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Problem: Please classify all groups which consists of 18 elements.
Solve:

$$18 = 2 \cdot 3^2$$

From The Third Theorem of Syllow, we have:

$$N_2 \equiv 1(mod\ 2) \quad N_2|9 \Rightarrow N_2 = 1, 3, 9$$
$$N_3 \equiv 1(mod\ 3) \quad N_3|2 \Rightarrow N_3 = 1$$

So we have 2 cases:

1. $N_2 = 1, N_3 = 1$ In this situation,there is only one 2-Syllow subgroup and one 3-Syllow subgroup, so $G \cong S_2 \oplus S_3$, where $|S_2| = 2, |S_3| = 9$.
   There will be 2 situations:

   (a)
   $$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{18}$$

   (b)
   $$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_6$$

2. $N_2 \neq 1, N_3 = 1$
   $S_3 \lhd G$
   So $G \cong S_3 \rightthreetimes K$,where $|K| = 2$.

   (a) $S_3 \cong \mathbb{Z}_9$ We can get automorphism of $S_3 \cong \mathbb{Z}_9$: $\mathrm{Aut}(\mathbb{Z}_9) = H \cong \mathrm{Aut}(S_3)$,where $|H| = 6$.
   So We can make a table of automorphism,we assume $S_3 = \langle a \rangle$:

ε	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
σ_1	e	a^2	a^4	a^6	a^8	a	a^3	a^5	a^7
σ_2	e	a^4	a^8	a^3	a^7	a^2	a^6	a	a^5
σ_3	e	a^5	a	a^6	a^2	a^7	a^3	a^8	a^4
σ_4	e	a^7	a^5	a^3	a	a^8	a^6	a^4	a^2
σ_5	e	a^8	a^7	a^6	a^5	a^4	a^3	a^2	a

   Where $\varepsilon, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are elements of $\mathrm{Aut}(S_3)$.
   Without difficut we can get rank of each map of isomorphism:

   $$O(\sigma_1) = 6, O(\sigma_2) = 3, O(\sigma_3) = 6, O(\sigma_4) = 3, O(\sigma_5) = 2$$

$\because$There is map with rank 6, so $\mathrm{Aut}(S_3) \cong \mathbb{Z}_6$.
Now we assume $K = \langle b \rangle, |K| = 2$, if we can find a homormorphism $\varphi : K \to \mathrm{Aut}(S_3)$, then we can establish a group.
$\because$ In subgroup K, there are only elemnts with rank 2 and zero.
So we can construct a map:

$$\begin{aligned} \varphi : K &\to \mathrm{Aut}(S_3) \\ e &\longmapsto \varepsilon \\ b &\longmapsto \sigma_5 \end{aligned}$$

So we can get a group with form:

$$G \cong \langle a, b | a^9 = b^2 = e, bab = a^{-1} \rangle \cong D_9$$

(b) $S_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\because |S_3| = 9 = 3^2, \therefore S_3$ is Abelian group. We can write S_3 in form

$$S_3 = \langle a, b | a^3 = b^3 = 1 \rangle$$

$\because S_3 \lhd G, \therefore cS_3c = S_3$,where $c \in K = \langle c \rangle, |K| = 2$.
So $\exists i_1, j_1, i_2, j_2 \in \{0, 1, 2\}$,and

$$cac = a^{i_1} b^{j_1}$$

$$cbc = a^{i_2} b^{j_2}$$

Therefore

$$\begin{aligned} cac &= a^{i_1} b^{j_1} \\ ccacc &= ca^{i_1} b^{j_1} c \\ a &= ca^{i_1} ccb^{j_1} c \\ a &= (cac)^{i_1} (cbc)^{j_1}, \text{ becasue } ca^n c = (cac)(cac) \cdots (cac) = (cac)^n \\ a &= (a^{i_1} b^{j_1})^{i_1} (a^{i_2} b^{j_2})^{j_1} \end{aligned}$$

$\because S_3$ is Abelian group, so

$$a = a^{i_1^2 + i_2 j_1} b^{i_1 j_1 + j_2 j_1}$$

Analogously, we can get

$$
\begin{aligned}
cbc &= a^{i_2}b^{j_2} \\
ccbcc &= ca^{i_2}b^{j_2}c \\
b &= ca^{i_2}ccb^{j_2}c \\
b &= (cac)^{i_2}(cbc)^{j_2} \\
b &= (a^{i_1}b^{j_1})^{i_2}(a^{i_2}b^{j_2})^{j_2} \\
b &= a^{i_1i_2+i_2j_2}b^{i_2j_1+j_2^2}
\end{aligned}
$$

Becase in S_3, $\langle a\rangle \cup \langle b\rangle = \emptyset$, so we can get a congruence equation:

$$
\begin{cases}
i_1^2 + i_2j_1 \equiv 1 (\text{mod } 3) \\
i_1j_1 + j_2j_1 \equiv 0 (\text{mod } 3) \\
i_1i_2 + i_2j_2 \equiv 0 (\text{mod } 3) \\
i_2j_1 + j_2^2 \equiv 1 (\text{mod } 3)
\end{cases}
$$

To solve this equation, we can respectively let $i_1 = 0, 1, 2$, and discuss them, but it is little complex. There will be a better method by using computer program. This is Python code.

```
DM = []
for i in range(0,3):
    for j in range(0,3):
        for k in range(0,3):
            for l in range(0,3):
                DM.append((i,j,k,l))

#       i1^2  + i2*j1 = 1 (mod 3)
#       i1*j1 + j2*j1 = 0 (mod 3)
#       i1*i2 + i2*j2 = 0 (mod 3)
#       i2*j1 + j2^2  = 1 (mod 3)
#   There I use representations: i[0] ~ i1; i[1]~j1; i[2]~i2; i[3]~j2

Ans = []
for i in DM:
    if (i[0]**2 + i[2]*i[1]) % 3 == 1 and\
    (i[0]*i[1] + i[3]*i[1]) % 3 == 0 and\
    (i[0]*i[2] + i[2]*i[3]) % 3 == 0 and\
    (i[2]*i[1] + i[3]**2) % 3 == 1:
        Ans.append(i)

print(Ans)
```

Then we will get answers: $(i_1, j_1, i_2, j_2) \in \{(0,1,1,0),(0,2,2,0),(1,0,0,1),(1,0,0,2),(1,0,1,2),$
$(1,0,2,2),(1,1,0,2),(1,2,0,2),(2,0,0,1),(2,0,0,2),(2,0,1,1),(2,0,2,1),$
$(2,1,0,1),(2,2,0,1)\}$
For these solutions, we can construct conjugation relationship:

1) $cac = b$	$cbc = a$
2) $cac = b^2$	$cbc = a^2$
3) $cac = a$	$cbc = b$
4) $cac = a$	$cbc = b^2$
5) $cac = a$	$cbc = ab^2$
6) $cac = a$	$cbc = a^2b^2$
7) $cac = ab$	$cbc = b^2$
8) $cac = ab^2$	$cbc = b^2$
9) $cac = a^2$	$cbc = b$
10)$cac = a^2$	$cbc = b^2$
11)$cac = a^2$	$cbc = ab$
12)$cac = a^2$	$cbc = a^2b$
13)$cac = a^2b$	$cbc = b$
14)$cac = a^2b^2$	$cbc = b$

But when we compare pairs [4), 9)],[5), 13)],[6), 14)],[7), 11)] and [8), 12)], we will discover that if we exchange status of a and b, and they are the same. So, in fact we could just explore cases 1), 2), 3), 4), 5), 6), 7), 8), 10).
Now we will consider them:

i. $cac = b, cbc = a$

$$G_1 = \langle a, b, c | a^3 = b^3 = c^2 = e, ab = ba, cac = b, cbc = a \rangle$$

ii. $cac = b^2, cbc = a^2$

$$cac = b^2, cbc = a^2 \Rightarrow cac = b^2, cb^2c = (a^2)^2 = a$$

$\therefore$ We can construct a map from G_2 onto G_1

$$\begin{aligned} \varphi : G_2 &\to G_1 \\ c &\longmapsto c \\ a &\longmapsto a \\ b^2 &\longmapsto b \end{aligned}$$

We can certify that c, a and b^2 all are generators,and we can use a and b^2 to generate group S_3, so this map will be isomorphism. The only difference between G_2 and G_1 is that we exchanged the symols of b and b^2. So $G_2 \cong G_1$.

iii. $cac = a, cbc = b$

In this situation, we will get

$$ca = ac, cb = bc, ab = ba$$

So G_3 will be an Abelian group, because$\forall a^{r_i} b^{s_i} c^{t_i} \in G_3, i = 1, 2,$

$$\begin{aligned} a^{r_1} b^{w_1} c^{t_1} \cdot a^{r_2} b^{s_2} c^{t_2} &= a^{r_1+r_2} b^{s_1+s_2} c^{t_1+t_2} \\ &= a^{r_2} b^{s_2} c^{t_2} \cdot a^{r_1} b^{s_1} c^{t_1} \end{aligned}$$

But $N_2 \neq 1$,so G_3 is impossible Abelian group, therefore it leads to contradiction. And G_3 does not exist.

iv. $cac = a, cbc = b^2$

$$cac = a, cbc = b^2 \Rightarrow cabc = caccbc = ab^2, cab^2c = ab$$

And now construct map from G_4 onto G_1:

$$\begin{aligned} \varphi : G_4 &\to G_1 \\ c &\longmapsto c \\ ab &\longmapsto a \\ ab^2 &\longmapsto b \end{aligned}$$

In this situation, we can proof that $\langle ab, ab^2 \rangle = \langle a, b \rangle = S_3$,because of $abab^2 = a^2, (a^2)^2 = a, (ab)^2ab^2 = b$. So ab and ab^2 are generators and they can generate S_3, and φ is isomorphism. We can change symols a to ab and b to ab^2, then we will get G_4 from G_1. So $G_4 \cong G_1$.

v. $cac = a, cbc = ab^2$

$$cbc = ab^2 \Rightarrow cbc = ab^2, cab^2c = b$$

For map:

$$\begin{aligned} \varphi : G_5 &\to G_1 \\ c &\longmapsto c \\ ab^2 &\longmapsto a \\ b &\longmapsto b \end{aligned}$$

Analogously,we can certify that $\langle ab^2, a \rangle = S_3, \because ab^2b = a$. Therefore, φ is isomorphism, and $G_5 \cong G_1$.