BoAz PoRAT

Solutions Manual to Accompany

A Course in Digital Signal Processing

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JOHN WILEY & SONS, INC.

New York • Chichester • Brisbane • Toronto • Singapore • Weinheim

To The Teacher

This manual includes full solutions to all problems in the book *A Course in Digital Signal Processing*, John Wiley & Sons, 1997, ISBN: 0-471-14961-6.

Equations in the solution manual are not numbered. All equation numbers refer to the corresponding equations in the book. Similarly, table numbers refer to the corresponding tables in the book. Figure and Program numbers, on the other hand, refer to the corresponding figures and programs in this manual.

When a problem asks the student to provide a plot that has been generated by a MATLAB program, I have given the MATLAB code as a solution in most cases, rather than the plot itself. This method is both more convenient and more flexible. You can obtain an ASCII file including all MATLAB procedures and code fragments in this solution manual by e-mail. Send e-mail to `boaz@ee.technion.ac.il`, and provide your full name and affiliation (only requests by teachers will be honored).

A PostScript file including all figures in the book, named `slides.ps`, is available by anonymous ftp from:
`ftp.wiley.com /public/college/math/matlab/bporat`
or from:
`ftp.technion.ac.il /pub/supported/ee/Signal_processing/B_Porat`
Change transfer mode to `binary`, get the file `slides.zip`, and unzip using `pkunzip` on a PC, or `unzip` on Unix. The file `slides.ps` can be printed on any PostScript printer and used to prepare transparencies for the course.

I welcome comments, corrections, suggestions, questions, and any other feedback on the solution manual; send e-mail to `boaz@ee.technion.ac.il`. Also send e-mail if you wish to get an updated Errata Sheet for this manual.

Boaz Porat

Haifa, September 1996

CONTENTS

Therefore, by the modulation property,

$$X^{F}(\omega) = 0.5\text{sinc}\left(\frac{\omega - \omega_0}{2\pi}\right) + 0.5\text{sinc}\left(\frac{\omega + \omega_0}{2\pi}\right).$$

2.5 No, since the integral

$$\int_{-\infty}^{\infty} 0.5(e^{t} + e^{-t})e^{-j\omega t}dt$$

diverges for all ω.

2.6 In this case the transform exists and is given by

$$\begin{aligned} X^{F}(\omega) &= \int_{-0.5}^{0.5} 0.5(e^{t} + e^{-t})e^{-j\omega t}dt \\ &= -\frac{e^{j0.5\omega}[j\omega\cosh(0.5) - \sinh(0.5)]}{1+\omega^2} + \frac{e^{-j0.5\omega}[j\omega\cosh(0.5) + \sinh(0.5)]}{1+\omega^2}. \end{aligned}$$

2.7 We have

$$G^{F}(\omega) = \sqrt{2\pi}e^{-0.5\omega^2}.$$

Therefore,

$$H^{F}(\omega) = [G^{F}(\omega)]^2 = 2\pi e^{-\omega^2} = \sqrt{\pi}\sqrt{2}\sqrt{2\pi}e^{-0.5(\sqrt{2}\omega)^2}.$$

We get, by the scale property of the Fourier transform,

$$h(t) = \sqrt{\pi}e^{-0.25t^2}.$$

2.8 We have

$$X^{F}(\omega) = \begin{cases} \frac{\pi}{\omega_0}\left(1 - \frac{|\omega|}{2\omega_0}\right), & |\omega| \le 2\omega_0, \\ 0, & \text{otherwise}, \end{cases} \qquad H^{F}(\omega) = \begin{cases} \frac{\pi}{\omega_0}, & |\omega| \le \omega_0, \\ 0, & \text{otherwise}, \end{cases}$$

Therefore,

$$Y^{F}(\omega) = \begin{cases} \frac{\pi^2}{\omega_0^2}\left(1 - \frac{|\omega|}{2\omega_0}\right), & |\omega| \le \omega_0, \\ 0, & \text{otherwise}. \end{cases}$$

So,

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} Y^{F}(\omega)e^{j\omega t}d\omega = \frac{\pi\sin(0.5\omega_0)[\omega_0 t\cos(0.5\omega_0 t) + \sin(0.5\omega_0 t)]}{t^2\omega_0^3} \\ &= \frac{\pi}{4\omega_0}\left[\text{sinc}^2\left(\frac{\omega_0 t}{2\pi}\right) + 2\text{sinc}\left(\frac{\omega_0 t}{\pi}\right)\right]. \end{aligned}$$

2.9 Comparing Figure 2.10 with $Y^{F}(\omega)$ in Solution 2.8, we see that

$$X^{F}(\omega) = \frac{\omega_0^2}{\pi^2}[Y^{F}(\omega - \omega_0) + Y^{F}(\omega + \omega_0)].$$

Therefore, by the modulation property of the Fourier transform,

$$x(t) = \frac{\omega_0^2}{\pi^2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]y(t) = \frac{\omega_0}{2\pi}\left[\text{sinc}^2\left(\frac{\omega_0 t}{2\pi}\right) + 2\text{sinc}\left(\frac{\omega_0 t}{\pi}\right)\right]\cos(\omega_0 t).$$

2.10 We get from the modulation property of the Fourier transform,

$$x(t) = \sqrt{\frac{2}{\pi}}e^{-0.5t^2}\cos(\omega_0 t),$$

so $x(t)$ is not a positive function.

2.11

(a) Differentiate the expression

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^{\mathrm{F}}(\omega)e^{j\omega t}d\omega$$

with respect to t to obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X^{\mathrm{F}}(\omega)]e^{j\omega t}d\omega.$$

(b) We get from Parseval's theorem and from part a,

$$\int_{-\infty}^{\infty} \left|\frac{dx(t)}{dt}\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2|X^{\mathrm{F}}(\omega)|^2 d\omega.$$

2.12

(a)

$$X^{\mathrm{F}}(\omega) = -\int_{-\infty}^{0} e^{(\alpha-j\omega)t}dt + \int_{0}^{\infty} e^{-(\alpha+j\omega)t}dt = -\frac{1}{\alpha-j\omega} + \frac{1}{\alpha+j\omega} = -\frac{j2\omega}{\alpha^2+\omega^2}.$$

(b) We have

$$\mathrm{sign}(t) = \lim_{\alpha\to 0} x(t),$$

where $x(t)$ is the signal in part a. Therefore, assuming that the order of the limit and integral operations can be interchanged, we get

$$\{\mathcal{F}\,\mathrm{sign}\}(\omega) = -\lim_{\alpha\to 0}\frac{j2\omega}{\alpha^2+\omega^2} = \frac{2}{j\omega}.$$

(c) We have

$$\upsilon(t) = 0.5\mathrm{sign}(t) + 0.5\mathcal{I}(t).$$

Therefore,

$$\mathcal{F}\{\upsilon\}(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

2.13 By duality to Solution 2.13 we can write

$$\{\mathcal{F}^{-1}\,\mathrm{sign}\}(t) = -\frac{1}{j\pi t}.$$

Therefore,

$$h(t) = -j\{\mathcal{F}^{-1}\,\mathrm{sign}\}(t) = \frac{1}{\pi t}.$$

2.14 From

$$y(t) = x(t) + j\{x * h\}(t)$$

it follows that

$$Y^{\mathrm{F}}(\omega) = X^{\mathrm{F}}(\omega) + jX^{\mathrm{F}}(\omega)H^{\mathrm{F}}(\omega) = [1 + j(-j)\mathrm{sign}(\omega)]X^{\mathrm{F}}(\omega) = 2\upsilon(\omega)X^{\mathrm{F}}(\omega).$$

2.15

(a) We have from Euler's formula for the cosine function,

$$x(t)\cos(\omega_0 t) = 0.5x(t)e^{j\omega_0 t} + 0.5x(t)e^{-j\omega_0 t}.$$

Therefore, by the modulation property (2.6),

$$X^{\mathrm{F}}(\omega) = 0.5X^{\mathrm{F}}(\omega-\omega_0) + 0.5X^{\mathrm{F}}(\omega+\omega_0).$$

(b) We have from Euler's formula for the sine function,

$$x(t)\sin(\omega_0 t) = -j0.5x(t)e^{j\omega_0 t} + j0.5x(t)e^{-j\omega_0 t}.$$

Therefore,

$$X^{\mathrm{F}}(\omega) = -j0.5X^{\mathrm{F}}(\omega - \omega_0) + j0.5X^{\mathrm{F}}(\omega + \omega_0).$$

(c) We have

$$y(t) = 0.5[x_1(t) + jx_2(t)]e^{j\omega_0 t} + 0.5[x_1(t) - jx_2(t)]e^{j\omega_0 t}.$$

Therefore,

$$Y^{\mathrm{F}}(\omega) = 0.5[X_1^{\mathrm{F}}(\omega - \omega_0) + jX_2^{\mathrm{F}}(\omega - \omega_0)] + 0.5[X_1^{\mathrm{F}}(\omega + \omega_0) - jX_2^{\mathrm{F}}(\omega + \omega_0)].$$

2.16 As is clear from Solution 2.14, the analytic signal's spectrum is twice the spectrum of the given signal at positive frequencies, and zero at negative frequencies. Since $\omega_0 > \omega_m$, $X^{\mathrm{F}}(\omega - \omega_0)$ is nonzero only for $\omega > 0$, whereas $X^{\mathrm{F}}(\omega + \omega_0)$ is nonzero only for $\omega < 0$. Therefore,

- For cosine modulation

$$Z^{\mathrm{F}}(\omega) = X^{\mathrm{F}}(\omega - \omega_0).$$

- For sine modulation

$$Z^{\mathrm{F}}(\omega) = -jX^{\mathrm{F}}(\omega - \omega_0).$$

- For QAM

$$Z^{\mathrm{F}}(\omega) = X_1^{\mathrm{F}}(\omega - \omega_0) + jX_2^{\mathrm{F}}(\omega - \omega_0).$$

2.17 If $x(t)$ is real then it is equal to its conjugate, so

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^{\mathrm{F}}(\omega)e^{-j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{X}^{\mathrm{F}}(\omega)e^{j\omega t}d\omega.$$

Now, if $x(-t) = x(t)$ then $\bar{X}^{\mathrm{F}}(\omega) = X^{\mathrm{F}}(\omega)$, implying that $X^{\mathrm{F}}(\omega)$ is real. If $x(-t) = -x(t)$ then $\bar{X}^{\mathrm{F}}(\omega) = -X^{\mathrm{F}}(\omega)$, implying that $X^{\mathrm{F}}(\omega)$ is imaginary.

2.18 We saw in the preceding problem that for real $x(t)$, the Fourier transform of $x(-t)$ is $\bar{X}^{\mathrm{F}}(\omega)$. Therefore,

$$X_{\mathrm{e}}^{\mathrm{F}}(\omega) = 0.5X^{\mathrm{F}}(\omega) + 0.5\bar{X}^{\mathrm{F}}(\omega) = \Re\{X^{\mathrm{F}}(\omega)\},$$

and

$$X_{\mathrm{o}}^{\mathrm{F}}(\omega) = 0.5X^{\mathrm{F}}(\omega) - 0.5\bar{X}^{\mathrm{F}}(\omega) = j\Im\{X^{\mathrm{F}}(\omega)\}.$$

2.19

(a) We have in this case,

$$x_{\mathrm{e}}(t) = \begin{cases} 0.5x(t), & t \ge 0, \\ 0.5x(-t), & t < 0, \end{cases} \qquad x_{\mathrm{o}}(t) = \begin{cases} 0.5x(t), & t \ge 0, \\ -0.5x(-t), & t < 0. \end{cases}$$

This proves that

$$x_{\mathrm{o}}(t) = \mathrm{sign}(t)x_{\mathrm{e}}(t).$$

(b) By Solution 2.18, we have that

$$\{\mathcal{F}x_{\mathrm{e}}\}(\omega) = \Re\{X^{\mathrm{F}}(\omega)\}, \quad -j\{\mathcal{F}x_{\mathrm{o}}\}(\omega) = \Im\{X^{\mathrm{F}}(\omega)\}.$$

Also, from part a we have that

$$-jx_{\mathrm{o}}(t) = -j\,\mathrm{sign}(t)x_{\mathrm{e}}(t).$$

Therefore, by duality to Solution 2.13, we get that $\Im\{X^{\mathrm{F}}(\omega)\}$ is the Hilbert transform of $\Re\{X^{\mathrm{F}}(\omega)\}$.

2.20

(a)

$$P_x = \int_0^\infty e^{-2t} dt = 0.5.$$

$$C_x = 2\int_0^\infty t e^{-2t} dt = 0.5.$$

$$W_x = \left[2\int_0^\infty (t-0.5)^2 e^{-2t} dt\right]^{1/2} = 0.5.$$

(b)

$$P_x = \int_{-0.5}^{0.5} 1 \cdot dt = 1.$$

$$W_x = \left[\int_{-0.5}^{0.5} t^2 dt\right]^{1/2} = \frac{1}{12^{1/2}}.$$

(c) For a real signal, $|X^{\mathrm{F}}(\omega)|^2$ is an even function of ω. Therefore, the integral of $\omega|X^{\mathrm{F}}(\omega)|^2$ on any interval symmetric with respect to the origin is zero.

2.21 The signal

$$y(t) = \sum_{n=-\infty}^{\infty} x(t-nT)$$

is obviously periodic, with period T. Therefore, by (2.39),

$$y(t) = \sum_{k=-\infty}^{\infty} Y^{\mathrm{S}}[k] \exp\left(\frac{j2\pi kt}{T}\right).$$

The Fourier coefficients of $y(t)$ are given by (2.40) as

$$\begin{aligned} Y^{\mathrm{S}}[k] &= \frac{1}{T}\int_0^T y(t)\exp\left(-\frac{j2\pi kt}{T}\right)dt = \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_0^T x(t-nT)\exp\left(-\frac{j2\pi kt}{T}\right)dt \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{nT}^{(n+1)T} x(t)\exp\left(-\frac{j2\pi k(t+nT)}{T}\right)dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\exp\left(-\frac{j2\pi kt}{T}\right)dt = \frac{1}{T}X^{\mathrm{F}}\left(\frac{2\pi k}{T}\right). \end{aligned}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} x(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{2\pi k}{T}\right)\exp\left(\frac{j2\pi kt}{T}\right),$$

as was required to prove.

2.22 It is seen from Problem 2.22 that The Fourier coefficients of $y(t)$ are

$$Y^{\mathrm{S}}[k] = \frac{1}{T}X^{\mathrm{F}}\left(\frac{2\pi k}{T}\right).$$

In our case we have

$$X^{\mathrm{F}}(\omega) = \frac{1}{\alpha + j\omega}.$$

Therefore,

$$Y^{\mathrm{S}}[k] = \frac{1}{\alpha T + j2\pi k}.$$

2.23 We can express the pulse train as

$$r_{T,\Delta}(t) = \{p_T * s\}(t), \quad \text{where} \quad s(t) = \frac{1}{\Delta}\mathrm{rect}\left(\frac{t}{\Delta}\right).$$

Therefore,

$$R^{\mathrm{F}}_{T,\Delta}(\omega) = \operatorname{sinc}\left(\frac{\omega\Delta}{2\pi}\right)\cdot\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\operatorname{sinc}\left(\frac{k\Delta}{T}\right)\delta\left(\omega-\frac{2\pi k}{T}\right).$$

2.24 The alternating impulse train can be expressed in terms of the conventional impulse train as

$$q_T(t) = \sum_{n=-\infty}^{\infty}\delta(t-2nT) - \sum_{n=-\infty}^{\infty}\delta(t-2nT-T) = p_{2T}(t) - p_{2T}(t-T).$$

Therefore,

$$Q^{\mathrm{F}}_T(\omega) = \frac{2\pi}{2T}\sum_{k=-\infty}^{\infty}(1-e^{-j\omega T})\delta\left(\omega-\frac{\pi k}{T}\right).$$

2.25 The frequency response of $\operatorname{sinc}(0.5t/T)$ is $2T$ in the frequency range $|\omega| \le 0.5\pi/T$ and 0 otherwise. Therefore, since $\cos(\pi t/T) = 0.5e^{j\pi t/T} + 0.5e^{-j\pi t/T}$, the frequency response of $h(t)$ is

$$H^{\mathrm{F}}(\omega) = \begin{cases} T, & \frac{0.5\pi}{T} \le |\omega| \le \frac{1.5\pi}{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Now, the frequency response of $p_T(t)$ is

$$P^{\mathrm{F}}_T(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T}\right).$$

Therefore $H^{\mathrm{F}}(\omega)P^{\mathrm{F}}_T(\omega) = 0$, so $y(t) = 0$.

2.26 Multiply both sides of (2.159) by $x(t)$ and integrate on $(-\infty,\infty)$:

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}x(t)\delta(t-nT)dt = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}x(t)\exp\left(j\frac{2\pi kt}{T}\right)dt = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}x(t)\exp\left(-j\frac{2\pi kt}{T}\right)dt,$$

which is the same as

$$\sum_{n=-\infty}^{\infty}x(nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X^{\mathrm{F}}\left(\frac{2\pi k}{T}\right).$$

2.27 We have

$$X^{\mathrm{f}}(\theta) = C\sum_{n_1}^{n_2}e^{-j\theta n} = C\frac{e^{-j\theta n_1} - e^{-j\theta(n_2+1)}}{1-e^{-j\theta}}.$$

2.28 We have

$$X^{\mathrm{f}}(\theta) = 0.5e^{j\theta} + 0.5e^{-j\theta} + j0.5e^{-j2\theta} - j0.5e^{j2\theta}.$$

Therefore,

$$x[n] = \begin{cases} \{j0.5, 0.5, 0, 0.5, -j0.5\}, & -2 \le n \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

2.29

(a)

$$X^{\mathrm{f}}(0) = \sum_{n=-3}^{3}x[n] = 0.$$

(b) Since $x[n]$ is a real symmetric sequence, its transform is real, so $\measuredangle X^{\mathrm{f}}(\theta) = 0$.

(c)

$$\int_{-\pi}^{\pi}X^{\mathrm{f}}(\theta)d\theta = 2\pi x[0] = 8\pi.$$

(d)

$$X^{\mathrm{f}}(\pi) = \sum_{n=-3}^{3} x[n](-1)^n = 4.$$

(e)

$$\int_{-\pi}^{\pi} |X^{\mathrm{f}}(\theta)|^2 d\theta = 2\pi \sum_{n=-3}^{3} (x[n])^2 = 56\pi.$$

(f)

$$\left.\frac{dX^{\mathrm{f}}(\theta)}{d\theta}\right|_{\theta=0} = -j \sum_{n=-3}^{3} nx[n] = 0.$$

2.30 No, since the period of $\cos(0.5\theta)$ is 4π, whereas the Fourier transform of a discrete-time signal must have a period 2π.

2.31
(a)

$$\sum_{n=-\infty}^{\infty} (-1)^n x[n] e^{-j\theta n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi n} e^{-j\theta n} = X^{\mathrm{f}}(\theta + \pi).$$

(b) We get from part a,

$$\sum_{n=-\infty}^{\infty} (-1)^n x[n] = X^{\mathrm{f}}(\pi).$$

2.32 Only the system whose impulse response is $h_1[n]$ is time invariant.

2.33

$$y[n+N] = \sum_{m=-\infty}^{\infty} h[m]x[n+N-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = y[n].$$

2.34 We have

$$Y^{\mathrm{f}}(\theta) = \sum_{n=-\infty}^{\infty} x[n](e^{-j\theta n} + e^{-j(\theta-\pi)n}) = \sum_{n=-\infty}^{\infty} x[n](1+(-1)^n)e^{-j\theta n} = 2\sum_{m=-\infty}^{\infty} x[2m]e^{-j2\theta m}.$$

As we see, $Y^{\mathrm{f}}(\theta)$ depends only on the even-indexed values $x[2m]$, and is independent of the odd-indexed values $x[2m+1]$.

2.35 When $\theta_0 = 2\pi p/q$ we have

$$\cos\left(\frac{2\pi p(n+q)}{q} + \phi_0\right) = \cos\left(\frac{2\pi pn}{q} + 2\pi p + \phi_0\right) = \cos\left(\frac{2\pi pn}{q} + \phi_0\right).$$

Therefore, the signal is periodic with period $N = q$. To prove the converse, assume that there exists an integer N such that

$$\cos(\theta_0 n + \phi_0) = \cos(\theta_0 n + \theta_0 N + \phi_0) = \cos(\theta_0 n + \phi_0)\cos(\theta_0 N) - \sin(\theta_0 n + \phi_0)\sin(\theta_0 N)$$

for all n. This is possible if and only if $\theta_0 N$ is an integer multiple of 2π, which happens if and only if θ_0 is a rational multiple of 2π.

2.36 By the shift property of the Fourier transform,

$$Z^{\mathrm{F}}(\omega) = \int_{-\infty}^{\infty} e^{j\omega\tau} X^{\mathrm{F}}(\omega)\bar{y}(\tau)d\tau = X^{\mathrm{F}}(\omega)\int_{-\infty}^{\infty} \bar{y}(\tau)e^{j\omega\tau}d\tau = X^{\mathrm{F}}(\omega)\bar{Y}^{\mathrm{F}}(\omega).$$

In the same way,

$$Z^{\mathrm{f}}(\theta) = \sum_{m=-\infty}^{\infty} e^{j\theta m} X^{\mathrm{f}}(\theta)\bar{y}[m] = X^{\mathrm{f}}(\theta)\sum_{m=-\infty}^{\infty} \bar{y}[m]e^{j\theta m} = X^{\mathrm{f}}(\theta)\bar{Y}^{\mathrm{f}}(\theta).$$

2.37

$$E\{[x(t+\tau)-\mu_x][x(t)-\mu_x]\} = E\{x(t+\tau)x(t)\} - \mu_x E\{x(t)\} - \mu_x E\{x(t+\tau)\} + \mu_x^2$$
$$= E\{x(t+\tau)x(t)\} - 2\mu_x^2 + \mu_x^2 = E\{x(t+\tau)x(t)\} - \mu_x^2.$$

2.38

(a) We have

$$\mathrm{NG}_i = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_i^{\mathrm{f}}(\theta)|^2 d\theta, \quad i = 1,2, \qquad \mathrm{NG} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_1^{\mathrm{f}}(\theta)|^2 |H_2^{\mathrm{f}}(\theta)|^2 d\theta.$$

In general, $\mathrm{NG} \neq \mathrm{NG}_1\mathrm{NG}_2$. For example, suppose that $H_1^{\mathrm{f}}(\theta)$ is nonzero only for $|\theta| < \theta_0$, whereas $H_2^{\mathrm{f}}(\theta)$ is nonzero only for $|\theta| > \theta_0$. Then both NG_1 and NG_2 are nonzero, but $H_1^{\mathrm{f}}(\theta)H_2^{\mathrm{f}}(\theta) = 0$, so $\mathrm{NG} = 0$.

(b) If $H_1^{\mathrm{f}}(\theta) = C$ then $\mathrm{NG}_1 = C^2$. Also, $H^{\mathrm{f}}(\theta) = CH_2^{\mathrm{f}}(\theta)$, so

$$\mathrm{NG} = C^2\mathrm{NG}_2 = \mathrm{NG}_1\mathrm{NG}_2$$

in this case.

2.39

(a) We have

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda, \quad y(t+\tau) = \int_{-\infty}^{\infty} h(\eta)x(t+\tau-\eta)d\eta,$$

so

$$\kappa_y(\tau) = E\{y(t+\tau)y(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda)h(\eta)E\{x(t+\tau-\eta)x(t-\lambda)\}d\lambda d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda)h(\eta)N_0\delta(\tau+\lambda-\eta)d\lambda d\eta = N_0\int_{-\infty}^{\infty} h(\lambda)h(\lambda+\tau)d\lambda.$$

(b) In the discrete-time case we have

$$y[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i], \quad y[n+m] = \sum_{l=-\infty}^{\infty} h[l]x[n+m-l],$$

so

$$\kappa_y[m] = E\{y[n+m]y[n]\} = \sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h[i]h[l]E\{x[n+m-l]x[n-i]\}$$
$$= \sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h[i]h[l]\gamma_x\delta[m+i-l] = \gamma_x\sum_{i=-\infty}^{\infty} h[i]h[i+m].$$

2.40 We have

$$K_y^{\mathrm{f}}(\theta) = K_x^{\mathrm{f}}(\theta)|H^{\mathrm{f}}(\theta)|^2 = \gamma_x \mathrm{rect}\left(\frac{\theta}{2\theta_c}\right).$$

Therefore,

$$\kappa_y[m] = \frac{\gamma_x}{2\pi}\int_{-\theta_c}^{\theta_c} e^{j\theta m}d\theta = \frac{\theta_c}{\pi}\mathrm{sinc}\left(\frac{m\theta_c}{\pi}\right).$$

2.41 Let x and y have the joint probability density function $P(x,y)$. Then

$$E(x^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2P(x,y)dxdy, \quad E(y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2P(x,y)dxdy, \quad E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyP(x,y)dxdy.$$

Define $\xi(x,y) = [P(x,y)]^{1/2}x$, $\eta(x,y) = [P(x,y)]^{1/2}y$, and substitute $\xi(x,y)$, $\eta(x,y)$ for the two functions in (2.147). This immediately gives (2.148).

2.42

(a)

$$X^S[k] = \frac{1}{T}\int_{-0.5T}^{0.5T}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^F(\omega)e^{j\omega t}d\omega\right]\exp\left(-\frac{j2\pi kt}{T}\right)dt$$
$$= \frac{1}{2\pi T}\int_{-\infty}^{\infty} X^F(\omega)\left\{\int_{-0.5T}^{0.5T}\exp\left[j\left(\omega - \frac{2\pi k}{T}\right)t\right]dt\right\}d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^F(\omega)\operatorname{sinc}\left(\frac{\omega T}{2\pi} - k\right)d\omega.$$

(b)

$$X^F(\omega) = \int_{-0.5T}^{0.5T}\left[\sum_{k=-\infty}^{\infty} X^S[k]\exp\left(\frac{j2\pi kt}{T}\right)\right]e{-}j\omega t dt$$
$$= \sum_{k=-\infty}^{\infty} X^S[k]\left\{\int_{-0.5T}^{0.5T}\exp\left[-j\left(\omega - \frac{2\pi k}{T}\right)t\right]dt\right\}$$
$$= T\sum_{k=-\infty}^{\infty} X^S[k]\operatorname{sinc}\left(\frac{\omega T}{2\pi} - k\right).$$

2.43

(a) Define $t_0 = (2\pi)^{-1}\arccos(A)$. The signal $x(t)$ is symmetric, so its Fourier coefficients are real and are given by

$$X^S[k] = -A\int_{-0.5}^{-0.5+t_0}\cos(2\pi kt)dt + \int_{-0.5+t_0}^{-t_0}\cos(2\pi t)\cos(2\pi kt)dt + A\int_{-t_0}^{t_0}\cos(2\pi kt)dt$$
$$+ \int_{t_0}^{0.5-t_0}\cos(2\pi t)\cos(2\pi kt)dt - A\int_{0.5-t_0}^{0.5}\cos(2\pi kt)dt$$
$$= \begin{cases} 0, & k \text{ even}, \\ 0.5 + 2t_0[A\operatorname{sinc}(2t_0) - 1], & k = \pm 1, \\ 2t_0\{2A\operatorname{sinc}(2kt_0) - \operatorname{sinc}[2(k+1)t_0] - \operatorname{sinc}[2(k-1)t_0]\}, & k = \pm 3, \pm 5, \ldots \end{cases}$$

(b) For $A = 0.9$ we get

$$\left\{\frac{X^S[k]}{X^S[1]},\ k = 3,5,7,9\right\} = \{-0.0329, -0.0229, -0.0120, -0.0031\}.$$

For $A = 0.5$ we get

$$\left\{\frac{X^S[k]}{X^S[1]},\ k = 3,5,7,9\right\} = \{-0.2263, 0.0453, 0.0162, -0.0226\}.$$

For $A = 0.1$ we get

$$\left\{\frac{X^S[k]}{X^S[1]},\ k = 3,5,7,9\right\} = \{-0.3289, 0.1921, -0.1317, 0.0968\}.$$

For $A = 0.01$ we get

$$\left\{\frac{X^S[k]}{X^S[1]},\ k = 3,5,7,9\right\} = \{-0.3333, 0.1999, -0.1427, 0.1110\}.$$

(c) For a pure sinusoidal signal, only the coefficients $X^S[\pm 1]$ are nonzero. When the sinusoid is distorted by hard limiting, harmonics are generated and appear as nonzero Fourier coefficients for $|k| > 1$. In this case, because of symmetry, only odd harmonics are present. As A decreases, the signal looks more and more like a square wave. In the limit, as A tends to zero, the ratio $X^S[2m+1]/X^S[1]$ approaches $(-1)^m/(2m+1)$, which is the ratio of Fourier coefficients of a square wave.

2.44

(a) We have

$$y(t) = \sin(M\omega_m t) + 0.5a\sin[(M+1)\omega_m t] + 0.5a\sin[(M-1)\omega_m t].$$

This shows that $y(t)$ is periodic with period $2\pi/\omega_m$.

(b) We get immediately from part a that

$$Y^S[\pm(M-1)] = \mp j0.25a, \quad Y^S[\pm M] = \mp j0.5, \quad Y^S[\pm(M+1)] = \mp j0.25a,$$

and $Y^S[k] = 0$ for all other k.

(c) This is obvious, since both $y(t)$ and $\sin(\omega_c t)$ are periodic with period $2\pi/\omega_m$.

(d)

$$Z^S[k] = \frac{\omega_m}{2\pi}\int_0^{2\pi/\omega_m} z(t)e^{-jk\omega_m t}dt = \frac{\omega_m}{2\pi}\sum_{m=0}^{M-1}\int_{2\pi m/M\omega_m}^{2\pi(m+0.5)/M\omega_m} y(t)e^{-jk\omega_m t}dt$$

$$= \frac{\omega_m}{2\pi}\sum_{m=0}^{M-1}\int_0^{\pi/M\omega_m} y\left(\tau + \frac{2\pi m}{M\omega_m}\right)\exp\left[-jk\omega_m\left(\tau + \frac{2\pi m}{M\omega_m}\right)\right]d\tau$$

$$= \frac{\omega_m}{2\pi}\sum_{m=0}^{M-1}\exp\left(-\frac{j2\pi km}{M}\right)\int_0^{\pi/M\omega_m} y\left(\tau + \frac{2\pi m}{M\omega_m}\right)e^{-jk\omega_m\tau}d\tau.$$

Let us further express $y(\cdot)$ in terms of its Fourier series, that is

$$y\left(\tau + \frac{2\pi m}{M\omega_m}\right) = \sum_{l=-\infty}^{\infty} Y^S[l]\exp\left[jl\omega_m\left(\tau + \frac{2\pi m}{M\omega_m}\right)\right] = \sum_{l=-\infty}^{\infty} Y^S[l]\exp\left(\frac{j2\pi lm}{M}\right)e^{jl\omega_m\tau}.$$

Then,

$$Z^S[k] = \frac{\omega_m}{2\pi}\sum_{l=-\infty}^{\infty} Y^S[l]\sum_{m=0}^{M-1}\exp\left(\frac{j2\pi(l-k)m}{M}\right)\int_0^{\pi/M\omega_m} e^{j(l-k)\omega_m\tau}d\tau.$$

We have

$$\sum_{m=0}^{M-1}\exp\left(\frac{j2\pi(l-k)m}{M}\right) = \begin{cases} M, & (l-k) \bmod M = 0, \\ 0, & \text{otherwise,} \end{cases} = M\delta[(l-k) \bmod M],$$

and

$$\int_0^{\pi/M\omega_m} e^{j(l-k)\omega_m\tau}d\tau = \frac{\pi}{M\omega_m}\operatorname{sinc}\left(\frac{l-k}{2M}\right)\exp\left(\frac{j\pi(l-k)}{2M}\right).$$

Therefore,

$$Z^S[k] = 0.5\sum_{l=-\infty}^{\infty} Y^S[l]\delta[(l-k) \bmod M]\operatorname{sinc}\left(\frac{l-k}{2M}\right)\exp\left(\frac{j\pi(l-k)}{2M}\right).$$

Note that the sinc term is nonzero only if $l-k=0$ or $l-k$ is an *odd* multiple of M. Recall that $Y^S[l]$ is nonzero only for $l = \pm(M-1), \pm M, \pm(M+1)$. Therefore, $Z^S[k]$ is nonzero in the range of interest only for $k = 0, \pm1, \pm(M-1), \pm M, \pm(M+1)$. In summary,

$$Z^S[0] = 0.5\operatorname{sinc}(0.5), \quad Z^S[\pm1] = 0.25a\operatorname{sinc}(0.5),$$

$$Z^S[\pm(M-1)] = \mp j0.125a, \quad Z^S[\pm M] = \mp j0.25, \quad Z^S[\pm(M+1)] = \mp j0.125a.$$

(e) The signal $x(t)$ is, up to a constant scale factor, the sum of the harmonics corresponding to $k = \pm1$. Therefore, we can reconstruct $x(t)$ by passing $z(t)$ through a band-pass filter that will eliminate frequencies considerably higher than ω_m, as well as the DC term, but will pass frequencies around ω_m. This is, in fact, the standard scheme for demodulating AM signals. The operation of generating $z(t)$ from $y(t)$ is called *half-wave rectification*, and is followed by low-pass filtering and DC blocking.

2.45

(a)

$$-\int_{-\infty}^{\infty} t\frac{d|x(t)|^2}{dt}dt = -t|x(t)|^2\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} |x(t)|^2 dt = 0 - 0 + P_x = P_x.$$

(b) We get from Solution 2.11 and from the definition of W_X,

$$W_X = \left[P_x^{-1}\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |X^{\mathrm{F}}(\omega)|^2 d\omega\right]^{1/2} = \left[P_x^{-1}\int_{-\infty}^{\infty}\left|\frac{dx(t)}{dt}\right|^2 dt\right]^{1/2}$$

(c) By the Cauchy-Schwarz inequality (2.147),

$$W_x W_X = \left[P_x^{-1}\int_{-\infty}^{\infty} t^2|x(t)|^2 dt\right]^{1/2}\left[P_x^{-1}\int_{-\infty}^{\infty}\left|\frac{dx(t)}{dt}\right|^2 dt\right]^{1/2} \ge \left|P_x^{-1}\int_{-\infty}^{\infty} t x(t)\frac{dx(t)}{dt}dt\right|.$$

(d) We have

$$x(t)\frac{dx(t)}{dt} = 0.5\frac{d|x(t)|^2}{dt},$$

therefore, by part a,

$$W_x W_X \ge \left|P_x^{-1} 0.5\int_{-\infty}^{\infty} t\frac{d|x(t)|^2}{dt}dt\right| = 0.5 \cdot |-P_x^{-1}P_x| = 0.5.$$

2.46 We have

$$X^{\mathrm{f}}(\theta) = \sum_{n=0}^{N-1} x[n]e^{-j\theta n} = \sum_{n=0}^{N-1} x[n](e^{-j\theta})^n.$$

This is a polynomial of degree $N-1$ in the variable $e^{-j\theta}$ with coefficients $\{x[n],\ 0 \le n \le N-1\}$. Therefore, it cannot have more than $N-1$ zeros. Each zero whose magnitude is 1 (if there are such zeros) will give a single value of θ in the range $[-\pi, \pi)$ for which $X^{\mathrm{f}}(\theta) = 0$. Therefore, the total number of such θ is finite.

2.47 The claim is false, for $\{x * y\} = 0$ if and only if $X^{\mathrm{f}}(\theta)Y^{\mathrm{f}}(\theta) = 0$, and this is possible even if none of the multiplicands is identically zero. For example,

$$X^{\mathrm{f}}(\theta) = \begin{cases} 1, & |\theta| \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad Y^{\mathrm{f}}(\theta) = \begin{cases} 1, & 2 \le |\theta| \le \pi, \\ 0, & \text{otherwise}. \end{cases}$$

2.48 If $x[n]$ and $y[n]$ are finite sequences, we know by Problem 2.46 that $X^{\mathrm{f}}(\theta)$ and $Y^{\mathrm{f}}(\theta)$ are zero on a finite set of θ at most. Therefore, in this case $X^{\mathrm{f}}(\theta)Y^{\mathrm{f}}(\theta) = 0$ is impossible, unless one of the two is identically zero. Therefore, in this case the claim is true.

2.49

(a) Let

$$X^{\mathrm{f}}(\theta) = \mathrm{rect}\left(\frac{\theta}{\pi}\right).$$

Then

$$x[n] = \frac{\sin(0.5\pi n)}{\pi n} = \begin{cases} 0.5, & n = 0, \\ \dfrac{(-1)^m}{\pi(2m+1)}, & n = 2m+1, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{4} + \frac{2}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2},$$

so

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{2}\left[\sum_{n=-\infty}^{\infty}|x[n]|^2 - \frac{1}{4}\right] = \frac{\pi^2}{2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|X^{\mathrm{f}}(\theta)|^2 d\theta - \frac{1}{4}\right] = \frac{\pi^2}{8}.$$

(b) We have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} + \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{m=1}^{\infty} \frac{1}{m^2},$$

hence

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Chapter 3

Sampling and Reconstruction

3.1 If $x(t) = \cos(0.5\pi t)$ and $T = 1$ then $x[n]$ is not aliased. If $x(t) = \cos(2.5\pi t)$ and $T = 1$ then $x[n]$ is aliased.

3.2 It follows from the result for $X^{\mathrm{f}}(\theta)$ that $x[n] = \delta[n]$. Since $e^{-0.02t^2} > 0$ for all t, we must have that the sampling of $\mathrm{sinc}(t)$ is $\delta[n]$. The minimum T for which this happens is $T = 1$.

3.3 It can be verified, by plotting $X^{\mathrm{F}}(\omega)$, that

$$X^{\mathrm{f}}(\theta) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{\theta - 2\pi k}{T}\right) = 1.$$

Therefore $x[n] = \delta[n]$.

3.4

(a) The signal $y(t)$ can be expressed as

$$y(t) = \{x * p_T\}(t),$$

where $p_T(t)$ is the impulse train (2.47). Therefore,

$$Y^{\mathrm{F}}(\omega) = X^{\mathrm{F}}(\omega)P_T^{\mathrm{F}}(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} X^{\mathrm{F}}(\omega)\delta(\omega - n\omega_0).$$

But, since $X^{\mathrm{F}}(\omega)$ is nonzero only on $[-1.5\omega_0, 1.5\omega_0]$, the infinite sum contains only three nonzero terms, that is,

$$Y^{\mathrm{F}}(\omega) = \frac{2\pi}{T}[X^{\mathrm{F}}(0)\delta(\omega) + X^{\mathrm{F}}(\omega_0)\delta(\omega - \omega_0) + X^{\mathrm{F}}(-\omega_0)\delta(\omega + \omega_0)].$$

Taking the inverse Fourier transform gives

$$y(t) = \frac{1}{T}[X^{\mathrm{F}}(0) + X^{\mathrm{F}}(\omega_0)e^{j\omega_0 t} + X^{\mathrm{F}}(-\omega_0)e^{-j\omega_0 t}] = C_0 + C_1\cos(\omega_0 t + \phi).$$

(b) Continuing the preceding derivation, we get

$$y(t) = \frac{1}{T}[X^{\mathrm{F}}(0) + 2\Re\{X^{\mathrm{F}}(\omega_0)e^{j\omega_0 t}\}].$$

Therefore,

$$C_0 = \frac{1}{T}X^{\mathrm{F}}(0), \quad C_1 = \frac{2}{T}|X^{\mathrm{F}}(\omega_0)|, \quad \phi_0 = \measuredangle X^{\mathrm{F}}(\omega_0).$$

3.5 Both formulas follow directly from (3.19). First,

$$\int_{-\infty}^{\infty} x(t)dt = X^{\mathrm{F}}(0), \quad \sum_{n=-\infty}^{\infty} x(nT) = X^{\mathrm{f}}(0).$$

This gives (3.65). Next, by Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |x(nT)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X^{\mathrm{f}}(\theta)|^2 d\theta = \frac{1}{2\pi T^2}\int_{-\pi}^{\pi} |X^{\mathrm{F}}(\theta/T)|^2 d\theta = \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T} |X^{\mathrm{F}}(\omega)|^2 d\omega = \frac{1}{T}\int_{-\infty}^{\infty} |x(t)|^2 dt.$$

This gives (3.66).

3.6

$$X^{\mathrm{F}}(\omega) = \int_{-\infty}^{0} e^{(\alpha-j\omega)t}dt + \int_{0}^{\infty} e^{-(\alpha+j\omega)t}dt = \frac{1}{\alpha-j\omega} + \frac{1}{\alpha+j\omega} = \frac{2\alpha}{\alpha^2+\omega^2}.$$

$$x[n] = e^{-\alpha T|n|}, \quad -\infty < n < \infty.$$

$$X^{\mathrm{f}}(\theta) = \sum_{n=-\infty}^{-1} e^{(\alpha T-j\theta)n} + \sum_{n=0}^{\infty} e^{-(\alpha T+j\theta)n} = \frac{e^{-(\alpha T-j\theta)}}{1-e^{-(\alpha T-j\theta)}} + \frac{1}{1-e^{-(\alpha T+j\theta)}} = \frac{1-e^{-2\alpha T}}{1+e^{-2\alpha T}-2e^{-\alpha T}\cos\theta}.$$

Hence we get from the sampling theorem

$$\sum_{k=-\infty}^{\infty} \frac{2\alpha T}{(\alpha T)^2+(\theta-2\pi k)^2} = \frac{1-e^{-2\alpha T}}{1+e^{-2\alpha T}-2e^{-\alpha T}\cos\theta}.$$

3.7 The continuous-time signal is

$$x(t) = \sum_{k=-\infty}^{\infty} X^{\mathrm{S}}[k]\exp\left(\frac{j2\pi kt}{T_0}\right).$$

Therefore, the sampled signal is

$$x(nT) = \sum_{k=-\infty}^{\infty} X^{\mathrm{S}}[k]\exp\left(\frac{j2\pi knT}{T_0}\right).$$

Define

$$\theta_k = 2\pi\left(\frac{kT}{T_0} - m_k\right),$$

where m_k is an integer chosen such that $\theta_k \in [-\pi, \pi)$. Then

$$\exp\left(\frac{j2\pi knT}{T_0}\right) = e^{j\theta_k n},$$

so

$$x(nT) = \sum_{k=-\infty}^{\infty} X^{\mathrm{S}}[k]e^{j\theta_k n}.$$

Therefore,

$$X^{\mathrm{f}}(\theta) = 2\pi\sum_{k=-\infty}^{\infty} X^{\mathrm{S}}[k]\delta(\theta-\theta_k).$$

3.8

(a) The minimum N is $N = 7$. Taking $N = 6$ is not enough, since then the harmonics $k = 3$ and $k = -3$ would add up after sampling, and $X^{\mathrm{S}}[3], X^{\mathrm{S}}[-3]$ would become ambiguous.

(b) Seven samples are enough, since they would enable us to construct seven equations in seven unknowns as follows:

$$\sum_{k=-3}^{3} X^{\mathrm{S}}[k]\exp\left(j\frac{2\pi knT}{T_0}\right) = x(nT), \quad 0 \le n \le 6.$$

These equations can be solved for $X^{\mathrm{S}}[k]$, since the samples $x(nT)$ are known. Note that $T/T_0 = 1/N = 1/7$.

(c) In this case there will be aliasing, as shown in Figure 3.1 (in this figure we define $\theta_0 = 2\pi/5.5$). However, the aliased harmonics do not overlap with any of the other harmonics, so unambiguous computation of the $X^{\mathrm{S}}[k]$ is still possible. Seven samples are again sufficient, and the equations are exactly as before, except that $T/T_0 = 1/5.5$.

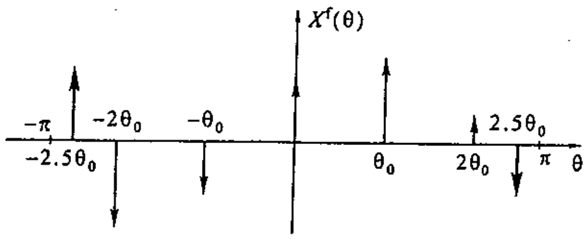

Figure 3.1 Pertaining to Solution 3.8; the arrows shown in heavier lines indicate aliased harmonics.

3.9 The Fourier transform of the signal at the output of the analog filter is

$$Y^F(\omega) = H^F(\omega)X^F(\omega).$$

Therefore, the frequency response of the sampled signal $y[n]$ is given by the sampling theorem

$$Y^f(\theta) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H^F\left(\frac{\theta - 2\pi k}{T}\right) X^F\left(\frac{\theta - 2\pi k}{T}\right).$$

For the system with the digital filter we have

$$Z^f(\theta) = \frac{1}{T}\sum_{l=-\infty}^{\infty} H^F\left(\frac{\theta - 2\pi l}{T}\right) \sum_{k=-\infty}^{\infty} X^F\left(\frac{\theta - 2\pi k}{T}\right).$$

The two outputs will be identical if and only if

$$H^F\left(\frac{\theta - 2\pi l}{T}\right) X^F\left(\frac{\theta - 2\pi k}{T}\right) = 0, \quad k \neq l.$$

An equivalent condition is

$$X^F\left(\frac{\theta}{T}\right) H^F\left(\frac{\theta - 2\pi k}{T}\right) = 0, \quad k \neq 0.$$

We have

$$X^F\left(\frac{\theta}{T}\right) = 0, \quad |\theta| > \omega_1 T,$$

$$H^F\left(\frac{\theta - 2\pi k}{T}\right) = 0, \quad |\theta - 2\pi k| > \omega_2 T.$$

Therefore, the condition will be met if and only if

$$\omega_1 T < 2\pi - \omega_2 T,$$

that is,

$$T < \frac{2\pi}{\omega_1 + \omega_2}.$$

Note that, in this case, aliasing is permitted in the sampling of either $x(t)$ or $h(t)$ (but not both).

3.10

(a) We get after carrying out the integration

$$y(0) = 1 - \alpha + \frac{4\alpha}{\pi},$$

$$y\left(\frac{\pm 1}{4\alpha f_0}\right) = \alpha\cos\left(\frac{(1-\alpha)\pi}{4\alpha}\right) + \frac{3\alpha}{\pi}\sin\left(\frac{(1-\alpha)\pi}{4\alpha}\right) + \frac{\alpha}{\pi}\sin\left(\frac{(1+3\alpha)\pi}{4\alpha}\right), \text{ and}$$

$$y(t) = \frac{\sin[(1-\alpha)\pi f_0 t] + 4\alpha f_0 t\cos[(1+\alpha)\pi f_0 t]}{\pi f_0 t[1 - (4\alpha f_0 t)^2]}$$

for other values of t.

(b) No, the sampled signal is aliased, since the bandwidth of $y(t)$ is $(1+\alpha)\pi f_0$, which is larger than πf_0.

(c) No, since it is easy to verify that $Y^{f}(\theta)$ is not identically 1.

(d) Since $\cos^2 x = 0.5(1 + \cos 2x)$, we have

$$[Y^{F}(\omega)]^2 = X^{F}(\omega),$$

where $X^{F}(\omega)$ is the raised cosine spectrum (3.20). Therefore $x[n]$ is aliased, but $x(t)$ is Nyquist-T. The spectrum $Y^{F}(\omega)$ is called *excess bandwidth root raised cosine.* In a digital communication channel, when the transmitted signal has a root raised cosine spectrum and the receiver has a root raised cosine filter with the same α, the signal at the output of the filter will be Nyquist-T, so it will be free of intersymbol interference after sampling.

3.11 Using the result in Solution 2.24, we get

$$\begin{aligned}
Y^{F}(\omega) &= \frac{1}{2\pi}\{X^{F} * Q_T^{F}\}(\omega) \\
&= \frac{1}{2T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X^{F}(\lambda)(1 - e^{-j(\omega-\lambda)T})\delta\left(\omega - \lambda - \frac{\pi k}{T}\right)d\lambda \\
&= \frac{1}{2T}\sum_{k=-\infty}^{\infty} X^{F}\left(\omega - \frac{\pi k}{T}\right)(1 - e^{-j\pi k}) = \frac{1}{2T}\sum_{k=-\infty}^{\infty} X^{F}\left(\omega - \frac{\pi k}{T}\right)[1 - (-1)^k] \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{F}\left(\omega - \frac{\pi(2k+1)}{T}\right).
\end{aligned}$$

The spectrum of $y(t)$ is plotted in Figure 3.2.

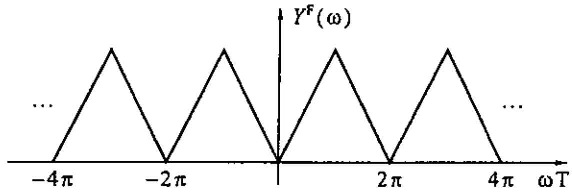

Figure 3.2 Pertaining to Solution 3.11.

3.12

(a) Let us compute the Fourier series of $x(t)$:

$$\begin{aligned}
X^{S}[k] &= \frac{1}{T}\int_0^T x(t)\exp\left(-\frac{j2\pi kt}{T}\right)dt \\
&= \frac{1}{T}\int_0^{0.5T}\exp\left(-\frac{j2\pi kt}{T}\right)dt - \frac{1}{T}\int_{0.5T}^{T}\exp\left(-\frac{j2\pi kt}{T}\right)dt \\
&= \frac{1}{j\pi k}[1 - (-1)^k].
\end{aligned}$$

The series is infinite, therefore the signal $x(t)$ is not band limited, so it is impossible to avoid aliasing no matter how high the sampling rate.

(b) By Parseval's theorem for Fourier series,

$$\sum_{k=-\infty}^{\infty} |X^{S}[k]|^2 = \frac{1}{T}\int_0^T x^2(t)dt = 1.$$

Therefore, we are looking for K such that

$$\sum_{k=-K}^{K} |X^{S}[k]|^2 = \frac{1}{\pi^2}\sum_{k=-K}^{K}\frac{[1-(-1)^k]^2}{k^2} \geq 0.99.$$

We find by trial and error that $K = 41$ is the minimum value that meets this inequality. Since the even harmonics are zero, it is convenient to choose $K = 42$. Therefore, we should choose $T = T_0/84$. The cutoff frequency of the antialiasing filter is π/T, which is $84\pi/T_0$ (or 42 times the frequency of the square wave).

3.13

(a) The impulse-sampled signal is

$$x_p(t) = \sum_{n=-\infty}^{\infty} [a\sin(0.5\pi n) + b\cos(\pi n)]\delta(t - nT).$$

The period of this signal is $T_0 = 4T$. The Fourier series coefficients are

$$\begin{aligned} X_p^S[k] &= \frac{1}{4T}\int_0^{4T} [b\delta(t) + (a-b)\delta(t-T) + b\delta(t-2T) - (a+b)\delta(t-3T)]e^{-j2\pi kt/(4T)}dt \\ &= \frac{1}{4T}[b + (a-b)(-j)^k + b(-1)^k - (a+b)j^k] \\ &= \begin{cases} 0, & k \bmod 4 = 0, \\ -\frac{ja}{2T}, & k \bmod 4 = 1, \\ \frac{b}{T}, & k \bmod 4 = 2, \\ \frac{ja}{2T}, & k \bmod 4 = 3. \end{cases} \end{aligned}$$

(b) Define

$$q_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) - \sum_{n=-\infty}^{\infty} \delta(t - 3nT).$$

Then, according to the description in the problem,

$$y(t) = x(t)q_T(t).$$

We have

$$Q_T^F(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - 4k\omega_0) - \frac{2\pi}{3T}\sum_{k=-\infty}^{\infty} \delta(\omega - (4/3)k\omega_0).$$

Therefore,

$$Y^F(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^F(\omega - 4k\omega_0) - \frac{1}{3T}\sum_{k=-\infty}^{\infty} X^F(\omega - (4/3)k\omega_0),$$

where

$$X^F(\omega) = \pi b[\delta(\omega - 2\omega_0) + \delta(\omega + 2\omega_0)] - j\pi a[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)].$$

Selecting the delta functions in the interval $[-1.5\omega_0, 1.5\omega_0]$ gives

$$\begin{aligned} Z^F(\omega) = &-\frac{2\pi b}{3}[\delta(\omega - (2/3)\omega_0) + \delta(\omega + (2/3)\omega_0)] \\ &- \frac{j2\pi a}{3}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \\ &- \frac{j\pi a}{3}[\delta(\omega - (1/3)\omega_0) - \delta(\omega + (1/3)\omega_0)], \end{aligned}$$

so finally

$$z(t) = \frac{2a}{3}\sin(\omega_0 t) + \frac{a}{3}\sin\left(\frac{\omega_0 t}{3}\right) - \frac{2b}{3}\cos\left(\frac{2\omega_0 t}{3}\right).$$

3.14 We have

$$\frac{\beta}{\beta^2 + t^2} = \frac{0.5}{\beta - jt} + \frac{0.5}{\beta + jt}.$$

Also, by duality to the signal $x_1(t)$ in Problem 2.3,

$$\left\{\mathcal{F}\frac{1}{\beta - jt}\right\}(\omega) = \begin{cases} 2\pi e^{-\beta\omega}, & \omega > 0, \\ \pi, & \omega = 0, \\ 0, & \omega < 0. \end{cases}$$

Therefore,

$$X^{\mathrm{F}}(\omega) = \pi e^{-\beta|\omega|}, \quad -\infty < \omega < \infty.$$

Finally, by the sampling theorem,

$$X^{\mathrm{f}}(0) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{2\pi k}{T}\right) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty} \exp\left(-\frac{2\pi\beta|k|}{T}\right) = \frac{\pi}{T}\cdot\frac{1+\exp\left(-\frac{2\pi\beta}{T}\right)}{1-\exp\left(-\frac{2\pi\beta}{T}\right)}.$$

3.15

(a) The sampled signal is

$$x(nT) = \begin{cases} 1, & |n| \le N, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$X^{\mathrm{f}}(\theta) = \sum_{n=-N}^{N} e^{-j\theta n} = \frac{\sin(N+0.5)\theta}{\sin 0.5\theta}.$$

(b) The Fourier transform of $x(t)$ is

$$X^{\mathrm{F}}(\omega) = \int_{-(N+0.5)T}^{(N+0.5)T} e^{-j\omega t}dt = (2N+1)\operatorname{sinc}\left[\frac{(N+0.5)\omega T}{\pi}\right].$$

Therefore we get by the sampling theorem,

$$\sum_{k=-\infty}^{\infty} \operatorname{sinc}\left[\frac{(N+0.5)(\omega T - 2\pi k)}{\pi}\right] = \frac{1}{2N+1}\sum_{k=-\infty}^{\infty} X^{\mathrm{F}}(\omega - 2\pi k/T) = \frac{T}{2N+1}\cdot\frac{\sin(N+0.5)\theta}{\sin 0.5\theta}.$$

3.16

(a) The minimum sampling frequency is $\omega_{\mathrm{sam}} = 1.5\omega_0$.

(b) The ideal reconstruction filter is

$$H^{\mathrm{F}}(\omega) = \begin{cases} \frac{2\pi}{\omega_0}, & 0 \le |\omega| \le 0.5\omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}\hat{x}(t) &= \frac{1}{\omega_0}\int_{-0.5\omega_0}^{0.5\omega_0} \cos\left(\frac{\pi\omega}{\omega_0}\right) e^{j\omega t}d\omega = \frac{1}{2\omega_0}\int_{-0.5\omega_0}^{0.5\omega_0} [e^{j\omega(t+\pi/\omega_0)} + e^{j\omega(t-\pi/\omega_0)}]d\omega \\ &= \frac{1}{\omega_0}\cdot\frac{\sin[0.5\omega(t+\pi/\omega_0)]}{t+\pi/\omega_0} + \frac{1}{\omega_0}\cdot\frac{\sin[0.5\omega(t-\pi/\omega_0)]}{t-\pi/\omega_0} \\ &= \frac{1}{\omega_0}\cdot\frac{\cos(0.5\omega t)}{t+\pi/\omega_0} - \frac{1}{\omega_0}\cdot\frac{\cos[(0.5\omega t)}{t-\pi/\omega_0} = 2\pi\frac{\cos(0.5\omega t)}{\pi^2 - (\omega_0 t)^2}.\end{aligned}$$

3.17

(a) The signal $x[n]$ is periodic, since $x(t)$ is sampled an integer number of times per period. Therefore $\hat{x}(t)$ is periodic as well.

(b) No. The signal $x(t)$ is not band limited, so sampling at any interval would lead to aliasing.

(c) We have

$$x[n] = \begin{cases} 0, & n \bmod 3 = 0, \\ 1, & n \bmod 6 = 1, 2, \\ -1, & n \bmod 6 = 4, 5. \end{cases}$$

Therefore,

$$X^{\mathrm{f}}(\theta) = (e^{-j\theta} + e^{-j2\theta}) \sum_{m=-\infty}^{\infty} e^{-j6m\theta} - (e^{j\theta} + e^{j2\theta}) \sum_{m=-\infty}^{\infty} e^{-j6m\theta} = -2j(\sin\theta + \sin 2\theta) \sum_{m=-\infty}^{\infty} e^{-j6m\theta}$$

$$= -2j(\sin\theta + \sin 2\theta)\frac{\pi}{3} \sum_{n=-\infty}^{\infty} \delta\left(\theta - \frac{n\pi}{3}\right) = -\frac{j2\pi}{3} \sum_{n=-\infty}^{\infty} \left[\sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right] \delta\left(\theta - \frac{n\pi}{3}\right).$$

Passing $x[n]$ through an ideal reconstructor gives

$$\hat{X}^{\mathrm{F}}(\omega) = -\frac{j2\pi\sqrt{3}}{3}\left[\delta\left(\omega - \frac{\pi}{3T}\right) - \delta\left(\omega + \frac{\pi}{3T}\right)\right].$$

Therefore,

$$\hat{x}(t) = \frac{2\sqrt{3}}{3} \sin\left(\frac{\pi t}{3}\right).$$

3.18 By the condition on the Fourier coefficients, we know that the signal is band limited to $2\pi K/T_0$, which is less than π/T. Therefore, there is no aliasing when sampling at an interval T. We have, by Shannon's interpolation formula,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}\left(\frac{(t-nT)}{T}\right).$$

Using the periodicity of $x(t)$, we can transform the sum over n to a double sum

$$x(t) = \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} x(mNT + lT)\operatorname{sinc}\left(\frac{(t - mNT - lT)}{T}\right)$$

$$= \sum_{l=0}^{N-1} x(lT) \sum_{m=-\infty}^{\infty} \operatorname{sinc}\left(\frac{(t - mT_0 - lT)}{T}\right).$$

By (2.157) we have

$$\sum_{m=-\infty}^{\infty} \operatorname{sinc}\left(\frac{(t - mT_0 - lT)}{T}\right) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} S^{\mathrm{F}}\left(\frac{2\pi k}{T_0}\right) \exp\left(j\frac{2\pi k(t - lT)}{T_0}\right),$$

where $S^{\mathrm{F}}(\omega)$ is the Fourier transform of $\operatorname{sinc}(t/T)$, given by

$$S^{\mathrm{F}}(\omega) = \begin{cases} T, & |\omega| \le \frac{\pi}{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the terms in sum over k vanish outside the range $|k| \le K$, yielding

$$\sum_{m=-\infty}^{\infty} \operatorname{sinc}\left(\frac{(t - mT_0 - lT)}{T}\right) = \frac{T}{T_0} \sum_{k=-K}^{K} \exp\left(j\frac{2\pi k(t - lT)}{T_0}\right) = \frac{1}{N} D\left(\frac{t}{T_0} - \frac{l}{N}, N\right),$$

where

$$D(\omega, N) = \frac{\sin(0.5N\omega)}{\sin(0.5\omega)}.$$

Finally,

$$x(t) = \frac{1}{N} \sum_{l=0}^{N-1} x(lT) D\left(\frac{t}{T_0} - \frac{l}{N}, N\right),$$

as stated.

3.19
(a)

$$q(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) + \sum_{n=-\infty}^{\infty} \delta(t - nT - a).$$

(b) We have, by the shift property of the Fourier transform,

$$Q^{\mathrm{F}}(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} (1 + e^{-ja\omega})\delta\left(\omega - \frac{2\pi k}{T}\right).$$

Therefore,

$$X_q^{\mathrm{F}}(\omega) = \{Q^{\mathrm{F}} * X^{\mathrm{F}}\}(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X^{\mathrm{F}}(\omega-\lambda)(1+e^{-ja\lambda})\delta\left(\lambda - \frac{2\pi k}{T}\right)d\lambda$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\left[1+\exp\left(-j\frac{2\pi ka}{T}\right)\right]X^{\mathrm{F}}\left(\omega - \frac{2\pi ka}{T}\right).$$

So, with

$$\omega_{\mathrm{sam}} = \frac{2\pi}{T}, \quad A_k = 1 + e^{-jka\omega_{\mathrm{sam}}},$$

we get

$$X_q^{\mathrm{F}}(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} A_k X^{\mathrm{F}}(\omega - k\omega_{\mathrm{sam}}).$$

(c) Sampling is at the Nyquist rate, so there is no aliasing. Therefore, the shape of the Fourier transform in the range

$$(k-0.5)\omega_{\mathrm{sam}} \le \omega \le (k+0.5)\omega_{\mathrm{sam}}$$

is the kth term in the sum. For $a = T/4$ we have $a\omega_{\mathrm{sam}} = 0.5\pi$, so $A_k = 1 + (-j)^k$. We are interested in $-2 \le k \le 2$, for which

$$A_{-2} = 0,\ A_{-1} = 1 + j,\ A_0 = 2,\ A_1 = 1 - j,\ A_2 = 0.$$

The real and imaginary parts of $X_{\mathrm{p}}^{\mathrm{F}}(\omega)$ are as shown in Figure 3.3.

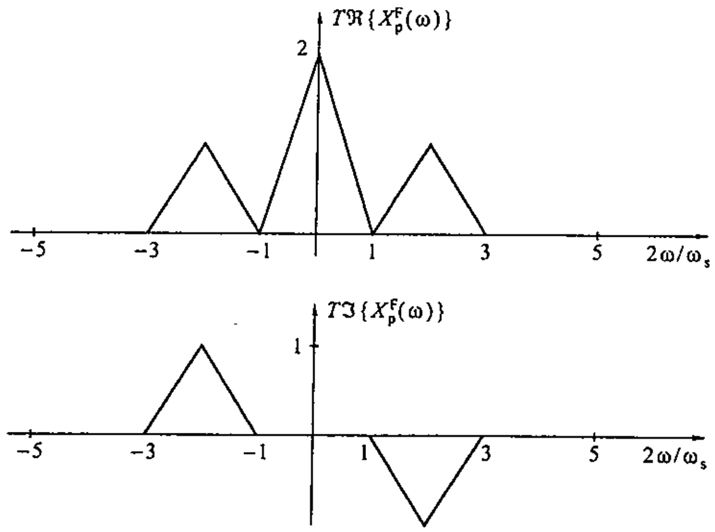

Figure 3.3 Pertaining to Solution 3.19.

(d) It follows from Figure 3.3 that the reconstruction filter is an ideal low-pass filter with cut-off frequency $0.5\omega_{\mathrm{sam}}$ and gain $0.5T$.

3.20 Let us denote $y(t) = x^3(t) + 0.5x(t)$. Then

$$Y^{\mathrm{F}}(\omega) = \frac{1}{4\pi^2}\{X^{\mathrm{F}} * X^{\mathrm{F}} * X^{\mathrm{F}}\}(\omega) + 0.5X^{\mathrm{F}}(\omega),$$

$$Y^{\mathrm{f}}(\theta) = \frac{1}{4\pi^2}\{X^{\mathrm{f}} * X^{\mathrm{f}} * X^{\mathrm{f}}\}(\theta) + 0.5X^{\mathrm{f}}(\theta).$$

The bandwidth of $Y^{\mathrm{F}}(\omega)$ is 24 kHz. Therefore $Y^{\mathrm{f}}(\omega T)$ will be equal to $Y^{\mathrm{F}}(\omega)$ (up to a scale factor) if $1/T >$ 48 kHz. Since the reconstructor is ideal, we will get equality between $Y^{\mathrm{f}}(\omega T)$ and $\hat{Y}^{\mathrm{F}}(\omega)$ (up to a scale factor), as required. On the other hand, if $1/T <$ 48 kHz, $Y^{\mathrm{f}}(\omega T)$ will be aliased in general, although in rare cases it may not be so.

3.21

(a) It follows from Solution 3.7 that the signal $\hat{x}(t)$ reconstructed from $x(nT)$ is

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} X^{S}[k] \exp\left(\frac{j\theta_k t}{T}\right),$$

where

$$\theta_k = 2\pi\left(\frac{kT}{T_0} - m_k\right),$$

and m_k is an integer. Let us show that θ_k/θ_l is not a rational number for $k \neq l$. Indeed, if

$$\frac{\theta_k}{\theta_l} = \frac{\frac{kT}{T_0} - m_k}{\frac{lT}{T_0} - m_l} = \frac{r}{s},$$

where r, s are integers, then

$$\frac{T}{T_0} = \frac{m_k s - m_l r}{ks - lr}.$$

The right side is rational, in contradiction to the assumption that T/T_0 is irrational. Since we have shown that θ_k/θ_l is irrational for all $k \neq l$, the θ_k cannot all be integer multiples of a single number. Therefore $\hat{x}(t)$ is not periodic.

(b) If

$$\frac{T}{T_0} = \frac{p}{q},$$

where p, q are coprime integers, then

$$x(nT) = \sum_{k=-\infty}^{\infty} X^{S}[k] \exp\left(\frac{j2\pi knp}{q}\right).$$

Let

$$k = m + lq, \text{ where } 0 \le m \le q-1, \ -\infty < l < \infty.$$

Then

$$x(nT) = \sum_{m=0}^{q-1} \left[\sum_{l=-\infty}^{\infty} X^{S}[m+lq]\right] \exp\left(\frac{j2\pi mnp}{q}\right).$$

Since p is coprime to q, the numbers $\{e^{j2\pi mp/q},\ 0 \le m \le q-1\}$ are the same as the numbers $\{e^{j2\pi m'/q},\ 0 \le m' \le q-1\}$ in a different order. Therefore, in this case $x(nT)$ is a periodic signal with period q. It consists of a sum of q sinusoids whose frequencies are the integer multiples of $2\pi/q$. Consequently, $\hat{x}(t)$ is a sum of q continuous-time sinusoids whose frequencies are integer multiples of $2\pi/qT$. The period of $\hat{x}(t)$ is qT.

3.22

(a) The sampled signal is

$$x[n] = 3\cos(0.25\pi n) + 2\sin(0.625\pi n).$$

The reconstructor assumes that the relationship between the digital and the analog frequencies is $\theta = 0.005\omega$. The output of the reconstructor is therefore

$$\hat{x}(t) = 3\cos(50\pi n) + 2\sin(125\pi n).$$

(b) Now the sampled signal is

$$x[n] = 3\cos(0.5\pi n) + 2\sin(1.25\pi n) = 3\cos(0.5\pi n) - 2\sin(0.75\pi n).$$

Note that the second term is aliased. The reconstructor assumes that the relationship between the digital and the analog frequencies is $\theta = 0.0025\omega$. The output of the reconstructor is therefore

$$\hat{x}(t) = 3\cos(200\pi n) - 2\sin(300\pi n).$$

3.23

(a) Define

$$u(t) = x(0.5t).$$

Then $y(t)$ is the impulse sampling of $u(t)$ at interval $2T$. The Fourier transform of $u(t)$ is

$$U^{\mathrm{F}}(\omega) = 2X^{\mathrm{F}}(2\omega).$$

Therefore, by the sampling theorem,

$$Y^{\mathrm{F}}(\omega) = \frac{1}{2T}\sum_{k=-\infty}^{\infty} U^{\mathrm{F}}\left(\omega - \frac{2\pi k}{2T}\right) = \frac{1}{2T}\sum_{k=-\infty}^{\infty} U^{\mathrm{F}}(\omega - k\omega_{\mathrm{m}}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^{\mathrm{F}}(2\omega - 2k\omega_{\mathrm{m}}).$$

(b) As we see, the kth replica is nonzero for

$$-\omega_{\mathrm{m}} \le 2\omega - 2k\omega_{\mathrm{m}} \le \omega_{\mathrm{m}},$$

or

$$(k-0.5)\omega_{\mathrm{m}} \le \omega \le (k+0.5)\omega_{\mathrm{m}}.$$

The replicas do not overlap, so the spectrum is not aliased. This result is obvious, since $y(t)$ contains all the samples of $x(t)$ at *its own Nyquist rate*, so no information about $x(t)$ is lost.

3.24

(a) Let $y(t)$ be the signal defined in Problem 3.23 and let $h(t) = \mathrm{sinc}(t/T)$. Then

$$\{y * h\}(t) = \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t-\tau-2nT)\right]\mathrm{sinc}\left(\frac{\tau}{T}\right)d\tau = \sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t-2nT}{T}\right) = z(t).$$

Therefore,

$$Z^{\mathrm{F}}(\omega) = TY^{\mathrm{F}}(\omega)\mathrm{rect}\left(\frac{\omega}{2\omega_{\mathrm{m}}}\right).$$

Using the result in Solution 3.23, we get

$$Z^{\mathrm{F}}(\omega) = \left[\sum_{k=-1}^{1} X^{\mathrm{F}}(2\omega - 2k\omega_{\mathrm{m}})\right]\mathrm{rect}\left(\frac{\omega}{2\omega_{\mathrm{m}}}\right).$$

The spectra of $x(t)$ and $z(t)$ are shown in Figure 3.4.

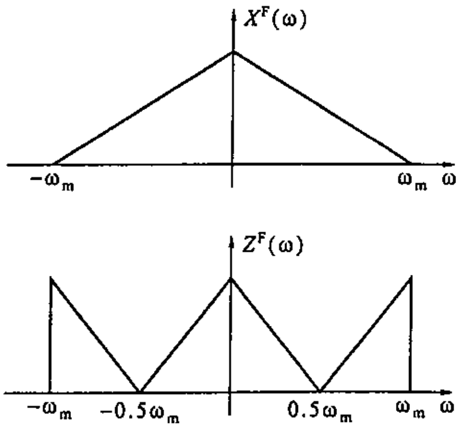

Figure 3.4 Pertaining to Solution 3.24.

(b) We have in this case

$$X^{\mathrm{F}}(\omega) = \pi[\delta(\omega - 0.6\omega_{\mathrm{m}}) + \delta(\omega + 0.6\omega_{\mathrm{m}})],$$

so

$$Z^{\mathrm{F}}(\omega) = \pi\left\{\sum_{k=-1}^{1}[\delta(2\omega - 2k\omega_{\mathrm{m}} - 0.6\omega_{\mathrm{m}}) + \delta(2\omega - 2k\omega_{\mathrm{m}} + 0.6\omega_{\mathrm{m}})]\right\}\mathrm{rect}\left(\frac{\omega}{2\omega_{\mathrm{m}}}\right)$$
$$= 0.5\pi\left\{\sum_{k=-1}^{1}[\delta(\omega - k\omega_{\mathrm{m}} - 0.3\omega_{\mathrm{m}}) + \delta(\omega - k\omega_{\mathrm{m}} + 0.3\omega_{\mathrm{m}})]\right\}\mathrm{rect}\left(\frac{\omega}{2\omega_{\mathrm{m}}}\right)$$
$$= 0.5\pi[\delta(\omega - 0.3\omega_{\mathrm{m}}) + \delta(\omega + 0.3\omega_{\mathrm{m}}) + \delta(\omega - 0.7\omega_{\mathrm{m}}) + \delta(\omega + 0.7\omega_{\mathrm{m}})].$$

Therefore,

$$z(t) = 0.5\cos(0.3\omega_{\mathrm{m}}t) + 0.5\cos(0.7\omega_{\mathrm{m}}t).$$

(c) Shannon's reconstruction of $u[n]$ gives

$$\sum_{n=-\infty}^{\infty} u[n]\mathrm{sinc}\left(\frac{t-nT}{T}\right) = \sum_{n=-\infty}^{\infty} u[2n]\mathrm{sinc}\left(\frac{t-2nT}{T}\right) = \sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t-2nT}{T}\right) = z(t).$$

3.25 The operation described in the problem is equivalent to taking every other sample of $x[n]$. Therefore

$$z[n] = x[2n] = \cos\left(\frac{2\pi n}{8}\right).$$

3.26 The Fourier transform of $x[n]$ is aliased because of the addition of the replicas $k = -1, 0, 1$. We get

$$X^{\mathrm{f}}(\theta) = \begin{cases} 1, & 0.25\pi \le |\theta| \le 0.75\pi, \\ 2, & 0.75\pi \le |\theta| \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$y(t) = \frac{T}{2\pi}\int_{-\pi}^{\pi} X^{\mathrm{f}}(\omega T)e^{j\omega t}d\omega = 2\mathrm{sinc}(t) - 0.75\mathrm{sinc}(0.75t) - 0.25\mathrm{sinc}(0.25t).$$

3.27 The signal can be written as

$$x(t) = 0.25\cos(1.2\pi t) + 0.75\cos(0.4\pi t).$$

Therefore,

$$x[n] = 0.25\cos(1.2\pi n) + 0.75\cos(0.4\pi n) = 0.25\cos(0.8\pi n) + 0.75\cos(0.4\pi n).$$

The reconstructed signal is

$$y(t) = 0.25\cos(1.6\pi t) + 0.75\cos(0.8\pi t).$$

3.28 We have

$$x(t) = 0.5\cos(1.3\pi t) + 0.5\cos(0.5\pi t).$$

After sampling,

$$x[n] = 0.5\cos(1.3\pi n) + 0.5\cos(0.5\pi n) = 0.5\cos(0.7\pi n) + 0.5\cos(0.5\pi n).$$

(note the aliasing of the frequency 1.3π). The output of the high-pass filter is

$$y[n] = 0.5\cos(0.7\pi n).$$

Finally, the reconstructed signal is

$$y(t) = 0.5\cos(0.7\pi t).$$

3.29

(a) The impulse response of the first-order hold is shown in Figure 3.5. The corresponding formula is

$$h(t) = \begin{cases} 1 + \frac{t}{T}, & 0 \le t < T, \\ 1 - \frac{t}{T}, & T \le t < 2T, \\ 0, & \text{otherwise.} \end{cases}$$

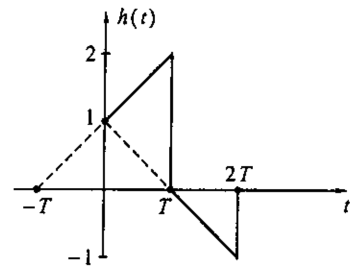

Figure 3.5 Pertaining to Solution 3.29.

(b) The frequency response is

$$H^{\mathrm{F}}(\omega) = \int_0^T \left(1+\frac{t}{T}\right)e^{-j\omega t}dt + \int_T^{2T}\left(1-\frac{t}{T}\right)e^{-j\omega t}dt = T(1+j\omega T)\operatorname{sinc}^2\left(\frac{\omega T}{2\pi}\right)e^{-j\omega T}.$$

The magnitude and phase responses are

$$|H^{\mathrm{F}}(\omega)| = T[1+(\omega T)^2]^{1/2}\operatorname{sinc}^2\left(\frac{\omega T}{2\pi}\right),$$

$$\measuredangle H^{\mathrm{F}}(\omega) = \arctan(\omega T) - \omega T.$$

3.30 Let us compute the Fourier transform of $\hat{x}[n]$:

$$\hat{X}^{\mathrm{F}}(\theta) = \sum_{n=-\infty}^{\infty}\hat{x}[n]e^{-j\theta m} = \sum_{m=-\infty}^{\infty}y[2m]e^{-j2\theta m} + \sum_{m=-\infty}^{\infty}y[2m]e^{-j\theta(2m+1)}$$

$$= (1+e^{-j\theta})\sum_{m=-\infty}^{\infty}y[2m]e^{-j2\theta m} = (1+e^{-j\theta})\sum_{n=-\infty}^{\infty}y[n]e^{-j\theta n} = (1+e^{-j\theta})Y^{\mathrm{f}}(\theta).$$

Therefore, the operation is equivalent to passing $y[n]$ through the filter

$$H^{\mathrm{f}}(\theta) = 1+e^{-j\theta} \implies H^{\mathrm{z}}(z) = 1+z^{-1}.$$

The frequency response of the filter can also be written as

$$H^{\mathrm{f}}(\theta) = 2\cos(0.5\theta)e^{-j0.5\theta}.$$

Therefore,

$$|H^{\mathrm{f}}(\theta)| = 2\cos(0.5\theta), \quad \measuredangle H^{\mathrm{f}}(\theta) = -0.5\theta.$$

3.31

(a) Figure 3.6 shows the impulse response of the reconstructor.

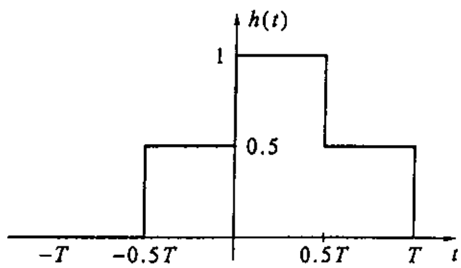

Figure 3.6 Pertaining to Solution 3.31.

(b)

$$H^{\mathrm{F}}(\omega) = \frac{0.5(e^{j0.5\omega T}-1)+(1-e^{-j0.5\omega T})+0.5(e^{-j0.5\omega T}-e^{-j\omega T})}{j\omega}.$$

3.32 Let ω_0 be such that $\omega_0 \ge \omega_2$ and

$$\omega_0 = L(\omega_0 - \omega_1),$$

where L is an integer. Then

$$\omega_0 = \frac{L\omega_1}{L-1} \ge \omega_2,$$

from which we get

$$L = \left\lfloor \frac{\omega_2}{\omega_2 - \omega_1} \right\rfloor.$$

This is the same as (3.54). However, the sampling interval is now

$$T = \frac{\pi}{\omega_0 - \omega_1} = \frac{\pi L}{\omega_0},$$

which is smaller than the sampling interval obtained in Section 3.6.

3.33 Practical signals have finite duration, so they cannot be band pass in the ideal sense. In other words, they always have some energy outside their effective bandwidth. Similarly to conventional sampling, it is desirable to use an antialiasing filter before the sampler and to leave some safety margin. In this case, since the replicas appear on both the right and the left of the original spectrum, it is advisable to extend the bandwidth on both right and left.

3.34

$$h(t) = \frac{T}{2\pi}\int_{-\omega_2}^{-\omega_0} e^{j\omega t}d\omega + \frac{T}{2\pi}\int_{\omega_0}^{\omega_2} e^{j\omega t}d\omega = \frac{\sin(\omega_2 t) - \sin(\omega_0 t)}{(\omega_2 - \omega_0)t} = \operatorname{sinc}\left(\frac{(\omega_2 - \omega_0)t}{2\pi}\right)\cos\left(\frac{(\omega_2 + \omega_0)t}{2}\right).$$

3.35

(a) Since the bands $[3, 5]$ and $[7, 9]$ contain only noise, they may be aliased by sampling without harming the useful information in the band $[5, 7]$. We can therefore treat the signal as if it were band limited to $[4, 8]$, and sample it at a rate $\omega_{\text{sam}} = 8$. As Figure 3.7 illustrates, this will leave the range $[5, 7]$ nonaliased. The useful signal can be reconstructed by an ideal band-pass filter

$$H^{\text{f}}(\omega) = \begin{cases} 1, & 5 \le |\omega| \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

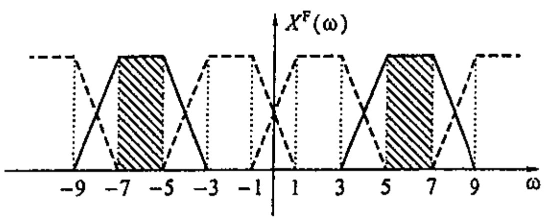

Figure 3.7 Pertaining to Solution 3.35.

(b) If we are permitted to use a filter, it is best to filter out the noise, so that the remaining useful signal will be band limited to the range $5 \le |\omega| \le 7$. Then we use the band-pass sampling formula to get

$$\omega_{\text{sam}} = \frac{2 \times 7}{\lfloor 7/2 \rfloor} = \frac{14}{3}.$$

3.36 Let us assume that the signal $x(t)$ has a flat spectrum in the frequency range $3.5 \le |\omega| \le 4.5$. Then its Fourier transform is as shown in Figure 3.8(a). The Fourier transform of $y(t)$ is given by

$$Y^{\text{F}}(\omega) = \frac{1}{2\pi}\{X^{\text{F}} * X^{\text{F}}\}(\omega).$$

Therefore, the Fourier transform is as shown in Figure 3.8(b). It is now clear that the minimum sampling rate is $\omega_{\text{sam}} = 6$. The spectrum of the sampled signal is as shown in Figure 3.8(c).

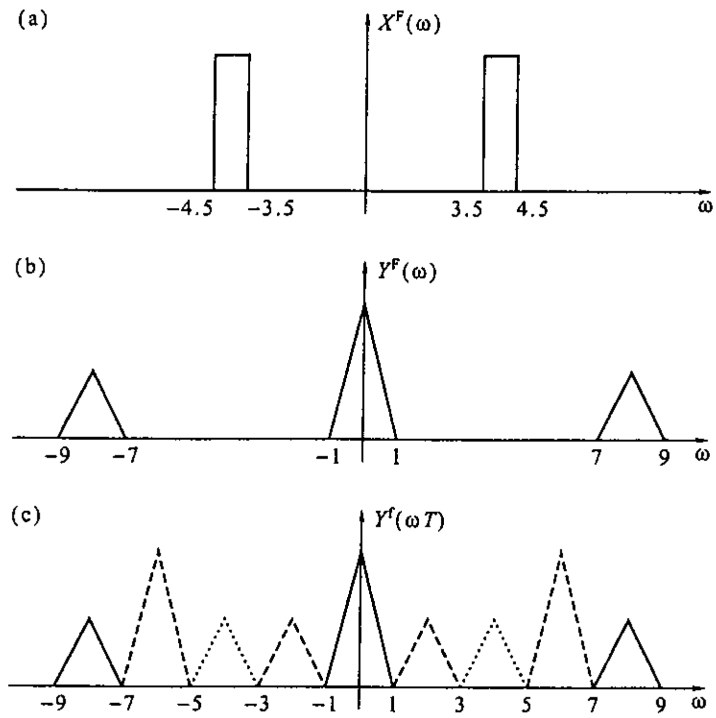

Figure 3.8 Pertaining to Solution 3.36.

3.37

(a) Denote

$$a(t) = \sum_{n=1}^{\infty} \lambda^n \sin(n\omega_0 t).$$

Then

$$A^F(\omega) = -j\pi \sum_{n=1}^{\infty} \lambda^n[\delta(\omega - n\omega_0) - \delta(\omega + n\omega_0)],$$

and

$$Y^F(\omega) = \{X^F * A^F\}(\omega) = -0.5j \sum_{n=1}^{\infty} \lambda^n[X^F(\omega - n\omega_0) - X^F(\omega + n\omega_0)].$$

(b) Aliasing is avoided if $\omega_0 \geq \omega_m$, since in this case

$$Y^F(\omega) = \begin{cases} -0.5j\lambda X^F(\omega - \omega_0), & 0 \leq \omega \leq \omega_0, \\ 0.5j\lambda X^F(\omega + \omega_0), & -\omega_0 \leq \omega \leq 0. \end{cases}$$

(c) As we see from part b, the first step is to pass $y(t)$ through an ideal low-pass filter

$$H_1^F(\omega) = \text{rect}\left(\frac{\omega}{2\omega_0}\right).$$

Denote the output of the filter by $z(t)$. Then $Z^F(\omega)$ is identical to $Y^F(\omega)$ in the frequency range $[-\omega_0, \omega_0]$ and zero elsewhere. The next step is to form the signal

$$u(t) = z(t)\sin(\omega_0 t).$$

Then,

$$U^{\mathrm{F}}(\omega) = -j0.5[Y^{\mathrm{F}}(\omega-\omega_0) - Y^{\mathrm{F}}(\omega+\omega_0)]$$

$$= \begin{cases} -0.25\lambda X^{\mathrm{F}}(\omega+2\omega_0), & -2\omega_0 \le \omega \le -\omega_0, \\ 0.25\lambda X(\omega), & -\omega_0 \le \omega \le 0, \\ 0.25\lambda X(\omega), & 0 \le \omega \le \omega_0, \\ -0.25\lambda X^{\mathrm{F}}(\omega-2\omega_0), & \omega_0 \le \omega \le 2\omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, passing $u(t)$ through an ideal low-pass filter

$$H_2^{\mathrm{F}}(\omega) = 4\lambda^{-1}\mathrm{rect}\left(\frac{\omega}{2\omega_0}\right)$$

will yield $x(t)$.

3.38

(a) The A/D must have 14 bits.

(b) The maximum number of samples per second is $19,200/14 = 1371$. Therefore, the maximum signal bandwidth is 685.5 Hz. Allowing 10 percent margin for an antialias filtering, we get a practical maximum bandwidth of about 600 Hz.

3.39 The Fourier transform of $x(t)$ is

$$X^{\mathrm{F}}(\omega) = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)].$$

Therefore [cf. (3.46)],

$$\begin{aligned}\tilde{X}^{\mathrm{F}}(\omega) &= \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]\mathrm{sinc}\left(\frac{\omega\Delta}{2\pi}\right)\exp\left(-\frac{j\omega\Delta}{2}\right) \\ &= \pi\mathrm{sinc}\left(\frac{\omega_0\Delta}{2\pi}\right)\exp\left(-\frac{j\omega_0\Delta}{2}\right)\delta(\omega-\omega_0) + \pi\mathrm{sinc}\left(-\frac{\omega_0\Delta}{2\pi}\right)\exp\left(\frac{j\omega_0\Delta}{2}\right)\delta(\omega+\omega_0).\end{aligned}$$

Since the sampling rate meets Nyquist's condition,

$$X^{\mathrm{f}}(\theta) = \pi\mathrm{sinc}\left(\frac{\omega_0\Delta}{2\pi}\right)\exp\left(-\frac{j\omega_0\Delta}{2}\right)\delta(\theta-\omega_0 T) + \pi\mathrm{sinc}\left(-\frac{\omega_0\Delta}{2\pi}\right)\exp\left(\frac{j\omega_0\Delta}{2}\right)\delta(\theta+\omega_0 T).$$

Taking the inverse Fourier transform gives

$$\begin{aligned}x[n] &= 0.5\mathrm{sinc}\left(\frac{\omega_0\Delta}{2\pi}\right)\exp\left(j\omega_0 Tn - \frac{j\omega_0\Delta}{2}\right) + 0.5\mathrm{sinc}\left(-\frac{\omega_0\Delta}{2\pi}\right)\exp\left(-j\omega_0 Tn + \frac{j\omega_0\Delta}{2}\right) \\ &= \mathrm{sinc}\left(\frac{\omega_0\Delta}{2\pi}\right)\cos(\omega_0 Tn - 0.5\omega_0\Delta).\end{aligned}$$

Finally, since the reconstruction is ideal,

$$\hat{x}(t) = \mathrm{sinc}\left(\frac{\omega_0\Delta}{2\pi}\right)\cos(\omega_0 t - 0.5\omega_0\Delta).$$

3.40

(a) No, the signal resembles a double side-band modulated sinusoid.

(b) Yes, in this case the sinusoidal shape is evident.

(c) As stated in the problem, plots of signals sampled only slightly above the Nyquist rate can be misleading, since they are likely to hide the true shape of the continuous-time signal. Although theoretically no information is lost in the sampling, in practice the plots do not perform ideal reconstruction, so information *is* lost in the plots.

3.41

(a) By the information given to us, we can deduce that

$$f_0 = 50 + 150k_1, \quad f_0 = 20 + 240k_2,$$

where k_1 and k_2 are unknown integers. Eliminating f_0, we get the following relationship between these integers:

$$8k_2 - 5k_1 = 1.$$

One solution to this equation is $k_1 = 3$, $k_2 = 2$, which gives $f_0 = 500\,\text{Hz}$. However, this is only one solution out of infinitely many. The complete set of solutions is

$$k_1 = 3 + 8n, \quad k_2 = 2 + 5n,$$

for all $n \geq 0$. Correspondingly,

$$f_0 = 500 + 1200n, \quad n \geq 0,$$

are all possible values of f_0.

(b) With the additional information, f_0 must be 500 Hz, since the next solution, 1700 Hz, is outside the range.

3.42

(a) For the first signal:

$$f_{\text{sam},1} = \frac{2700}{\lfloor 1350 - 1000 \rfloor} = 900\,\text{Hz}.$$

For the second signal:

$$f_{\text{sam},2} = \frac{4800}{\lfloor 2400 - 2000 \rfloor} = 800\,\text{Hz}.$$

(b) The channel must be capable of transmitting 1700 samples per second, which is the sum of the rates of the two signals.

(c) It is most convenient to take $N_1 = 9$ samples from the first signal and $N_2 = 8$ samples from the second signal every 10 milliseconds. Then the packet size is 17 and 100 packets per second are transmitted.

3.43

(a) Sampling $y(t)$ at an interval T gives

$$y(nT) = x(nT)\cos(\omega_0 nT).$$

If we choose T such that $\omega_0 T = 2\pi k$, with k an integer, we will get

$$y(nT) = x(nT)\cos(2\pi kn) = x(nT),$$

so the sampled modulated signal will be identical to the sampled demodulated and low-pass filtered signal. To find the largest admissible T, recall that $x(nT)$ should be nonaliased with respect to $x(t)$, so

$$T \leq \frac{\pi}{\omega_{\text{m}}},$$

or

$$\frac{\omega_0 T}{2\pi} \leq \frac{\omega_0}{2\omega_{\text{m}}}.$$

Since the left side must be an integer, we get

$$\frac{\omega_0 T}{2\pi} = \left\lfloor \frac{\omega_0}{2\omega_{\text{m}}} \right\rfloor,$$

and finally

$$T = \frac{2\pi}{\omega_0}\left\lfloor \frac{\omega_0}{2\omega_{\text{m}}} \right\rfloor.$$

Explanation: The modulated signal $y(t)$ is a cosine whose envelope is $x(t)$. By choosing T as above we are sampling the envelope directly, that is, we sample at the peaks of the cosine.
Frequency domain interpretation: the spectrum of $y(t)$ is

$$Y^{\mathrm{F}}(\omega) = 0.5[X^{\mathrm{F}}(\omega - \omega_0) + X^{\mathrm{F}}(\omega - \omega_0)].$$

Sampling of $y(t)$ at interval T gives

$$Y^{\mathrm{f}}(\theta) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{\theta - 2\pi k - \omega_0 T}{T}\right) + \frac{1}{2T} \sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{\theta - 2\pi k + \omega_0 T}{T}\right).$$

But, since $\omega_0 T$ is an integer, the two terms on the right side are equal and

$$Y^{\mathrm{f}}(\theta) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X^{\mathrm{F}}\left(\frac{\theta - 2\pi k}{T}\right) = X^{\mathrm{f}}(\theta).$$

Another way to think of this result is: the replicas generated by $X^{\mathrm{F}}(\omega - \omega_0)$ after sampling and those generated $X^{\mathrm{F}}(\omega + \omega_0)$ coincide, and since they are identical in shape, they add up. Finally, since $T \le \pi/\omega_{\mathrm{m}}$, the result is nonaliased with respect to $X^{\mathrm{F}}(\omega)$.

(b) Demodulating with frequency error gives

$$z(t) = 2x(t)\cos(\omega_0 t)\cos[(\omega_0 + \Delta\omega)t] = x(t)\{\cos(\Delta\omega t) + \cos[(2\omega_0 + \Delta\omega)t]\}.$$

After passing $z(t)$ through a low-pass filter, we will get $x(t)\cos(\Delta\omega t)$. This still has a residual modulation, which is not what we want. In particular, if $\Delta\omega < \omega_{\mathrm{m}}$, the two frequency-domain components $X^{\mathrm{F}}(\omega - \Delta\omega)$ and $X^{\mathrm{F}}(\omega + \Delta\omega)$ will overlap partially, thus distorting the signal.

A similar phenomenon will occur in case of direct sampling. The receiver, since it does not know ω_0 accurately, will use a sampling interval

$$T = \frac{2\pi}{\omega_0 + \Delta\omega} \left\lfloor \frac{\omega_0 + \Delta\omega}{2\omega_{\mathrm{m}}} \right\rfloor.$$

Now $\omega_0 T$ will not be an integer multiple of 2π, so the frequency-domain replicas of the sampled signal will not coincide, only partially overlap. This will give rise to distortion similar to the one encountered in the first scheme.

3.44 First we determine the worst-case frequency deviation as a result of all four sources:

(a) Error in f_0 due to the frequency generator at the satellite:

$$\Delta f_0 = 0.5 \times 10^{-6} \times 160 \times 10^{6} = 80\,\mathrm{Hz}.$$

(b) Error in f_{d} due to the frequency generator at the receiver:

$$\Delta f_{\mathrm{d}} = 3 \times 10^{-6} \times 160 \times 10^{6} = 480\,\mathrm{Hz}.$$

Remark: The frequency of the demodulator will not be 160 MHz exactly but it will be close enough, so we can use this value as first approximation.

(c) Doppler effect resulting from the satellite motion. The gravity at the satellite's orbit will be

$$g(h) = 9.81 \cdot \left(\frac{r_0}{r_0 + h}\right)^2 = 9.81 \times 0.955^2 = 8.95\,\mathrm{m/sec}^2.$$

Therefore, the speed of the satellite is

$$v_{\mathrm{sat}} = \sqrt{(r_0 + h)g(h)} = \sqrt{8.95 \times 6.67 \times 10^{6}} = 7730\,\mathrm{m/sec}.$$

The frequency shift resulting from the motion of the satellite is thus

$$\Delta f_{\mathrm{sat}} = \frac{f_0 v_{\mathrm{sat}}}{c} = 4120\,\mathrm{Hz}.$$

(d) Doppler effect resulting from Earth motion. The speed of the Earth at the equator is its angular velocity times its radius, that is

$$v_{\mathrm{e}} = \Omega r_0 = \frac{2\pi}{24 \times 3600} \times 6.37 \times 10^{6} = 463\,\mathrm{m/sec}.$$

The corresponding frequency shift is

$$\Delta f_e = \frac{f_0 v_e}{c} = 247\,\text{Hz}.$$

The total worst-case frequency error is therefore

$$\Delta f = 80 + 480 + 4120 + 247 \approx 4930\,\text{Hz}.$$

The worst-case deviation will occur with a very low probability, that is, if both frequency generators are at their extreme deviations in the same direction, the satellite moves directly toward (or away from) the receiver, the receiver is located on the equator and its motion resulting from Earth rotation adds in magnitude to the motion of the satellite.

Now, since the information bandwidth is 19.2 kHz, it follows from the above calculation that using $f_d = f_0 - 25 \times 10^3$ will guarantee that the demodulated signal always remains band pass. The worst case is depicted in Figure 3.9(a). As we see, there is a margin of 870 Hz at either side of $f = 0$.

In order to determine the cutoff frequency of the low-pass filter and the sampling interval, we need to look at the opposite worst case, namely when all deviations add up to increase the difference between f_0 and f_d. This worst case is depicted in Figure 3.9(b). As we see, the demodulated signal has bandwidth 49130 Hz in this case. Therefore, a low-pass filter with cutoff frequency $f_c = 50$ kHz is adequate in this case. The sampling interval should be $T = 10$ microseconds or slightly less.

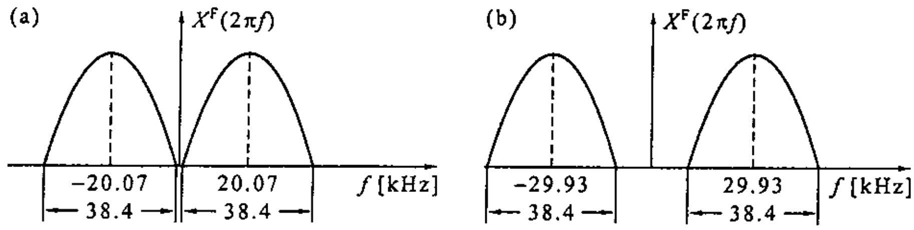

Figure 3.9 Pertaining to Solution 3.44.

3.45

(a) The signal $x(t)$ obtained from $x[n]$ by Shannon's reconstruction formula (3.23) qualifies as a T-parent for any fixed T. The solution is unique, since if $x_1(t)$ is a T-parent of $x[n]$, then $x_1[n] = x[n]$ and, since $x_1(t)$ is band limited to $|\omega| \le \pi/T$, Shannon's reconstruction of $x_1[n]$ must yield $x_1(t)$.

(b)

$$x(t) = \operatorname{sinc}\left(\frac{t+T}{T}\right) + \operatorname{sinc}\left(\frac{t}{T}\right) + \operatorname{sinc}\left(\frac{t-T}{T}\right).$$

(c)

$$x(t) = \sum_{n=n_1}^{n_2} x[n]\operatorname{sinc}\left(\frac{t-nT}{T}\right).$$

(d) No, because $e^{-t/T}$ is not band limited, so it cannot be the T-parent of any discrete-time signal.

Chapter 4

The Discrete Fourier Transform

4.1 It follows by duality to (4.40) that $x[n]$ has conjugate symmetry, that is,

$$x[(N-n) \bmod N] = \bar{x}[n].$$

4.2 If $x[n]$ is real, $X^{\mathrm{d}}[k]$ satisfies the conjugate symmetry property $\bar{X}^{\mathrm{d}}[N-k] = X^{\mathrm{d}}[k]$. But, if $X^{\mathrm{d}}[k]$ is real, then $X^{\mathrm{d}}[N-k] = X^{\mathrm{d}}[k]$. Therefore $X^{\mathrm{d}}[k]$ is symmetric and, by a dual argument, $x[n]$ is symmetric as well, that is, $x[(N-n) \bmod N] = x[n]$.

If $X^{\mathrm{d}}[k]$ is purely imaginary, then $jX^{\mathrm{d}}[k]$ is real. Therefore its inverse DFT, which is $jx[n]$, satisfies the conjugate symmetry property, that is,

$$\overline{jx[(N-n) \bmod N]} = jx[n].$$

From this we get, using the fact that $x[n]$ is real,

$$x[(N-n) \bmod N] = -x[n],$$

so $x[n]$ is antisymmetric.

4.3

(a) Since

$$X^{\mathrm{d}}[k] = \bar{X}^{\mathrm{d}}[N-k],$$

the signal $x[n]$ is real.

(b)

$$\begin{aligned} x[n] &= \frac{0.5}{N}(W_{4M}^{Mn} + W_{4M}^{3Mn}) = \frac{0.5}{N}(W_{4M}^{Mn} + W_{4M}^{-Mn}) \\ &= \frac{0.5}{N}\left[\exp\left(j\frac{2\pi Mn}{4M}\right) + \exp\left(-j\frac{2\pi Mn}{4M}\right)\right] = \frac{1}{N}\cos(0.5\pi n). \end{aligned}$$

(c)

$$y[n] = \frac{1}{N}(-1)^n \cos(\pi n) = \frac{1}{N}(-1)^n(-1)^n = \frac{1}{N}.$$

Therefore,

$$Y^{\mathrm{d}}[k] = \frac{1}{N}\sum_{n=0}^{2M-1} W_{2M}^{-kn} = 0.5\delta[k].$$

4.4 We have, by the shift property of the DFT,

$$Y^{\mathrm{d}}[k] = X^{\mathrm{d}}[k] + W_{4N}^{-kN}X^{\mathrm{d}}[k] = [1 + (-j)^k]X^{\mathrm{d}}[k].$$

Since the signal is real,

$$Y^{\mathrm{d}}[4N-k] = \bar{Y}^{\mathrm{d}}[k].$$

Therefore,

$$Z^{\mathrm{d}}[k] = [1 + (-j)^k]X^{\mathrm{d}}[k] + [1 + (j)^k]\bar{X}^{\mathrm{d}}[k] = \begin{cases} 4\Re\{X^{\mathrm{d}}[k]\}, & k \bmod 4 = 0, \\ 2\Re\{X^{\mathrm{d}}[k]\} + 2\Im\{X^{\mathrm{d}}[k]\}, & k \bmod 4 = 1, \\ 0, & k \bmod 4 = 2, \\ 2\Re\{X^{\mathrm{d}}[k]\} - 2\Im\{X^{\mathrm{d}}[k]\}, & k \bmod 4 = 3. \end{cases}$$

4.5 The frequency of the discrete-time signal is 0.25π. A 12-point DFT samples the frequency axis at points $\{2\pi k/12,\ 0 \le k \le 11\}$. The frequency 0.25π is exactly in the middle between $k = 1$ and $k = 2$. Therefore, $|X^{\mathrm{d}}[k]|$ will be maximum for $k = 1, 2$, as well as for $k = 10, 11$.

4.6

(a) Compute the conjugate of $Y^{\mathrm{d}}[k]$:

$$\bar{Y}^{\mathrm{d}}[k] = W_N^{0.5k}\bar{X}^{\mathrm{d}}[k] = \sum_{n=0}^{N-1} x[n]W_N^{(n+0.5)k} = \sum_{m=0}^{N-1} x[N-m-1]W_N^{(N-m-0.5)k} = \sum_{m=0}^{N-1} x[m]W_N^{-(m+0.5)k} = Y^{\mathrm{d}}[k].$$

Therefore $Y^{\mathrm{d}}[k]$ is real.

(b)

$$Y^{\mathrm{d}}[N-k] = \sum_{n=0}^{N-1} x[n]W_N^{-(n+0.5)(N-k)} = \sum_{n=0}^{N-1} x[n]W_N^{-(n+0.5)N}W_N^{(n+0.5)k} = -\bar{Y}^{\mathrm{d}}[k] = -Y^{\mathrm{d}}[k].$$

4.7 We have

$$X^{\mathrm{f}}(\theta) = \sum_{n=0}^{\infty} x[n]e^{-j\theta n}.$$

So,

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{f}}\left(\frac{2\pi k}{N}\right)\exp\left(j\frac{2\pi kn}{N}\right) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{\infty} x[m]\exp\left(j\frac{2\pi k(n-m)}{N}\right) = \sum_{m=0}^{\infty} x[m]\sum_{k=0}^{N-1} W_N^{k(n-m)}$$

$$= \sum_{m=0}^{\infty} x[m]\delta[(n-m) \bmod N] = \sum_{l=0}^{\infty} x[n+lN] = \sum_{l=0}^{\infty} \alpha^{n+lN} = \alpha^n\sum_{l=0}^{\infty}(\alpha^N)^l = \frac{\alpha^n}{1-\alpha^N}.$$

4.8

$$\begin{aligned} Y^{\mathrm{d}}[k] &= \sum_{n=0}^{MN-1} y[n]W_{MN}^{-kn} = \sum_{m=0}^{M-1}\sum_{l=0}^{N-1} y[Nm+l]W_{MN}^{-k(Nm+l)} \\ &= \sum_{l=0}^{N-1} x[l]W_{NM}^{-kl}\sum_{m=0}^{M-1} W_M^{-km} = M\sum_{l=0}^{N-1} x[l]W_{NM}^{-kl}\delta[k \bmod M]. \end{aligned}$$

Therefore,

$$Y^{\mathrm{d}}[k] = \begin{cases} MX^{\mathrm{d}}[k/M], & k \bmod M = 0, \\ 0, & \text{otherwise.} \end{cases}$$

4.9 The claim is right. A discrete-time periodic signal $x[n]$ with period N can be expressed by its inverse DFT formula

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k]W_N^{nk} = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k]\left[\cos\left(\frac{2\pi kn}{N}\right) + j\sin\left(\frac{2\pi kn}{N}\right)\right],$$

which is a finite sum of discrete-time sinusoidal signals.

4.10

$$Y^{\mathrm{d}}[k] = \sum_{n=0}^{3N-1} y[n]W_{3N}^{-nk} = \sum_{i=0}^{N-1} x_1[i]W_{3N}^{-3ik} + \sum_{i=0}^{N-1} x_2[i]W_{3N}^{-(3i+2)k} = X_1^{\mathrm{d}}[k \bmod N] + W_{3N}^{-2k}X_2^{\mathrm{d}}[k \bmod N].$$

4.11

$$Y_1^{\mathrm{d}}[k] = \sum_{n=0}^{N-1} x_1[n] W_{2N}^{-nk}.$$

$$Y_2^{\mathrm{d}}[k] = \sum_{n=N}^{2N-1} x_2[n-N] W_{2N}^{-nk} = -\sum_{n=0}^{N-1} x_2[n] W_{2N}^{-nk}.$$

Therefore,

$$Y_1^{\mathrm{d}}[2k] - Y_2^{\mathrm{d}}[2k] = \sum_{n=0}^{N-1} (x_1[n] + x_2[n]) W_{2N}^{-2nk} = Z^{\mathrm{d}}[k].$$

4.12 The information is insufficient for conclusion. Since we are not given the sequence of phase angles $\{\measuredangle X^{\mathrm{d}}[k],\ 0 \le k \le 5\}$, they can be completely arbitrary. Accordingly, $x[n]$ can be real, imaginary, or complex.

4.13

(a)

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} (-1)^k W_N^{nk} = \frac{1}{N} \sum_{k=0}^{N-1} (-W_N^n)^k = \delta[n - 0.5N].$$

(b) Express $Y^{\mathrm{d}}[k]$ as

$$Y^{\mathrm{d}}[k] = 0.5(1 + (-1)^k) X^{\mathrm{d}}[k] = H^{\mathrm{d}}[k] X^{\mathrm{d}}[k].$$

Then, by part (a),

$$h[n] = 0.5(1 + \delta[n - 0.5N]),$$

and

$$y[n] = \{h \circledast x\}[n] = 0.5x[n] + 0.5x[(n - 0.5N) \bmod N].$$

(c) Since circular convolution is equal to linear convolution of the periodic extensions over one period, we get from part b,

$$\tilde{y}[n] = \{h * \tilde{x}\}[n],$$

so $\{h[n],\ 0 \le n \le N-1\}$ is the impulse response of the filter in question.

(d) Since $x[n]$ is real, $X^{\mathrm{d}}[k]$ satisfies the conjugate symmetry property

$$\bar{X}^{\mathrm{d}}[N-k] = X^{\mathrm{d}}[k].$$

Therefore,

$$\bar{Y}^{\mathrm{d}}[N-k] = 0.5(1 + (-1)^{N-k}) \bar{X}^{\mathrm{d}}[N-k] = 0.5(1 - (-1)^k) X^{\mathrm{d}}[k] \ne Y^{\mathrm{d}}[k],$$

unless $X^{\mathrm{d}}[k] \equiv 0$. Therefore, $Y^{\mathrm{d}}[k]$ is not real valued in general when N is odd.

4.14 Express the inverse DFT as

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X^{\mathrm{d}}[k] W_N^{nk} = \frac{1}{N} \sum_{k=0}^{N-1} X^{\mathrm{d}}[k] W_N^{-n(N-k)} = \frac{1}{N} \sum_{k=0}^{N-1} X^{\mathrm{d}}[(N-k) \bmod N] W_N^{-kn}.$$

Therefore the solution is to perform direct DFT on the sequence

$$\{X^{\mathrm{d}}[0], X^{\mathrm{d}}[N-1], \ldots, X^{\mathrm{d}}[1]\},$$

and divide the result by N.

4.15

(a)

$$y[n] = \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x[m] W_N^{-km} W_N^{-nk} = \sum_{m=0}^{N-1} x[m] \left[\sum_{k=0}^{N-1} W_N^{-k(m+n)} \right] = N \sum_{m=0}^{N-1} x[m] \delta[n+m] = N x[(N-n) \bmod N].$$

(b) It is clear that repeating part a twice will result in $w[n] = N^2 x[n]$. Therefore $P = 4$ and $A = N^2$.

4.16

(a) It was not said that the signal is band limited. Therefore, the sampled signal may be aliased, in which case unambiguous reconstruction is impossible.

(b) If the signal is band limited to ±2 Hz, so it includes only the harmonics $0, \pm 1, \pm 2$, we can reconstruct the signal unambiguously. In this case

$$X^{\mathrm{s}}[k] = 0.2\{1, -j, 2, j, 1\},$$

so

$$\begin{aligned} x(t) &= 0.4 + 0.2j[e^{j2\pi t} - e^{-j2\pi t}] + 0.2[e^{j4\pi t} + e^{-j4\pi t}] \\ &= 0.4[1 - \sin(2\pi t) + \cos(4\pi t)]. \end{aligned}$$

4.17

(a) We have

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k] W_N^{nk},$$

and

$$X^{\mathrm{f}}(\theta) = \sum_{n=0}^{N-1} x[n] e^{-j\theta n}.$$

Therefore,

$$X^{\mathrm{f}}(\theta) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k] W_N^{nk} e^{-jn\theta} = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k] \sum_{n=0}^{N-1} e^{-jn(\theta - 2\pi k/N)} = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k] \frac{1 - e^{-jN\theta}}{1 - e^{-j(\theta - 2\pi k/N)}}.$$

(b) Program 4.1 implements the computation.

Program 4.1 Fourier transform at arbitrary frequencies from the DFT.

```
function Xf = dft2four(Xd,theta);
% Synopsis: Xf = dft2four(Xd,theta).
% Computes the Fourier transform of a finite-duration sequence
% directly from the DFT.
% Input parameters:
% Xd: the vector of DFT values
% theta: the vector of frequencies at which the Fourier transform
%        is to be computed.
% Output:
% Xf: the result.

N = length(Xd); M = length(theta);
Xd = reshape(Xd,N,1); theta = reshape(theta,1,M);
Xf = (1/N)*sum((Xd*ones(1,M)).*(ones(N,1)*(1-exp(-j*N*theta))./ ...
     (1-exp(-j*((2*pi/N)*(0:N-1)'*ones(1,M)-ones(N,1)*theta))));
```

4.18 From the definition of $X^{\mathrm{e}}[k]$ we get

$$X^{\mathrm{e}}[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{\pi(2k+1)n}{N}\right) = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{\pi n}{N}\right) W_N^{-nk} = \mathrm{DFT}_N\{\hat{x}[n]\},$$

where

$$\hat{x}[n] = x[n]\exp\left(-j\frac{\pi n}{N}\right).$$

Therefore,

$$\hat{z}[n] = \{\hat{x} \circledast \hat{y}\}[n],$$

or

$$z[n]\exp\left(-j\frac{\pi n}{N}\right) = \sum_{m=0}^{N-1} x[m]\exp\left(-j\frac{\pi m}{N}\right\} y[(n-m) \bmod N]\exp\left\{-j\frac{\pi((n-m) \bmod N)}{N}\right)$$

$$= \sum_{m=0}^{n} x[m]\exp\left(-j\frac{\pi m}{N}\right) y[n-m]\exp\left(-j\frac{\pi(n-m)}{N}\right)$$

$$+ \sum_{m=n+1}^{N-1} x[m]\exp\left(-j\frac{\pi m}{N}\right) y[N+n-m]\exp\left(-j\frac{\pi(n-m+N)}{N}\right)$$

$$= \exp\left(-j\frac{\pi n}{N}\right)\sum_{m=0}^{n} x[m]y[n-m] + \exp\left(-j\frac{\pi n}{N}\right)\sum_{m=n+1}^{N-1}(-1)x[m]y[N+n-m].$$

Finally, cancellation of $e^{-j\pi n/N}$ on both sides gives

$$z[n] = \sum_{m=0}^{n} x[m]y[n-m] - \sum_{m=n+1}^{N-1} x[m]y[N+n-m].$$

4.19

(a)

$$X^{g}[k] = W_N^{-0.5k}\sum_{n=0}^{N-1} x[n]W_N^{-kn} = W_{2N}^{-k}X^{d}[k].$$

(b)

$$Y^{g}[k] = \sum_{n=0}^{N-1} x[n]W_{2N}^{-k(n+0.5)} + \sum_{n=N}^{2N-1} x[2N-n-1]W_{2N}^{-k(n+0.5)} = \sum_{n=0}^{N-1} x[n](W_{2N}^{-k(n+0.5)} + W_{2N}^{-k(2N-n-0.5)})$$

$$= \sum_{n=0}^{N-1} x[n](W_{2N}^{-k(n+0.5)} + W_{2N}^{k(n+0.5)}).$$

Assume that k is even and let $k = 2l$. Then

$$Y^{g}[2l] = \sum_{n=0}^{N-1} x[n](W_{2N}^{-2l(n+0.5)} + W_{2N}^{2l(n+0.5)}) = W_{2N}^{-l}X^{d}[l] + W_{2N}^{l}\bar{X}^{d}[l] = 2\Re\{W_{2N}^{-l}X^{d}[l]\}.$$

The result is always real valued.

4.20 We have

$$X_a^{d}[2l+1] = \sum_{n=0}^{N-1} x[n]W_{2N}^{-n(2l+1)} = \sum_{n=0}^{N-1} x[n]W_{2N}^{-n}W_N^{-nl}.$$

Therefore, the inverse DFT gives the sequence $x[n]W_{2N}^{-n}$.

4.21

(a) Let M be an integer multiple of N, say $M = NI$. Then

$$X^{d}[k] = \sum_{n=0}^{N-1} x[n]W_N^{-nk} = \sum_{n=0}^{N-1} x[n]W_M^{-nkI} = X_a^{d}[kI].$$

In this case, all points of $X^{d}[k]$ appear as points of $X_a^{d}[l]$, for $l = kI$, $0 \le k \le N-1$.

(b) The equality $X^{\mathrm{d}}[k] = X_{\mathrm{a}}^{\mathrm{d}}[l]$ implies

$$\sum_{n=0}^{N-1} x[n](W_N^{-nk} - W_M^{-nl}) = 0.$$

Since $x[n]$ is a general sequence, this holds if and only if $W_N^{-nk} = W_M^{-nl}$ for all n. Equivalently,

$$\exp\left[j2\pi n\left(\frac{k}{N} - \frac{l}{M}\right)\right] = 1, \quad \text{for all } n.$$

Let I be the greatest common divisor of N and M, so

$$N = N'I, \quad M = M'I.$$

Then

$$\exp\left(j2\pi n\frac{kM' - lN'}{N'M'I}\right) = 1, \quad \text{for all } n.$$

The solutions of this equation are

$$k = N'i, \quad l = M'i, \quad 0 \le i \le I-1.$$

When N and M are coprime (that is, $I = 1$), the only solution is $k = l = 0$. When they are not, there is a total of I different solutions.

4.22

(a) Define $L = \lceil N/M \rceil$ and zero-pad $x[n]$ to a length LM. Then,

$$\begin{aligned} X^{\mathrm{f}}(\theta_k) &= \sum_{n=0}^{N-1} x[n]\exp\left(-j\frac{2\pi nk}{M}\right) = \sum_{l=0}^{L-1}\sum_{m=0}^{M-1} x[Ml+m]\exp\left(-j\frac{2\pi(Ml+m)k}{M}\right) \\ &= \sum_{m=0}^{M-1}\left(\sum_{l=0}^{L-1} x[Ml+m]\right) W_M^{-mk}. \end{aligned}$$

The procedure is therefore:

i. Fold $x[n]$ on itself to a length M by forming

$$y[m] = \sum_{l=0}^{L-1} x[Ml+m], \quad 0 \le m \le M-1.$$

ii. Compute the length-M DFT of $y[m]$.

(b) We have in this case

$$\begin{aligned} y[0] &= x[0] + x[5] + x[10] + x[15], \\ y[1] &= x[1] + x[6] + x[11] + x[16], \\ y[2] &= x[2] + x[7] + x[12] + x[17], \\ y[3] &= x[3] + x[8] + x[13], \\ y[4] &= x[4] + x[9] + x[14]. \end{aligned}$$

Note that zero padding of $x[n]$ is needed only for the concise writing of the procedure, but not for the actual computation.

4.23

(a) The DFT frequencies in the θ domain are

$$\theta_k = \frac{2\pi k}{100}.$$

The corresponding physical frequencies are

$$f_k = \frac{k}{100T} = 12.8k.$$

Substituting $f_k = 305\,\mathrm{Hz}$ gives $k = 23.83$. The nearest integer is $k = 24$. Correspondingly, $f_k = 307.2\,\mathrm{Hz}$.

(b) We look for k and N such that

$$\frac{k}{TN} = 305.$$

Therefore,

$$\frac{k}{N} = 305T = \frac{305}{1280} = \frac{61}{256}.$$

Therefore, we have to pad the signal by 156 zeros to get $N = 256$. Then the element $X^{\mathrm{d}}[61]$ of the DFT will give the Fourier transform at $f = 305$ Hz.

4.24 Pierre's zero-padded DFT is

$$\sum_{n=0}^{N-1} x[n]W_{NL}^{-nLk} = \sum_{n=0}^{N-1} x[n]W_N^{-nk} = X^{\mathrm{d}}[k].$$

This is just the usual DFT of $x[n]$, extended periodically L times.

4.25 Let

$$z[n] = \{x * y\}[n].$$

The nth element of the circular convolution $x \circledast y$ is given by

$$\begin{aligned}\{x \circledast y\}[n] &= \sum_{k=0}^{N-1} x[k]y[(n-k) \bmod N] = \sum_{k=0}^{n} x[k]y[n-k] + \sum_{k=n+1}^{N-1} x[k]y[N+n-k] \\ &= z[n] + \begin{cases} z[n+N], & n < N-1, \\ 0, & n = N-1. \end{cases}\end{aligned}$$

This is precisely what Program 4.2 computes.

4.26 The DFTs of x and w are

$$X^{\mathrm{d}} = \{4, 8j, 0, -8j\},$$
$$W^{\mathrm{d}} = \{-12, 16+4j, 8, 16-4j\}.$$

We are looking for a sequence y such that $W^{\mathrm{d}}[k] = X^{\mathrm{d}}[k]Y^{\mathrm{d}}[k]$. But this is impossible, since $X^{\mathrm{d}}[2] = 0$, whereas $W^{\mathrm{d}}[2] \neq 0$.

4.27 The length of the sequence y is

$$4 + 1 - 2 = 3.$$

The DFTs of x and w are

$$X^{\mathrm{d}} = \{1, 2+j, 3, 2-j\},$$
$$W^{\mathrm{d}} = \{8, -1-3j, 18, -1+3j\}.$$

We get the DFT of y by division (component by component)

$$Y^{\mathrm{d}} = \{8, -1-j, 6, -1+j\}.$$

Taking the inverse DFT gives

$$y = \{3, 1, 4\}.$$

4.28 Commutativity is obvious, since if we substitute $k = (n-m) \bmod N$, we will have $m = (n-k) \bmod N$. For associativity:

$$\{x \circledast (y \circledast z)\}[n] = \sum_{k=0}^{N-1} x[k] \sum_{m=0}^{N-1} y[m]z[(n-k-m) \bmod N] = \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x[k]y[m]z[(n-k-m) \bmod N],$$

and

$$\{(x \circledast y) \circledast z\}[n] = \sum_{m=0}^{N-1} z[(n-k) \bmod N] \sum_{k=0}^{N-1} x[k]y[(m-k) \bmod N]$$
$$= \sum_{k=0}^{N-1} x[k] \sum_{l=0}^{N-1} x[k]y[l]z[(n-k-l) \bmod N],$$

so the two are equal.

4.29 Let us compute the circular convolution separately for three ranges of n:

(a) For $0 \le n \le 7$:

$$y[n] = \sum_{m=8}^{59} x_1[m]x_2[n+75-m] = z[n+75].$$

So, we have $z[n'] = y[n' \bmod 75]$ for $75 \le n' \le 82$.

(b) For $8 \le n \le 59$:

$$y[n] = \sum_{m=8}^{n} x_1[m]x_2[n-m] + \sum_{m=n+1}^{59} x_1[m]x_2[n+75-m].$$

We do not have equality for this range of n.

(c) For $60 \le n \le 74$:

$$y[n] = \sum_{m=8}^{59} x_1[m]x_2[n-m] = z[n].$$

So, we have $z[n'] = y[n' \bmod 75]$ for $60 \le n' \le 75$.

In summary, equality holds for $60 \le n \le 82$.

4.30 We have

$$X^{\mathrm{d}}[k] = \{2, 1+W_3^2, 1+W_3\} = \{2, -W_3, -W_3^2\}.$$

Therefore, since $Y^{\mathrm{d}}[k] = Z^{\mathrm{d}}[k]/X^{\mathrm{d}}[k]$,

$$Y^{\mathrm{d}}[k] = \{1, W_3, W_3^2\}.$$

Therefore

$$y[n] = \{0, 0, 1\}.$$

4.31 The first $N_1 + N_2 - 1$ terms of the result will hold the correct linear convolution and the remaining terms will be zero.

4.32 Augment each sequence to length $N_1 + N_2 + N_3 - 2$ by zero padding, compute the DFTs of the augmented sequences, multiply the DFTs point by point, and finally compute the inverse DFT of the product.

4.33 Performing the DFT operation on the circular convolution and using the circular shift property gives

$$Z^{\mathrm{d}}[k] = z[n] = \sum_{m=0}^{N-1} W_N^{mk} X^{\mathrm{d}}[k]\bar{y}[m] = X^{\mathrm{d}}[k] \sum_{m=0}^{N-1} \bar{y}[m]W_N^{mk} = X^{\mathrm{d}}[k]\bar{Y}^{\mathrm{d}}[k].$$

4.34 The shifted sequence is

$$x[n] = (n-N)^3 + 1, \quad 0 \le n \le 2N.$$

Therefore,

$$X^{\mathrm{d}}[0] = \sum_{n=0}^{2N} x[n] = 2N + 1.$$

Note that $\sum_{n=0}^{2N}(n-N)^3 = 0$, because of antisymmetry.

4.35

(a) When f varies from 1000 Hz to 1600 Hz, the bandwidth of the signal remains within the limits 1000 to 2000 Hz. Therefore, using the rules of pass-band signal sampling, the minimal sampling rate is 2000 Hz.

(b) Since the ratio between the frequencies of the two components is rational, the signal is periodic. Its period is the least common multiple of the two periods. Since the periods are $1/f$ and $0.8/f$, the period of the sum is $4/f$, so the frequency is $f/4$.

(c) The frequency of the signal is now 300 Hz and the two components are the 4th and 5th harmonics. It is convenient to sample at an integer number of samples per period, since then the Fourier coefficients are proportional to the DFT values of the samples. Since the highest harmonic is the 5th, we need $N = 11$ samples per period, so the sampling interval is $T = 1/3300$ second. The sampled signal is

$$x(nT) = a_1 \cos\left(2\pi\frac{4n}{11} + \phi_1\right) + a_2 \cos\left(2\pi\frac{5n}{11} + \phi_2\right).$$

The DFT values are

$$X^{\mathrm{d}}[k] = \begin{cases} 0.5Na_1 e^{j\phi_1}, & k = 4, \\ 0.5Na_2 e^{j\phi_2}, & k = 5, \\ 0.5Na_2 e^{-j\phi_2}, & k = 6, \\ 0.5Na_1 e^{-j\phi_1}, & k = 7, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$a_1 = (2/N)|X^{\mathrm{d}}[4]|,\ a_2 = (2/N)|X^{\mathrm{d}}[5]|,\ \phi_1 = \measuredangle X^{\mathrm{d}}[4],\ \phi_2 = \measuredangle X^{\mathrm{d}}[5].$$

(d) We have

$$\begin{aligned} y(t) &= [a_1\cos(2\pi\cdot 1200t + \phi_1) + a_2\cos(2\pi\cdot 1500t + \phi_2)]\cos(2\pi\cdot 1000t) \\ &= 0.5a_1\cos(2\pi\cdot 2200t + \phi_1) + 0.5a_1\cos(2\pi\cdot 200t + \phi_1) \\ &\quad + 0.5a_2\cos(2\pi\cdot 2500t + \phi_2) + 0.5a_2\cos(2\pi\cdot 500t + \phi_2). \end{aligned}$$

The output of the low-pass filter is

$$z(t) = 0.5a_1\cos(2\pi\cdot 200t + \phi_1) + 0.5a_2\cos(2\pi\cdot 500t + \phi_2).$$

Now the signal frequency is 100 Hz, and it consists of the 2nd and 5th harmonics. The number of samples is again $N = 11$, but the sampling rate is $T = 1/1100$ second. Similarly to part c, we get

$$a_1 = (4/N)|Z^{\mathrm{d}}[2]|,\ a_2 = (4/N)|Z^{\mathrm{d}}[5]|,\ \phi_1 = \measuredangle Z^{\mathrm{d}}[2],\ \phi_2 = \measuredangle Z^{\mathrm{d}}[5].$$

4.36

$$X^{\mathrm{d}}[k] = 2 + 2W_N^{kL} + 2W_N^{k(N-L)}.$$

Therefore,

$$x[n] = \frac{2}{N}\{\delta[n] + \delta[n-L] + \delta[n-N+L]\}.$$

Either $L = N/3$ or $L = 2N/3$ gives periodicity, provided N is divisible by 3. The period is $N/3$ in either case.

4.37

$$F_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_5^{-1} & W_5^{-2} & W_5^{-3} & W_5^{-4} \\ 1 & W_5^{-2} & W_5^{-4} & W_5^{-1} & W_5^{-3} \\ 1 & W_5^{-3} & W_5^{-1} & W_5^{-4} & W_5^{-2} \\ 1 & W_5^{-4} & W_5^{-3} & W_5^{-2} & W_5^{-1} \end{bmatrix},$$

where

$$W_5^{-1} = \cos(0.4\pi) - j\sin(0.4\pi),\quad W_5^{-2} = \cos(0.8\pi) - j\sin(0.8\pi),$$
$$W_5^{-3} = \cos(0.8\pi) + j\sin(0.8\pi),\quad W_5^{-4} = \cos(0.4\pi) + j\sin(0.4\pi).$$

4.38 Assume that some power of W_N^{-1} appears twice in a certain row, that is,

$$W_N^{-kn} = W_N^{-km}, \quad m \neq n.$$

Then

$$W_N^{k(m-n)} = 1,$$

so

$$k(m-n) \bmod N = 0.$$

But, since N is prime, this is possible only if $k \bmod N = 0$ or $(m-n) \bmod N = 0$. The first case corresponds to the first row. The second case gives a contradiction. Finally, since the sequence of powers of W_N has length N, which is also the length of the rows of F_N, each element of the sequence must appear exactly once.

4.39 The basis vectors of the natural basis are shown in Figure 4.1.

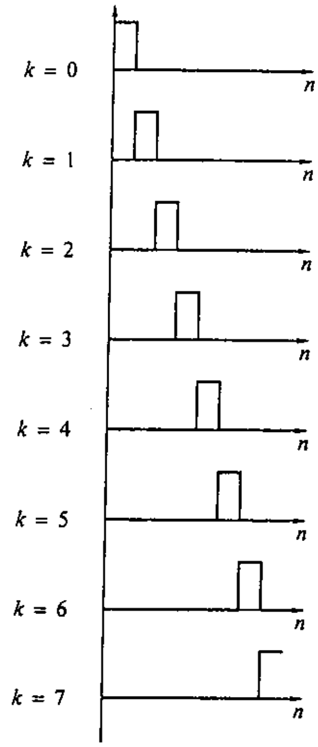

Figure 4.1 Pertaining to Solution 4.39.

4.40

(a) The (k,l)th element of Y is

$$Y_{k,l} = y[(k-l) \bmod N].$$

(b) Let us define

$$S_N = F_N Y \bar{F}'_N.$$

Then the (k,l)th element of S_N is

$$
\begin{aligned}
(S_N)_{k,l} &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}(F_N)_{k,m}Y_{m,n}(\bar{F}_N)_{l,n} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}W_N^{-km}y[(m-n) \bmod N]W_N^{nl} \\
&= \sum_{i=0}^{N-1}\sum_{n=0}^{N-1}W_N^{-k(n+i)}y[i]W_N^{nl} = \sum_{i=0}^{N-1}W_N^{-ki}y[i]\sum_{n=0}^{N-1}W_N^{n(l-k)} \\
&= N\sum_{i=0}^{N-1}W_N^{-ki}y[i]\delta[k-l] = \begin{cases} NY^{\mathrm{d}}[k], & k = l, \\ 0, & k \neq l. \end{cases}
\end{aligned}
$$

(c) The operation (4.53) can be written as

$$z_N = Yx_N.$$

Premultiplying both sides by F_N gives

$$F_N z_N = F_N Y x_N.$$

Now, since $N^{-1}\bar{F}'_N F_N$ is the identity matrix, we can insert it between Y and x on the right side without affecting the result. This gives

$$F_N z_N = N^{-1}F_N Y \bar{F}'_N F_N x_N,$$

as stated in (4.115).

(d) By (4.20), we can express (4.115) as

$$Z_N^{\mathrm{d}} = N^{-1}S_N X_n^{\mathrm{d}}$$

But, by part b, $N^{-1}S_N$ is a diagonal matrix whose diagonal elements are the components of Y_N^{d}. Therefore,

$$Z^{\mathrm{d}}[k] = Y^{\mathrm{d}}[k]X^{\mathrm{d}}[k], \quad 0 \le k \le N-1.$$

This is precisely the convolution property of the DFT.

4.41 Program 4.2 implements the interpolation. For now, ignore the parameter w; see Solution 6.14.

Program 4.2 Signal interpolation by zero padding the DFT.

```
function xi = intfd(x,M,w);
% Synopsis: xi = intfd(x,M,w).
% Signal interpolation by zero padding in the frequency domain,
% with optional windowing.
% Input parameters:
% x: the given sequence of length N
% M: the desired length of the interpolated signal.
% w: an optional window, whose length must N if  N is odd,
%    or N+1 if N is even.
% Output:
% xi: the interpolated signal.

N = length(x);
if (M <= N), error('M must be > N in INTFD'), end

X = reshape(fftshift(fft(x)),1,N);
if (rem(N,2) == 0), X = [0.5*X(1),X(2:N),0.5*X(1)]; end
N = length(X); Nz = M-N; N2 = floor(N/2);
if (nargin == 3), X = X.*reshape(w,1,N); end
Xi = (M/N)*[X(N2+1:N),zeros(1,Nz),X(1:N2)];
xi = ifft(Xi);
if (max(abs(imag(x))) < 2*eps), xi = real(xi); end
```

4.42 When N is even, we must split $X^{d}[N/2]$ evenly between $X_{i}^{d}[N/2]$ and $X_{i}^{d}[M-N/2]$, so we define

$$X_{i}^{d}[k]=\begin{cases}\frac{M}{N}X^{d}[k], & 0\le k\le \frac{N}{2}-1,\\ \frac{M}{2N}X^{d}[k], & k=\frac{N}{2},\\ \frac{M}{2N}X^{d}[k-M+N], & k=M-\frac{N}{2},\\ \frac{M}{N}X^{d}[k-M+N], & M-\frac{N}{2}+1\le k\le M-1,\\ 0, & \text{otherwise},\end{cases} \tag{4.1}$$

4.43 The results are shown in Figure 4.2. As we see, the DCT-II of a cosine signal whose frequency is an integer multiple of π/N is nonzero only at the frequency of the signal, that is, it is free of the ripple present in DCT-I. The DCT-II of a sine signal behaves similarly to DCT-I.

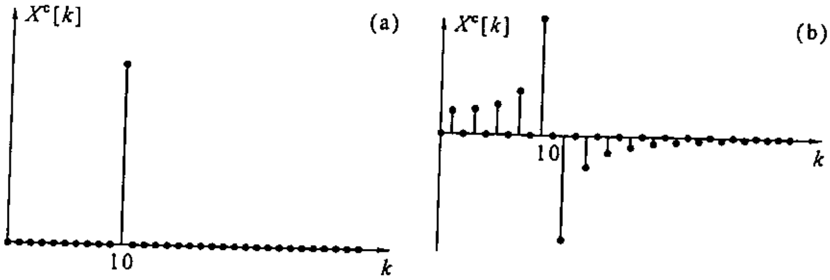

Figure 4.2 Pertaining to Solution 4.43.

4.44

$$\begin{aligned}X_{3}^{d}[k]&=\sqrt{2}x[0]+\sum_{n=1}^{N-1}x[n]W_{2N}^{-(k+0.5)n}-\sum_{n=N+1}^{2N-1}x[2N-n]W_{2N}^{-(k+0.5)n}\\ &=\sqrt{2}x[0]+\sum_{n=1}^{N-1}x[n]W_{2N}^{-(k+0.5)n}+\sum_{n=1}^{N-1}x[n]W_{2N}^{(k+0.5)n}=2\sum_{n=0}^{N-1}b[n]x[n]\cos\left(\frac{\pi n(k+0.5)}{N}\right).\end{aligned}$$

DCT-III is defined as

$$X^{c3}[k]=\frac{1}{\sqrt{2N}}X_{3}^{d}[k].$$

4.45 Program 4.3 computes the four DCT types.

4.46 Program 4.4 computes the four DST types.

4.47 The DFT of $x_5[n]$ is

$$\begin{aligned}X_{5}^{d}[k]&=2^{1/2}x[0]+\sum_{n=1}^{N-1}x[n]W_{2N-1}^{-nk}+\sum_{n=N}^{2N-2}x[2N-n-1]W_{2N-1}^{-nk}=2^{1/2}x[0]+\sum_{n=1}^{N-1}x[n](W_{2N-1}^{-nk}+W_{2N-1}^{nk})\\ &=2^{1/2}x[0]+2\sum_{n=1}^{N-1}x[n]\cos\left(\frac{\pi nk}{N-0.5}\right)=2\sum_{n=0}^{N-1}a[n]x[n]\cos\left(\frac{\pi nk}{N-0.5}\right),\end{aligned}$$

where

$$a[n]=\begin{cases}2^{-1/2}, & n=0,\\ 1, & \text{otherwise}.\end{cases}$$

Since the DFT is real, we have $X_{5}^{d}[k]=X_{5}^{d}[2N-k-1]$. We therefore get, using the same derivation as before,

$$x_{5}[n]=\frac{1}{2N-1}X_{5}^{d}[0]+\frac{2}{2N-1}\sum_{k=1}^{N-1}X_{5}^{d}[k]\cos\left(\frac{\pi nk}{N-0.5}\right)=\frac{2}{2N-1}\sum_{k=0}^{N-1}X_{5}^{d}[k]a^{2}[k]\cos\left(\frac{\pi nk}{N-0.5}\right).$$

Program 4.3 Direct implementation of the DCT.

```
function y = dct(x,typ);
% Synopsis: y = dct(x,typ).
% Discrete cosine transform by direct computation
% Input parameters:
% x: the input vector
% type: the DCT type, 1 through 4
% Output:
% y: the output vector

N = length(x); x = reshape(x,N,1); fac = 1/sqrt(2);
norfac = sqrt(2/N);
if (typ == 1), norfac = sqrt(2/(N-1)); end
y = zeros(1,N);
for k = 0:N-1,
   if (typ == 1),
      row = cos((pi*k/(N-1))*(0:N-1));
      row([1,N]) = fac*row([1,N]);
      if (k == 0 | k == N-1), row = fac*row; end
   elseif (typ == 2),
      row = cos((pi*k/N)*((0:N-1)+0.5));
      if (k == 0), row = fac*row; end
   elseif (typ == 3),
      row = cos((pi*(k+0.5)/N)*(0:N-1));
      row(1) = fac*row(1);
   elseif (typ == 4),
      row = cos((pi*(k+0.5)/N)*((0:N-1)+0.5));
   end
   y(k+1) = norfac*row*x;
end
```

The new DCT is defined as

$$X^{c5}[k] = \frac{1}{\sqrt{2N-1}} a[k] X_5^{d}[k] = \sqrt{\frac{2}{N-0.5}} \sum_{n=0}^{N-1} a[k]a[n]x[n] \cos\left(\frac{\pi nk}{N-0.5}\right).$$

The inverse transform is

$$x[n] = a[n]x_5[n] = \sqrt{\frac{2}{N-0.5}} \sum_{n=0}^{N-1} a[k]a[n]X^{c5}[k] \cos\left(\frac{\pi nk}{N-0.5}\right).$$

The new DCT matrix is

$$[C_N^{\mathrm{V}}]_{k,n} = \sqrt{\frac{2}{N-0.5}} a[k]a[n] \cos\left(\frac{\pi nk}{N-0.5}\right), \quad 0 \le k, n \le N-1.$$

The new DCT is, like DCT-I, real, orthonormal, and symmetric.

4.48

(a) We have by (4.116),

$$\sum_{n=0}^{N-1} \operatorname{cas}\left(\frac{2\pi kn}{N}\right) \operatorname{cas}\left(\frac{2\pi nl}{N}\right) = \sum_{n=0}^{N-1} 2\cos\left(\frac{2\pi kn}{N} - 0.25\pi\right)\cos\left(\frac{2\pi nl}{N} - 0.25\pi\right)$$
$$= \sum_{n=0}^{N-1} \sin\left(\frac{2\pi n(k+l)}{N}\right) + \sum_{n=0}^{N-1} \cos\left(\frac{2\pi n(k-l)}{N}\right) = N\delta[k-l].$$

Program 4.4 Direct implementation of the DST.

```
function y = dst(x,typ);
% Synopsis: y = dst(x,typ).
% Discrete sine transform by direct computation
% Input parameters:
% x: the input vector
% type: the DST type, 1 through 4
% Output:
% y: the output vector

N = length(x); x = reshape(x,N,1); fac = 1/sqrt(2);
norfac = sqrt(2/N);
if (typ == 1), norfac = sqrt(2/(N+1)); end
y = zeros(1,N);
for k = 0:N-1,
   if (typ == 1),
      row = sin((pi*(k+1)/(N+1))*(1:N));
   elseif (typ == 2),
      row = sin((pi*(k+1)/N)*((0:N-1)+0.5));
      if (k == N-1), row = fac*row; end
   elseif (typ == 3),
      row = sin((pi*(k+0.5)/N)*(1:N));
      row(N) = fac*row(N);
   elseif (typ == 4),
      row = sin((pi*(k+0.5)/N)*((0:N-1)+0.5));
   end
   y(k+1) = norfac*row*x;
end
```

(b) We can write the DHT in matrix-vector form

$$X_N^{\mathrm{h}} = H_N x_N.$$

Multiplying both sides by H_N and using the orthogonality and symmetry of this matrix gives

$$H_N X_N^{\mathrm{h}} = N x_N,$$

which is the same as (4.119).

(c) We have

$$X^{\mathrm{h}}[k] = \Re\{X^{\mathrm{d}}[k]\} - \Im\{X^{\mathrm{d}}[k]\} = 0.5(X^{\mathrm{d}}[k] + \bar{X}^{\mathrm{d}}[k]) + 0.5j(X^{\mathrm{d}}[k] - \bar{X}^{\mathrm{d}}[k])$$
$$= 0.5(1+j)X^{\mathrm{d}}[k] + 0.5(1-j)\bar{X}^{\mathrm{d}}[k] = \Re\{(1+j)X^{\mathrm{d}}[k]\}.$$

(d) We have

$$\Re\{X^{\mathrm{d}}[k]\} = \sum_{n=0}^{N-1} x[n]\cos\left(\frac{2\pi nk}{N}\right) = 0.5(X^{\mathrm{h}}[k] + X^{\mathrm{h}}[(N-k) \bmod N]),$$
$$\Im\{X^{\mathrm{d}}[k]\} = -\sum_{n=0}^{N-1} x[n]\sin\left(\frac{2\pi nk}{N}\right) = 0.5(-X^{\mathrm{h}}[k] + X^{\mathrm{h}}[(N-k) \bmod N]).$$

Therefore,

$$X^{\mathrm{d}}[k] = 0.5(1-j)X^{\mathrm{h}}[k] + 0.5(1+j)X^{\mathrm{h}}[(N-k) \bmod N] = \Re\{(1-j)X^{\mathrm{h}}[k]\}.$$

4.49

(a) We have from Solution 4.48,

$$
\begin{aligned}
Y^{\mathrm{h}}[k] &= \Re\{(1+j)Y^{\mathrm{d}}[k]\} = \Re\{(1+j)W_N^{-mk}X^{\mathrm{h}}[k]\}\\
&= \Re\{0.5(1+j)(1-j)W_N^{-mk}X^{\mathrm{h}}[k] + 0.5(1+j)^2 W_N^{-mk}X^{\mathrm{h}}[(N-k) \bmod N]\}\\
&= X^{\mathrm{h}}[k]\cos\left(\frac{2\pi mk}{N}\right) + X^{\mathrm{h}}[(N-k) \bmod N]\sin\left(\frac{2\pi mk}{N}\right).
\end{aligned}
$$

(b) Let

$$
z[n] = \sum_{m=0}^{N-1} x[m]y[(n-m) \bmod N].
$$

Then, by part a,

$$
\begin{aligned}
Z^{\mathrm{h}}[k] &= \sum_{m=0}^{N-1} x[m]\left\{Y^{\mathrm{h}}[k]\cos\left(\frac{2\pi mk}{N}\right) + Y^{\mathrm{h}}[(N-k) \bmod N]\sin\left(\frac{2\pi mk}{N}\right)\right\}\\
&= Y^{\mathrm{h}}[k]\sum_{m=0}^{N-1} x[m]\cos\left(\frac{2\pi mk}{N}\right) + Y^{\mathrm{h}}[(N-k) \bmod N]\sum_{m=0}^{N-1} x[m]\sin\left(\frac{2\pi mk}{N}\right)\\
&= 0.5Y^{\mathrm{h}}[k](X^{\mathrm{h}}[k] + X^{\mathrm{h}}[(N-k) \bmod N])\\
&\quad + 0.5Y^{\mathrm{h}}[(N-k) \bmod N](X^{\mathrm{h}}[k] - X^{\mathrm{h}}[(N-k) \bmod N]).
\end{aligned}
$$

As we see, $Z^{\mathrm{h}}[k]$ can be computed with three real multiplications and three real additions for each k. If we discount the multiplication by 0.5, which can be performed by right shift, there are only two real multiplications. By comparison, $Z^{\mathrm{d}}[k]$ can be computed from $X^{\mathrm{d}}[k]Y^{\mathrm{d}}[k]$ with four real multiplications and two real additions.

Chapter 5

The Fast Fourier Transform

5.1 We have

$$W_N^N = \exp\left(j\frac{2\pi N}{N}\right) = 1, \quad W_N^Q = \exp\left(j\frac{2\pi Q}{PQ}\right) = \exp\left(j\frac{2\pi}{P}\right) = W_P, \quad W_N^P = \exp\left(j\frac{2\pi P}{PQ}\right) = \exp\left(j\frac{2\pi}{Q}\right) = W_Q.$$

5.2

```
q = floor(n/P); p = n-P*q;
```

5.3

$$\mathcal{A}_r(N) = 3.5N\log_2 N - 3N + 3,$$
$$\mathcal{M}_r(N) = 1.5N(\log_2 N - 2) + 3.$$

5.4 The verification is straightforward and is left to the reader.

5.5 The order of the indices is: 0, 9, 18, 3, 12, 21, 6, 15, 24, 1, 10, 19, 4, 13, 22, 7, 16, 25, 2, 11, 20, 5, 14, 23, 8, 17, 26.

5.6 Program 5.1 implements the bit-reversal procedure.

Program 5.1 Bit reversal.

```
function n = bitrev(r);
% Synopsis:  n = bitrev(r).
% Performs bit reversal.
% Input:
% r: the number of bits.
% Output:
% n: the numbers [0..2^r-1] in a bit-reversed order.

if (r < 1),
   error('Input to BITREV must be greater than 0');
elseif (r == 1),
   n = [0,1];
else,
   temp = 2*bitrev(r-1);
   n = [temp, temp+1];
end
```

5.7

$$\mathcal{A}_c(N) = (N/p)p(p-1) + p\mathcal{A}_c(N/p) = p\mathcal{A}_c(N/p) + N(p-1),$$
$$\mathcal{M}_c(N) = (N/p)(p-1)^2 + p\mathcal{M}_c(N/p) + (p-1)((N/p)-1) = p\mathcal{M}_c(N/p) + (N-1)(p-1).$$

Let us guess for $\mathcal{A}_c(N)$ a solution

$$\mathcal{A}_c(N) = aN\log_p N + bN + c,$$

where a, b, c are constants to be determined. With this guess,

$$\mathcal{A}_c(N/p) = a(N/p)(\log_p N - 1) + b(N/p) + c,$$

so

$$p\mathcal{A}_c(N/p) + N(p-1) = aN(\log_p N - 1) + bN + cp + N(p-1) = aN\log_p N + (b - a + p - 1)N + cp.$$

We therefore get that $c = 0$ and $b = b - a + p - 1$. This gives $a = p - 1$ and leaves b unspecified. However, substituting $N = p$ and requiring that $\mathcal{A}_c(p) = (p-1)p$ gives $b = 0$. In summary,

$$\mathcal{A}_c(N) = (p-1)N\log_p N.$$

Next we guess for $\mathcal{M}_c(N)$ a solution

$$\mathcal{M}_c(N) = aN\log_p N + bN + c,$$

where a, b, c are constants to be determined. With this guess,

$$\mathcal{M}_c(N/p) = a(N/p)(\log_p N - 1) + b(N/p) + c,$$

so

$$p\mathcal{M}_c(N/p) + (N-1)(p-1) = aN(\log_p N - 1) + bN + cp + (N-1)(p-1) = aN\log_p N + (b-a+p-1)N + cp - p + 1.$$

We therefore get $a = p - 1$, $c = 1$, and b is unspecified. However, substituting $N = p$ and requiring that $\mathcal{M}_c(p) = (p-1)^2$ gives $b = -1$. In summary,

$$\mathcal{M}_c(N) = (p-1)N\log_p N - N + 1.$$

5.8

(a)

$$\mathcal{A}_c = N(2r+1), \quad \mathcal{M}_c = 0.5N(2r-1) + 1.$$

(b)

$$\mathcal{A}_c = N(2r+1), \quad \mathcal{M}_c = 0.5N(1.5r-1) + 1.$$

(c)

$$\mathcal{A}_c = 2N(2r+2), \quad \mathcal{M}_c = 0.5N(3r-1) + 1.$$

Therefore, method (b) is the most efficient.

5.9

(a)

$$\mathcal{M}_c(3\times 4^r) = 3\mathcal{M}_c(4^r) + 4^r\mathcal{M}_c(3) + 2(4^r - 1) = 3(0.75r4^r - 4^r + 1) + 4\times 4^r + 2(4^r - 1) = (2.25r + 3)4^r + 1.$$

(b)

$$\mathcal{M}_c(4^{r+1}) = 0.75(r+1)4^{r+1} - 4^{r+1} + 1 = (3r-1)4^r + 1.$$

The first scheme is more efficient than the second if $3r - 1 > 2.25r + 3$, that is, if $r > 5$. If $r \le 5$, the second scheme is more efficient.

5.10

(a) We have $240 = 3 \times 5 \times 2^4$, so

$$\mathcal{M}_c(3) = 4, \quad \mathcal{M}_c(5) = 16, \quad \mathcal{M}_c(2^4) = 17,$$

$$\mathcal{A}_c(3) = 6, \quad \mathcal{A}_c(5) = 20, \quad \mathcal{A}_c(2^4) = 64,$$

$$\mathcal{M}_c(48) = 3 \times 17 + 16 \times 4 + 2 \times 15 = 145, \quad \mathcal{A}_c(48) = 3 \times 64 + 16 \times 6 = 288,$$

$$\mathcal{M}_c(240) = 5 \times 145 + 48 \times 16 + 4 \times 47 = 1681, \quad \mathcal{A}_c(240) = 5 \times 288 + 48 \times 20 = 2400.$$

(b)

$$\mathcal{M}_c(256) = 128 \times 6 + 1 = 769, \quad \mathcal{A}_c(256) = 128 \times 8 = 1024.$$

The zero-padding scheme is more efficient in this case.

5.11

(a) We have $24 = 3 \times 2^3$, so

$$\mathcal{M}_c(24) = 3\mathcal{M}_c(8) + 8\mathcal{M}_c(3) + 14 = 3 \times 5 + 8 \times 4 + 14 = 61.$$

(b)

$$\mathcal{M}_c(25) = 2 \times 5\mathcal{M}_c(5) + 16 = 176.$$

(c)

$$\mathcal{M}_c(32) = 49.$$

(d)

$$\mathcal{M}_c(64) = 81.$$

Zero padding to 32 is the most efficient in this case.

5.12 Denote $c = 2^{-1/2}$. Then the sequence of W_8^n is

$$1, c + jc, j, -c + jc, -1, -c - jc, -j, c - jc.$$

If n is even, then $(a + jb)W_8^n$ requires no multiplications. If n is odd, then

$$(a + jb)W_8^n = (a + jb)(\pm c \pm jc) = (ca + jcb)(\pm 1 \pm j),$$

and this requires two multiplications.

5.13

(a) The length of $y[n]$ is 3×2^r, so it is zero-padded to length 2^{r+2}, that is, to $4N$. Therefore,

$$\mathcal{M}_{c1} = 2rN + 1.$$

The second method requires 2 FFTs of length N and $3N$ additional multiplications by W_{3N}^{-2k}. Therefore,

$$\mathcal{M}_{c2} = N(r - 2) + 1 + 3N = N(r + 1) + 1.$$

Therefore $\mathcal{M}_{c2} \leq \mathcal{M}_{c1}$.

(b) No, as is seen from the following counterexample. Let $N = 5$, then $3N = 15$. In the first method we zero-pad to 16, hence

$$\mathcal{M}_{c1} = 17.$$

In the second method we zero-pad each sequence to 8, hence

$$\mathcal{M}_{c2} = 10 + 24 = 34.$$

5.14 Since convolution is a commutative operation we can assume, without loss of generality, that $N_1 \geq N_2$. By 1.16, we can write the convolution as

$$\{x * y\}[n] = \begin{cases} \sum_{m=0}^{n} x[m]y[n-m], & 0 \leq n \leq N_2 - 1, \\ \sum_{m=n-N_2+1}^{n} x[m]y[n-m], & N_2 \leq n \leq N_1 - 1, \\ \sum_{m=n-N_2+1}^{N_1-1} x[m]y[n-m], & N_1 \leq n \leq N_1 + N_2 - 1. \end{cases}$$

The first case requires $n+1$ multiplications for each n; the second requires N_2 multiplications for each n; the third requires $N_1 + N_2 - 1 - n$ multiplications for each n. The total is

$$\sum_{n=0}^{N_2-1}(n+1) + \sum_{n=N_2}^{N_1-1} N_2 + \sum_{n=N_1}^{N_1+N_2-1}(N_1+N_2-1-n) = 0.5N_2(N_2+1) + N_2(N_1-N_2) + 0.5N_2(N_2-1) = N_1N_2.$$

The number of additions is one less for each value of n, so the total number of additions is $N_1N_2 - (N_1 + N_2 - 1)$.

5.15 Direct convolution requires 2^{2r} real multiplications (the product of the lengths of the two sequences). The DFTs of the two sequences can be computed with $2(r-2)2^r + 4$ real multiplications, as explained in Section 5.5. Multiplication of the two DFTs can be done with 2×2^r real multiplications if we take advantage of the conjugate symmetry of the two sequences. Computation of the inverse DFT of the product requires $2(r-2)2^r + 4$ real multiplications. The total is $(4r-6)2^r + 8$ real multiplications.

5.16 We have

$$|X^{\mathrm{d}}[k]| = \left|\sum_{n=0}^{N-1} x[n]W_N^{-nk}\right| \leq \sum_{n=0}^{N-1} |x[n]| \cdot |W_N^{-nk}| \leq N.$$

We have $X^{\mathrm{d}}[k] = N$ if and only if $x[n] = W_N^{nk}$.

5.17 Define

$$Z^{\mathrm{d}}[k] = X^{\mathrm{d}}[k] + jY^{\mathrm{d}}[k].$$

Then

$$z[n] = x[n] + jy[n],$$

so

$$x[n] = \Re\{z[n]\}, \quad y[n] = \Im\{z[n]\}.$$

5.18 Define

$$x_{\mathrm{e}}[n] = x[2n], \quad x_{\mathrm{o}}[n] = x[2n+1], \quad y[n] = x_{\mathrm{e}}[n] + jx_{\mathrm{o}}[n], \quad 0 \leq n \leq 0.5N - 1.$$

Then

$$Y^{\mathrm{d}}[k] = \sum_{n=0}^{0.5N-1}(x_{\mathrm{e}}[n] + jx_{\mathrm{o}}[n])W_{0.5N}^{-nk} = X_{\mathrm{e}}^{\mathrm{d}}[k] + jX_{\mathrm{o}}^{\mathrm{d}}[k],$$

$$\bar{Y}^{\mathrm{d}}[N-k] = \sum_{n=0}^{0.5N-1}(x_{\mathrm{e}}[n] - jx_{\mathrm{o}}[n])W_{0.5N}^{-nk} = X_{\mathrm{e}}^{\mathrm{d}}[k] - jX_{\mathrm{o}}^{\mathrm{d}}[k].$$

Therefore,

$$X_{\mathrm{e}}^{\mathrm{d}}[k] = 0.5(Y^{\mathrm{d}}[k] + \bar{Y}^{\mathrm{d}}[N-k]), \quad X_{\mathrm{o}}^{\mathrm{d}}[k] = -j0.5(Y^{\mathrm{d}}[k] - \bar{Y}^{\mathrm{d}}[N-k]).$$

Finally, we get from the time-decimated radix-2 FFT that

$$X^{\mathrm{d}}[k] = X_{\mathrm{e}}^{\mathrm{d}}[k] + W_N^{-k}X_{\mathrm{o}}^{\mathrm{d}}[k], \quad X^{\mathrm{d}}[k+0.5N] = X_{\mathrm{e}}^{\mathrm{d}}[k] - W_N^{-k}X_{\mathrm{o}}^{\mathrm{d}}[k], \quad 0 \leq k \leq 0.5N - 1.$$

Computation of $Y^{\mathrm{d}}[k]$ requires approximately $0.25N(\log_2 N - 3)$ multiplications and $0.5N(\log_2 N - 1)$ additions. Computation of $X_{\mathrm{e}}^{\mathrm{d}}[k]$ and $X_{\mathrm{o}}^{\mathrm{d}}[k]$ requires N additions. The final step requires $0.5N$ multiplications and N additions. The total is $0.25N(\log_2 N - 1)$ complex multiplications and additions.

5.19

(a) This is just a restatement of Theorem 2.5. Note that the period of the output is N and not $N + L - 1$, as one might erroneously assume.

(b) The crucial observation is that one period of the output is obtained by circular convolution of one period of the input with the sequence $h[n]$ zero-padded to length N. This is shown as follows:

$$y[n] = \sum_{l=0}^{L-1} h[l]x[n-l] = \sum_{l=0}^{N-1} h_a[l]x[n-l] = \sum_{l=0}^{N-1} h_a[l]x[(n-l) \bmod N],$$

where the last equality follows from the periodicity of $x[n]$. Therefore,

$$y_p[n] = \{h_a \circledast x_p\}[n],$$

where $h_a[n]$ is the zero-padded sequence $h[n]$ and $x_p[n]$, $y_p[n]$ are one period of the input and output sequences, respectively. An efficient way to compute $y_p[n]$ is through

$$y_p = \mathrm{IFFT}\{\mathrm{FFT}\{h_a\} \cdot \mathrm{FFT}\{x_p\}\}.$$

All FFTs are of length N.

(c) We have in this case

$$y[n] = \sum_{m=0}^{M-1}\sum_{k=0}^{N-1} h[mN+k]x[n-mN-k] = \sum_{m=0}^{M-1}\sum_{k=0}^{N-1} h[mN+k]x[n-k] = \sum_{k=0}^{N-1}\left(\sum_{m=0}^{M-1} h[mN+k]\right)x[n-k].$$

We therefore have to form the length-N sequence

$$\sum_{m=0}^{M-1} h[mN+k], \quad 0 \le k \le N-1,$$

and then carry out the computation as in part b. Only about L more additions (and no multiplications) are needed beyond the operations in part b.

(d) In this case we zero-pad $h[n]$ until its length is an integer multiple of N and proceed as in part c.

5.20 We have,

$$F_8 = G_{8,5}G_{8,4}G_{8,3}G_{8,2}G_{8,1},$$

where

$$G_{8,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad G_{8,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_4^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & W_4^{-1} \end{bmatrix},$$

$$G_{8,3} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}, \quad G_{8,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_8^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_8^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & W_8^{-3} \end{bmatrix},$$

$$G_{8,5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

5.21

(a) The difference equation (5.48a) shows that $\mathcal{A}_c[0] = \mathcal{A}_c[-1] + 1$. However, we know that $\mathcal{A}_c[0] = 0$; therefore, we must assume that $\mathcal{A}_c[-1] = -1$ to make the equation correct at $r = 0$. A mathematically equivalent approach is to assume $\mathcal{A}_c[-1] = 0$ and add $-\delta[r]$ to the right side. With this modification, the equation becomes

$$\mathcal{A}_c[r] = 2\mathcal{A}_c[r-1] + 2^r - \delta[r],$$

with zero initial conditions. We now take the z-transform:

$$\mathcal{A}_c^z(z) = 2z^{-1}\mathcal{A}_c^z(z) + \frac{1}{1-2z^{-1}} - 1 = 2z^{-1}\mathcal{A}_c^z(z) + \frac{2z^{-1}}{1-2z^{-1}}.$$

Therefore,

$$\mathcal{A}_c^z(z) = \frac{2z^{-1}}{(1-2z^{-1})^2}.$$

From Table 7.1 we get that

$$\mathcal{A}_c[r] = r \cdot 2^r,$$

which gives (5.23a) when substituting $r = \log_2 N$.

The difference equation (5.48b) similarly requires modification to

$$\mathcal{M}_c[r] = 2\mathcal{M}_c[r-1] + 0.5 \times 2^r - 1 + 0.5\delta[r].$$

Taking the z-transform of this equation gives

$$\mathcal{M}_c^z(z) = 2z^{-1}\mathcal{M}_c^z(z) + \frac{0.5}{1-2z^{-1}} - \frac{1}{1-z^{-1}} + 0.5 = \frac{z^{-1}}{1-2z^{-1}} - \frac{z^{-1}}{1-z^{-1}}.$$

Therefore,

$$\mathcal{M}_c^z(z) = \frac{z^{-1}}{(1-2z^{-1})^2} - \frac{z^{-1}}{(1-z^{-1})(1-2z^{-1})} = \frac{z^{-1}}{(1-2z^{-1})^2} - \frac{1}{1-2z^{-1}} + \frac{1}{1-z^{-1}}.$$

From Table 7.1 we get that

$$\mathcal{M}_c[r] = 0.5r \cdot 2^r - 2^r + 1 = 0.5(r-2)2^r + 1,$$

which gives (5.23b) when substituting $r = \log_2 N$.

(b) With $r = \log_4 N$, the difference equations are

$$\mathcal{A}_c[r] = 4\mathcal{A}_c[r-1] + 2 \times 4^r,$$
$$\mathcal{M}_c[r] = 4\mathcal{M}_c[r-1] + 0.75 \times 4^r - 3.$$

In order to properly set $\mathcal{A}_c[0] = 0$ while avoiding initial condition, we modify the first equation to

$$\mathcal{A}_c[r] = 4\mathcal{A}_c[r-1] + 2 \times 4^r - 2\delta[r].$$

Taking the z-transform, we get

$$\mathcal{A}_c^z(z) = 4z^{-1}\mathcal{A}_c^z(z) + \frac{2}{1-4z^{-1}} - 2 = 4z^{-1}\mathcal{A}_c^z(z) + \frac{8z^{-1}}{1-4z^{-1}}.$$

Therefore,

$$\mathcal{A}_c^z(z) = \frac{8z^{-1}}{(1-4z^{-1})^2}.$$

From Table 7.1 we get that

$$\mathcal{A}_c[r] = 2r \cdot 4^r,$$

which gives (5.27a) when substituting $r = \log_4 N$.

The second equation similarly requires modification to

$$\mathcal{M}_c[r] = 4\mathcal{M}_c[r-1] + 0.75 \times 4^r - 3 + 2.25\delta[r].$$

Taking the z-transform, we get

$$\mathcal{M}_c^z(z) = 4z^{-1}\mathcal{M}_c^z(z) + \frac{0.75}{1-4z^{-1}} - \frac{3}{1-z^{-1}} + 2.25 = \frac{3z^{-1}}{1-4z^{-1}} - \frac{3z^{-1}}{1-z^{-1}}.$$

Therefore,

$$\mathcal{M}_c^z(z) = \frac{3z^{-1}}{(1-4z^{-1})^2} - \frac{3z^{-1}}{(1-z^{-1})(1-4z^{-1})} = \frac{3z^{-1}}{(1-4z^{-1})^2} - \frac{1}{1-4z^{-1}} + \frac{1}{1-z^{-1}}.$$

From Table 7.1 we get that

$$\mathcal{M}_c[r] = 0.75r \cdot 4^r - 4^r + 1,$$

which gives (5.27b) when substituting $r = \log_4 N$.

5.22

(a) We have,

$$F^d[k] = \sum_{m=0}^{0.5N-1} x[2m]W_{0.5N}^{-mk} = \sum_{m=0}^{0.5N-1} x[2m]W_N^{-2mk},$$

$$G^d[k] = \sum_{m=0}^{0.25N-1} x[4m+1]W_{0.25N}^{-mk} = \sum_{m=0}^{0.25N-1} x[4m+1]W_N^{-4mk},$$

$$H^d[k] = \sum_{m=0}^{0.25N-1} x[4m+3]W_{0.25N}^{-mk} = \sum_{m=0}^{0.25N-1} x[4m+3]W_N^{-4mk}.$$

Therefore, noting that $F^d[k]$ has period $0.5N$ and $G^d[k]$ and $H^d[k]$ have period $0.25N$, we can write

$$\begin{aligned}
X^d[k] &= F^d[k] + W_N^{-k}G^d[k] + W_N^{-3k}H^d[k], & 0 \le k \le 0.25N-1,\\
X^d[k+0.25N] &= F^d[k+0.25N] - jW_N^{-k}G^d[k] + jW_N^{-3k}H^d[k], & 0 \le k \le 0.25N-1,\\
X^d[k+0.5N] &= F^d[k] - W_N^{-k}G^d[k] - W_N^{-3k}H^d[k], & 0 \le k \le 0.25N-1,\\
X^d[k+0.75N] &= F^d[k+0.25N] + jW_N^{-k}G^d[k] - jW_N^{-3k}H^d[k], & 0 \le k \le 0.25N-1.
\end{aligned}$$

(b) The split-radix butterfly is shown in Figure 5.1. It includes two twiddle-factor multiplications and six additions.

(c) The DFT_{16} is split into one DFT_8 and two DFT_4; this requires four split-radix butterflies. The DFT_8, in turn, is split into one DFT_4 and two DFT_2; this requires two split-radix butterflies. In total we have six split-radix butterflies, two DFT_4 butterflies, and two DFT_2 butterflies. The number of complex multiplications is 12, because DFT_4 and DFT_2 require no multiplications.

(d) The DFT_{32} is split into one DFT_{16} and two DFT_8; this requires eight split-radix butterflies. As we saw, the DFT_{16} requires six split-radix butterflies, two DFT_4 butterflies, and two DFT_2 butterflies. Each of the two DFT_8 requires two split-radix butterflies, one DFT_4 and two DFT_2. The total is 18 split-radix butterflies, four DFT_4 and six DFT_2. The number of complex multiplications is 36.

Guessing the coefficient of $N\log_2 N$ in the count of complex multiplications is not easy. However, using the technique developed in Solution 5.21, this coefficient is found to be 1/3. Recall, for comparison, that the coefficient of $N\log_2 N$ in the count of complex multiplications in radix-4 FFT is 3/8. Therefore, split-radix FFT is slightly more efficient than radix-4 FFT for large N.

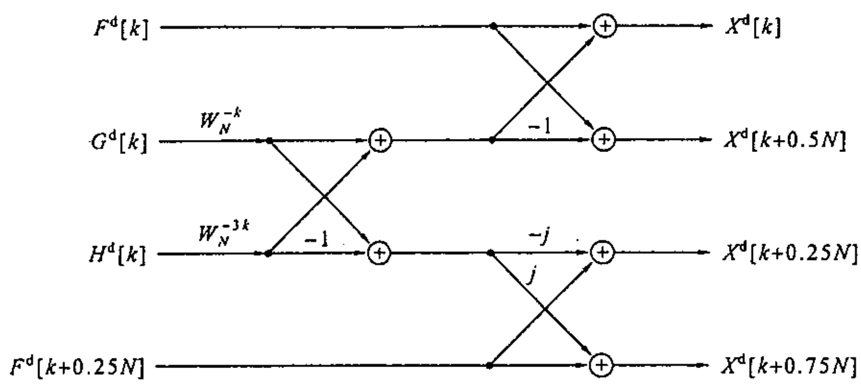

Figure 5.1 Pertaining to Solution 5.22.

5.23

(a) Let $x[n]$ have length $N = N_1 + N_2 - 1$ and let $y[n]$ have length N_2. The circular convolution of $x[n]$ and the zero-padded $y[n]$ can be expressed as

$$z[n] = \sum_{m=0}^{N_2-1} y[m]x[(n-m) \bmod N].$$

If $N_2 - 1 \le n \le N - 1$ then $0 \le 0n - m \le N - 1$ for all $0 \le m \le N_2 - 1$. Therefore, for this range of n, the $z[n]$ will be equal to the corresponding values of the linear convolution.

(b)

$$x_i[n] = x[n + i(N - N_2 + 1)] = x[n + iN_1].$$

(c) When $x_i[n]$ is convolved with the zero-padded $y[n]$, the points from 0 to $N_2 - 2$ should be discarded, whereas the points from $N_2 - 1$ to $N - 1$ should be retained. The points to be retained correspond to $iN_1 + N_2 - 1 \le n \le iN_1 + N - 1$ in the result. When the retained segments for all i are patched together, they yield the desired convolution $\{x * y\}[n]$. This is the overlap-save method.

5.24 Program 5.2 implements the zoom FFT.

Program 5.2 Zoom FFT.

```
function X = zoomfft(x,k0,K);
% Synopsis: X = zoomfft(x,k0,K).
% Computes the zoom FFT of x.
% Input parameters:
% x: the input vector
% k0: the initial value of k
% K: the length of the output vector.
% Output:
% X: the zoom FFT.

L = ceil(length(x)/K); N = L*K;
if (N > length(x)), x = [x,zeros(1,N-length(x))]; end
X = fft((reshape(x,L,K)).').';
k = k0:(k0+K-1);
X = X(:,rem(k,K)+1);
X = sum(X.*exp(-j*(2*pi/N)*(0:L-1)'*k));
```

5.25 Program 5.3 implements the fast DCTs of four types.

Program 5.3 Fast DCT through FFT.

```
function y = fdct(x,typ);
% Synopsis: y = fdct(x,typ).
% Discrete cosine transform by FFT
% Input parameters:
% x: the input vector
% typ: the DCT type, 1 through 4
% Output:
% y: the output vector

N = length(x); x = reshape(x,1,N); fac = sqrt(2);
norfac = sqrt(1/(2*N));
if (typ == 1), norfac = sqrt(1/(2*(N-1))); end
if (typ == 1),
   x([1,N]) = fac*x([1,N]);
   x1 = [x, fliplr(x(2:N-1))];
   X1 = fft(x1); y = real(norfac*X1(1:N));
   y([1,N]) = (1/fac)*y([1,N]);
elseif (typ == 2),
   x2 = [x, fliplr(x)];
   X2 = fft(x2); X2 = X2(1:N);
   y = real(norfac*exp(-(j*0.5*pi/N)*(0:N-1)).*X2);
   y(1) = y(1)/fac;
elseif (typ == 3),
   x(1) = fac*x(1);
   x3 = [x, 0, -fliplr(x(2:N))];
   x3 = exp(-(j*0.5*pi/N)*(0:2*N-1)).*x3;
   X3 = fft(x3);
   y = norfac*X3(1:N);
elseif (typ == 4),
   x4 = [x, -fliplr(x)];
   x4 = exp(-(j*0.5*pi/N)*(0:2*N-1)).*x4;
   X4 = fft(x4); X4 = X4(1:N);
   y = real(norfac*exp(-(j*0.5*pi/N)*((0:N-1)+0.5)).*X4);
end
```

Chapter 6

Practical Spectral Analysis

6.1 The magnitudes of the side lobes of the Dirichlet kernel decay monotonically as θ increases. Recall that $D(\theta, N) = \sin(0.5\theta N)/\sin(0.5\theta)$. If N is odd, the smallest side lobe peaks at $\theta = \pi$ and

$$D(\pi, N) = \frac{\sin(0.5\pi N)}{\sin(0.5\pi)} = 1.$$

In this case, $D(\pi, N)/D(0, N) = 1/N$. If N is even, the smallest side lobe peaks at $\theta = (N-1)\pi/N$ and

$$D((N-1)\pi/N, N) = \frac{\sin(0.5\pi(N-1))}{\sin(0.5(N-1)\pi/N)} \approx 1.$$

In this case, $D(\pi, N)/D(0, N) \approx 1/N$.

6.2 Let $w_{r1}[n]$ be a rectangular window of length $N/2$ and w_{r2} a rectangular window of length $(N+2)/2$. Then

$$\begin{aligned}\{w_{r1} * w_{r2}\}[n] &= 1 + \min\{0.5N - 1, n\} - \max\{0, n - 0.5N - 1 + 1\} \\ &= \begin{cases} n + 1, & 0 \le n \le 0.5N - 1, \\ N - n, & 0.5N \le n \le N - 1 \end{cases} = 0.5(N + 1 - |2n - N + 1|).\end{aligned}$$

Therefore

$$\frac{2}{N+1}\{w_{r1} * w_{r2}\}[n] = w_t[n]$$

in the case of even N. Now

$$W_{r1}^f(\theta) = D(\theta, 0.5N)e^{-j(0.25N-0.5)\theta},$$
$$W_{r2}^f(\theta) = D(\theta, 0.5(N+2))e^{-j0.25N\theta}.$$

Therefore $W_t^f(\theta)$ is as given in (6.12).

6.3 The window $w[n]$ is given by $\{w_r * w_r * w_r\}[n]$, where $w_r[n]$ is a rectangular window of length N. Therefore, $W^f(\theta) = [W_r^f(\theta)]^3$. Therefore, the length of $w[n]$ is $3N - 2$, the main-lobe width is $4\pi/N$, and the side-lobe level is -40.5 dB.

6.4 For a rectangular window of length N we have

$$W_r^f(\theta) = D(\theta, N)e^{-j0.5\theta(N-1)}.$$

Therefore, the Fourier transform of a K-fold convolution of a rectangular window is

$$W^f(\theta) = [D(\theta, N)]^K e^{-j0.5\theta(N-1)K}.$$

The side-lobe level of $W^f(\theta)$ is $-13.5K$ dB. The main-lobe width is still $4\pi/N$; however, the length of the window is $(N-1)K + 1$, which is approximately equal to NK. Therefore, relative to the window length, the main-lobe width is almost $4K\pi/((N-1)K+1)$, or almost K times larger than that of a rectangular window.

6.5

(a) We have from (4.32),

$$y_1[n] = 0.5x[n] - 0.25W_N^n x[n] - 0.25W_N^{-n}x[n] = x[n]w_1[n],$$

where

$$w_1[n] = 0.5\left[1 - \cos\left(\frac{2\pi n}{N}\right)\right].$$

The window $w_1[n]$ differs from the Hann window only in the replacement of $N-1$ by N in the cosine.

(b) We have

$$y_2[n] = 0.42x[n] - 0.25W_N^n x[n] - 0.25W_N^{-n}x[n] + 0.04W_N^{2n}x[n] + 0.04W_N^{-2n}x[n] = x[n]w_2[n],$$

where

$$w_2[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right).$$

The window $w_2[n]$ differs from Blackman window only in the replacement of $N-1$ by N in the cosines.

(c) By performing the DFT first and then using (6.79) or (6.80), we save the need for computing and storing the window coefficients. Also, in the case of Hann window we can eliminate the need for multiplications, since multiplications by 0.5 and 0.25 can be performed by bit shifts. As we have seen in this chapter, some applications require computation of a windowed DFT at frequency points whose number is less than N. In such cases, the use of (6.79) or (6.80) may be more economical than windowing in the time domain.

(d) The idea is also applicable to Hamming and Blackman-Harris windows (see Problem 6.8).

6.6 By the identity $\sin^2\alpha = 0.5[1 - \cos(2\alpha)]$ we can write

$$w[n] = 0.25\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right]^2.$$

Therefore, by (6.14) we have

$$w[n] = (w_{\text{hn}}[n])^2.$$

This implies that

$$W^{\text{f}}(\theta) = \frac{1}{2\pi}\{W_{\text{hn}}^{\text{f}} * W_{\text{hn}}^{\text{f}}\}(\theta).$$

Using MATLAB, we can see that the main-lobe width of this window is $12\pi/N$. This is 1.5 times wider then the main-lobe width of the Hann window. The side-lobe level is approximately -47 dB.

6.7

(a) We recall from Solution 6.1 that the side lobe of the rectangular window nearest to $\theta = \pi$ has magnitude N times smaller than the main lobe. In this case the ratio is approximately 0.01. Since the second sinusoidal component has amplitude 0.001, it is completely masked by the side lobe of the rectangular window.

(b) The side lobes of the Hann window decay much faster than those of the Hamming window, therefore the Hann window is much better for detecting the second component in this case. This is confirmed by testing the windowed DFTs on MATLAB.

6.8 The side-lobe level of the Blackman-Harris window is about -70 dB. This window has the lowest side-lobe level among all windows based on two cosine terms.

6.9 The proper procedure is to apply a window of length N first, then do zero padding. Doing it the opposite way is equivalent to using a window which is the truncation of a length-M window $\{w[n],\ 0 \le n \le M-1\}$ to length N. Such a window is neither symmetric nor effective. We could take the middle part of $w[n]$ instead (again of length M), but this would be very similar to a rectangular window if $M \gg N$.

6.10 The best performance is obtained when both windowing schemes are used: 20 windows of length 80 each applied to the individual chops, *and* a single window of length 2000 applied to the entire data sequence. We can see by experimenting with MATLAB that the Fourier transform of the data is shaped like a comb. The local behavior around each tooth of the comb is controlled by the large window, whereas the rate of decay of the magnitudes of the teeth is controlled by the short windows.

6.11

(a) Nonzero frequency is discernible with rectangular, Bartlett, Hamming, and Kaiser window with $\alpha < 6.5$. It is not discernible with Hann, Blackman, and Kaiser window with $\alpha > 6.5$. The (somewhat unexpected) behavior of the Hamming window is a result of its peculiar side-lobe structure (see Figure 6.13).

(b) This time all windows show the existence of a nonzero frequency, including Kaiser window with all practical values of α. Only at $\alpha > 32$ does the Kaiser window stop discerning the existence of nonzero frequency.

6.12

(a) The verification is left to the reader.

(b) For the Bartlett window, the roll-off rate is 12 dB/octave and $K \approx 0.07$.

(c) The verification is left to the reader. The parameter K is approximately 0.003.

(d) For the Hamming window, the roll-off rate is 6 dB/octave and $K \approx 0.15$. The roll-off rate is only 6 dB/octave because, unlike the Hann window, the Hamming window has discontinuities at the end points.

(e) For the Blackman window, the roll-off rate is 18 dB/octave and $K \approx 0.001$. A Kaiser window has a roll-off rate 6 dB/octave regardless of the value of α, because it has discontinuities at the end points.

(f) The Dolph window has a roll-off rate 0 dB/octave, because its kernel function is equiripple.

6.13 In the case of a single real-valued sinusoid, we get from (6.42),

$$X^{\mathrm{f}}(\theta_1) = 0.5A_1 e^{j\phi_1} W^{\mathrm{f}}(0, N) + 0.5A_1 e^{-j\phi_1} W^{\mathrm{f}}(2\theta_1, N).$$

As we see, there is interference resulting from the term $0.5A_1 e^{-j\phi_1} W^{\mathrm{f}}(2\theta_1, N)$. The situation here is similar to that shown in Figures 6.18(a), (b), (c) (for unwindowed DFT) or in Figures 6.19(a), (b), (c) (for windowed DFT). If no window is used, we can measure the frequency as long as $\pi/N \le \theta_1 \le \pi(1 - 1/N)$, but there is bias when θ_1 is close to either end of this interval. Using a window narrows the range of θ_1 that can be measured, but reduces the bias near the ends of the interval.

6.14 Program 4.2 already includes the window option. Note that the window has to act on $X^{\mathrm{d}}[k]$ *before* zero padding, and that it should preserve the conjugate symmetry property of $X^{\mathrm{d}}[k]$. The window always has an odd length. Windowing helps reducing the wiggles in the interpolated signal, but it does not reduce the distortion at the ends of the interval, the reason of which was explained in Section 4.5.

6.15 The Fourier transform computed locally around θ_m without windowing is still not free of the interfering terms. The interfering terms will lead to

$$E\hat{\theta}_m \neq \theta_m,$$

the difference being dependent on the distances and amplitudes of the interfering terms. Since they are not well attenuated, the adverse effect of the interfering terms is likely to be stronger than in the case of windowed Fourier transform.

6.16

(a) The radix-2 frequency-decimated FFT algorithm computes the sequence $X^{\mathrm{d}}[k]$ in a bit-reversed order. Therefore, all we have to do is eliminate all butterflies leading to the $0.5N$ bottom outputs of the frequency-decimated scheme; see Figure 5.6. This will save about half the number of butterflies, hence about half the number of computations. We can also explain this mathematically as follows:

$$X^{\mathrm{d}}[2m] = \sum_{n=0}^{N-1} x[n]W_N^{-2mn} = \sum_{n=0}^{N-1} x[n]W_{0.5N}^{-mn} = \sum_{n=0}^{0.5N-1} (x[n] + x[n+0.5N])W_{0.5N}^{-mn},$$

so we can perform the computation with an FFT of length $0.5N$ and $0.5N$ additions.

(b) By skipping the odd-indexed frequency points, the worst-case coherent gain and processing gain occur at $\Delta\theta = 4\pi/N$, rather than at $\Delta\theta = 2\pi/N$. By running the procedure cpgains for this value of $\Delta\theta$, we discover that they are worse by a few dB than those given in Table 6.2. Therefore, the idea proposed by student is not a good one.

6.17

(a) The Fourier transform of $y(t)$ is not a delta function, but is spread over a large bandwidth, as can be seen in Figure 3.29. If $f_0 = 0$, the Fourier transform is centered at zero frequency, otherwise it is centered at frequency f_0. Accurate determination of the center of the Fourier transform is difficult, especially when noise is added.

(b) Because the NRZ signal is either $+1$ or -1 at all times, we have

$$z(t) = y^2(t) = e^{j4\pi f_0 t}.$$

Therefore, the Fourier transform of $z(t)$ is a delta function at frequency $2f_0$. If follows that we can determine $2f_0$ (hence f_0) unambiguously if we sample $z(t)$ at a frequency at least twice the maximum possible value of $2|f_0|$; that is, at 4000 Hz or higher.

(c) When noise is present we have

$$z[n] = (e^{j\theta_0 n} + v[n])^2 \approx e^{j2\theta_0 n} + 2e^{j\theta_0 n}v[n].$$

Repeating the derivation that has led to (6.52), we get

$$\mathrm{SNR_o} = \frac{(NA \cdot \mathrm{CG})^2}{4\gamma_v \sum_{n=0}^{N-1} w^2[n]}.$$

(d) As we see, we lose a factor of 4, or about 6 dB in SNR. This loss is the main limiting factor of frequency estimation based on squaring.

6.18

(a) We proceed as suggested in the hint:

$$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\kappa[n-m]e^{-j\theta_0(n-m)} = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\left[\frac{\gamma_v}{2\pi}\int_{-\theta_c}^{\theta_c} e^{j\theta(n-m)}d\theta\right]e^{-j\theta_0(n-m)}$$

$$= \frac{\gamma_v}{2\pi}\int_{-\theta_c}^{\theta_c}\left[w[n]e^{j(\theta-\theta_0)n}\right]\overline{\left[w[m]e^{j(\theta-\theta_0)m}\right]}d\theta = \frac{\gamma_v}{2\pi}\int_{-\theta_c}^{\theta_c}|W^{\mathrm{f}}(\theta-\theta_0)|^2 d\theta = \frac{\gamma_v}{2\pi}\int_{-\theta_c-\theta_0}^{\theta_c-\theta_0}|W^{\mathrm{f}}(\theta)|^2 d\theta.$$

(b) For $\theta_c = \pi$ we get, by periodicity of $W^{\mathrm{f}}(\theta)$ and by Parseval's theorem,

$$\frac{\gamma_v}{2\pi}\int_{-\pi-\theta_0}^{\pi-\theta_0}|W^{\mathrm{f}}(\theta)|^2 d\theta = \frac{\gamma_v}{2\pi}\int_{-\pi}^{\pi}|W^{\mathrm{f}}(\theta)|^2 d\theta = \gamma_v\sum_{n=0}^{N-1} w^2[n].$$

(c) If the window is good, most of the energy of its kernel is concentrated in the main lobe and the first few side lobes at each side of the main lobe. Therefore, unless $|\theta_0|$ is very close to θ_0, the integral in part a will be only slightly smaller than the integral in part b. Another interpretation of this result is: The window acts to filter the noise so as to eliminate most of its energy outside the vicinity of θ_0. The effect of the additional low-pass filter is negligible compared with the effect of the window.

6.19 The following MATLAB code implements the experiment:

```
N = 512; omega0 = pi/2; epsmax = 0.1; w = window(N,'blac');
t = 0:N-1; tj = t + 2*epsmax*(rand(1,N)-1);
x = cos(omega0*t); xj = cos(omega0*tj);
X = 20*log10(abs(fft(x.*w))); X = X(1:0.5*N);
XJ = 20*log10(abs(fft(xj.*w))); XJ = XJ(1:0.5*N);
plot([X;XJ]'),grid,figure(1)
```

The jitter appears as additive white noise whose energy is proportional to $\varepsilon_{\max}^2$. The mathematical explanation for this behavior is as follows:

$$\cos(\omega_0 Tn + \omega_0\varepsilon[n]) \approx \cos(\omega_0 Tn) - \omega_0\varepsilon[n]\sin(\omega_0 Tn).$$

The jitter term $\omega_0\varepsilon[n]\sin(\omega_0 Tn)$ is uncorrelated since the sequence $\varepsilon[n]$ is uncorrelated; its mean is zero and its variance is

$$E\{\omega_0\varepsilon[n]\sin(\omega_0 Tn)\}^2 = \omega_0^2\sin^2(\omega_0 Tn)E\{\varepsilon[n]\}^2 = \frac{\omega_0^2\varepsilon_{\max}^2}{6}[1-\cos(2\omega_0 Tn)].$$

Note that the variance of the jitter noise is not constant, but periodically time varying, with period $\pi/\omega_0 T$.

6.20

(a) Denote

$$y_x = 20\log_{10}|X^f(\theta_x)|, \quad x = a, b, c.$$

The parabola we seek is

$$y = a_0\theta^2 + a_1\theta + a_2,$$

where $\{a_0, a_1, a_2\}$ are to be found. Since this parabola must pass through the three given points, we get the three equations

$$\begin{bmatrix}\theta_a^2 & \theta_a & 1\\ \theta_b^2 & \theta_b & 1\\ \theta_c^2 & \theta_c & 1\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix} = \begin{bmatrix}y_a\\ y_b\\ y_c\end{bmatrix}.$$

The point of maximum of the parabola is

$$\hat\theta = -\frac{a_1}{2a_0}.$$

Solving the equations and substituting in $\hat\theta$ gives

$$\hat\theta = 0.5\frac{y_a(\theta_b^2-\theta_c^2)+y_b(\theta_c^2-\theta_a^2)+y_c(\theta_a^2-\theta_b^2)}{y_a(\theta_b-\theta_c)+y_b(\theta_c-\theta_a)+y_c(\theta_a-\theta_b)}.$$

Substituting the values of the θ_x gives

$$\hat\theta = \frac{\pi}{N}\cdot\frac{-(2k_0+1)y_a+4k_0y_b-(2k_0-1)y_c}{-y_a+2y_b-y_c}.$$

(b) The procedure `freqint` (for *frequency interpolation*) shown in Program 6.1 implements the computation. Note the treatment of the special cases $k = 0$, $k = N-1$.

(c) The procedure `maxdft1` shown in Program 6.2 implements the computation.

Program 6.1 Pertaining to Solution 6.20(a).

```
function theta = freqint(X,k);
% Synopsis: theta = freqint(X,k).
% Estimates the point of local maximum of a windowed DFT by
% parabolic interpolation.
% Input parameters:
% X: a vector of windowed DFT magnitudes.
% k: index of a local peak around which to interpolate.
% Output parameters:
% theta: the interpolated frequency.

N = length(X); kp1 = k+1; km1 = k-1;
if (kp1 > N), kp1 = kp1-N; end
if (km1 < 1), km1 = km1+N; end
ya = 20*log10(X(km1));
yb = 20*log10(X(k));
yc = 20*log10(X(kp1));
k = k-1;
theta = (pi/N)*(4*k*yb-(2*k+1)*ya-(2*k-1)*yc)/(2*yb-ya-yc);
if (theta > pi), theta = 2*pi-theta; end
if (theta < 0), theta = -theta; end
```

Program 6.2 Pertaining to Solution 6.20(b).

```
function [theta,val] = maxdft1(x,M,name,alpha);
% Synopsis: [t,val] = maxdft1(x,M,name,alpha).
% Finds the M largest maxima of the DFT of the real vector x.
% Input parameters:
% x: the input vector
% M: number of local maxima to be found
% name: an optional window for x; one of the names in window.m
% alpha: needed if name = 'kaiser'.
% Output parameters:
% theta: vector of thetas at the local maxima
% val: vector of corresponding values of DFT(x).

N = length(x); x = reshape(x,1,N);
if (nargin == 3), x = x.*window(N,name);
elseif (nargin == 4), x = x.*window(N,name,alpha); end
X = abs(fft(x));
[y,ind] = locmax(X(1:floor((N+1)/2)));
ind = ind(1:M); val = zeros(1,M);
for m = 1:M,
   theta(m) = freqint(X,ind(m));
   val(m) = abs(sum(x.*exp(-j*theta(m)*(0:N-1))));
end
```

Chapter 7

Review of z-Transforms and Difference Equations

7.1 We have

$$\cos\theta_0 n = 0.5e^{j\theta_0 n} + 0.5e^{-j\theta_0 n}.$$

Therefore, by (7.10),

$$X_1^z(z) = \frac{0.5}{1-e^{j\theta_0}z^{-1}} + \frac{0.5}{1-e^{-j\theta_0}z^{-1}} = \frac{1-(\cos\theta_0)z^{-1}}{1-2(\cos\theta_0)z^{-1}+z^{-2}}.$$

Similarly,

$$X_2^z(z) = -\frac{0.5j}{1-e^{j\theta_0}z^{-1}} + \frac{0.5j}{1-e^{-j\theta_0}z^{-1}} = \frac{(\sin\theta_0)z^{-1}}{1-2(\cos\theta_0)z^{-1}+z^{-2}}.$$

The region of convergence is $|z| > 1$ in both cases.

7.2

$$\begin{aligned}
X^z(z) &= \sum_{n=1}^{N} nz^{-n} + \sum_{n=N+1}^{2N-1}(2N-n)z^{-n} = \sum_{n=1}^{N} nz^{-n} + \sum_{n=1}^{N-1} nz^{-(2N-n)} \\
&= -z\frac{d}{dz}\sum_{n=1}^{N} z^{-n} + z^{-2N}z\frac{d}{dz}\sum_{n=1}^{N-1} nz^{n} \\
&= -z\frac{d}{dz}\left(\frac{z^{-1}-z^{-(N+1)}}{1-z^{-1}}\right) + z^{-2N+1}\frac{d}{dz}\left(\frac{z-z^{N}}{1-z}\right) \\
&= \frac{1-z^{-(N+1)}[N-1+2z+(N+1)z^2]-z^{-2N+1}(z-2)}{(z-1)^2}.
\end{aligned}$$

7.3 We get from (7.10)

$$X^z(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}.$$

The region of convergence is $|z| > |a|$.

7.4 For $m = 1$ we get, by adding the third and fifth entries of Table 7.1,

$$(n+1)a^n v[n] \implies \frac{az^{-1}}{(1-az^{-1})^2} + \frac{1}{1-az^{-1}} = \frac{1}{(1-az^{-1})^2},$$

as required. Now assume that the formula holds for m, so

$$\sum_{n=0}^{\infty}\frac{(n+1)(n+2)\cdots(n+m)a^n}{m!}z^{-n} = \frac{1}{(1-az^{-1})^{m+1}}.$$

Differentiating with respect to z yields

$$-\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)\cdots(n+m)a^n}{m!}z^{-(n+1)} = -az^{-2}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)\cdots(n+m+1)a^n}{m!}z^{-n}$$
$$=\frac{d}{dz}\frac{1}{(1-az^{-1})^{m+1}} = -\frac{(m+1)az^{-2}}{(1-az^{-1})^{m+2}}.$$

Dividing both sides by $-(m+1)az^{-2}$ gives

$$\frac{1}{(1-az^{-1})^{m+2}} = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)\cdots(n+m+1)a^n}{(m+1)!}z^{-n},$$

which is the required equality for $m+1$.

7.5 The function $X^z(z) = \bar{z}$ is not analytic, because it does not satisfy the Cauchy-Riemann conditions. Therefore, it cannot be the z-transform of any sequence.

7.6

(a)

$$X^d[k] = X^z(e^{j2\pi k/N}).$$

(b) Substitute the inverse DFT formula for $x[n]$ in the definition of $X^z(z)$:

$$X^z(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}X^d[k]W_N^{nk}\right]z^{-n} = \frac{1}{N}\sum_{k=0}^{N-1}X^d[k]\sum_{n=0}^{N-1}(W_N^k z^{-1})^k$$
$$=\frac{1}{N}\sum_{k=0}^{N-1}X^d[k]\frac{1-(W_N^k z^{-1})^N}{1-W_N^k z^{-1}} = \frac{1}{N}\sum_{k=0}^{N-1}X^d[k]\frac{1-z^{-N}}{1-W_N^k z^{-1}}.$$

7.7

(a) If $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$, then $X^z(z)$ exists on the unit circle. Conversely, suppose that $X^z(z)$ exists for some z_0. Then, since the sequence is symmetric,

$$\sum_{n=-\infty}^{\infty}x[n]z_0^{-n} = \sum_{n=-\infty}^{\infty}x[n](1/z_0)^{-n},$$

so $X^z(z)$ exists for $1/z_0$ as well. Therefore, $X^z(z)$ exists on $|z_0| \le |z| \le |z_0|^{-1}$ (or $|z_0|^{-1} \le |z| \le |z_0|$). This implies that $X^z(z)$ exists on $|z| = 1$, so $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$.

(b)

$$X^z(z) = \frac{2}{3}\left[\frac{1}{1-2z^{-1}} - \frac{1}{1-0.5z^{-1}}\right].$$

The region of convergence is $0.5 < |z| < 2$ and the sequence is

$$x[n] = -\frac{2}{3}0.5^{|n|}.$$

7.8 Any sequence that grows faster than exponential does not have a z-transform. For example:

$$x[n] = \begin{cases} 2^{n^2}, & n \ge 0, \\ 0, & n < 0. \end{cases}$$

7.9 Differentiate the z-transform formula once to get

$$\frac{dX^z(z)}{dz} = -\sum_{n=-\infty}^{\infty}nx[n]z^{-(n+1)}.$$

Differentiate once again to get

$$\frac{d^2X^z(z)}{dz^2} = \sum_{n=-\infty}^{\infty} (n+1)nx[n]z^{-(n+2)} = z^{-2}\sum_{n=-\infty}^{\infty} (n+1)nx[n]z^{-n}.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} n^2x[n]z^{-n} = z^2\frac{d^2X^z(z)}{dz^2} - \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = z^2\frac{d^2X^z(z)}{dz^2} + z\frac{dX^z(z)}{dz}.$$

7.10 Let $w[n] = \{x * y\}[n]$. Then, since the two sequences are causal, it follows immediately that $w[0] = x[0]y[0]$. However, by the inverse z-transform formula (7.6), the equality given in the problem is simply $w[0] = x[0]y[0]$, so this equality is true.

7.11 Let $\{x_1[n], y_1[n]\}$ and $\{x_2[n], y_2[n]\}$ two pairs of input-output sequences, and α, β real constants. Then

$$\sum_{k=0}^{p} a_k y_1[n-k] + \sum_{k=0}^{q} b_k x_1[n-k] = 0,$$
$$\sum_{k=0}^{p} a_k y_2[n-k] + \sum_{k=0}^{q} b_k x_2[n-k] = 0.$$

Multiply the first equation by α, the second by β, and add to get

$$\sum_{k=0}^{p} a_k(\alpha y_1[n-k] + \beta y_2[n-k]) + \sum_{k=0}^{q} b_k(\alpha x_1[n-k] + \beta x_2[n-k]) = 0.$$

Therefore the response of the system to the input sequence $\alpha x_1[n] + \beta x_2[n]$ is $\alpha y_1[n] + \beta y_2[n]$. This proves linearity. To prove time invariance, assume that $x_2[n] = x_1[n-m]$ for some constant m. Then

$$\sum_{k=0}^{p} a_k y_2[n-k] + \sum_{k=0}^{q} b_k x_2[n-k] = \sum_{k=0}^{p} a_k y_2[n-k] + \sum_{k=0}^{q} b_k x_1[n-m-k].$$

But

$$\sum_{k=0}^{p} a_k y_1[n-m-k] + \sum_{k=0}^{q} b_k x_1[n-m-k] = 0.$$

Therefore

$$y_2[n] = y_1[n-m].$$

This proves time invariance.

7.12

(a) The common denominator of the sum on the right side is

$$\prod_{k=0}^{2}(1 - W_3^k z^{-1}) = 1 - z^{-3}.$$

Since all three poles are on the unit circle, the only value of A which makes the system stable is the one that cancels the common denominator, that is, $A = 1$.

(b) For $A = 1$ we get

$$H^z(z) = (b_0 + b_1 + b_2) + (b_0 + W_3 b_1 + W_3^2 b_2)z^{-1} + (b_0 + W_3^2 b_1 + W_3^4 b_2)z^{-2}$$
$$= \sum_{n=0}^{2}\left[\sum_{k=0}^{2} b_k W_3^{nk}\right] z^{-n}.$$

Therefore:

i.

$$h[n] = \begin{cases} \sum_{k=0}^{2} b_k W_3^{nk}, & 0 \le n \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

ii.

$$H^f(0) = H^z(1) = 3b_0, \quad H^f(2\pi/3) = H^z(W_3) = 3b_1, \quad H^f(4\pi/3) = H^z(W_3^2) = 3b_2.$$

iii.

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix}.$$

Therefore $\{b_0, b_1, b_2\}$ is the IDFT$_3$ of $\{h[n],\ 0 \le n \le 2\}$.

7.13 The impulse response of the continuous-time system described by the differential equation is

$$h(t) = e^{-\alpha t}, \quad t \ge 0.$$

Therefore,

$$y(t) = \int_0^\infty e^{-\alpha\tau} x(t-\tau)d\tau.$$

Substituting $t = nT$ gives

$$\begin{aligned} y(nT) &= \int_0^\infty e^{-\alpha\tau} x(nT-\tau)d\tau \\ &= \int_0^T e^{-\alpha\tau} x(nT-\tau)d\tau + \int_T^\infty e^{-\alpha\tau} x(nT-\tau)d\tau \\ &= \int_0^T e^{-\alpha\tau} x(nT-\tau)d\tau + \int_0^\infty e^{-\alpha(\sigma+T)} x(nT-T-\sigma)d\sigma \\ &= \int_0^T e^{-\alpha\tau} x(nT-\tau)d\tau + e^{-\alpha T}\int_0^\infty e^{-\alpha\sigma} x(nT-T-\sigma)d\sigma \\ &= \int_0^T e^{-\alpha\tau} x(nT-\tau)d\tau + e^{-\alpha T} y(nT-T). \end{aligned}$$

Defining

$$a = e^{-\alpha T}, \quad u[n] = \int_0^T e^{-\alpha\tau} x(nT-\tau)d\tau$$

gives

$$y[n] = ay[n-1] + u[n].$$

7.14

(a) If the system is causal, we must have $|z| > k$ and $|z| > k^{-1}$ for all $2 \le k \le 10$. This implies $|z| > 10$. The impulse response is

$$h[n] = \sum_{k=2}^{10} (k^n + k^{-n}), \quad n \ge 0.$$

(b) If the system is stable, the ROC must include the unit circle. This happens if and only if $|z| > k^{-1}$ and $|z| < k$ for all $2 \le k \le 10$. This implies $0.5 < |z| < 2$. The impulse response is [cf. Example 7.1 (2), (3)]

$$h[n] = \begin{cases} \sum_{k=2}^{10} k^{-n}, & n \ge 0, \\ -\sum_{k=2}^{10} k^n, & n < 0. \end{cases}$$

7.15

(a) In order for the system to be causal, it must have two poles at $z = 0$. Its transfer function is then

$$H^z(z) = \frac{(z^2+z+0.5)(z^2-z+0.5)}{z^2(z^2+0.25)}.$$

(b) The magnitude response is shown in Figure 7.1.

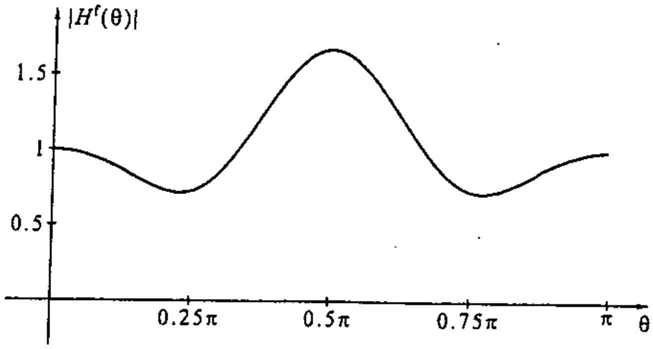

Figure 7.1 Pertaining to Solution 7.15.

(c) We have

$$G^z(z) = \sum_{n=0}^{\infty}(-1)^n h[n]z^{-n} = \sum_{n=0}^{\infty} h[n](-z)^{-n} = H^z(-z).$$

It is now easy to verify that the poles and zeros of $G^z(z)$ are identical to those of $H^z(z)$. In fact, the two systems are identical.

7.16 We have

$$\frac{3(1-z^{-1})}{(1-0.5z^{-1})(1-2z^{-1})} = \frac{1}{1-0.5z^{-1}} + \frac{2}{1-2z^{-1}}.$$

Since we know that the system is stable, the region of convergence must be $0.5 < |z| < 2$. Therefore,

$$h[n] = \begin{cases} -2 \times 2^n, & n < 0, \\ 0.5^n, & n \geq 0. \end{cases}$$

7.17

(a) The z-transform of the unit-step sequence is $1/(1-z^{-1})$. Therefore, the step response of the system is the inverse z-transform of $H^z(z)/(1-z^{-1})$, where $H^z(z)$ is the transfer function of the difference equation. The inverse z-transform can be computed by partial fraction expansion. Since we have assumed that $H^z(z)$ does not have a pole at $z = 1$, all the poles of $H^z(z)/(1-z^{-1})$ are simple. Note that, if $H^z(z)$ has $q = p$, then $H^z(z)/(1-z^{-1})$ will not have the coefficients c_k in its partial fraction expansion.

(b)

$$\frac{2+2.7z^{-1}-0.36z^{-2}}{(1+0.5z^{-1}-0.36z^{-2})(1-z^{-1})} = \frac{217}{57(1-z^{-1})} - \frac{4}{3(1-0.4z^{-1})} - \frac{9}{19(1+0.9z^{-1})}.$$

Therefore, the step response is

$$y[n] = \left[\frac{217}{57} - \frac{4}{3}0.4^n - \frac{9}{19}(-0.9)^n\right]u[n].$$

7.18

(a) We have from property (7.29) of the z-transform

$$H_1^z(z) = H^z(z/\gamma) = \frac{1+\gamma z^{-1}+\gamma^2 z^{-2}}{(1-0.8\gamma z^{-1})(1+0.5\gamma z^{-1})} = \frac{1+\gamma z^{-1}+\gamma^2 z^{-2}}{1-0.3\gamma z^{-1}-0.4\gamma^2 z^{-2}}.$$

The difference equation is

$$y[n] = 0.3\gamma y[n-1] + 0.4\gamma^2 y[n-2] + x[n] + \gamma x[n-1] + \gamma^2 x[n-2].$$

(b) If $|\gamma| < 1$ the new system is stable. If $|\gamma| > 1$, but $|\gamma| < 1.25$, it is still stable. If $|\gamma| \geq 1.25$, the new system is unstable.

(c) Let

$$H^z(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{\sum_{k=0}^{p} a_k z^{-k}}.$$

Then

$$H_1^z(z) = \frac{\sum_{k=0}^{q} b_k \gamma^k z^{-k}}{\sum_{k=0}^{p} a_k \gamma^k z^{-k}}.$$

The difference equation is

$$y[n] = -\sum_{k=1}^{p} a_k \gamma^k y[n-k] + \sum_{k=0}^{q} b_k \gamma^k x[n-k].$$

The system $H_1^z(z)$ is stable as long as $|\gamma|$ is smaller than the inverse of the pole of $H^z(z)$ nearest to the unit circle.

7.19

$$H^z(z) = \frac{a^{-1}(1 - a\cos\theta_0 z^{-1})}{1 - 2a\cos\theta_0 z^{-1} + a^2 z^{-2}} - a^{-1} = \frac{\cos\theta_0 z^{-1} - a z^{-2}}{1 - 2a\cos\theta_0 z^{-1} + a^2 z^{-2}}.$$

Therefore,

$$y[n] = 2a\cos\theta_0 y[n-1] - a^2 y[n-2] + \cos\theta_0 x[n-1] - a x[n-2].$$

7.20 The transfer function from $x[n]$ to $y[n]$ is

$$\frac{Y^z(z)}{X^z(z)} = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}}.$$

Also,

$$X^z(z) = K(1 - 0.5z^{-1})[U^z(z) - Y^z(z)].$$

Therefore,

$$Y^z(z) = \frac{K(z^{-1} - 0.5z^{-2})[U^z(z) - Y^z(z)]}{1 - 2z^{-1} + z^{-2}},$$

$$\frac{1 + (K-2)z^{-1} + (1 - 0.5K)z^{-2}}{1 - 2z^{-1} + z^{-2}} Y^z(z) = \frac{K(z^{-1} - 0.5z^{-2})U^z(z)}{1 - 2z^{-1} + z^{-2}},$$

$$\frac{Y^z(z)}{U^z(z)} = \frac{K(z^{-1} - 0.5z^{-2})}{1 + (K-2)z^{-1} + (1 - 0.5K)z^{-2}}.$$

We get from (7.63) that the system is stable if and only if

$$-1 < 1 - 0.5K < 1, \quad 0.5K > 0, \quad 4 - 1.5K > 0.$$

This gives

$$0 < K < \frac{8}{3}.$$

7.21 We have from Bob's report,

$$b_0 = h[0] = 1.$$

From Nick's report,

$$H^z(1) = \frac{1 + b_1}{1 + a_1} = 4.$$

From Dave's report,

$$|H^f(\pi/3)|^2 = \left|\frac{1 + b_1 e^{-j\pi/3}}{1 + a_1 e^{-j\pi/3}}\right|^2 = \frac{1 + 2b_1\cos(\pi/3) + b_1^2}{1 + 2a_1\cos(\pi/3) + a_1^2} = \frac{1 + b_1 + b_1^2}{1 + a_1 + a_1^2} = 4.$$

We get from Nick's equation

$$b_1 = 3 + 4a_1.$$

Substitution in Dave's equation gives

$$4a_1^2 + 8a_1 + 3 = 0.$$

The solutions of this equation are -0.5 and -1.5. Since we are given that the system is stable,

$$a_1 = -0.5, \quad b_1 = 1.$$

7.22 We have

$$X^z(z) = \frac{1}{1-0.5z^{-1}}, \quad Y^z(z) = \frac{1}{1-0.25z^{-1}}, \quad H^z(z) = \frac{1-0.5z^{-1}}{1-0.25z^{-1}}.$$

Therefore,

$$h[n] = 2\delta[n] - 0.25^n, \quad n \ge 0.$$

7.23

$$H^z(z) = \frac{1+z^{-1}}{1+0.5z^{-1}}, \quad X^z(z) = \frac{1}{1-z^{-2}} = \frac{1}{(1+z^{-1})(1-z^{-1})}.$$

Therefore,

$$Y^z(z) = \frac{1}{(1+0.5z^{-1})(1-z^{-1})} = \frac{1}{3}\left[\frac{1}{1+0.5z^{-1}} + \frac{2}{1-z^{-1}}\right],$$

$$y[n] = \frac{1}{3}[(-0.5)^n + 2], \quad n \ge 0.$$

7.24

(a) Since $\delta[n] = 0$ for all $n > 0$, $\delta[n-i] = 0$ for all $n > q$ and $0 \le i \le q$. Therefore, substitution of $x[n] = \delta[n]$ in the difference equation (7.45) gives zero on the right side; this implies (7.119).

(b) Equation (7.120) is obtained from (7.119) by using the latter p times, for $q+1 \le n \le q+p$.

(c) We get from (7.47) that

$$b_0 + b_1 z^{-1} + \cdots + b_q z^{-q} = (h[0] + h[1]z^{-1} + \cdots h[q]z^{-q} + \cdots)(1 + a_1 z^{-1} + \cdots + a_p z^{-p}).$$

If we expand the parentheses on the right side and equate powers of z^{-i} for $0 \le i \le q$, we get the system of equations (7.121).

7.25 We have

$$h[n] = 0.5(\delta[n] + 0.5^n), \quad n \ge 0,$$

so

$$g[0] = h[0] = 1, \quad g[1] = h[1] = 0.25, \quad g[2] = h[2] = 0.125.$$

Therefore, by (7.120),

$$\begin{bmatrix} g[0] & 0 \\ g[1] & g[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} g[1] \\ g[2] \end{bmatrix},$$

or

$$\begin{bmatrix} 1 & 0 \\ 0.25 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 0.25 \\ 0.125 \end{bmatrix}.$$

This gives

$$a_1 = -0.25, \quad a_2 = -0.0625.$$

Also,

$$g[3] = -a_1 g[2] - a_2 g[1] = \frac{3}{64}, \quad h[3] = \frac{1}{16},$$

so $g[3] \ne h[3]$.

7.26 Program 7.1 implements the Schur-Cohn test without recursive calls.

7.27 The three functions can be expressed by the power series

$$\exp(z^{-1}) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}, \quad \cos(z^{-1}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n}, \quad \sin(z^{-1}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-(2n+1)}.$$

Therefore (with $n \ge 0$ in all cases),

$$x_1[n] = \frac{1}{n!}, \quad x_2[n] = \begin{cases} \frac{(-1)^{(n/2)}}{n!}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \quad x_3[n] = \begin{cases} \frac{(-1)^{(n-1)/2}}{n!}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Program 7.1 Schur-Cohn test without recursive calls.

```
function s = scnonrec(a);
% Synopsis: s = scnonrec(a).
% Schur-Cohn stability test without recursive calls.
% Input:
% a: coefficients of polynomial to be tested.
% Output:
% s: 1 if stable, 0 if unstable.

s = 1;
for i = 1:length(a)-1,
   temp = a(length(a));
   if (abs(temp) >= 1), s = 0; break; end
   a = (1/(1-temp^2))*(a - temp*fliplr(a));
   a = a(1:length(a)-1);
end
```

7.28 The z-transform of the signal is obtained from the Fourier transform as

$$X^z(z) = \frac{1}{z+\frac{1}{6}-\frac{1}{6}z^{-1}} = \frac{z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{6}{5}\left(\frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1+\frac{1}{2}z^{-1}}\right).$$

Therefore, the corresponding causal sequence $x[n]$ is

$$x[n] = \frac{6}{5}\left[\left(\frac{1}{3}\right)^n - \left(-\frac{1}{2}\right)^n\right].$$

7.29

$$H^{\mathrm{f}}(\theta) = \frac{a_p + a_{p-1}e^{-j\theta} + \cdots + a_1 e^{-j\theta(p-1)} + e^{-j\theta p}}{1 + a_1 e^{-j\theta} + \cdots + a_{p-1}e^{-j\theta(p-1)} + a_p e^{-j\theta p}} = e^{-j\theta p}\frac{1 + a_1 e^{j\theta} + \cdots + a_{p-1}e^{j\theta(p-1)} + a_p e^{j\theta p}}{1 + a_1 e^{-j\theta} + \cdots + a_{p-1}e^{-j\theta(p-1)} + a_p e^{-j\theta p}}.$$

The numerator of the fraction on the right side is the conjugate of the denominator, so the absolute value of the fraction is 1. The absolute value of $e^{-j\theta p}$ is also 1, so the conclusion is that $|H^{\mathrm{f}}(\theta)| = 1$. This means that the magnitude response of the system is constant for all frequencies. However, the impulse response of the system is *not* $\delta[n]$, because the phase response is not constant.

7.30

(a) The poles are at

$$\alpha_k = (1-\varepsilon)^{1/p}e^{j(2k+1)\pi/p}, \quad 0 \le k \le p-1.$$

The zeros are at

$$\beta_k = (1-\varepsilon)^{1/p}e^{j2k\pi/p}, \quad 0 \le k \le p-1.$$

The system is stable.

(b) The following MATLAB code draws the pole-zero map.

```
p = 5; e = 0.1;
a = [1,zeros(1,p-1),1-e];
b = [1,zeros(1,p-1),-(1-e)];
plot(roots(a),'x')
hold on
plot(roots(b),'o'),grid,figure(1),pause
hold off
H = frqresp(b,a,501);
theta = (1/500)*(0:500);
plot(theta,abs(H)),grid,figure(1)
```

(c) The above MATLAB code also draws the magnitude response.

7.31 Program 7.2 implements the computation. Note that it uses the program `frqresp` given in the book.

Program 7.2 Frequency response of a rational transfer function from pole-zero decomposition.

```
function H = frqrspzp(v,u,C,K,theta);
% Synopsis: H = frqresp(v,u,C,K,theta).
% Freqeuncy response of b(z)/a(z) on a given frequency
% interval, where b(z)/a(z) is given in factored form.
% Input parameters:
% v,u,C: poles, zeros, and constant gain
% K: the number of frequency response points to compute
% theta: if absent, the K points are uniformly spaced on [0, pi];
%        if present and theta is a 1-by-2 vector, its entries are
%        taken as the end points of the interval on which K evenly
%        spaced points are placed; if the size of theta is different
%        from 2, it is assumed to be a vector of frequencies for which
%        the frequency response is to be computed, and K is ignored.
% Output:
% H: the frequency response vector.

p = length(v); q = length(u);
if (p > q), u = [u, zeros(1,p-q)]; end
if (q > p), v = [v, zeros(1,q-p)]; end
H = C;
for i = 1:max(p,q),
   if (nargin == 4),
   H = H.*frqresp([1,-u(i)],[1,-v(i)],K);
   else,
   H = H.*frqresp([1,-u(i)],[1,-v(i)],K,theta);
   end
end
```

7.32

(a) We have

$$x[n] = \sum_{k=-\infty}^{\infty} y[n-kN],$$

so

$$X_+^z(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{k=-\infty}^{\infty} z^{-kN} Y_+^z(z) = \frac{Y_+^z(z)}{1-z^{-N}}.$$

The region of convergence is $|z| > 1$.

(b)

$$Y_+^z(z) = 1 + z^{-1} + z^{-2} - z^{-3} - z^{-4} - z^{-5} = (1 + z^{-1} + z^{-2})(1 - z^{-3}).$$

Therefore,

$$X_+^z(z) = \frac{(1 + z^{-1} + z^{-2})(1 - z^{-3})}{1 - z^{-6}} = \frac{1 + z^{-1} + z^{-2}}{1 + z^{-3}}.$$

7.33

(a) The transfer functions of the three LTI systems are

$$H_1^z(z) = \frac{1}{1-0.5z^{-1}}, \quad H_2^z(z) = \frac{-2z^{-1}}{1-0.5z^{-1}}, \quad H_3^z(z) = \frac{1}{1-2.5z^{-1}+z^{-2}}.$$

The transfer function from $x[n]$ to $y_3[n]$ is

$$H^z(z) = [H_1^z(z) + H_2^z(z)]H_3^z(z) = \frac{1-2z^{-1}}{1-0.5z^{-1}} \cdot \frac{1}{(1-2z^{-1})(1-0.5z^{-1})} = \frac{1}{(1-0.5z^{-1})^2}.$$

We know from Table 7.1 that the inverse z-transform of $0.5z^{-1}/(1-0.5z^{-1})^2$ is $n0.5^n$. Therefore, the inverse z-transform of $1/(1-0.5z^{-1})^2$ is

$$h[n] = 2(n+1)0.5^{n+1}, \quad n \ge 0.$$

(b) Yes, because the poles are inside the unit circle.

(c)

$$\lim_{n\to\infty} y_3[n] = H^z(1) = 4.$$

(d) By (7.117),

$$Y_{\mathrm{zir}}^z(z) = -\frac{y[-2]a_2 + y[-1]a_1 + y[-1]a_2z^{-1}}{1+a_1z^{-1}+a_2z^{-2}} = \frac{1.5-z^{-1}}{1-2.5z^{-1}+z^{-2}}$$
$$= \frac{1}{6(1-0.5z^{-1})} + \frac{4}{3(1-2z^{-1})}.$$

Therefore,

$$y_{\mathrm{zir}}[n] = \frac{1}{6}0.5^n + \frac{4}{3}2^n, \quad n \ge 0.$$

(e) As we see from part d, the response to nonzero initial condition diverges, although the system is BIBO-stable. This is because $H_3^z(z)$ has an unstable pole at $z = 2$, which is canceled by a zero of $H_1^z(z) + H_2^z(z)$ at the same location.

7.34

(a) We have

$$\mathrm{sinc}(0.5n) = \frac{\sin(0.5\pi n)}{0.5\pi n} = \begin{cases} 1, & n = 0, \\ 0, & n = 2m,\ m \ne 0, \\ \dfrac{(-1)^m}{0.5\pi(2m-1)}, & n = 2m-1. \end{cases}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |\mathrm{sinc}(0.5n)) = 1 + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1} = \infty.$$

It is also easy to verify that

$$\sum_{n=-\infty}^{\infty} r^n|\mathrm{sinc}(0.5n)| = \infty$$

for all $r \ne 1$. Therefore, the sequence does not possess a z-transform.

(b) Let $r = p/q$, where p, q are coprime integers. Let

$$n = mq + i, \quad 0 \le m < \infty,\ 0 \le i \le q-1.$$

Then

$$\sum_{n=-\infty}^{\infty} |\mathrm{sinc}(rn)| = \frac{2q}{\pi p}\sum_{i=0}^{q-1}\sum_{m=0}^{\infty}\frac{\left|\sin\left(\pi pm + \frac{\pi pi}{q}\right)\right|}{mq+i} - 1 = \frac{2q}{\pi p}\sum_{i=0}^{q-1}\left|\sin\left(\frac{\pi pi}{q}\right)\right|\sum_{m=0}^{\infty}\frac{1}{mq+i} - 1 = \infty.$$

It is also easy to verify that

$$\sum_{n=-\infty}^{\infty} r^n |\text{sinc}(rn)| = \infty$$

for all $r \neq 1$. Therefore the sequence does not possess a z-transform.

7.35

(a) We have

$$\text{sinc}^2(0.5n) = \begin{cases} 1, & n = 0, \\ 0, & n = 2m,\ m \neq 0, \\ \dfrac{1}{0.25\pi^2(2m-1)^2}, & n = 2m - 1. \end{cases}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |\text{sinc}(0.5n)| = 1 + \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} < \infty.$$

Therefore, the z-transform exists on $|z| = 1$. On the other hand, it is easy to verify that

$$\sum_{n=-\infty}^{\infty} r^n |\text{sinc}(0.5n)|^2 = \infty$$

for all $r \neq 1$. Therefore the sequence possesses a z-transform only on the circle $|z| = 1$.

(b) Let $r = p/q$, where p, q are coprime integers. Then, as in Solution 7.35,

$$\sum_{n=-\infty}^{\infty} |\text{sinc}(rn)|^2 = \frac{2q^2}{\pi^2 p^2} \sum_{i=0}^{q-1} \left| \sin\left(\frac{\pi p i}{q}\right) \right|^2 \sum_{m=0}^{\infty} \frac{1}{(mq+i)^2} - 1 < \infty.$$

As before, the sequence possesses a z-transform only on the circle $|z| = 1$.

7.36

(a) We have

$$X^z(z) = \frac{2}{1 - 0.8z^{-1}} + \frac{1}{1 + 0.4z^{-1}}, \quad x[n] = 2 \times 0.8^n + (-0.4)^n.$$

Therefore,

$$y[n] = 2 \times 0.8^{2n} + (-0.4)^{2n} = 2 \times 0.64^n + 0.16^n,$$

and

$$Y^z(z) = \frac{2}{1 - 0.64z^{-1}} + \frac{1}{1 - 0.16z^{-1}} = \frac{3 - 0.96z^{-1}}{1 - 0.8z^{-1} + 0.1024z^{-2}}.$$

(b) We have

$$X^z(z) = \frac{1}{1 - (0.5 + j0.5)z^{-1}} + \frac{1}{1 - (0.5 - j0.5)z^{-1}}, \quad x[n] = (0.5 + j0.5)^n + (0.5 - j0.5)^n.$$

Therefore,

$$y[n] = (0.5 + j0.5)^{2n} + (0.5 - j0.5)^{2n} = (j0.5)^n + (-j0.5)^n,$$

and

$$Y^z(z) = \frac{1}{1 - j0.5z^{-1}} + \frac{1}{1 + j0.5z^{-1}} = \frac{1}{1 + 0.25z^{-2}}.$$

(c) Yes. To prove this, consider the partial fraction expansion of $X^z(z)$:

$$X^z(z) = \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k z^{-1}}.$$

Therefore,

$$x[n] = \sum_{k=1}^{p} A_k \alpha_k^n, \quad y[n] = \sum_{k=1}^{p} A_k (\alpha_k^2)^n.$$

The z-transform of $y[n]$ is

$$Y^z(z) = \sum_{k=1}^{p} \frac{A_k}{1 - (\alpha_k^2)z^{-1}},$$

which is a rational function.

(d) In the same way as in part c we get

$$Y^z(z) = \sum_{k=1}^{p} \frac{A_k}{1 - (\alpha_k^M)z^{-1}},$$

which is a rational function.

(e) There is no way to tell, since the odd-index points $x[2n+1]$ could have any values.

7.37 The linear equations (7.72) are

$$\begin{bmatrix} 2 & 2a_1 \\ a_1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} b_0 \\ 0 \end{bmatrix}.$$

The solution for c_0 is

$$c_0 = \frac{0.5b_0^2}{1 - a_1^2}.$$

Therefore,

$$\text{NG} = \frac{b_0^2}{1 - a_1^2}.$$

Chapter 8

Introduction to Digital Filters

8.1 If we eliminate all noise energy in the stop band and 0.75 of the noise energy in the transition band, we will be left with 2.625 percent of the original noise energy. Therefore, the theoretical achievable output SNR is 25.8 dB. Suppose that we agree to leave 3 percent of the noise energy, resulting in output SNR of 25.2 dB. The total noise in the stop band is not higher than $(0.97)\delta_s^2$, and this should be equal to 0.00375. Therefore, we get that $\delta_s = 0.0622$, or $A_s = 24.1$ dB. Also, $A_p = 0.05$ dB, $\theta_p = 0.025\pi$, and $\theta_s = 0.03\pi$.

8.2

(a) The frequency response of $G^z(z)$ is

$$G^f(\theta) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\theta n} = [1 + (-1)^n]h[n]e^{-j\theta n} = H^f(\theta) + H^f(\theta - \pi).$$

The filter $H^f(\theta)$ is low pass with cutoff frequency 0.25π and the filter $H^f(\theta - \pi)$ is high pass with cutoff frequency 0.75π. Therefore, $G^f(\theta)$ is band stop.

(b) The specifications of $G^f(\theta)$ are

$$\theta_{p,1} = 0.25\pi, \quad \theta_{s,1} = 0.35\pi, \quad \theta_{s,2} = 0.65\pi, \quad \theta_{p,2} = 0.75\pi,$$
$$\delta_{p,1} = \delta_{p,2} = 0.025, \qquad \delta_s = 0.01.$$

8.3 Suppose $z = e^{j\zeta}$ is a zero of $H^z(z)$ on the unit circle. Then $H^z(e^{j\zeta}) = 0$, implying that $H^f(\zeta) = 0$. Therefore, zeros on the unit circle can appear only at phase angles within the stop band of the filter. For convenience, we consider zeros only on the upper half of the unit circle, because those on the lower half are their conjugates. A low-pass filter can have zeros from a certain angle upward, possibly including the point $z = -1$ (or $\theta = \pi$). A high-pass filter can have zeros from a certain angle downward, possibly including the point $z = 1$ (or $\theta = 0$). The set of zeros of a band-pass filter on the unit circle appears as a union of zeros of a low-pass filter and those of a high-pass filter. The zeros of a band-pass filter on the unit circle must occupy only the midrange, without reaching either $z = 1$ or $z = -1$.

8.4 The discrete-time signal is $x[n] = \cos(0.25\pi n)$. The magnitude response of the filter at frequency $\theta = 0.25\pi$ is 3, and the phase response is -0.25π. Therefore,

$$y[n] = 3\cos(0.25\pi n - 0.25\pi).$$

8.5

(a) we have

$$H^z(1) = \sum_{n=0}^{N} h[n], \quad H^z(-1) = \sum_{n=0}^{N} (-1)^n h[n].$$

Therefore, a necessary and sufficient condition for a zero at $z = 1$ is $\sum_{n=0}^{N} h[n] = 0$, and a necessary and sufficient condition for a zero at $z = -1$ is $\sum_{n=0}^{N}(-1)^n h[n] = 0$.

(b)

Filter type	I	II	III	IV
Must have zero at $z = 1$	no	no	yes	yes
Must have zero at $z = -1$	no	yes	yes	no

Explanation:

- a type-I filter satisfies $h[n] = h[N-n]$ with N even, so neither condition holds in general.
- a type-II filter satisfies $h[n] = h[N-n]$ with N odd, so necessarily $\sum_{n=0}^{N}(-1)^n h[n] = 0$.
- a type-III filter satisfies $h[n] = -h[N-n]$ with N even, so necessarily $\sum_{n=0}^{N} h[n] = 0$ as well as $\sum_{n=0}^{N}(-1)^n h[n] = 0$.
- a type-IV filter satisfies $h[n] = -h[N-n]$ with N odd, so necessarily $\sum_{n=0}^{N} h[n] = 0$.

8.6 A type-I filter does not necessarily have zeros either at $z = 1$ or at $z = -1$, so neither form is applicable to it. A type-II filter must have a zero at $z = -1$, so it can be expressed as $H^z(z) = (1 + z^{-1})F^z(z)$. A type-III filter must have zeros at both $z = 1$ and $z = -1$, so it can be expressed as $H^z(z) = (1 - z^{-2})F^z(z)$. Finally, a type-IV filter must have a zero at $z = 1$, so it can be expressed as $H^z(z) = (1 - z^{-1})F^z(z)$. For a type-II filter:

$$h[0] + h[1]z^{-1} + \cdots + h[N]z^{-N} = (1 + z^{-1})(f[0] + f[1]z^{-1} + \cdots + f[N-1]z^{-(N-1)}),$$

so

$$h[0] = f[0], \quad , h[N] = f[N-1], \quad h[n] = f[n] + f[n-1], \; 1 \le n \le N-1.$$

For a type-III filter:

$$h[0] + h[1]z^{-1} + \cdots + h[N]z^{-N} = (1 - z^{-2})(f[0] + f[1]z^{-1} + \cdots + f[N-2]z^{-(N-2)}),$$

so

$$h[0] = f[0], \quad , h[1] = f[1], \quad h[N-1] = -f[N-3], \quad h[N] = -f[N-2], \quad h[n] = f[n] - f[n-2], \; 2 \le n \le N-2.$$

For a type-IV filter:

$$h[0] + h[1]z^{-1} + \cdots + h[N]z^{-N} = (1 - z^{-1})(f[0] + f[1]z^{-1} + \cdots + f[N-1]z^{-(N-1)}),$$

so

$$h[0] = f[0], \quad , h[N] = -f[N-1], \quad h[n] = f[n] - f[n-1], \; 1 \le n \le N-1.$$

8.7

(a) We have

$$H^f(\theta) = b_0 \frac{\prod_{i=1}^{q}(1 - \beta_i e^{-j\theta})}{\prod_{i=1}^{p}(1 - \alpha_i e^{-j\theta})}.$$

Therefore,

$$G^f(\theta) = b_0 \frac{\prod_{i=1}^{q}(1 + \beta_i e^{-j\theta})}{\prod_{i=1}^{p}(1 + \alpha_i e^{-j\theta})} = b_0 \frac{\prod_{i=1}^{q}(1 - \beta_i e^{-j(\theta-\pi)})}{\prod_{i=1}^{p}(1 - \alpha_i e^{-j(\theta-\pi)})} = H^f(\theta - \pi).$$

(b) In this case $G^f(\theta)$ is identical to $H^f(\theta)$, so it follows from part a that

$$H^f(\theta) = H^f(\theta - \pi).$$

In particular, $H^f(0) = H^f(\pi)$. Thus, the filter can be neither low-pass nor high-pass. It can be band-pass or band-stop. In either case, its magnitude response is symmetric around 0.5π.

(c)

$$G^z(z) = \frac{(z+1)^2}{z^2 + 1.212436z + 0.49}.$$

The magnitude of $G^f(\theta)$ is nonzero at $\theta = 0$ and zero at $\theta = \pi$. The two poles are not very close to the unit circle. Therefore, the filter is low-pass.

8.8 We have

$$H^z(z) = 1 - 1.5z^{-1} + 0.5z^{-2} = (1 - 0.5z^{-1})(1 - z^{-1}).$$

Therefore,

$$H^f(\theta) = (1 - 0.5e^{-j\theta})(1 - e^{-j\theta}) = (1.25 - \cos\theta)^{1/2} \arctan\frac{\sin\theta}{1 - 0.5\cos\theta} \cdot 2 \cdot \sin(0.5\theta)e^{j(0.5\pi - 0.5\theta)}.$$

Note that $\arctan[\sin\theta/(1 - 0.5\cos\theta)]$ is a continuous function of θ, because $1 - 0.5\cos\theta$ is strictly positive for all θ. Therefore,

$$A(\theta) = 2(1.25 - \cos\theta)^{1/2}\sin(0.5\theta), \quad \phi(\theta) = \arctan\frac{\sin\theta}{1 - 0.5\cos\theta} + 0.5\pi - 0.5\theta.$$

8.9 Consider a filter $H^z(z)$ whose zeros are all on the unit circle. According to Theorem 8.5 such a filter is minimum phase, since each zero is equal to its conjugate inverse. Such a filter is also linear phase. To show this, write $H^z(z)$ as

$$H^z(z) = h[0](1 - z^{-1})^{N_1}(1 + z^{-1})^{N_2}\prod_{k=1}^{N_3}(1 - \cos\zeta_k z^{-1} + z^{-2}),$$

where N_1 is the number of zeros at $z = 1$, N_2 is the number of zeros at $z = -1$, and N_3 is the number of complex conjugate pairs of zeros on the unit circle. Then

$$H^f(\theta) = h[0](1 - e^{-j\theta})^{N_1}(1 + e^{-j\theta})^{N_2}\prod_{k=1}^{N_3}(1 - \cos\zeta_k e^{-j\theta} + e^{-j2\theta}).$$

We have,

$$\begin{aligned}
(1 - e^{-j\theta})^{N_1} &= e^{-j0.5N_1\theta} j^{N_1}(2\sin 0.5\theta)^{N_1}, \\
(1 + e^{-j\theta})^{N_2} &= e^{-j0.5N_2\theta}(2\cos 0.5\theta)^{N_2}, \\
(1 - \cos\zeta_k e^{-j\theta} + e^{-j2\theta}) &= e^{-j\theta}(\cos\theta - \cos\zeta_k).
\end{aligned}$$

Therefore,

$$H^f(\theta) = j^{N_1}e^{-j(0.5N_1 + 0.5N_2 + N_3)\theta}(2\sin 0.5\theta)^{N_1}(2\cos 0.5\theta)^{N_2}\prod_{k=1}^{N_3}(\cos\theta - \cos\zeta_k).$$

The factor j^{N_1} is either ± 1 or $\pm e^{j0.5\pi}$. Therefore, $H^f(\theta)$ is linear phase.

8.10 As said in the hint, $H^z(z)$ can be factored as

$$H^z(z) = h[0](1 - z^{-1})^{N_1}(1 + z^{-1})^{N_2}\prod_{k=1}^{N_3}(1 - 2\cos\zeta_k z^{-1} + z^{-2}).$$

Now,

$$1 - e^{-j\theta} = 2\sin(0.5\theta)e^{j(0.5\pi - 0.5\theta)}, \quad 1 + e^{-j\theta} = 2\cos(0.5\theta)e^{-j0.5\theta},$$

and

$$1 - 2\cos\zeta_k e^{-j\theta} + e^{-j2\theta} = 2(\cos\theta - \cos\zeta_k)e^{-j\theta}.$$

As we see, each factor has linear phase. The product is

$$H^f(\theta) = 2^{N_1 + N_2 + N_3}h[0][\sin(0.5\theta)]^{N_1}[\cos(0.5\theta)]^{N_2}\prod_{k=1}^{N_3}(\cos\theta - \cos\zeta_k)e^{j[0.5N_1\pi - 0.5(N_1 + N_2 + 2N_3)\theta]}.$$

As we see, $H^f(\theta)$ has linear phase.

8.11

(a) We have

$$H^{\mathrm{f}}(\theta) + H^{\mathrm{f}}(\theta - \pi) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\theta n} + \sum_{n=-\infty}^{\infty} h[n]e^{-j(\theta-\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} h[n](1 + e^{j\pi n})e^{-j\theta n} = \sum_{n=-\infty}^{\infty} h[n](1 + (-1)^n)e^{-j\theta n}$$

$$= 2\sum_{n=-\infty}^{\infty} h[2n]e^{-j2\theta n} = c.$$

This equality holds if and only if

$$h[2n] = \begin{cases} 0.5c, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Note that the odd-indexed coefficients are arbitrary. However, for a zero-phase half-band filter they have to be symmetric.

(b)

$$H^{\mathrm{f}}(0.5\pi) = H^{\mathrm{z}}(j) = 0.5c + \sum_{n=0}^{\infty} h[2n+1](j)^{-(2n+1)} + \sum_{n=0}^{\infty} h[-(2n+1)](j)^{(2n+1)}$$

$$= 0.5c + \sum_{n=0}^{\infty} \{h[2n+1] - h[-(2n+1)]\}(j)^{-(2n+1)} = 0.5c.$$

Note that we have used the symmetry of the coefficients.

(c) Using the same derivation as above, we get for this case

$$h[2n] = \begin{cases} 0.5c, & n = M, \\ 0, & n \neq M. \end{cases}$$

(d) By part b, both $h[0]$ and $h[4M]$ must be zero. But then we can define

$$g[n] = h[n+1], \quad 0 \le n \le 4M - 2.$$

The filter $g[n]$ is of order $4M - 2$ and its amplitude is equal to that of $h[n]$. Its phase term is $e^{-j(2M-1)\theta}$, so its group delay is $2M - 1$. All odd-indexed coefficients of this filter are zero, except $g[2M-1]$, which is 0.5.

8.12

(a) We have

$$\frac{Y^{\mathrm{z}}(z)}{X^{\mathrm{z}}(z)} = \frac{1}{a(z)} \implies X^{\mathrm{z}}(z) = a(z)Y^{\mathrm{z}}(z).$$

Therefore, we can always recover $x[n]$ by passing it through the causal FIR filter $a(z)$, that is

$$x[n] = y[n] + a_1 y[n-1] + \cdots + a_p y[n-p].$$

(b) Now we have

$$\frac{Y^{\mathrm{z}}(z)}{X^{\mathrm{z}}(z)} = b(z) \implies X^{\mathrm{z}}(z) = \frac{Y^{\mathrm{z}}(z)}{b(z)}.$$

It is possible to recover $x[n]$ if and only if $1/b(z)$ is RCSR; that is, if and only if $b_0 \neq 0$ and all the zeros of $b(z)$ are inside the unit circle. When this condition holds we have

$$x[n] = b_0^{-1}(y[n] - b_1 x[n-1] - \cdots - b_q x[n-q]).$$

(c) In this case we have

$$\frac{Y^{\mathrm{z}}(z)}{X^{\mathrm{z}}(z)} = \frac{b(z)}{a(z)} \implies X^{\mathrm{z}}(z) = \frac{a(z)Y^{\mathrm{z}}(z)}{b(z)}.$$

Now it is possible to recover $x[n]$ if and only if $a(z)/b(z)$ is RCSR, that is, if and only if $b_0 \neq 0$ and all the zeros of $b(z)$ are inside the unit circle. When this condition holds we have

$$x[n] = b_0^{-1}(y[n] + a_1 y[n-1] + \cdots + a_p y[n-p] - b_1 x[n-1] - \cdots - b_q x[n-q]).$$

8.13 Let $H^f(\theta) = b(e^{j\theta})/a(e^{j\theta})$ and let the continuous-phase representations of the numerator and denominator be

$$b(e^{j\theta}) = A_{\text{num}}(\theta)\exp\{j\phi_{\text{num}}(\theta)\}, \quad a(e^{j\theta}) = A_{\text{den}}(\theta)\exp\{j\phi_{\text{den}}(\theta)\}.$$

Therefore, the continuous-phase representation of $H^f(\theta)$ is

$$A(\theta) = \frac{A_{\text{num}}(\theta)}{A_{\text{den}}(\theta)}, \quad \phi(\theta) = \phi_{\text{num}}(\theta) - \phi_{\text{den}}(\theta).$$

Finally, differentiating $\phi(\theta)$ with respect to θ gives

$$\tau_g(\theta) = \tau_{g,\text{num}}(\theta) - \tau_{g,\text{den}}(\theta).$$

8.14 For a minimum-phase filter, the early impulse response coefficients (the ones nearer to $n = 0$) have relatively large magnitudes. For the nonminimum-phase filters, impulse response coefficients further away from $n = 0$ have relatively large magnitudes. Put in other words, most of the energy of the impulse response sequence of a minimum-phase filter is concentrated at the initial part of the sequence. Nonminimum-phase filters do not have this property.

8.15

(a) The zeros of $H_1^z(z)$ are at $-1/4$ and $1/16$. The zeros of $H_2^z(z)$ are at -4 and $1/16$. The zeros of $H_3^z(z)$ are at $-1/4$ and 16. The zeros of $H_4^z(z)$ are at -4 and 16.

(b) The following MATLAB code plots the phase responses.

```
h = [1,3/16,-1/64;
     1/4,63/64,-1/16;
    -1/16,63/64,1/4;
    -1/64, 3/16, 1];
H = []; D = [];
for i = 1:4,
   H = [H; frqresp(h(i,:),1,501)];
   D = [D; grpdly(h(i,:),1,501)];
end
theta = (1/500)*(0:500);
plot(theta,unwrap(angle(H)')),grid,figure(1),pause
plot(theta,D'),grid,figure(1)
```

(c) The above MATLAB code also plots the group delays.

(d) The filter with both zeros inside the unit circle has the minimum group delay and the minimum value of $-\psi(\theta)$ at all frequencies. The filter with both zeros outside the unit circle has the maximum group delay and the maximum value of $-\psi(\theta)$ at all frequencies. The other two filters are between the two extremes.

8.16

(a) We have

$$H^z(z) = \prod_{k=1}^{p} \frac{\bar{\alpha}_k - z^{-1}}{1 - \alpha_k z^{-1}} = (-1)^p \frac{z^{-p}\prod_{k=1}^{p}(1 - \bar{\alpha}_k z)}{\prod_{k=1}^{p}(1 - \alpha_k z^{-1})} = (-1)^p \frac{z^{-p}\prod_{k=1}^{p}(1 - \alpha_k z)}{\prod_{k=1}^{p}(1 - \alpha_k z^{-1})}.$$

The last equality holds since for each complex α_k, $\bar{\alpha}_k$ also appears in the product and vice versa. Now observe that the numerator on the right side [except for the sign term $(-1)^p$] is obtained from the denominator by replacing z^{-1} by z and multiplying the result by z^{-p}. This is exactly the defining relationship of the numerator of $H^z(z)$ in Problem 7.29 to its denominator. Therefore, $H^z(z)$ in (8.69) has the form shown in Problem 7.29, except for the sign factor $(-1)^p$.

(b) By the same argument, if $a(z)$ is factored as

$$a(z) = \prod_{k=1}^{p}(1 - \alpha_k z^{-1})$$

and we replace z^{-1} by z and multiply the result by z^{-p}, we will get $(-1)^p \prod_{k=1}^{p}(\bar{\alpha}_k - z^{-1})$, which is identical to the numerator of (8.69), except for the sign factor $(-1)^p$.

8.17 The filter whose transfer function is

$$H^L(s) = \frac{1 - e^{-sT}}{s}$$

is FIR. This is exactly the zero-order hold, studied in Section 3.4.

8.18

(a) The given filter satisfies (see Example 8.11)

$$|H^f(\theta)| = 1.$$

Therefore, the amplitudes of the two output signals are 1 and 3, respectively.

(b) We have

$$\measuredangle H^f(\theta) = \arctan\frac{-\sin\theta}{0.5 + \cos\theta} - \arctan\frac{-0.5\sin\theta}{1 + 0.5\cos\theta}.$$

So,

$$\measuredangle H^f(0.1\pi) = -0.1055, \quad \measuredangle H^f(0.4\pi) = -0.4752.$$

(c) No, since no rational IIR filter can be distortion free. We also see from part b that

$$\tau_p(0.1\pi) = 1.055, \quad \tau_p(0.4\pi) = 1.188,$$

so the phase delay is not constant.

8.19

(a) Express the factors corresponding to zeros inside the unit circle as

$$z - \beta_{i,k} = z(1 - \beta_{i,k}z^{-1}).$$

Express the factors corresponding to zeros outside the unit circle as

$$z - \beta_{o,k} = -\beta_{o,k}(1 - \beta_{o,k}^{-1}z).$$

Express the factors corresponding to poles as

$$z - \alpha_k = z(1 - \alpha_k z^{-1}).$$

Substitution of these expressions in (8.76) gives (8.77).

(b) Let $\alpha_k = \alpha_{k,r} + j\alpha_{k,i}$. Then

$$1 - \alpha_k e^{-j\theta} = 1 - \alpha_{k,r}\cos\theta - \alpha_{k,i}\sin\theta + j(\alpha_{k,r}\sin\theta - \alpha_{k,i}\cos\theta).$$

Since $|\alpha_k| < 1$, we have

$$\alpha_{k,r}\cos\theta + \alpha_{k,i}\sin\theta < 1 \quad \text{for all } \theta.$$

Therefore, the real part of $1 - \alpha_k e^{-j\theta}$ is positive for all θ, implying that the phase of $1 - \alpha_k e^{-j\theta}$ is in the range $(-0.5\pi, 0.5\pi)$ for all θ. Since the phase does not pass through π and since $1 - \alpha_k e^{-j\theta}$ is never zero, the phase of this factor is continuous in θ. The same argument applies to the factor $1 - \beta_{i,k}e^{-j\theta}$.

(c) Since $|\beta_{i,k}| > 1$, we have that $|\beta_{i,k}^{-1}| < 1$, so the same argument as in part b shows that the phase of $1 - \beta_{i,k}^{-1}e^{j\theta}$ is in the range $(-0.5\pi, 0.5\pi)$ for all θ, hence it is continuous in θ.

(d) Let $\beta_{u,k} = e^{j\zeta_{u,k}}$. Then

$$e^{j\theta} - \beta_{u,k} = e^{j\theta} - e^{j\zeta_{u,k}} = e^{j0.5(\theta+\zeta_{u,k})}[e^{j0.5(\theta-\zeta_{u,k})} - e^{-j0.5(\theta-\zeta_{u,k})}] = 2je^{j0.5(\theta+\zeta_{u,k})}\sin(0.5\theta - 0.5\zeta_{u,k})$$
$$= 2e^{j0.5(\pi+\theta+\zeta_{u,k})}\sin(0.5\theta - 0.5\zeta_{u,k}).$$

The right side is a product of a real function of θ and a linear-phase factor, as required.

(e) Since the zeros outside the unit circle must either be real or appear in conjugate pairs, their product is real. Also, $e^{j\theta(-q+r_i)}$ is obviously a linear-phase factor.

(f) It follows from the preceding parts that the continuous-phase representation of $H^f(\theta)$ is

$$A(\theta) = b_{q-r}\prod_{k=1}^{r_o}(-\beta_{o,k}) \cdot \frac{\left[\prod_{k=1}^{r_i}|1-\beta_{i,k}e^{-j\theta}|\right]\left[\prod_{k=1}^{r_o}|1-\beta_{o,k}^{-1}e^{j\theta}|\right]\left[\prod_{k=1}^{r_u}2\sin(0.5\theta-0.5\zeta_{u,k})\right]}{\prod_{k=1}^{p}|1-\alpha_k e^{-j\theta}|},$$

$$\phi(\theta) = (-q+r_i)\theta + \sum_{k=1}^{r_i}\measuredangle(1-\beta_{i,k}e^{-j\theta}) + \sum_{k=1}^{r_o}\measuredangle(1-\beta_{o,k}^{-1}e^{j\theta}) + \sum_{k=1}^{r_u}0.5(\pi+\theta+\zeta_{u,k}) - \sum_{k=1}^{p}\measuredangle(1-\alpha_k e^{-j\theta}).$$

8.20 The covariance sequence of the output is given by

$$\kappa_y[m] = \gamma_v\sum_{n=-\infty}^{\infty}h[n]h[n+m].$$

However, since $h[n]$ is nonzero only for $0 \le n \le N$, we get

$$\kappa_y[m] = \begin{cases}\gamma_v\sum_{n=0}^{N-m}h[n]h[n+m], & 0 \le m \le N,\\ \gamma_v\sum_{n=-m}^{N}h[n]h[n+m], & -N \le m < 0,\\ 0, & |m| > N.\end{cases}$$

Therefore, the covariance sequence is nonzero only on a finite range of indices m, that is, on $-N \le m \le N$.

8.21

(a) The signal and noise outputs at time $N-1$ are given by

$$y_1[N-1] = \sum_{m=0}^{N-1}h[m]x[N-1-m], \quad y_2[N-1] = \sum_{m=0}^{N-1}h[m]v[N-1-m].$$

We have by the Cauchy-Schwarz inequality (2.145),

$$|y_1[N-1]|^2 \le \left(\sum_{m=0}^{N-1}h^2[m]\right)\left(\sum_{m=0}^{N-1}x^2[N-1-m]\right).$$

Also,

$$E(|y_2[N-1]|^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|H^f(\theta)|^2 d\theta = \sum_{m=0}^{N-1}h^2[m].$$

Therefore,

$$\mathrm{SNR_o} = \frac{|y_1[N-1]|^2}{E(|y_2[N-1]|^2} \le \sum_{m=0}^{N-1}x^2[N-1-m].$$

Furthermore, choosing

$$h[n] = cx[N-1-n], \quad 0 \le n \le N-1,$$

where c is an arbitrary nonzero constant, gives

$$\mathrm{SNR_o} = \frac{\left(\sum_{m=0}^{N-1}cx^2[N-1-m]\right)^2}{\sum_{m=0}^{N-1}c^2x^2[N-1-m]} = \sum_{m=0}^{N-1}x^2[N-1-m],$$

so $h[n]$ thus defined maximizes $\mathrm{SNR_o}$.

(b) As we saw in part a,

$$(\mathrm{SNR_o})_{\max} = \sum_{m=0}^{N-1}x^2[N-1-m].$$

(c) Using the same argument as in part a, but with the Cauchy-Schwarz inequality (2.146), gives

$$h[n] = cx[N-1-n], \quad -\infty < n \le N-1,$$

where c is an arbitrary nonzero constant. In this case the filter is causal IIR, while in part a it was causal FIR.

8.22

(a) We have

$$x[n] = 0.5\cos[(\theta_c - \theta_m)n] + 0.5\cos[(\theta_c + \theta_m)n].$$

Therefore, the output of the filter is

$$y[n] = 0.5\cos[(\theta_c + \theta_m)n].$$

(b) This time the output will be

$$y[n] = A_1\cos[(\theta_c + \theta_m)n + \phi_1] + A_2\cos[(\theta_c - \theta_m)n + \phi_2],$$

where $0.495 \le A_1 \le 0.505$ and $A_2 \le 0.0005$. The first component is distorted because of the pass-band tolerance and the second is present because of the finite stop-band attenuation.

(c) We have

$$X^f(\theta) = 0.5S^f(\theta - \theta_c) + 0.5S^f(\theta + \theta_c).$$

An ideal filter would eliminate the energy in the frequency range $[\theta_c - 1.5\theta_m, \theta_c - 0.5\theta_m]$ and would retain the energy in the range $[\theta_c + 0.5\theta_m, \theta_c + 1.5\theta_m]$. In other words, it would retain only the positive part of the spectrum of $s[n]$. A practical filter would distort the positive part somewhat and would pass about 0.1 percent of the energy of the negative part.

Chapter 9

Finite Impulse Response Filters

9.1 The frequency response of a type-II filter is zero at $\theta = 0$, so $H^z(z)$ must have a zero at $z = 1$. The frequency response of a type-IV filter is zero at $\theta = \pi$, so $H^z(z)$ must have a zero at $z = -1$. The frequency response of a type-III filter is zero at both $\theta = 0$ and $\theta = \pi$, so $H^z(z)$ must have zeros at $z = \pm 1$. Therefore, the three filter types must have factors as stated in the problem.

9.2

(a) Define

$$g_1[n] = 0.5(h[n] + h[N-n]), \quad g_2[n] = 0.5(h[n] - h[N-n]).$$

Then the order of $g_1[n]$ and $g_2[n]$ is clearly not greater than N. Also,

$$g_1[n] + g_2[n] = h[n],$$

as required. The filter $g_1[n]$ is symmetric, so it has exact linear phase. The filter $g_2[n]$ is antisymmetric, so it has generalized linear phase.

(b) When N is even, $G_1^z(z)$ is type I, and $G_2^z(z)$ is type III. When N is odd, $G_1^z(z)$ is type II, and $G_2^z(z)$ is type IV.

(c) We have

$$G_1^f(\theta) = A_1(\theta)e^{-j0.5N\theta},$$
$$G_2^f(\theta) = A_2(\theta)e^{j(0.5\pi - 0.5N\theta)} = jA_2(\theta)e^{-j0.5N\theta}.$$

Therefore,

$$H^f(\theta) = [A_1(\theta) + jA_2(\theta)]e^{-j0.5N\theta},$$

and

$$|H^f(\theta)| = [A_1^2(\theta) + A_2^2(\theta)]^{1/2}.$$

9.3 We have

$$\begin{aligned} H^f(\theta) &= H_1^f(\theta)H_2^f(\theta) + H_3^f(\theta) \\ &= A_1(\theta)e^{j(\phi_{0,1} - 0.5N_1\theta)}A_2(\theta)e^{j(\phi_{0,2} - 0.5N_2\theta)} + A_3(\theta)e^{j(\phi_{0,3} - 0.5N_3\theta)} \\ &= [A_1(\theta)A_2(\theta)e^{j(\phi_{0,1} + \phi_{0,2})} + A_3(\theta)e^{j(\phi_{0,3})}]e^{-j0.5N_3\theta}. \end{aligned}$$

In four out of the eight cases we will get linear phase, according to the following table:

$\phi_{0,1}$	$\phi_{0,2}$	$\phi_{0,3}$	symmetry type
0	0	0	symmetric
0.5π	0	0.5π	antisymmetric
0	0.5π	0.5π	antisymmetric
0.5π	0.5π	0	symmetric

9.4 Type I:

$$h[n] = \{1, 3, -2, 3, 1\}.$$

Type II:

$$h[n] = \{1, 3, -2, -2, 3, 1\}.$$

Type III:

$$h[n] = \{1, 3, -2, 0, 2, -3, -1\}.$$

Type IV:

$$h[n] = \{1, 3, -2, 2, -3, -1\}.$$

9.5 The zero β_1 can stand by itself. The zero β_2 entails three additional zeros, at $0.5e^{-j\pi/3}$ and $2e^{-j\pi/3}$. The zero β_3 entails an additional zero at -0.2. Finally, the zero β_4 entails an additional zero at $-j$. Therefore the total number of zeros is at least nine, and this is also the minimal order of the filter. Multiplying out all first-order factors gives the transfer function (up to a constant gain)

$$\begin{aligned} H^z(z) = 1 &+ 1.7z^{-1} - 8.45z^{-2} + 30.75z^{-3} - 38.5z^{-4} \\ &+ 38.5z^{-5} - 30.75z^{-6} + 8.45z^{-7} - 1.7z^{-8} - z^{-9}. \end{aligned}$$

The filter has odd order and is antisymmetric, so it is type IV.

9.6 A linear-phase FIR filter cannot possibly have all its zeros inside the unit circle. We distinguish between two cases:

(a) If $H^z(z)$ has at least one zero on the unit circle, then $1/H^z(z)$ is not stable, since its region of convergence cannot possibly include the unit circle.

(b) If $H^z(z)$ has no zeros on the unit circle, it must have $N/2$ zeros inside the unit circle and $N/2$ zeros outside it. Then $1/H^z(z)$ cannot be both causal and stable. However, $1/H^z(z)$ is stable, but not causal, if we choose its region of convergence as a disc including the unit circle.

9.7

(a) Let $H_1^z(z)$, $H_2^z(z)$ be linear-phase filters, so

$$H_k^f(\theta) = A_k(\theta)e^{j(\phi_{0,k} - 0.5N_k\theta)}, \quad k = 1, 2.$$

Then

$$H_1^f(\theta)H_2^f(\theta) = A_1(\theta)A_2(\theta)\exp\{j[(\phi_{0,1} + \phi_{0,2}) - 0.5(N_1 + N_2)\theta)]\}.$$

Therefore, the cascade connection is a linear-phase filter of order $N_1 + N_2$. Its amplitude function is

$$A(\theta) = \begin{cases} -A_1(\theta)A_2(\theta), & \phi_{0,1} = \phi_{0,2} = 0.5\pi, \\ A_1(\theta)A_2(\theta), & \text{otherwise.} \end{cases}$$

Its initial phase is

$$\phi_0 = \begin{cases} 0, & \phi_{0,1} = \phi_{0,2}, \\ 0.5\pi, & \text{otherwise.} \end{cases}$$

(b) If the two filters have different orders, it is easy to see that the parallel connection does not have linear phase in general. If they have the same order then the parallel connection is linear phase if and only if $\phi_{0,1} = \phi_{0,2}$; that is, the filters are either both symmetric or both antisymmetric. In this case the amplitude function of the parallel connection is $A_1(\theta) + A_2(\theta)$ and its initial phase is the same as that of the two filters.

9.8 First let us consider simple filters, whose zeros are limited to the individual cases.

(a) The filter $H^z(z) = 1 - z^{-1}$ is linear phase of type IV.

(b) The filter $H^z(z) = 1 + z^{-1}$ is linear phase of type II.

(c) The filter

$$H^z(z) = (1 - e^{j\zeta}z^{-1})(1 - e^{-j\zeta}z^{-1}) = 1 - 2\cos\zeta z^{-1} + z^{-2}$$

is linear phase of type I.

(d) The filter

$$H^z(z) = (1 - rz^{-1})(1 - r^{-1}z^{-1}) = 1 - (r + r^{-1})z^{-1} + z^{-2}$$

is linear phase of type I.

(e) The filter

$$\begin{aligned}
H^z(z) &= (1 - re^{j\zeta}z^{-1})(1 - re^{-j\zeta}z^{-1})(1 - r^{-1}e^{j\zeta}z^{-1})(1 - r^{-1}e^{-j\zeta}z^{-1}) \\
&= (1 - 2r\cos\zeta z^{-1} + r^2z^{-2})(1 - 2r^{-1}\cos\zeta z^{-1} + r^{-2}z^{-2}) \\
&= (1 + z^{-4}) - 2(r + r^{-1})\cos\zeta(z^{-1} + z^{-3}) + (r^2 + r^{-2} + 4\cos^2\theta)z^{-2}
\end{aligned}$$

is linear phase of type I.

Now, any filter whose zeros are only of the above types is necessarily a cascade connection of such filters. By Problem 9.7, we know that the cascade connection has (generalized) linear phase.

9.9 It follows from the given equality that the filter has a zero

$$\beta_1 = 0.8e^{j\pi/3} = 0.4(1 + j\sqrt{3}).$$

Therefore, the filter must have three other zeros as follows:

$$\begin{aligned}
\beta_2 &= 0.4(1 - j\sqrt{3}), \\
\beta_3 &= 0.625(1 + j\sqrt{3}), \\
\beta_4 &= 0.625(1 - j\sqrt{3}).
\end{aligned}$$

The minimal order is $N = 4$, and the filter's transfer function is

$$\begin{aligned}
H^z(z) &= (1 - 0.8z^{-1} + 0.64z^{-2})(1 - 1.25z^{-1} + 1.5625z^{-2}) \\
&= 1 - 2.05z^{-1} + 3.2025z^{-2} - 2.05z^{-3} + z^{-4}.
\end{aligned}$$

The coefficients $h[n]$ are $\{1, -2.05, 3.2025, -2.05, 1\}$.

9.10 The filter must have a zero at $z = 1$, a zero at $z = -1$, and zeros at $\{z = e^{\pm j\theta_k},\ 1 \le k \le 7\}$. Therefore, the filter has order 16 and its transfer function is

$$H^z(z) = (1 - z^{-1})(1 + z^{-1})\prod_{k=1}^{7}(1 - 2\cos(0.125k\pi)z^{-1} + z^{-2}).$$

All factors but $(1 - z^{-1})$ have exact linear phase and this factor has generalized linear phase. Therefore, $H^z(z)$ has generalized linear phase.

9.11 The transfer function of the given filter is

$$H_1^z(z) = C(1 - z^{-1} + 0.8125z^{-2})^2,$$

where C is an unspecified gain. Therefore, the filter

$$H_2^z(z) = C(1 - z^{-1} + 0.8125z^{-2})(0.8125 - z^{-1} + z^{-2})$$

has the same magnitude response as $H_1^z(z)$ and linear phase.

9.12 The signal has sinusoidal components at $\theta_1 = 0.9273$ and $\theta_2 = \pi$. In order to completely eliminate the signal, the filter must have zeros at

$$\beta_{1,2} = e^{\pm j0.9273} = 0.6 \pm j0.8, \quad \beta_3 = -1.$$

Therefore

$$H^z(z) = (1 - 1.2z{-}1 + z^{-2})(1 + z^{-1}) = 1 - 0.2z^{-1} - 0.2z^{-2} + z^{-3}.$$

The coefficients $h[n]$ are $\{1, -0.2, -0.2, 1\}$.

9.13 Since $0.5 - j0.5$ is a zero, there must be zeros at $0.5 + j0.5$, $1 + j$, and $1 - j$. Therefore,

$$\begin{aligned} H^z(z) &= K[1-(0.5-j0.5)z^{-1}][1-(0.5+j0.5)z^{-1}][1-(1-j)z^{-1}][1-(1+j)z^{-1}][1-\beta_5 z^{-1}] \\ &= K[1-3z^{-1}+4.5z^{-2}-3z^{-3}+z^{-4}][1-\beta_5 z^{-1}]. \end{aligned}$$

The zero β_5 must be either 1 or -1. The former choice gives a type-II filter and the latter gives a type-IV filter. Therefore, $\beta_5 = 1$ and

$$H^z(1) = K = 6.$$

Finally,

$$H^z(-j) = 6(1 - 3j - 4.5 + 3j + 1)(1 + j) = -15(1 + j).$$

9.14 The coefficients of the filter are $h[n] = \{1, 2, 2, 1\}$; its transfer function is $H^z(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$; its zeros are $\beta_1 = -1$, $\beta_{2,3} = 0.5 \pm j0.5\sqrt{3}$; it is a type-II linear-phase filter; its frequency response is $H^f(\theta) = \cos\theta(1 + 2\cos\theta)e^{-j1.5\theta}$; it is a low-pass filter.

9.15 We have

$$H_2^f(\theta) = \left(\frac{1 - e^{-j4\theta}}{1 - e^{-j\theta}}\right)^2,$$

so

$$\begin{aligned} H_2^z(z) &= \left(\frac{1 - z^{-4}}{1 - z^{-1}}\right)^2 = (1 + z^{-1} + z^{-2+}z^{-3})^2 \\ &= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}. \end{aligned}$$

Also,

$$H_1^z(z) = \sum_{n=0}^{\infty} 0.5^n z^{-n}.$$

Therefore,

$$h[n] = h_1[n]h_2[n] = \left\{1, 1, \frac{3}{4}, \frac{1}{2}, \frac{3}{16}, \frac{1}{16}, \frac{1}{64}\right\}.$$

9.16

We get from (9.31), by substituting $N = 0$ and $\theta_2 = 0.2\pi$, that $X_1^f(\theta)$ is an ideal low-pass filter with cutoff frequency 0.2π. Similarly, we get from (9.32), by substituting $N = 0$ and $\theta_1 = 0.6\pi$, that $Y_1^f(\theta)$ is an ideal high-pass filter with cutoff frequency 0.6π. Therefore $X_1^f(\theta)Y_1^f(\theta) = 0$, implying that $\{x_1 * y_1\}[n] = 0$.

In a similar manner, $X_2^f(\theta)$ is an ideal low-pass filter with cutoff frequency 0.6π and $Y_2^f(\theta)$ is an ideal high-pass filter with cutoff frequency 0.2π. Therefore $X_2^f(\theta)Y_2^f(\theta)$ is a band-pass filter with cutoff frequencies 0.2π and 0.6π. It follows from (9.30) that

$$\{x_2 * y_2\}[n] = 0.6\mathrm{sinc}(0.6n) - 0.2\mathrm{sinc}(0.2n).$$

9.17

(a) From the general multiband filter formula we get

$$h_d[n] = \frac{\sin[(n - 0.5N)(\pi/3)]}{\pi(n - 0.5N)} - 0.5\frac{\sin[(n - 0.5N)(2\pi/3)]}{\pi(n - 0.5N)}, \quad n \neq 0.5N,$$

and

$$h_d[0.5N] = 0.5.$$

(b) Using the Hamming window parameters, we get

$$\begin{aligned} \theta_{p,1} &= \frac{\pi}{3} - \frac{4\pi}{41}, \quad \theta_{s,1} = \frac{\pi}{3} + \frac{4\pi}{41}, \\ \theta_{s,2} &= \frac{2\pi}{3} - \frac{4\pi}{41}, \quad \theta_{p,2} = \frac{2\pi}{3} + \frac{4\pi}{41}, \\ \delta_{p,1} &= \delta_s = 0.0022, \quad \delta_{p,2} = 0.0011. \end{aligned}$$

(c) No, since a type II filter cannot meet the pass-band specifications near $\theta = \pi$.

9.18

(a)

$$A = -20\log_{10} 0.0021 = 53.55.$$
$$\alpha = 0.1102(A - 8.7) = 4.943.$$
$$N = \lceil 31.755 \rceil = 32.$$

(b) For $\tilde{G}_1^z(z)$ we have

$$|\tilde{G}_1^f(\theta)| = |\tilde{H}^f(\theta)|^2,$$

so this is a low-pass filter. For $\tilde{G}_1^z(z)$ we have

$$|\tilde{G}_1^f(\theta)| = |\tilde{H}^f(\theta)| \cdot |2 - \tilde{H}^f(\theta)|,$$

so this is also a low-pass filter.

(c) For $\tilde{G}_1^f(\theta)$ we have in the pass band

$$(1 - \delta_p)^2 \le |\tilde{G}_1^f(\theta)| \le (1 + \delta_p)^2,$$

so its pass-band ripple is $2\delta_p + \delta_p^2 \approx 2\delta_p$. Its stop-band attenuation is clearly δ_s^2. Let us write $\tilde{G}_1^f(\theta)$ as

$$\tilde{H}^f(\theta) = 1 + \Delta(\theta),$$

where $\Delta(\theta)$ is real valued. This is possible, since $\tilde{H}^f(\theta)$ is zero-phase. Then we have

$$\tilde{G}_2^f(\theta) = [1 + \Delta(\theta)][1 - \Delta(\theta)] = 1 - \Delta^2(\theta).$$

Therefore we have in the pass-band

$$1 - \delta_p^2 \le |\tilde{G}_2^f(\theta)| \le 1,$$

so the pass-band ripple is δ_p^2. The stop-band attenuation is $\delta_s(2 - \delta_s) \approx 2\delta_s$.

(d) We build the two filters as

$$G_1^z(z) = [H^z(z)]^2,$$

and

$$G_2^z(z) = H^z(z)[2z^{-N/2} - H^z(z)].$$

The factor $z^{-N/2}$ is needed to equalize the group delays of the two terms in the brackets.

9.19

(a) The desired frequency response is

$$H_d^f(\theta) = e^{-j(L+0.25)\theta}, \quad -\pi \le \theta \le \pi.$$

(b) It makes sense to choose $N = 2L$, since then the integer part of the delay will be L, as required.

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(n-L-0.25)\theta}d\theta = \frac{\sin[(n-L-0.25)\pi]}{(n-L-0.25)\pi}, \quad 0 \le n \le N.$$

(c) The filter does not have linear phase, since linear phase is possible only for a group delay which is an integer multiple of 0.5, whereas here the desired group delay has a fractional part 0.25. We also see from part b that the impulse response is neither symmetric nor antisymmetric.

9.20

(a) The desired frequency response is

$$H_d^f(\theta) = \begin{cases} j\dfrac{\theta}{\theta_0}, & |\theta| \le \theta_0, \\ 1, & |\theta| > \theta_0. \end{cases}$$

Therefore, the desired impulse response is

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{-\theta_0} e^{j\theta n}d\theta + \frac{j}{2\pi\theta_0}\int_{-\theta_0}^{\theta_0} \theta e^{j\theta n}d\theta + \frac{1}{2\pi}\int_{\theta_0}^{\pi} e^{j\theta n}d\theta$$
$$= \frac{1}{\pi(n-0.5N)}\left[\operatorname{sinc}\left(\frac{\theta_0(n-0.5N)}{\pi}\right) - \cos(\theta_0(n-0.5N))\right]$$
$$+ \delta[n-0.5N] - \frac{\theta_1}{\pi}\operatorname{sinc}\left[\frac{\theta_0(n-0.5N)}{\pi}\right].$$

The impulse response is neither symmetric nor antisymmetric, therefore the IRT filter does not have linear phase. This is also evident from $H_d^f(\theta)$, since this frequency response is neither purely real nor purely imaginary.

(b) The following MATLAB code performs the required computations and plots the results. The line computing the window w should be edited as needed. Note that the phase is advanced by $\theta(n-0.5N)$ before plotting, to compensate for the filter's group delay.

```
w = window(129,'rect'); % change as needed to 'hamm'; 'kais',6; 'kais',12.
theta0 = 0.5*pi;
n = -64:64;
h = (sinc((theta0/pi)*n)-cos(theta0*n)) ./ (pi*n) - ...
    (theta0/pi)*sinc((theta0/pi)*n);
h(65) = 1 - (theta0/pi);
h = h.*w;
H = frqresp(h,1,501);
theta = (1/500)*(0:500);
plot(theta,abs(H)),grid,figure(1),pause
plot(theta,(180/pi)*angle(H.*exp(j*64*pi*theta)),grid,figure(1)
```

9.21 The frequency response of $g[n]$ is

$$G^f(\theta) = \sum_{n=0}^{N} g[n]e^{-j\theta n} = (-1)^{N/2}e^{-j0.5\theta N} - \sum_{n=0}^{N}(-1)^n h[n]e^{-j\theta n}$$
$$= e^{-j0.5(\theta+\pi)N} - \sum_{n=0}^{N} h[n]e^{-j(\theta+\pi)n} = e^{-j0.5(\theta+\pi)N} - H^f(\theta+\pi).$$

Since the given filter is type I, its frequency response can be written in the form

$$H^f(\theta) = A(\theta)e^{-j0.5\theta N}.$$

Therefore,

$$G^f(\theta) = [1 - A(\theta+\pi)]e^{-j0.5(\theta+\pi)N}.$$

The filter is low pass, with parameters

$$\tilde{\theta}_p = \pi - \theta_s, \quad \tilde{\theta}_s = \pi - \theta_p, \quad \tilde{\delta}_p = \delta_s, \quad \tilde{\delta}_s = \delta_p.$$

9.22

(a) The continuous-time signal $y(t)$ corresponding to (3.67) was computed in Solution 3.10. With $f_0T = 0.5$, the bandwidth of the signal is $0.5(1+\alpha)\pi/T$, and this is less than π/T. Therefore, there is no aliasing when sampling at interval T; hence

$$h_d[n] = 0.5y((n-0.5N)T).$$

We get from Solution 3.10,

$$h_d[0.5N] = 0.5\left(1 - \alpha + \frac{4\alpha}{\pi}\right),$$

$$h_d\left[0.5N \pm \frac{1}{2\alpha}\right] = 0.5\alpha\cos\left(\frac{(1-\alpha)\pi}{4\alpha}\right) + \frac{1.5\alpha}{\pi}\sin\left(\frac{(1-\alpha)\pi}{4\alpha}\right) + \frac{0.5\alpha}{\pi}\sin\left(\frac{(1+3\alpha)\pi}{4\alpha}\right), \text{ and}$$

$$h_d[n] = \frac{\sin[0.5(1-\alpha)\pi(n-0.5N)] + 2\alpha(n-0.5N)\cos[0.5(1+\alpha)\pi(n-0.5N)]}{\pi(n-0.5N)[1-(2\alpha(n-0.5N))^2]}$$

for other values of n. This filter is not half band, since it does not satisfy the requirement $h_d[2n] = 0$ for all $n \neq 0.25N$.

(b) We know from Solution 3.10, part d, that $[Y^F(\omega)]^2$ is a raised-cosine spectrum, therefore the same is true for $[H_d^f(\theta)]^2$. The latter can be easily verified to be half band, therefore $g_d[n] = \{h_d * h_d\}[n]$ is a half band filter.

(c) An IRT filter or a window-based filter will approximate the desired frequency response. The following MATLAB code implements the computation of $h[n]$ and its magnitude response.

```
N = 38; n = -19:19; alpha = 0.4;
h = (sin(0.5*(1-alpha)*pi*n) + 2*alpha*n.*cos(0.5*(1+alpha)*pi*n)) ./ ...
    (pi*n.*(1-(2*alpha*n).^2));
N2 = 0.5*N;
if (N2 == round(N2)), h(N2+1) = 0.5*(1-alpha+(4*alpha/pi)); end
N2ma = 0.5*N2-(0.5/alpha); N2pa = 0.5*N2+(0.5/alpha);
if (N2ma == round(N2ma)),
   h(N2ma+1) = 0.5*alpha*cos((1-alpha)*pi/(4*alpha)) ...
             + (1.5*alpha/pi)*sin((1-alpha)*pi/(4*alpha)) ...
             + (0.5*alpha/pi)*sin((1+3*alpha)*pi/(4*alpha));
end
if (N2pa == round(N2pa)),
   h(N2pa+1) = 0.5*alpha*cos((1-alpha)*pi/(4*alpha)) ...
             + (1.5*alpha/pi)*sin((1-alpha)*pi/(4*alpha)) ...
             + (0.5*alpha/pi)*sin((1+3*alpha)*pi/(4*alpha));
end
H = frqresp(h,1,501);
theta = (1/500)*(0:500);
plot(theta,abs(H)),grid,figure(1)
```

(d) No, $g[n]$ is not a half-band filter, because $G^f(\theta) = [H^f(\theta)]^2$ is only an approximation of the raised-cosine spectrum, and the half-band property is lost in the approximation.

9.23 Following (8.72), we define a zero-phase filter to be quarter band if it satisfies

$$H^f(\theta) + H^f(\theta - 0.5\pi) + H^f(\theta - \pi) + H^f(\theta - 1.5\pi) = c.$$

Similarly to Solution 8.11, we can show that a necessary and sufficient condition for a filter to be zero-phase quarter band is

$$h[4n] = \begin{cases} 0.25c, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

A causal quarter-band FIR filter should have an order $N = 8M$. Its coefficients $h[4n]$ must be zero, except for $h[4M]$. Its frequency response satisfies

$$H^f(\theta) + H^f(\theta - 0.5\pi) + H^f(\theta - \pi) + H^f(\theta - 1.5\pi) = ce^{-4\theta M}.$$

We can delete the two end points, thereby getting an FIR filter of order $8M - 2$.

A possible way of designing a quarter-band FIR filter is to take the desired frequency response as a rectangle on $-0.25\pi \leq \theta \leq 0.25\pi$. Then the desired impulse response is

$$h_d[n] = \frac{1}{2\pi}\int_{-0.25\pi}^{0.25\pi} e^{j\theta(n-0.5N)}d\theta = 0.25\,\text{sinc}[0.25(n-0.5N)], \quad N = 8M.$$

The desired impulse response can then be multiplied by any window. The window will preserve the condition $h[4n] = 0$ except for $h[4M]$; therefore, it will preserve the quarter-band property. The two end points can be deleted.

A quarter-band raised-cosine filter can be obtained by sampling (3.22) at interval $T = 0.25/f_0$ and multiplying by 0.25. This gives

$$h_d[n] = \frac{\sin[0.25\pi(n-0.5N)]\cos[0.25\pi\alpha(n-0.5N)]}{\pi(n-0.5N)[1-(0.5\alpha(n-0.5N))^2]}.$$

The impulse response is then truncated to $0 \le n \le N$, multiplied by a window and its two end points deleted. The following MATLAB code illustrates this procedure (using a rectangular window):

```
N = 40; n = -20:20; alpha = 0.4;
h = sin(0.25*pi*n).*cos(0.25*pi*alpha.*n) ./ ...
    (pi*n.*(1 - (0.5*alpha*n).^2));
h(0.5*N+1) = 0.25;
a2 = 2/alpha;
if (a2 == round(a2)),
   h(0.5*N+1+a2) = 0.25*pi*sin(0.5*pi/alpha)/(2*pi/alpha);
   h(0.5*N+1-a2) = h(0.5*N+1+a2);
end
h = h(2:N);
H = frqresp(h,1,501);
theta = (1/500)*(0:500);
plot(theta,abs(H)),grid,figure(1)
```

Note that $h[n]$ has to be computed separately for $n = 0.5N \pm 2/\alpha$ to avoid division of zero by zero.

9.24 Comparing the given $A_d(\theta)$ with (9.7), we see that it can be realized exactly by a type-II filter having $M = 1$ and $g[0] = g[1] = 1$. Therefore, we get from (9.9) that

$$h[0] = 0.25g[1] = 0.25, \quad h[1] = 0.5g[0] + 0.25g[1] = 0.75.$$

Therefore, $N = 3$ and the coefficients are $\{0.25, 0.75, 0.75, 0.25\}$. This filter obviously has $\varepsilon^2 = 0$.

9.25 We have

$$\sin(2\theta) = -0.5j(e^{j2\theta} - e^{-j2\theta}) = -je^{j2\theta}0.5(1 - e^{-j4\theta}).$$

Therefore, a possible choice is

$$H^z(z) = 0.5 - 0.5z^{-4}.$$

9.26 If the Fourier transform of $w[n]$ is

$$W^f(\theta) = A_w(\theta)e^{-j0.5\theta},$$

then that of $v[n]$ is

$$V^f(\theta) = [W^f(\theta)]^2 = A_w^2(\theta)e^{-j\theta}.$$

Therefore [cf. (9.54)],

$$\Gamma_v(x) = \frac{1}{L}\int_0^x A_w^2\left(\frac{2\mu}{L}\right)d\mu.$$

Since the integrand is a nonnegative function, the integral is a monotonically nondecreasing function.

9.27 The frequency response

$$H_d^f(\theta) = \operatorname{rect}\left(\frac{\theta}{\pi}\right)$$

has the half-band property. The corresponding impulse response is

$$h_d[n] = \frac{1}{2\pi}\int_{-0.5\pi}^{0.5\pi} e^{j\theta(n-0.5N)}d\theta = 0.5\operatorname{sinc}[0.5(n-0.5N)].$$

Taking $N = 4M$ and multiplying $h_d[n]$ by any window preserves the condition $h[2n] = 0$ except $n = 2M$; therefore it preserves the half-band property. The two end points of $h[n]$ are identically zero, so they can be deleted, thus yielding an FIR filter of order $4M - 2$.

9.28

(a) The ideal frequency response is

$$H_d^f(\theta) = \begin{cases} 1, & |\theta| \le \theta_p, \\ 1 - \dfrac{|\theta| - \theta_p}{\theta_s - \theta_p}, & \theta_p \le |\theta| \le \theta_s, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$h[n] = \begin{cases} \dfrac{\cos[\theta_p(n-0.5N)] - \cos[\theta_s(n-0.5N)]}{\pi(n-0.5N)^2(\theta_s - \theta_p)}, & n \ne 0.5N, \\ \dfrac{\theta_p + \theta_s}{2\pi}, & n = 0.5N. \end{cases}$$

(b) The following MATLAB code implements the computation.

```
thetap = 0.2*pi; thetas = 0.4*pi; N = 20; n = -10:10;
h = (cos(thetap*n)-cos(thetas*n)) ./ (pi*(thetas-thetap)*(n.^2));
N2 = N/2;
if (N2 == round(N2)),
   h(N2+1) = (thetas+thetap)/(2*pi);
end
h(1:11)',pause
H = frqresp(h,1,501);
theta = (1/500)*(0:500);
plot(theta,20*log10(abs(H))),grid,figure(1)
```

(c) The above MATLAB code also shows the magnitude response. The pass-band ripple of the filter is 0.09 dB and the stop-band attenuation is 40 dB.

(d) Let the band-edge frequencies of the band-pass filter be $\theta_{s,1}$, $\theta_{p,1}$, $\theta_{p,2}$, $\theta_{s,2}$. Then we define the ideal amplitude function $A_d(\theta)$ by connecting these points by straight lines, thus forming a trapezoidal shape (instead of a rectangular shape). The corresponding IRT filter will have smaller pass-band and stop-band tolerances than those obtained for a rectangular amplitude function.

9.29 The output of the Hilbert transformer is approximately the ideal Hilbert transform of the input, delayed by $N/2$ samples, the group delay of the filter. Therefore, we must delay the input by $N/2$ samples before using it to form the analytic signal. If we neglect to do this, we will fail to form the true analytic signal. Delaying the input by $N/2$ sample is simple if N is even, but complicated if N is odd. Therefore, a type-III Hilbert transformer is the right choice in this case.

9.30

(a) MATLAB deletes the last N points.

(b) When the filter is linear phase, its initial and final transients behave similarly, because of the symmetry of the impulse response. It then makes sense to delete $N/2$ points from the beginning and $N/2$ points from the end. This way, the transient responses of the filter will be removed equally from both ends of the data.

(c) When the filter is minimum phase, the initial transient tends to be short, whereas the final transient tends to be long. It therefore makes sense to delete the last N points in this case, which is what MATLAB does.

9.31 The ideal impulse response of the filter is

$$\begin{aligned} h_d[n] &= \frac{j}{2\pi}\int_{-0.75\pi}^{-0.25\pi} e^{j\theta(n-0.5N)}d\theta - \frac{j}{2\pi}\int_{0.25\pi}^{0.75\pi} e^{j\theta(n-0.5N)}d\theta \\ &= \begin{cases} \dfrac{\cos[0.25\pi(n-0.5N)] - \cos[0.75\pi(n-0.5N)]}{\pi(n-0.5N)}, & n \ne 0.5N, \\ 0, & n = 0.5N. \end{cases} \end{aligned}$$

We use a Kaiser window to meet the given specifications. By the Kaiser design equations (9.56), we get

$$\alpha = 3.4, \quad N = 45.$$

The following MATLAB code designs the filter, provides the coefficients, and plots the magnitude response.

```
N = 45; alpha = 3.4; n = (0:N)-0.5*N;
h = (cos(0.25*pi*n) - cos(0.75*pi*n)) ./ (pi*n);
N2 = 0.5*N;
if (N2 == round(N2)), h(N2+1) = 0.0; end
h = h.*window(N+1,'kais',alpha);
format long, h', pause
H = frqresp(h,1,501);
theta = (1/500)*(0:500);
plot(theta,20*log10(abs(H))),grid
axis([0,1,-80,0]),figure(1)
```

9.32

(a) For $f_{\text{sam}} = 120\,\text{Hz}$ we get

$$\theta_{\text{p}} = \frac{2\pi \cdot 20}{120}, \quad \theta_{\text{s}} = \frac{2\pi \cdot 40}{120}, \quad \theta_{\text{s}} - \theta_{\text{p}} = \frac{\pi}{3}.$$

Also, $A = 60\,\text{dB}$. Therefore,

$$N \geq \frac{60 - 7.95}{2.285(\pi/3)} = 21.75 \implies N = 22.$$

(b) For $f_{\text{sam}} = 100\,\text{Hz}$ we get aliasing of $x_2(t)$, so its magnitude Fourier transform in the range 40–50 will be potentially doubled. Therefore, we need stop-band attenuation $A = 66\,\text{dB}$ in this case. Also,

$$\theta_{\text{p}} = \frac{2\pi \cdot 20}{100}, \quad \theta_{\text{s}} = \frac{2\pi \cdot 40}{100}, \quad \theta_{\text{s}} - \theta_{\text{p}} = 0.4\pi.$$

Therefore,

$$N \geq \frac{66 - 7.95}{2.285 \times 0.4\pi} = 20.2 \implies N = 21.$$

In summary, the choice $f_{\text{sam}} = 100\,\text{Hz}$ leads to a filter of lower order.

9.33 Write $H^{\text{f}}(\theta)$ in terms of its continuous-phase representation:

$$H^{\text{f}}(\theta) = A(\theta)e^{-j0.5\theta N}.$$

Then,

$$G^{\text{f}}(\theta)[3 - 2A(\theta)][A(\theta)]^2 e^{-j1.5\theta N}.$$

In the pass band we have

$$A(\theta) = 1 + \Delta A(\theta), \quad \text{where} |\Delta A(\theta)| \leq \delta_{\text{p}}.$$

Therefore,

$$[3 - 2A(\theta)][A(\theta)]^2 = [1 - 2\Delta A(\theta)][1 + 2\Delta A(\theta) + (\Delta A(\theta))^2] = 1 - 2(\Delta A(\theta))^2 - 2(\Delta A(\theta))^3.$$

Therefore, assuming that $\delta_{\text{p}} \ll 1$, the pass-band ripple of $G^{\text{z}}(z)$ is approximately $2\delta_{\text{p}}^2$.

In the stop-band we have approximately

$$|G^{\text{f}}(\theta)| \leq 3\delta_{\text{s}}^2,$$

so the stop-band attenuation of $G^{\text{z}}(z)$ is approximately $3\delta_{\text{s}}^2$.

The order of $G^{\text{z}}(z)$ is $3N$. An application of this idea is as follows: Given an FIR filter $H^{\text{z}}(z)$ with small tolerances, we can construct a filter $G^{\text{z}}(z)$ whose order is three times that of the given filter, and whose tolerances are proportional to the squares of the tolerances of the given filter. For example, if the tolerances of $H^{\text{z}}(z)$ are 0.01 in both bands, then the pass-band ripple of $G^{\text{z}}(z)$ is 0.0002 and the stop-band attenuation of $G^{\text{z}}(z)$ is 0.0003.

9.34

(a) By choosing the weighting function $W(\theta) = \delta \cdot \theta$, we cause the *relative* error of the magnitude response to be bounded by a constant. It makes more sense in this case to bound the relative error, because the magnitude response is small at low frequencies and large at high frequencies.

(b) Program 9.1 implements least-squares differentiator design. Note that the order of the differentiator must be odd, due to reasons explained in Section 9.2.4.

Program 9.1 Least-squares differentiator design.

```
function h = diffls(N,delta);
% Synopsis: h = diffls(N,delta).
% Designs a differentiator by least-squares.
% Input parameters:
% N: the filter order; must be odd
% delta: the relative tolerance.
% Output:
% h: the filter coefficients.

thetai = (pi/(32*N)) + (pi/(16*N))*(0:(16*N-1));
K = (N-1)/2; F = sin(0.5*thetai);
Ad = thetai; V = delta*thetai;
carray = cos(thetai'*(0:K)).*((F.*V)'*ones(1,K+1));
darray = (V.*Ad)';
g = (carray\darray)';
h = 0.25*[g(K+1),fliplr(-g(3:K+1))+fliplr(g(2:K))];
h = [h,-0.25*g(2)+0.5*g(1)];
h = [h,-fliplr(h)];
```

(c) An order $N = 81$ meets the specification.

9.35

(a) The filter $H^z(z) = 0.5[G^z(z^2) + z^{-N}]$ has the coefficients

$$h[n] = \begin{cases} 0.5g[0.5n], & n \text{ even}, \\ 0.5, & n = N, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, by Solution 8.11, part d, $h[n]$ is a half-band filter.

(b) The order of $H^z(z)$ is $2N$.

(c) The pass-band and stop-band tolerances are both $0.5\delta_p$. The band-edge frequencies are $0.5\pi - 0.5\theta_p$ and $0.5\pi + 0.5\theta_p$.

(d) The following MATLAB code implements the design and shows the magnitude response.

```
N = 11;
g = remez(N,[0,0.9],[1,1]);
h = zeros(1,2*N+1);
for n = 0:N, h(2*n+1) = 0.5*g(n+1); end
h(N+1) = 0.5;
G = frqresp(g,1,501);
H = frqresp(h,1,501);
theta = (1/500)*(0:500);
```

```
plot(theta,abs(G)),grid,figure(1),pause
plot(theta,abs(H)),grid,figure(1)
```

9.36

(a) By the inverse DFT formula:

$$h[n] = \frac{1}{N+1}\sum_{k=0}^{N} H_{\mathrm{d}}^{\mathrm{f}}(\theta_k)e^{-j2\pi kn/(N+1)}.$$

Therefore,

$$H^{\mathrm{f}}(\theta) = \sum_{n=0}^{N} h[n]e^{-j\theta n} = \frac{1}{N+1}\sum_{k=0}^{N} H_{\mathrm{d}}^{\mathrm{f}}(\theta_k)\sum_{n=0}^{N} e^{-j[\theta-2\pi k/(N+1)]n} = \frac{1}{N+1}\sum_{k=0}^{N} H_{\mathrm{d}}^{\mathrm{f}}(\theta_k)\frac{1-e^{-j\theta(N+1)}}{1-e^{-j[\theta-2\pi k/(N+1)]}}.$$

(b) Let

$$K = \left\lfloor \frac{\theta_{\mathrm{c}}(N+1)}{2\pi} \right\rfloor.$$

Take

$$H_{\mathrm{d}}^{\mathrm{f}}(\theta_k) = \begin{cases} 1, & 0 \le k \le K \text{ and } N-K+1 \le k \le N, \\ 0, & \text{otherwise.} \end{cases}$$

This sequence is conjugate symmetric, so the corresponding $h[n]$ is real. The sequence $h[n]$ is obtained as the inverse DFT of $H_{\mathrm{d}}^{\mathrm{f}}(\theta_k)$.

(c) The following MATLAB code implements the design and shows the impulse and magnitude responses. Note that the two halves of $h[n]$ are swapped, to maintain causality.

```
N = 63; thetac = 0.3*pi;
K = floor(thetac*(N+1)/(2*pi));
H = ones(1,N+1);
H(K+2:N-K+1) = zeros(1,N-2*K);
h = real(fftshift(ifft(H)));
plot(h),grid,figure(1),pause
HH = frqresp(h,1,501);
theta = (1/500)*(0:500);
plot(theta,abs(HH)),grid,figure(1)
```

(d) We see that the frequency response exhibits Gibbs oscillations. These oscillations are typical of the frequency sampling method, since this method does not impose any constraints on the behavior of $H^{\mathrm{f}}(\theta)$ at frequencies different from θ_k. Consequently, the frequency sampling method is not commonly used.

Chapter 10

Infinite Impulse Response Filters

10.1 Equation (8.5) can be written as

$$s^{q-p}\frac{b_0 + b_1 s^{-1} + \cdots + b_q s^{-q}}{1 + a_1 s^{-1} + \cdots + a_p s^{-p}}.$$

When we substitute $s = j\omega$ and let ω tend to infinity, the fraction approaches 1, hence we get (10.4).

10.2

(a) The term $(1-\delta_{\rm p})^{-2} - 1$ is only slightly greater than zero, and typically much smaller than 1. On the other hand, the term $\delta_{\rm s}^{-2} - 1$ is typically much larger than 1. Therefore, d is typically much smaller than 1.

(b)

$$d = \left[\frac{(1-\delta_{\rm p})^{-2} - 1}{\delta_{\rm s}^{-2} - 1}\right]^{1/2} \approx \left[\frac{2\delta_{\rm p}}{\delta_{\rm s}^{-2}}\right]^{1/2} = (2\delta_{\rm p})^{1/2}\delta_{\rm s}.$$

10.3 The two inequalities in (10.19) are obtained from (10.14) by isolating ω_0 in both cases. Equality at the lower range means that the pass-band specification is met exactly. Equality at the higher range means that the stop-band specification is met exactly.

10.4

$$C = 1 + \delta_{\rm p}', \quad \delta_{\rm p} = \frac{2\delta_{\rm p}'}{1+\delta_{\rm p}'}, \quad \delta_{\rm s} = \frac{\delta_{\rm s}'}{1+\delta_{\rm p}'}.$$

10.5 This is a third-order normalized Butterworth filter. Therefore,

$$(1+\omega_{\rm p}^6)^{-1} = 10^{-0.05} = 0.8912 \implies \omega_{\rm p} = 0.7043,$$

$$(1+\omega_{\rm s}^6)^{-1} = 10^{-2} = 0.01 \implies \omega_{\rm s} = 2.1508,$$

so

$$k = \frac{0.7043}{2.1508} = 0.3275.$$

10.6

(a) The poles of a Butterworth filter must be uniformly spaced on a circle. The poles of $H^{\rm L}(s)$ are at angles 0°, ±45° and ±60°; these are not uniformly spaced.

(b) The poles of $H^{\rm L}(s)$ are located on a circle of radius 1. Its response is monotone decreasing. Its asymptotic attenuation is −100 dB/decade. It is as flat at $\omega = 0$ as a second-order Butterworth filter. However, at frequency $\omega = 1$ its attenuation is 6 dB rather than 3 dB.

10.7 We get from (10.21) that $T_0(x)$ is symmetric and $T_1(x)$ is antisymmetric. We also get from (10.21) that

$$T_{2M}(x) = 2xT_{2M-1}(x) - T_{2M-2}(x), \quad T_{2M+1}(x) = 2xT_{2M}(x) - T_{2M-1}(x).$$

Note also that $2xT_{2M-1}(x)$ is symmetric if $T_{2M-1}(x)$ is antisymmetric, and $2xT_{2M}(x)$ is antisymmetric if $T_{2M}(x)$ is symmetric. It follows by induction that $T_{2M}(x)$ is symmetric for all M and $T_{2M+1}(x)$ is antisymmetric for all M.

10.8 We get from (10.21) that $T_0(0) = 1$ and $T_1(0) = 0$. We also get from (10.21) that

$$T_{2M}(0) = -T_{2M-2}(0), \quad T_{2M+1}(0) = -T_{2M-1}(0).$$

It therefore follows by induction that $T_{2M}(0) = (-1)^M$ and $T_{2M+1}(0) = 0$ for all M.

10.9 The selectivity factor is

$$k = \frac{\omega_p}{\omega_s} = 0.3.$$

For a Chebyshev filter that meets the specifications exactly we have

$$N = \frac{\operatorname{arccosh}(1/d)}{\operatorname{arccosh}(1/k)},$$

from which we can solve for d:

$$d = [\cosh(3\operatorname{arccosh}(1/0.3))]^{-1} = 0.007238.$$

Therefore,

$$\varepsilon = d(\delta_s^{-2} - 1)^{1/2} = 0.36186.$$

Also, for a Chebyshev-I filter

$$\omega_0 = \omega_p = 3.$$

We now substitute in the design formulas to get

$$\frac{1}{N}\operatorname{arcsinh}(1/\varepsilon) = 0.5803.$$

$$\sinh(0.5803) = 0.6134, \quad \cosh(0.5803) = 1.1731.$$

$$s_{0,2} = -0.9201 \pm j3.0478, \quad s_1 = -1.8402.$$

Finally,

$$H^L(s) = \frac{18.652}{(s + 1.8402)(s^2 + 1.8402s + 10.1356)}.$$

10.10

(a) According to the L'Hospital rule, the limit of the ratio of the two functions is equal to the limit of the ratio of their derivatives. The ratio of the derivatives is the function

$$z(x) = \frac{1 + \frac{x}{\sqrt{x^2-1}}}{1 + \frac{\sqrt{x^2-1}}{x}}.$$

We see that

$$\lim_{x \to 1} z(x) = \infty, \quad \lim_{x \to \infty} z(x) = 1.$$

(b) The function is monotone decreasing.

(c) The function $y(x)$ can also be written as

$$y(x) = \frac{\arccos x}{\log_e x}.$$

For all practical IIR filters $1/d > 1/k$ (usually much greater). Therefore, since $y(x)$ is monotone decreasing,

$$\frac{\arccos(1/d)}{\log_e(1/d)} < \frac{\arccos(1/k)}{\log_e(1/k)}.$$

This implies

$$\frac{\arccos(1/d)}{\arccos(1/k)} < \frac{\log_e(1/d)}{\log_e(1/k)}.$$

As we see from (10.18) and (10.35), this implies that the order of a Chebyshev filter meeting given specifications will be smaller than that of Butterworth filter meeting the same specifications, or equal to it at most.

10.11 It follows from the given information that the filter has a zero at $\omega = 1$, or $s = j$. The zeros of a second-order Chebyshev-II low-pass filter are at $\pm j\omega_0/\cos(0.25\pi)$, therefore $\omega_0 = 1/\sqrt{2}$. Now that we know ε and ω_0, we can compute the poles of the filter. We get from (10.27) that

$$s_{1,2} = -1.0636 \pm j1.1753,$$

and from (10.40) that

$$v_{1,2} = -0.2117 \pm j0.2339.$$

Therefore,

$$H^{\mathrm{L}}(s) = \frac{K(s^2+1)}{s^2+0.4223s+0.0995}.$$

Also, we must have $H^{\mathrm{L}}(0) = 1/(1+\varepsilon^2)^{1/2} = 0.995$, so $K = 0.099$.

10.12 The filter has two zeros at positive frequencies, hence four zeros altogether, so it is necessarily Chebyshev-II, of order 4 or 5. From the formulas for the zeros of Chebyshev-II filter it follows that

$$\frac{\omega_0}{\cos(\pi/2N)} = 10.8239, \quad \frac{\omega_0}{\cos(3\pi/2N)} = 26.1313.$$

Therefore,

$$\frac{\cos(\pi/2N)}{\cos(3\pi/2N)} = 2.4142.$$

Checking with $N = 4, 5$ reveals that $N = 4$ is the correct order, and $\omega_0 = 10$.

The parameter ε is computed from

$$\frac{\varepsilon^2}{1+\varepsilon^2} = 0.2 \implies \varepsilon = 0.5.$$

Now we use the formula for the poles:

$$\frac{1}{4}\operatorname{arcsinh}(1/0.5) = 0.3609,$$

$$\sinh(0.3609) = 0.3688, \quad \cosh(0.3609) = 1.0658,$$

$$s_{0,3} = -1.4113 \pm j9.8471, \quad s_{1,2} = -3.4072 \pm j4.0788,$$

$$v_{0,3} = -1.4262 \pm j9.9509, \quad v_{1,2} = -12.0627 \pm j14.4405.$$

In summary,

$$H^{\mathrm{L}}(s) = \frac{0.4472(s^2+117.16)(s^2+682.84)}{(s^2+2.8524s+101.06)(s^2+24.125s+354.04)}.$$

10.13 Since the pass-band response has to be monotone, only Butterworth and Chebyshev-II filters are eligible. For the former, $\omega_0 = (2\pi \cdot 19.4756)10^3$ and $N = 41$. For the latter, $\omega_0 = (2\pi \cdot 24)10^3$, $N = 16$, and $\varepsilon = 10^{-4}$.

10.14 We get from the given information

$$\omega_{\mathrm{p}} = 1, \quad \omega_{\mathrm{s}} = \min\{2.505, 2.254\} = 2.254.$$

$$k = 0.444, \quad d = 0.0562.$$

The requirement that the filter have no ripple in the pass band eliminates Chebyshev-I and elliptic filters, so we must choose Chebyshev-II (which will have a lower order than Butterworth). We therefore get

$$N \geq 2.46 \implies N = 3.$$

The order of the band-pass filter is therefore 6.

10.15 Sharon's design is as follows:

$$\omega_{\mathrm{p}} = 1, \quad \omega_{\mathrm{s}} = \min\{2.5, 5.25\} = 2.5.$$

$$k = 0.4, \quad d = 0.001015.$$

$$N \geq 7.52 \implies N = 8.$$

Therefore, Sharon's band-pass filter has order 16.

Irwin's design is as follows. He has to choose $\delta_p = 0.01$, $\delta_s = 0.005$ in each of the filters in order to satisfy the overall tolerances. His high-pass design is

$$k = 0.5, \quad d = 0.000712, \quad N_1 = 11.$$

His low-pass design is

$$k = 0.25, \quad d = 0.000712, \quad N_2 = 6.$$

Therefore, Irwin's band-pass filter has order 17. Sharon wins in this case.

10.16 For Butterworth and Chebyshev-I filters $q = 0$, so the band-pass filter will have p zeros at $\tilde{s} = 0$. For Chebyshev-II and elliptic filters and even p, $q = p$. Therefore, there will be $2p$ zeros according to (10.83). To show that these zeros are purely imaginary, denote temporarily

$$u_k = ja, \quad \omega_h - \omega_l = b, \quad \omega_l\omega_h = c.$$

Then the second-order equation for the two corresponding zeros of the band-pass filter is

$$\tilde{s}^2 - jab\tilde{s} + c = 0.$$

The solutions are

$$\tilde{s}_{1,2} = 0.5[jab \pm (-a^2b^2 - 4c)^{1/2}].$$

Since a, b, c are real and c is positive, the solutions are purely imaginary.

If the low-pass filter is Chebyshev-II or elliptic and p is odd, then $q = p - 1$. In this case, the band-pass filter will have $2p - 1$ zeros: one at $\tilde{s} = 0$ and the rest as above.

10.17 For Butterworth and Chebyshev-I filters $q = 0$, so the band-stop filter will have p zeros at $\tilde{s} = j(\omega_l\omega_h)^{1/2}$ and p zeros at $\tilde{s} = -j(\omega_l\omega_h)^{1/2}$. For Chebyshev-II and elliptic filters and even p, $q = p$. Therefore, there will be $2p$ zeros according to (10.93). These zeros are shown to be purely imaginary as in Solution 10.16. If the low-pass filter is Chebyshev-II or elliptic and p is odd, then $q = p - 1$. In this case, the band-stop filter will have $2(p - 1)$ zeros as above and two zeros at $\tilde{s} = \pm j(\omega_l\omega_h)^{1/2}$.

10.18 In a Chebyshev-II or an elliptic low-pass filter, the number of zeros is equal to the number of poles. Therefore, such a filter is not strictly proper, so the impulse invariant transform is not applicable to it.

10.19 The poles of the digital filter are

$$\alpha_{0,2} = 0.8577 \pm 0.2347, \quad \alpha_1 = 0.7907.$$

The poles are mapped from those of the analog filter according to

$$\alpha_k = e^{\lambda_k T}.$$

Therefore,

$$\gamma_{0,2} = -1.1743 \pm 2.6715, \quad \gamma_1 = -2.3486.$$

We have for a third-order Chebyshev-I filter,

$$\gamma_0 = -\omega_0 \sinh\left(\frac{1}{3}\text{arcsinh}\frac{1}{\varepsilon}\right)\sin 30^\circ + j\omega_0 \cosh\left(\frac{1}{3}\text{arcsinh}\frac{1}{\varepsilon}\right)\cos 30^\circ,$$

$$\gamma_1 = -\omega_0 \sinh\left(\frac{1}{3}\text{arcsinh}\frac{1}{\varepsilon}\right).$$

Using the identity $\cosh^2 x - \sinh^2 x = 1$, we get from the value of γ_0 that

$$\omega_0^2 = 4 \implies \omega_0 = 2.$$

Now we get from the value of γ_1 that

$$\sinh\left(\frac{1}{3}\text{arcsinh}\frac{1}{\varepsilon}\right) = 1.1743.$$

Solving for ε gives $\varepsilon = 0.1$.

10.20

(a) We have from the given information

$$\sinh\left(\frac{1}{3}\text{arcsinh}\frac{1}{\varepsilon}\right) = 1,$$

which gives $\varepsilon = 1/7$. Also,

$$\cosh\left(\frac{1}{3}\text{arcsinh}\frac{1}{\varepsilon}\right) = \sqrt{2}.$$

Therefore, the other two poles are

$$s_{0,2} = -\sin 30^\circ \pm j\sqrt{2}\cos 30^\circ = 0.5(-1 \pm j\sqrt{6}).$$

(b)

$$H^{L}(s) = \frac{1.75}{(s+1)(s^2+s+1.75)}.$$

$$H^{z}(z) = \frac{0.2333(1+z^{-1})^3}{1+0.4z^{-1}+0.4666z^{-2}}.$$

10.21 The analog filter is

$$H^{L}(s) = \frac{s+2}{3s+2}.$$

Therefore, the digital filter is

$$H^{z}(z) = \frac{(T+1)z+(T-1)}{(T+3)z+(T-3)}.$$

This gives,

$$T = 1, \quad K = 0.5, \quad \alpha = 0.5.$$

10.22

(a) At frequency $\theta = 0$ we have

$$H^{f}(0) = H^{z}(1) = 1,$$

whereas at frequency $\theta = \pi$,

$$H^{f}(\pi) = H^{z}(-1) = 0.$$

Therefore, this is a low-pass filter.

(b) We can write the given transfer function as

$$H^{z}(z) = \frac{2\theta_0^2}{\frac{(z-1)^2}{(z+1)^2} + 2\theta_0\frac{(z-1)}{(z+1)} + 2\theta_0^2}.$$

Substituting $s = (z-1)/(z+1)$ (the bilinear transform with $T = 2$) gives

$$H^{L}(s) = \frac{2\theta_0^2}{s^2+2\theta_0 s+2\theta_0^2}.$$

The poles are at

$$s_{0,1} = -\theta_0 \pm j\theta_0.$$

There are no zeros.

(c) This is a second-order Butterworth filter.

(d)

$$\omega_{3\text{db}} = \sqrt{2}\theta_0.$$

(e)

$$\theta_{3\text{db}} = 2\arctan(\sqrt{2}\theta_0).$$

10.23

We have

$$H^z(z) = \frac{K(z+1)^2}{z^2+a^2},$$

where K is unknown. Therefore,

$$H^f(0) = \frac{4K}{1+a^2}, \quad |H^f(0.5\pi)| = \frac{2K}{1-a^2}, \quad H^f(\pi) = 0.$$

For $a > 1/\sqrt{3}$ we get $|H^f(0.5\pi)| > |H^f(0)|$. Therefore, the magnitude response is as shown in Figure 10.1.

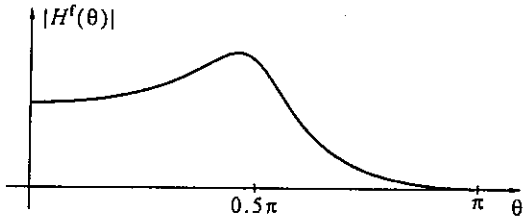

Figure 10.1 Pertaining to Solution 10.23.

(a) Substituting

$$z = \frac{1+0.5Ts}{1-0.5Ts}$$

in $H^z(z)$ gives

$$H^L(s) = \frac{4K}{0.25T^2(1+a^2)s^2 + (1-a^2)Ts + 1 + a^2}.$$

(b) We get for the given values of a and T,

$$H^L(s) = \frac{(32/3)K}{s^2 + (4/3)s + 4},$$

so the poles are

$$s_{0,1} = -\frac{2}{3} \pm j\frac{4\sqrt{2}}{3}.$$

The filter can be neither elliptic nor Chebyshev-II, because it has no zeros. It cannot be Butterworth, since its poles are not at angles $\pm 135°$. Therefore, it is Chebyshev-I.

10.24

(a)

$$H^z(z) = \frac{K(z+1)^2}{z^2 + (\sqrt{2}-1)^2}.$$

Substitute

$$z = \frac{1+0.5s}{1-0.5s}$$

to get

$$H^L(s) = \frac{4K}{(1-0.5\sqrt{2})s^2 + 2(\sqrt{2}-1)s + 4(1-0.5\sqrt{2})}.$$

The characteristic equation is

$$s^2 + 2\sqrt{2}s + 4 = 0.$$

The poles are

$$s_{0,1} = -\sqrt{2} \pm j\sqrt{2}.$$

This is a second-order Butterworth filter with $\omega_0 = 2$.

(b) Impossible, since all digital filters of the permitted kinds have their zeros only on the unit circle.

(c) Butterworth and Chebyshev-I filters are impossible here, since both have their zeros either at $z = 1$ or at $z = -1$. A second-order Chebyshev-II is possible with properly chosen α and β, since it has two zeros on the imaginary axis in the s-plane, which would be transformed to two zeros on the unit circle in the z-plane. It can be easily verified that, since the zeros are at $z = \pm j$, $\omega_0 = 1$ in this case. Depending on ε, the poles would be transformed to some points $z = -\alpha \pm j\beta$ inside the unit circle in the z-plane.

10.25 It follows from the second requirement that the filter must have zeros at $\beta_{1,2} = e^{\pm j\pi/3}$. There are no other requirements from the numerator. Therefore, $p = q = 2$, and the numerator is

$$K(1 - \beta_1 z^{-1})(1 - \beta_2 z^{-1}) = K(1 - z^{-1} + z^{-2}),$$

where K is yet to be determined.

The pole transformation of a second-order normalized Butterworth filter with $T = \sqrt{2}$ is

$$s^2 + \sqrt{2}s + 1 \implies (\sqrt{2})^2(z-1)^2 + (\sqrt{2})^2(z-1)(z+1) + (z+1)^2$$
$$= 5^2 - 2z + 1 \implies 1 - 0.4z^{-1} + 0.2z^{-2}.$$

Therefore,

$$H^z(z) = \frac{K(1 - z^{-1} + z^{-2})}{1 - 0.4z^{-1} + 0.2z^{-2}}.$$

Finally, the requirement that the DC gain be 1 gives

$$H^z(1) = \frac{K}{0.8} = 1 \implies K = 0.8.$$

10.26

(a) We have

$$s = \frac{2}{T}\frac{z-1}{z+1}, \quad \tilde{s} = \frac{2}{T}\frac{\tilde{z}-1}{\tilde{z}+1}, \quad \tilde{s} = \frac{\omega_0}{s}.$$

Therefore,

$$\frac{2}{T}\frac{\tilde{z}-1}{\tilde{z}+1} = \frac{\omega_0 T}{2} \cdot \frac{z+1}{z-1}.$$

Solving for $\tilde{z}$ gives

$$\tilde{z} = \frac{z - \alpha}{\alpha z - 1},$$

where

$$\alpha = \frac{1 - 0.25\omega_0 T^2}{1 + 0.25\omega_0 T^2}.$$

(b) Let $z = e^{j\theta}$. Then

$$\tilde{z} = \frac{e^{j\theta} - \alpha}{\alpha e^{j\theta} - 1}.$$

The magnitude of the right side is 1, hence $\tilde{z}$ is on the unit circle, that is $\tilde{z} = e^{j\tilde{\theta}}$. We get

$$\tilde{\theta} = \arctan\frac{\sin\theta}{\cos\theta - \alpha} - \arctan\frac{\alpha\sin\theta}{\alpha\cos\theta - 1}.$$

(c) We have

$$\theta = 0 \to z = 1 \to \tilde{z} = -1 \to \tilde{\theta} = \pi,$$

and

$$\theta = \pi \to z = -1 \to \tilde{z} = 1 \to \tilde{\theta} = 0.$$

This proves that the transformation is low pass to high pass.

(d) θ_p in the z domain translates to $(2/T)\tan(0.5\theta_p)$ in the s domain. Similarly, $\tilde{\theta}_p$ in the $\tilde{z}$ domain translates to $(2/T)\tan(0.5\tilde{\theta}_p)$ in the $\tilde{s}$ domain. Therefore,

$$\omega_0 = [(2/T)\tan(0.5\theta_p)] \cdot [(2/T)\tan(0.5\tilde{\theta}_p)].$$

10.27

(a) We have

$$\tilde{\theta} = 0 \iff \tilde{z} = 1 \iff z = 1 \iff \theta = 0,$$
$$\tilde{\theta} = \pi \iff \tilde{z} = -1 \iff z = -1 \iff \theta = \pi.$$

Therefore, $\tilde{H}^z(\tilde{z})$ is a low-pass filter.

(b) The requirement will be satisfied if

$$e^{-j\theta_0} = \frac{e^{-j\tilde{\theta}_0} - \alpha}{1 - \alpha e^{-j\tilde{\theta}_0}}.$$

Solving for α gives

$$\alpha = \frac{e^{-j\tilde{\theta}_0} - e^{-j\theta_0}}{1 - e^{-j(\tilde{\theta}_0+\theta_0)}}.$$

Multiplying the numerator and the denominator by $e^{j0.5(\tilde{\theta}_0+\theta_0)}$ gives

$$\alpha = \frac{e^{j0.5(\theta_0-\tilde{\theta}_0)} - e^{j0.5(\tilde{\theta}_0-\theta_0)}}{e^{j0.5(\theta_0+\tilde{\theta}_0)} - e^{-j0.5(\tilde{\theta}_0+\theta_0)}} = \frac{\sin[0.5(\theta_0 - \tilde{\theta}_0)]}{\sin[0.5(\theta_0 + \tilde{\theta}_0)]}.$$

10.28 We observe that the stop band ripple of each of the two filters will leak into the pass band of the other filter and increase its pass-band ripple. We also observe that the pass-band tolerances are much tighter than the stop-band tolerances. Therefore, we will design each filter with pass-band tolerance and stop-band tolerance equal to 0.005 each. This guarantees that the pass-band specifications of the parallel connection will be met, whereas the stop-band specifications will be exceeded. To keep the computational complexity as low as possible, we choose elliptic filters. The design is implemented in the following MATLAB code. This code also displays the magnitude response of the parallel connection in the entire frequency range and in the pass bands.

```
[b1,a1,v1,u1,C1] = iirdes('ell','l',pi*[0.2,0.25],0.005,0.005);
[b2,a2,v2,u2,C2] = iirdes('ell','p',pi*[0.45,0.5,0.7,0.75],0.005,0.005*[1,1]);
H = frqresp(b1,a1,501) + frqresp(b2,a2,501);
theta = (1/500)*(0:500);
plot(theta,20*log10(abs(H))),grid,figure(1),pause
plot(theta(1:100),abs(H(1:100))),grid,figure(1),pause
plot(theta(251:350),abs(H(251:350))),grid,figure(1)
```

10.29

(a) The zero-order and first-order polynomials are obtained directly from the definition. The general recursion is proved as follows:

$$\begin{aligned}(2N-1)B_{N-1}(s) + s^2 B_{N-2}(s) &= \sum_{k=0}^{N-1}(2N-1)b_{N-1,k}s^k + \sum_{k=0}^{N-2} b_{N-2,k}s^{k+2}\\ &= b_{N-2,N-2}s^N + \sum_{k=2}^{N-1}[b_{N-2,k-2} + (2N-1)b_{N-1,k}]s^k + (2N-1)b_{N-1,1}s + (2N-1)b_{N-1,0}.\end{aligned}$$

We have

$$b_{N-2,N-2} = 1 = b_{N,N}, \quad (2N-1)b_{N-1,1} = \frac{(2N-1)!}{2^{N-1}(N-1)!} = b_{N,1}, \quad (2N-1)b_{N-1,0} = \frac{(2N)!}{2^N N!} = b_{N,0},$$

and

$$\begin{aligned}
b_{N-2,k-2} + (N-1)b_{N-1,k} &= \frac{(2N-2-k)!}{2^{N-k}(k-2)!(N-k)!} + \frac{(N-1)(2N-2-k)!}{2^{N-k-1}k!(N-1-k)!} \\
&= \frac{(2N-2-k)!}{2^{N-k}k!(N-k)!}[(k-1)k + 2(2N-1)(N-k)] \\
&= \frac{(2N-2-k)!}{2^{N-k}k!(N-k)!}[k^2 + k + 4N^2 - 2N - 4Nk] \\
&= \frac{(2N-2-k)!}{2^{N-k}k!(N-k)!}(2N-k-1)(2N-k) = \frac{(2N-k)!}{2^{N-k}k!(N-k)!} = B_{N,k}.
\end{aligned}$$

(b) Program 10.1 implements the computation.

Program 10.1 Computation of the coefficients of Bessel polynomials.

```
function b = bespol(N);
% Synopsis: b = bespol(N).
% Generates the coefficients of Bessel polynomial of order N.

bold = 1; bnew = [1, 1];
if (N == 0), b = bold; return;
elseif (N == 1), b = bnew; return;
else
   for n = 2:N,
      b = [0, (2*n-1)*bnew] + [bold, 0, 0];
      bold = bnew; bnew = b;
   end
end
```

(c) The phase response is nearly linear at low frequencies, implying that the group delay is nearly constant at these frequencies. The magnitude response, however, is not as flat as that of a Butterworth filter of the same order. Bessel filters are used when an IIR filter is needed with group delay as nearly constant as can be conveniently achieved.

10.30

(a) From the given relationship between $g[n]$ and $h[n]$ we get

$$G^z(z) = \frac{H^z(z)}{1-z^{-1}},$$

therefore

$$H^z(z) = (1-z^{-1})G^z(z).$$

The sequence $g[n]$ is obtained by sampling the step response of $H^L(s)$. We have

$$g(t) = \int_0^t h(\tau)d\tau \quad \Longrightarrow \quad G^L(s) = \frac{H^L(s)}{s}.$$

The computational procedure is therefore as follows:

i. Compute the inverse Laplace transform of $H^L(s)/s$ to get $g(t)$.
ii. Sample $g(t)$ at interval T to obtain $g[n]$.
iii. Compute $G^z(z)$, the z-transform of the sequence $g[n]$.
iv. Take $H^z(z) = (1-z^{-1})G^z(z)$.

(b) As we see from part a, the step invariant transform is the impulse invariant transform of $G^L(s)$, multiplied by $(1-z^{-1})$. Note that $G^L(s)$ has all the poles of $H^L(s)$ and an additional pole at $s = 0$. This pole is transformed to a pole at $z = 1$ in $G^z(z)$, and is canceled when multiplied by $(1-z^{-1})$. Therefore, the step invariant transform preserves the stability and the order of the analog filter.

Like the impulse invariant transform, the step invariant transform leads to aliasing, because $G^{\mathrm{L}}(s)$ does not have a finite bandwidth. However, $G^{\mathrm{L}}(s)$ contains less energy at high frequencies, because $G^{\mathrm{F}}(\omega) = H^{\mathrm{F}}(\omega)/(j\omega)$. Therefore, aliasing affects the step invariant transform less than it affects the impulse invariant transform.

Thanks to the extra pole, $G^{\mathrm{L}}(s)$ is always strictly proper, even when $H^{\mathrm{L}}(s)$ is exactly proper. Therefore, $g[n]$ always exists. It follows that the step invariant transform can be used for high-pass and band-stop filters as well, contrary to the impulse invariant transform (which, as we recall, is limited to low-pass and band-pass filters).

10.31 The inverse transformation is $z = 1 + sT$, therefore the left half-plane $\Re\{s\} < 0$ is mapped to the region $\Re\{z\} < 1$. This region includes points outside the unit circle, therefore stability of $H^{\mathrm{z}}(z)$ is not guaranteed.

10.32 The bilinear transform gives

$$Y^{\mathrm{z}}(z) = 0.5T\frac{z+1}{z-1}X^{\mathrm{z}}(z),$$

or

$$(1 - z^{-1})Y^{\mathrm{z}}(z) = 0.5T(1 + z^{-1})X^{\mathrm{z}}(z).$$

The corresponding time-domain relationship is

$$y(nT) = y(nT) + 0.5T[x(nT) + x(nT - T)].$$

The exact relationship satisfied by $y(nT)$ is

$$y(nT) = y(nT - T) + \int_{nT-T}^{nT} x(\tau)d\tau.$$

The above difference equation, on the other hand, approximates the integral by the area of the trapeze having base of length T and parallel sides of lengths $x(nT)$ and $x(nT - T)$.

10.33

(a) We have

$$X^{\mathrm{f}}(\theta_0) = \sum_{n=0}^{N-1} x[n]e^{-j\theta_0 n} = e^{-j\theta_0(N-1)}\sum_{n=0}^{N-1} x[n]e^{j\theta_0(N-1-n)} = e^{-j\theta_0(N-1)}\{x * h\}[N-1],$$

where $h[n]$ is as given in the problem.

(b)

$$H^{\mathrm{z}}(z) = \sum_{n=0}^{N-1} e^{j\theta_0 n}z^{-n} = \frac{1}{1 - e^{j\theta_0}z^{-1}}.$$

(c) The convolution (10.130) can be performed with N complex multiplications, or $4N$ real multiplications per output point, and we only look at one output point.

(d) Multiplying the numerator and the denominator of $H^{\mathrm{z}}(z)$ by $1 - e^{-j\theta_0}z^{-1}$ gives the stated expression.

(e) This is obvious.

(f) Passing $x[n]$ through the IIR filter $H_1^{\mathrm{z}}(z)$ is equivalent to solving the difference equation

$$y[n] = 2\cos\theta_0 y[n-1] - y[n-2] + x[n].$$

This requires only one multiplication of the complex number $y[n-1]$ by the real number $2\cos\theta_0$ per time point, hence two real multiplications per time point. Therefore, the entire sequence $\{y[n],\ 0 \le n \le N-1\}$ can be computed with $2N$ real multiplications. Forming $X^{\mathrm{f}}(\theta_0)$ requires two more complex multiplications, or four more real multiplications. The total is $2N + 4$ real multiplications, or slightly over half the number of real multiplications in direct computation.

10.34

(a) The transfer function of the reconstructor, to be denoted by $H^{L}_{\text{rec}}(s)$, is

$$H^{L}_{\text{rec}}(s) = \frac{(1+sT)(1-e^{-sT})}{s^2T}$$

We have at the output of the reconstructor,

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x[k]h_{\text{rec}}(t-kT).$$

The Laplace transform of $\hat{x}(t)$ is therefore

$$\hat{X}^{L}(s) = \sum_{k=-\infty}^{\infty} x[k]H^{L}_{\text{rec}}(s)e^{-ksT} = \sum_{k=-\infty}^{\infty} x[k]\frac{(1+sT)(1-e^{-sT})}{s^2T}e^{-ksT}$$
$$= \sum_{k=-\infty}^{\infty} x[k]\frac{(1+sT)(e^{-ksT}-e^{-(k+1)sT})}{s^2T}.$$

The Laplace transform of $y(t)$, the output of the continuous-time system, is

$$Y^{L}(s) = G^{L}(s)\hat{X}^{L}(s) = \sum_{k=-\infty}^{\infty} x[k]\frac{(1+sT)G^{L}(s)(e^{-ksT}-e^{-(k+1)sT})}{s^2T}$$
$$= \sum_{k=-\infty}^{\infty} x[k]F^{L}(s)(e^{-ksT}-e^{-(k+1)sT}),$$

where we define

$$F^{L}(s) = \frac{(1+sT)G^{L}(s)}{s^2T}.$$

Let $f(t)$ be the inverse Laplace transform of $F^{L}(s)$. Then the inverse Laplace transform of $Y^{L}(s)$ is

$$y(t) = \sum_{k=-\infty}^{\infty} x[k]\{f(t-kT)-f(t-kT-T)\}.$$

The sampled sequence $y[n]$ is given by

$$y[n] = y(nT) = \sum_{k=-\infty}^{\infty} x[k]\{f(n-kT)-f(nT-kT-T)\} = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \{x*h\}[n],$$

where we define

$$h[n] = f(nT) - f(nT-T).$$

(b)

$$F^{L}(s) = \frac{(1+sT)}{s^3T} \implies f(t) = \frac{t^2}{2T} + t \implies f(nT) = 0.5Tn^2 + Tn.$$

$$F^{z}(z) = \frac{0.5Tz^{-1}(1+z^{-1})}{(1-z^{-1})^3} + \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{0.5Tz^{-1}(3-z^{-1})}{(1-z^{-1})^3}.$$

Finally,

$$H^{z}(z) = (1-z^{-1})F^{z}(z) = \frac{0.5Tz^{-1}(3-z^{-1})}{(1-z^{-1})^2}.$$

10.35 We have at the output of the ZOH,

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x[m]h_{\text{zoh}}(t-mT-\Delta).$$

The Laplace transform of $\hat{x}(t)$ is therefore

$$\hat{X}^{L}(s) = \sum_{m=-\infty}^{\infty} x[m]H^{L}_{\text{zoh}}(s)e^{-msT-s\Delta} = e^{-s\Delta}\sum_{m=-\infty}^{\infty} x[m]\frac{1-e^{-sT}}{s}e^{-msT}$$
$$= e^{-s\Delta}\sum_{m=-\infty}^{\infty} x[m]\frac{e^{-msT}-e^{-(m+1)sT}}{s}.$$

The Laplace transform of $y(t)$ is

$$Y^{\mathrm{L}}(s) = G^{\mathrm{L}}(s)e^{-s\Delta}\hat{X}^{\mathrm{L}}(s) = \sum_{m=-\infty}^{\infty} x[m]\frac{G^{\mathrm{L}}(s)e^{-s\Delta}(e^{-msT} - e^{-(m+1)sT})}{s}$$
$$= \sum_{m=-\infty}^{\infty} x[m]F^{\mathrm{L}}(s)(e^{-msT} - e^{-(m+1)sT}),$$

where we define

$$F^{\mathrm{L}}(s) = \frac{G^{\mathrm{L}}(s)e^{-s\Delta}}{s}.$$

Let $f(t)$ be the inverse Laplace transform of $F^{\mathrm{L}}(s)$. Then

$$y(t) = \sum_{m=-\infty}^{\infty} x[m]\{f(t - mT) - f(t - mT - T)\}.$$

The sampled sequence $y[n]$ is given by

$$y[n] = y(nT) = \sum_{m=-\infty}^{\infty} x[m]\{f(nT - mT) - f(nT - mT - T)\} = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \{x * h\}[n],$$

where we define

$$h[n] = f(nT) - f(nT - T).$$

Chapter 11

Digital Filter Realization and Implementation

11.1

(a) The transfer function of the realization is

$$H^z(z) = \frac{1-\alpha^4 z^{-4}}{1-\alpha z^{-1}} = 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3}.$$

Therefore, the impulse response is

$$h[n] = \{1, \alpha, \alpha^2, \alpha^3, 0, 0, \ldots\}.$$

(b) See Figure 11.1:

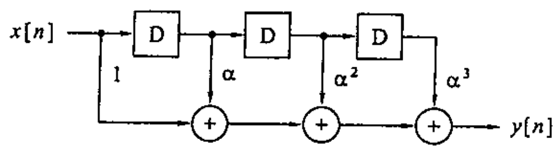

Figure 11.1 Pertaining to Solution 11.1.

(c) The filter has exact linear phase for $\alpha = 1$ and generalized linear phase for $\alpha = -1$.

(d)

$$\{x * h\}[n] = \{1, 1+\alpha, 1+\alpha+\alpha^2, 1+\alpha+\alpha^2+\alpha^3, 1+\alpha+\alpha^2+\alpha^3, \ldots\}.$$

11.2 Figure 11.2 shows a direct realization with $2N$ delay elements ($N = 3$ in the figure). A dual realization can be obtained by transposing the realization shown in Figure 11.2.

11.3 The transfer function of the filter is

$$H^z(z) = \sum_{m=0}^{\infty} \alpha^m z^{-mM} = \frac{1}{1-\alpha z^{-M}}.$$

The difference equation of the filter is

$$y[n] = \alpha y[n-M] + x[n].$$

One multiplication and one addition are needed per time point. Figure 11.3 shows a direct realization of the filter, for $M = 3$.

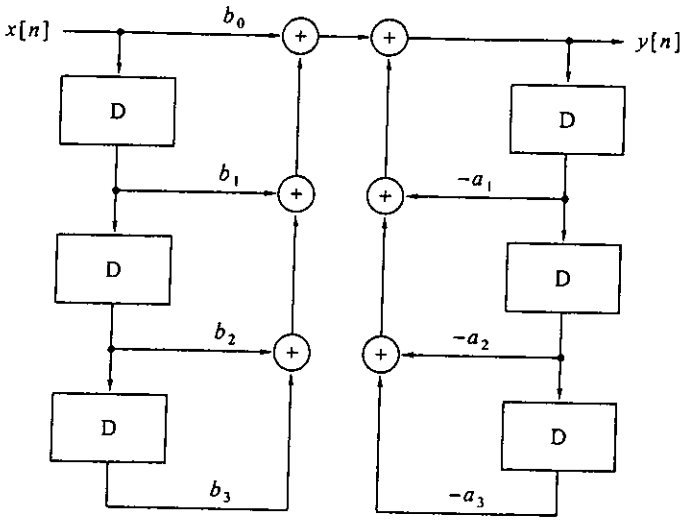

Figure 11.2 Pertaining to Solution 11.2.

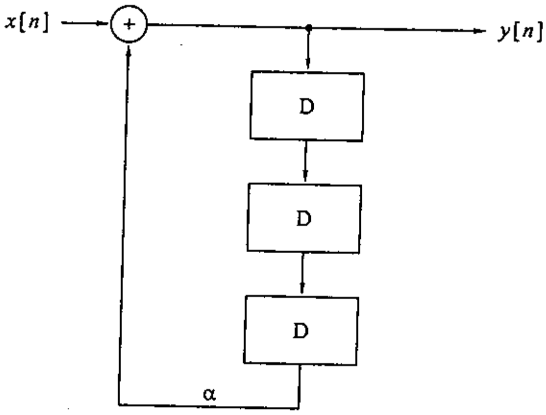

Figure 11.3 Pertaining to Solution 11.3.

11.4 Express the transfer function as

$$H^z(z) = 1 + z^{-1} + \cdots + z^{-(N-1)} = \frac{1 - z^{-N}}{1 - z^{-1}}.$$

Now we can realize $H^z(z)$ in either of the two direct realizations; see Figures 11.4 and 11.6. We have to substitute

$$a_1 = -1, \quad a_k = 0,\ 2 \le k \le N, \quad b_0 = 1, \quad b_k = 0,\ 1 \le k \le N-1, \quad b_N = -1.$$

The corresponding difference equation is

$$y[n] = y[n-1] + x[n] - x[n-N].$$

As we see, there are two additions and no multiplications per time point.

11.5

(a) Let the state variables $s_1[n], s_2[n]$ be the outputs of the top and bottom delay elements, respectively. Then

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 2\cos\theta_0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} A\delta[n], \quad y[n] = \begin{bmatrix} \cos\theta_0 & -1 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + A\delta[n].$$

In the z domain,

$$Y^z(z) = A[C(zI_2 - A)^{-1}B + 1].$$

We have

$$zI_2 - A = \begin{bmatrix} z - 2\cos\theta_0 & 1 \\ -1 & z \end{bmatrix}, \quad (zI_2 - A)^{-1} = \frac{1}{z^2 - 2\cos\theta_0 z + 1}\begin{bmatrix} z & -1 \\ 1 & z - 2\cos\theta_0 \end{bmatrix}.$$

Therefore,

$$Y^z(z) = A\frac{z\cos\theta_0 - 1}{z^2 - 2\cos\theta_0 z + 1} + A = A\frac{1 - \cos\theta_0 z^{-1}}{1 - 2\cos\theta_0 z^{-1} + z^{-2}}.$$

We see from Table 7.1 that

$$y[n] = A\cos(\theta_0 n), \quad n \geq 0.$$

(b) The realization can be used to generate a discrete-time cosine signal having a given frequency θ_0. It does not require any explicit cosine operations, only multiplications and additions, so it is very efficient in computations. The frequency can be controlled by setting the constant gain $\cos\theta_0$ according to the desired frequency.

11.6 The ideal impulse response is [see (9.30)],

$$h_d[n] = \begin{cases} \dfrac{\sin[(n-0.5N)(\theta_0+\Delta\theta)] - \sin[(n-0.5N)(\theta_0-\Delta\theta)]}{\pi(n-0.5N)}, & n \neq 0.5N, \\ \dfrac{2\Delta\theta}{\pi}, & n = 0.5N. \end{cases}$$

$$= \begin{cases} \dfrac{2\sin[(n-0.5N)\Delta\theta]\cos[(n-0.5N)\theta_0]}{\pi(n-0.5N)}, & n \neq 0.5N, \\ \dfrac{2\Delta\theta}{\pi}, & n = 0.5N. \end{cases}$$

Let $w[n]$ the selected window, and define

$$f[n] = \begin{cases} \dfrac{2w[n]\sin[(n-0.5N)\Delta\theta]}{\pi(n-0.5N)}, & n \neq 0.5N, \\ \dfrac{2w[0.5N]\Delta\theta}{\pi}, & n = 0.5N. \end{cases}$$

Then

$$h[n] = f[n]\cos[(n-0.5N)\theta_0].$$

The filter is implemented as

$$y[n] = f[0.5N]x[n-0.5N] + \sum_{k=0}^{0.5N-1} f[k]\cos[(k-0.5N)\theta_0](x[n-k] + x[n-N+k]).$$

This implementation requires $0.5N$ trigonometric function generators, for $\{\cos[(k-0.5N)\theta_0],\ 0 \leq k \leq 0.5N-1\}$, as well as $N+1$ multipliers and N adders.

11.7 The transfer function of the transposed realization is given by [cf. (11.48)]

$$H_{\mathrm{T}}^z(z) = B'(zI_N - A')^{-1}C' + D.$$

However, since $H_{\mathrm{T}}^z(z)$ is scalar it is equal to its own transpose. Therefore we get, by transposing $H_{\mathrm{T}}^z(z)$,

$$[H_{\mathrm{T}}^z(z)]' = C(zI_N - A)^{-1}B + D = H^z(z).$$

11.8 Observe first that if $\tilde{A} = T^{-1}AT$, then

$$zI_N - \tilde{A} = zT^{-1}T - T^{-1}AT = T^{-1}[zI_N - A]T.$$

Therefore,

$$[zI_N - \tilde{A}]^{-1} = T^{-1}AT = T^{-1}[zI_N - A]^{-1}T.$$

Finally,

$$\tilde{H}^z(z) = \tilde{C}[zI_N - \tilde{A}]^{-1}\tilde{B} + \tilde{D} = CTT^{-1}[zI_N - A]^{-1}TT^{-1}B + D = C[zI_N - A]^{-1}B + D = H^z(z).$$

11.9

(a) The state-space representation of the parallel connection is

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} x[n],$$

$$y[n] = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + (D_1 + D_2)x[n].$$

(b) The matrix F is zero; the matrices G and H are the identity matrix of dimension $N_1 + N_2$; the scalar K is equal to c_0.

11.10

(a) Denote by $u[n]$ the output of the first system. Then $u[n]$ is the input to the second system, therefore

$$s_1[n+1] = A_1 s_1[n] + B_1 x[n], \quad u[n] = C_1 s_1[n] + D_1 x[n],$$
$$s_2[n+1] = A_2 s_2[n] + B_2 u[n], \quad y[n] = C_2 s_2[n] + D_2 u[n].$$

Substitute $u[n]$ from the first equation to the second:

$$s_2[n+1] = A_2 s_2[n] + B_2 C_1 s_1[n] + B_2 D_1 x[n],$$
$$y[n] = C_2 s_2[n] + D_2 C_1 s_1[n] + D_2 D_1 x[n].$$

Therefore, the state-space representation of the cascade connection is

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} x[n],$$

$$y[n] = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + D_2 D_1 x[n].$$

(b) The entries of the matrix F are 1 along the first subdiagonal and zero elsewhere; the matrix G has b_0 as its first entry and is zero elsewhere; the matrix H has 1 as its last entry and is zero elsewhere; the scalar K is equal to 0.

11.11 The state-space matrices of the direct realization of an FIR filter are

$$A_1 = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} b_1 & b_2 & \dots & b_N \end{bmatrix}, \quad D_1 = b_0.$$

The state-space matrices of the transposed realization are

$$A_2 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \quad D_2 = b_0.$$

11.12

(a) See Figure 11.4.

(b) We derive the transfer function first:

$$zI - A = \begin{bmatrix} z-3 & 0.25 \\ -5 & z \end{bmatrix},$$

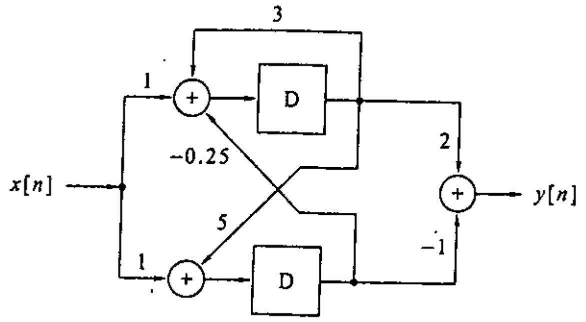

Figure 11.4 Pertaining to Solution 11.12(a).

$$(zI - A)^{-1} = \frac{1}{z^2 - 3z + 1.25}\begin{bmatrix} z & -0.25 \\ 5 & z-3 \end{bmatrix},$$

$$H^z(z) = C(zI - A)^{-1}B = \frac{1}{z^2 - 3z + 1.25}\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} z & -0.25 \\ 5 & z-3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{z - 2.5}{z^2 - 3z + 1.25}.$$

Therefore, the difference equation is

$$y[n] = 3y[n-1] - 1.25y[n-2] + x[n-1] - 2.5x[n-2].$$

(c) We have

$$H^z(z) = \frac{z - 2.5}{(z - 2.5)(z - 0.5)} = \frac{1}{z - 0.5}.$$

Therefore,

$$h[n] = 0.5^{n-1}, \quad n \geq 1.$$

(d) See Figure 11.5:

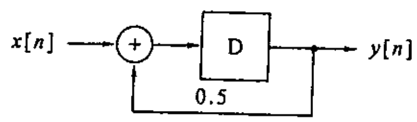

Figure 11.5 Pertaining to Solution 11.12(d).

11.13

(a) The approximation is obvious from Solution 10.32.

(b) We get from part a,

$$(I_N - 0.5TA)s(nT + T) = (I_N + 0.5TA)s(nT) + 0.5TBx(nT + T) + 0.5TBx(nT).$$

Therefore,

$$s(nT + T) = \tilde{A}s(nT) + \tilde{B}x(nT + T) + \tilde{B}x(nT),$$

where $\tilde{A}$ and $\tilde{B}$ are as defined in the problem. This is not yet a standard state-space form, because of the term $x(nT + T)$ on the write side. However, by defining the state vector $\tilde{s}(nT)$ as given in the problem, we can write the above as

$$\begin{aligned} \tilde{s}(nT + T) &= \tilde{A}[\tilde{s}(nT) + \tilde{B}x(nT)] + \tilde{B}x(nT) \\ &= \tilde{A}\tilde{s}(nT) + (I_N + \tilde{A})\tilde{B}x(nT). \end{aligned}$$

The output equation is obtained by using the same substitution:

$$\begin{aligned} y(nT) &= Cs(nT) + Dx(nT) = C[\tilde{s}(nT) + \tilde{B}x(nT)] + Dx(nT) \\ &= C\tilde{s}(nT) + (D + C\tilde{B})x(nT). \end{aligned}$$

(c) Program 11.1 implements the desired computation.

Program 11.1 Bilinear transform in state space.

```
function [bout,aout] = bilinss(bin,ain,T);
% Synopsis: [bout,aout] = bilinss(bin,ain,T).
% Bilinear transform in state space.
% Input parameters:
% bin,ain: the continuous-time polynomials
% T: the sampling interval.
% Output parameters:
% bout,aout: the discrete-time polynomials.

p = length(ain) - 1; q = length(bin) - 1;
if (q < p), bin = [zeros(1,p-q), bin]; end
[A,B,C,D] = tf2ss(bin,ain);
temp = eye(p) - 0.5*T*A;
Atilde = temp\(eye(p) + 0.5*T*A);
Btilde = temp\(0.5*T*B);
Ad = Atilde; Bd = (eye(p) + Atilde)*Btilde;
Cd = C; Dd = D + C*Btilde;
[bout,aout] = ss2tf(Ad,Bd,Cd,Dd);
```

11.14

(a) By the definition of ρ_2, the condition $-1 < \rho_2 < 1$ is equivalent to $-1 < a_2 < 1$. When this holds, then by the definition of ρ_1, the condition $-1 < \rho_1 < 1$ is equivalent to $1 + a_1 + a_2 > 0$ and $1 - a_1 + a_2 > 0$. By Example 7.2, these conditions are necessary and sufficient for stability of a discrete-time second-order system.

(b) The state matrices for $y_1[n]$ are read from the block diagram as

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -\rho_1 & -\rho_2 \end{bmatrix}, \quad D_1 = 1.$$

Similarly, the state matrices for $y_2[n]$ are

$$A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} \rho_1\rho_2 & 1 \end{bmatrix}, \quad D_2 = -\rho_2.$$

(c)

$$\begin{aligned} H_1^z(z) &= C_1(zI_2 - A_1)^{-1}B_1 + D_1 = \begin{bmatrix} -\rho_1 & -\rho_2 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix} + 1 \\ &= 1 - \rho_1(1-\rho_2)z^{-1} - \rho_2 z^{-2} = 1 + a_1 z^{-1} + a_2 z^{-2} = a(z). \end{aligned}$$

(d)

$$\begin{aligned} H_2^z(z) &= C_2(zI_2 - A_2)^{-1}B_2 + D_2 = \begin{bmatrix} \rho_1\rho_2 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix} - \rho_2 \\ &= -\rho_2 - \rho_1(1-\rho_2)z^{-1} + z^{-2} = a_2 + a_1 z^{-1} + z^{-2}. \end{aligned}$$

As we see, the transfer function from $x[n]$ to $y_2[n]$ is the reversed-order polynomial.

11.15

(a) The state matrices for $y_1[n]$ are read from the block diagram as

$$A_1 = \begin{bmatrix} \rho_1 & \rho_2 \\ 1-\rho_1^2 & -\rho_1\rho_2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \rho_1 & \rho_2 \end{bmatrix}, \quad D_1 = 1.$$

Similarly, the state matrices for $y_2[n]$ are

$$A_2 = \begin{bmatrix} \rho_1 & \rho_2 \\ 1-\rho_1^2 & -\rho_1\rho_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1-\rho_2^2 \end{bmatrix}, \quad D_2 = -\rho_2.$$

(b)

$$zI_2 - A_1 = \begin{bmatrix} z-\rho_1 & -\rho_2 \\ -(1-\rho_1^2) & z+\rho_1\rho_2 \end{bmatrix}, \quad (zI_2 - A_1)^{-1} = \frac{1}{a(z)}\begin{bmatrix} z+\rho_1\rho_2 & \rho_2 \\ 1-\rho_1^2 & z-\rho_1 \end{bmatrix},$$

$$H_1^z(z) = C_1(zI_2 - A_1)^{-1}B_1 + D_1 = \frac{1}{z^2-\rho_1(1-\rho_2)z-\rho_2}\begin{bmatrix} \rho_1 & \rho_2 \end{bmatrix}\begin{bmatrix} z+\rho_1\rho_2 & \rho_2 \\ 1-\rho_1^2 & z-\rho_1 \end{bmatrix}\begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix} + 1$$
$$= \frac{\rho_1 z - \rho_1\rho_2 z + \rho_2}{z^2-\rho_1(1-\rho_2)z-\rho_2} + 1 = \frac{1}{a(z)}.$$

(c)

$$H_2^z(z) = C_2(zI_2 - A_2)^{-1}B_2 + D_2 = \frac{1}{z^2-\rho_1(1-\rho_2)z-\rho_2}\begin{bmatrix} 0 & 1-\rho_2^2 \end{bmatrix}\begin{bmatrix} z+\rho_1\rho_2 & \rho_2 \\ 1-\rho_1^2 & z-\rho_1 \end{bmatrix}\begin{bmatrix} 1 \\ -\rho_1 \end{bmatrix} - \rho_2$$
$$= \frac{1-\rho_2^2-\rho_1(1-\rho_2^2)z}{z^2-\rho_1(1-\rho_2)z-\rho_2} - \rho_2 = \frac{a_2 + a_1 z^{-1} + z^{-2}}{a(z)}.$$

As we recall from Problem 7.29, the transfer function $H_2^z(z)$ is all-pass: its magnitude response is 1 at all frequencies.

11.16

(a) We see, by comparing Figure 11.30 with Figure 11.29, that

$$y[n] = c_0 y_1[n] + c_1 s_2[n+1] + c_2 y_2[n],$$

therefore,

$$Y^z(z) = c_0 Y_1^z(z) + c_1 z S_2^z(z) + c_2 Y_2^z(z).$$

Also,

$$S_2^z(z) = \begin{bmatrix} 0 & 1 \end{bmatrix}(zI_2 - A_2)^{-1}B_2 X^z(z) = \frac{1-\rho_1 z}{z^2-\rho_1(1-\rho_2)z-\rho_2}X^z(z).$$

Using the results for $Y_1^z(z)$ and $Y_2^z(z)$ from Solution 11.15, we can write

$$\frac{Y^z(z)}{X^z(z)} = \frac{c_0 + c_1(z^{-1}-\rho_1) + c_2(a_2 + a_1 z^{-1} + z^{-2})}{a(z)} = \frac{(c_0 - \rho_1 c_1 + a_2 c_2) + (c_1 + a_1 c_2)z^{-1} + c_2 z^{-2}}{a(z)}.$$

(b) We have to satisfy the three equalities

$$c_0 - \rho_1 c_1 + a_2 c_2 = b_0, \quad c_1 + a_1 c_2 = b_1, \quad c_2 = b_2.$$

This gives

$$c_2 = b_2, \quad c_1 = b_1 - a_1 c_2, \quad c_0 = b_0 + \rho_1 c_1 - a_2 c_2.$$

11.17

(a) We get from the sum representation

$$\frac{\partial a(z)}{\partial a_k} = z^{-k},$$

and from the product representation

$$\frac{\partial a(z)}{\partial \alpha_l} = -z^{-1}\prod_{i\neq l}(1-\alpha_i z^{-1}).$$

(b) Substituting $z = \alpha_m$, $m \neq l$ in the second result of part a makes the right hand zero. Substituting $z = \alpha_l$ gives

$$\left.\frac{\partial a(z)}{\partial \alpha_l}\right|_{z=\alpha_l} = -\alpha_l^{-1}\prod_{i\neq l}(1-\alpha_i \alpha_l^{-1}).$$

(c) We get, using the results of parts a and b,

$$\alpha_m^{-k} = -\alpha_m^{-1}\prod_{i\neq m}(1-\alpha_i\alpha_m^{-1})\frac{\partial \alpha_m}{\partial a_k} = -\alpha_m^{-p}\prod_{i\neq m}(\alpha_m-\alpha_i)\frac{\partial \alpha_m}{\partial a_k},$$

hence

$$\frac{\partial \alpha_m}{\partial a_k} = -\frac{\alpha_m^{p-k}}{\prod_{i\neq m}(\alpha_m-\alpha_i)}.$$

(d) First-order Taylor-series approximation of $\hat{\alpha}_m$ around the point α_m gives

$$\hat{\alpha}_m - \alpha_m \approx \sum_{k=1}^{p}\frac{\partial \alpha_m}{\partial a_k}(\hat{a}_k - a_k) = -\sum_{k=1}^{p}\frac{\alpha_m^{p-k}(\hat{a}_k - a_k)}{\prod_{i\neq m}(\alpha_m-\alpha_i)}.$$

This is identical to (11.77), as was required to prove.

11.18

(a) If p_k appears only in $a(z)$ we have

$$\frac{\partial H^{\mathrm{f}}(\theta)}{\partial p_k} = -\frac{b(e^{j\theta})}{a^2(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial p_k} = -H^{\mathrm{f}}(\theta)\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial p_k} = -|H^{\mathrm{f}}(\theta)|e^{j\psi(\theta)}\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial p_k}.$$

Substitution of this result in (11.79) gives the required expression.

(b) If p_k appears only in $b(z)$ we have

$$\frac{\partial H^{\mathrm{f}}(\theta)}{\partial p_k} = \frac{1}{a(e^{j\theta})}\frac{\partial b(e^{j\theta})}{\partial p_k} = H^{\mathrm{f}}(\theta)\frac{1}{b(e^{j\theta})}\frac{\partial b(e^{j\theta})}{\partial p_k} = |H^{\mathrm{f}}(\theta)|e^{j\psi(\theta)}\frac{1}{b(e^{j\theta})}\frac{\partial b(e^{j\theta})}{\partial p_k}.$$

Substitution of this result in (11.79) gives the required expression.

11.19

(a) The partial derivatives are

$$\frac{\partial a(e^{j\theta})}{\partial \alpha_{\mathrm{r}}} = -2e^{-j\theta} + 2\alpha_{\mathrm{r}}e^{-j2\theta}, \quad \frac{\partial a(e^{j\theta})}{\partial \alpha_{\mathrm{i}}} = 2\alpha_{\mathrm{i}}e^{-j2\theta}.$$

(b)

$$\Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial \alpha_{\mathrm{r}}}\right\} = \Re\left\{\frac{-2e^{-j\theta}+2\alpha_{\mathrm{r}}e^{-j2\theta}}{1+a_1e^{-j\theta}+a_2e^{-j2\theta}}\right\}, \quad \Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial \alpha_{\mathrm{i}}}\right\} = \Re\left\{\frac{2\alpha_{\mathrm{i}}e^{-j2\theta}}{1+a_1e^{-j\theta}+a_2e^{-j2\theta}}\right\},$$

$$\Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial a_1}\right\} = \Re\left\{\frac{e^{-j\theta}}{1+a_1e^{-j\theta}+a_2e^{-j2\theta}}\right\}, \quad \Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial a_2}\right\} = \Re\left\{\frac{e^{-j2\theta}}{1+a_1e^{-j\theta}+a_2e^{-j2\theta}}\right\}.$$

(c) For $\theta = 0$,

$$\Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial \alpha_{\mathrm{r}}}\right\} = \frac{2(-1+\alpha_{\mathrm{r}})}{1+a_1+a_2}, \quad \Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial \alpha_{\mathrm{i}}}\right\} = \frac{2\alpha_{\mathrm{i}}}{1+a_1+a_2}, \qquad (*)$$

$$\Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial a_1}\right\} = \frac{1}{1+a_1+a_2}, \quad \Re\left\{\frac{1}{a(e^{j\theta})}\frac{\partial a(e^{j\theta})}{\partial a_2}\right\} = \frac{1}{1+a_1+a_2}.$$

All four expressions have the same denominator. When the poles are close to $z = 1$, then α_r is close to 1 and α_i is close to 0. Therefore, the numerators in $(*)$ are much smaller than 1 in magnitude. This implies that the sensitivity of the coupled realization to coefficient quantization is much smaller than that of the direct realization at frequency $\theta = 0$. The same conclusion holds for nonzero, but close to zero frequencies.

11.20 Coefficient quantization preserves the relationship $h[N-n] = \pm h[n]$. This relationship is necessary and sufficient for an FIR filter to have linear phase.

11.21 Since all coefficients are 1, there is neither coefficient quantization nor quantization noise. Overflow is possible and should be prevented by proper scaling. Zero-frequency limit cycle may also be present, because in the absence of input we have

$$y[n] = y[n-1].$$

11.22 Program 11.2 implements the procedure qtf.

11.23 Figure 11.6 shows the possible pole locations when ρ_1 and ρ_2 are quantized to 5 bits. Similarly to the direct realization, there are only few possible pole locations near $z = \pm 1$.

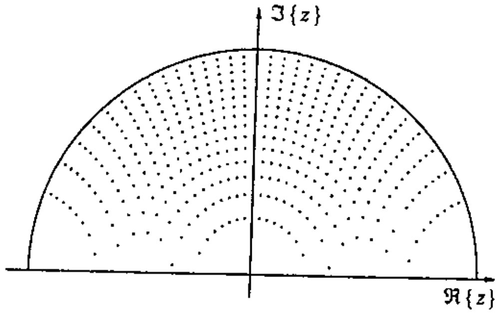

Figure 11.6 Pertaining to Solution 11.23.

11.24

(a) Quantization of $\cos\theta_0$ will change the frequency of the cosine waveform. Quantization of the multiplier's output will add noise to $y[n]$. As a result, $y[n]$ will deviate from a true cosine waveform.

(b) The following MATLAB code implements the simulation:

```
s1 = 0; s2 = 0;
for n = 1:N+1,
   if (n == 1), u = 0.875; else, u = 0; end
   temp = quant(ct0*s1,'t',10);
   s1in = 2*temp - s2 + u;
   y(n) = s1in - temp;
   s2 = s1; s1 = s1in;
end
```

The variable ct0 has to be preset to $\cos\theta_0$; the variable N has to be preset to the desired number of points.

(c) The error is oscillatory and its envelope increases over time in both cases. However, in the second case, the error envelope increases at a much faster rate. The reason is that, in the first case, the parameter $\cos\theta_0$ is represented exactly (so it is not subject to quantization), and the error is caused only by the multiplier's quantization noise. In the second case $\cos\theta_0$ is quantized, so we get both frequency error and quantization noise. We note that 10 bits is a poor resolution for such an application, and at least 16 bits should be used in practice. Also, in practice the circuit should be reset and re-initialized periodically, to avoid error build-up of the kind we have seen.

Program 11.2 Conversion of a transfer function to a parallel or a cascade realization with coefficient quantization and scaling.

```
function [c,nsec,dsec,sc,sn,sd] = qtf(b,a,typ,B);
% Synopsis: [c,nsec,dsec,sc,sn,sd] = qtf(b,a,typ,B).
% Converts a transfer function to either parallel or cascade
% realization, scales and quantizes the coefficients to
% a desired number of bits.
% Input parameters:
% b, a: the numerator and denominator polynomials
% typ: 'c' for cascade, 'p' for parallel
% B: number of bits
% Output parameters:
% c: the constant coefficient
% nsec: matrix of numerators of second-order sections
% dsec: matrix of denominators of second-order sections
% sc: scale factor for c
% sn: scale factors for numerators
% sd: scale factors for denominators

if (typ == 'p'),
   [c,nsec,dsec] = tf2rpf(b,a);
   sc = scale2(c); sct = (2^(B-1))/sc;
   c = (1/sct)*round(sct*c);
   [M,junk] = size(nsec);
   sn = zeros(1,M); sd = zeros(1,M);
   for k = 1:M,
      nt = nsec(k,:); dt = dsec(k,:);
      if (dt(3) == 0), dt = dt(1:2); nt = nt(1); end
      sn(k) = scale2(nt); snt = (2^(B-1))/sn(k);
      nsec(k,1:length(nt)) = (1/snt)*round(snt*nt);
      sd(k) = scale2(dt); sdt = (2^(B-1))/sd(k);
      dsec(k,1:length(dt)) = (1/sdt)*round(sdt*dt);
   end
elseif (typ == 'c'),
   c = b(1); v = roots(a); u = roots(b);
   sc = scale2(c); sct = (2^(B-1))/sc;
   c = (1/sct)*round(sct*c);
   [nsec,dsec] = pairpz(v,u);
   [M,junk] = size(nsec); H = c;
   sn = zeros(1,M); sd = zeros(1,M);
   for k = 1:M,
      nt = nsec(k,:); dt = dsec(k,:);
      if (dt(3) == 0), dt = dt(1:2); nt = nt(1:2); end
      sn(k) = scale2(nt); snt = (2^(B-1))/sn(k);
      nsec(k,1:length(nt)) = (1/snt)*round(snt*nt);
      sd(k) = scale2(dt); sdt = (2^(B-1))/sd(k);
      dsec(k,1:length(dt)) = (1/sdt)*round(sdt*dt);
   end
else,
   error('Unrecognized type in QTF');
end
```

11.25 Equation (11.126) is equivalent to

$$-0.5 \le -2^{B-1}ay - 2^{B-1}y \le 0.5,$$

or

$$-0.5 \le 2^{B-1}(1+a)y \le 0.5.$$

This is the same as (11.127).

11.26 Assume that there are complex poles at $z = \rho e^{\pm j\zeta}$, close to $z = -1$. Then ρ is nearly 1 and ζ is nearly π. Since $a_1 = -2\rho\cos\zeta$ and $a_2 = \rho^2$, a_1 is nearly 2 and a_2 is nearly 1. Write the difference equation for the auxiliary variable $u[n]$ of the direct realization as [cf. 11.2]

$$u[n] = x[n] - u[n-1] - (u[n-1] + u[n-2]) - (a_1 - 2)u[n-1] - (a_2 - 1)u[n-2].$$

Now the modified coefficients $(a_1 - 2)$ and $(a_2 + 1)$ are both small numbers, so we can scale them up by a few bits, thus retaining a few of their lower-order bits. Since the filter is necessarily high pass, the signal $u[n]$ changes fast in time, so $|u[n-1] + u[n-2]|$ is typically small compared with the full scale of $u[n]$.

Chapter 12

Multirate Signal Processing

12.1 Let $x_3[n] = a_1x_1[n] + a_2x_2[n]$. Then we get from (12.1),

$$y_3[n] = x_3[nM] = a_1x_1[nM] + a_2x_2[nM] = a_1y_1[n] + a_2y_2[n].$$

This proves that decimation is linear. We similarly get from (12.2),

$$\begin{aligned} y_3[n] &= \begin{cases} x_3[n/L], & \text{if } n/L \text{ is an integer,} \\ 0, & \text{if } n/L \text{ is noninteger,} \end{cases} \\ &= \begin{cases} a_1x_1[n/L] + a_2x_2[n/L], & \text{if } n/L \text{ is an integer,} \\ 0, & \text{if } n/L \text{ is noninteger,} \end{cases} \\ &= a_1y_1[n] + a_2y_2[n]. \end{aligned}$$

12.2 The decimated signal is

$$y[n] = \cos(M\theta_0 n).$$

The decimated signal is nonaliased if $\theta_0 < \pi/M$ and aliased otherwise.

12.3 As we know from frequency-domain theory of expansion, $Y^f(\theta)$ will exhibit L pairs of delta functions, at frequencies $\{l\theta_0/L,\ 1 \le L\}$. There will be no aliasing.

12.4 We get from (12.12) and (12.15) together,

$$Y^z(z) = \frac{1}{M}\sum_{m=0}^{M-1} X^z(zW_M^{-m}).$$

12.5 By property (7.29) of the z-transform, the inverse z-transform of $X^z(zW_M^{-m})$ is the sequence $W_M^{mn}x[n]$. In particular, the first component, corresponding to $m = 1$, is the signal itself. The second component, in case of $M = 2$, is $(-1)^n x[n]$.

12.6 Let $h[n]$ be the triangularly-shaped FIR filter

$$h[n] = \begin{cases} \dfrac{n+L}{L}, & -(L-1) \le n \le -1, \\ \dfrac{L-n}{L}, & 0 \le n \le L-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have from (12.19),

$$y[nL+i] = \sum_m x[m]h[nL+i-Lm] = \sum_m x[m]h[(n-m)L+i].$$

Chapter 13

Analysis and Modeling of Random Signals

13.1 The Welch periodogram for K overlapping points is

$$\hat{K}_x^{\mathrm{f}}(\theta) = \frac{1}{L}\sum_{l=0}^{L-1}\left\{\frac{1}{N}\left|\sum_{n=0}^{N-1} w[n]x[n+l(N-K)]e^{-j\theta n}\right|^2\right\}.$$

13.2 The signal $p(t)$ satisfies

$$p(t) = \pm 1 \pm j \quad \text{for all } t.$$

Therefore,

$$p^2(t) = \pm j2, \quad \text{and} \quad p^4(t) = -1.$$

Therefore, if $y(t) = p(t)e^{j2\pi f_0 t}$, then

$$z(t) = y^4(t) = -e^{j8\pi f_0 t}.$$

The Fourier transform of $z(t)$ has a delta function at $4f_0$. Therefore, if we sample $z(t)$ at a rate higher than eight times the maximum value of $|f_0|$, we will be able to determine f_0 unambiguously.

Repeating the computation in Solution 6.17, we get that the loss in SNR to due raising $y(t)$ to the fourth power is by a factor of 16, or 12 dB.

13.3

(a) We have from (13.11), using the symmetry of the sequence $\hat{\kappa}_x[m]$,

$$\begin{aligned}
S^{\mathrm{d}}[k] &= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{m=-(N-1)}^{-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) \\
&= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{m=1)}^{N-1}\hat{\kappa}_x[m]\exp\left(\frac{j2\pi m}{N}\right) \\
&= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{m=1)}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi (N-m)}{N}\right) \\
&= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{n=1)}^{N-1}\hat{\kappa}_x[N-n]\exp\left(-\frac{j2\pi n}{N}\right).
\end{aligned}$$

Therefore,

$$s[n] = \begin{cases}\hat{\kappa}_x[0], & n = 0, \\ \hat{\kappa}_x[n] + \hat{\kappa}_x[N-n], & 1 \le n \le N-1.\end{cases}$$

(b) We get by the same derivation as before,

$$\frac{M}{N}s_{\mathrm{a}}[n] = \begin{cases}\hat{\kappa}_{x_{\mathrm{a}}}[0], & n = 0, \\ \hat{\kappa}_{x_{\mathrm{a}}}[n] + \hat{\kappa}_{x_{\mathrm{a}}}[M-n], & 1 \le n \le M-1.\end{cases}$$

12.9

(a) Let us assume that the signal $x[n]$ has been obtained by sampling a continuous-time signal $x(t)$ at interval T. Then an ideal fractional-delay filter would have output

$$y[n] = x\left(nT - \frac{lT}{M}\right).$$

Interpolating $x[n]$ by a factor M gives approximately the sequence $x(nT/M)$. Errors are introduced at this stage by the nonideal frequency response of the interpolation filter. Delaying this sequence by l time units gives $x((n-l)T/M) = x(nT/M - lT/M)$. Finally, M-fold decimation gives $x(nT - lT/M)$, which is the required result. This scheme only requires a finite number of rational operations per time point, therefore it is necessarily only an approximation of $e^{-j\theta\tau}$. As we see, the only source of error in this scheme is the interpolation filter.

(b) The output of the interpolation filter is

$$v[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - mM].$$

Therefore, the delayed and decimated output is

$$y[k] = v[kM - l] = \sum_{m=-\infty}^{\infty} x[m]h[kM - l - mM] = \sum_{m=-\infty}^{\infty} x[m]h[kM + M - l - (m+1)M]$$

$$= \sum_{m=-\infty}^{\infty} x[m-1]h[(k-m)M + M - l].$$

The right side is equivalent to delaying the signal $x[n]$ by one sample and then convolving with the polyphase component $q_{l-1}[n]$.

12.10 We have

$$\hat{x}\left(\frac{nT}{L}\right) = \sum_{m=-\infty}^{\lfloor n/L \rfloor} x[m]h\left(\left(\frac{n}{L} - m\right)T\right) = \sum_{m=-\infty}^{\lfloor n/L \rfloor} x[m]h\left((n - mL)\frac{T}{L}\right).$$

The right side can be interpreted as follows: Obtain a digital filter by the impulse invariant transform of $h(t)$, and use this filter as an interpolation filter for L-fold interpolation of $x[m]$.

12.11 In general, the answer is negative. Consider, however, the special case $M = 2$. Assume first that N is even, say $N = 2K$. Then

$$p_0[n] = h[2n] = \pm h[N - 2n] = \pm h[2(K - n)] = \pm p_0[K - n],$$

therefore $p_0[n]$ is linear phase. Also,

$$p_1[n] = h[2n+1] = \pm h[N - 2n - 1] = \pm h[2(K - 1 - n) + 1] = \pm p_0[K - 1 - n],$$

therefore $p_1[n]$ is linear phase. Next assume that N is odd, say $N = 2K + 1$. Then

$$p_0[n] = h[2n] = \pm h[N - 2n] = \pm h[2K + 1 - 2n] = \pm p_1[K - n],$$

and

$$p_1[n] = h[2n+1] = \pm h[N - 2n - 1] = \pm h[2K - 2n] = \pm p_0[K - n].$$

Therefore, neither polyphase component is linear phase. However, each is the reversed-order copy of the other, up to a sign.

12.12

(a) This is obvious, since the pass-band ripple of two filters in cascade is always bounded from above by the sum of the individual pass-band ripples. The decimation and expansion operations only stretch and compress the spectra along the θ axis, so they do not change the pass-band ripple.

for some constant $\{c_1, c_2\}$ and some integers $\{l_1, l_2\}$. Solving these two equations gives

$$G_0^z(z) = 0.5c_1 z^{-l_1} + 0.5c_2 z^{-l_2}, \quad G_0^z(-z) = 0.5c_1 z^{-l_1} - 0.5c_2 z^{-l_2}.$$

Therefore, $G_0^z(z)$ has only two nonzero coefficients. Furthermore, we see that l_1 must be even and l_2 must be odd.

12.17

(a) The QMF filters are expressed in terms of the polyphase components of $G_0^z(z)$ as follows:

$$G_0^z(z) = P_{0,0}^z(z^2) + z^{-1}P_{0,1}^z(z^2),$$
$$G_1^z(z) = P_{0,0}^z(z^2) - z^{-1}P_{0,1}^z(z^2),$$
$$H_0^z(z) = 2P_{0,0}^z(z^2) + 2z^{-1}P_{0,1}^z(z^2),$$
$$H_1^z(z) = -2P_{0,0}^z(z^2) + 2z^{-1}P_{0,1}^z(z^2).$$

(b) Figure 12.1 shows the analysis and synthesis QMF banks constructed from their polyphase components.

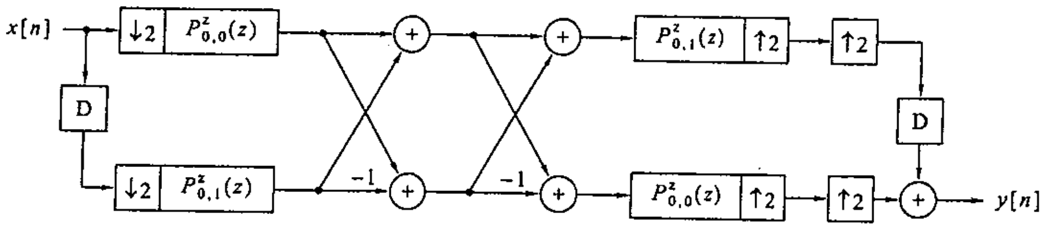

Figure 12.1 Pertaining to Solution 12.17.

12.18 Figure 12.2 shows the solutions to both parts of the problem.

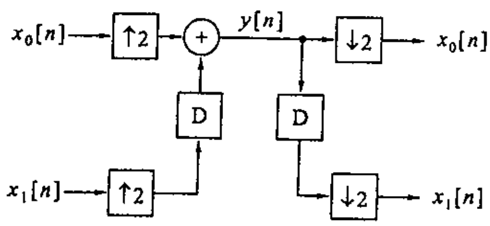

Figure 12.2 Pertaining to Solution 12.18.

12.19 Figure 12.3 shows the solutions to both parts of the problem.

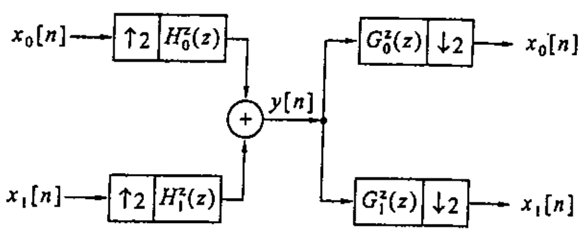

Figure 12.3 Pertaining to Solution 12.19.

12.20

(a) The transmultiplexer first splits the input signal $u[n]$ into two subsequences: the even-indexed $x_0[n]$ and the odd-indexed $x_1[n]$, as explained in Problem 12.18, part b. It then performs frequency division multiplexing of $x_0[n]$ and $x_1[n]$, as explained in Problem 12.19, part a. The right half of the transmultiplexer reverses these operations. The frequency division multiplexed signal $v[n]$ is demultiplexed into its components $y_0[n]$ and $y_1[n]$, as explained in Problem 12.19, part b. Finally, $w[n]$ is constructed by time division multiplexing of $y_0[n]$ and $y_1[n]$, as explained in Problem 12.18, part a.

(b) We have,

$$V^z(z) = H_0^z(z)X_0^z(z^2) + H_1^z(z)X_1^z(z^2),$$

$$\begin{aligned} Y_0^z(z^2) &= 0.5G_0^z(z)V^z(z) + 0.5G_0^z(-z)V^z(-z) \\ &= 0.5[G_0^z(z)H_0^z(z) + G_0^z(-z)H_0^z(-z)]X_0^z(z^2) + 0.5[G_0^z(z)H_1^z(z) + G_0^z(-z)H_1^z(-z)]X_1^z(z^2), \\ Y_1^z(z^2) &= 0.5G_1^z(z)V^z(z) + 0.5G_1^z(-z)V^z(-z) \\ &= 0.5[G_1^z(z)H_0^z(z) + G_1^z(-z)H_0^z(-z)]X_0^z(z^2) + 0.5[G_1^z(z)H_1^z(z) + G_1^z(-z)H_1^z(-z)]X_1^z(z^2). \end{aligned}$$

(c) We get

$$\begin{aligned} Y_0^z(z^2) &= z^{-1}[G_0^z(z)G_1^z(-z) - G_0^z(-z)G_1^z(z)]X_0^z(z^2) + z^{-1}[-G_0^z(z)G_0^z(-z) + G_0^z(-z)G_0^z(z)]X_1^z(z^2) \\ &= z^{-1}[G_0^z(z)G_1^z(-z) - G_0^z(-z)G_1^z(z)]X_0^z(z^2), \\ Y_1^z(z^2) &= z^{-1}[G_1^z(z)G_1^z(-z) - G_1^z(-z)G_1^z(z)]X_0^z(z^2) + z^{-1}[-G_1^z(z)G_0^z(-z) + G_1^z(-z)G_0^z(z)]X_1^z(z^2) \\ &= z^{-1}[-G_1^z(z)G_0^z(-z) + G_1^z(-z)G_0^z(z)]X_1^z(z^2). \end{aligned}$$

(d) The transfer function from $X_0^z(z^2)$ to $Y_0^z(z^2)$, as well as from $X_1^z(z^2)$ to $Y_1^z(z^2)$, is precisely $F^z(z)$ defined in (12.61). As we saw in Section 12.7.3, taking $G_0^z(z)$ and $G_1^z(z)$ to be conjugate quadrature filters makes $F^z(z)$ a pure delay, therefore it leads to perfect reconstruction.

12.21 We have

$$G_1^z(z) = (-z)^{-N}G_0^z(-z^{-1}) = (-z)^{-N}\sum_{n=0}^{N} g_0[n](-z)^n = \sum_{n=0}^{N} g_0[n](-z)^{-(N-n)} = \sum_{n=0}^{N}(-1)^n g_0[N-n]z^{-n}.$$

Therefore,

$$g_1[n] = (-1)^n g_0[N-n].$$

12.22 We get

$$\begin{aligned} \tilde{\delta}_s &= \frac{0.5(1+\mu)\delta_s}{0.5+\mu\delta_s}, \\ \tilde{\delta}_p &= \frac{0.5(\delta_p+\mu\delta_s)}{0.5+\mu\delta_s}. \end{aligned}$$

Therefore,

$$\begin{aligned} \delta_s &= \frac{0.5\tilde{\delta}_s}{0.5(1+\mu)-\mu\tilde{\delta}_s}, \\ \delta_p &= (1+2\mu\delta_s)\tilde{\delta}_p - \mu\delta_s. \end{aligned}$$

12.23 The lth level of the tree has relative input rate $2^{-(l-1)}$, and relative output rate 2^{-l}. Therefore, each filter at this level performs $2^{-l}N$ operations per input data point. There are 2^l filters at the lth level, so the total number of operations at each level is N per input data point. Therefore, the total number of operations in L levels is LN per input data point.

12.24 Each of filters in the highest-level two-channel bank requires $N/2$ operations per input point, so the highest level requires N operations per input point in total. The signal rate at the input of the next level is half that of the input, so this level requires $N/2$ operations per input point in total. Continuing this way, we get that the total number of operations in the bank is

$$\sum_{l=0}^{L-1} 2^{-l} N \approx 2N.$$

12.25 Program 12.1 implements an octave-band analysis filter bank.

Program 12.1 An octave-band analysis filter bank.

```
function u = octbafb(g0,g1,L,x);
% Synopsis: u = octbfb(g0,g1,L,x).
% An octave-band analysis filter bank.
% Input parameters:
% g0, g1: the two analysis filters
% L: the number of octaves
% x: the input signal.
% Output:
% u: a matrix of L+1 rows, each row of which is an output signal.

u = zeros(L+1,ceil(0.5*(length(x)+length(g1))));

for i = 1:L,
   temp = ppdec(x,g1,2);
   u(i,1:length(temp)) = temp;
   x = ppdec(x,g0,2);
end
u(L+1,1:length(x)) = x;
```

Chapter 13

Analysis and Modeling of Random Signals

13.1 The Welch periodogram for K overlapping points is

$$\hat{K}_x^{\mathrm{f}}(\theta) = \frac{1}{L}\sum_{l=0}^{L-1}\left\{\frac{1}{N}\left|\sum_{n=0}^{N-1} w[n]x[n+l(N-K)]e^{-j\theta n}\right|^2\right\}.$$

13.2 The signal $p(t)$ satisfies

$$p(t) = \pm 1 \pm j \quad \text{for all } t.$$

Therefore,

$$p^2(t) = \pm j2, \quad \text{and} \quad p^4(t) = -1.$$

Therefore, if $y(t) = p(t)e^{j2\pi f_0 t}$, then

$$z(t) = y^4(t) = -e^{j8\pi f_0 t}.$$

The Fourier transform of $z(t)$ has a delta function at $4f_0$. Therefore, if we sample $z(t)$ at a rate higher than eight times the maximum value of $|f_0|$, we will be able to determine f_0 unambiguously.

Repeating the computation in Solution 6.17, we get that the loss in SNR to due raising $y(t)$ to the fourth power is by a factor of 16, or 12 dB.

13.3

(a) We have from (13.11), using the symmetry of the sequence $\hat{\kappa}_x[m]$,

$$\begin{aligned}
S^{\mathrm{d}}[k] &= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{m=-(N-1)}^{-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right)\\
&= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{m=1)}^{N-1}\hat{\kappa}_x[m]\exp\left(\frac{j2\pi m}{N}\right)\\
&= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{m=1)}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi (N-m)}{N}\right)\\
&= \sum_{m=0}^{N-1}\hat{\kappa}_x[m]\exp\left(-\frac{j2\pi m}{N}\right) + \sum_{n=1)}^{N-1}\hat{\kappa}_x[N-n]\exp\left(-\frac{j2\pi n}{N}\right).
\end{aligned}$$

Therefore,

$$s[n] = \begin{cases}\hat{\kappa}_x[0], & n = 0,\\ \hat{\kappa}_x[n] + \hat{\kappa}_x[N-n], & 1 \le n \le N-1.\end{cases}$$

(b) We get by the same derivation as before,

$$\frac{M}{N}s_{\mathrm{a}}[n] = \begin{cases}\hat{\kappa}_{x_{\mathrm{a}}}[0], & n = 0,\\ \hat{\kappa}_{x_{\mathrm{a}}}[n] + \hat{\kappa}_{x_{\mathrm{a}}}[M-n], & 1 \le n \le M-1.\end{cases}$$

However, it is easy to verify that

$$\hat{\kappa}_{x_a}[n] = \begin{cases} \frac{M}{N}\hat{\kappa}_x[n], & 0 \le n \le N-1, \\ 0, & n \ge N. \end{cases}$$

Therefore, if $M \ge 2N-1$, then $\hat{\kappa}_{x_a}[M-n] = 0$ for all $0 \le n \le N-1$. This means that

$$s_a[n] = \hat{\kappa}_x[n], \quad 0 \le n \le N-1.$$

(c) Let M be a number not smaller than $2N-1$, preferably such that M is an integer power of 2. The following MATLAB code accomplishes the required operation:

```
s = real(ifft((1/N)*(abs(fft([x,zeros(1,M-N)])).^2));
kappa = s(1:p+1);
```

13.4 The system of equations is

$$\begin{bmatrix} \kappa_x[0] & \kappa_x[1] \\ \kappa_x[1] & \kappa_x[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} \kappa_x[1] \\ \kappa_x[2] \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} \kappa_x[0] & \kappa_x[1] \\ \kappa_x[1] & \kappa_x[0] \end{bmatrix}^{-1} \begin{bmatrix} \kappa_x[1] \\ \kappa_x[2] \end{bmatrix} = -\frac{1}{\kappa_x^2[0] - \kappa_x^2[1]} \begin{bmatrix} \kappa_x[0] & -\kappa_x[1] \\ -\kappa_x[1] & \kappa_x[0] \end{bmatrix} \begin{bmatrix} \kappa_x[1] \\ \kappa_x[2] \end{bmatrix}$$

$$= \frac{1}{\kappa_x^2[0] - \kappa_x^2[1]} \begin{bmatrix} (\kappa_x[2] - \kappa_x[0])\kappa_x[1] \\ \kappa_x^2[1] - \kappa_x[0]\kappa_x[2] \end{bmatrix}.$$

13.5 The equations are

$$\kappa_x[0]a_1 = -\kappa_x[1], \quad \gamma_v = \kappa_x[0] + a_1\kappa_x[1].$$

These can be written as

$$\begin{bmatrix} 1 & a_1 \\ a_1 & 1 \end{bmatrix} \begin{bmatrix} \kappa_x[0] \\ \kappa_x[1] \end{bmatrix} = \begin{bmatrix} \gamma_v \\ 0 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \kappa_x[0] \\ \kappa_x[1] \end{bmatrix} = \begin{bmatrix} 1 & a_1 \\ a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_v \\ 0 \end{bmatrix} = \frac{1}{1-a_1^2} \begin{bmatrix} 1 & -a_1 \\ -a_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_v \\ 0 \end{bmatrix} = \frac{\gamma_v}{1-a_1^2} \begin{bmatrix} 1 \\ -a_1 \end{bmatrix}.$$

13.6 The equations are

$$\begin{bmatrix} 1 & a_1 & a_2 \\ a_1 & 1+a_2 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} \kappa_x[0] \\ \kappa_x[1] \\ \kappa_x[2] \end{bmatrix} = \begin{bmatrix} \gamma_v \\ 0 \\ 0 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} \kappa_x[0] \\ \kappa_x[1] \\ \kappa_x[2] \end{bmatrix} = \frac{\gamma_v}{(1+a_1+a_2)(1-a_1+a_2)(1-a_2)} \begin{bmatrix} 1+a_2 \\ -a_1 \\ a_1^2 - a_2 - a_2^2 \end{bmatrix}.$$

13.7 The equations are

$$\left\{ \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ a_p & a_{p-1} & \dots & 1 \end{bmatrix} + \begin{bmatrix} 1 & a_1 & \dots & a_p \\ a_1 & a_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ a_p & 0 & \dots & 0 \end{bmatrix} \right\} \begin{bmatrix} 0.5\kappa_x[0] \\ \kappa_x[1] \\ \vdots \\ \kappa_x[p] \end{bmatrix} = \begin{bmatrix} \gamma_v \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

13.8 Program 13.1 implements the computation.

13.9 We have from (13.36),

$$v_i[n] = x[n] + a_{i,1}x[n-1] + \cdots + a_{i,i}x[n-i].$$

Program 13.1 Computation of the covariance sequence of an AR model.

```
function kappa = atokappa(a,gammav);
% Synopsis: kappa = atokappa(a,gammav).
% Computes the covariance sequence corresponding to a given polynomial
% a(z) and a given gamma_v.
% Input parameters:
% a:      the coefficients of the polynomial a(z)
% gammav: the innovation variance.
% Output:
% kappa:  the covariance sequence.

p = length(a)-1;
a = reshape(a,p+1,1);
A1 = toeplitz(a,[1,zeros(1,p)]);
A2 = hankel(a);
kappa = ((A1+A2)\[gammav; zeros(p,1)])';
kappa(1) = 2*kappa(1);
```

Multiplying by $x[n-i-1]$ and taking expected values gives

$$E(v_i[n]x[n-i-1]) = \kappa_x[i+1] + a_{i,1}\kappa_x[i] + \cdots + a_{i,i}\kappa_x[1].$$

The right side is equal to the numerator of (13.43).

13.10 We have from (13.46),

$$E(\tilde{v}_i[n])^2 = \sum_{l=0}^{i}\sum_{m=0}^{i} a_{i,i-l}a_{i,i-m}E(x[n-l]x[n-m]) = \sum_{l=0}^{i}\sum_{m=0}^{i} a_{i,i-l}a_{i,i-m}\kappa_x[l-m]$$
$$= \sum_{l=0}^{i}\sum_{m=0}^{i} a_{i,l}a_{i,m}\kappa_x[(i-l)-(i-m)] = \sum_{l=0}^{i}\sum_{m=0}^{i} a_{i,l}a_{i,m}\kappa_x[m-l] = E(v_i[n])^2.$$

13.11 We have from (13.46),

$$E(v_i[n]\tilde{v}_i[n-1]) = \sum_{l=0}^{i}\sum_{m=0}^{i} a_{i,l}a_{i,i-m}E(x[n-l]x[n-1-m]) = \sum_{l=0}^{i}\sum_{m=0}^{i} a_{i,l}a_{i,i-m}\kappa_x[m+1-l].$$

By the ith-order Yule-Walker equations, we have

$$\sum_{m=0}^{i} a_{i,i-m}\kappa_x[m+1-l] = 0, \quad \text{for} \quad 0 \le m \le i-1.$$

Therefore, only the term corresponding to $m = i$ is nonzero, and we get

$$E(v_i[n]\tilde{v}_i[n-1]) = \sum_{l=0}^{i} a_{i,l}\kappa_x[i+1-l]$$

This is again equal to the numerator of (13.43). Also, we know that the denominator of (13.43) is equal to $E(v_i[n])^2$ and we saw in Solution 13.10 that this is equal to $E(\tilde{v}_i[n])^2$. Therefore, the denominator of (13.43) is also equal to $[E(v_i[n])^2 E(\tilde{v}_i[n-1])^2]^{1/2}$. In summary,

$$\rho_{i+1} = \frac{E(v_i[n]\tilde{v}_i[n-1])}{[E(v_i[n])^2 E(\tilde{v}_i[n-1])^2]^{1/2}}.$$

13.12

(a) We have,

$$x[n] = \sin(\theta_0 n + \phi_0) = \sin(\theta_0 n)\cos\phi_0 + \cos(\theta_0 n)\sin\phi_0.$$

Therefore,

$$E(x[n]) = \sin(\theta_0 n)E(\cos\phi_0) + \cos(\theta_0 n)E(\sin\phi_0) = 0.$$

(b) We have,

$$x[n]x[n-m] = \sin(\theta_0 n + \phi_0)\sin(\theta_0 n - \theta_0 m + \phi_0) = 0.5\cos(\theta_0 m) + 0.5\cos(2\theta_0 n - \theta_0 m + 2\phi_0).$$

Therefore,

$$\kappa_x[m] = E(x[n]x[n-m]) = 0.5\cos(\theta_0 m) + 0.5E[\cos(2\theta_0 n - \theta_0 m + 2\phi_0)] = 0.5\cos(\theta_0 m) + 0 = 0.5\cos(\theta_0 m).$$

(c) We have,

$$x[n] + x[n-2] = \sin(\theta_0 n + \phi_0) + \sin(\theta_0 n - 2\theta_0 + \phi_0) = 2\sin(\theta_0 n - \theta_0 + \phi_0)\cos\theta_0 = 2\cos\theta_0 x[n-1].$$

Therefore,

$$x[n] - 2\cos\theta_0 x[n-1] + x[n-2] = 0.$$

(d) Suppose we predict the value of $x[n]$ from $x[n-1]$ and $x[n-2]$ using the predictor

$$\hat{x}[n] = -a_{2,1}x[n-1] - a_{2,2}x[n-2], \quad \text{where} \quad a_{2,1} = -2\cos\theta_0,\ a_{2,2} = 1.$$

Then we get from part c that $\hat{x}[n] = x[n]$, implying that $v_2[n] = 0$. Since $x[n]$ can be predicted exactly from its two past values, it is deterministic in the sense defined in Section 13.4.3.

(e) The Levinson-Durbin algorithm gives

$$s_0 = 0.5, \quad \rho_1 = \cos\theta_0, \quad s_1 = 0.5(1 - \cos^2\theta_0) = 0.5\sin^2\theta_0, \quad a_{1,1} = -\cos\theta_0,$$

$$\rho_2 = \frac{\cos(2\theta_0) - \cos^2\theta_0}{\sin^2\theta_0} = -1, \quad s_2 = 0, \quad , a_{2,1} = -2\cos\theta_0, \quad a_{2,2} = 1.$$

Since $s_2 = 0$, necessarily $v_2[n] = 0$, in agreement with the result of part d.

13.13 An all-pass filter must have a transfer function

$$H^z(z) = \frac{a_p + a_{p-1}z^{-1} + \cdots + z^{-p}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}.$$

This transfer function can be realized by building an IIR lattice filter for the denominator polynomial (which is necessarily stable); the transfer function from the upper-right input to the lower-right output is the required $H^z(z)$.

13.14

(a) The difference between the two frequencies is $2\pi\cot 0.01$, which is only slightly above the resolution of an unwindowed DFT (this resolution being $2\pi/128$) and below the resolution of any windowed DFT. Therefore, DFT-based techniques are not useful for measuring the frequencies in this problem.

(b) The following MATLAB code implements the computation. The variable `sig` has to be set to the desired noise level.

```
x  = sin(2*pi*0.17*(0:127)) + 0.5*sin(2*pi*0.18*(0:127));
v = sig*randn(1,128);
y = x + v;
kappa = kappahat(y,50);
[a,gv] = yw(kappa);
H = frqresp(gv,a,1001);
theta = (1/1000)*(0:1000);
plot(theta,20*log10(abs(H))),grid,figure(1)
```

The AR spectrum does show the two peaks, therefore it enables measurement of the two frequencies.

(c) It is possible to measure the two frequencies at noise levels 0.1 and 0.2, but not with 0.4. At this noise level, the weaker component is masked by the noise.

(d) A spectrum based on an autoregressive model for sinusoids in noise has better resolution than a windowed DFT. It therefore enables frequency measurement when a windowed DFT fails. However, large AR orders may be required and success is not guaranteed if the noise level is high.

13.15 Program 13.2 implements the Schur algorithm.

Program 13.2 The Schur algorithm.

```
function [rho,s] = schuralg(kappa);
% Synopsis: [rho,s] = schuralg(kappa).
% The Schur algorithm.
% Input:
% kappa: the covariance sequence values from 0 to p.
% Output parameters:
% rho: the set of p reflection coefficients
% s: the innovation variance.

p = length(kappa)-1;
c = kappa; ctilde = kappa;
for i = 0:p-1,
   rhoi = c(i+2)/ctilde(i+1); rho = [rho,rhoi];
   temp1 = c(i+3:p+1) - rhoi*ctilde(i+2:p);
   temp2 = ctilde(i+1:p) - rhoi*c(i+2:p+1);
   c(i+3:p+1) = temp1; ctilde(i+2:p+1) = temp2;
end
s = ctilde(p+1);
```

13.16 We have

$$\frac{\partial E(e[n])^2}{\partial b_l} = 2E\left(\frac{\partial e[n]}{\partial b_l}e[n]\right) = -2E\left\{x[n-l]\left(y[n] - \sum_{k=0}^{q} b_k x[n-k]\right)\right\}$$

$$= -2E(x[n-l]y[n]) + 2\sum_{k=0}^{q} b_k E(x[n-l]x[n-k]) = -2\kappa_{yx}[l] + 2\sum_{k=0}^{q} b_k \kappa_x[l-k] = 0, \quad 0 \le l \le q.$$

The resulting set of equations is identical to (13.80).

13.17 Let

$$e[n] = y[n] - \sum_{k=0}^{q} b_k x[n-k], \quad q \le n \le N-1.$$

Define the average square error

$$V = \frac{1}{N-q}\sum_{n=q}^{N-1} e^2[n].$$

The least-squares model of order q is the set of parameters $\{b_k,\ 0 \le k \le q\}$ for which V is minimum.

Differentiate with respect to the unknown parameters and equate the partial derivatives to zero. We get

$$\frac{\partial V}{\partial b_l} = \frac{2}{N-q}\sum_{n=q}^{N-1}\frac{\partial e[n]}{\partial b_l}e[n] = -\frac{2}{N-q}\sum_{n=q}^{N-1} x[n-l]\left(y[n] - \sum_{k=0}^{q} b_k x[n-k]\right)$$

$$= -\frac{2}{N-q}\sum_{n=q}^{N-1} y[n]x[n-l] + \frac{2}{N-q}\sum_{k=0}^{q} b_k\left(\sum_{n=q}^{N-1} x[n-l]x[n-k]\right) = 0, \quad 0 \le l \le q.$$

Define

$$\eta_l = \frac{1}{N-q}\sum_{n=q}^{N-1} y[n]x[n-l], \quad \xi_{l,k} = \frac{1}{N-q}\sum_{n=q}^{N-1} x[n-l]x[n-k].$$

Let Ξ_q be the $(q+1) \times (q+1)$ matrix constructed from $\{\xi_{l,k},\ 0 \le l, k \le q\}$ and let η_q be the $(q+1)$-dimensional vector constructed from $\{\eta_l,\ 0 \le l \le q\}$. Then we can rewrite the foregoing equation as

$$\Xi_p \boldsymbol{b}_q = \boldsymbol{\eta}_q \implies \boldsymbol{b}_q = \Xi_q^{-1} \boldsymbol{\eta}_q.$$

The vector of parameters $\boldsymbol{b}_q$ obtained by solving this equation is the qth-order, least-squares estimate of the model parameters.