Correction of Homework on 12.02.2019
Tanglin

PROBLEM 1. Correct.

PROBLEM 2. Is group $\mathbb{Z}_2 \times D_3$ and D_6 isomorphic?
Solution:
We can describe group D_3 and D_6 in following forms:

$$D_3 = \{\langle \varphi_{D_3}, d_{D_3} \rangle | \varphi_{D_3}^3 = d_{D_3}^2 = e_{D_3}, d_{D_3}\varphi_{D_3}d_{D_3} = \varphi_{D_3}^{-1}\}$$

$$D_6 = \{\langle \varphi_{D_6}, d_{D_6} \rangle | \varphi_{D_6}^6 = d_{D_6}^2 = e_{D_6}, d_{D_6}\varphi_{D_6}d_{D_6} = \varphi_{D_6}^{-1}\}$$

So we can write $\mathbb{Z}_2 \times D_3$ as form:

$$\mathbb{Z}_2 \times D_3 = \{(a, b) | a \in \mathbb{Z}_2, b \in D_3\}$$

Where $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ and $a_1, a_2 \in \mathbb{Z}_2, b_1b_2 \in D_3$.
We will consider a map:

$$\begin{aligned} f : \mathbb{Z}_2 \times D_3 &\to D_6 \\ (z, e_{D_3}) &\longmapsto \varphi_{D_6}^3 \\ (e_Z, d_{D_3}) &\longmapsto d_{D_6} \\ (e_Z, \varphi_{D_3}) &\longmapsto \varphi_{D_6}^4 \\ f(r_1r_2) &= f(r_1)f(r_2) \end{aligned}$$

We can prove that f is a function.
$\forall r_1, r_2 \in \mathbb{Z}_2 \times D_3$
$\exists l_1, m_1, n_1$ and $l_2, m_2, n_2 \in \mathbb{Z}$, s.t. $r_1 = (z^{l_1}, \varphi_{D_3}^{m_1}d_{D_3}^{n_1}), r_2 = (z^{l_2}, \varphi_{D_3}^{m_2}d_{D_3}^{n_2})$
If $r_1 = r_2$, we can get:

$$l_1 \equiv l_2 (\text{mod } 2) \tag{1}$$
$$m_1 \equiv m_2 (\text{mod } 3) \tag{2}$$
$$n_1 \equiv n_2 (\text{mod } 2) \tag{3}$$

Then

$$\begin{aligned}
f(r_1) &= f((z^{l_1}, \varphi_{D_3}^{m_1} d_{D_3}^{n_1})) \\
&= f((z^{l_1}, e_{D_3})(e_Z, \varphi_{D_3}^{m_1})(e_Z, d_{D_3}^{n_1})) \\
&= f((z^{l_1}, e_{D_3}))f((e_Z, \varphi_{D_3}^{m_1}))f((e_Z, d_{D_3}^{n_1})) \\
&= f((z, e_{D_3}))^{l_1} f((e_Z, \varphi_{D_3}))^{m_1} f((e_Z, d_{D_3}))^{n_1} \\
&= \varphi_{D_6}^{3l_1} \varphi_{D_6}^{4m_1} d_{D_6}^{n_1} \\
&= \varphi_{D_6}^{3l_1+4m_1} d_{D_6}^{n_1}
\end{aligned}$$

In the same way, we can get:

$$f(r_2) = \varphi_{D_6}^{3l_2+4m_2} d_{D_6}^{n_2}$$

From (1),(2),(3), we have:

$$\begin{aligned}
l_1 &= 2k_1 + l_2 \\
m_1 &= 3k_2 + m_2 \\
n_1 &= 2k_3 + n_2
\end{aligned}$$

So

$$\begin{aligned}
3l_1 &= 3 \cdot 2k_1 + 3l_2 = 6k_1 + 3l_2 \\
4m_1 &= 4 \cdot 3k_2 + 4m_2 = 6 \cdot 2k_2 + 4m_2 \\
n_1 &= 2k_3 + n_2
\end{aligned}$$

And Then

$$\begin{aligned}
3l_1 &\equiv 3l_2 (\mathrm{mod}\ 6) \\
4m_1 &\equiv 4m_2 (\mathrm{mod}\ 6) \\
n_1 &\equiv n_2 (\mathrm{mod}\ 2)
\end{aligned}$$

So we reach our purpose:

$$\begin{aligned}
3l_1 + 4m_1 &\equiv 3l_2 + 4m_2 (\mathrm{mod}\ 6) \\
n_1 &\equiv n_2 (\mathrm{mod}\ 2)
\end{aligned}$$

Based on this system, we can assert:

$$\varphi_{D_6}^{3l_1+4m_1} d_{D_6}^{n_1} = \varphi_{D_6}^{3l_2+4m_2} d_{D_6}^{n_2}$$

and

$$f(r_1) = f(r_2)$$

So f is really a function.

Then we consider another function h:

$$\begin{aligned} h : D_6 &\to \mathbb{Z}_2 \times D_3 \\ d_{D_6} &\longmapsto (e_Z, d_{D_3}) \\ \varphi_{D_6} &\longmapsto (z, \varphi_{D_3}) \\ h(s_1 s_2) &= h(s_1)h(s_2) \end{aligned}$$

According to the same way, we can prove that h is also a function.
Now we can see $f \circ h$:
$\forall g \in D_6, \exists m_1, n_1 \in \mathbb{Z}$ s.t. $g = \varphi_{D_6}^{m_1} d_{D_6}^{n_1}$

$$\begin{aligned} f \circ h(g) &= f(h(\varphi_{D_6}^{m_1} d_{D_6}^{n_1})) \\ &= f(h(\varphi_{D_6}^{m_1}) h(d_{D_6}^{n_1}))) \\ &= f((z, \varphi_{D_3})^{m_1} (e_Z, d_{D_3})^{n_1}) \\ &= f((z, \varphi_{D_3}))^{m_1} f((e_Z, d_{D_3}))^{n_1} \\ &= \varphi_{D_6}^{7m_1} d_{D_6}^{n_1} \\ &= \varphi_{D_6}^{m_1} d_{D_6}^{n_1} \\ &= g \end{aligned}$$

$\therefore f \circ g = \varepsilon_{D_6}$ On contrary, we will see $h \circ f$: $\forall g \in \mathbb{Z}_2 \times D_3, \exists l_1, m_1, n_1 \in \mathbb{Z}$ s.t. $g = (z^{l_1}, \varphi_{D_3}^{m_1} d_{D_3}^{n_1})$

$$
\begin{aligned}
h \circ f(g) &= h(f((z^{l_1}, \varphi_{D_3}^{m_1} d_{D_3}^{n_1}))) \\
&= h(f((z^{l_1}, e_{D_3}))f((e_Z, \varphi_{D_3}^{m_1}))f((e_Z, d_{D_3}^{n_1}))) \\
&= h(f((z, e_{D_3}))^{l_1} f((e_Z, \varphi_{D_3}))^{m_1} f((e_Z, d_{D_3}))^{n_1}) \\
&= h(\varphi_{D_6}^{3l_1} \varphi_{D_6}^{4m_1} d_{D_6}^{n_1}) \\
&= h(\varphi_{D_6})^{3l_1+4m_1} h(d_{D_6})^{n_1} \\
&= (z^{3l_1+4m_1}, \varphi_{D_3}^{3l_1+4m_1})(e_Z, d_{D_3}^{n_1}) \\
&= (z^{3l_1} z^{4m_1}, \varphi_{D_3}^{3l_1} \varphi_{D_3}^{4m_1} d_{D_3}^{n_1}) \\
&= (z^{l_1}, \varphi_{D_3}^{m_1} d_{D_3}^{n_1}) \\
&= g
\end{aligned}
$$

$\therefore h \circ f = \varepsilon_{\mathbb{Z}_2 \times D_3}$
Therefore we can get consequence, functions f and h are inverse functions of each other.
So, f is an isomorphism from $\mathbb{Z}_2 \times D_3$ onto D_6.
$\therefore \mathbb{Z}_2 \times D_3 \cong D_6$
The proof has been completed.

Problem 3. Correct.