementioned claim is essentially correct, and frequency-domain operations with the DFT *can be used* for convolution and filtering. The proper way of doing this is elaborated in the next section.

4.7 Linear Convolution via Circular Convolution

Suppose we are given two finite-duration sequences having different lengths, say $\{x[n],\ 0 \le n \le N_1 - 1\}$, and $\{y[n],\ 0 \le n \le N_2 - 1\}$, and we wish to perform their (conventional) discrete-time convolution

$$z[n] = \sum_{m=m_1}^{m_2} x[m]y[n-m], \tag{4.59}$$

where

$$m_1 = \max\{0, n + 1 - N_2\}, \quad m_2 = \min\{N_1 - 1, n\}.$$

To avoid confusion, we will henceforth refer to conventional convolution as *linear convolution.*

Let us zero-pad $x[n]$ and $y[n]$ to a length $N = N_1 + N_2 - 1$ and denote the zero-padded sequences by $x_a[n]$, $y_a[n]$, respectively. Now we can express (4.59) as

$$z[n] = \sum_{m=0}^{n} x_a[m]y_a[n-m], \quad 0 \le n \le N-1. \tag{4.60}$$

The zero-padded sequences have the same length; let us compute their circular convolution:

$$\begin{aligned}\{x_a \circledast y_a\}[n] &= \sum_{m=0}^{N-1} x_a[m]y_a[(n-m) \bmod N] \\ &= \sum_{m=0}^{n} x_a[m]y_a[n-m] + \sum_{m=n+1}^{N-1} x_a[m]y_a[n-m+N]. \end{aligned} \tag{4.61}$$

In the second sum, the lengths of the two sequences $x[n]$ and $y[n]$ impose the following limits on the summation index m:

$$\begin{aligned} n+1 &\le m \le N_1 - 1, \\ N_1 + n &\le m \le N + n. \end{aligned}$$

The intersection of these two limits is empty, so the second term on the right side of (4.61) is zero. The first term is identical to the right side of (4.60), so the conclusion is that

$$\{x * y\}[n] = \{x_a \circledast y_a\}[n], \quad 0 \le n \le N-1. \tag{4.62}$$

Therefore, we can perform linear convolution of two finite-length sequences by computing the circular convolution of the corresponding zero-padded sequences, provided zero padding is made to the sum of the lengths minus one. Figure 4.10 illustrates this procedure for $N_1 = 4$, $N_2 = 3$. In this figure we use $x[n] = y[n] = 1$ for convenience. The small solid circles (4 in the sequence $x[n]$, 3 in the sequence $y[n]$) represent the value 1. The small open circles represent the value 0 (i.e., the zero-padded points). As we see, the zero-valued points are properly aligned for each n so as to make the circular convolution equal to the desired linear convolution.

The circular convolution $\{x_a \circledast y_a\}$ can be performed by computing the DFTs of both sequences, multiplying the resulting vectors point by point, and then computing the inverse DFT of the result. The advantage of performing the convolution this way will become clear in the next chapter, when we study the fast Fourier transform.

Example 4.8 We wish to compute the linear convolution of the two sequences

$$x[n] = \{2, 3\}, \quad y[n] = \{1, -4, 5\}.$$

Direct computation of the linear convolution gives

$$z[n] = \{2, -5, -2, 15\}.$$

Computation by circular convolution and DFT proceeds as follows. The zero-padded sequences are

$$x_a[n] = \{2, 3, 0, 0\}, \quad y_a[n] = \{1, -4, 5, 0\}.$$

The DFTs of the zero-padded sequences are

$$X_a^d[k] = \{5, 2-j3, -1, 2+j3\}, \quad Y_a^d[k] = \{2, -4+j4, 10, -4+j4\}.$$

Therefore,

$$Z_a^d[k] = \{10, 4+j20, -10, 4-j20\}.$$

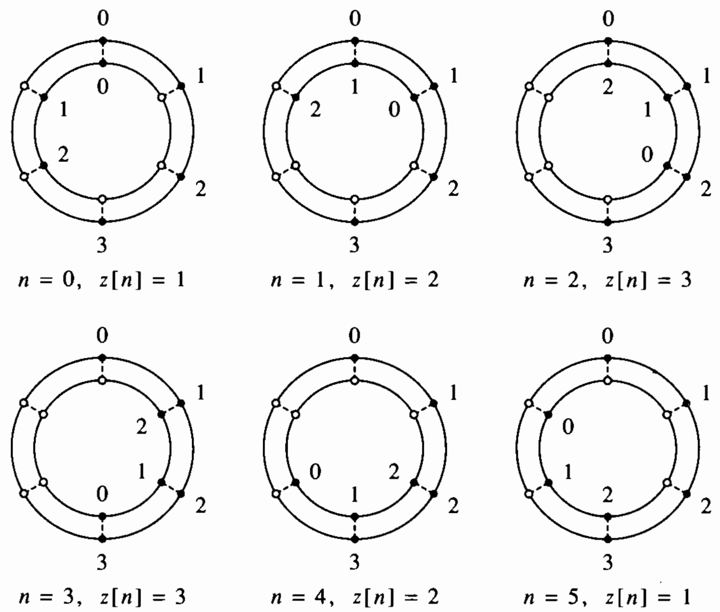

Figure 4.10 Illustration of linear convolution via circular convolution for $N = 6$. Outer circles: $x[m]$; inner circles: $y[n - m]$. Solid circles represent original sequence point; open circles represent zero-padded points.

Finally, taking the inverse DFT yields

$$z_a[k] = \{2, -5, -2, 15\},$$

which is the same as the result obtained by direct computation. □

4.8 The DFT of Sampled Periodic Signals

We already mentioned, in Chapter 3, that no continuous-time signal can be simultaneously limited in time and in frequency; see page 56. To use the DFT, we need to sample a finite-duration signal (or to use a finite number of samples from an infinite-duration signal, which amounts to the same). But then the signal bandwidth must be infinite, so sampling will involve aliasing. The DFT cannot repair this aliasing, since it represents only sampling in the frequency domain of the Fourier transform of the sampled signal. The conclusion is that, in general, the DFT *does not* provide an exact description of a continuous-time signal in the frequency domain.

There is an exception to the preceding conclusion. When we sample a continuous-time *periodic* signal we can, under certain restrictions, get an exact frequency-domain description from the DFT of a finite number of samples. To see this, let $x(t)$ be a periodic signal with period T_0. Then the signal admits a Fourier series

$$x(t) = \sum_{m=-\infty}^{\infty} X^S[m] \exp\left(\frac{j2\pi mt}{T_0}\right). \tag{4.63}$$

Suppose we sample the signal an integer number of samples over a single period, say N samples. The corresponding sampling interval is $T = T_0/N$. The samples then make up the finite sequence

$$x[n] = x(nT) = \sum_{m=-\infty}^{\infty} X^{\mathrm{s}}[m] \exp\left(\frac{j2\pi mn}{N}\right), \quad 0 \le n \le N-1. \tag{4.64}$$

Write m as

$$m = k + lN, \quad 0 \le k \le N-1, \ -\infty < l < \infty,$$

and substitute in (4.64) to get

$$x[n] = \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} X^{\mathrm{s}}[k+lN] \exp\left(\frac{j2\pi kn}{N}\right) = \frac{1}{N} \sum_{k=0}^{N-1} \left(N \sum_{l=-\infty}^{\infty} X^{\mathrm{s}}[k+lN]\right) W_N^{nk}. \tag{4.65}$$

This has the form of an inverse DFT, so we conclude that

$$X^{\mathrm{d}}[k] = N \sum_{l=-\infty}^{\infty} X^{\mathrm{s}}[k+lN]. \tag{4.66}$$

As we see, each DFT coefficient of the one-period sampled sequence is an infinite sum of Fourier coefficients of the continuous-time signal. This is again a manifestation of aliasing: In general, we cannot recover the individual Fourier coefficients $X^{\mathrm{s}}[m]$ from the DFT coefficients $X^{\mathrm{d}}[k]$. There exists, however, a special case in which there is no aliasing. Suppose the signal $x(t)$ is band limited and its nonzero Fourier coefficients are only $\{X^{\mathrm{s}}[m], \ -M \le m \le M\}$. Note that there is no contradiction to the theorem quoted in the beginning of the section, since any periodic signal has an infinite duration. If we choose $N \ge 2M+1$, we will get

$$X^{\mathrm{d}}[k] = \begin{cases} NX^{\mathrm{s}}[k], & 0 \le k \le M, \\ NX^{\mathrm{s}}[k-N], & N-M \le k \le N-1. \end{cases} \tag{4.67}$$

In this special case we can recover the $2M+1$ nonzero Fourier coefficients from the corresponding values of the DFT coefficients. In summary, if a periodic band-limited signal is sampled an integer number of times N over one period, such that N is at least $2M+1$ (where M is the index of the highest nonzero harmonic), the DFT coefficients of the sampled sequence are equal, up to a constant factor N, to the corresponding Fourier coefficients of the signal harmonics.

Example 4.9 Eliza and Beth are two engineers in a DSP company. Eliza, the senior, generates a continuous-time triangular signal whose fundamental frequency is 1 Hz; one period of this signal is shown in Figure 4.11. She samples the signal at interval T, an integer number of samples per period, and gives the values of the sampled signal to Beth, the junior. Eliza tells Beth that the signal is periodic, gives its period and the sampling interval T, and asks Beth to reconstruct the signal. Beth, after an hour's work, reports the following answer:

$$\hat{x}(t) = 0.25(\sqrt{2}+2)\sin(2\pi t) + 0.25(\sqrt{2}-2)\sin(6\pi t).$$

What was the value of T and how did Beth arrive at the result?

Since Beth knows both the period and the sampling interval, she knows the number of samples per period. It is obvious from her result that she was able to get only the first and third harmonics (recall that a symmetric triangular wave, like the one shown in Figure 4.11, has only odd harmonics). Thus, the number of samples per period is at least 7 (since with 6 samples or less she would not have found a third harmonic) at most 10 (since with 11 samples she would have found a fifth harmonic).

Let us try $T = 1/7$ first. With this sampling interval, the 7 samples of the triangular wave are

$$x[n] = \{0, 4/7, 6/7, 2/7, -2/7, -6/7, -4/7\}.$$

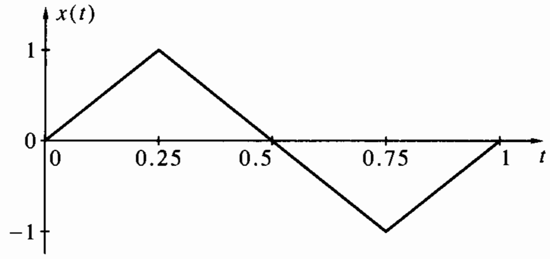

Figure 4.11 The signal in Example 4.9.

The DFT of $x[n]$ is

$$X^{\mathrm{d}}[k] = \{0, -j2.8128, j0.0764, j0.2873, -j0.2873, -j0.0764, +j2.8128\}.$$

As we see, there is a second harmonic in this case, which was evidently introduced by aliasing. Since we know that Beth did not find a second harmonic, $T = 1/7$ is not the right answer.

Next let us try $T = 1/8$. With this sampling interval, the 8 samples of the triangular wave are

$$x[n] = \{0, 0.5, 1, 0.5, 0, -0.5, -1, -0.5\}.$$

The DFT of $x[n]$ is now

$$X^{\mathrm{d}}[k] = \{0, -j(\sqrt{2}+2), 0, -j(\sqrt{2}-2), 0, j(\sqrt{2}-2), 0, j(\sqrt{2}+2)\}.$$

Now we have only the first and third harmonics, as found by Beth. The signal reconstructed from these harmonics is

$$\begin{aligned}\hat{x}(t) &= -j0.125(\sqrt{2}+2)e^{j2\pi t} - j0.125(\sqrt{2}-2)e^{j6\pi t} \\ &\quad + j0.125(\sqrt{2}-2)e^{-j6\pi t} + j0.125(\sqrt{2}+2)e^{-j2\pi t} \\ &= 0.25(\sqrt{2}+2)\sin(2\pi t) + 0.25(\sqrt{2}-2)\sin(6\pi t),\end{aligned}$$

which is precisely what Beth found; therefore $T = 1/8$. □

4.9 The Discrete Cosine Transform*

The discrete cosine transform (DCT) is a kin of the DFT. It can be regarded as a discrete-time version of the Fourier cosine series, described in Section 2.5. Like the DFT, the DCT provides information about the signal in the frequency domain. Unlike the DFT, the DCT of a real signal is real valued. The transform is linear, so it can be expressed in a matrix-vector form

$$\boldsymbol{X}_N^{\mathrm{c}} = \boldsymbol{C}_N \boldsymbol{x}_N, \tag{4.68}$$

where $\boldsymbol{x}_N$ is the N-vector describing the signal, $\boldsymbol{X}_N^{\mathrm{c}}$ is the N-vector describing the result of the transform, and $\boldsymbol{C}_N$ is a square nonsingular $N \times N$ matrix describing the transform itself. The matrix $\boldsymbol{C}_N$ is real valued.

Recall that, to obtain the Fourier cosine series, we extended the signal symmetrically around the origin. We wish to do the same in the discrete-time case. It turns out that symmetric extension of a discrete-time signal is not as obvious as in continuous time, and there are several ways of proceeding. Each symmetric extension gives rise to a different transform. In total there are four types of discrete cosine transform, as described next.

4.9.1 Type-I Discrete Cosine Transform

Let $x[n]$ be a discrete-time real signal of length N. A minimal symmetric extension of this signal involves adding $N-2$ more points: $x[0]$ and $x[N-1]$ remain unique, whereas $x[1]$ through $x[N-2]$ are duplicated symmetrically around $x[N-1]$. The points $x[0]$ and $x[N-1]$ are not duplicated, but are multiplied by $2^{1/2}$. We thus define

$$x_1[n] = \begin{cases} 2^{1/2}x[n], & n = 0, N-1 \\ x[n], & 1 \le n \le N-2, \\ x[2N-n-2], & N \le n \le 2N-3. \end{cases} \tag{4.69}$$

Figure 4.12 illustrates this definition for $N = 10$.

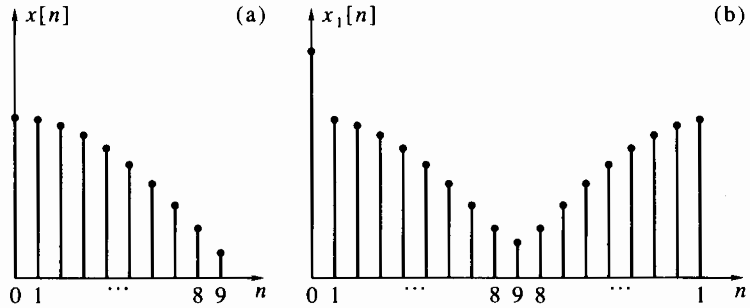

Figure 4.12 The symmetric extension used in the definition of DCT-I: (a) the given sequence; (b) the symmetrically extended sequence (for clarity, the time indices are marked not by the actual n but in a way that shows the symmetry).

The DFT of $x_1[n]$ is given by

$$\begin{aligned} X_1^{\mathrm{d}}[k] &= \sum_{n=0}^{2N-3} x_1[n]W_{2N-2}^{-nk} = 2^{1/2}x[0] + 2^{1/2}x[N-1](-1)^k \\ &\quad + \sum_{n=1}^{N-2} x[n]W_{2N-2}^{-nk} + \sum_{n=N}^{2N-3} x[2N-n-2]W_{2N-2}^{-nk} \\ &= 2^{1/2}x[0] + 2^{1/2}x[N-1](-1)^k + \sum_{n=1}^{N-2} x[n](W_{2N-2}^{-nk} + W_{2N-2}^{nk}) \\ &= 2^{1/2}x[0] + 2^{1/2}x[N-1](-1)^k + 2\sum_{n=1}^{N-2} x[n]\cos\left(\frac{\pi nk}{N-1}\right) \\ &= 2\sum_{n=0}^{N-1} a[n]x[n]\cos\left(\frac{\pi nk}{N-1}\right), \quad 0 \le k \le 2N-3, \end{aligned} \tag{4.70}$$

where

$$a[n] = \begin{cases} 2^{-1/2}, & n = 0, N-1, \\ 1, & \text{otherwise.} \end{cases} \tag{4.71}$$

The sequence $X_1^{\mathrm{d}}[k]$ is real and satisfies $X_1^{\mathrm{d}}[k] = X_1^{\mathrm{d}}[2N-2-k]$. Therefore,

$$\begin{aligned} x_1[n] &= \frac{1}{2N-2}\sum_{k=0}^{2N-3} X_1^{\mathrm{d}}[k]W_{2N-2}^{nk} = \frac{1}{2N-2}X_1^{\mathrm{d}}[0] \\ &\quad + \frac{1}{2N-2}X_1^{\mathrm{d}}[N-1](-1)^n + \frac{1}{N-1}\sum_{k=1}^{N-2} X_1^{\mathrm{d}}[k]\cos\left(\frac{\pi nk}{N-1}\right). \end{aligned} \tag{4.72}$$

The discrete cosine transform of type I (DCT-I) is defined in terms of the DFT of $x_1[n]$ as follows:

$$X^{\mathrm{c1}}[k] = \frac{1}{\sqrt{2(N-1)}} a[k] X_1^{\mathrm{d}}[k] = \sqrt{\frac{2}{N-1}} \sum_{n=0}^{N-1} a[k]a[n]x[n] \cos\left(\frac{\pi nk}{N-1}\right). \tag{4.73}$$

The inverse transform is obtained from (4.72) as

$$x[n] = \sqrt{\frac{2}{N-1}} \sum_{k=0}^{N-1} a[k]a[n] X^{\mathrm{c1}}[k] \cos\left(\frac{\pi nk}{N-1}\right). \tag{4.74}$$

We can express both DCT-I and its inverse as matrix-vector operations:

$$\boldsymbol{X}_N^{\mathrm{c1}} = \boldsymbol{C}_N^{\mathrm{I}} \boldsymbol{x}_N, \quad \boldsymbol{x}_N = \boldsymbol{C}_N^{\mathrm{I}} \boldsymbol{X}_N^{\mathrm{c1}}, \tag{4.75}$$

where

$$[\boldsymbol{C}_N^{\mathrm{I}}]_{k,n} = \sqrt{\frac{2}{N-1}} a[k]a[n] \cos\left(\frac{\pi nk}{N-1}\right), \quad 0 \le k, n \le N-1. \tag{4.76}$$

$\boldsymbol{C}_N^{\mathrm{I}}$ is called the DCT-I matrix. It is clear from the definition that this matrix is symmetric. It also follows from (4.75) that $\boldsymbol{C}_N^{\mathrm{I}}$ is its own inverse, that is,

$$(\boldsymbol{C}_N^{\mathrm{I}})^{-1} = \boldsymbol{C}_N^{\mathrm{I}}. \tag{4.77}$$

The DCT-I matrix is a symmetric orthonormal matrix. Recall, for comparison, that the normalized Fourier matrix $N^{-1/2}\boldsymbol{F}_N$ is a symmetric unitary matrix, that is,

$$(N^{-1/2}\boldsymbol{F}_N)^{-1} = N^{-1/2}\bar{\boldsymbol{F}}_N. \tag{4.78}$$

Example 4.10 Let us test the DCT-I transform on a sinusoidal waveform, as we did in Example 4.3 for the DFT. A sinusoidal waveform with an arbitrary initial phase can always be written as a linear combination of a cosine and a sine, because

$$\cos(\theta_0 n + \phi_0) = \cos\phi_0 \cos(\theta_0 n) - \sin\phi_0 \sin(\theta_0 n).$$

Since the DCT-I is a real transform, we need to consider the transforms of a cosine and a sine separately. Let the frequency be an integer multiple of $\pi/(N-1)$, which is matched to DCT-I as seen from (4.73). The signals to be tested are thus

$$x_{\cos}[n] = \cos\left(\frac{\pi mn}{N-1}\right), \quad x_{\sin}[n] = \sin\left(\frac{\pi mn}{N-1}\right).$$

Figure 4.13(a) shows $X_{\cos}^{\mathrm{c1}}[k]$ for $N = 32$ and $m = 10$; Figure 4.13(b) shows $X_{\sin}^{\mathrm{c1}}[k]$. As we see, the transform of the cosine signal exhibits a distinct peak at the right frequency, but is not free from ripple at other frequencies. The transform of the sine signal is not as informative. It is zero at the frequency of the signal, exhibits positive and negative peaks at the neighboring frequencies, and has a sizable ripple at other frequencies. For a sinusoidal signal of an arbitrary initial phase, the transform will be a weighted sum of the graphs shown in the two parts of Figure 4.13. □

4.9.2 Type-II Discrete Cosine Transform

Type-II DCT is based on nonminimal symmetrization of the signal $x[n]$ which duplicates all N elements. Let us define

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ x[2N-1-n], & N \le n \le 2N-1. \end{cases} \tag{4.79}$$

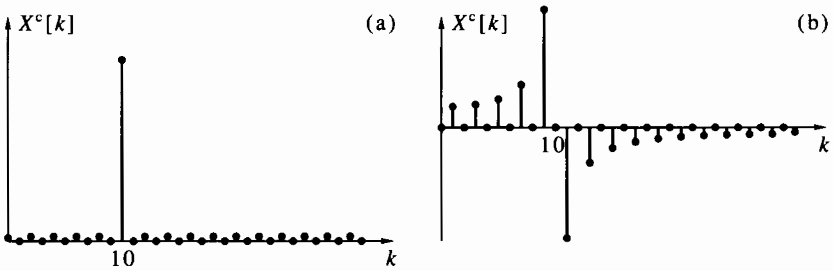

Figure 4.13 DCT-I of a sinusoidal waveform, $N = 32$: (a) pure cosine; (b) pure sine.

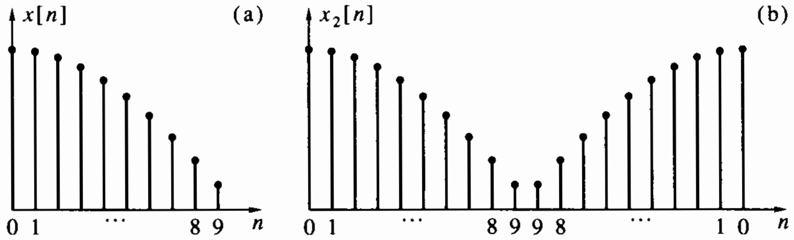

Figure 4.14 The symmetric extension used in the definition of DCT-II: (a) the given sequence; (b) the symmetrically extended sequence (for clarity, the time indices are marked not by the actual n but in a way that shows the symmetry).

Figure 4.14 illustrates this definition for $N = 10$. Part a shows the sequence $x[n]$ and part b the symmetrically extended sequence $x_2[n]$.

The DFT of $x_2[n]$ is given by

$$\begin{aligned} X_2^{\mathrm{d}}[k] &= \sum_{n=0}^{N-1} x[n]W_{2N}^{-nk} + \sum_{n=N}^{2N-1} x[2N-1-n]W_{2N}^{-nk} \\ &= \sum_{n=0}^{N-1} x[n](W_{2N}^{-nk} + W_{2N}^{(n+1)k}), \quad 0 \le k \le 2N-1. \end{aligned} \tag{4.80}$$

This sequence is neither real nor symmetric in k. However, if we multiply it by $W_{2N}^{-0.5k}$ we will get a real sequence, since

$$\begin{aligned} U[k] = W_{2N}^{-0.5k}X_2^{\mathrm{d}}[k] &= \sum_{n=0}^{N-1} x[n](W_{2N}^{-(n+0.5)k} + W_{2N}^{(n+0.5)k}) \\ &= 2\sum_{n=0}^{N-1} x[n]\cos\left[\frac{\pi(n+0.5)k}{N}\right], \quad 0 \le k \le 2N-1. \end{aligned} \tag{4.81}$$

We make two observations:

1. The sequence $U[k]$ satisfies the antisymmetry relation

$$U[2N-k] = -U[k], \tag{4.82}$$

since

$$\cos\left[\frac{\pi(n+0.5)(2N-k)}{N}\right] = -\cos\left[\frac{\pi(n+0.5)k}{N}\right]. \tag{4.83}$$

2. $U[N] = 0$, since

$$\cos\left[\frac{\pi(n+0.5)N}{N}\right] = \cos[\pi(n+0.5)] = 0. \tag{4.84}$$

The original sequence can be expressed in terms of $U[k]$ as follows:

$$\begin{aligned}
x[n] &= \frac{1}{2N}\sum_{k=0}^{2N-1} X_2^{\mathrm{d}}[k]W_{2N}^{nk} = \frac{1}{2N}\sum_{k=0}^{2N-1} U[k]W_{2N}^{0.5k}W_{2N}^{nk} \\
&= \frac{1}{2N}\sum_{k=0}^{N-1} U[k]W_{2N}^{(n+0.5)k} - \frac{1}{2N}\sum_{k=N+1}^{2N-1} U[2N-k]W_{2N}^{(n+0.5)k} \\
&= \frac{1}{2N}U[0] + \frac{1}{2N}\sum_{k=1}^{N-1} U[k](W_{2N}^{(n+0.5)k} + W_{2N}^{-(n+0.5)k}) \\
&= \frac{1}{2N}U[0] + \frac{1}{N}\sum_{k=1}^{N-1} U[k]\cos\left[\frac{\pi(n+0.5)k}{N}\right].
\end{aligned} \tag{4.85}$$

The transform DCT-II is defined as

$$X^{\mathrm{c2}}[k] = \frac{1}{\sqrt{2N}}b[k]U[k], \tag{4.86}$$

where

$$b[k] = \begin{cases} 2^{-1/2}, & k = 0, \\ 1, & k \neq 0. \end{cases} \tag{4.87}$$

Substitution of (4.81) and (4.87) in (4.86) gives

$$X^{\mathrm{c2}}[k] = \sqrt{\frac{2}{N}}\sum_{n=0}^{N-1} b[k]x[n]\cos\left[\frac{\pi(n+0.5)k}{N}\right]. \tag{4.88}$$

Substitution of (4.86) and (4.87) in (4.85) gives

$$x[n] = \sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} b[k]X^{\mathrm{c2}}[k]\cos\left[\frac{\pi(n+0.5)k}{N}\right]. \tag{4.89}$$

Equations (4.88) and (4.89) are the DCT-II transform pair. The matrix-vector representation of the DCT-II transform pair is

$$\boldsymbol{X}_N^{\mathrm{c2}} = \boldsymbol{C}_N^{\mathrm{II}}\boldsymbol{x}_N, \quad \boldsymbol{x}_N = (\boldsymbol{C}_N^{\mathrm{II}})'\boldsymbol{X}_N^{\mathrm{c2}}, \tag{4.90}$$

where

$$[\boldsymbol{C}_N^{\mathrm{II}}]_{k,n} = \sqrt{\frac{2}{N}}b[k]\cos\left[\frac{\pi(n+0.5)k}{N}\right], \quad 0 \le k, n \le N-1. \tag{4.91}$$

It follows from (4.90) that $\boldsymbol{C}_N^{\mathrm{II}}$ is a real orthonormal matrix, that is, it satisfies

$$(\boldsymbol{C}_N^{\mathrm{II}})^{-1} = (\boldsymbol{C}_N^{\mathrm{II}})'. \tag{4.92}$$

4.9.3 Type-III Discrete Cosine Transform

The type-III DCT is the inverse, or transpose, of the type-II transform. It is defined by

$$X^{\mathrm{c3}}[k] = \sqrt{\frac{2}{N}}\sum_{n=0}^{N-1} b[n]x[n]\cos\left[\frac{\pi(k+0.5)n}{N}\right], \tag{4.93}$$

$$x[n] = \sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} b[n]X^{\mathrm{c3}}[k]\cos\left[\frac{\pi(k+0.5)n}{N}\right]. \tag{4.94}$$

The DCT-III matrix is

$$\boldsymbol{C}_N^{\mathrm{III}} = (\boldsymbol{C}_N^{\mathrm{II}})'. \tag{4.95}$$

Problem 4.44 discusses a direct construction of the DCT-III.

4.9.4 Type-IV Discrete Cosine Transform

Both DCT-II and DCT-III are not symmetrical in k and n. Type-IV DCT is, like DCT-I, symmetric. Its construction is similar to that of DCT-II. However, in addition to extending $x[n]$ symmetrically we modulate it as follows:

$$x_4[n] = \begin{cases} W_{2N}^{-0.5n}x[n], & 0 \le n \le N-1, \\ -W_{2N}^{-0.5n}x[2N-1-n], & N \le n \le 2N-1. \end{cases} \tag{4.96}$$

The DFT of $x_4[n]$ is given by

$$\begin{aligned} X_4^{\mathrm{d}}[k] &= \sum_{n=0}^{N-1} x[n]W_{2N}^{-0.5n}W_{2N}^{-nk} - \sum_{n=N}^{2N-1} x[2N-1-n]W_{2N}^{-0.5n}W_{2N}^{-nk} \\ &= \sum_{n=0}^{N-1} x[n]W_{2N}^{-n(k+0.5)} - \sum_{n=0}^{N-1} x[n]W_{2N}^{-(2N-1-n)(k+0.5)} \\ &= \sum_{n=0}^{N-1} x[n](W_{2N}^{-n(k+0.5)} + W_{2N}^{(n+1)(k+0.5)}), \quad 0 \le k \le 2N-1. \end{aligned} \tag{4.97}$$

This sequence is neither real nor symmetric in k. However, if we multiply it by $W_{2N}^{-0.5(k+0.5)}$ we will get a real sequence, since

$$\begin{aligned} V[k] = W_{2N}^{-0.5(k+0.5)}X_4^{\mathrm{d}}[k] &= \sum_{n=0}^{N-1} x[n](W_{2N}^{-(n+0.5)(k+0.5)} + W_{2N}^{(n+0.5)(k+0.5)}) \\ &= 2\sum_{n=0}^{N-1} x[n]\cos\left[\frac{\pi(n+0.5)(k+0.5)}{N}\right], \quad 0 \le k \le 2N-1. \end{aligned} \tag{4.98}$$

The sequence $V[k]$ satisfies the antisymmetry relation

$$V[2N-1-k] = -V[k], \tag{4.99}$$

since

$$\cos\left[\frac{\pi(n+0.5)(2N-1-k+0.5)}{N}\right] = -\cos\left[\frac{\pi(n+0.5)(k+0.5)}{N}\right]. \tag{4.100}$$

The original sequence can be expressed in terms of $V[k]$ as follows:

$$\begin{aligned} x[n] &= W_{2N}^{0.5n}\frac{1}{2N}\sum_{k=0}^{2N-1} X_4^{\mathrm{d}}[k]W_{2N}^{nk} = W_{2N}^{0.5n}\frac{1}{2N}\sum_{k=0}^{2N-1} V[k]W_{2N}^{0.5(k+0.5)}W_{2N}^{nk} \\ &= \frac{1}{2N}\sum_{k=0}^{N-1} V[k]W_{2N}^{(n+0.5)(k+0.5)} - \frac{1}{2N}\sum_{k=N}^{2N-1} V[2N-1-k]W_{2N}^{(n+0.5)(k+0.5)} \\ &= \frac{1}{2N}\sum_{k=0}^{N-1} V[k](W_{2N}^{(n+0.5)(k+0.5)} + W_{2N}^{-(n+0.5)(k+0.5)}) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} V[k]\cos\left[\frac{\pi(n+0.5)(k+0.5)}{N}\right]. \end{aligned} \tag{4.101}$$

The transform DCT-IV is defined as

$$X^{\mathrm{c}4}[k] = \frac{1}{\sqrt{2N}}V[k] = \sqrt{\frac{2}{N}}\sum_{n=0}^{N-1} x[n]\cos\left[\frac{\pi(n+0.5)(k+0.5)}{N}\right]. \tag{4.102}$$

The inverse transform is obtained from (4.101) as

$$x[n] = \sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} X^{\mathrm{c}4}[k]\cos\left[\frac{\pi(n+0.5)(k+0.5)}{N}\right]. \tag{4.103}$$

Equations (4.102) and (4.103) are the DCT-IV transform pair. The matrix-vector representation of the DCT-IV transform pair is

$$\boldsymbol{X}_N^{c4} = \boldsymbol{C}_N^{\mathrm{IV}} \boldsymbol{x}_N, \quad \boldsymbol{x}_N = \boldsymbol{C}_N^{\mathrm{IV}} \boldsymbol{X}_N^{c4}, \tag{4.104}$$

where

$$[\boldsymbol{C}_N^{\mathrm{IV}}]_{k,n} = \sqrt{\frac{2}{N}} \cos\left[\frac{\pi(n+0.5)(k+0.5)}{N}\right], \quad 0 \le k, n \le N-1. \tag{4.105}$$

It follows from (4.104) that $\boldsymbol{C}_N^{\mathrm{IV}}$ is a symmetric orthonormal matrix, that is, it satisfies

$$(\boldsymbol{C}_N^{\mathrm{IV}})^{-1} = \boldsymbol{C}_N^{\mathrm{IV}}. \tag{4.106}$$

4.9.5 Discussion

All four DCTs are real orthonormal transforms. As we saw in Section 4.2, the columns of the inverse of an orthonormal transform can be regarded as basis vectors for the N-dimensional vector space, and the components of the transform of a given signal are the coordinates of the signal in this basis. Thus, for any of the four transforms we have

$$\boldsymbol{x}_N = \sum_{k=0}^{N-1} X^c[k] \boldsymbol{c}_{N,k}, \tag{4.107}$$

where $\boldsymbol{c}_{N,k}$ is the kth column of $\boldsymbol{C}_N'$. Since the DCTs are real, their corresponding vectors form bases to the real N-dimensional vector space (unlike the DFT vectors, which form a basis for the complex N-dimensional space). Figures 4.15 and 4.16 show the basis vectors of the four transforms for $N = 8$. As we explained in Section 4.2, each vector is interpreted as a discrete-time signal, that is, its components are interpreted as points in time.

The DCT is not as useful as the DFT for frequency-domain signal analysis, due to its deficiencies in representing pure sinusoidal waveforms—recall Example 4.10. The main application of the DCT is in signal compression. We shall discuss this application in Section 14.1; see page 551.

4.10 The Discrete Sine Transform*

The discrete sine transform (DST) is similar to the DCT. It can be regarded as a discrete-time version of the Fourier sine series, which we described in Section 2.5. There are four types of DST, which differ in the way the given signal is extended antisymmetrically. Since their derivation is similar to the derivation of the corresponding DCTs, and since in practice they are less important than the DCTs, we skip the details and give only the DST definitions. We describe the transforms in terms of the (k, n)th element of the corresponding matrix, where n is the column index (corresponding to $x[n]$) and k is the row index (corresponding to $X^{si}[k]$, where $i = 1, 2, 3, 4$). In all cases $0 \le n, k \le N-1$.

1. DST-I

$$[\boldsymbol{S}_N^{\mathrm{I}}]_{k,n} = \sqrt{\frac{2}{N+1}} \sin\left[\frac{\pi(k+1)(n+1)}{N+1}\right], \quad (\boldsymbol{S}_N^{\mathrm{I}})^{-1} = \boldsymbol{S}_N^{\mathrm{I}}. \tag{4.108}$$

2. DST-II

$$[\boldsymbol{S}_N^{\mathrm{II}}]_{k,n} = \sqrt{\frac{2}{N}} c[k] \sin\left[\frac{\pi(k+1)(n+0.5)}{N}\right], \quad (\boldsymbol{S}_N^{\mathrm{II}})^{-1} = (\boldsymbol{S}_N^{\mathrm{II}})'. \tag{4.109}$$

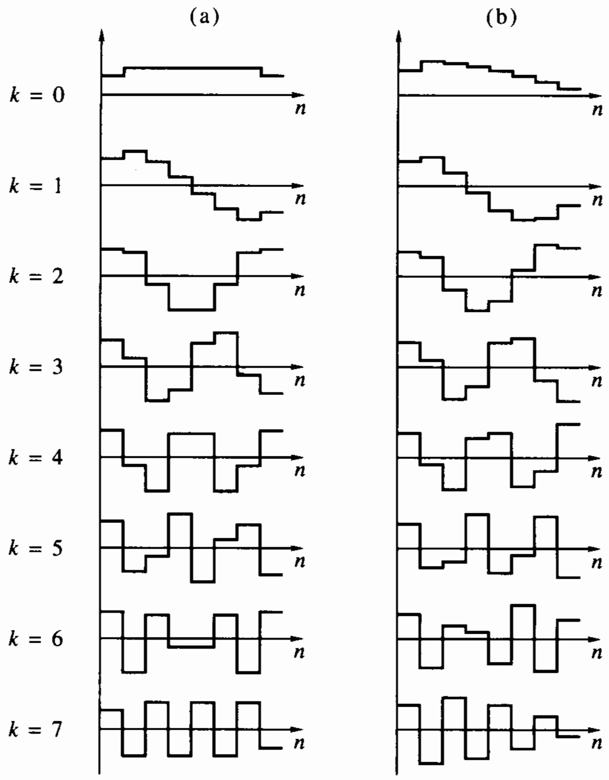

Figure 4.15 The DCT basis vectors for $N = 8$: (a) DCT-I; (b) DCT-II.

3. DST-III

$$[S_N^{\mathrm{III}}]_{k,n} = \sqrt{\frac{2}{N}} c[n] \sin\left[\frac{\pi(k+0.5)(n+1)}{N}\right], \quad (S_N^{\mathrm{III}})^{-1} = (S_N^{\mathrm{III}})'. \tag{4.110}$$

4. DST-IV

$$[S_N^{\mathrm{IV}}]_{k,n} = \sqrt{\frac{2}{N}} \sin\left[\frac{\pi(k+0.5)(n+0.5)}{N}\right], \quad (S_N^{\mathrm{IV}})^{-1} = S_N^{\mathrm{IV}}. \tag{4.111}$$

In these formulas we have defined

$$c[n] = \begin{cases} 2^{-1/2}, & n = N-1, \\ 1, & n \neq N-1. \end{cases} \tag{4.112}$$

4.11 Summary and Complement

4.11.1 Summary

In this chapter we introduced the discrete Fourier transform (4.3). The DFT is defined for discrete-time, finite-duration signals and is a uniform sampling of the Fourier transform of a signal, with a number of samples equal to the length of the signal. The signal can be uniquely recovered from its DFT through the inverse DFT formula (4.16). The DFT can be represented as a matrix-vector multiplication. The DFT matrix is unitary, except for a constant scale factor. The columns of the IDFT matrix can be interpreted

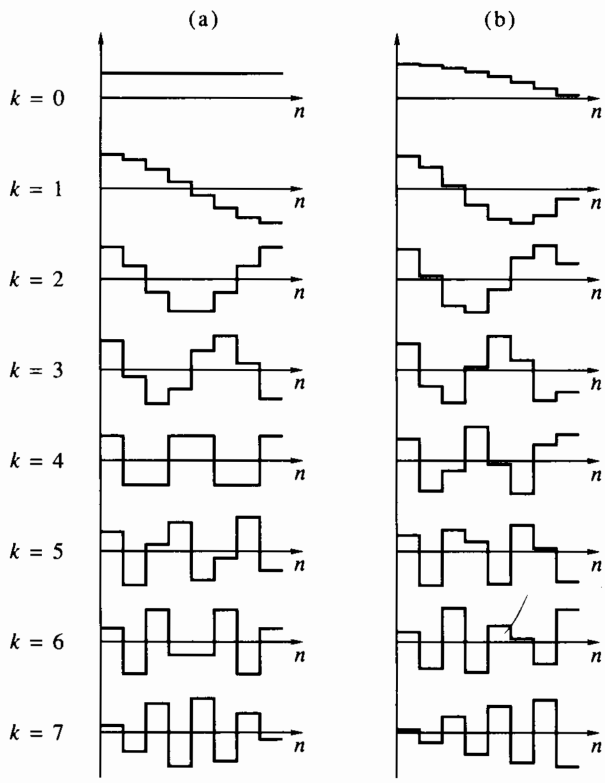

Figure 4.16 The DCT basis vectors for $N = 8$: (a) DCT-III; (b) DCT-IV.

as a basis for the N-dimensional vector space, and the DFT values can be interpreted as the coordinates of the signal in this basis. The DFT shares many of the properties of the usual Fourier transform.

The DFT of a signal of length N can also be defined at M frequency points, with $M > N$. This is done by padding the signal with $M - N$ zeros and computing the M-point DFT of the zero-padded signal. Zero padding is useful for refining the display of the frequency response of a finite-duration signal, thus improving its visual appearance. Frequency-domain zero padding allows interpolation of a finite-duration signal.

Closely related to the DFT is the operation of circular convolution (4.51). Circular convolution is defined between two signals of the same (finite) length, and the result has the same length. It can be thought of as a conventional (linear) convolution of the periodic extensions of the two signals over one period. The DFT of the circular convolution of two signals is the product of the individual DFTs. The DFT of the product of two signals is the circular convolution of the individual DFTs (up to a factor N^{-1}).

Circular convolution can be used for computing the linear convolution of two finite-duration signals, not necessarily of the same length. This is done by zero padding the two sequences to a common length, equal to the sum of the lengths minus 1, followed by a circular convolution of the zero-padded signals. The latter can be performed by multiplication of the DFTs and taking the inverse DFT of the product.

In general, the DFT does not give an exact picture of the frequency-domain characteristics of a signal, only an approximate one. An exception occurs when the signal is periodic and band limited, and sampling is at an integer number of samples per period, at a rate higher than the Nyquist rate. In this case the DFT values are equal, up to a constant factor, to the Fourier series coefficients of the signal.

In this chapter we also introduced the discrete cosine and sine transforms. Contrary to the DFT, these transforms are real valued (when applied to real-valued signals) and orthonormal. There are four types of each. The DCT has become a powerful tool for image compression applications. We shall describe the use of the DCT for compression in Section 14.1; see page 551. Additional material on the DCT can be found in Rao and Yip [1990].

4.11.2 Complement

1. [p. 99] *Resolution* here refers to the spacing of frequency points; it does not necessarily determine the *accuracy* to which the frequency of a sinusoidal signal can be determined from the DFT. We shall study the accuracy issue in Chapter 6.

4.12 MATLAB Programs

Program 4.1 Brute-force DFT and IDFT.

```
function y = bfdft(x,swtch);
% Synopsis: y = bfdft(x,swtch).
% Brute-force DFT or IDFT.
% Input parameters:
% x: the input vector
% swtch: 0 for DFT, 1 for IDFT.
% Output parameters:
% y: the output vector.

N = length(x);
x = reshape(x,1,N);
n = 0:N-1;
if (swtch), W = exp((j*2*pi/N)*n);
else, W = exp((-j*2*pi/N)*n), end
y = zeros(1,N);
for k = 0:N-1,
   y(k+1) = sum(x.*(W.^k));
end
if (swtch), y = (1/N)*y; end
```

Program 4.2 Circular convolution via linear convolution.

```
function z = circonv(x,y);
% Synopsis: z = circonv(x,y).
% Performs circular convolution by means of linear convolution.
% Input parameters:
% x, y; the two vectors to be convolved.
% Output parameter:
% z: the result of the circular convolution.

N = length(x);
if (length(y) ~= N),
   error('Vectors of unequal lengths in circonv');
end
z = conv(reshape(x,1,N),reshape(y,1,N));
z = z(1:N) + [z(N+1:2*N-1),0];
```

4.13 Problems

4.1 The sequence $x[n]$ has length N, and $X^{\mathrm{d}}[k]$ is known to be real. What does this imply about $x[n]$?

4.2 If both $x[n]$ and $X^{\mathrm{d}}[k]$ are real, what else can we say about these two sequences? If $x[n]$ is real and $X^{\mathrm{d}}[k]$ is purely imaginary, what else can we say about $x[n]$?

4.3 Let $N = 4M$, where M is an integer. Let

$$X^{\mathrm{d}}[k] = \begin{cases} 0.5, & k = M, \\ 0.5, & k = 3M, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Is the signal $x[n]$ (the inverse DFT of $X^{\mathrm{d}}[k]$) real?

(b) Compute $x[n]$ explicitly.

(c) Let

$$y[n] = (-1)^n x[2n], \quad 0 \le n \le 2M - 1.$$

Compute $Y^{\mathrm{d}}[k]$.

4.4 We are given a real sequence $x[n]$ of length $4N$. Define $y[n]$ as

$$y[n] = x[(n - N) \bmod 4N] + x[n].$$

Also define

$$Z^{\mathrm{d}}[k] = Y^{\mathrm{d}}[k] + Y^{\mathrm{d}}[4N - k].$$

Express $Z^{\mathrm{d}}[k]$ as a function of $X^{\mathrm{d}}[k]$ for each of the following four cases separately: $k \bmod 4 = 0, 1, 2, 3$.

4.5 The signal $x[n]$ is obtained by sampling the continuous-time signal $\sin(2\pi t)$ at interval $T = 0.125$ and taking 12 consecutive samples. At which values of k will $|X^{\mathrm{d}}[k]|$ be maximum?

4.6 Let $x[n]$ be a real sequence of even-length N satisfying

$$x[n] = x[N - 1 - n].$$

Let

$$Y^{\mathrm{d}}[k] = W_N^{-0.5k} X^{\mathrm{d}}[k].$$

(a) Show that $Y^{\mathrm{d}}[k]$ is real.

(b) Show that $Y^{\mathrm{d}}[k]$ is antisymmetric modulo N, that is,

$$Y^{\mathrm{d}}[k] = -Y^{\mathrm{d}}[N - k].$$

4.7 Let $x[n]$ be the sequence

$$x[n] = \begin{cases} \alpha^n, & n \ge 0, \\ 0, & n < 0, \end{cases}$$

where α is real, $|\alpha| < 1$. We are given another sequence $y[n]$ of length N such that

$$Y^{\mathrm{d}}[k] = X^{\mathrm{f}}\left(\frac{2\pi k}{N}\right).$$

Give an explicit expression for $y[n]$.

4.8 Let $x[n]$ be a signal of length N and let $y[n]$ be the periodic extension of $x[n]$ to M periods, so the length of $y[n]$ is MN. Express Y^{d} in terms of X^{d}.

4.9 Examine the following claim: "Every real periodic discrete-time signal is necessarily a finite sum of discrete-time sinusoidal signals." Is the claim right or wrong? If it is right, explain why. If it is wrong, give a counterexample.

4.10 The sequences $x_1[n]$, $x_2[n]$ have length N each. Define the sequence $y[n]$ of length $3N$ as

$$\{y[n]\}_{n=0}^{3N-1} = \{x_1[0], 0, x_2[0], x_1[1], 0, x_2[1], \ldots, x_1[N-1], 0, x_2[N-1]\}.$$

Express Y^{d} in terms of X_1^{d} and X_2^{d}.

4.11 The sequences $x_1[n]$, $x_2[n]$ have length N each. Define the sequences $y_1[n]$, $y_2[n]$ as

$$y_1[n] = \begin{cases} x_1[n], & 0 \le n \le N-1, \\ 0, & N \le n \le 2N-1, \end{cases}$$

$$y_2[n] = \begin{cases} 0, & 0 \le n \le N-1, \\ x_2[n-N], & N \le n \le 2N-1. \end{cases}$$

Let

$$z[n] = x_1[n] + x_2[n].$$

Express Z^{d} in terms of Y_1^{d} and Y_2^{d}.

4.12 We are given a sequence $x[n]$ of length 6 and we know that

$$\{|X^{\mathrm{d}}[k]|,\ 0 \le k \le 5\} = \{12, 7, 3, 0, 3, 7\}.$$

Which of the following is correct? (1) The sequence is necessarily real; (2) the sequence is necessarily imaginary; (3) the sequence is necessarily complex; (4) the information is insufficient for conclusion. Explain your answer.

4.13 Let N be an even number.

(a) Let $f[n]$ be a sequence of length N. We are given that

$$F^{\mathrm{d}}[k] = (-1)^k, \quad 0 \le k \le N-1.$$

Find $f[n]$.

(b) Let $x[n]$ be a sequence of length N and let

$$Y^{\mathrm{d}}[k] = \begin{cases} X^{\mathrm{d}}[k], & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Express $y[n]$, the inverse DFT of $Y^{\mathrm{d}}[k]$, as a function of the sequence $x[n]$. Hint: Use the result of part a.

(c) Let $\tilde{x}[n]$ and $\tilde{y}[n]$ be the periodic extensions of $x[n]$ and $y[n]$, respectively, that is,

$$\tilde{x}[n] = x[n \bmod N], \quad \tilde{y}[n] = y[n \bmod N].$$

Find a filter $h[n]$ such that

$$\tilde{y} = h * \tilde{x}.$$

(d) Return to part b and assume that N is odd, and that $x[n]$ is real valued. Is $y[n]$ real valued in general?

4.14 Suppose you wish to implement an inverse DFT, but your computer program (or hardware device) is capable of computing only a direct DFT. Show how to use the direct DFT to perform inverse DFT.

4.15 This problem explores the result of successive applications of the DFT operation.

(a) Let $x[n]$ have length N and let

$$y = \text{DFT}_N\{\text{DFT}_N\{x\}\}.$$

Express the sequence $y[n]$ in terms of the elements of the sequence $x[n]$ in the simplest possible form.

(b) Suppose we perform the operation DFT_N P times in cascade on the input sequence $x[n]$ and obtain the output sequence $w[n]$. What is the minimum value of P for which $w[n] = Ax[n]$, where A is a constant? What is the value of A?

4.16 The sequence $\{x[n],\ 0 \le n \le 4\}$ was obtained from a periodic signal $x(t)$ whose period is $T_0 = 1$ second by sampling at an interval $T = 0.2$ second. The DFT of the sequence is

$$X^{\text{d}}[k] = \{2, j, 1, 1, -j\}.$$

(a) Explain why this information is not sufficient to reconstruct $x(t)$ unambiguously.

(b) What additional information will make it possible to reconstruct $x(t)$ unambiguously? With this additional information, write the expression for $x(t)$.

4.17 This problem shows that the DFT values of a finite-duration signal uniquely determine the Fourier transform of the signal at all frequency points.

(a) We are given the DFT of a length-N sequence, $X^{\text{d}}[k]$. Express the Fourier transform of the sequence, $X^{\text{f}}(\theta)$, as a function of $\{X^{\text{d}}[k],\ 0 \le k \le N-1\}$.

(b) Write a MATLAB program that implements the result in part a. The inputs to the program are the vector of $X^{\text{d}}[k]$ and a vector of frequencies θ at which the frequency response is to be calculated.

4.18 Define the transform $X^{\text{e}}[k]$ of the length-N sequence $x[n]$ as the samples of the Fourier transform of $x[n]$ at the frequency points

$$\theta[k] = \frac{\pi(2k+1)}{N}, \quad 0 \le k \le N-1.$$

Let $x[n], y[n], z[n]$ be three sequences of length N each, and suppose we know that

$$Z^{\text{e}}[k] = X^{\text{e}}[k]Y^{\text{e}}[k].$$

Express the sequence $z[n]$ in terms of the sequences $x[n], y[n]$. Hint: Write $X^{\text{e}}[k]$ as the DFT of a sequence related to $x[n]$.

4.19 Define the transform $X^{\text{g}}[k]$ of a real sequence $\{x[n],\ 0 \le n \le N-1\}$ as

$$X^{\text{g}}[k] = \sum_{n=0}^{N-1} x[n]W_N^{-k(n+0.5)}. \tag{4.113}$$

(a) Compute $X^{\text{g}}[k]$ as a function of $X^{\text{d}}[k]$.

(b) Let $y[n]$ be the sequence

$$y[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ x[2N-n-1], & N \le n \le 2N-1. \end{cases}$$

Compute $Y^{\mathrm{g}}[k]$ and express it as a function of $X^{\mathrm{d}}[k]$ for even values of k. Is the result always real or complex in general?

4.20 A length-N signal $x[n]$ is zero padded to length $2N$ and the corresponding DFT $X_{\mathrm{a}}^{\mathrm{d}}[k]$ is computed. The length-N inverse DFT of the sequence of odd-index components $X_{\mathrm{a}}^{\mathrm{d}}[2l+1]$ is then computed. What is the result of this operation?

4.21 Let $X^{\mathrm{d}}[k]$ be the DFT of a length-N signal $x[n]$, and $X_{\mathrm{a}}^{\mathrm{d}}[l]$ the DFT of the signal zero padded to length M. What points $0 \le k \le N-1$ have corresponding points $0 \le l \le M-1$ such that $X^{\mathrm{d}}[k] = X_{\mathrm{a}}^{\mathrm{d}}[l]$:

(a) When M is an integer multiple of N?

(b) When M is not an integer multiple of N?

4.22 In Section 4.4 we saw how to use the DFT for computing the Fourier transform of a length-N sequence at M equally spaced frequency points when $M > N$. Suppose we wish to do this when $M < N$, that is, to compute

$$X^{\mathrm{f}}(\theta[k]), \quad \theta[k] = \frac{2\pi k}{M}, \quad 0 \le k \le M-1.$$

(a) Show how to do this using DFT. Remark: At the time of writing, MATLAB did not perform this computation correctly. When called with an argument N shorter than the sequence length, `fft` gave the DFT of a truncated sequence, rather than the Fourier transform of the given sequence. Check your current version of MATLAB to see if this problem has been corrected!

(b) Illustrate the procedure you developed in part a for $N = 17$ and $M = 5$.

4.23 A continuous-time signal $x(t)$ is sampled at interval $T = 1/1.28$ millisecond, a total of 100 samples. We wish to compute the Fourier transform of the sampled signal at $f = 305$ Hz using DFT.

(a) Using DFT of the 100 samples, what is the frequency nearest to f at which the DFT is computed, and what is the corresponding k?

(b) What is the minimum number of zeros with which we must pad the 100 samples to obtain the DFT at $f = 305$ Hz exactly?

4.24 Pierre, a programmer in a DSP company, was instructed to write a computer program that zero-pads a sequence of length N to length LN (L a positive integer), then computes the DFT. Misunderstanding the concept of zero padding, he padded each point $x[n]$ of the given sequence by $L-1$ zeros. What did his program compute?

4.25 Show, by carrying out the necessary derivation, that Program 4.2 indeed performs circular convolution.

4.26 We are given the two sequences

$$x = \{1, -3, 1, 5\}, \quad w = \{7, -7, -9, -3\}.$$

Does there exist a sequence y of length 4 such that

$$x \circledast y = w?$$

If so, find y; if not, prove that such y does not exist. Hint: Solve using DFT.

4.27 Given that $x = \{2, -1\}$, $w = x * y$, and $w = \{6, -1, 7, -4\}$, compute the sequence y using DFT.

4.28 Prove the commutativity and associativity of circular convolution (4.54).

4.29 We are given two sequences $x_1[n]$, $x_2[n]$ of length 75 each. It is known that $x[n] = 0$ for all $0 \le n \le 7$ and for all $60 \le n \le 74$, but is nonzero otherwise; $x_2[n]$ is nonzero in general. Let

$$y[n] = \{x_1 \circledast x_2\}[n], \quad z[n] = \{x_1 * x_2\}[n].$$

For what values of n will $y[n \bmod 75] = z[n]$ hold?

4.30 The sequences $x[n]$, $y[n]$, $z[n]$ have length 3 each, and it is known that

$$z[n] = \{x \circledast y\}[n].$$

We are given that

$$x[n] = \{1, 1, 0\}, \quad Z^{\mathrm{d}}[k] = \{2, -W_3^2, -W_3\}.$$

Find the sequence $y[n]$.

4.31 In Section 4.7 we showed how to perform linear convolution of sequences of lengths N_1, N_2 by zero padding to length $N_1 + N_2 - 1$. Suppose, instead, that we zero-pad the two sequences to length $M > N_1 + N_2 - 1$. How then is the circular convolution of the zero-padded sequences related to the linear convolution?

4.32 Suggest a procedure for performing linear convolution of *three* sequences of lengths N_1, N_2, N_3 by DFT.

4.33 The *circular correlation* (or *circular cross-correlation*) of two discrete-time signals, both of length N, is defined as (cf. Problem 2.36)

$$z[n] = \sum_{m=0}^{N-1} x[(m+n) \bmod N]\bar{y}[m]. \tag{4.114}$$

Express the circular correlation operation in the frequency domain.

4.34* Let

$$x(t) = t^3 + 1, \quad -\infty < t < \infty.$$

The signal is sampled at interval T, for $-N \le n \le N$, and shifted to the right by N, yielding a sequence of length $(2N + 1)$. Find $X^{\mathrm{d}}[0]$ of this sequence.

4.35* Let $x(t)$ be the signal

$$x(t) = a_1 \cos(2\pi f t + \phi_1) + a_2 \cos(2\pi \cdot 1.25 f t + \phi_2).$$

The frequency f is unknown, but it is known to lie in the range

$$1000\,\text{Hz} < f < 1600\,\text{Hz}$$

(with strict inequalities). The parameters a_1, a_2, ϕ_1, ϕ_2 are real and unknown.

(a) It is required to sample the signal such that for all f in the specified range it will be possible to reconstruct $x(t)$ from its samples unambiguously. What is the smallest sampling rate that meets this requirement?

(b) Is the signal $x(t)$ periodic? If so, what is its period? If not, explain why.

(c) Now suppose we know that $f = 1200$ Hz, and we wish to find a_1 and a_2. Suggest a convenient way of doing it using DFT (not necessarily as efficiently as possible). Choose a sampling interval T and a number of sample N, and show the details of the calculation.

(d) Consider the following alternative to the operation in part c. Generate the signal

$$y(t) = x(t)\cos(2\pi 1000t).$$

Pass $y(t)$ through an ideal low-pass filter having a cutoff frequency of 1000 Hz. Let $z(t)$ denote the output of the filter. Now repeat part c for $z(t)$. What then do you choose for N and T?

4.36* Let $x[n]$ be a signal on $0 \le n \le N-1$ and $\tilde{x}[n]$ its periodic extension with period N. It is known that

$$X^{\mathrm{d}}[k] = 4\cos\left(\frac{2\pi kL}{N}\right) + 2$$

for some integer L in the range $0 < L < N$. Find the values of L for which $\tilde{x}[n]$ is periodic with period $N' < 0.5N$. Specify the constraint on N for this problem to have a solution.

4.37* Write F_5 in full, similarly to the matrices given in (4.21), (4.22).

4.38* Let N be a prime integer. Prove that each row of F_N, with the exception of the first, is a permutation of the sequence $\{W_N^{-n},\ 0 \le n \le N-1\}$ (i.e., each element of this sequence appears in the row exactly once).

4.39* Plot the basis vectors of the natural basis for $N = 8$, similarly to Figure 4.5.

4.40* This problem examines the representation of circular convolution by matrix notation.

(a) Denote by Y the matrix of elements $y[n]$ appearing in (4.53). Write a general expression for the (k, l)th element of this matrix.

(b) Let F_N be the DFT matrix, as given in (4.19). Derive a general expression for the (k, l)th element of the matrix product $F_N Y \bar{F}'_N$ and bring it to the simplest form.

(c) Return to (4.53) and use the property (4.23) of the DFT matrix to write it in the form

$$F_N z_N = N^{-1}(F_N Y \bar{F}'_N) F_N x_N. \tag{4.115}$$

Justify the equivalence of this form to (4.53).

(d) Substitute the result of part b in (4.115), then write (4.115) in terms of the DFTs of the three sequences. What property of the DFT have you thus proved?

4.41* Write a MATLAB program that implements interpolation of a finite sequence $x[n]$ by zero padding in the frequency domain. The program should treat both even and odd lengths of $x[n]$ and not be limited to M which is an integer multiple of N.

4.42* Modify the definition of zero padding in the frequency domain (4.47) to the case of even N. You are not requested to derive (4.49) for this case. Hint: Take care to preserve the conjugate symmetry property of $X_i^d[k]$.

4.43* Repeat Example 4.10 for DCT-II. Match the signals to this transform by taking

$$x_{\cos}[n] = \cos\left[\frac{\pi m(n+0.5)}{N-1}\right], \quad x_{\sin}[n] = \sin\left[\frac{\pi m(n+0.5)}{N-1}\right].$$

Use $N = 32$ and $m = 10$. Plot the results, similarly to Figure 4.13. What is the main difference with respect to the results in Example 4.10?

4.44* Construct the DCT-III directly, similarly to the other three DCTs. Hint: Define $x_3[n]$ as

$$x_3[n] = \begin{cases} \sqrt{2}x[n], & n = 0, \\ x[n]W_{2N}^{-0.5n}, & 1 \le n \le N-1, \\ 0, & n = N, \\ -x[2N-n]W_{2N}^{-0.5n}, & N+1 \le n \le 2N-1. \end{cases}$$

4.45* Write a MATLAB program that implements the four types of DCT.

4.46* Write a MATLAB program that implements the four types of DST.

4.47* Define a new discrete cosine transform, based on the following symmetric extension of $x[n]$:

$$x_5[n] = \begin{cases} 2^{1/2}x[0], & n = 0, \\ x[n], & 1 \le n \le N-1, \\ x[2N-n-1], & N \le n \le 2N-2. \end{cases}$$

The transform is to be real and orthonormal. Derive the formulas for the transform and its inverse.

4.48* Define a new trigonometric function

$$\text{cas}(\theta) = \cos\theta + \sin\theta = \sqrt{2}\cos(\theta - 0.25\pi) \tag{4.116}$$

(the notation *cas* is short for "cosine and sine"). For a real length-N sequence $x[n]$, define

$$X^{\text{h}}[k] = \{\mathcal{H}x\}[k] = \sum_{n=0}^{N-1} x[n]\text{cas}\left(\frac{2\pi nk}{N}\right). \tag{4.117}$$

$X^{\text{h}}[k]$ is called the *discrete Hartley transform*, or DHT for short. The DHT is a *real transform*, that is, the transform of a real sequence is real.

(a) Prove that the function cas satisfies the orthogonality property

$$\sum_{n=0}^{N-1} \text{cas}\left(\frac{2\pi kn}{N}\right)\text{cas}\left(\frac{2\pi nl}{N}\right) = N\delta[k-l], \quad 0 \le k,l \le N-1. \tag{4.118}$$

(b) Explain why (4.118) implies that the inverse DHT is

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^{\text{h}}[k]\text{cas}\left(\frac{2\pi nk}{N}\right). \tag{4.119}$$

From this we conclude that the same computer program that computes the DHT can be used for computing the inverse DHT.

(c) Express the DHT in terms of the DFT of $x[n]$ (assuming that this sequence is real).

(d) Express the DFT of $x[n]$ in terms of the DHT.

4.49* This problem explores certain properties of the DHT.

(a) Establish the following shift property of the DHT:

$$y[n] = x[(n-m) \bmod N] \implies$$
$$Y^{\mathrm{h}}[k] = X^{\mathrm{h}}[k]\cos\left(\frac{2\pi mk}{N}\right) + X^{\mathrm{h}}[(N-k) \bmod N]\sin\left(\frac{2\pi mk}{N}\right). \quad (4.120)$$

(b) Use (4.120) for expressing the DHT of the circular convolution of two sequences $x[n], y[n]$ as a function of the DHTs of the two sequences. The formula looks more complicated than the corresponding formula for the DFTs, but it does not require more computations. Explain why.