Frame-based Completeness of Intermediate Logics

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Abstract. In this paper, the duality between descriptive frames and Heyting algebras is proved in detail. This, together with the standard result on the completeness of intermediate logics with respect to Heyting algebras, is used to obtain the completeness of intermediate logics with respect to descriptive frames.

1 Introduction

It is well known that propositional logics are complete with respect to the appropriate algebraic semantics. For example, classical propositional logic (**CPC**) is complete with respect to the class of Boolean algebras and intuitionistic propositional logic (**IPC**) is complete with respect to the class of Heyting algebras. These proofs can be modified to obtain similar results about certain classes of extensions. For example, every modal logic (which can be considered an extension of **CPC**) is complete with respect to a certain class of Boolean algebras with operators. And every intermediate logic (which is an extension of **IPC**) is complete with respect to a certain class of Heyting algebras.

Some would prefer, however, completeness with respect to a frame-based semantics. This can be done easily in the cases of **CPC** (which is complete with respect to the single reflexive point) and **IPC** (which is complete with respect to Kripke frames). Similar results can be obtained for particular extensions. (For example, the modal logic **S4** is complete with respect to reflexive, transitive Kripke frames.) But this cannot be done directly for all extensions, so a new method is needed. The new method relies on a broader notion of frame which provides a link to the algebraic semantics. In the case of modal logics, this notion is that of the general frame, and the link with algebraic semantics comes in the form of the Stone Representation Theorem. The final result is that every modal logic is complete with respect to a certain class of general frames.

The goal of this paper is to prove the result for intermediate logics analogous to the one just mentioned for modal logics. That is, we will prove the completeness of intermediate logics with respect to frame-based semantics. We will do this by introducing the notion of a descriptive frame, relating these to Heyting algebras, and then transferring the completeness from the algebraic side to the frame-based side.

2 Preliminaries

In this section, we introduce the preliminary notions needed for the proof, along with some results to be assumed.

2.1 Logics, Frames, and Algebras

We here recall some familiar definitions, mainly to set the notation for the rest of the paper.

Definition 1. The *intuitionistic propositional calculus* **IPC** is the smallest set of formulas (of a propositional language $\mathcal{L}$ containing $\vee, \wedge, \rightarrow, \bot$, and infinitely many propositional letters Prop) containing

1. $p \rightarrow (q \rightarrow p)$,
2. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,
3. $p \wedge q \rightarrow p$,
4. $p \wedge q \rightarrow q$,
5. $p \rightarrow p \vee q$,
6. $q \rightarrow p \vee q$,
7. $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r)))$,
8. $\bot \rightarrow p$,

and closed under modus ponens and substitution.

The logics we will be concerned with in this paper are extensions of intuitionistic logic:

Definition 2. An *intermediate logic* is any consistent (i.e., not containg $\bot$) logic (i.e., set of formulas closed under modus ponens and substitution) of $\mathcal{L}$ containing **IPC**.

The intuitive semantics for modal and intuitionistic logics is based on Kripke frames.

Definition 3. An intuitionistic *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ where R is a partial order on $W \neq \emptyset$. An intuitionistic *Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F} = (W, R)$ is a Kripke frame and V, an *intuitionistic valuation*, is a map from Prop to

$$Up(\mathfrak{F}) = \{X \in \mathcal{P}(W) : w \in X \wedge wRv \rightarrow v \in X\},$$

the *upsets* of $\mathfrak{F}$.

The notions of truth and validity are standard, except for the implication clause of the truth defintion.

Definition 4. We define by recursion φ *is true in* $\mathfrak{M}$ *at* w (notation $\mathfrak{M}, w \models \varphi$):

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$,

2. $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$,
3. $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$,
4. $\mathfrak{M}, w \models \varphi \to \psi$ iff for all v such that wRv, if $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, v \models \psi$,
5. $\mathfrak{M}, w \not\models \bot$.

We say that φ is valid on a frame $\mathfrak{F}$, and write $\mathfrak{F} \models \varphi$, if $(\mathfrak{F}, V), w \models \varphi$ for every valuation V and world w.

With the frame-based semantics in hand, we now recall the algebraic semantics.

Definition 5. A structure $\mathfrak{A} = (A, \vee, \wedge, \to, \bot, \top)$ is a *Heyting algebra* iff $A \neq \emptyset$, $\vee$, $\wedge$, and $\to$ are binary operations on A, and $\bot, \top \in A$ such that for every $a, b, c \in A$:

(i) $\mathfrak{A}$ is a *bounded lattice*:

1. $a \vee a = a$,	$a \wedge a = a$,
2. $a \vee b = b \vee a$,	$a \wedge b = a \wedge b$,
3. $a \vee (b \vee c) = (a \vee b) \vee c$,	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
4. $a \vee \bot = a$,	$a \wedge \top = a$,
5. $a \vee (b \wedge a) = a$,	$a \wedge (b \vee a) = a$,

(ii) $\mathfrak{A}$ is *distributive*:
1. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
2. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(iii) and $\to$ is *Heyting implication*:
1. $a \to a = \top$,
2. $a \wedge (a \to b) = a \wedge b$,
3. $b \wedge (a \to b) = b$,
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

A useful semantic characterization of Heyting implication is

$$c \leq a \to b \text{ iff } a \wedge c \leq b$$

where $a \leq b$ iff $a \wedge b = a$ (see [3, Theorem 7.10]). To define truth in a Heyting algebra $\mathfrak{A}$, we define a valuation $v : \textsc{Prop} \to A$ and extend it to all formulas of $\mathcal{L}$ by the obvious recursion. Then

Definition 6. φ is *valid* in $\mathfrak{A}$ iff $v(\varphi) = \top$ for every valuation v.

One last notion that will be needed is that of a filter.

Definition 7. Let $\mathfrak{A}$ be a Heyting algebra. A nonempty, proper subset $F \subset A$ is a *filter* of $\mathfrak{A}$ if

1. $a, b \in F$ implies $a \wedge b \in F$,
2. $a \in F$ and $a \leq b$ implies $b \in F$,

and a *prime* filter if in addition:

3. $a \vee b \in F$ implies $a \in F$ or $b \in F$.

A sometimes useful equivalent (see [3, Theorem 7.23]) definition of filter replaces conditions 1 and 2 with $\top \in F$ and

$$a \in F \text{ and } a \to b \in F \text{ implies } b \in F.$$

We state here a result, sometimes referred to as the Prime Filter Theorem, about filters that will be needed in §3. It is a minor generalization of [3, Theorem 7.41].

Proposition 8. *Let F be a filter of $\mathfrak{A}$ and $X \subset A$ such that $F \cap X = \emptyset$. Then there is a prime filter F' of $\mathfrak{A}$ such that $F \subseteq F'$ and $F' \cap X = \emptyset$.*

2.2 Algebraic Completeness

We can associate to each intermediate logic L the class $\mathbf{V}_L$ of those Heyting algebras in which all theorems of L are valid. $\mathbf{V}_L$ will be a variety by Birkhoff's Theorem, which states that a class of algebras is equationally defined iff it is a variety. Then, by a Lindenbaum-Tarski type construction, the following can be proved (as in [3, Theorem 7.73(iv)]).

Theorem 9. *Every intermediate logic L is sound and complete with respect to $\mathbf{V}_L$.*

This gives us the completeness with respect to algebraic semantics that we will try to transfer to the frame-based semantics. Before we can do that, though, we must define the frame-based semantics.

2.3 Descriptive Frames

We define here the notion that will give us an adequate frame-based semantics for completeness. It is a generalization of the Kripke frame:

Definition 10. An intuitionistic *general frame* is a triple $\mathfrak{F} = (W, R, \mathcal{P})$ where (W, R) is a Kripke frame, $\mathcal{P} \subseteq Up(\mathfrak{F})$ containing $\emptyset$ and W, and $\mathcal{P}$ is closed under $\cup$, $\cap$, and $\to$ defined by

$$U_1 \to U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W \backslash R^{-1}(U_1 \backslash U_2),$$

where $R^{-1}(U) = \bigcup_{w \in U}\{v \in W : vRw\}$.

Definition 11. An intuitionistic *descriptive frame* is a general frame that is refined and compact, where:

1. $\mathfrak{F}$ is *refined* if for every $w, v \in W$, $\neg(wRv)$ implies that there is $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$, and
2. $\mathfrak{F}$ is *compact* if for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \backslash U : U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

Definition 12. An intuitionistic *descriptive model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F} = (W, R, \mathcal{P})$ is a descriptive frame and $V : \textsc{Prop} \to \mathcal{P}$.

Truth and validity are defined as usual.

3 Duality

In this section, we prove a duality theorem for Heyting algebras and descriptive frames. This will provide us with the link needed to infer frame-based completeness from algebraic completeness.

3.1 From Frames to Algebras

We first define an operator * from descriptive frames to Heyting algebras.

Definition 13. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. Then $\mathfrak{F}^* := (\mathcal{P}, \cup, \cap, \rightarrow, \emptyset, W)$ (where $\rightarrow$ is the operation on $\mathcal{P}$ defined in the previous section).

Lemma 14. *For every descriptive frame $\mathfrak{F}$, $\mathfrak{F}^*$ is a Heyting algebra.*

Proof. That $\mathfrak{F}^*$ is a distributive lattice follows directly from the fact that any set of sets forms a distributive lattice. So we only need to show that $\rightarrow$ satisfies the axioms for Heyting implication. Let $X, Y, Z \in \mathcal{P} \subset Up(\mathfrak{F})$.

1. $a \rightarrow a = \top$:

$$\begin{aligned} X \rightarrow X &= W \backslash R^{-1}(X \backslash X) \\ &= W \backslash R^{-1}(\emptyset) \\ &= W \backslash \emptyset \\ &= W. \end{aligned}$$

2. $a \wedge (a \rightarrow b) = a \wedge b$:

$$\begin{aligned} X \cap (X \rightarrow Y) &= X \cap (W \backslash R^{-1}(X \backslash Y)) \\ &= X \backslash R^{-1}(X \backslash Y) \\ &= X \cap Y. \end{aligned}$$

   To see that the last equality holds, notice that if $w \in X \cap Y$, then $u \in Y$ if wRu, since Y is an upset. So $\neg wRu$ for any $u \in X \backslash Y$. For the other containment, let $w \in X$ and $w \notin R^{-1}(X \backslash Y)$. Then $w \notin X \backslash Y$, since R is reflexive, showing that $w \in Y$.

3. $b \wedge (a \rightarrow b) = b$:

$$\begin{aligned} Y \cap (X \rightarrow Y) &= Y \cap (W \backslash R^{-1}(X \backslash Y)) \\ &= Y \backslash R^{-1}(X \backslash Y) \\ &= Y, \end{aligned}$$

   where the last equality holds because $Y \cap R^{-1}(X \backslash Y) = \emptyset$, since Y is an upset.

4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$:

$$\begin{aligned} W \backslash R^{-1}(X \backslash (Y \cap Z)) &= W \backslash R^{-1}((X \backslash Y) \cup (X \backslash Z)) \\ &= W \backslash (R^{-1}(X \backslash Y) \cup R^{-1}(X \backslash Z)) \\ &= W \backslash R^{-1}(X \backslash Y) \cap W \backslash R^{-1}(X \backslash Z) \\ &= (X \to Y) \wedge (X \to Z). \end{aligned}$$

The second equality holds because

$$\begin{aligned} w \in R^{-1}((X \backslash Y) \cup (X \backslash Z)) &\text{ iff } \exists u \in (X \backslash Y) \cup (X \backslash Z) : wRu \\ &\text{ iff } \exists u \in X \backslash Y : wRu \text{ or } \exists u \in X \backslash Z : wRu \\ &\text{ iff } w \in R^{-1}(X \backslash Y) \text{ or } w \in R^{-1}(X \backslash Z). \end{aligned}$$

Therefore $\mathfrak{F}^*$ is a Heyting algebra.

Note that we didn't use that $\mathfrak{A}$ was descriptive. This works for general frames as well, but the importance of using descriptive frames will become clear in the proof of Theorem 17.

3.2 From Algebras to Frames

Now we define an operator $_*$ in the other direction.

Definition 15. Let $\mathfrak{A} = (A, \vee, \wedge, \to, \bot, \top)$ be a Heyting algebra.

1. $W_{\mathfrak{A}} = \{F \subset A : F \text{ is a prime filter of } \mathfrak{A}\}$,
2. $FR_{\mathfrak{A}}F'$ iff $F \subseteq F'$,
3. $\mathcal{P}_{\mathfrak{A}} = \{\widehat{a} : a \in A\}$ where $\widehat{a} = \{F \in W_{\mathfrak{A}} : a \in F\}$.

Then $\mathfrak{A}_* := (W_{\mathfrak{A}}, R_{\mathfrak{A}}, \mathcal{P}_{\mathfrak{A}})$.

Lemma 16. *For every Heyting algebra $\mathfrak{A}$, $\mathfrak{A}_*$ is a descriptive frame.*

Proof. That $R_{\mathfrak{A}}$ is a partial order, and hence that $\mathfrak{F}_{\mathfrak{A}} := (W_{\mathfrak{A}}, R_{\mathfrak{A}})$ is a Kripke frame, follows directly from the fact that $\subseteq$ is a partial order on sets. If $F \in \widehat{a}$ and $FR_{\mathfrak{A}}F'$, then $a \in F$ and $F \subseteq F'$, and so $a \in F'$ and $F' \in \widehat{a}$. So each $\widehat{a}$ is an upset of $F_{\mathfrak{A}}$, giving us $\mathcal{P}_{\mathfrak{A}} \subseteq Up(\mathfrak{F}_{\mathfrak{A}})$. Because filters are upsets, $\bot \notin F$ (else $F = A$, contradicting that filters are proper) and $\top \in F$ (else $F = \emptyset$, contradicting that filters are nonempty) for every (prime) filter F. Thus $\widehat{\bot} = \emptyset$ and $\widehat{\top} = W_{\mathfrak{A}}$ are in $\mathcal{P}_{\mathfrak{A}}$.

We next have to check that $\mathcal{P}_{\mathcal{A}}$ is closed under $\cup$, $\cap$, and $\to$. Let $\widehat{a}, \widehat{b} \in \mathcal{P}_{\mathfrak{A}}$. Then

$$\begin{aligned} \widehat{a} \cup \widehat{b} &= \{F \in W_{\mathfrak{A}} : a \in F\} \cup \{F \in W_{\mathfrak{A}} : b \in F\} \\ &= \{F \in W_{\mathfrak{A}} : a \in F \text{ or } b \in F\} \\ &= \{F \in W_{\mathfrak{A}} : a \vee b \in F\} \\ &= \widehat{a \vee b} \\ &\in \mathcal{P}_{\mathfrak{A}}, \end{aligned}$$

where the third equality holds because F is a prime filter. Also

$$\begin{aligned}\widehat{a} \cap \widehat{b} &= \{F \in W_{\mathfrak{A}} : a \in F\} \cap \{F \in W_{\mathfrak{A}} : b \in F\} \\ &= \{F \in W_{\mathfrak{A}} : a \in F \text{ and } b \in F\} \\ &= \{F \in W_{\mathfrak{A}} : a \wedge b \in F\} \\ &= \widehat{a \wedge b} \\ &\in \mathcal{P}_{\mathfrak{A}},\end{aligned}$$

where the third equality holds because F is a filter. And in the definition of descriptive frame, we defined $\rightarrow$ precisely to make the following work:

$$\begin{aligned}\widehat{a} \rightarrow \widehat{b} &= \{F \in W_{\mathfrak{A}} : \forall F'(FR_{\mathfrak{A}}F' \wedge F' \in \widehat{a} \rightarrow F' \in \widehat{b})\} \\ &= \{F \in W_{\mathfrak{A}} : \forall F'(F \subseteq F' \wedge a \in F' \rightarrow b \in F')\} \\ &= \widehat{a \rightarrow b} \\ &\in \mathcal{P}_{\mathfrak{A}}.\end{aligned}$$

To show the right is contained in the left in the last equality, let $F \in \widehat{a \rightarrow b}$ and suppose that $F \subseteq F'$ with $a \in F'$. Then, as $a \rightarrow b \in F$, $a \rightarrow b \in F'$. And so, as F' is a filter containing a, $b \in F'$. Thus $F \in \{F \in W_{\mathfrak{A}} : \forall F'(F \subseteq F' \wedge a \in F' \rightarrow b \in F')\}$. For the reverse containment, let $F \in \{F \in W_{\mathfrak{A}} : \forall F'(F \subseteq F' \wedge a \in F' \rightarrow b \in F')\}$. We want to show that $a \rightarrow b \in F$. If $b \in F$, then $a \rightarrow b \in F$, so we suppose $b \notin F$. If there is $c \in F$ such that $c \wedge a = 0$, then $c \wedge a \leq b$, so $c \leq a \rightarrow b$ by the semantic characterization of $\rightarrow$, and so $a \rightarrow b \in F$. So assume there is no such c. Let F_a be the filter generated by F and a.[1] This exists since $c \wedge a \neq 0$ for every $c \in F$ by assumption. If $b \notin F_a$, then there is a prime filter F' extending F_a with $b \notin F'$ (by Proposition 8). Since $F \subseteq F_a \subseteq F'$ and $a \in F'$, $b \in F'$ (by our original supposition about F), a contradiction. So $b \in F_a$. That is, there is a $c \in F$ such that $c \wedge a \leq b$. Then $c \leq a \rightarrow b$, by the semantic characterization of $\rightarrow$, and so $a \rightarrow b \in F$, since filters are upsets. Therefore, $F \in \widehat{a \rightarrow b}$.

Thus $\mathfrak{A}_*$ is a general frame. It remains to show that it is descriptive. To see that $\mathfrak{A}_*$ is refined, suppose that $\neg(FR_{\mathfrak{A}}F')$, that is $F \not\subseteq F'$. Then there is an $a \in A$ such that $a \in F \wedge a \notin F'$. So there is an $\widehat{a} \in \mathcal{P}_{\mathfrak{A}}$ such that $F \in \widehat{a} \wedge F' \notin \widehat{a}$.

For compactness, let $\mathcal{X} \subseteq \mathcal{P}_{\mathfrak{A}}$, $\mathcal{Y} \subseteq \{W_{\mathfrak{A}} \backslash \widehat{b} : \widehat{b} \in \mathcal{P}_{\mathfrak{A}}\}$, and $\mathcal{X} \cup \mathcal{Y}$ have the finite intersection property. We want to show $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$. Let $F = [\{a : \widehat{a} \in \mathcal{X}\})$ be the filter in $\mathfrak{A}$ generated by those $a \in A$ such that $\widehat{a} \in \mathcal{X}$ and $I = (\{b : W_{\mathfrak{A}} \backslash \widehat{b} \in \mathcal{Y}\}]$ the ideal[2] generated by those $b \in A$ such that $W_{\mathfrak{A}} \backslash \widehat{b} \in \mathcal{Y}$. To see that F is a proper subset of A, and hence exists, suppose not. Then there

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[1] If the closure under $\wedge$ of the set $\{a \in A : \exists x \in X(x \leq a)\}$ is a proper subset of A, then it is a filter called *the filter generated by* X and denoted $[X)$.

[2] An *ideal* is a nonempty, proper subset F of a Heyting algebra $\mathfrak{A}$ such that (1) $a, b \in F$ implies $a \vee b \in F$, and (2) $a \in F$ and $a \geq b$ implies $b \in F$. The ideal generated by a set X, which is defined analogously to the filter generated by a set, but replacing $\wedge$ by $\vee$ and $\leq$ by $\geq$, is denoted by $(X]$.

are $\widehat{a}_1, \ldots, \widehat{a}_n \in \mathcal{X}$ such that $a_1 \wedge \cdots \wedge a_n = 0$. But then $\widehat{a}_1 \cap \cdots \cap \widehat{a}_n = \emptyset$, contradicting that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property. Similarly, I is a proper subset since, if not, then there are $W_{\mathfrak{A}} \backslash \widehat{b}_1, \ldots, W_{\mathfrak{A}} \backslash \widehat{b}_m \in \mathcal{Y}$ such that $b_1 \vee \cdots \vee b_m = 1$. But then $\widehat{b}_1 \cup \cdots \cup \widehat{b}_m = W_{\mathfrak{A}}$, and so $W_{\mathfrak{A}} \backslash \widehat{b}_1 \cap \cdots \cap W_{\mathfrak{A}} \backslash \widehat{b}_m = \emptyset$, again contradicting that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property.

So F is a filter and I is an ideal. We now show they are disjoint. Suppose $a \in F \cap I$. Then, since $a \in F$, there are $\widehat{a}_1, \ldots, \widehat{a}_n \in \mathcal{X}$ such that $a_1 \wedge \cdots \wedge a_n \leq a$. Also, since $a \in I$, there are $W_{\mathfrak{A}} \backslash \widehat{b}_1, \ldots, W_{\mathfrak{A}} \backslash \widehat{b}_m \in \mathcal{Y}$ such that $b_1 \vee \cdots \vee b_m \geq a$. So

$$
\begin{aligned}
\widehat{a}_1 \cap \cdots \cap \widehat{a}_n &= a_1 \widehat{\wedge \cdots \wedge} a_n \\
&\subseteq \widehat{a} \\
&\subseteq b_1 \widehat{\vee \cdots \vee} b_m \\
&= \widehat{b}_1 \cup \cdots \cup \widehat{b}_m \\
&= W_{\mathfrak{A}} \backslash (W_{\mathfrak{A}} \backslash \widehat{b}_1 \cap \cdots \cap W_{\mathfrak{A}} \backslash \widehat{b}_m).
\end{aligned}
$$

But then $\widehat{a}_1 \cap \cdots \cap \widehat{a}_n \cap W_{\mathfrak{A}} \backslash \widehat{b}_1 \cap \cdots \cap W_{\mathfrak{A}} \backslash \widehat{b}_m = \emptyset$, contradicting that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property. So $F \cap I = \emptyset$, and we can apply the Prime Filter Theorem to get a prime filter F' of $\mathfrak{A}$ such that $F \subseteq F'$ and $F' \cap I = \emptyset$.

It is this F' we will show to be in $\bigcap(\mathcal{X} \cup \mathcal{Y})$, completing the proof. Let $\widehat{a} \in \mathcal{X}$. Then $a \in F$ by the definition of F, and so $a \in F'$ since it contains F, giving us $F' \in \widehat{a}$. Now let $W_{\mathfrak{A}} \backslash \widehat{b} \in \mathcal{Y}$. Then $b \in I$ by the definition of I, and so $b \notin F'$ since it is disjoint from I, giving us $F' \notin \widehat{b}$ and thus $F' \in W_{\mathfrak{A}} \backslash \widehat{b}$. So $F' \in \bigcap(\mathcal{X} \cup \mathcal{Y})$. Thus $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$, so $\mathfrak{A}_*$ is compact and therefore a descriptive frame.

3.3 Back and Forth

Results analogous to the previous two lemmas hold for regular Kripke frames. But what we gain with the added generality of descriptive frames is that every Heyting algebra can be obtained from a descriptive frame via the * operation, and, vice versa, every descriptive frame can be obtained from a Heyting algebra via $_*$. This fact, plus some extra symmetry, is expressed by the following duality.

Theorem 17. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathfrak{F}$ a descriptive frame. Then*

1. $\mathfrak{A} \cong (\mathfrak{A}_*)^*$,
2. $\mathfrak{F} \cong (\mathfrak{F}^*)_*$.

Proof. For part 1, we define a map $f : \mathfrak{A} \to (\mathfrak{A}_*)^*$ by

$$f(a) = \widehat{a}.$$

f is bijective since the map from a to $\widehat{a}$ is a bijection between A and $\mathcal{P}_{\mathfrak{A}}$, which is the domain of $(\mathfrak{A}_*)^*$. The proof that f is a homomorphism is contained in the proof of Lemma 16. From that proof, we get the second equality in each of the following:

$$
\begin{aligned}
f(\bot) &= \widehat{\bot} \qquad & f(\top) &= \widehat{\top} \\
&= \emptyset & &= W_{\mathfrak{A}}
\end{aligned}
$$

and

$$\begin{aligned} f(a \vee b) &= \widehat{a \vee b} \\ &= \widehat{a} \cup \widehat{b} \\ &= f(a) \cup f(b) \end{aligned} \qquad \begin{aligned} f(a \wedge b) &= \widehat{a \wedge b} \\ &= \widehat{a} \cap \widehat{b} \\ &= f(a) \cap f(b) \end{aligned} \qquad \begin{aligned} f(a \to b) &= \widehat{a \to b} \\ &= \widehat{a} \to \widehat{b} \\ &= f(a) \to f(b). \end{aligned}$$

So f is an isomorphism.

For part 2, we define a map $g : \mathfrak{F} \to (\mathfrak{F}^*)_*$ by

$$g(w) = \widehat{w} = \{U \in P : w \in U\}.$$

We start by showing that $g(w) \in W_{\mathfrak{F}^*}$, that is, that $\widehat{w}$ is a prime filter of $\mathfrak{F}^*$. By definition, $\widehat{w} \subseteq \mathcal{P}$. Since $w \notin \emptyset$ and $w \in W$, we have $\emptyset \notin \widehat{w}$ and $W \in \widehat{w}$. Let $X, Y \in \widehat{w}$. Then $w \in X, Y$. So $w \in X \cap Y$, giving $X \cap Y \in \widehat{w}$. Now let $X \in \widehat{w}$ and $X \subseteq Y$. Then $w \in X \subseteq Y$, so $Y \in \widehat{w}$. Now let $X \cup Y \in \widehat{w}$. Then $w \in X \cup Y$. So $w \in X$ or $w \in Y$, and hence $X \in \widehat{w}$ or $Y \in \widehat{w}$. So $\widehat{w}$ is an element of $W_{\mathfrak{F}^*}$. To see that g is injective, let $w \neq v$. Then either $\neg(wRv)$ or $\neg(vRw)$, since Kripke frames are partial orders. Without loss of generality, assume the first. Then, since $\mathfrak{F}$ is descriptive and hence refined, there is an upset $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$. Then $U \in \widehat{w}$ and $U \notin \widehat{v}$, giving us $\widehat{w} \neq \widehat{v}$. We now show that g is a homomorphism: wRv iff every upset in $\mathcal{P}$ containing w contains v iff $\widehat{w} \subseteq \widehat{v}$ iff $\widehat{w} R_{\mathfrak{F}^*} \widehat{v}$ (where the first equivalence holds from right to left because $\mathfrak{F}$ is refined). To show that g is a homomorphism, we must also prove that for every $U \subseteq W$, $U \in \mathcal{P}$ iff $g(U) \in \mathcal{P}_{\mathfrak{F}^*}$. Note that $\mathcal{P}_{\mathfrak{F}^*} = \{\{\widehat{w} : w \in U\} : U \in \mathcal{P}\}$. Then $U \in \mathcal{P}$ implies $g(U) = \{\widehat{w} : w \in U\} \in \mathcal{P}_{\mathfrak{F}^*}$. And $g(U) \in \mathcal{P}_{\mathfrak{F}^*}$ implies $g(U) = \{\widehat{w} : w \in U'\}$ for some $U' \in \mathcal{P}$. But then $g(U) = g(U')$, and so $U = U'$ since g is injective. So $U \in \mathcal{P}$.

It remains to show that g is surjective. This is where that fact that $\mathfrak{F}$ is descriptive, and compact in particular, is essential. We must show that every element of $W_{\mathfrak{F}^*}$ is of the form $\widehat{w}$ for some $w \in W$. So let $\mathcal{X} \in W_{\mathfrak{F}^*}$. Then $\mathcal{X}$ is a prime filter in $\mathfrak{F}^*$. Then $\mathcal{Y} = \mathcal{P} \backslash \mathcal{X}$ is a prime ideal, and so $\bigcup \mathcal{Y}_0 \neq W$ for any finite subset $\mathcal{Y}_0 \subseteq \mathcal{Y}$. We will show that the set $\mathcal{X} \cup \mathcal{Y}'$ has the finite intersection property, where $\mathcal{Y}' = \{W \backslash Y : Y \in \mathcal{Y}\}$. Let $Z = X_1 \cap \cdots \cap X_n \cap Y_1 \cap \cdots \cap Y_m$ where each $X_i \in \mathcal{X}$, $Y_i \in \mathcal{Y}'$. Let X and Y be the intersections of the X_i and Y_i, respectively. Then $X \in \mathcal{X}$, since $\mathcal{X}$ is a filter, and $Y \in \mathcal{Y}'$ since $W \backslash Y \in \mathcal{Y}$ (since $\mathcal{Y}$ is an ideal). If $Z = \emptyset$, then $X \subseteq W \backslash Y \in \mathcal{Y}$. Since $\mathcal{X}$ is a filter, this gives $W \backslash Y \in \mathcal{X}$, yielding $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$, contradicting the definition of $\mathcal{Y}$. So $Z \neq \emptyset$, showing that $\mathcal{X} \cup \mathcal{Y}'$ has the finite intersection property. Since $\mathfrak{F}$ is descriptive and hence compact, there is a $w \in \bigcap(\mathcal{X} \cup \mathcal{Y}')$. Finally, we show that $\mathcal{X} = \widehat{w}$. Let $U \in P$. If $U \in \mathcal{X}$, then $w \in \bigcap(\mathcal{X} \cup \mathcal{Y}') \subseteq U$, so $U \in \widehat{w}$, giving $\mathcal{X} \subseteq \widehat{w}$. Now let $U \in \widehat{w}$. So $w \in U$. Suppose $U \in \mathcal{Y}$. Then $W \backslash U \in \mathcal{Y}'$ giving $w \notin \bigcap \mathcal{Y}'$, contradicting $w \in \bigcap(\mathcal{X} \cup \mathcal{Y}')$. So $U \in \mathcal{X}$, giving $\widehat{w} \subseteq \mathcal{X}$. Thus $\mathcal{X} = \widehat{w}$, establishing that g is surjective, and therefore an isomorphism.

4 Completeness

This section contains the final result. Having obtained a link between the algebraic and frame-based semantics, we can obtain the more intuitive version of completeness we have been looking for.

Lemma 18. *Let $\mathfrak{A}$ be a Heyting algebra. Then*

$$\mathfrak{A}, v \models \varphi \text{ iff } \mathfrak{A}_*, v_* \models \varphi,$$

where $v_*(p) = \widehat{v(p)}$.

Proof. We first prove that $v_*(\varphi) = \widehat{v(\varphi)}$ for all formulas by induction on φ:

PROP: $v_*(p) = \widehat{v(p)}$ by definition.

$\wedge$:

$$\begin{aligned} \widehat{v(\varphi \wedge \psi)} &= \widehat{v(\varphi) \wedge v(\psi)} \\ &= \widehat{v(\varphi)} \cap \widehat{v(\psi)} \\ &= v_*(\varphi) \cap v_*(\psi) \\ &= v_*(\varphi \wedge \psi) \end{aligned}$$

$\vee$:

$$\begin{aligned} \widehat{v(\varphi \vee \psi)} &= \widehat{v(\varphi) \vee v(\psi)} \\ &= \widehat{v(\varphi)} \cup \widehat{v(\psi)} \\ &= v_*(\varphi) \cup v_*(\psi) \\ &= v_*(\varphi \vee \psi) \end{aligned}$$

$\rightarrow$:

$$\begin{aligned} \widehat{v(\varphi \rightarrow \psi)} &= \widehat{v(\varphi) \rightarrow v(\psi)} \\ &= \widehat{v(\varphi)} \rightarrow \widehat{v(\psi)} \\ &= v_*(\varphi) \rightarrow v_*(\psi) \\ &= v_*(\varphi \rightarrow \psi) \end{aligned}$$

where the second equality of each induction step was proved in Lemma 16. Using this, we get that

$$\begin{aligned} \mathfrak{A}, v \models \varphi &\text{ iff } v(\varphi) = \top \\ &\text{ iff } v_*(\varphi) = \widehat{\top} = \{F \subset W_{\mathfrak{A}} : \top \in F\} = W_{\mathfrak{A}} \\ &\text{ iff } \mathfrak{A}_*, v_* \models \varphi \end{aligned}$$

since every (prime) filter is a nonempty upset and hence contains the top element $\top$.

Notice that this lemma only makes sense given the correspondence between descriptive frames and Heyting algebras proved in the previous section. Combining this lemma with the algebraic completeness theorem of §2 will give us the desired result. For a class of algebras C, we write $C_* := \{\mathfrak{A}_* : \mathfrak{A} \in C\}$.

Theorem 19. *Every intermediate logic L is sound and complete with respect to $(\mathbf{V}_L)_*$.*

Proof. Let L be an intermediate logic. Then

$$
\begin{aligned}
L \vdash \varphi \text{ iff } & \mathbf{V}_L \models \varphi && (4.1)\\
\text{iff } & \mathfrak{A} \models \varphi \text{ for every } \mathfrak{A} \in \mathbf{V}_L && (4.2)\\
\text{iff } & \mathfrak{A}, v \models \varphi \text{ for every } \mathfrak{A} \in \mathbf{V}_L \text{ and valuation } v && (4.3)\\
\text{iff } & \mathfrak{A}_*, v_* \models \varphi \text{ for every } \mathfrak{A}_* \in (\mathbf{V}_L)_* \text{ and valuation } v_* && (4.4)\\
\text{iff } & \mathfrak{A}_* \models \varphi \text{ for every } \mathfrak{A}_* \in (\mathbf{V}_L)_* && (4.5)\\
\text{iff } & (\mathbf{V}_L)_* \models \varphi. && (4.6)
\end{aligned}
$$

(1) is just the algebraic completeness theorem from §2.2, (4) follows from the previous lemma, and the rest follow from the definition of $\models$. So provability in L corresponds with validity in the class $(\mathbf{V}_L)_*$.

Thus we have shown that every intermediate logic is complete with respect to a class of descriptive frames. That is, we have found our complete frame-based semantics.

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