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Two-Colouring All Two-Element Maximal Antichains

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A *fibre* in a partially ordered set P is a subset of P meeting every maximal antichain of P. We give an example of a finite poset P with no one-element maximal antichain and containing no fibre of size at most $|P|/2$, thus answering a question of Aigner–Andreae and disproving a conjecture of Lonc–Rival. We also prove

THEOREM 1. *The elements of an arbitrary partially ordered set can be coloured with two colours such that every two-element maximal antichain receives both colours.*

1. INTRODUCTION

An element of a partially ordered set P is called *splitting* if it is comparable to all elements of P. A *fibre* is a subset of P meeting every maximal antichain of P (maximal with respect to set inclusion). In an unpublished manuscript [1], Aigner and Andreae asked whether every poset P with no splitting element contains a fibre of cardinality at most $|P|/2$. In [3], Lonc and Rival proposed the stronger conjecture that P always contains a fibre whose complement is also a fibre; or (to put it another way) that the elements of P can be coloured with two colours such that every maximal antichain receives both colours.

In this paper we give a counterexample to both these statements. We will, however, prove a restricted version of the conjecture, where only *two-element* maximal antichains are considered.

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THEOREM 1. *The elements of an arbitrary partially ordered set can be coloured with two colours such that every two-element maximal antichain is two-coloured.*

It will be convenient to define the following graph, which we call $G(P)$, associated with P. The vertices of $G(P)$ are the elements of P, and two vertices a, b are adjacent in $G(P)$ if and only if $\{a, b\}$ is a maximal antichain of P. It is clear that the above theorem is equivalent to the following: *$G(P)$ is bipartite for all P.*

Having found the above theorem, in our subsequent attempts to prove or disprove the conjecture we looked at posets P whose graphs $G(P)$ were connected, and whose two-colourings were therefore forced. Eventually we found such a poset containing a monochromatic three-element maximal antichain, and it is shown in Fig. 1. The graph $G(P)$ of this poset P is just the path $1, 2, \ldots, 17$, and $\{1, 9, 17\}$ is a maximal antichain which is monochromatic under the forced two-colouring of P. Moreover $G(P)$ shows that the only possible way to obtain a fibre of size at most $|P|/2$, i.e., at most eight, is to take the eight-element set $\{2, 4, 6, \ldots, 16\}$, and this set misses $\{1, 9, 17\}$. Thus no fibre can have less than nine elements.

$G(P)$ satisfying additional properties can have unusual and perhaps interesting consequences for P. For example, we prove:

THEOREM 2. *If P is finite with at least two elements and $G(P)$ is connected, then P has exactly two maximal elements and two minimal elements.*

At the end of this paper we discuss some open problems.

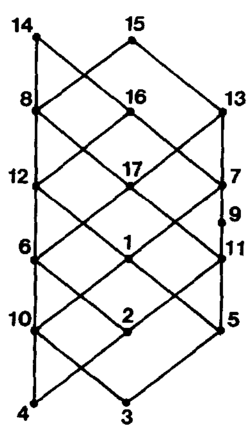

FIG. 1. A poset P having no fibre of size at most $|P|/2$.

2. Lemmas

In this section P will be an arbitrary poset and $G(P)$ its graph of two-element maximal antichains. By a *path* in $G(P)$ we will mean a finite string $abc...$ of *distinct* elements of P such that a is adjacent to b (in $G(P)$), b is adjacent to c, and so on. First an obvious fact.

Lemma 1. *If abc is a path in $G(P)$, then a and c are comparable in P.*

Proof. Since $\{a, b\}$ is a maximal antichain of P, and $c \| b$, c must be comparable to a. ∎

It is interesting to note that a similar result holds for five-element paths; that is, *if abcde is a path in $G(P)$, then the endpoints a and e are comparable in P.* We will not need this, though, and leave the proof for the reader.

For four-element paths the situation is not quite as nice.

Lemma 2. *Let abcd be a path in $G(P)$.*

(a) *If $a<d$ (in P), then $a<c$ and $b<d$. Dually, if $a>d$, then $a>c$ and $b>d$.*

(b) *If $a \| d$ but $\{a, d\}$ is not an edge of $G(P)$, then either*

(i) *$a<c$ and $d<b$, or*

(ii) *$a>c$ and $d>b$.*

Proof. (a) Suppose $a<d$. We know that a and c are comparable by Lemma 1. If $a>c$, then $c<a<d$, a contradiction; thus $a<c$. Similarly, b and d are comparable and it follows that $b<d$.

(b) If neither (i) nor (ii) hold, by Lemma 1 we may assume that $a<c$ and $b<d$ (the other case is dual). Since $a \| d$, but a and d are not adjacent in $G(P)$ and thus do not form a maximal antichain of P, there is some element $k \in P$ such that $a \| k$ and $d \| k$. Furthermore k must be comparable to both b and c since ab and cd are edges of $G(P)$. It follows, since $a<c$ and $b<d$, that $k<c$ and $b<k$, but then $b<k<c$, a contradiction. ∎

A path $abcd$ of $G(P)$ such that $\{a, d\}$ is not an edge, $a \| d$, $a<c$ and $d<b$, i.e., satisfying Lemma 2(b)(i), will be called an *attractor*. The set $\{b, c\}$ will be called the *interior* of the attractor $abcd$.

Now for the most important lemma in the proof of our theorem.

Lemma 3. *If abcd is an attractor then there exists $k \in P$ such that*

(a) *$k \| a$ and $k \| d$,*

(b) *$k<b$ and $k<c$,*

(c) *if klm is a path in $G(P)$ then $m<k$.*

Proof. By the definition of an attractor, $\{a, d\}$ is an incomparable pair of elements which do not form a maximal antichain, so there exists an element k in P satisfying (a). We choose k to be maximal with this property. Since $k \parallel a$ and $\{a, b\}$ is a maximal antichain, k must be comparable to b. Moreover, since $d < b$ we must have $k < b$. Similarly, $k < c$.

It remains to show (c). We know from Lemma 1 that k and m are comparable, so suppose $k < m$. By the maximality of k, and by symmetry, we can assume m and a are comparable. Certainly $m < a$ would imply $k < a$, contradicting (a), so $m > a$. Meanwhile, $\{k, l\}$ is a maximal antichain so by (a) l must be comparable to both a and d. Since $a < m$, $l < a$ is impossible, so $a < l$. Then $a \parallel d$ means that $d < l$ as well. Since $\{b, c\}$ is a maximal antichain, l must be comparable to either b or c. From (b) and $k \parallel l$ we see that l cannot be greater than b or c. But $l < b$ is impossible since $l > a$, and $l < c$ is impossible since $l > d$. This contradiction establishes (c). ∎

3. Proof of Theorem 1

We want to show that $G(P)$ is bipartite for every poset P.

First, it is obvious that $G(P)$ contains no triangle, since a triangle would correspond to a three-element antichain in P while each pair of vertices in the triangle would have to be a two-element maximal antichain of P.

Next note that if there were a poset P for which $G(P)$ is not bipartite, then $G(P)$ contains an odd cycle. Restricting P to the elements of this odd cycle gives us a finite poset whose graph will still contain this odd cycle, since two-element maximal antichains will remain so in the restriction. Thus we need only consider the case when P is finite, with an odd number of elements (hence at least five elements), and $G(P)$ is Hamiltonian.

Further, we may assume that $G(P)$ *is* an odd cycle, with no additional edges. For if $\{u, v\}$ were an edge of $G(P)$ with u and v not neighbouring vertices of the Hamiltonian cycle, then we could form a smaller odd cycle by using the edge $\{u, v\}$ together with whichever of the paths in $G(P)$ connecting u and v is of even length; restricting P to the elements of this odd cycle would again yield a smaller counterexample.

To finish the proof we wish to show that $G(P)$ cannot be an odd cycle. In fact we prove:

Claim. If $G(P)$ is a cycle then $G(P)$ is a four-element cycle.

It follows that the only poset P for which $G(P)$ is a cycle is the four-element poset consisting of two unrelated two-element chains.

From the above, we may by way of contradiction assume that the vertices of $G(P)$, and the elements of P, are precisely

$$\{1, 2, \ldots, n\},$$

where $n \geqslant 5$, and that the edges of $G(P)$, i.e., the two-element maximal antichains of P, are precisely the consecutive pairs $\{i, i+1\}$ $(i = 1, 2, ..., n$, the last pair taken mod $n)$.

First note that, *without loss of generality, $G(P)$ contains an attractor.* If all pairs $\{i, i+3\}$ were comparable in P, we may suppose $1 < 4$. (Here of course $<$ denotes "less than" in P, not in the integers!) Then by Lemma 2(a), $1 < 3$ and $2 < 4$, from which $2 < 5$ and thus $3 < 5$ follow, and so on. We conclude that either

$$1 < 3 < 5 < \cdots < 1 \qquad \text{or} \qquad 1 < 3 < 5 < \cdots < n.$$

Both are impossible, the first trivially, the second since 1 and n are adjacent in $G(P)$ and hence incomparable in P. Thus we may assume that $1 \| 4$ say, but since $n \geqslant 5$ $\{1, 4\}$ is not a maximal antichain. By Lemma 2(b) (and dualizing if necessary) 1234 is an attractor.

Abbreviate the path $i, i+1, ..., i+j$ by $[i, i+j]$. If $[i, i+3]$ is an attractor, let $t(i)$ denote the element k from Lemma 3. Thus

(a) $t(i) \| i$, $t(i) \| i+3$,

(b) $t(i) < i+1$, $t(i) < i+2$, and

(c) $t(i) > t(i)+2$, $t(i) > t(i)-2$.

We claim that *$t(i)$ lies in the interior of an attractor.* For by Lemma 1 and symmetry we can suppose that $t(i)-1 < t(i)+1$. Then by Lemma 2(a), and since $t(i)+2 < t(i)$, $t(i)-1$ and $t(i)+2$ are incomparable. This means that $[t(i)-1, t(i)+2]$ is an attractor, with $t(i)$ in its interior.

Now the proof of the claim follows easily. Starting with an attractor $[a_1, a_1+3]$, we obtain the corresponding element $t(a_1)$, which lies in the interior of an attractor, call it $[a_2, a_2+3]$. Thus $t(a_1)$ equals a_2+1 or a_2+2, and furthermore $t(a_1) < a_1+1$ and a_1+2. Next find the element $t(a_2)$ corresponding to $[a_2, a_2+3]$; it lies in an attractor $[a_3, a_3+3]$, and $t(a_2) < a_2+1$ and a_2+2, in particular $t(a_2) < t(a_1)$. Continuing in this way, we get an infinite chain

$$t(a_1) > t(a_2) > t(a_3) > \cdots,$$

contradicting the finiteness of P. This completes the proof of Theorem 1. ▮

4. Proof of Theorem 2

Suppose that $G(P)$ is connected, where P is finite, $|P| > 1$. P obviously cannot have a unique maximal element or a unique minimal element, for such an element would be an isolated vertex of $G(P)$, contradicting the assumption that $G(P)$ is connected. Thus, by way of contradiction, we may by duality choose P to be the *smallest* poset containing at least three maximal elements for which $G(P)$ is connected.

Clearly, the maximal elements of P form an independent set in $G(P)$. By the minimality of P, $G(P)$ cannot contain any proper connected subgraph containing three of the maximal elements of P, for otherwise the poset corresponding to this subgraph would be smaller than P but would still have three maximal elements. It follows easily that $G(P)$ is a tree with at most three endvertices and that each endvertex of $G(P)$ is a maximal element of P. We have two cases.

(i) *$G(P)$ has three endvertices a, b and c.*

Then a, b and c are precisely the maximal elements of P. We claim that every other element of P is less than at least two of a, b, c. For suppose $x \in P$ is such that $x < a$, $x \parallel b$, $x \parallel c$. We may assume that a *covers* x, that is, there are no elements of P between x and a. Thus x is a maximal element of $P - \{a\}$, which therefore has at least three maximal elements. Moreover, $G(P - \{a\})$ is connected, since it will consist of just $G(P) - \{a\}$, with maybe some edges added but none lost, and $G(P) - \{a\}$ is connected because a is an endvertex of $G(P)$. This contradicts the minimality of P, so x cannot exist.

Now let a' be the unique element adjacent to a in $G(P)$. Since $\{a', a\}$ is a maximal antichain of P, and $\{a, b, c\}$ are distinct maximal elements of P and hence incomparable, we must have $a' < b$ and $a' < c$. By the previous paragraph, we can find an element $a_1 \geqslant a'$ such that $a_1 \parallel a$, $a_1 < b$, $a_1 < c$, and both b and c cover a_1. Similarly we find $b_1, c_1 \in P$ such that $b_1 \parallel b$, a and c cover b_1, $c_1 \parallel c$, and a and b cover c_1. Thus a_1, b_1, c_1 will be distinct maximal elements of $P - \{a, b, c\}$. Also, $G(P - \{a, b, c\})$ is connected since it contains all the edges of the connected graph $G(P) - \{a, b, c\}$. This again contradicts the minimality of P, so case (i) is impossible.

(ii) *$G(P)$ has only two endvertices a, b, i.e., $G(P)$ is a path.*

Then the third maximal element c lies somewhere in the interior of this path. Clearly (by Lemma 1) c will have at least two vertices between it and a or b, so we can find a path $wxcyz$ in $G(P)$.

We claim that c lies in the interior of an attractor. By Lemma 1, and since c is a maximal element, $w < c$ and $z < c$. Also x and y are comparable, and if $x < y$, say, then $xcyz$ will be an attractor by Lemma 2(a), with c in its interior.

Now we apply Lemma 3 to this attractor to obtain an element k_1 such that $k_1 \parallel x$, $k_1 \parallel z$, $k_1 < c$, $k_1 < y$, and such that if $k_1 lm$ is a path in $G(P)$ then $m < k_1$. It follows as above that if k_1 is in the centre of a five-element path then k_1 is in the interior of an attractor. Applying Lemma 3 again we obtain an element k_2 which among other properties satisfies $k_2 < k_1 < c$. If k_2 lies in the centre of a five-element path it will be in the interior of an attractor, and so on. Were this procedure to continue indefinitely we would obtain an infinite chain $c > k_1 > k_2 > k_3 > \cdots$, contradicting the finiteness of P. Therefore some k_n must be an endvertex of $G(P)$ or the neighbour of an endvertex of $G(P)$. By symmetry we can suppose that either $k_n = a$ or $k_n = a'$, where a' is the unique element adjacent to a in $G(P)$. Since $c > k_n$, $k_n = a$ is impossible. So $k_n = a'$. However, $\{y, c\}$ is a maximal antichain in P, so $y < a$ since a is a maximal element. Since $k_n \leqslant k_1 < y$, this contradicts the fact that a and $k_n = a'$ form an antichain. ▌

5. Remarks and Open Problems

Let λ be the smallest positive number such that every finite poset P with no splitting elements has a fibre of size at most $\lambda|P|$. Figure 1 shows that $\lambda \geqslant 9/17$. R. Maltby [4] has observed that if one stacks up n copies of the poset of Fig. 1 and identifies maximals with minimals in adjacent copies, one obtains a poset with $15n+2$ elements in which every fibre contains at least $8n+1$ elements. Letting $n \to \infty$ then shows $\lambda \geqslant 8/15$. On the other hand, in [2] Duffus, Kierstead, and Trotter have recently shown that the elements of any finite partial order can be coloured with three colours so that each maximal antichain with more than one element receives at least two colours, and thus (by choosing the two smallest colour classes) $\lambda \leqslant 2/3$. *What is the exact value of* λ?

We can see that the Aigner–Andreae question has a positive answer for posets of dimension 2. Take $P = [n]$ and let L_1 and L_2 be linear extensions whose intersection is P, with L_1 equal to $1 < 2 < \cdots < n$ without loss of generality. Then maximal antichains of P are just maximal decreasing subsequences in L_2. Choose the *least* element of each such maximal decreasing subsequence. The collection S of these elements contains exactly one element from each maximal antichain of P, and therefore is a fibre (which in fact is a chain). Moreover, since P has no splitting elements, $|S| \leqslant |P|/2$, and the complement of S is also a fibre, verifying the Lonc–Rival conjecture in this case.

One could introduce width parameters, generalizing the condition that P have no splitting element at the same time. Given arbitrary positive integers m, w with $m \leqslant w$, what is the smallest positive number $\lambda = \lambda(m, w)$ such that every finite poset with smallest maximal antichain of size m and largest antichain of size w ($=$width of P) has a fibre of size at most $\lambda|P|$? It is easy to see that in general

$$\lambda(m, w) \leqslant 1 - \frac{m-1}{w};$$

for by Dilworth's theorem P is the union of w chains, and choosing all elements from the smallest $w-m+1$ of them yields a fibre. It is also easy to see that for $m=1$ this bound, namely $\lambda(1, w) \leqslant 1$, is best possible for all w. For $m=2$ and $w=3$, the above results give

$$\tfrac{8}{15} \leqslant \lambda(2, 3) \leqslant \tfrac{2}{3}.$$

On another topic, Theorem 2 shows that the connectedness of $G(P)$ may impose certain properties on P. In an earlier version of this paper we asked: if $G(P)$ is connected, must the width of P be bounded? R. Maltby [4] has since shown the answer is no. In fact, he constructs finite posets P of arbitrary width with $G(P)$ a path, and an infinite poset Q of infinite width with $G(Q)$ connected.

Finally, we mention a more general hypergraph colouring problem. A (good) *n-colouring* of a hypergraph $\mathscr{H}$ is a vertex colouring with n colours such that each edge of $\mathscr{H}$ receives at least two colours. Let $\mathscr{C}(P)$ be the hypergraph of maximal chains and $\mathscr{A}(P)$ the hypergraph of maximal antichains of a finite poset P. If P has no isolated points then $\mathscr{C}(P)$ is trivially two-colourable—just colour the minimals with one colour and the rest of the elements with the second. On the other hand, we know that, with P having no splitting elements, $\mathscr{A}(P)$ need not be two-colourable, but is three-colourable [2]. All this could be recast into the terminology of graph theory, whereby we would consider the hypergraphs of maximal complete subgraphs and of maximal independent sets of a comparability graph. What about more general graphs? It is known that for arbitrary graphs the chromatic number of neither hypergraph is bounded. Is there a bound for the chromatic number of those hypergraphs which arise from, say, perfect graphs? In this case only one kind of hypergraph (maximal complete subgraph, or maximal independent set) need be considered, since the complement of a perfect graph is perfect.

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