Some thoughts on SNO+ Reconstruction

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1 Energy

We can estimate a lower limit to the energy resolution of SNO+ by imagining a likelihood analysis independent of the position of the event, which depends only on the number of detected photons N. The likelihood function $\mathscr{L}$ is given by:

$$\mathscr{L}(E) \equiv P(\mathrm{N}|E) = \sum_{j \geq \mathrm{N}} \sum_{k \geq j} P(j|E) P(k|j) P(\mathrm{N}|k) \tag{1}$$

where j is the number of photons created, k is the number of photons that reach a PMT, and N is the number of photons detected. The number of photons created follows a Poisson distribution

$$P(j|E) = e^{-\mu} \frac{\mu^j}{j!}$$

where μ is usually assumed to depend linearly on the energy E.

For isotropic light, the number of photons reaching the PMTs, k, follows a binomial distribution:

$$P(k|j) = \binom{k}{j} p^j (1-p)^{k-j}$$

where p is the probability that any one photon hits a PMT. For an event at the center, p would be equal to the coverage of the detector. The probability that N photons are then detected out of the k that hit a PMT also follows a binomial distribution with p being the quantum efficiency of the PMT (I'll call it q).

The sum in Equation 1 reduces to a single Poisson distribution

$$\mathscr{L}(E) = e^{-\lambda} \frac{\lambda^{\mathrm{N}}}{\mathrm{N}!}$$

with a mean λ equal to the product of p, q, and μ. We know from simulations that λ is approximately equal to 400 hits per MeV.

We can estimate the resolution by looking at the standard deviation of the likelihood function for a given N.

$$\mathbb{E}(\lambda) = \int \lambda \mathscr{L}(E) = \mathrm{N} + 1$$

$$\mathbb{E}(\lambda^2) = \int \lambda^2 \mathscr{L}(E) = (\mathrm{N}+2)(\mathrm{N}+1)$$

$$\mathrm{Var}(\lambda) = \mathbb{E}(\lambda^2) - \mathbb{E}(\lambda)^2 = \mathrm{N} + 1$$

Therefore, we see that the energy resolution σ is given by

$$\sigma = \frac{\sqrt{\mathrm{N}+1}}{400}$$

For a 3 MeV event, the lowest achievable energy resolution is therefore around 0.09 MeV, or 3%.

2 Position Reconstruction

We can try a similar method to estimate a lower limit to the position resolution achievable in SNO+. For simplicity, assume a perfectly spherical detector and only direct light, without any spread in time due to the PMTs. Scintillation light typically contains multiple components, each component emitting light with a different time constant. For the purpose of estimating a lower bound on the position resolution, I will assume the light only comes from a single component. In this case the likelihood can be written as [1],

$$\mathscr{L}(\mathbf{x}) = \prod_i e^{-(t-t_0-s_i)} \frac{\cos\phi}{4\pi s_i^2}$$

where s_i is equal to the distance between the proposed event $\mathbf{x}$ and the PMT $\mathbf{x_i}$, and ϕ is the angle between the normal of the sphere and the vector $\mathbf{s_i}$. In reference [1], McCarty estimates the position resolution by assuming a *gaussian* time profile of emission, and finding the second derivative of the likelihood function at the maximum, and obtains

$$\sigma \approx \sqrt{\frac{3}{\mathrm{N}}}\frac{c\tau}{n} \tag{2}$$

where n is the refractive index of the scintillator material, and N is the number of hits detected. For SNO+, this estimate gives a resolution of 5 cm for a 3 MeV event.

To verify this, I wrote a simple Monte Carlo simulation and fit for the position of an event generated at the center of the detector for different N. A plot of the resolution as a function of N is shown in Figure 1.

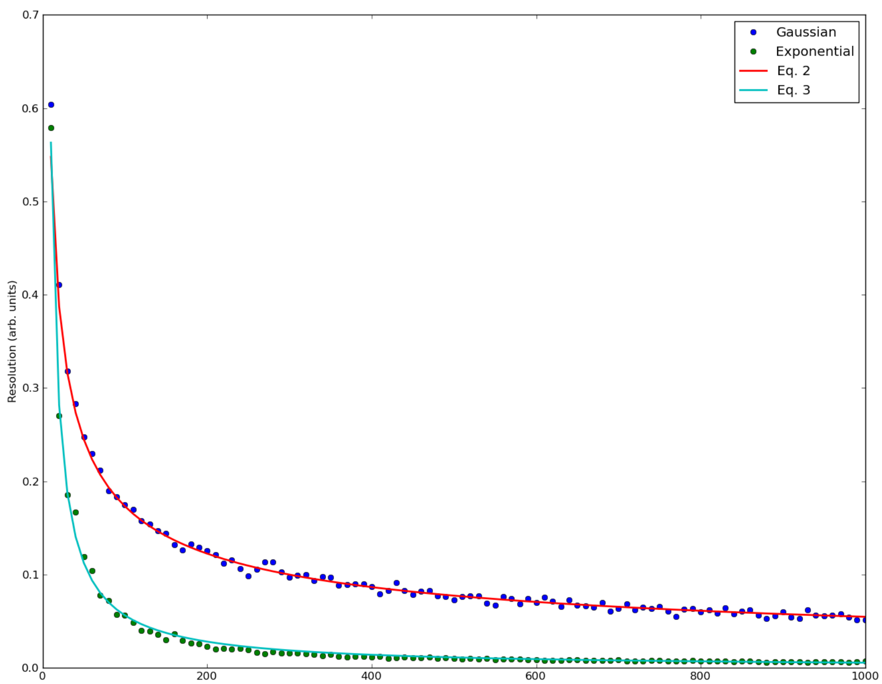

Figure 1: Resolution vs. Hits

The gaussian time profile case is modeled well by the formula in Equation 2, but the resolution for the exponential time profile is much better! The exponential resolution is modeled best by

$$\sigma \approx \frac{5.63}{\mathrm{N}} \frac{c\tau}{n} \tag{3}$$

This would predict a lower limit for position resolution at 0.5 cm for a 3 MeV event. To check that the fitter I wrote was actually fitting events correctly, I made a pull plot, shown in Figure 2. It fits well to a standard normal distribution and so I have confidence that these results are meaningful. In order to understand this difference, I think it's instructive to consider the one dimensional case.

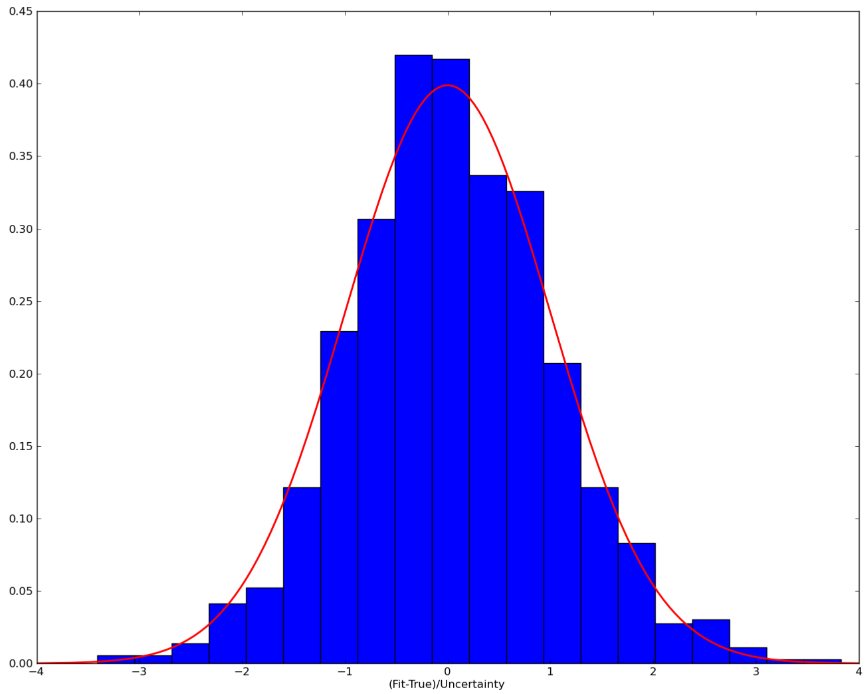

Figure 2: Pull Plot

2.1 1D Position Reconstruction

In the one dimensional case, assume we have a single PMT with perfect timing, and we want to estimate the distance to some scintillation event.

2.1.1 Gaussian Time Profile

If the light is emitted with a Gaussian time distribution, we can estimate the position by maximizing the likelihood

$$\mathscr{L}(x) = \prod_i e^{-\frac{(t_i - x)^2}{2\tau^2}}$$

We can find the maximum by setting the first derivative to 0,

$$\frac{d\mathscr{L}}{dx} = -\left(\sum_i \frac{t_i - x}{\tau^2}\right) \prod_i e^{-\frac{(t-x)^2}{2\tau^2}} = 0$$

The maximum is therefore simply the mean of the times

$$\hat{x} = \frac{1}{\mathrm{N}} \sum_i t_i$$

The resolution at the maximum is found easiest by expanding $\log \mathscr{L}$ in a Taylor series about the maximum.

$$\log \mathscr{L} = -\sum_i \frac{(t_i - x)^2}{2\tau^2}$$

$$\frac{d^2\mathscr{L}(\hat{x})}{dx^2} = -\frac{\mathrm{N}}{\tau^2}$$

Therefore,

$$\log \mathscr{L} \approx \text{const} - \frac{\mathrm{N}}{\tau^2} x^2$$

and

$$\mathscr{L} \approx e^{-\frac{\mathrm{N}}{\tau^2} x^2}$$

from which, the resolution σ is given by

$$\sigma = \frac{\tau}{\sqrt{\mathrm{N}}}$$

2.1.2 Exponential Time Profile

The case of light emitted with an exponential time distribution can be treated in a similar manner. The likelihood function is given by

$$\mathscr{L} = \prod_i e^{-\frac{t_i - x}{\tau}}$$

In this case however, the best estimate for x is given by

$$x = \min t_i$$

The resolution is therefore found from the distribution for the smallest value selected from N exponentially distributed numbers. This distribution is simply an exponential with mean $\frac{\mathrm{N}}{\tau}$ [2].

Therefore, the resolution is given by

$$\sigma = \frac{\tau}{\mathrm{N}}$$

The resolution scales with N and *not* $\sqrt{\mathrm{N}}$ like in the Gaussian case.

References

[1] Kevin McCarty, *The Borexino Nylon Film and the Third Counting Test Facility*. Chapter 5.

[2] Kyle Siegrist, *http://www.math.uah.edu/stat/sample/OrderStatistics2.html*