Tang Lin
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Problem: Please classify all groups which consists of 18 elements.
Solve:

$$18 = 2 \cdot 3^2$$

From The Third Theorem of Syllow, we have:

$$N_2 \equiv 1(mod\ 2) \quad N_2|9 \Rightarrow N_2 = 1, 3, 9$$
$$N_3 \equiv 1(mod\ 3) \quad N_3|2 \Rightarrow N_3 = 1$$

So we have 2 cases:

1. $N_2 = 1, N_3 = 1$ In this situation,there is only one 2-Syllow subgroup and one 3-Syllow subgroup, so $G \cong S_2 \oplus S_3$, where $|S_2| = 2, |S_3| = 9$.
   There will be 2 situations:
   (a)
   $$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{18}$$
   (b)
   $$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_6$$

2. $N_2 \neq 1, N_3 = 1$
   $S_3 \lhd G$
   So $G \cong S_3 \leftthreetimes K$,where $|K| = 2$.

   (a) $S_3 \cong \mathbb{Z}_9$ We can get automorphism of $S_3 \cong \mathbb{Z}_9$: $\mathrm{Aut}(\mathbb{Z}_9) = H \cong \mathrm{Aut}(S_3)$,where $|H| = 6$.
   So We can make a table of automorphism,we assume $S_3 = \langle a \rangle$:

ε	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
σ_1	e	a^2	a^4	a^6	a^8	a	a^3	a^5	a^7
σ_2	e	a^4	a^8	a^3	a^7	a^2	a^6	a	a^5
σ_3	e	a^5	a	a^6	a^2	a^7	a^3	a^8	a^4
σ_4	e	a^7	a^5	a^3	a	a^8	a^6	a^4	a^2
σ_5	e	a^8	a^7	a^6	a^5	a^4	a^3	a^2	a

   Where $\varepsilon, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are elements of $\mathrm{Aut}(S_3)$.
   Without difficut we can get rank of each map of isomorphism:

$$O(\sigma_1) = 6, O(\sigma_2) = 3, O(\sigma_3) = 6, O(\sigma_4) = 3, O(\sigma_5) = 2$$

$\because$There is map with rank 6, so $\text{Aut}(S_3) \cong \mathbb{Z}_6$.
Now we assume $K = \langle b \rangle, |K| = 2$, if we can find a homormorphism $\varphi : K \to \text{Aut}(S_3)$, then we can establish a group.
$\because$ In subgroup K, there are only elemnts with rank 2 and zero.
So we can construct a map:

$$\begin{aligned}\varphi : K &\to \text{Aut}(S_3)\\ e &\longmapsto \varepsilon\\ b &\longmapsto \sigma_5\end{aligned}$$

So we can get a group with form:

$$G \cong \langle a, b | a^9 = b^2 = e, bab = a^{-1} \rangle \cong D_9$$

(b) $S_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\because |S_3| = 9 = 3^2, \therefore S_3$ is Abelian group. We can write S_3 in form

$$S_3 = \langle a, b | a^3 = b^3 = 1 \rangle$$

$\because S_3 \lhd G, \therefore cS_3c = S_3$,where $c \in K = \langle c \rangle, |K| = 2$.
So $\exists i_1, j_1, i_2, j_2 \in \{0, 1, 2\}$,and

$$cac = a^{i_1} b^{j_1}$$

$$cbc = a^{i_2} b^{j_2}$$

Therefore

$$\begin{aligned}cac &= a^{i_1} b^{j_1}\\ ccacc &= ca^{i_1} b^{j_1} c\\ a &= ca^{i_1} ccb^{j_1} c\\ a &= (cac)^{i_1} (cbc)^{j_1}, \text{ becasue } ca^n c = (cac)(cac)\cdots(cac) = (cac)^n\\ a &= (a^{i_1} b^{j_1})^{i_1} (a^{i_2} b^{j_2})^{j_1}\end{aligned}$$

$\because S_3$ is Abelian group, so

$$a = a^{i_1^2 + i_2 j_1} b^{i_1 j_1 + j_2 j_1}$$

Analogously, we can get

$$\begin{aligned} cbc &= a^{i_2}b^{j_2} \\ ccbcc &= ca^{i_2}b^{j_2}c \\ b &= ca^{i_2}ccb^{j_2}c \\ b &= (cac)^{i_2}(cbc)^{j_2} \\ b &= (a^{i_1}b^{j_1})^{i_2}(a^{i_2}b^{j_2})^{j_2} \\ b &= a^{i_1i_2+i_2j_2}b^{i_2j_1+j_2^2} \end{aligned}$$

Becase in S_3, $\langle a\rangle \cup \langle b\rangle = \emptyset$, so we can get a congruence equations system:

$$\begin{cases} i_1^2 + i_2j_1 \equiv 1(\text{mod } 3) \\ i_1j_1 + j_2j_1 \equiv 0(\text{mod } 3) \\ i_1i_2 + i_2j_2 \equiv 0(\text{mod } 3) \\ i_2j_1 + j_2^2 \equiv 1(\text{mod } 3) \end{cases}$$

To solve this equation, we can respectively let $i_1 \equiv 0, 1, 2(\text{mod } 3)$, and discuss them, but it is little complex.

i. $i_1 \equiv 0(\text{mod } 3)$

We can simplify this congruence equation system to:

$$\begin{cases} i_2j_1 \equiv 1(\text{mod } 3) \\ j_2j_1 \equiv 0(\text{mod } 3) \\ i_2j_2 \equiv 0(\text{mod } 3) \\ i_2j_1 + j_2^2 \equiv 1(\text{mod } 3) \end{cases}$$

It is not difficut to konw that $\jmath_2 \equiv 0(\text{mod } 3)$ and $j_1 \not\equiv 0(\text{mod } 3)$, and we can more simplify it to just one equation $i_2j_1 \equiv 1(\text{mod } 3)$

And now we can get the first two solutions $(i_1, j_1, i_2, j_2) = (0, 1, 1, 0)\text{or}(0, 2, 2, 0)$.

ii. $i_1 \equiv 1(\text{mod } 3)$

We can simplify this congruence equation system to:

$$\begin{cases} i_2j_1 \equiv 0(\text{mod } 3) \\ j_1(j_2 + 1) \equiv 0(\text{mod } 3) \\ i_2(j_2 + 1) \equiv 0(\text{mod } 3) \\ i_2j_1 + j_2^2 \equiv 1(\text{mod } 3) \end{cases}$$

If $j_1 \equiv 0(\text{mod } 3)$, then we can get system of two equations, $i_2(j_2 + 1) \equiv 0(\text{mod } 3)$ and $i_2j_1 + j_2^2 \equiv 1(\text{mod } 3)$, so in this case we can get answers will have form $(i_1, j_1, i_2, j_2) = (1, 0, *, *)$, if $i_2 \equiv 0(\text{mod } 3)$, we will get answers $(1, 0, 0, 1)$ and $(1, 0, 0, 2)$, and if $i_2 \equiv 1(\mathrm{mod}\ 3)$, we will get answer $(1,0,1,2)$, if $i_2 \equiv 2(\mathrm{mod}\ 3)$, get $(1,0,2,2)$.

If $j_1 \not\equiv 0(\mathrm{mod}\ 3)$, and must $i_2 \equiv 0(\mathrm{mod}\ 3)$, we can get system with 2 equations $j_1(j_2+1) \equiv (\mathrm{mod}\ 3)$ and $j_2^2 \equiv 1(\mathrm{mod}\ 3)$, but if $j_2 \equiv 1(\mathrm{mod}\ 3)$,we will find that $j_1 \equiv 0(\mathrm{mod}\ 3)$, so there will only be $j_2 \equiv 2(\mathrm{mod}\ 3)$,and we can get solutions $(1,1,0,2),(1,2,0,2)$.

iii. $i_1 \equiv 2(\mathrm{mod}\ 3)$

We can simplify system to:

$$\begin{cases} i_2 j_1 \equiv 0(\mathrm{mod}\ 3) \\ j_1(j_2+2) \equiv 0(\mathrm{mod}\ 3) \\ i_2(j_2+2) \equiv 0(\mathrm{mod}\ 3) \\ i_2 j_1 + j_2^2 \equiv 1(\mathrm{mod}\ 3) \end{cases}$$

If $j_1 \equiv 0(\mathrm{mod}\ 3)$, we can get system of 2 equations,$i_2(j_2 + 2) \equiv 0(\mathrm{mod}\ 3)$ and $j_2^2 \equiv 1(\mathrm{mod}\ 3)$, in this case we will get answers with form $(2,0,*,*)$,if $j_2 \equiv 1(\mathrm{mod}\ 3)$, we will get answers $(2,0,0,1),(2,0,1,1)$ and $(2,0,2,1)$, if $j_2 \equiv 2(\mathrm{mod}\ 3)$, we will get answer $(2,0,0,2)$. If $j_1 \not\equiv 0(\mathrm{mod}\ 3)$, so there must be $i_2 \equiv 0(\mathrm{mod}\ 3)$ and $j_2+2 \equiv 0(\mathrm{mod}\ 3)$, so answer are $(2,1,0,1)$ and $(2,2,0,1)$.

So we have solved this congruence equations system.

But there is a better method by using computer program. This is Python code.

```
DM = []
for i in range(0,3):
    for j in range(0,3):
        for k in range(0,3):
            for l in range(0,3):
                DM.append((i,j,k,l))

#       i1^2  + i2*j1 = 1 (mod 3)
#       i1*j1 + j2*j1 = 0 (mod 3)
#       i1*i2 + i2*j2 = 0 (mod 3)
#       i2*j1 + j2^2  = 1 (mod 3)
#   There I use representations: i[0] ~ i1; i[1]~j1; i[2]~i2; i[3]~j2

Ans = []
for i in DM:
    if (i[0]**2 + i[2]*i[1]) % 3 == 1 and\
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    (i[0]*i[1] + i[3]*i[1]) % 3 == 0 and\
    (i[0]*i[2] + i[2]*i[3]) % 3 == 0 and\
    (i[2]*i[1] + i[3]**2) % 3 == 1:
        Ans.append(i)

print(Ans)
```

Then we will get answers: $(i_1, j_1, i_2, j_2) \in \{(0,1,1,0), (0,2,2,0), (1,0,0,1), (1,0,0,2), (1,0,1,2), (1,0,2,2), (1,1,0,2), (1,2,0,2), (2,0,0,1), (2,0,0,2), (2,0,1,1), (2,0,2,1), (2,1,0,1), (2,2,0,1)\}$

For these solutions, we can construct conjugation relationship:

1) $cac = b$	$cbc = a$
2) $cac = b^2$	$cbc = a^2$
3) $cac = a$	$cbc = b$
4) $cac = a$	$cbc = b^2$
5) $cac = a$	$cbc = ab^2$
6) $cac = a$	$cbc = a^2b^2$
7) $cac = ab$	$cbc = b^2$
8) $cac = ab^2$	$cbc = b^2$
9) $cac = a^2$	$cbc = b$
10)$cac = a^2$	$cbc = b^2$
11)$cac = a^2$	$cbc = ab$
12)$cac = a^2$	$cbc = a^2b$
13)$cac = a^2b$	$cbc = b$
14)$cac = a^2b^2$	$cbc = b$

But when we compare pairs [4), 9)],[5), 13)],[6), 14)],[7), 11)] and [8), 12)], we will discover that if we exchange status of a and b, and they are the same. So, in fact we could just explore cases 1), 2), 3), 4), 5), 6), 7), 8), 10).

Now we will consider them:

i. $cac = b, cbc = a$

$$G_1 = \langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = b, cbc = a \rangle$$

ii. $cac = b^2, cbc = a^2$

$$cac = b^2, cbc = a^2 \Rightarrow cac = b^2, cb^2c = (a^2)^2 = a$$

$\therefore$ We can construct a map from G_2 onto G_1

$$\begin{aligned}\varphi : G_2 &\to G_1 \\ c &\longmapsto c_* \\ a &\longmapsto a_* \\ b^2 &\longmapsto b_* \\ \varphi(g_1 g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

Where $g_* \in G_1$.
We can certify that c, a and b^2 all are generators,and we can use a and b^2 to generate group S_3, so next, we will prove that the map will be isomorphism. First of all we have to certify that it is a function:
Becasue G includes 18 elements, and $|a| * |b| * |c| = 3 * 3 * 2 = 18$, we will spontaneously consider that every element of G can be represented as form $a^r b^s c^t$, now we can see:

$$\begin{aligned}1) \quad a^r b^s c^t &= a^r b^s c^t = b^s a^r c^t \\ 2) \quad a^r c b^s &= a^r c b^s c c \\ &= a^r (cbc)^s c \\ &= a^r a^{2s} c = a^{2s+r} c \\ 3) \quad c a^r b^s &= c a^r c c b^s c c \\ &= b^{2r} a^{2s} c = c b^s a^r\end{aligned}$$

Therefore, it indeed exits with this form. For group G_1, analogously, we can get the same consequence.
Therefore for any elment g in G, we can assume there will be $r, s, t \in \mathbb{Z}$, and $g = a^r b^s c^t$,so for any $g_1 = a^{r_1} b^{s_1} c^{t_1}, g_2 = a^{r_2} b^{s_2} c^{t_2}$ and if $g_1 = g_2$, then we will get

$$\begin{aligned}a^{r_1} b^{s_1} c^{t_1} &= a^{r_2} b^{s_2} c^{t_2} \\ b^{-s_2} a^{r_1 - r_2} b^{s_1} &= c^{t_2 - t_1} \\ a^{r_1 - r_2} b^{s_1 - s_2} &= c^{t_2 - t_1}\end{aligned}$$

$\because S_3 \cap \langle c \rangle = \{e\}$, so $a^{r_1 - r_2} b^{s_1 - s_2} = c^{t_2 - t_1} = e$, and infer that $s_1 \equiv s_2 (\text{mod } 3), r_1 \equiv r_2 (\text{mod } 3), t_1 \equiv t_2 (\text{mod } 2)$

Now we can verify it is really function.

$$\begin{aligned}
\varphi(g_1) &= \varphi(a^{r_1}b^{s_1}c^{t_1}) \\
&= \varphi(a^{r_1})\varphi(b^{s_1})\varphi(c^{t_1}) \\
&= (\varphi(a))^{r_1}(\varphi(b))^{s_1}(\varphi(c))^{t_1} \\
&= {a_*}^{r_1}(\varphi(b^4))^{s_1}{c_*}^{t_1} \\
&= {a_*}^{r_1}(\varphi(b^2))^{2s_1}c_*^{t_1} \\
&= {a_*}^{r_1}{b_*}^{2s_1}c^{t_1}
\end{aligned}$$

and

$$\varphi(g_2) = {a_*}^{r_2}{b_*}^{2s_2}c^{t_2}$$

But as we know, they are equal; becasue $a^{r_1} = a^{r_2}, b^{2s_1} = b^{2s_2}, c^{t_1} = c^{t_2}$. And $\varphi(g_1) = \varphi(g_2)$. So map φ is function,and according to our assumption, it is also homormorphism; now we will prove that it is a bijection. To see other function ψ:

$$\begin{aligned}
\psi : G_1 &\to G_2 \\
c_* &\longmapsto c \\
a_* &\longmapsto a \\
b_* &\longmapsto b^2 \\
\psi(g_{*1}g_{*2}) &= \varphi(g_{*1})\varphi(g_{2*})
\end{aligned}$$

Where $g_{*1}, g_{*2} \in G_1$.
For every element $g_* = a_*^r b_* s c_*^t$ in G_1, we have:

$$\begin{aligned}
\varphi(\psi(a_*^r b_*^s c_*^t)) &= \varphi(\psi(a_*)^r\psi(b_*)^s\psi(c_*)^t) \\
&= \varphi(a^r b^{2s} c^t) \\
&= \varphi(a)^r\varphi(b)^{2s}\varphi(c)^t \\
&= a_*^r\varphi(b^4)^{2s}c_*^t \\
&= a_*^r\varphi(b^2)^{4s}c_*^t \\
&= a_*^r b_*^{4s} c_*^t \\
&= a_*^r b_*^s c_*^t \\
&= g_*
\end{aligned}$$

So $\varphi \circ \psi = e_{G_1}$. Analogously, for $\psi \circ \varphi$, if we assume that

$g = a^r b^s c^t$ then we have:

$$\begin{aligned}\psi(\varphi(a^r b^s c^t)) &= \psi(\varphi(a)^r \varphi(b)^s \varphi(c)^t) \\ &= \psi(a_*^r b_*^{2s} c_*^t) \\ &= \psi(a_*)^r \psi(b_*)^{2s} \psi(c_*)^t \\ &= a^r b^{4s} c^t \\ &= a^r b^s c^t \\ &= g\end{aligned}$$

Therefore $\psi \circ \varphi = e_{G_2}$, so they are inverse mapping for each other, and φ is acctually bijection. We have proven that it is isomorphism.The only difference between G_2 and G_1 is that we exchanged the symols of b and b^2. So $G_2 \cong G_1$.

iii. $cac = a, cbc = b$

In this situation, we will get

$$ca = ac, cb = bc, ab = ba$$

So G_3 will be an Abelian group, because$\forall a^{r_i} b^{s_i} c^{t_i} \in G_3, i = 1, 2,$

$$\begin{aligned}a^{r_1} b^{w_1} c^{t_1} \cdot a^{r_2} b^{s_2} c^{t_2} &= a^{r_1+r_2} b^{s_1+s_2} c^{t_1+t_2} \\ &= a^{r_2} b^{s_2} c^{t_2} \cdot a^{r_1} b^{s_1} c^{t_1}\end{aligned}$$

But $N_2 \neq 1$,so G_3 is impossible Abelian group, therefore it leads to contradiction. And G_3 does not exist.

iv. $cac = a, cbc = b^2$

$$cac = a, cbc = b^2 \Rightarrow cabc = caccbc = ab^2, cab^2c = ab$$

And now construct map from G_4 onto G_1:

$$\begin{aligned}\varphi : G_4 &\to G_1 \\ c &\longmapsto c_* \\ ab &\longmapsto a_* \\ ab^2 &\longmapsto b_* \\ \varphi(g_1 g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

In the same way as we used for proof in previous situation, we can get analogous result. We can change symols a to ab and b to ab^2, then we will get G_4 from G_1. So $G_4 \cong G_1$.

v. $cac = a, cbc = ab^2$

$$cbc = ab^2 \Rightarrow cbc = ab^2, cab^2c = b$$

For map:

$$\begin{aligned}\varphi : G_5 &\to G_1\\ c &\longmapsto c\\ ab^2 &\longmapsto a\\ b &\longmapsto b\\ \varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

In the same way we used,we can prove that φ is isomorphism $G_5 \cong G_1$.

vi. $cac = a, cbc = a^2b^2$

$$cac = a, cbc = a^2b^2 \Rightarrow cbc = a^2b^2, ca^2b^2c = b$$

And consider this map:

$$\begin{aligned}\varphi : G_6 &\to G_1\\ c &\longmapsto c_*\\ a^2b^2 &\longmapsto a_*\\ b &\longmapsto b_*\\ \varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

Similarly, we can prove that φ is isomorphism from G_6 onto G_1, we need just exchange status of a and a^2b^2. So $G_6 \cong G_1$.

vii. $cac = ab, cbc = b^2$

$$cac = ab, cbc = b^2 \Rightarrow cac = ab, cabc = a$$

And map:

$$\begin{aligned}\varphi : G_7 &\to G_1\\ c &\longmapsto c_*\\ a &\longmapsto a_*\\ ab &\longmapsto b_*\\ \varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

Similarly, we can prove that φ is isomorphism from G_6 onto G_1, we need just exchange status of b and ab. So $G_7 \cong G_1$.

viii. $cac = ab^2, cbc = b^2$ $cac = ab^2, cbc = b^2 \Rightarrow cac = ab^2, cab^2c = a$

And map:

$$\begin{aligned}\varphi : G_8 &\to G_1\\ c &\longmapsto c_*\\ a &\longmapsto a_*\\ ab^2 &\longmapsto b_*\\ \varphi(g_1 g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

Similarly, we can prove that φ is isomorphism from G_6 onto G_1, we need just exchange status of b and ab^2. So $G_8 \cong G_1$.

ix. $cac = a^2, cbc = b^2$

In this situation, spontaneously, we will think of the same way as we used for previous situations. But soon we will find that it becomes useless.

We can see 3 cases of them:

A. $cac = a^2, ca^2c = a$

If we make map like:

$$\begin{aligned}\varphi : G_9 &\to G_1\\ c &\longmapsto c_*\\ a &\longmapsto a_*\\ a^2 &\longmapsto b_*\\ \varphi(g_1 g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

we will find that $a_* = \varphi(a^4) = \varphi(a^2)^2 = b_*^2$, but we know that $\langle a_*\rangle \cap \langle b_*\rangle = \{e_*\}$, so it is impossible. And this φ is not isomorphism.

B. $cbc = b^2, cb^2c = b$

Similarly, if we construct map like:

$$\begin{aligned}\varphi : G_9 &\to G_1\\ c &\longmapsto c_*\\ b^2 &\longmapsto a_*\\ b &\longmapsto b_*\\ \varphi(g_1 g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

we will find that $b_* = \varphi(b^4) = \varphi(b^2)^2 = a_*^2$, but we know that $\langle a_*\rangle \cap \langle b_*\rangle = \{e_*\}$, so it is impossible. And this φ is not isomorphism.

C. $cabc = caccbc = a^2b^2, ca^2b^2c = ab$

Analogously, if map like:

$$\begin{aligned}\varphi : G_9 &\to G_1\\ c &\longmapsto c_*\\ ab &\longmapsto a_*\\ a^2b^2 &\longmapsto b_*\\ \varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)\end{aligned}$$

we wil find that $e_* = \varphi(e) = \varphi(a^3b^3) = \varphi(aba^2b^2) = \varphi(ab)\varphi(a^2b^2) = a_*b_*$, this is also impossible, and φ is not isomorphism.

Now we have enough reasons to consider that there is not isomorphism between G_9 and G_1.

We will prove our conjecture. Assume that there is an isomorphism φ

Because $O(c_*) = 2, O(c) = 2$,and they are the only element with rank 2 in respective group, so $\varphi(c) = c_*$.

So $\exists a^rb^sc^t : \varphi(a^rb^sc^t) = a_*$,where $r, s \in \{0, 1, 2\}, t \in \{0, 1\}$,there will be 2 cases:

A. $t = 0$

In this case, $\varphi(a^rb^s) = a_*$

$$\begin{aligned}\varphi(a^rb^s) &= a_*\\ \varphi(ca^rb^sc) &= \varphi(c)a_*\varphi(c)\\ \varphi(ca^rc)\varphi(cb^sc) &= c_*a_*c_*\\ \varphi(cac)^r\varphi(cbc)^s &= b_*\\ \varphi(a^2)^r\varphi(b^2)^s &= b_*\\ \varphi((a^rb^s)^2) &= b_*\\ \varphi(a^rb^s)^2 &= b_*\\ a_*^2 &= b_*\end{aligned}$$

It is impossible, because in group G_1, $\langle a_*\rangle \cap \langle b_*\rangle = \{e_*\}$, and it causes contradiction.

B. $t=1$

In this case, $\varphi(a^r b^s c) = a_*$

$$\begin{aligned}
\varphi(a^r b^s c) &= a_* \\
\varphi(a^r b^s)\varphi(c) &= a_* \\
\varphi(ca^r b^r c)c_* &= \varphi(c)a_*\varphi(c) \\
\varphi(ca^r c)\varphi(cb^s c) &= c_* a_* \\
\varphi(a^r b^s)^2 &= c_* a_* \\
\varphi(a^r b^s)\varphi(c)\varphi(c)\varphi(a^r b^s)\varphi(c)\varphi(c) &= c_* a_* \\
\varphi(a^r b^s c)\varphi(c)\varphi(a^r b^s c)\varphi(c) &= c_* a_* \\
a_* c_* a_* c_* &= c_* a_* \\
a_* b_* &= c_* a_* \\
a_* b_* a_*^{-1} &= c_* \\
b_* &= c_*
\end{aligned}$$

It is no impossible, and causes contradiction.

And it has been proven that there is not isomorphism between G_9 and G_1. So $G_9 \ncong G_1$.
So

$$G_9 = \langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = a^2, cbc = b^2 \rangle$$

is another type of group which consists of 18 elemnts.

And we had find all the types of group which consists of 18 elements.

They are

1)$\mathbb{Z}_{18}$

2)$\mathbb{Z}_3 \oplus \mathbb{Z}_6$

3)D_9

4)$\langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = b, cbc = a \rangle$

5)$\langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = a^2, cbc = b^2 \rangle$

We have completed classification of groups which consists of 18 elemnts.