Correction of Homework on 05.03.2019

Tanglin

Problem 1. Study all sunsets of some set with operations symmetric difference and intersection. Is it a ring? If it is, then is it commutative? Does it include identity element? Which elements are reversible, zero divisor, idempotent?

Solution:

Assume that set S is an abitrary non-empty set. And let:

$$\mathscr{M}(A) := \{D | D \subseteq A\}$$

First of all, we will verify some operations of symmetric difference.

Definition of symmetric difference:

$$R \triangle S := \{x | (x \in R \& x \notin S) \vee (x \in S \& x \notin R)\}$$

1. $R \triangle S = S \triangle R$

$$\begin{aligned} R \triangle S &= \{x | (x \in R \& x \notin S) \vee (x \in S \& x \notin R)\} \\ &= \{x | (x \in S \& x \notin R) \vee (x \in R \& x \notin S)\} \\ &= S \triangle R \end{aligned}$$

2. $(R \triangle S) \triangle T = R \triangle (S \triangle T)$

$$
\begin{aligned}
(R \triangle S) \triangle T &= \{x|(x \in R \& x \notin S) \vee (x \in S \& x \notin R)\} \triangle T \\
&= \{x|((x \in R \& x \notin S) \vee (x \in S \& x \notin R)) \& x \notin T \\
&\vee x \in T \& (\neg((x \in R \& x \notin S) \vee (x \in S \& x \notin R)))\} \\
&= \{x|(x \in R \& x \notin S \& x \notin T) \vee (x \in S \& x \notin R \& x \notin T) \\
&\vee (x \in T \& (x \notin R \vee x \in S) \& (x \notin S \vee x \in R))\} \\
&= \{x|(x \in R \& x \notin S \& x \notin T) \vee (x \in S \& x \notin R \& x \notin T) \\
&\vee (x \in T \& (x \notin R \& x \notin S \vee x \in R \& x \in S))\} \\
&= \{x|(x \in R \& x \notin S \& x \notin T) \vee (x \in S \& x \notin R \& x \notin T) \\
&\vee (x \in T \& x \notin R \& x \notin S) \vee (x \in T \& x \in R \& x \in S)\} \\
&= \{x|(x \in R \& x \notin S \& x \notin T) \vee (x \notin R \& x \in S \& x \notin T) \\
&\vee (x \notin R \& x \notin S \& x \in T) \vee (x \in R \& x \in S \& x \in T)\} \\
&= \{x|(x \in T \& x \notin S \& x \notin R) \vee (x \notin T \& x \in S \& x \notin R) \\
&\vee (x \notin T \& x \notin S \& x \in R) \vee (x \in T \& x \in S \& x \in R)\} \\
&= (T \triangle S) \triangle R \\
&= (S \triangle T) \triangle R \\
&= R \triangle (S \triangle T)
\end{aligned}
$$

3. $R \cap (S \triangle T) = (R \cap S) \triangle (R \cap T)$
   For $R \cap (S \triangle T)$, we have:

$$
\begin{aligned}
R \cap (S \triangle T) &= \{x|x \in R \& (x \in S \& x \notin T \vee x \in T \& x \notin S)\} \\
&= \{x|x \in R \& x \in S \& x \notin T \vee x \in R \& x \in T \& x \notin S\}
\end{aligned}
$$

   For $(R \cap S) \triangle (R \cap T)$, we have:

$$
\begin{aligned}
(R \cap S) \triangle (R \cap T) &= \{(x \in R \& x \in S) \& (\neg(x \in R \& x \in T)) \\
&\vee (x \in R \& x \in T) \& (\neg(x \in R \& x \in S))\} \\
&= \{x|x \in R \& x \in S \& (x \notin R \vee x \notin T) \\
&\vee x \in R \& x \in T \& (x \notin R \vee x \notin S)\} \\
&= \{x|x \in R \& x \in S \& x \notin T \vee x \in R \& x \in T \& x \notin S\}
\end{aligned}
$$

   So we can assert that $R \cap (S \triangle T) = (R \cap S) \triangle (R \cap T)$.

Now we will prove that set $\mathscr{M}(A)$ with operations symmetric difference and intersection is a ring.

1. $(\mathscr{M}(A), \triangle)$ is Abelian group.

   (a) Associativity:
   $\because \forall R, S, T \in \mathscr{M}(A), (R \triangle S) \triangle T = R \triangle (S \triangle T)$

   (b) Commutativity:
   $\because \forall R, S \in \mathscr{M}(A), R \triangle S = S \triangle R$

   (c) Closure:
   $\forall R, S \in \mathscr{M}(A) \Rightarrow R \subseteq A, S \subseteq A \Rightarrow R \triangle S \subseteq A \Rightarrow R \triangle S \in \mathscr{M}(A)$

   (d) Zero element exists: Consider empty set in $\mathscr{M}(A)$
   $\forall R \in \mathscr{M}(A), R \triangle \emptyset = \emptyset \triangle R = R$
   We have proved that it is a group.

2. Satisfy the law of distribution.
   From nature 3 of operation symmetric difference, we can get it.

3. Closure of operation intersection. $\forall R, S \subset A \Rightarrow R \cap S \subset A$

We have proved that $(\mathscr{M}(A), \triangle, \cap)$ is a ring.