JOURNAL OF COMBINATORIAL THEORY, Series A **57**, 151–156 (1991)

Note

An Infinite Number of Plane Figures with Heesch Number Two

ANNE FONTAINE

Department of Mathematics and Statistics, State University of New York at Albany, Albany, New York 12222

Communicated by the Managing Editors

Received November 28, 1988

It has been a long standing problem to find a plane figure that does not tile, yet can be completely surrounded twice by congruent copies of itself. An infinite family of plane figures with this property is given.

Some figures will tile the plane, most will not. The Heesch number of a nontiling figure can be a measure of how close the figure comes to tiling. A plane figure will have Heesch number k if it can be completely surrounded by congruent copies of itself k times, but not $k+1$ times. There are many examples of figures with Heesch number 1; however, it has been a long standing problem [2] to find an example of a figure with Heesch number 2. See Tilings and Patterns by Branko Grünbaum and G. C. Shephard [1] for a complete description of Heesch's problem.

THEOREM. *A U-frame polyomino with sides lettered as in Fig.* 1 *will have Heesch number* 2 *if*

$$a = g + h - e,$$

$$b = g + h,$$

$$c = h,$$

$$d = 2g + 3h,$$

$$f = g + h,$$

when e, g, and h are integers and $0 < g < e < h < 2e$. (For brevity, both the side of the polyomino and its length are given the same letter.)

582a/57/1-11

0097-3165/91 $3.00

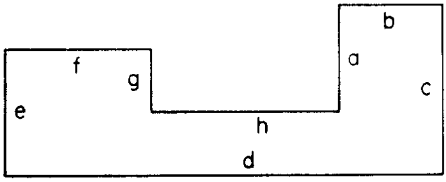

FIG. 1. A U-frame polyomino.

Proof. An ordered set of eight positive integers, a through h, can be used as the length of sides of a U-frame polyomino if and only if

$$a+e=c+g,$$

$$b+f+h=d \qquad \text{and} \qquad a<c.$$

The equations are necessary to have closure; the inequality prevents self intersection of the figure.

The patch in Fig. 2 shows a polyomino that can be surrounded twice. It can be checked that the patch can be formed by any U-frame polyomino such that

$$c=h,$$

$$f=b,$$

$$b=c+g,$$

$$b=a+e,$$

$$2c>f \qquad \text{and} \qquad a>g.$$

These linear requirements together with those for a U-frame polyomino imply

$$a=g+h-e,$$

$$b=g+h,$$

$$c=h,$$

$$d=2g+3h,$$

$$f=g+h,$$

$$0<g<e<h \qquad g, e, h \text{ integers.}$$

Therefore every polyomino with our original restrictions will make such a patch and thus has a Heesch number that is at least 2.

If we add the stronger requirement that h is less than $2e$, we can readily

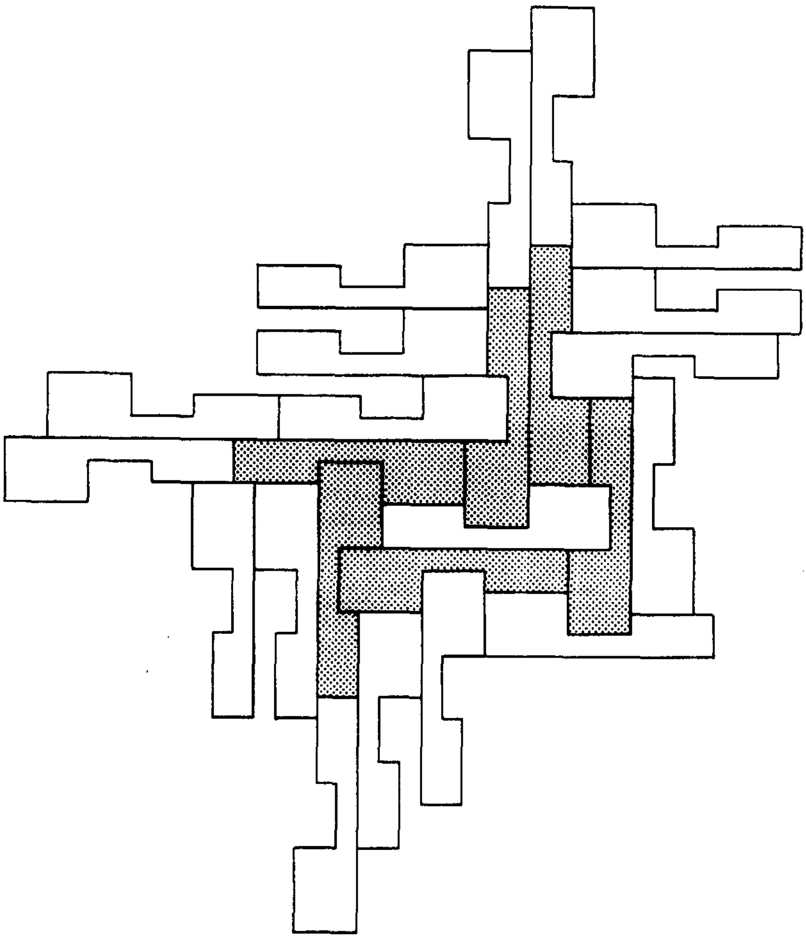

FIG. 2. A polyomino completely surrounded twice.

show that the polyomino cannot be completely surrounded three times and therefore has Heesch number at most 2. Side h can be covered only by combinations of sides b, c, e or f, placed directly or indirectly. By the requirements of the theorem, f and b are greater than h; therefore, sides f and b cannot be placed against side h. Side e is shorter than h, but there is no side that can combine with side e to cover side h. This leaves side c placed directly or indirectly as the only way to completely cover side h. Construct all possible patches of four pieces in which the sides labeled h of the first three pieces are completely covered. Such a patch must be contained in any larger patch that would demonstrate that the polyomino has Heesch number at least 3. Although each time we cover side h completely

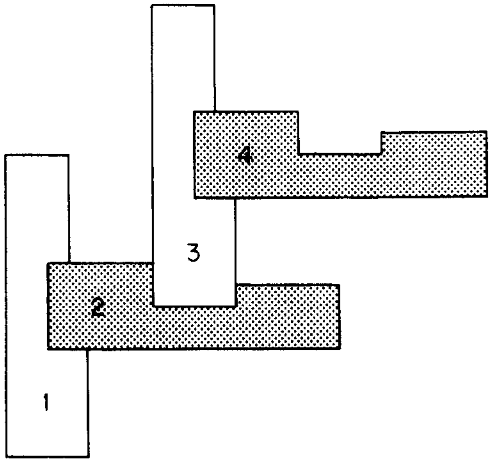

FIG. 3. A patch in which sides h of pieces 1, 2, and 3 are completely covered.

we have two choices, only the two patches in Fig. 3 and Fig. 4 have any hope of being continued so that the first piece is completely surrounded three times. (The shading in Figs. 3 and 4 indicates that the pieces have been placed indirectly.) If we try to continue the patch in Fig. 3 so that piece 1 can be surrounded three times, pieces 5 through 11 in Fig. 5 are determined; but then it is impossible to cover both piece 1 and piece 11.

The patch in Fig. 4 will be blocked in much the same way. Therefore the polyominoes in the family must have Heesch number at most 2. This completes the proof of the theorem.

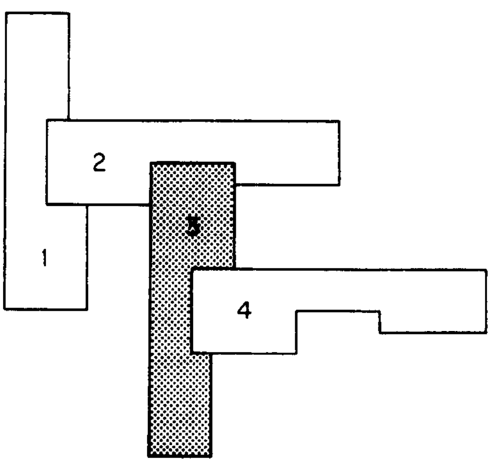

FIG. 4. A patch in which sides h of pieces 1, 2, and 3 are completely covered.

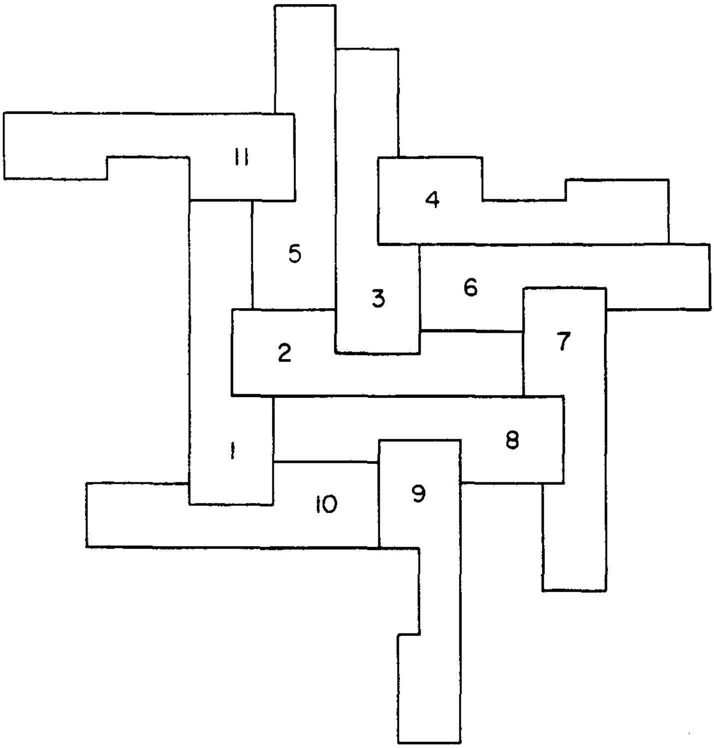

FIG. 5. Continuation of Fig. 3.

This infinite family of U-frame polyominoes with Heesch number 2 was found because they almost tile. A U-frame polyomino with

$$c = h,$$

$$f = b,$$

$$b = c + g, \qquad \text{and}$$

$$e = h \quad \text{will tile.}$$

However, if $e > h$, the polyomino has Heesch number 1. Further, if $e < h$, the polyomino has Heesch number 2. The subset in the theorem was chosen so that the proof that the polyominoes had Heesch number at most 2 would be easier.

There are other polyominoes known to the author to have Heesch number 2. For example, see the Z-frame polyomino in Fig. 6.

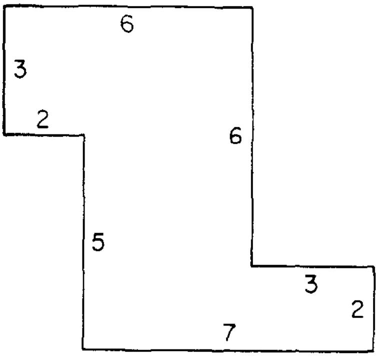

FIG. 6. A Z-frame polyomino with Heesch Number 2.

REFERENCES

1. B. GRÜNBAUM AND G. C. SHEPHARD, "Tilings and Patterns," Freeman, New York, 1986.
2. H. HEESCH, "Regülares Parkettierungsproblem," Westdeutscher Verlag, Cologne and Opladen, 1968.