

## The Spectrum for 2-Perfect 6-Cycle Systems

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Recently, the spectrum problem for 2-perfect  $m$ -cycle systems has been studied by several authors. In this paper we find the spectrum for 2-perfect 6-cycle systems with two possible exceptions. The connection between these systems and quasigroups satisfying some 2 variable identities is discussed. © 1991 Academic Press, Inc.

### 1. INTRODUCTION

In the last 30 years, there has been much interest in decomposing a complete graph  $K_v$  into edge-disjoint copies of a graph  $G$ . The most popular choices for  $G$  have been a complete graph (block designs) and a cycle. When  $G$  is a cycle of length  $m$ , such a decomposition is called an  $m$ -cycle system of  $K_v$ .

More recently, additional structure has been asked of the decomposition: a graph  $H$  that is closely related to  $G$  is defined; then the decomposition into copies of  $G$  is constructed in such a way that replacing each copy of  $G$  by a corresponding copy of  $H$  results in a decomposition of the complete graph into edge-disjoint copies of  $H$ . The problem of most interest then is to find the *spectrum* of such a decomposition; that is, the values of  $v$  for which there exists such a decomposition of  $K_v$ . For example, the following are such problems that have been solved.

The nesting of a Steiner triple system is such a problem, where  $G$  is  $K_3$  and  $H$  is the complement of  $G$  in  $K_4$  (so  $H$  is a star, joining a fourth vertex to each of the three vertices in  $G$ ). The problem of finding the spectrum of Steiner triple systems that have a nesting has been completely solved [3, 8, 16]. A generalization of this problem is the nesting of a cycle system: in this case  $G$  is a cycle of length  $m$  and  $H$  is the star consisting of  $m+1$  vertices, one vertex being joined to each of the vertices in  $G$ . Again, the

spectrum for which there exists such a decomposition of  $K_v$  has been found [7, 9], with a few exceptions for each value of  $m$ .

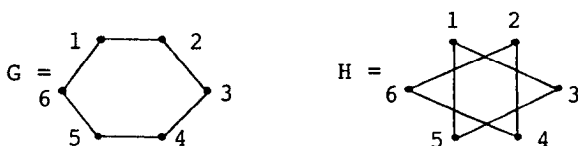
If  $G$  consists of a path containing 3 edges and  $H$  is the complement of  $G$  (so is a path of length 3) then the spectrum for this decomposition has been settled [5] (see also [14]).

A third setting for this problem has been considered, where again  $G$  is a cycle of length  $m$ , but now  $H$  is formed by joining two vertices if they are distance  $i$  apart in  $G$ , for some  $1 < i < m/2$ . So for  $m = 5$ ,  $i$  must be 2 and so  $G = (1, 2, 3, 4, 5)$  and  $H = (1, 3, 5, 2, 4)$ .



Such a decomposition of  $K_v$  when  $m = 5$  is called a Steiner pentagon system and its spectrum has been found [10]. The problem when  $m = 7$  and  $i = 2$  has also been essentially settled [11]. This problem has also been studied in the context of directed graphs [1, 6, 12]. Following the notation used in that setting, if  $G$  is an  $m$ -cycle and  $H$  is formed from  $G$  by joining two vertices if they are distance  $i$  apart in  $G$ , then such a decomposition of  $K_v$  we define to be an  $i$ -perfect  $m$ -cycle system of  $K_v$ .

In this paper the spectrum for this problem is settled in the case where  $m = 6$  (and so  $i = 2$ ) except for two values of  $v$ . Note that  $G$  is a 6-cycle and so  $H$  is the union of two 3-cycles.



Therefore, when  $m = 6$  the decomposition of  $K_v$  into edge-disjoint copies of  $H$  is a Steiner triple system that contains an even number of  $K_3$ s. Clearly a necessary condition for the existence of such a decomposition of  $K_v$  is that  $v \equiv 1$  or  $9 \pmod{12}$  (since the number of edges in  $K_v$  must be divisible by 6 and each vertex must have even degree). The rest of this paper is devoted to showing that this condition is also sufficient for the existence of a 2-perfect 6-cycle system, except for  $v = 9$  and possibly except for  $v \in \{45, 57\}$ .

One of the main tools used in proving this result is to find an edge-disjoint decomposition of the complete tripartite graph  $K_{2x, 2x, 2x}$  into 6-cycles so that the triangles formed by joining vertices distance 2 apart in the

6-cycles also form an edge-disjoint decomposition of  $K_{2x, 2x, 2x}$ . Naturally we define such a decomposition to be a *2-perfect 6-cycle decomposition of  $K_{2x, 2x, 2x}$* .

Finally, it is worth noting that a 2-perfect 6-cycle system can be used to define a quasigroup as follows: Let  $C$  be a 2-perfect 6-cycle system of  $K_v$  defined on the vertex set  $Q$ . Define a binary operation " $\circ$ " on  $Q$  by:  $x \circ x = x$  for all  $x \in Q$ , and  $x \circ y = z$  if and only if  $(x, y, z, a, b, c) \in C$ . Then  $(Q, \circ)$  is a quasigroup. A little reflection (but not too much) shows that  $(Q, \circ)$  satisfies the 2-variable identities  $(yx)x = y$  and  $(xy)(y(xy)) = x(yx)$ . On the other hand, a quasigroup  $(Q, \circ)$  satisfying the above 2-variable identities which is *also* antisymmetric defines a 2-perfect 6-cycle system  $C$  of  $K_v$ , where  $C = \{(x, y, x \circ y, y \circ (x \circ y), x \circ (y \circ x), y \circ x) \mid x, y \in Q, x \neq y\}$ . So a 2-perfect 6-cycle system is equivalent to an anti-symmetric quasigroup satisfying the three 2-variable identities in  $I = \{x^2 = x, (yx)x = y, (xy)(y(xy)) = x(yx)\}$ . Whether or not there exists a finite collection of 2-variable quasigroup identities  $K$  so that a 2-perfect 6-cycle system is equivalent to a quasigroup satisfying  $I \cup K$  is an open and (so it seems to the authors) interesting problem.

## 2. PRELIMINARY RESULTS

We begin with some notation. Let  $(v_0, v_1, \dots, v_{m-1})$  denote the  $m$ -cycle consisting of the edges  $v_i v_{i+1}$  for  $0 \leq i \leq m-1$ , reducing the subscript modulo  $m$ . Corresponding to the 6-cycle  $G = (1, 2, 3, 4, 5, 6)$  is the subgraph  $H$  consisting of the two 3-cycles  $(1, 3, 5)$  and  $(2, 4, 6)$ . Also, define  $Z_n = \{0, 1, \dots, n-1\}$ .

We need to know that 2-perfect 6-cycle systems exist for some small values.

EXAMPLE 2.1. For  $v = 13$  define

$$C_{13} = \{(0, 5, 2, 8, 7, 9) + i \mid 0 \leq i \leq 12\},$$

where  $(v_0, v_1, v_2, v_3, v_4, v_5) + i = (v_0 + i, v_1 + i, v_2 + i, v_3 + i, v_4 + i, v_5 + i)$ , reducing each component modulo 13. Then  $C_{13}$  is a 2-perfect 6-cycle system of  $K_{13}$  defined on the vertex set  $Z_{13}$ .

For  $v = 21$  define

$$\begin{aligned} C_{21} = \{ & ((0, 0), (0, 6), (0, 1), (2, 1), (0, 3), (1, 0)) + (0, i), \\ & ((0, 0), (1, 2), (1, 0), (0, 4), (2, 0), (1, 1)) + (0, i), \\ & ((0, 0), (0, 3), (1, 2), (2, 6), (2, 1), (2, 2)) + (0, i), \\ & ((0, 0), (2, 6), (1, 3), (1, 2), (1, 6), (2, 4)) + (0, i), \\ & ((0, 0), (2, 1), (2, 4), (1, 3), (2, 5), (1, 5)) + (0, i) \mid 0 \leq i \leq 6 \} \end{aligned}$$

reducing the second component modulo 7. Then  $C_{21}$  is a 2-perfect 6-cycle system of  $K_{21}$  defined on the vertex set  $Z_3 \times Z_7$ .

For  $v = 25$  define

$$C_{25} = \{(0, 8, 2, 24, 12, 7) + i, (0, 4, 3, 18, 7, 23) + i \mid 0 \leq i \leq 24\},$$

reducing each component modulo 25. Then  $C_{25}$  is a 2-perfect 6-cycle system of  $K_{25}$  defined on the vertex set  $Z_{25}$ .

For  $v = 33$  define

$$\begin{aligned} C_{33} = \{ & ((0, 0), (0, 2), (0, 1), (0, 5), (1, 6), (1, 9)) + (0, i), \\ & ((1, 0), (0, 4), (1, 1), (2, 3), (2, 6), (2, 5)) + (0, i), \\ & ((1, 0), (1, 2), (1, 3), (2, 3), (2, 7), (1, 4)) + (0, i), \\ & ((0, 6), (1, 1), (2, 0), (1, 5), (2, 1), (2, 3)) + (0, i), \\ & ((0, 7), (1, 7), (2, 0), (0, 4), (2, 3), (0, 10)) + (0, i), \\ & ((0, 0), (2, 5), (0, 2), (0, 7), (1, 1), (2, 9)) + (0, i), \\ & ((0, 0), (1, 3), (0, 4), (1, 8), (1, 2), (2, 0)) + (0, i), \\ & ((0, 8), (1, 10), (2, 0), (0, 10), (2, 5), (2, 10)) \\ & + (0, i) \mid 0 \leq i \leq 10\}, \end{aligned}$$

reducing the second component modulo 11. Then  $C_{33}$  is a 2-perfect 6-cycle system of  $K_{33}$  defined on the vertex set  $Z_3 \times Z_{11}$ .

Before moving on to the main construction, we need some preliminary results.

**LEMMA 2.2.** *For any integer  $x \geq 2$ , there exists a 2-perfect 6-cycle system of  $K_{2x, 2x, 2x}$ .*

*Proof.* Let  $L_1$  be a latin square of order  $x$  on the symbols  $\{0, \dots, x-1\}$  and define a latin square  $L$  of order  $2x$  on the symbols  $\{0, \dots, 2x-1\}$  as follows: if cell  $(i, j)$  of  $L_1$  contains symbol  $k$  then we have cells  $(2i, 2j)$ ,  $(2i, 2j+1)$ ,  $(2i+1, 2j)$  and  $(2i+1, 2j+1)$  of  $L$  containing symbols  $2k$ ,  $2k+1$ ,  $2k+3$  and  $2k+2$ , respectively. Let  $(\{0, \dots, 2x-1\}, \cdot)$  be the quasigroup corresponding to  $L$ .

Let the vertex set of  $K_{2x, 2x, 2x}$  be  $\{(i, j) \mid 0 \leq i \leq 2x-1, 0 \leq j \leq 2\}$ . Define the 2-perfect 6-cycle system of  $K_{2x, 2x, 2x}$  by  $C = \{((2i, 0), (2j, 1), (2i \cdot (2j+1), 2), (2i+1, 0), (2j+1, 1), ((2i+1) \cdot 2j), 2)), ((2i, 0), (2j+1, 1), (2i \cdot 2j, 2), (2i+1, 0), (2j, 1), ((2i+1) \cdot (2j+1), 2)) \mid 0 \leq i \leq x-1, 0 \leq j \leq x-1\}$ . It is easy to check this is a 2-perfect 6-cycle system, though it may

help to note that by the construction of  $L$ ,  $2i \cdot (2j+1) = 2k+1 \neq 2k+3 = (2i+1) \cdot 2j$  and  $2i \cdot 2j = 2k \neq 2k+2 = (2i+1) \cdot (2j+1)$ , so the 6-cycles are defined on 6 distinct vertices. ■

**LEMMA 2.3.** *There exists a pairwise balanced design (PBD) of order  $v$  with block sizes 3 and 4 that has a parallel class for all  $v \equiv 1 \pmod{3}$  except for  $v \in \{7, 10\}$ .*

*Proof.* Brouwer [2] has shown that for all  $u \equiv 0$  or  $1 \pmod{3}$  there exists a PBD of order  $u$  with block sizes 3 and 4. Furthermore, when  $u \equiv 0 \pmod{3}$  his construction yields a design which contains at least one block of size 4 unless  $u \in \{3, 6, 9\}$ . When  $u \in \{3, 6, 9\}$  such a design with a block of size four does not exist.

We use Brouwer's result to make sure that when  $v \equiv 1 \pmod{3}$  the PBD's contain a parallel class.

If  $v = 9x+1$  and  $x \geq 2$ , then construct a PBD with the set of blocks being  $B$  and on the set of symbols  $\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 3, 1 \leq j \leq 3x\}$  as follows. Place the blocks from a PBD of order  $3x$  with block sizes 3 and 4 on the vertex set  $\{(i, j) \mid 1 \leq j \leq 3x\}$  in  $B$  for each  $i \in \{1, 2, 3\}$ . Let  $(Q, \cdot)$  be an idempotent quasigroup of order  $3x$  with a transversal  $T$  intersecting the diagonal in just the cell  $(1, 1)$  (a pair of idempotent orthogonal latin squares of order  $3x$  is sufficient, (see [17], for example) and it is easy to construct one of order 6, so such a latin square exists for all  $x \geq 2$ ). Let  $B$  also contain the blocks  $\{\infty, (1, i), (2, i), (3, i)\}$  for  $1 \leq i \leq 3x$  and, for  $i \neq j$ , the blocks  $\{(1, i), (2, j), (3, i \cdot j)\}$ . This defines a PBD with block sizes 3 and 4 with a parallel class consisting of the block  $\{\infty, (1, 1), (2, 1), (3, 1)\}$  and the blocks  $\{(1, i), (2, j), (3, i \cdot j)\}$  for each cell  $(i, j)$  in  $T$  except for  $(1, 1)$ .

If  $v = 9x+4$  and  $x \geq 1$ , then the same construction as in the case when  $v = 9x+1$  will work if a PBD of order  $3x+1$  and an idempotent quasigroup of order  $3x+1$  with a transversal intersecting the diagonal in just the cell  $(1, 1)$  is used.

If  $v = 9x+7$  and  $x \geq 3$ , then define a PBD with set of blocks  $B$  on the symbols  $\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 3, 1 \leq j \leq 3x+2\}$  as follows. Place the blocks of a PBD of order  $3x+3$  with block sizes 3 and 4, at least one block having size 4 (so  $x \geq 3$ ), and on the vertex set  $\{\infty\} \cup \{(i, j) \mid 1 \leq j \leq 3x+2\}$  in  $B$  for each  $i \in \{1, 2, 3\}$ , except that the block  $\{\infty, (i, 1), (i, 2), (i, 3)\}$  is omitted. Let  $L$  be a latin square of order  $3x+2$  with a subsquare of order 3 (on rows, columns and symbols 1, 2 and 3) which contains a transversal  $T$  that intersects the subsquare in precisely the cell  $(1, 1)$  and let  $(\{1, \dots, 3x+2\}, \cdot)$  be the corresponding quasigroup. (Such a latin square exists of order 8 and for all other orders, the incomplete latin square

1	2	3	7	6
3	1	2	4	8
2	3	1	8	5
8	6	4	5	7
5	8	7	6	4

can be embedded in a latin square of any order  $n \geq 11$  in which cells  $(6, 6)$ ,  $(7, 7)$  and  $(i, i)$  for  $8 \leq i \leq n$  contain symbols 2, 3 and  $i$ , respectively [15]; then select  $T$  to consist of cells  $(1, 1)$ ,  $(2, 4)$ ,  $(3, 5)$ ,  $(4, 2)$ ,  $(5, 3)$  and  $(i, i)$  for  $6 \leq i \leq n$ ). Place the blocks of a PBD with block sizes 3 and 4 of order 10 on the vertex set  $\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 3, 1 \leq j \leq 3\}$  in  $B$ , making sure that  $b = \{\infty, (1, 1), (2, 1), (3, 1)\}$  is a block. Finally, for  $1 \leq i \leq 3x + 2$ ,  $1 \leq j \leq 3x + 2$  but not both  $i$  and  $j$  in  $\{1, 2, 3\}$ , add the block  $\{(1, i), (2, j), (3, i \cdot j)\}$  to  $B$ . Then the blocks of  $B$  form a PBD of block sizes 3 and 4 and the blocks arising from the cells in  $T$  without cell  $(1, 1)$  together with  $b$  form a parallel class.

Suppose  $v = 9x + 7$  and  $x \in \{1, 2\}$ . For  $v = 16$  there exists a resolvable BIBD with block size 4. For  $v = 25$ , introduce four new points  $\infty_1, \infty_2, \infty_3$ , and  $\infty_4$  to a Kirkman triple system of order 21 (see [17], for example), add  $\infty_i$  to the blocks in parallel class  $i$  for  $1 \leq i \leq 4$  and add the block  $b = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ . Then parallel class 5 of the Kirkman triple system together with  $b$  forms a parallel class in the PBD of order 25 constructed in this way. ■

Of course, the PBD's constructed in Lemma 2.3 are group divisible designs (GDD's) with groups and blocks of sizes 3 and 4. We now obtain some more GDD's.

**LEMMA 2.4.** *For all  $x \geq 2$  there exists a GDD of order  $6x + 5$  with blocks of size 3, one group of size 5 and the remaining groups of size 3.*

*Proof.* We make use of Wilson's construction for (partial) Steiner triple systems [18]. Let  $G$  be the graph with  $V(G) = \{0, 1, \dots, 6x + 2\} - \{0, 2x + 1, 4x + 2\}$ , and  $E(G) = \{\{y, -y\}, \{y, 2y\} \mid y \in V(G)\}$ , all elements being reduced modulo  $6x + 3$ . Wilson shows that this graph has a one-factorization with 1-factors  $F_1, F_2$ , and  $F_3$ , and so the following define a PBD with one block of size 5 and the rest of size 3:

- (a)  $\{\infty_1, \infty_2, 0, 2x + 1, 4x + 2\}$  is a block,
- (b) if  $\{u, v\} \in F_1, F_2$  or  $F_3$  then  $\{\infty_1, u, v\}$ ,  $\{\infty_2, u, v\}$  or  $\{0, u, v\}$  is a block respectively, and

(c) if  $u \neq v \neq w \neq u$ ,  $u + v + w \equiv 0 \pmod{6x+3}$ ,  $0 \notin \{u, v, w\}$  and  $\{y, 2y\} \not\subseteq \{u, v, w\}$  for any  $y \in \{1, \dots, 6x+2\}$ , then  $\{u, v, w\}$  is a block.

Therefore the result follows if  $\{1, \dots, 6x+2\} - \{2x+1, 4x+2\}$  can be partitioned into sets  $\{u, v, w\}$  such that  $u + v + w \equiv 0 \pmod{6x+3}$ , for then these blocks together with the block of size 5 form a parallel class. This set can be so partitioned. Cyclic triple systems of all orders  $6x+3 \neq 9$  have been formed [13] (or see [4]) by partitioning  $\{1, \dots, 3x+1\} - \{2x+1\}$  into sets  $\{u, v, w\}$  satisfying either  $u + v + w \equiv 0 \pmod{6x+3}$  or  $u + v = w$ . For each triple  $\{u, v, w\}$  with  $u + v + w \equiv 0 \pmod{6x+3}$  take the triples  $\{u, v, w\}$  and  $\{-u, -v, -w\}$  and for each triple with  $u + v = w$  take the triples  $\{u, v, -w\}$  and  $\{-u, -v, w\}$  (reducing elements modulo  $6x+3$ ) to produce the desired partition. ■

The following result is the main construction for 2-perfect 6-cycle systems.

**LEMMA 2.5.** *If there exists a GDD of order  $t$  with blocks of size 3 and groups  $h$  in the set  $H$ , and if for  $s \geq 2$  there exists a 2-perfect 6-cycle system of  $K_{2s|h|+1}$  for all  $h \in H$ , then there exists a 2-perfect 6-cycle system of  $K_{2st+1}$ .*

*Proof.* Define a 2-perfect 6-cycle system of  $K_{2st+1}$  on the vertex set  $\{\infty\} \cup (Z_{2s} \times Z_t)$  as follows:

- (1) for each block  $\{u, v, w\}$  define a 2-perfect 6-cycle system of  $K_{2s, 2s, 2s}$  on the vertex set  $Z_{2s} \times \{u, v, w\}$  (this exists by Lemma 2.2), and
- (2) for each group  $h \in H$  define a 2-perfect 6-cycle system of order  $2s|h|+1$  on the vertex set  $\{\infty\} \cup (Z_{2s} \times h)$ . ■

### 3. CONSTRUCTIONS OF 2-PERFECT 6-CYCLE SYSTEMS

**THEOREM 3.1.** *For all  $v \equiv 1$  or  $9 \pmod{12}$  except for  $v = 9$  and possibly  $v \in \{45, 57\}$ , there exists a 2-perfect 6-cycle system of  $K_v$ .*

*Proof.* We consider the cases  $v \equiv 1, 9, 13$ , and  $21 \pmod{24}$  in turn.

*Case 1.*  $v \equiv 1 \pmod{24}$ .

Let  $v = 24x + 1$ . By removing one symbol from a Steiner triple system of order  $24x + 1$  (see [17], for example), a GDD with blocks of size 3 and groups of size 2 results. We shall use such a design many times!

If  $2x = 6y$  then  $v = 12(6y) + 1$  and there exists a GDD of order  $6y$  with blocks of size 3 and groups of size 2. Apply Lemma 2.5 with  $t = 6y$  and  $s = 6$ . Since  $|h| = 2$  for all  $h \in H$  and since a 2-perfect 6-cycle system of  $K_{2s|h|+1}$  is constructed in Example 2.1, the result follows.

If  $2x = 6y + 2$  then  $v = 12(6y + 2) + 1$  and there exists a GDD of order  $6y + 2$  with blocks of size 3 and groups of size 2. Apply Lemma 2.5 with  $t = 6y + 2$  and  $s = 6$ . A 2-perfect 6-cycle system of  $K_{25}$  is constructed in Example 2.1.

If  $2x = 6y + 4$  then  $v = 6(12y + 8) + 1$  and there exists a GDD of order  $12y + 8$  with blocks of size 3 and groups of size 2. Apply Lemma 2.5 with  $t = 12y + 8$  and  $s = 3$ . A 2-perfect 6-cycle system of  $K_{13}$  is constructed in Example 2.1.

*Case 2.*  $v \equiv 9 \pmod{24}$ .

Let  $v = 24x + 9 = 4(6x + 2) + 1$ . For  $x \neq \{2, 3\}$ , from Lemma 2.3, there exists a PBD with set of blocks  $B$  on the set of symbols  $Z_{3x+1}$  with block sizes 3 and 4 which has a parallel class. Define a 2-perfect 6-cycle system of  $K_v$  on the vertex set  $\{\infty\} \cup (Z_4 \times (Z_{3x+1} \times Z_2))$  as follows:

(1) for each block  $\{u, v, w\}$  of  $B$  which does not lie in the parallel class, place four 2-perfect 6-cycle systems of  $K_{4,4,4}$ , one on each of the following vertex sets:  $Z_4 \times \{(u, 0), (v, 0), (w, 0)\}$ ,  $Z_4 \times \{(u, 0), (v, 1), (w, 1)\}$ ,  $Z_4 \times \{(u, 1), (v, 0), (w, 1)\}$  and  $Z_4 \times \{(u, 1), (v, 1), (w, 0)\}$ ;

(2) for each block  $\{t, u, v, w\}$  of  $B$  which does not lie in the parallel class, place eight 2-perfect 6-cycle systems of  $K_{4,4,4}$ , one on each of the following vertex sets:

$$\begin{aligned} &Z_4 \times \{(t, 0), (u, 0), (v, 0)\}, Z_4 \times \{(t, 0), (u, 1), (w, 0)\}, \\ &Z_4 \times \{(t, 0), (v, 1), (w, 1)\}, Z_4 \times \{(u, 0), (v, 1), (w, 0)\} \\ &Z_4 \times \{(t, 1), (u, 1), (v, 1)\}, Z_4 \times \{(t, 1), (u, 0), (w, 1)\}, \\ &Z_4 \times \{(t, 1), (v, 0), (w, 0)\}, Z_4 \times \{(u, 1), (v, 0), (w, 1)\}; \end{aligned}$$

and

(3) for each block  $h$  in the parallel class of  $B$ , place a 2-perfect 6-cycle system of order  $8|b| + 1$  on the vertex set  $\{\infty\} \cup (Z_4 \times (b \times Z_2))$  (2-perfect 6-cycle systems of  $K_{25}$  and of  $K_{33}$  are constructed in Example 2.1).

If  $x = 3$  then  $v = 81$  and so the construction used in Case 1 starting with a GDD of order 8 with blocks of size 3 and groups of size 2 and then using 2-perfect 6-cycle systems of  $K_{10,10,10}$ , and finally placing a 2-perfect 6-cycle system of  $K_{21}$  (see Example 2.1) on the vertex set  $\{\infty\} \cup (Z_{10} \times h)$  for each group  $h$  gives a 2-perfect 6-cycle systems of  $K_{81}$ .

*Case 3.*  $v \equiv 13 \pmod{24}$ .

Let  $v = 24x + 13$ . Then  $v = 4(6x + 3) + 1$  and there exists a GDD of order  $6x + 3$  with groups and blocks of size 3 (just remove the blocks in a parallel



class from a Kirkman triple system of order  $6x + 3$  to form the groups). Apply Lemma 2.5 with  $t = 6x + 3$  and  $s = 2$ . A 2-perfect 6-cycle system of  $K_{13}$  is constructed in Example 2.1.

*Case 4.*  $v \equiv 21 \pmod{24}$ .

Let  $v = 24x + 21$ . Then  $v = 4(6x + 5) + 1$  and for  $x \geq 2$  there exists a GDD with blocks and groups of size 3 except for one group of size 5 (this is constructed in Lemma 2.4). Apply Lemma 2.5 with  $t = 6x + 5$  and  $s = 2$ . Examples of 2-perfect 6-cycle systems of  $K_{13}$  and  $K_{21}$  are constructed in Example 2.1.

#### 4. OPEN PROBLEMS

Clearly many problems related to Theorem 3.1 still remain. To find the spectrum of  $i$ -perfect  $m$ -cycle systems in general is an extremely difficult problem. At this stage, the spectrum for  $m$ -cycle systems has not yet been found, and it is likely that this spectrum will be essentially the same as the spectrum for  $i$ -perfect  $m$ -cycle systems. Then of course there is the more difficult problem of constructing  $I$ -perfect  $m$ -cycle systems; that is, an  $m$ -cycle system that is  $i$ -perfect for all  $i \in I$ . In the case where  $I = \{1, \dots, \lfloor m/2 \rfloor\}$  such a system is a Steiner  $m$ -cycle system. Finally an easier problem: find 2-perfect 6-cycle systems of  $K_{45}$  and  $K_{57}$ .

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*Note added in proof.* Recently Elizabeth Billington, University of Queensland, has found 2-perfect 6-cycle systems of orders 45 and 57, thereby removing the possible exceptions to Theorem 3.1.

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