Electrical and Thermal Waves

Last update: 27 Sep 2019 by Dr Francisco Suzuki-Vidal (HoE)

This second year laboratory experiment consists of two independent parts:

Electrical waves: You will investigate how voltage propagates in a medium made of inductors and capacitors, known as a "lumped transmission line".

Thermal waves: You will investigate the conduction of heat in PTFE, a plastic insulator.

The overall aims of the experiments are to demonstrate the physics of wave propagation, heat propagation, and their associated mathematical descriptions (wave and diffusion equations). By performing these experiments you will enhance skills such as real-time data taking, note keeping, and data analysis, e.g. by performing **Fourier analysis**.

The experiments will be conducted over a four week lab cycle with half of the time (4 sessions over two weeks) allocated to each part. Note that your equipment is used by a different group of students (Monday-Tuesday/Thursday-Friday of each week), thus do not rely on finding it exactly as you left it in the previous week. **This is when having a clear and accurate lab book pays out!**

> As you work through the script, boxes like this one will highlight specific tasks/results which should be performed/recorded and questions which should be answered in your lab book. As always in the lab, you should include in your book enough information on how you set up the apparatus and on your experimental techniques. **It should contain enough details so that others could repeat your procedure by following your notes**. This should be done as you work through the experiment. You will find it helpful to refer to the handout "Introduction to the Second Year Laboratory" for detailed advice on this.

You are strongly encouraged to make full use of your demonstrators. They will be more than happy to talk to you and answer your questions, no matter how silly. However, we also expect you to take a **proactive approach into problem solving**, i.e. try working out why something does not work by yourself first! Remember, however, that **time in the lab is limited**, thus avoid getting stuck in one single problem.

Further relevant information is given in the **Supplementary Note** available from the website and the posters on the walls of the lab.

Part 1: Fourier analysis of a square wave

In the very first session and regardless on whether you are starting with the **Electrical Waves** or the **Thermal Waves** experiment, you will perform **Fourier analysis of a periodic, square wave function**. This will help you visualise how an ideal square wave can be expressed as the sum of different Fourier components (harmonics), and their dependence on amplitude and frequency.

A periodic function in time $T(t)$ with period τ can be expressed as a Fourier series, i.e. an infinite sum of *sine* and *cosine* functions:

$$T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n t}{\tau}\right) + b_n \sin\left(\frac{2\pi n t}{\tau}\right) \right] \quad (1)$$

where the coefficients (amplitudes) a_n and b_n are given by:

$$a_n = \frac{2}{\tau} \int_0^{\tau} T(t) \cos\left(\frac{2\pi n t}{\tau}\right) \mathrm{d}t \quad (2)$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} T(t) \sin\left(\frac{2\pi n t}{\tau}\right) \mathrm{d}t \quad (3)$$

Use **Microsoft Excel** to perform Fourier analysis of a square function:

- Plot a positive square function with amplitude $T(t)$=100 and period τ=240. Choose an appropriate temporal step (i.e. this is your resolution).
- Calculate expressions for the coefficients $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ from Eqs. (2) and (3).
- Use the coefficients to plot the Fourier analysis of the square function using Eq. (1) and plot on top of your square function (See Fig. 1). The n=1 component is known as the **fundamental frequency**, why?
- Is your Fourier analysis able to reproduce the square wave? Explain.

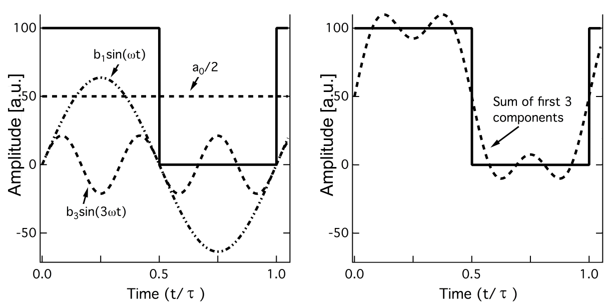

Figure 1: Fourier analysis of a periodic square function

Part 2: Electrical Waves Experiment

§2.1 Introduction and aims

This experiment studies the propagation of electrical signals (square pulses and sine waves) along a *lumped transmission line*, a series of discrete electronic components that serve as a medium for the signals to travel, i.e. a model of a cable. This experiment will make you more familiar with:

- The reflection of waves at interfaces between media of different impedances
- The concept of *dispersion*, phase speed and group speed
- Some understanding of the relationships between distortion and band-width.
- Estimation and propagation of errors.

These phenomena, studied here in the context of electrical signals in transmission lines, have wider application in many fields of physics, e.g. propagation of information in fibre optics.

§2.2 Safety

This experiment involves using a model of an electrical transmission line which has standard low voltage electronic components.

The equipment to be used has been safety tested whilst the mains supply to the equipment is provided with an automatic safety cut-out. Nevertheless, for good safety practice follow basic **common sense**, i.e. **do not open any of the equipment**. If there is any fault, report it to a demonstrator.

§2.3 Equipment

You will use a lumped transmission line consisting of 40 sections as illustrated in Fig. 1.

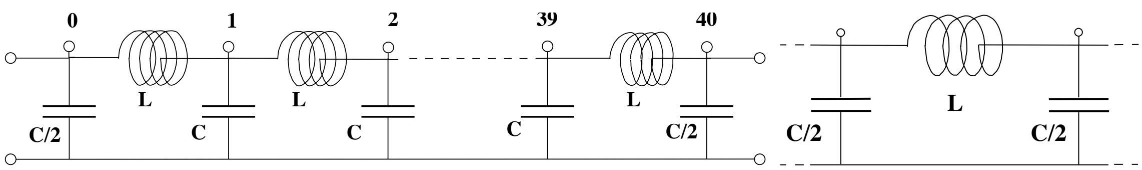

Figure 1: Left: A lumped transmission line. Right: An individual section.

"Lumped" means that L and C occur as individual inductors and capacitors. This is in contrast to a "continuous" line, such as a coaxial cable or a "twisted pair" of conductors where L and C are distributed along the conductors. This line can be thought of as 40 identical single sections in series (Fig. 1, right). Note that our transmission line as realised in hardware that is "folded" into a hairpin configuration in a perspex box. The input terminal is labelled "0" and the output at the end is labelled "40". The signal terminals are coloured red, the green terminals are the common ground line. The pins at positions 0, 1, 2...40 provide points at which the oscilloscope probe can be attached to measure the signals at these intermediate points down the line.

In addition you will use signal generators to drive electrical square and sine waves in the transmission line, a oscilloscope to probe the behaviour of the signals, a digital multimeter and a frequency counter. You will use coaxial cables to transport electrical signals. The exterior cylindrical conductor made of copper braid is usually at 0 volts = "ground", while the central inner wire conductor carries the signal. Hence if by mistake you connect the central inner conductor to any ground line, the signal will disappear (but it is unlikely that any permanent damage will be done)

§2.4 Theory

Electrical signals propagate along a medium (e.g. a transmission line) according to the **wave equation** (see the **Supplementary Note** for a mathematical derivation):

$$\frac{\partial^2 V(\hat{x},t)}{\partial t^2} = v_{phase}^2 \frac{\partial^2 V(\hat{x},t)}{\partial \hat{x}^2} \tag{1}$$

where $V(\hat{x},t)$ is the voltage, v_{phase} is the phase velocity (measured in sections/sec) and $\hat{x}$ refers to the node/section number (analogue to a spatial coordinate), along the transmission line.

A solution for $V(\hat{x},t)$ is oscillatory in space and time with a frequency w and wavenumber $\hat{k}$:

$$V(\hat{x},t) = V_0 \exp\Big(i\ (\hat{k}\hat{x} - \omega t)\Big) \tag{2}$$

where V_0 is a constant dependant on initial and boundary conditions.

Verify that Eq. (2) is a solution to the **wave equation**.

Like Schrödinger's equation and the **diffusion equation** (in the **Thermal Wave** experiment), the **wave equation** is a **linear differential equation**.

§2.4.1 Characteristic impedance

Continuous transmission lines have inductance L, capacitance C and resistance R distributed along their length. L and C are measured in *Henrys* and *Farads* per meter ([H/m] and [F/m]).

If any series R is negligibly small and any leakage R in parallel between the conductors is negligibly high, the **characteristic impedance** is

$$Z_0 = \sqrt{L/C} \tag{3}$$

and the phase speed

$$v_{phase} = 1/\sqrt{LC} \tag{4}$$

Lumped transmission lines are made of separate discrete L and C components, with each measured per section.

§2.4.2 Reflections at changes of impedance

If a signal propagating in a medium A of impedance Z_a meets an interface with medium B where the impedance changes to Z_b the signal will be only partially transmitted from A to B with the rest being reflected back into A (even if both A and B are lossless).

If the size of the signal voltage incident from A onto the interface is V_i and the amplitude of the transmitted signal into B is V_t with V_r reflected from the interface back into A, the voltage reflection coefficient

$$R = V_r/V_i = (Z_b - Z_a)/(Z_b + Z_a) \tag{5}$$

(by the requirements of continuity of voltage and current at the interface).

R is plotted as a function of Z_b/Z_a in Figure 2 and some key values are tabulated below.

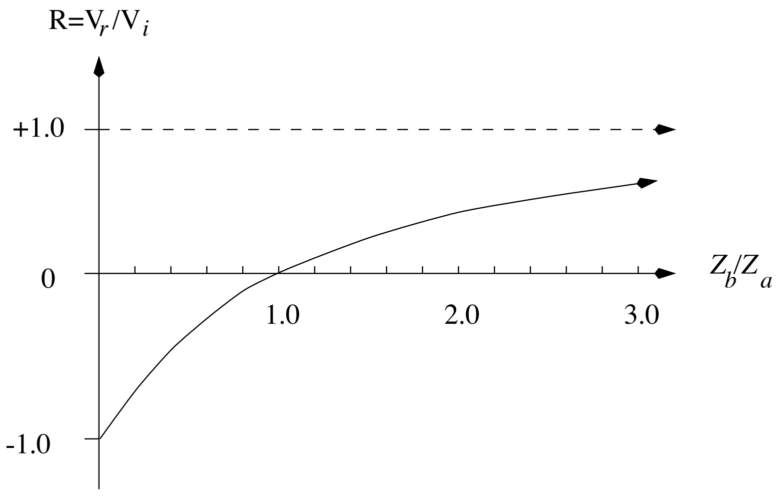

Figure 2: The voltage reflection coefficient V_r/V_i as a function of Z_b/Z_a.

So	if $Z_a = Z_b$	$R = 0$	i.e. there is no reflection
	if $Z_b = 0$	$R = -1$	i.e. complete reflection with inversion
	if $Z_b = \infty$	$R = +1$	i.e. complete reflection without inversion

§2.5 Measurements with square pulses

§2.5.1 Initial investigations

First connect up the pulse generator directly to the oscilloscope so that you are familiar with the pulse generator and scope controls and also the main pulse features, e.g. size, shape, etc. Use the trigger pulse from the *Sync Out* socket of the pulse generator to trigger the scope *externally*. Set the pulse generator to output 4 Volt positive pulses about 100 μs long, with a pulse repetition frequency of 100 Hz.

Then connect the pulses to the input of the transmission line leaving its output end open circuit. Look at the signals at the input, output and several intermediate points along the line.

Sketch in your lab book the gross features of the signals seen at these locations and explain the time delays, several polarity inversions and changes of signal size seen.

Note that signal reflections will take place at sudden impedance changes - such as the open circuit end of the line. This also occurs at the beginning of the line, as seen by pulses sampled in the line. The impedance before the line (the 50 Ω output impedance of the pulse generator) is very different from the **characteristic impedance** of the line itself.

Test your understanding by predicting the pulse polarities you expect to see if you short circuit the far end of the line. Make a note of it in the lab book. Then try and see.

§2.5.2 Measure the characteristic impedance of the line

Now terminate the output end of the line with a variable resistor, which can be varied from about 0 Ω to 1000 Ω. Observe the signal reflected from the end of the line as the variable resistor is adjusted. Make adjustments such that the reflected signal is as near 0 as possible. Then measure the value of the variable resistor which give rise to zero reflection. There may be some uncertainty in judging what adjustment of the resistor gives lowest reflections. Try it more than once. Is your partner's judgement the same as yours?

Hence obtain the characteristic impedance Z_0 of the line together with an estimate of its experimental uncertainty of δZ_0.

The resistance of the resistor can be measured with the digital multimeter supplied. The resistor must be isolated from the source of the pulses when its resistance is being measured.

§2.5.3 Measure the output impedance of the pulse generator

With the end of the transmission line short circuited again, such that pulses propagate back up the line towards the input end. Measure the reflection coefficient of these pulses at the **beginning of the line** where it is connected to the pulse generator.

Hence obtain a value for the output impedance of the pulse generator, and its uncertainty using your measured value of Z_0. A general expression for the voltage reflection coefficient is in §2.4.2.)

When calculating the uncertainty, remember to propagate properly both the uncertainty δZ_0 as well as the experimental uncertainty of your measured voltage reflection coefficient. There are some useful notes on the lab poster **Treatment of experimental errors**

§2.5.4 Measure the speed of propagation of pulses

Use the oscilloscope sweep to measure the time taken for pulses to pass through say 40 or 80 sections of the line.

Record the speed as accurately as you can and its error. The units of this speed are "line sections"/second. Section §2.4.1 gives an expression for this speed in terms of the L/section and C/section of the line. A value for C is indicated on the perspex box which you can assume is known accurately.

Hence determine from this signal speed (and its measurement uncertainty), a value of L per line section and an estimate of the uncertainty δL of L. Section §2.4.1 also gives an expression for Z_0 in terms of L/section and C/section.

From this expression obtain a separate estimate of L and its uncertainty δL. Are your two values of L compatible?

§2.5.5 Pulse distortion

You have seen how the pulse degrades and attenuates as it propagates down the line. The distortion of the pulse can be approximately characterised by the four parameters indicated in Fig. 3.

Make measurements of these four parameters (for the "ringing" oscillation measure its frequency) at two points on the line (e.g. after the input pulse has travelled through say 12 and 24 sections of the line). It is not necessary to make very detailed drawings of the pulses themselves. Just measure the five quantities from the scope trace and tabulate them.

After completing the sine wave measurements in Section §2.6, come back and explain (qualitatively) what determines the values of these parameters in terms of the frequency response of the line.

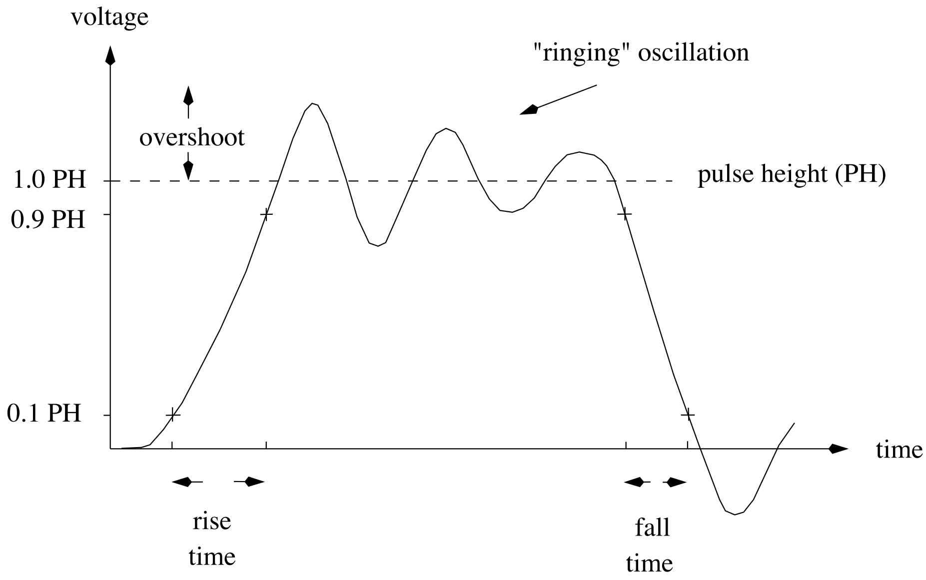

Figure 3: Schematic of pulse distortion.

§2.6 Measurements with sine waves

§2.6.1 Propagation of sinusoidal signals

"Match" the line by terminating it with a resistor which gives no reflections of pulses. e.g. terminate the line with a suitable variable resistor and adjust this as in §2.5.2 so that there are minimal reflections of pulses injected from the pulse generator at the beginning of the line. i.e. the line is terminated by its characteristic impedance so it seems to be infinitely long, and pulses propagate down it but no reflections return.

Matching is easier using pulse signals than using sine wave signals. Overlapping reflections of sine waves are difficult to disentangle but pulses and their reflections are more easily distinguished. Now disconnect the pulse signal generator. With the line still terminated, introduce a 5 kHz input signal from the sine wave generator. Trigger the oscilloscope "externally" with the "sync" signal from the sine wave generator so that the oscilloscope sweep always starts at the same phase of input sine wave.

Now look at the signal displayed at various points along the line. Satisfy yourself that the electrical sine wave signal is indeed propagating down the line from input to output.

§2.6.2 Speed of propagation as a function of signal frequency

You are strongly advised to read this **whole** section and to take some practice measurements before starting to collect all the data for this part. A careful run and analysis will take at least one lab session.

Keep the line matched so that signals propagate from input to output without any returning reflections. The configuration of the oscilloscope and its display could in principle be used to measure the signal speed. But it is more accurate to judge relative phases when two sine wave signals are "in-phase" (relative phase difference = 0 or $2N\pi$ radians) or "out-of-phase" (relative phase = π or $(2N + 1)\pi$), by superimposing the signals on the double beam scope or, even better, by using the scope in the X-Y mode to draw Lissajous figures. Having set the scope into X-Y mode, you connect the sinusoidal signals (you wish to compare phase relations) to the two inputs of the scope. The in or out of phase conditions are achieved by scanning the frequency of the signal and identified on the display of the scope.

Determine the characteristics of the scope display which correspond to the in or out of phase conditions. Can you see them as parametric equations of x and y in terms of time t?

In doing this scan the frequency range from near 0 Hz (where the signals must be nearly in phase) to the maximum frequency at which signals will propagate down the line. For this line at frequencies above about 53 kHz, the output signals become too small to measure.

Use this method, compare the sinusoidal signals at the beginning and the end of the line. Measure <u>ALL</u> the frequencies at which the signals at the input and output ends are exactly in phase or exactly out of phase. Make a note against each frequency whether it is in- or out-of-phase.

For the calculations in this part, it is important to measure the signal frequency to a high accuracy (thus differences between frequencies are also fairly accurate). Sufficient accuracy is not obtainable by reading the dials of the sine wave generator (or measured simply using the time base of the oscilloscope). Use the digital frequency meter supplied which is accurate to better than 1 part in 10000.

Also at frequency interval of about every 5 to 10 kHz, measure the voltage amplitudes of the input and output signals and hence calculate the "transmission ratio" of the line = output/input as a function of frequency.

Note: This presents an exercise in co-operation and organisation between partners. After a little practice it should be possible to make the measurements at the 40 or so frequencies required in a little over an hour. Ideally one partner should be calculating some check, such as the frequency difference between successive measurements, as the measurements are made, to catch any errors.)

A sinusoidal wave is frequently expressed as $\phi(x,t) = \phi_0 \exp[i(k \cdot x - \omega t)]$. Identify which part of the expression is "phase". Write down the definition of phase speed and show that $V_{phase} = \omega/k$. Write down the expression giving the phase difference between the end and the beginning of the transmission line.

A little thought should convince you that at the particular frequencies described above, when the phase difference between input and output signals is an even or odd integral number times

π, the propagation delay time along the line is an integral number of half signal periods, or equivalently that the length of the line is an integral number of half wavelengths of the electrical wave, with the integer being 0 at DC, 1 at the lowest (non-DC) frequency measured (around 2 kHz) and increasing by 1 for each higher frequency in turn (so long as you don't miss any out!).

Show that the frequency interval $\Delta\omega$ between two successive measurements in the procedure outlined above corresponds to a Δk which is constant $= 2\pi/80$ (for a line with 40 sections).

Thus the average group velocity in this frequency interval between the nth and $(n+1)$th measurement is given by

$$V_{group} = \Delta\omega/\Delta k = (\omega_{n+1} - \omega_n) \times 80/2\pi \quad \text{sections per second.} \tag{6}$$

Use your results to plot graphs of 1) transmission factor versus frequency, 2) V_{phase} and V_{group} as a function of frequency, both on the same graph, and 3) a "dispersion diagram" of ω versus k for your measurements.

Study these graphs and explain qualitatively the principal features of each. Compare the measured speeds with the speed of square pulses in the appropriate frequency range.

Calculate a theoretical "cut off frequency" at which the transmission factor tends to 0 from your earlier measurements of L and the given C.

Part 3: Thermal Waves Experiment

§3.1 Introduction and aims

In this experiment you will investigate the propagation, by heat conduction, of a *temperature wave* through a solid. The propagation is analysed in the time domain in terms of the Fourier components of an initial square temperature wave.

As in the **Electrical Waves** experiment, the propagation of these components is frequency dependent. The frequencies of the temperature wave are about 0.01 Hz instead of the 10 kHz of the voltage wave in the *lumped transmission line*.

Through this experiment you will learn about:

- Quantitative analysis of heat conduction
- The linear differential equation for heat conduction (the diffusion equation)
- Quantitative Fourier analysis
- Propagation of errors
- Real-time data acquisition and analysis

The "final result" of this experiment is a measurement of the thermal diffusivity D of a material (called PTFE) with its experimental error.

The main heat transfer mechanism in solid materials is by conduction. The thermal diffusivity D is a constant for the material and is related to the thermal conduction coefficient, density and specific heat of the material. It is a measure of the solid material's ability to transmit temperature waves through heat conduction.

This is discussed in detail in the Supplementary Note and wall poster "The propagation of thermal waves by conduction".

To measure D, we establish a regular temperature fluctuation at one point in a solid (the outside surface of a cylinder), and measure the variation of temperature at another point (the centre of the cylinder).

The solid in this case is a cylinder of PTFE (poly-tetra-fluoro-ethylene). A temperature wave is launched at the outer surface of this cylinder by dipping it alternately in boiling water and melting ice. Equal times (of the order of minutes) are spent in the hot and cold baths. Thus a "square" (equal times hot and cold) temperature wave going from 0°C to 100°C is established at the outer surface.

The parts of the cylinder inside the surface take some time to heat and cool. A wave of temperature variation propagates radially inwards in the cylinder. A concentric central hole in the cylinder contains a small electronic thermometer which has a low heat capacity, thus responding rapidly to the temperature on the walls of the central hole.

This thermometer is connected to a computer so that the inner wall temperature can be measured frequently during the cyclic temperature changes. A computer program **Measurement** plots the variation of temperature of the inner wall on the screen and captures the data digitally. The value of D is obtained by analysis of the shape of this inner temperature variation.

§3.2 Safety

This experiment involves using **boiling water** as well as **electrical apparatus** (computers and heaters). **Only the probe should be in contact with water**.

The hot glass beaker will easily crack if it comes into contact with cold water on the outside. Take care **not to drip ice-cold water onto the outside of the flask when you transfer the probe.**

Remember that another group is working close next to you. Be aware of hazards from their experiment and avoid your experiment creating a hazard for them.

§3.3 Equipment

A cross section of the PTFE cylinder is shown in Figure 1. Note the position of the electronic thermometer. To ensure good thermal contact with the walls of the hole, it is permanently fixed in. To ensure that the heat flow is radial (as is assumed in the analysis), make sure that the boiling water and the melting ice reach up well above the level of the thermometer, but not so far up in the hot flask that water splashes onto the flask heater.

Note that there are a number of issues that may affect the ideal behaviour of the temperature bath, particularly the cold one. Water's density is largest at 4°C. In a water ice mixture, the temperature of the liquid may be higher at the bottom if there is insufficient heat exchange through movement of water. When a hot cylinder is introduced to the cold bath, the water in the vicinity is heated by the hot cylinder. If there is no effective flow of water, then the water temperature in the vicinity of the cylinder rises above 0°C. Thus the average temperature in the central hole, which would be expected to be 50°C, can be few °C higher. To reduce this effect while thermal waves are being recorded, use the stirrer in the ice-water beaker to remove this layer of warmer water, and hence make sure the outer surface of the cylinder is close to 0°C.

In the hot flask, the boiling water (lower viscosity) is thoroughly mixed by the rising bubbles, so no corresponding layer of cooler water builds up round the cylinder

You should include a drawing of the important features of the apparatus you use in your lab book, and refer to it in your summary.

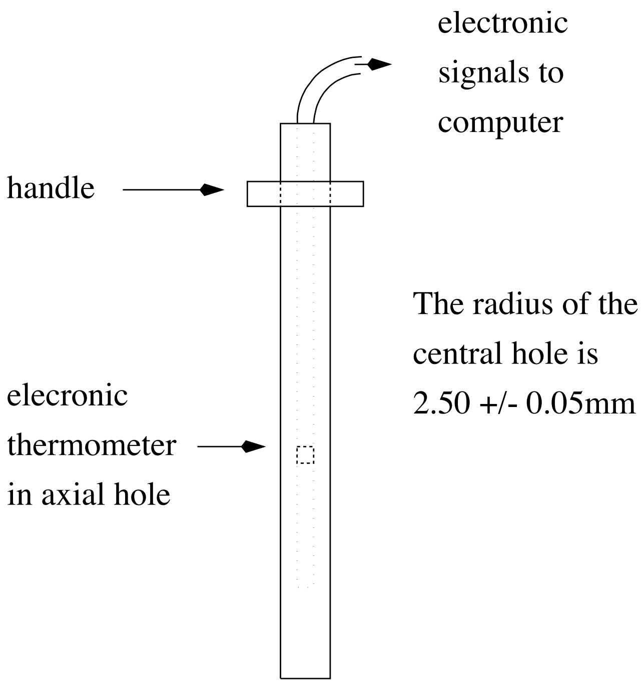

Figure 1: A cross-section of the PTFE cylinder showing the internal thermometer.

The computer next to these apparatus is used to assist you in the acquisition of the data, extraction of useful parameters from the acquired data and optionally calculating the value of D based on the more sophisticated **cylindrical model** using Bessel functions (see the **Supplementary Note** for more details).

§3.4 Theory

Temperature as a function of position and time $T(x,t)$ propagates in a medium with **thermal diffusion** D according to the **diffusion equation**, which in 1-dimension is given by (see the **Supplementary Note** for a mathematical derivation):

$$\frac{\partial T(x,t)}{\partial t} = D\frac{\partial^2 T(x,t)}{\partial x^2} \tag{1}$$

A solution for $T(x,t)$ decays in space and oscillates in time, with a frequency ω:

$$T(x,t) = C\exp\left(-\sqrt{\omega/2D}\cdot x\right)\sin\left(\sqrt{\omega/2D}\cdot x - \omega t\right) \tag{2}$$

where C is a constant dependant on initial and boundary conditions.

Verify that Eq. (2) is a solution to the diffusion equation (Eq. (1)).

Like Schrödinger's equation and the **wave equation** (in the **Electrical Wave** experiment), the **diffusion equation** is a **linear differential equation**.

§3.4.1 Simplified "Plane slab" model

In the plane slab model, the cylinder is crudely treated as a 1-dimensional slab (x is in the direction of the radius, pointing in) and consider a sinusoidal temperature wave of amplitude C, frequency ω and phase lag 0 at the outer surface $x = 0$. At depth x (= the difference between the outer and inner radii, i.e. on the wall of the central hole) the temperature wave has a smaller amplitude $C \exp(-\sqrt{\omega/2D} \cdot x)$ and a phase lag of $\sqrt{\omega/2D} \cdot x$ radian. **If either the attenuation or phase shift is measured for a known x and ω, then D can be found**. This is the principle of the analysis.

Note that this 1-dimensional model can give only an approximate answer, since a semi-infinite slab is not a very good approximation to a cylinder. The necessary mathematics for the proper cylindrical geometry are covered in the **Supplementary Note** and on the poster "An introduction to Bessel functions".

In practice, the heating and cooling method we use does **not** launch a sinusoidally varying temperature wave at the outer surface of the cylinder. It would be difficult to do this accurately. Recall from Part 1 that a periodic function such as our square temperature wave can be analysed into its Fourier components.

The wave propagates as if each Fourier component propagates independently due to orthogonality between Fourier components and the linear nature of the diffusion equation. Note that the attenuation factor of the amplitude of the components of different frequencies $C \exp(-\sqrt{\omega/2D} \cdot x)$ depends on the frequency ω, thus **higher harmonics with higher frequencies are more attenuated**.

If we work at a frequency which is high enough then effectively all the frequency components except the fundamental will be attenuated to "negligible" values. In this case the temperature wave registered by the thermometer will be sinusoidal. The transmission factor of a sinusoidal wave at the **fundamental frequency** can be measured by the ratio of the amplitude of the temperature oscillations at the thermometer to the amplitude of the fundamental contained in the square wave at the outer surface. Note that from the **Fourier analysis of a square wave** in Part 1, the amplitude of the fundamental at the outer surface is 63.7°C, not 50°C.

If you make measurements with **long temperature wave periods** (i.e. short ω) near 8 minutes or longer then you will see that the shape of the internal temperature waveform **is not sinusoidal**. Thus harmonic components other than the fundamental is seen at the centre. In this case you do not expect the calculated values of D to be accurate, independent of any plane slab - cylinder question. But it is not obvious by inspection whether this is true for periods much shorter (higher ω) than 8 minutes. Fourier analysis of your digitally recorded data can overcome problem associated with presence of multiple harmonics.

Moreover the amplitude of the temperature wave measured by the thermometer becomes small at shorter periods. This becomes difficult to measure accurately, as does the phase shift. So there is a compromise to be struck here: ideally your measurements should be done over a range of periods.

The resulting values of D as evaluated by any one analysis method (plane slab or Bessel function; attenuation or phase shift) being consistent over a range of fairly short periods may demonstrate that the effects of harmonics are negligible.

Explain why the measured thermal wave at the centre is sinusoidal when the input at the surface input at the surface is a square wave.

§3.4.2 Plane slab transmission factor calculations

At a frequency ω, the thermal wave transmission factor is written as:

$$\begin{aligned}\text{Transmission factor} &= \frac{\text{measured temperature amplitude inside}}{\text{Fourier fundamental frequency temperature amplitude outside}} \\ &= \exp(-\sqrt{\omega/2D}\cdot x) \qquad (3)\end{aligned}$$

then
$$D = \frac{\omega x^2}{2\cdot[\ln(\text{transmission factor})]^2} \qquad (5)$$

where x = the thickness of the PTFE cylinder = difference between the cylinder outer radius and the radius of the central hole.

§3.4.3 Plane slab phase calculation

The phase lag $\Delta\phi$ of this sinusoidal wave can also be measured relative to the known phase of the fundamental component at the outer surface (see the poster on Fourier analysis to make sure you measure this phase lag correctly). If the temperature wave inside is $\Delta\phi$ radian behind the fundamental at the outside then

$$D = \frac{\omega x^2}{2(\Delta\phi)^2} \qquad (6)$$

giving a second independent measurements of D.

§3.5 Measurement of the thermal diffusivity D

If this is your first lab session on this experiment, to save time it is suggested that you:

1. put water in the hot flask and set it to boil. Turn the thermostat full up, then when it is boiling turn down to about 5 or 6 to keep it on the boil, but slowly, so that you do not have to replenish the water too often.

   Beware! If water splashes down onto the electrical heating element for the hot flask, safety trips switch off all the computers! (not just yours)

2. put a mixture of ice and water in the cold beaker. Ice is available in the brown cabinet on the bench at the window side.

3. put the plastic cylinder into the central well in the cold beaker and leave it to cool to 0°C ready for thermometer calibration.

Expect to wait at least 15 minutes for the cooling to complete.

§3.5.1 Calibration of the electronic thermometer

The electronic thermometer (transducer) needs to be calibrated. It contains an electronic component which regulate the current flowing through itself with a temperature coefficient of 1μA/K. The computer measures the voltage developed when this current flows through a resistor of approximately 10kΩ. Thus the computer read voltage is a linear response to the temperature. It has been tested that it is sufficiently linear between 0°C and 100°C for the needs of this experiment. The thermometer is permanently fixed in the central hole of the plastic. You will need to decide the gradient and intercept in the linear relation between the actual temperature and voltage read by computer. Calibration temperatures of 0°C and 100°C can be conveniently achieved by just leaving the the PTFE cylinder immersed in the ice or boiling water until the thermometer response is stable. Frequent stirring is recommended during the 0°C calibration.

Run the program **Calibration** to display the thermometer response on the screen. After around 25 minutes the inner temperature reading should become stable to an accuracy of typically ±5 parts in 10000. The calibration can be visualised as in Figure 2.

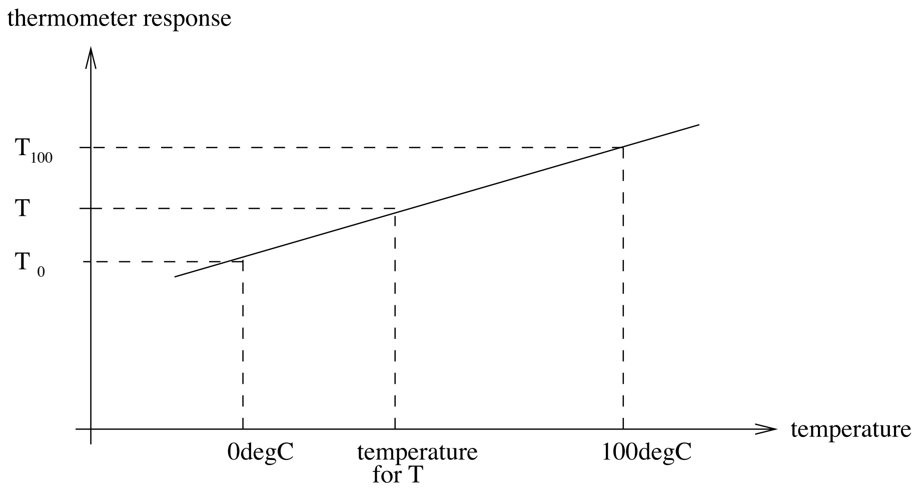

Figure 2: Electronic thermometer calibration.

Then the temperature corresponding to a measured voltage T is

$$\text{temperature} = (T - T_0) \times 100/(T_{100} - T_0) \tag{7}$$

Record your measured calibration constants T_0 and T_{100} in your lab book together with your best estimates of their uncertainty.

When running the program **Measurement** you will be asked to type in these two values. If the thermometer is working properly the calibration should yield stable values. However it is probably worthwhile rechecking the value of T_0 or T_{100} when you have 15 minutes or so to spare.

§3.5.2 Acquisition of thermal wave data

The computer program **Measurement** is used to acquire all the data digitally. Start the program on the computer and make sure the 'Show Context Help' under the 'Help' menu is selected (having a tick just before it). Run the cursor over items on the front panel to familiarise yourself with the functionality of each of the controls and displays. Once you are familiar with the controls, uncheck the 'Show Context Help' option. Your are discouraged from running other programs (e.g. the web browser) while you are acquiring data. This is to prevent interruption of real time acquisition of data by the relevant programs.

Make sure you enter the T_0 and T_{100} obtained from the calibration procedure in the relevant boxes and start the operation of the program by clicking the ⇒ at the top left of the control panel. The computer attempts to regulate the switching between the hot and cold baths **through your action**. Your will hear audible tones which warn you of impending (<5 sec) swap (500 Hz), and the moment of swap (1000 Hz). It is up to you to follow the directions of the computer. Keep in mind that the hot/cold status should also match which bath the PTFE cylinder is immersed in. You may change any of the parameters/controls at any time (except when they are disabled), and they take immediate effect.

§3.5.3 Eliminating the effect of temperature "transients"

During the temperature cycling, the average temperature at the centre of the cylinder is expected to be 50°C. If the cylinder was at room temperature (or 100°C just after calibration) immediately prior to initiation of the temperature wave at the outside surface, the heat needed to warm (or cool) it to about 50°Cwill take some minutes or so to be conducted to the centre of the cylinder. This will give rise to initial temperature transients which partially obscure the sinusoidal temperature wave which is to be measured at the centre of the cylinder.

One way of avoiding this transient is by cycling the outer surface temperature for some 5 or 6 cycles as directed by the computer before you start to acquire the inner temperature wave. You will be able to monitor the temperature history display on the computer. Once your are satisfied that the initial transient has died down, you can initiate data collection by clicking the 'Take Data' button. Acquisition of data actually commences at the next cold to hot transition and continues for the next four cycles. While data acquisition is in progress, the colour of temperature display is changed to green.

When data acquisition is complete, you should click the 'Save Data' button and try to save the data in your own home directory (H: drive) with file extension '.xls'. The 'Trash Data' button deletes the data currently stored in the computer's volatile memory.

You (or your partner) should consider continuous cycling the PTFE cylinders between the hot/cold bath while the data is saved. It will reduce the temperature transient when you change the period for the next data point.

§3.5.4 Initial suggestion for data collection

During the first lab session, a reasonable aim is to finish off the thermometer calibration and then do a test data run with **Measurement** for a period 2 minutes, aiming to gain experience of making these measurements so that the future success rate is high.

During the second session you should aim at optimising your data acquisition for a period of 2 minutes and perform a first quick calculation of D using the "Physical" method in §3.6.2. Following this, you should record your best, final data for 2 minutes and perform Fourier analysis on these data to compare your values of D with different methods.

In the following sessions, you should aim at taking data with longer periods, say, 4 and 8 minutes. This would cover a range of frequency for the calculation of D and reveal any apparently variations due to longer periods.

Some of the approximations of the model are valid only at short periods (high frequencies ω). Experimentation becomes more difficult as the period decreases because the amplitude of the observed thermal wave falls and transmission factors and phase lags become more uncertain to measure.

You should also attempt to obtain data with a period of around 1 minute (= 30 seconds hot + 30 seconds cold, or perhaps 40 + 40). There is nothing magic in this choice of periods. You should feel free to choose your own, but bear in mind the criteria for your choice. You may wish to take measurements at more than 1 short period (the approximations become valid with apparatus of this size with periods of 1 or 2 minutes or less), and also some longer periods which give lower statistical errors, but perhaps larger systematic errors.

It is easy to fall into the trap of forgetting to keep notes in your lab book during this part. You should produce graphs with appropriate scale / grid / tick marks from your data using Excel or other software, and paste them in your lab book. Keep a record of which runs you have taken, any problems encountered, etc. These graphs will be used later in the analysis.

Time permitting, you may want to refine your estimates of D using "Bessel function" analysis of your data (see the Supplementary Note). These are not compulsory. The Bessel function calculations use the same experimental data, take an hour or so, and need access to the computers in the lab (unless you have written your own code. Fun!).

§3.6 Analysis

§3.6.1 Extraction of parameters from digital data

From the discussion on previous section, you need to extract the amplitude and phase lag of the measured temperature variation at the centre of the cylinder compare them with that at the surface. This section outlines two methods you need to put into practice.

§3.6.2 "Physical" method

Measure the amplitude and phase lag from plotted data, e.g. measure with a ruler from a printout, on screen (e.g. Excel), or any other method you consider valid (time permitting obviously!).

> Obtain values of transmission factor and phase lag of the measured temperature variation at the centre of the cylinder from all your data. Evaluate the values of D and uncertainties. Pay particular attention of the phase of the input thermal wave at the surface as this is the reference from which you measure the phase lag.

§3.6.3 Fourier analysis of your result

In the first analysis step, it is assumed that the effect of higher harmonics can be ignored and Fourier analysis was performed on the input square wave to obtain the amplitude of the fundamental of 63.7°C.

The effect of higher harmonic is evident when the period is long as the measured result becomes non-sinusoidal. You probably noticed this among your results already. In the time domain, the measured periodic temperature changes at the centre of the cylinder can be expressed as a Fourier series (Eq. (8)). This can also be expressed as a series (Eq. (9) below) of terms similar to the solution outlined in section §3.4 (Eq. (2)):

$$T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n t}{\tau}\right) + b_n \sin\left(\frac{2\pi n t}{\tau}\right)\right] \tag{8}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\beta_n \sin\left(\frac{2\pi n t}{\tau} - \Delta\phi_n\right) \tag{9}$$

where

$$a_n = \frac{2}{\tau}\int_0^{\tau} T(t)\cos\left(\frac{2\pi n t}{\tau}\right)\mathrm{d}t \tag{10}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} T(t)\sin\left(\frac{2\pi n t}{\tau}\right)\mathrm{d}t \tag{11}$$

where τ is the period, β_n and $\Delta\phi_n$ are the amplitude and phase lag relevant to the harmonics.

The harmonics are **orthogonal** functions and Fourier analysis can be performed on the measured results to extract amplitude and phase lag at the fundamental angular frequency $\omega = 2\pi/\tau$ and higher harmonics 3ω, 5ω. This method of analysis overcomes the problem of presence of multiple harmonics.

Show that the amplitude β_n and phase lag $\Delta\phi_n$ are related to the standard Fourier series coefficient a_n and b_n by:

$$\beta_n = \sqrt{a_n^2 + b_n^2} \tag{12}$$

$$\Delta\phi_n = -\arctan\left(a_n/b_n\right) \tag{13}$$

Perform Fourier analysis on your data (Excel recommended) with the longest period and one around 2 minutes. Find the coefficients a_0, a_1, b_1 and hence calculate β_1 and ϕ_1. If you are uncertain on how to perform numerical integration using, e.g. Excel, seek assistance from a demonstrator. Bear in mind the size of the time steps you used.

Calculate the transmission factor, phase lag and values of D using the plane slab model. Compare your answer with your results obtained using the physical method. What are the impact of higher harmonics on the value of D you obtained?

§4 Extracting the final, 'best' value of D from the results

Having completed the calculations there will be two different sets of values for D from transmission factor and also phase lag. You should also have Fourier analysis results from data at certain periods (Note: you may also have additional data from the 'Bessel function' analysis, again independently from transmission ratio and phase lag). You will also have estimates of the experimental uncertainties of these separate results.

The results from the plane slab model are expected to contain systematic errors because of the difference between the plane slab and the physical shape of the thermal probe (in fact, the systematic error in the phase lag plane slab results is quite small). However there should be a smooth variation in the values of D with period. It is expected that for the longer period waves they will be further in error because of higher harmonics. Because of the known systematic error involved in the plane slab physical model, do not be surprised if the transmission factor and phase lag results differ from one another even at short periods.

It is hoped that the optional Bessel results, coupled with transmission factor & phase lag extracted using Fourier analysis, will be unbiased because the physical model used is realistic. So we expect that the transmission factor and phase lag results agree with one another provided appropriate Fourier analysed data is used.

Closer agreement between results from transmission factor and phase lag (from plane-slab model) at shorter period suggest the systematic error is period dependent. Of course the errors need to be considered when judging compatibility of different values of D, and the final overall result, with the smallest error, does not necessarily come from the shortest period alone.

Produce a graph of all your separate results of D as a function of period. Depending on what you find, you should report a single experimental value of D based on this graph and analysis and discussions above.

Report for assessment

At the end of the four week experiment cycle you are required to write a 4 page report. For details on the requirements of the report, please refer to the documentation available on the 2nd year lab website.

The content of the report will be discussed with your **assigned demonstrator**, and **you will need to choose** between the following options:

- A report on the Electrical Waves part only
- A report on the Thermal Waves part only
- A report on both Electrical and Thermal Waves parts

Once you agree on the subject of your report with your assigned demonstrator, **it is not possible to change it**, thus do a careful assessment of which experiment you feel more confident to write about (**having a good lab book should be very helpful for this purpose!**)

The same page limit applies if you decide to write about both parts of the experiment, thus you will need to **be very concise**.

RISK ASSESSMENT AND STANDARD OPERATING PROCEDURE

1. PERSON(S) CARRYING OUT THIS ASSESSMENT – This assessment has been carried out by the head of experiment.	
Name (Head of Experiment)	Francisco Suzuki-Vidal
Date	25 September 2019

2. PROJECT DETAILS.						
Project Name	Electrical and Thermal Waves				Experiment Code	W
Brief Description Of Project Outline	Wave propagation experiment in the 2nd year undergraduate teaching laboratory					
Location	Campus	South Ken	Building	Blackett	Room	415B (inside 415)

3. HAZARD SUMMARY – Think carefully about all aspects of the experiment and what the work could entail. Write down any potential hazards you can think of under each section – this will aid you in the next section. If a hazard does not apply then leave blank.			
Manual Handling		Electrical	X
Mechanical		Hazardous Substances	Boiling water
Lasers		Noise	
Extreme Temperature		Pressure/Steam	
Trip Hazards	X	Working At Height	
Falling Objects		Accessibility	X
Other			

4. CONTROLS – List the multiple procedures which may be carried out during the experiment along with the controls/ precautions that you will use to minimise any risks. Remember to take into consideration who may be harmed and how – other people such as students, support staff, cleaners etc will be walking past the experimental setup even when you aren't around.	
Brief description of the procedure and the associated hazards	**Controls to reduce the risk as much as possible**
Accessibility	All bags, coats, jumpers, etc. to be placed away from aisles, doors and walkways to have have clear evacuation paths.
Electrical	No adjustment of electrical, mechanical or other parts of the experiments by anyone other than designated technicians, e.g. Mr Graham Axtell
Hazardous substances	Boiling water: no handling by students. Sturdy shoes to be worn for the 'Thermal Waves' part of the experiment. Students not wearing suitable footwear will be asked to change or remove themselves from the likelihood of harm.
Trip hazards	No cables are to be left on the floor of the lab

5. EMERGENCY ACTIONS – What to do in case of an emergency, for example, chemical spillages, pressure build up in a system, overheating in a system etc. Think ahead about what should be done in the worst case scenario.
In event of scalds or burns seek assistance from First Aiders - Mr. G. Axtel [Rm 406] is qualified FAW. All present must be aware of the available escape routes and follow instructions in the event of an evacuation.