

Tang Lin
April 22, 2019

Problem: Please classify all groups which consists of 18 elements.
Solve:

$$18 = 2 \cdot 3^2$$

From The Third Theorem of Sylow, we have:

$$\begin{aligned} N_2 &\equiv 1 \pmod{2} \quad N_2 | 9 \Rightarrow N_2 = 1, 3, 9 \\ N_3 &\equiv 1 \pmod{3} \quad N_3 | 2 \Rightarrow N_3 = 1 \end{aligned}$$

So we have 2 cases:

1. $N_2 = 1, N_3 = 1$ In this situation, there is only one 2-Sylow subgroup and one 3-Sylow subgroup, so $G \cong S_2 \oplus S_3$, where $|S_2| = 2, |S_3| = 9$.
There will be 2 situations:

(a)

$$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{18}$$

(b)

$$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_6$$

2. $N_2 \neq 1, N_3 = 1$

$$S_3 \triangleleft G$$

So $G \cong S_3 \rtimes K$, where $|K| = 2$.

- (a) $S_3 \cong \mathbb{Z}_9$ We can get automorphism of $S_3 \cong \mathbb{Z}_9$: $\text{Aut}(\mathbb{Z}_9) = H \cong \text{Aut}(S_3)$, where $|H| = 6$.

So We can make a table of automorphism, we assume $S_3 = \langle a \rangle$:

ε	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
σ_1	e	a^2	a^4	a^6	a^8	a	a^3	a^5	a^7
σ_2	e	a^4	a^8	a^3	a^7	a^2	a^6	a	a^5
σ_3	e	a^5	a	a^6	a^2	a^7	a^3	a^8	a^4
σ_4	e	a^7	a^5	a^3	a	a^8	a^6	a^4	a^2
σ_5	e	a^8	a^7	a^6	a^5	a^4	a^3	a^2	a

Where $\varepsilon, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are elements of $\text{Aut}(S_3)$.

Without difficult we can get rank of each map of isomorphism:

$$O(\sigma_1) = 6, O(\sigma_2) = 3, O(\sigma_3) = 6, O(\sigma_4) = 3, O(\sigma_5) = 2$$

\therefore There is map with rank 6, so $\text{Aut}(S_3) \cong \mathbb{Z}_6$.

Now we assume $K = \langle b \rangle$, $|K| = 2$, if we can find a homomorphism

$\varphi : K \rightarrow \text{Aut}(S_3)$, then we can establish a group.

\therefore In subgroup K , there are only elements with rank 2 and zero.

So we can construct a map:

$$\varphi : K \rightarrow \text{Aut}(S_3)$$

$$e \mapsto \varepsilon$$

$$b \mapsto \sigma_5$$

So we can get a group with form:

$$G \cong \langle a, b | a^9 = b^2 = e, bab = a^{-1} \rangle \cong D_9$$

(b) $S_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$\therefore |S_3| = 9 = 3^2$, $\therefore S_3$ is Abelian group. We can write S_3 in form

$$S_3 = \langle a, b | a^3 = b^3 = 1 \rangle$$

$\therefore S_3 \triangleleft G$, $\therefore cS_3c = S_3$, where $c \in K = \langle c \rangle$, $|K| = 2$.

So $\exists i_1, j_1, i_2, j_2 \in \{0, 1, 2\}$, and

$$cac = a^{i_1} b^{j_1}$$

$$cbc = a^{i_2} b^{j_2}$$

Therefore

$$cac = a^{i_1} b^{j_1}$$

$$ccacc = ca^{i_1} b^{j_1} c$$

$$a = ca^{i_1} cb^{j_1} c$$

$$a = (cac)^{i_1} (cbc)^{j_1}, \text{ because } ca^n c = (cac)(cac) \cdots (cac) = (cac)^n$$

$$a = (a^{i_1} b^{j_1})^{i_1} (a^{i_2} b^{j_2})^{j_1}$$

$\therefore S_3$ is Abelian group, so

$$a = a^{i_1^2 + i_2 j_1} b^{i_1 j_1 + j_2 j_1}$$

Analogously, we can get

$$\begin{aligned}
cbc &= a^{i_2} b^{j_2} \\
ccbcc &= ca^{i_2} b^{j_2} c \\
b &= ca^{i_2} ccb^{j_2} c \\
b &= (cac)^{i_2} (cbc)^{j_2} \\
b &= (a^{i_1} b^{j_1})^{i_2} (a^{i_2} b^{j_2})^{j_2} \\
b &= a^{i_1 i_2 + i_2 j_2} b^{i_2 j_1 + j_2^2}
\end{aligned}$$

Because in S_3 , $\langle a \rangle \cup \langle b \rangle = \emptyset$, so we can get a congruence equations system:

$$\begin{cases} i_1^2 + i_2 j_1 \equiv 1 \pmod{3} \\ i_1 j_1 + j_2 j_1 \equiv 0 \pmod{3} \\ i_1 i_2 + i_2 j_2 \equiv 0 \pmod{3} \\ i_2 j_1 + j_2^2 \equiv 1 \pmod{3} \end{cases}$$

To solve this equation, we can respectively let $i_1 \equiv 0, 1, 2 \pmod{3}$, and discuss them, but it is little complex.

i. $i_1 \equiv 0 \pmod{3}$

We can simplify this congruence equation system to:

$$\begin{cases} i_2 j_1 \equiv 1 \pmod{3} \\ j_2 j_1 \equiv 0 \pmod{3} \\ i_2 j_2 \equiv 0 \pmod{3} \\ i_2 j_1 + j_2^2 \equiv 1 \pmod{3} \end{cases}$$

It is not difficult to know that $j_2 \equiv 0 \pmod{3}$ and $j_1 \not\equiv 0 \pmod{3}$, and we can more simplify it to just one equation $i_2 j_1 \equiv 1 \pmod{3}$

And now we can get the first two solutions $(i_1, j_1, i_2, j_2) = (0, 1, 1, 0)$ or $(0, 2, 2, 0)$.

ii. $i_1 \equiv 1 \pmod{3}$

We can simplify this congruence equation system to:

$$\begin{cases} i_2 j_1 \equiv 0 \pmod{3} \\ j_1(j_2 + 1) \equiv 0 \pmod{3} \\ i_2(j_2 + 1) \equiv 0 \pmod{3} \\ i_2 j_1 + j_2^2 \equiv 1 \pmod{3} \end{cases}$$

If $j_1 \equiv 0 \pmod{3}$, then we can get system of two equations, $i_2(j_2 + 1) \equiv 0 \pmod{3}$ and $i_2 j_1 + j_2^2 \equiv 1 \pmod{3}$, so in this case we can get answers will have form $(i_1, j_1, i_2, j_2) = (1, 0, *, *)$, if $i_2 \equiv 0 \pmod{3}$, we will get answers $(1, 0, 0, 1)$ and $(1, 0, 0, 2)$, and

if $i_2 \equiv 1 \pmod{3}$, we will get answer $(1, 0, 1, 2)$, if $i_2 \equiv 2 \pmod{3}$, get $(1, 0, 2, 2)$.

If $j_1 \not\equiv 0 \pmod{3}$, and must $i_2 \equiv 0 \pmod{3}$, we can get system with 2 equations $j_1(j_2 + 1) \equiv 0 \pmod{3}$ and $j_2^2 \equiv 1 \pmod{3}$, but if $j_2 \equiv 1 \pmod{3}$, we will find that $j_1 \equiv 0 \pmod{3}$, so there will only be $j_2 \equiv 2 \pmod{3}$, and we can get solutions $(1, 1, 0, 2)$, $(1, 2, 0, 2)$.

iii. $i_1 \equiv 2 \pmod{3}$

We can simplify system to:

$$\begin{cases} i_2 j_1 \equiv 0 \pmod{3} \\ j_1(j_2 + 2) \equiv 0 \pmod{3} \\ i_2(j_2 + 2) \equiv 0 \pmod{3} \\ i_2 j_1 + j_2^2 \equiv 1 \pmod{3} \end{cases}$$

If $j_1 \equiv 0 \pmod{3}$, we can get system of 2 equations, $i_2(j_2 + 2) \equiv 0 \pmod{3}$ and $j_2^2 \equiv 1 \pmod{3}$, in this case we will get answers with form $(2, 0, *, *)$, if $j_2 \equiv 1 \pmod{3}$, we will get answers $(2, 0, 0, 1)$, $(2, 0, 1, 1)$ and $(2, 0, 2, 1)$, if $j_2 \equiv 2 \pmod{3}$, we will get answer $(2, 0, 0, 2)$. If $j_1 \not\equiv 0 \pmod{3}$, so there must be $i_2 \equiv 0 \pmod{3}$ and $j_2 + 2 \equiv 0 \pmod{3}$, so answer are $(2, 1, 0, 1)$ and $(2, 2, 0, 1)$.

So we have solved this congruence equations system.

But there is a better method by using computer program. This is Python code.

```
DM = []
for i in range(0,3):
    for j in range(0,3):
        for k in range(0,3):
            for l in range(0,3):
                DM.append((i,j,k,l))

#      i1^2  + i2*j1 = 1 (mod 3)
#      i1*j1 + j2*j1 = 0 (mod 3)
#      i1*i2 + i2*j2 = 0 (mod 3)
#      i2*j1 + j2^2  = 1 (mod 3)
#  There I use representations: i[0] ~ i1; i[1]~j1; i[2]~i2; i[3]~j2

Ans = []
for i in DM:
    if (i[0]**2 + i[2]*i[1]) % 3 == 1 and \
```

```

(i[0]*i[1] + i[3]*i[1]) % 3 == 0 and\
(i[0]*i[2] + i[2]*i[3]) % 3 == 0 and\
(i[2]*i[1] + i[3]**2) % 3 == 1:
    Ans.append(i)

print(Ans)

```

Then we will get answers: $(i_1, j_1, i_2, j_2) \in \{(0, 1, 1, 0), (0, 2, 2, 0), (1, 0, 0, 1), (1, 0, 0, 2), (1, 0, 1, 2), (1, 0, 2, 2), (1, 1, 0, 2), (1, 2, 0, 2), (2, 0, 0, 1), (2, 0, 0, 2), (2, 0, 1, 1), (2, 0, 2, 1), (2, 1, 0, 1), (2, 2, 0, 1)\}$

For these solutions, we can construct conjugation relationship:

1) $cac = b$	$cbc = a$
2) $cac = b^2$	$cbc = a^2$
3) $cac = a$	$cbc = b$
4) $cac = a$	$cbc = b^2$
5) $cac = a$	$cbc = ab^2$
6) $cac = a$	$cbc = a^2b^2$
7) $cac = ab$	$cbc = b^2$
8) $cac = ab^2$	$cbc = b^2$
9) $cac = a^2$	$cbc = b$
10) $cac = a^2$	$cbc = b^2$
11) $cac = a^2$	$cbc = ab$
12) $cac = a^2$	$cbc = a^2b$
13) $cac = a^2b$	$cbc = b$
14) $cac = a^2b^2$	$cbc = b$

But when we compare pairs $[4), 9)], [5), 13)], [6), 14)], [7), 11)]$ and $[8), 12)]$, we will discover that if we exchange status of a and b, and they are the same. So, in fact we could just explore cases 1), 2), 3), 4), 5), 6), 7), 8), 10).

Now we will consider them:

i. $cac = b, cbc = a$

$$G_1 = \langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = b, cbc = a \rangle$$

ii. $cac = b^2, cbc = a^2$

$$cac = b^2, cbc = a^2 \Rightarrow cac = b^2, cb^2c = (a^2)^2 = a$$

\therefore We can construct a map from G_2 onto G_1

$$\begin{aligned}\varphi : G_2 &\rightarrow G_1 \\ c &\longmapsto c_* \\ a &\longmapsto a_* \\ b^2 &\longmapsto b_* \\ \varphi(g_1 g_2) &= \varphi(g_1) \varphi(g_2)\end{aligned}$$

Where $g_* \in G_1$.

We can certify that c, a and b^2 all are generators, and we can use a and b^2 to generate group S_3 , so next, we will prove that the map will be isomorphism. First of all we have to certify that it is a function:

Because G includes 18 elements, and $|a| * |b| * |c| = 3 * 3 * 2 = 18$, we will spontaneously consider that every element of G can be represented as form $a^r b^s c^t$, now we can see:

$$\begin{aligned}1) \quad a^r b^s c^t &= a^r b^s c^t = b^s a^r c^t \\ 2) \quad a^r c b^s &= a^r c b^s c c \\ &= a^r (c b c)^s c \\ &= a^r a^{2s} c = a^{2s+r} c \\ 3) \quad c a^r b^s &= c a^r c c b^s c c \\ &= b^{2r} a^{2s} c = c b^s a^r\end{aligned}$$

Therefore, it indeed exists with this form. For group G_1 , analogously, we can get the same consequence.

Therefore for any element g in G , we can assume there will be $r, s, t \in \mathbb{Z}$, and $g = a^r b^s c^t$, so for any $g_1 = a^{r_1} b^{s_1} c^{t_1}$, $g_2 = a^{r_2} b^{s_2} c^{t_2}$ and if $g_1 = g_2$, then we will get

$$\begin{aligned}a^{r_1} b^{s_1} c^{t_1} &= a^{r_2} b^{s_2} c^{t_2} \\ b^{-s_2} a^{r_1 - r_2} b^{s_1} &= c^{t_2 - t_1} \\ a^{r_1 - r_2} b^{s_1 - s_2} &= c^{t_2 - t_1}\end{aligned}$$

$\therefore S_3 \cap \langle c \rangle = \{e\}$, so $a^{r_1 - r_2} b^{s_1 - s_2} = c^{t_2 - t_1} = e$, and infer that $s_1 \equiv s_2 \pmod{3}$, $r_1 \equiv r_2 \pmod{3}$, $t_1 \equiv t_2 \pmod{2}$

Now we can verify it is really function.

$$\begin{aligned}
\varphi(g_1) &= \varphi(a^{r_1} b^{s_1} c^{t_1}) \\
&= \varphi(a^{r_1}) \varphi(b^{s_1}) \varphi(c^{t_1}) \\
&= (\varphi(a))^{r_1} (\varphi(b))^{s_1} (\varphi(c))^{t_1} \\
&= a_*^{r_1} (\varphi(b^4))^{s_1} c_*^{t_1} \\
&= a_*^{r_1} (\varphi(b^2))^{2s_1} c_*^{t_1} \\
&= a_*^{r_1} b_*^{2s_1} c_*^{t_1}
\end{aligned}$$

and

$$\varphi(g_2) = a_*^{r_2} b_*^{2s_2} c_*^{t_2}$$

But as we know, they are equal; because $a^{r_1} = a^{r_2}, b^{2s_1} = b^{2s_2}, c^{t_1} = c^{t_2}$. And $\varphi(g_1) = \varphi(g_2)$. So map φ is function, and according to our assumption, it is also homomorphism; now we will prove that it is a bijection. To see other function ψ :

$$\begin{aligned}
\psi : G_1 &\rightarrow G_2 \\
c_* &\mapsto c \\
a_* &\mapsto a \\
b_* &\mapsto b^2 \\
\psi(g_{*1} g_{*2}) &= \varphi(g_{*1}) \varphi(g_{*2})
\end{aligned}$$

Where $g_{*1}, g_{*2} \in G_1$.

For every element $g_* = a_*^r b_*^s c_*^t$ in G_1 , we have:

$$\begin{aligned}
\varphi(\psi(a_*^r b_*^s c_*^t)) &= \varphi(\psi(a_*)^r \psi(b_*)^s \psi(c_*)^t) \\
&= \varphi(a^r b^{2s} c^t) \\
&= \varphi(a)^r \varphi(b)^{2s} \varphi(c)^t \\
&= a_*^r \varphi(b^4)^{2s} c_*^t \\
&= a_*^r \varphi(b^2)^{4s} c_*^t \\
&= a_*^r b_*^{4s} c_*^t \\
&= a_*^r b_*^s c_*^t \\
&= g_*
\end{aligned}$$

So $\varphi \circ \psi = e_{G_1}$. Analogously, for $\psi \circ \varphi$, if we assume that

$g = a^r b^s c^t$ then we have:

$$\begin{aligned}
\psi(\varphi(a^r b^s c^t)) &= \psi(\varphi(a)^r \varphi(b)^s \varphi(c)^t) \\
&= \psi(a_*^r b_*^{2s} c_*^t) \\
&= \psi(a_*)^r \psi(b_*)^{2s} \psi(c_*)^t \\
&= a^r b^{4s} c^t \\
&= a^r b^s c^t \\
&= g
\end{aligned}$$

Therefore $\psi \circ \varphi = e_{G_2}$, so they are inverse mapping for each other, and φ is actually bijection. We have proven that it is isomorphism. The only difference between G_2 and G_1 is that we exchanged the symbols of b and b^2 . So $G_2 \cong G_1$.

iii. $cac = a, cbc = b$

In this situation, we will get

$$ca = ac, cb = bc, ab = ba$$

So G_3 will be an Abelian group, because $\forall a^{r_i} b^{s_i} c^{t_i} \in G_3, i = 1, 2$,

$$\begin{aligned}
a^{r_1} b^{s_1} c^{t_1} \cdot a^{r_2} b^{s_2} c^{t_2} &= a^{r_1+r_2} b^{s_1+s_2} c^{t_1+t_2} \\
&= a^{r_2} b^{s_2} c^{t_2} \cdot a^{r_1} b^{s_1} c^{t_1}
\end{aligned}$$

But $N_2 \neq 1$, so G_3 is impossible Abelian group, therefore it leads to contradiction. And G_3 does not exist.

iv. $cac = a, cbc = b^2$

$$cac = a, cbc = b^2 \Rightarrow cabc = caccbc = ab^2, cab^2c = ab$$

And now construct map from G_4 onto G_1 :

$$\begin{aligned}
\varphi : G_4 &\rightarrow G_1 \\
c &\mapsto c_* \\
ab &\mapsto a_* \\
ab^2 &\mapsto b_* \\
\varphi(g_1 g_2) &= \varphi(g_1) \varphi(g_2)
\end{aligned}$$

In the same way as we used for proof in previous situation, we can get analogous result. We can change symbols a to ab and b to ab^2 , then we will get G_4 from G_1 . So $G_4 \cong G_1$.

v. $cac = a, cbc = ab^2$

$$cbc = ab^2 \Rightarrow cbc = ab^2, cab^2c = b$$

For map:

$$\begin{aligned}
\varphi : G_5 &\rightarrow G_1 \\
c &\mapsto c \\
ab^2 &\mapsto a \\
b &\mapsto b \\
\varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)
\end{aligned}$$

In the same way we used, we can prove that φ is isomorphism $G_5 \cong G_1$.

vi. $cac = a, cbc = a^2b^2$

$$cac = a, cbc = a^2b^2 \Rightarrow cbc = a^2b^2, ca^2b^2c = b$$

And consider this map:

$$\begin{aligned}
\varphi : G_6 &\rightarrow G_1 \\
c &\mapsto c_* \\
a^2b^2 &\mapsto a_* \\
b &\mapsto b_* \\
\varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)
\end{aligned}$$

Similarly, we can prove that φ is isomorphism from G_6 onto G_1 , we need just exchange status of a and a^2b^2 . So $G_6 \cong G_1$.

vii. $cac = ab, cbc = b^2$

$$cac = ab, cbc = b^2 \Rightarrow cac = ab, cabc = a$$

And map:

$$\begin{aligned}
\varphi : G_7 &\rightarrow G_1 \\
c &\mapsto c_* \\
a &\mapsto a_* \\
ab &\mapsto b_* \\
\varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)
\end{aligned}$$

Similarly, we can prove that φ is isomorphism from G_7 onto G_1 , we need just exchange status of b and ab . So $G_7 \cong G_1$.

viii. $cac = ab^2, cbc = b^2 \Rightarrow cac = ab^2, cabc = a$

And map:

$$\begin{aligned}
\varphi : G_8 &\rightarrow G_1 \\
c &\mapsto c_* \\
a &\mapsto a_* \\
ab^2 &\mapsto b_* \\
\varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)
\end{aligned}$$

Similarly, we can prove that φ is isomorphism from G_6 onto G_1 , we need just exchange status of b and ab^2 . So $G_8 \cong G_1$.

ix. $cac = a^2, cbc = b^2$

In this situation, spontaneously, we will think of the same way as we used for previous situations. But soon we will find that it becomes useless.

We can see 3 cases of them:

A. $cac = a^2, ca^2c = a$

If we make map like:

$$\begin{aligned}
\varphi : G_9 &\rightarrow G_1 \\
c &\mapsto c_* \\
a &\mapsto a_* \\
a^2 &\mapsto b_* \\
\varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)
\end{aligned}$$

we will find that $a_* = \varphi(a^4) = \varphi(a^2)^2 = b_*^2$, but we know that $\langle a_* \rangle \cap \langle b_* \rangle = \{e_*\}$, so it is impossible. And this φ is not isomorphism.

B. $cbc = b^2, cb^2c = b$

Similarly, if we construct map like:

$$\begin{aligned}
\varphi : G_9 &\rightarrow G_1 \\
c &\mapsto c_* \\
b^2 &\mapsto a_* \\
b &\mapsto b_* \\
\varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)
\end{aligned}$$

we will find that $b_* = \varphi(b^4) = \varphi(b^2)^2 = a_*^2$, but we know that $\langle a_* \rangle \cap \langle b_* \rangle = \{e_*\}$, so it is impossible. And this φ is not isomorphism.

C. $cabc = caccbc = a^2b^2, ca^2b^2c = ab$

Analogously, if map like:

$$\begin{aligned}
\varphi : G_9 &\rightarrow G_1 \\
c &\longmapsto c_* \\
ab &\longmapsto a_* \\
a^2b^2 &\longmapsto b_* \\
\varphi(g_1g_2) &= \varphi(g_1)\varphi(g_2)
\end{aligned}$$

we will find that $e_* = \varphi(e) = \varphi(a^3b^3) = \varphi(aba^2b^2) = \varphi(ab)\varphi(a^2b^2) = a_*b_*$, this is also impossible, and φ is not isomorphism.

Now we have enough reasons to consider that there is not isomorphism between G_9 and G_1 .

We will prove our conjecture. Assume that there is an isomorphism φ

Because $O(c_*) = 2, O(c) = 2$, and they are the only element with rank 2 in respective group, so $\varphi(c) = c_*$.

So $\exists a^r b^s c^t : \varphi(a^r b^s c^t) = a_*$, where $r, s \in \{0, 1, 2\}, t \in \{0, 1\}$, there will be 2 cases:

A. $t = 0$

In this case, $\varphi(a^r b^s) = a_*$

$$\begin{aligned}
\varphi(a^r b^s) &= a_* \\
\varphi(ca^r b^s c) &= \varphi(c)a_*\varphi(c) \\
\varphi(ca^r c)\varphi(cb^s c) &= c_*a_*c_* \\
\varphi(cac)^r \varphi(cbc)^s &= b_* \\
\varphi(a^2)^r \varphi(b^2)^s &= b_* \\
\varphi((a^r b^s)^2) &= b_* \\
\varphi(a^r b^s)^2 &= b_* \\
a_*^2 &= b_*
\end{aligned}$$

It is impossible, because in group G_1 , $\langle a_* \rangle \cap \langle b_* \rangle = \{e_*\}$, and it causes contradiction.

B. $t = 1$

In this case, $\varphi(a^r b^s c) = a_*$

$$\begin{aligned}
\varphi(a^r b^s c) &= a_* \\
\varphi(a^r b^s) \varphi(c) &= a_* \\
\varphi(ca^r b^r c) c_* &= \varphi(c) a_* \varphi(c) \\
\varphi(ca^r c) \varphi(cb^s c) &= c_* a_* \\
\varphi(a^r b^s)^2 &= c_* a_* \\
\varphi(a^r b^s) \varphi(c) \varphi(c) \varphi(a^r b^s) \varphi(c) \varphi(c) &= c_* a_* \\
\varphi(a^r b^s c) \varphi(c) \varphi(a^r b^s c) \varphi(c) &= c_* a_* \\
a_* c_* a_* c_* &= c_* a_* \\
a_* b_* &= c_* a_* \\
a_* b_* a_*^{-1} &= c_* \\
b_* &= c_*
\end{aligned}$$

It is no impossible, and causes contradiction.

And it has been proven that there is not isomorphism between G_9 and G_1 . So $G_9 \not\cong G_1$.

So

$$G_9 = \langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = a^2, cbc = b^2 \rangle$$

is another type of group which consists of 18 elemnts.

And we had find all the types of group which consists of 18 elements.

They are

- 1) \mathbb{Z}_{18}
- 2) $\mathbb{Z}_3 \oplus \mathbb{Z}_6$
- 3) D_9
- 4) $\langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = b, cbc = a \rangle$
- 5) $\langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = a^2, cbc = b^2 \rangle$

We have completed classification of groups which consists of 18 elemnts.