Magnetometer Calibration
using ellipsoid fitting

1 Hard-iron and soft-iron calibration

Magnetometers are subject to hard-iron and soft-iron perturbations [4].
Hard-iron perturbations are due to permanently magnetized objects whose magnetic field adds itself to the Earth's magnetic field. It shifts the measured magnetic field.
Soft-iron perturbations are due to ferrous non-magnetized objects that distorts the Earth's magnetic field. It gives the measured magnetic field an ellipsoid shape.

Let's note:

- B_m the measured magnetic vector
- B_r the real magnetic vector we desire to measure

and introduce:

- v a (3,1) vector corresponding to the shift due to hard-iron perturbations
- W a (3,3) matrix corresponding to the distortions due to soft-iron perturbations

Then, we have:

$$B_m = WB_r + v \tag{1}$$

and

$$B_r = W^{-1}(B_m - v) \tag{2}$$

From the knowledge of hard-iron and soft-iron perturbations we could retrieve the real Earth's magnetic field the magnetometer should measure. At least its direction, as it is the only information we need from the magnetometer to estimate the heading.

$$\begin{cases} ||B_r||^2 = B^2 \\ ||B_r||^2 = {B_r}^T B_r \end{cases}$$

$$\begin{aligned} {B_r}^T B_r &= (W^{-1}(B_m - v))^T (W^{-1}(B_m - v)) \\ &= (B_m - v)^T \, (W^{-1})^T \, W^{-1} \, (B_m - v) \end{aligned}$$

So

$$(B_m - v)^T \, A_0 \, (B_m - v) = B^2 \tag{3}$$

with $A_0 = (W^{-1})^T \, W^{-1}$

According to [3], equation 3 describes an ellipsoid of parameters $A_e = \frac{1}{B^2} \, A_0$ and v. The equation of this ellipsoid is:

$$(B_m - v)^T \, A_e \, (B_m - v) = 1 \tag{4}$$

where A_e is a real symmetric positive-definite matrix.

Unfortunately, we can't estimate independently A_0 and B, because an infinite number of couples (A_0,B) could represent the same ellipsoid of parameter $A_e = \frac{1}{B^2} \, A_0$. Thus, we can't deduce W either.

However, we are only interested in the direction of the magnetic vector, not its norm.
The ellipsoid can also be represented as

$${A_e}^{-\frac{1}{2}} \, S_{0,1} + v \tag{5}$$

So, in order to calibrate the magnetometer and correct the measurements to fit the unit sphere, we need to estimate A_e and v respecting equation 4.

From these estimates we can calibrate the measurements using:

$$B_{cal} = \hat{M}(B_m - \hat{v}) \tag{6}$$

where $\hat{M} = \hat{A_e}^{\frac{1}{2}}$

M will be called the **soft-iron matrix**
v will be called the **hard-iron offset**

In order to estimate A_e and v, we use a data set of raw measurements. Then, using linear least squares methods, we find the ellipsoid that best fits the data set and deduce its parameters.

2 Ellispoid fitting

Let's consider the ellipsoid described by the equation

$$(x - v)^T \, A_e \, (x - v) = 1$$

where A_e is a real symmetric definite-positive matrix. The goal of this section is to find the parameters $\hat{A}_e$ and $\hat{v}$ of the ellipsoid that best fits a given data set of size N.

An ellipsoid is a type of quadric surface. The equation for a quadric surface is:

$$x^T Ax + B^T x + C = 0 \tag{7}$$

where A is a (n,n) matrix, B is a (n,1) vector and C is a scalar.
In 3 dimensions, the equation 7 for a quadric in cartesian coordinates is

$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j = 0 \tag{8}$$

such as

$$\begin{cases} A = \begin{pmatrix} a & d/2 & f/2 \\ d/2 & b & e/2 \\ f/2 & e/2 & c \end{pmatrix} \\ B = \begin{pmatrix} g & h & i \end{pmatrix}^T \\ C = j \end{cases} \tag{9}$$

Let's note:

$$p = \begin{pmatrix} a & b & c & d & e & f & g & h & i & j \end{pmatrix}^T$$

From our data set we create the matrix:

$$D = \begin{pmatrix} {x_1}^2 & {y_1}^2 & {z_1}^2 & x_1y_1 & y_1z_1 & z_1x_1 & x_1 & y_1 & z_1 & 1 \\ {x_2}^2 & {y_2}^2 & {z_2}^2 & x_2y_2 & y_2z_2 & z_2x_2 & x_2 & y_2 & z_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {x_N}^2 & {y_N}^2 & {z_N}^2 & x_Ny_N & y_Nz_N & z_Nx_N & x_N & y_N & z_N & 1 \end{pmatrix}$$

We want p that minimizes $||D\,p||^2$.

A simple algorithm of minimization will find the trivial solution where $p = 0$, so we must add a constraint to the parameters. For example, we can see that any multiplication by a constant of equation 8 still represents the same quadric surface, so we can set the constraint $||p|| = 1$.

The second issue is that equation 8 represents quadric surfaces, so the solution $\hat{p}$ may represent any type of quadric surfaces, and eventually not an ellipsoid. We must find a constraint on the parameters so the solution represent an ellipsoid.

Let's note

$$P = \begin{pmatrix} a & d/2 & f/2 & g/2 \\ d/2 & b & e/2 & h/2 \\ f/2 & e/2 & c & i/2 \\ g/2 & h/2 & i/2 & j \end{pmatrix}$$

and

$$Q = \begin{pmatrix} a & d/2 & f/2 \\ d/2 & b & e/2 \\ f/2 & e/2 & c \end{pmatrix} = A$$

[1] gives a classification of quadratic surfaces depending on the rank of Q, the signature of Q and the rank of P. From this, we know the constraints on the parameters p, for the quadratic surface to be an ellipsoid:

- $\text{rank}(Q) = 3$
- Signature of Q = (3,0,0) or (0,3,0), ie eigenvalues of Q must be all strictly positives or all strictly negatives.
- $\text{rank}(P) = 4$

Unfortunately, these constraints are cubic or quadric, and make the least squares minimization problem very difficult to solve.

A method of direct least square fitting of ellipsoids is given in [2]. In this paper, the authors give a strategy to use a simple and accessible constraint based on the following necessary condition: any plane intersecting an ellipsoid is an ellipse.

We can use the algorithm described in this article to perform direct least squares ellipsoid fitting.

Thus, we obtain the parameters $\hat{p}$ describing the best ellipsoid fitting our data set. From $\hat{p}$, we deduce $\hat{A}$, $\hat{B}$ and $\hat{C}$.

If the parameters describe an ellipsoid, the determinant of A is non null and the quadric equation can be written as

$$(x - v)^T A_0 (x - v) = k$$

where A_0 is a real symmetric matrix and k a scalar.
We have

$$(x - v)^T A_0 (x - v) = x^T A_0 x - 2v^T A_0 x + v^T A_0 v \tag{10}$$

So

$$x^T A_0 x - 2v^T A_0 x + v^T A_0 v - k = 0 \tag{11}$$

We identify:

$$\begin{cases} A = A_0 \\ B = -2vA_0 \\ C = v^T A_0 v - k \end{cases} \tag{12}$$

From which we deduce:

$$\begin{cases} \hat{A_0} = \hat{A} \\ \hat{v} = -\frac{1}{2}\hat{A_0}^{-1}\hat{B} \\ \hat{k} = v^T \hat{A_0} v - \hat{C} \end{cases} \tag{13}$$

In addition, there exists a unique real symmetric definite-positive matrix noted $\hat{A_e}$ describing the ellipse according to equation 4. So $\hat{A_e} = \frac{1}{\hat{k}}\hat{A_0}$.

Thus, we obtain our ellipsoid parameters from the quadric parameters:

$$\hat{v} = -\frac{1}{2}\hat{A}^{-1}\hat{B} \tag{14}$$

and

$$\hat{A_e} = \frac{1}{v^T \hat{A_0} v - \hat{C}}\hat{A} \tag{15}$$

References

[1] Pauline Rüegg-Reymond. "Classification of Quadratic Surfaces". In: (June 2012). URL: https://lcvmwww.epfl.ch/teaching/projects/classification_quad.pdf.

[2] Ying Xianghua et al. "Direct Least Square Fitting of Ellipsoids". In: *21st International Conference on Pattern Recognition* (Nov. 2012). URL: https://projet.liris.cnrs.fr/imagine/pub/proceedings/ICPR-2012/media/files/0437.pdf.

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