Dynamic Topological Logic

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explaining work by Philip Kremer and Grigori Mints

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Basic Definitions

Topological dynamics studies the asymptotic properties of continuous maps on topological spaces.

- Trimodal Language L
- Set PV of propositional variables
- Boolean connectives ∨ and ¬
- One-place modalities (interior), O (next), * (henceforth)
- L^{O} is the O (next-interior) fragment of L

Def 1.1: A *topological model* is an ordered pair M=<X, V>.

X topological space

V:PV —>*P* (X)

M(B) defined inductively

Standard validity definition

Def 1.2: A *Kripke frame* is an order pair <W, R>.

W non-empty set

R reflexive, transitive relation on W (S4, Int)

$S \subseteq W$ is open iff S is closed under R

For every Kripke frame <W, R> define corresponding topological space:
the open sets of W are those closed under R

Def 1.3: A *Kripke space* is a topological space in which the intersection of arbitrary open sets is open.

Given any topological space X, define R_X on X:
xR_Xy iff $x \in Cl(y)$

Then $\langle X, R_X \rangle$ is a Kripke frame.

Def 1.6: A *Kripke model* for S4 is a model $M = <X, V>$.

X a Kripke space

This is equivalent to the usual definition of Kripke model given duality of Kripke models and Kripke frames.

Thm 1.7: Suppose that X is a dense-in-itself metric space and A is a formula in the language L . Then the following are equivalent:

i) $A \in \mathrm{S4}$

ii) $\models A$

iii) $X \models A$

iv) $\mathbf{R} \models A$

v) $Y \models A$ for every finite topological space Y

vi) $Y \models A$ for every Kripke space Y

Proved by McKinsey, Tarski, and Kripke.

Def 1.9: A *dynamic topological system* (DTS) is an ordered pair <X, T>.

X a topological space

T a continuous function on X

A *dynamic topological model* (DTM) is an ordered triple M = <X, T, V>.

<X, T> a DTS

V:PV —> $\mathcal{P}$(X)

For each formula *B*, standard definition of M(*B*) plus:

$M(OB) = T^{-1}(B)$

$M(*B) = \bigcap_{n\geq 0} T^{-1}(B)$

Def 1.10: A *dynamic Kripke system* is an ordered pair <X, T>.

X a Kripke space

T a continuous function on X

A *dynamic Kripke model* is a DTM <X, T, V>.

X a Kripke space

Def 1.12: Let $M = \langle X, T, V \rangle$ be a DTM.

Define validity relations:

$M \models B$ iff $M(B) = X$

$\langle X, T \rangle \models B$ iff $M \models B$ $\forall$ model $M = \langle X, T, V \rangle$

$X \models B$ iff $\langle X, T \rangle \models B$ $\forall$ continuous function T

$\models B$ iff $X \models B$ $\forall$ topological space X

Def 1.13: Suppose $\mathcal{F}$ a class of functions with each $T \in \mathcal{F}$ continuous on some

Define validity relations:

$\tau, \mathcal{F} \models B$ iff $\forall\ T \in \mathcal{F}$, $\forall\ X \in \tau$,

if T continuous on X then $\langle X, T \rangle \models B$

$\mathcal{F} \models B$ iff $\forall\ T \in \mathcal{F}$, $\forall$ topological space X,

if T continuous on X then $\langle X, T \rangle \models B$

$\tau \models B$ iff $\forall\ X \in \tau\ \ X \models B$

Preliminaries to Main Proof

We are interested in dynamical topological systems <X, T> where T is in the class $\mathcal{H}$ of homeomorphisms (continuous bijections with continuous inverses). This way we can keep track of time with functions that are continuous in both directions.

In the resulting logic, $\mathrm{DTL}_{\mathcal{H}}$, the connective O commutes with everything.

The restriction to $\mathcal{H}$ can be expressed by $(\quad \mathrm{O}p \rightarrow \mathrm{O} \quad p)$.

Axiomatization of S4O (S4C + $\mathcal{H}$ axiom):

classical tautologies

S4 axioms for

$(\mathrm{O}(A \vee B) \leftrightarrow (\mathrm{O}A \vee \mathrm{O}B))$

$(\mathrm{O}\neg A \leftrightarrow \neg \mathrm{O}A)$

$(\mathrm{O} \quad A \leftrightarrow \quad \mathrm{O}A)$

rules of modus ponens and necessitation for O and .

Def 3.4: For each formula B, let g(B) be the result of pushing all occurences of O to the atomic formulas.

Def 3.5: A *near-atom* is a formula of the form $O^n p$ where $p \in$ PV.

Def 3.6: A formula is *simple* iff it is built up from near-atoms using Boolean connectives and   . Simple formulas are the range of g.

Convention 3.7: We treat the simple formulas as formulas and the near atoms as atomic formulas. This slightly changes the definitions of topological model and validity.

That is, in the definition of topological model, V assigns a subset of X to each near atom. And for the base of the inductive definition of validity, $M(O^n p) = V(O^n p)$.

Theorem 1.7 still holds.

Lemma 3.8: The following are equivalent:

i) B is a theorem of S4O

ii) $g(B)$ is a theorem of S4

iii) $g(B)$ is a theorem of S4O.

Proof: (i) $\Rightarrow$ (ii) Standard induction on the proof of B in S4O

(ii) $\Rightarrow$ (iii) Obvious

(iii) $\Rightarrow$ (i) Since $(B \leftrightarrow g(B))$ is a theorem of S4O ■

Lemma 3.9: For every formula B, $g(B)$ is a theorem of S4 iff $(0,1) \models g(B)$ where $(0,1)$ is the open unit interval

This follows from Theorem 1.7 and Lemma 3.8.

We are now ready to prove the main theorem.

Thm 3.10: $(\mathbf{R}, \mathsf{H} \models A) \Rightarrow$ (A is a theorem of S4O)

Proof: Suppose A not a theorem of S4O

$M \sim\models g(A)$ for some topological model $M = \langle(0,1), V\rangle$ (by 3.8 and 3.9)

Let M' be the DTM $\langle \mathbf{R}, T, V' \rangle$

$Tx = x + 1$, a homeomorphism

$V'(p) = \{x \in \mathbf{R}: x - m \in V(O^m p) \text{ for some } m \in \mathbf{N}\}$

Need to show $M' \sim\models A$

$\Leftarrow M' \sim\models g(A)$ (by 3.8 and soundness)

$\Leftarrow$ For every simple formula B, $M(B) = (0,1) \cap M'(B)$

We proceed by induction on the construction of B.

Base case:

B is a near-atom, say $O^n p$

$$
\begin{aligned}
& x \in (0,1) \cap M`(B) \\
& \Rightarrow x \in (0,1) \text{ and } x \in M`(O^n p) \\
& \Rightarrow x \in (0,1) \text{ and } x + n \in M`(p) \\
& \Rightarrow x \in (0,1) \text{ and } x + n \in V`(p) \\
& \Rightarrow x \in (0,1) \text{ and } x + n - m \in V(O^m p) \text{ for some m} \\
& \Rightarrow m = n, \text{ since } x \in (0,1) \text{ and} \\
& \qquad\qquad\qquad x + n - m \in V(O^m p) \subseteq (0,1) \qquad\qquad \ast \\
& \Rightarrow x \in (0,1) \text{ and } x \in V(O^n p) \\
& \Rightarrow x \in V(B) \\
& \Rightarrow x \in M(B)
\end{aligned}
$$

Reverse implications hold, omitting $\ast$.

Inductive step $B = C \vee D$:

$$\mathrm{M}(C \vee D) = \mathrm{M}(C) \vee \mathrm{M}(D)$$

$$= ((0,1) \cap \mathrm{M}`(\mathrm{C})) \cap ((0,1) \cap \mathrm{M}`(\mathrm{D}))$$

$$= (0,1) \cap \mathrm{M}`\ (C \vee D)$$

Inductive step $B = \neg C$:

$$\mathrm{M}(\neg C) = (0,1) - \mathrm{M}(C)$$

$$= (0,1) - ((0,1) \cap \mathrm{M}`(C))$$

$$= (0,1) - (\mathbf{R} \cap \mathrm{M}`(C))$$

$$= (0,1) \cap \mathrm{M}`(\neg C)$$

Inductive step $B = \quad C$:

$$\mathrm{M}(\quad C) = \mathrm{Int}(\mathrm{M}(C))$$

$$= \mathrm{Int}((0,1) \cap \mathrm{M}`(C))$$

$$= \mathrm{Int}((0,1)) \cap \mathrm{Int}(\mathrm{M}`(C))$$

$$= (0,1) \cap \mathrm{M}`(\quad C) \qquad \blacksquare$$

$DTL_0^{O*/}$ $=_{df}$ {$A \in DTL_0$: A contains no occurrences of either O or * in the scope of an occurrence of }

Axiomatic system for W0/S4:

i) classical tautologies

ii) S4 axioms for , for formulas A and B in L :

$(A \to B) \to (A \to B)$

$A \to A$

$A \to A$

iii) W0 axioms for O and *

$*(A \to B) \to (*A \to *B))$

$*A \to A$

$*A \to **A$

$(O(A \vee B) \leftrightarrow (OA \vee OB))$

$(O\neg A \leftrightarrow \neg OA)$

$(O{*}A \leftrightarrow {*}OA)$

$({*}A \rightarrow OA)$

$(A \wedge {*}(A \rightarrow OA) \rightarrow {*}A)$ induction

iv) modus ponens

v) rule of necessitation for each modality:

From A infer OA

From A infer ${*}A$

From A infer A

A is a *theorem* iff $A \in$ W0/S4.

A is *consistent* iff $\neg A \notin$ W0/S4.

A W0/S4-*theory* is a set of formulas containing the theorems of W0/S4 and closed under modus ponens.

A is an S4-*theorem* iff *A* has no occurrence of O or * and $A \in$ S4.

An S4-*theory* is a set of formulas in L containing the theorems of S4 and closed under modus ponens.

A W0/S4-theory S is *complete* iff for every formula *A*, either $A \in$ S or $\neg A \in$ S.

A W0/S4-theory S is *consistent* iff every formula in S is consistent.

A W0/S4-theory S is ω-*closed* iff for any formula *A* we have the following: if $O^n A \in S$ for every $n \in \omega$ then $*A \in S$.

Thm 4.1: W0/S4 = $DTL_0^{O*/}$