Homework 3
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TangLin

PROBLEM 1. Construct formula of language $(+,0,1,=)$ whose length is $O(\log n)$ with free variable x for any natural number n, and it is true in Module $(\mathbb{N},+,0,1)$ if and only if $x=n$.

Solution:

Let the formula which we want to build be $R_n(x)$. We can build it inductively. First, let

$$R_0(x) = (x = 0)$$

then, for any natural number $n \leqslant 1$,

$$R_n(y) = \begin{cases} \exists x \left(R_{\frac{n}{2}}(x) \wedge y = x + x\right), & \text{if } n \text{ is even number.} \\ \exists x \left(R_{n-1}(x) \wedge y = x + 1\right), & \text{if } n \text{ is odd number.} \end{cases}$$

Obviously, it is true for all natural number n

$$R_n(x) = 1 \Leftrightarrow x = n.$$

Now we need to analyse its number of symbols. We use function $l(R)$ to indicate the number of symbols of formula R. $l(R_1(x)) = 5$, when $n = 0$; $l(R_n(x)) = l(R_{\text{previous}}(x)) + 10$ when n is odd or even. So, we can get the relation

$$\begin{aligned} l(R_n(x)) \quad &= l(R_{\lfloor \frac{n}{2} \rfloor}(x)) + 10 \\ &= l(R_{\lfloor \frac{n}{2^2} \rfloor}(x)) + 2 \cdot 10 \\ &= l(R_{\lfloor \frac{n}{2^3} \rfloor}(x)) + 3 \cdot 10 \\ &= \cdots \\ &= l(R_0(x)) + (\log n + 1) \cdot 10 \\ &= 10 \log n + 15 \in O(\log n). \end{aligned}$$

So, we design a formula $R_n(x)$ with number of symbols $O(\log n)$ which can decide $x = n$ and the number of variables are $O(1)$.

PROBLEM 2. Construct formula of language $(+,0,1,=)$ whose length is $O(n)$ with three free variables x,y,z, and it is true in Module $(\mathbb{N},+,0,1,=)$ if and only if $x = y \cdot z$ and $y < 2^n$.

Solution:

Let the formula which we want to find be $P_n(x,y,z)$. We can also build it inductively. First, let

$$P_0(x,y,z) = (y = 0 \wedge z = 0) \vee (y = 1 \wedge z = x).$$

Then we are building the inductive part.
Because we have the limiting condition $y < 2^n$, so we have to design the recursive process from n to $n+1$ such that y satisfies corresponding condition. We can do it, let

$$y = y_1 + y_2 + 1$$

where $y < 2^{n+1}$) and $y_1, y_2 < 2^n$. So the maximum of $y + 1$ is

$$1 + y = y_1 + y_2 + 1 + 1 = 2^n - 1 + 2^n - 1 + 2 = 2^{n+1}$$

i.e. y has maximum $2^{n+1} - 1$, it satisfies our requirement. For any z, we have $zy = z(y_1 + y_2 + 1) = zy_1 + zy_2 + z$. Then we build the inductive formula,

$$P_{n+1}(x, y, z) = \exists y_1, y_2, r, s\, (y = y_1 + y_2 + 1 \wedge x = r + s + z \wedge P_n(r, y_1, z) \wedge P_n(s, y_2, z))$$

But there are two P_n occurrences, and the total length will be the exponential of n. So we have to do some transformation.

$$\begin{aligned} P_{n+1}(x, y, z) = \quad & \exists y_1, y_2, r, s\Big(y = y_1 + y_2 + 1 \wedge x = r + s + z \wedge \\ & \wedge \forall t \exists u (t = y_1 \wedge u = r) \vee (t = y_2 \wedge u = s) \rightarrow P_n(u, t, z)\Big) \end{aligned}$$

Now we analyse the number of symbols of formula $P_n(x, y, z)$. We use notation $l(P_n)$ to indicate the number of symbols of P_n.

$$l(P_0(x, y, z)) = 19$$

then for any $n >= 1$,

$$l(P_n(x, y, z)) = l(P_{n-1}(x, y, z)) + 50$$

So

$$l(P_n(x, y, z)) = 19 + 50n \in O(n).$$

We have built the formula which satisfies the condition with 9 variables.