Homework 2
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PROBLEM 1. Prove that the function $x, y \mapsto \lfloor x^{1/y} \rfloor$ (for positive numbers) belongs to the class FP.

Solution:

I will prove it by constructing Random Access Machine program which can find result in polynomial time, and according to the theorem of translation from RAM to MT, the complexity remains polynomial if the problems can be solved by RAM in polynomial time.

Then I give the algorithm.

```
// Initialize all the registors we will use.
a <- 2;
min_a <- 1;
max_a <- 2;
s <- 1;
tmp <- 1
t <- 0
s <- 0

x <- input_x;
y <- input_y;

// Use dichotomy to find suitable results.
while true:
    t <- input_x;
    s <- 1;
    tmp <- a;

    // Fast exponentiation using dichotomy.
    while t != 0:
        if t % 2 == 0 then {tmp <- tmp*tmp; t <- t/2}
        else {s <- s*tmp; t <- t-1}

if s > x then {
    tmp <- min_a;
    tmp <- tmp + max_a;
    tmp <- tmp/2;
    a <- tmp;
    max_a <- a;
```

```
        if a == min_a then print a;
    }
    else {
        if s == x then {print a;}
        else {
            min_a <- a;
            a <- 2*a;
            max_a <- a;
        }
    }
```

Let $n = \log\max(x, y)$, where x and y are positive integer numbers. In this algorithm, I need 9 registers to store binary data, and let all the registers have the same bits n, so $l = O(n)$ and $S = O(1)$.
Now we need to analyze the time complexity. In the main while loop, there is another while loop for calculating register a of power register y. Becase I need compare a^y with b, and find the final answer. Let the time complexity of this loop be w_2, we have

$$w_2 \leqslant 2(\log(y) + 1)$$
$$w_2 = O(n)$$

In main while loop, I use a dichotomy to find number a, the largest number whose y-th power dose not exceed x. Let the time complexity of main while loop be w_1,so

$$w_1 \leqslant 2(\log(x) + 1) \cdot w_2 \cdot (3 + 6) + 1$$
$$w_1 = O(\log(\max(x, y)) \cdot w_2) = O(n) \cdot O(n) = O(n^2)$$

So $t = O(n^2)$. And this algorithm can be executed by RAM in $O(n^2)$ time, and according to the theorem we can obtain the time complexity on TM with the same algorithm,

$$t' = O(tSl^2 + n) = O(n^2 \cdot 1 \cdot n^2 + n) = O(n^4) \in \text{FP}.$$

So, we can find the required number in polynomial time by Turing Machine. And this problem belongs to the class FP.

Problem 2. Function EVAL which receives propositional formula with logical connectors $\wedge, \vee, \neg$ and values of variables of this formula as arguments gives truth value of the formula on the variables, prove that function EVAL belongs to the class FP.

Solution:

The proof for this problem is similar to the first problem. I will build an algorithm which can be executed by RAM, and explain that it can be half in polynomial time on RAM, and print the right answer.

The algorithm:

```
\\ Initial all the rigestors we will use.

i <- 0                          // The index of character of expression.
stack_i = [0,0,...,0]           // Rigisters for i.
stack_ex = ['', ..., '']        // Rigisters for subexpressions.
stack_paren = [0,...,0]         // Denotes the number of left parenthesises.
stack_label = ['', ..., ''] // Denotes the next lable.
stack_left = [0,...,0]
stack_right = [0,...,0]
k_left <- 0                  // index of left
k_right <- 0                 // index of right
j <- 0                          // The stack index.
tmp <- 0                    // The value of expreesion or subexpression
parent <- 0                 // The number of parenthesised.
tmplabel <- 'loop'

expr <- input

label_loop:
    if expr[i] == '(' then { paren <- paren + 1 }
    else if expr[i] == ')' then{ paren <- paren - 1 }
    else {
        if expr[i] == '~' then {
            if expr[i+1] in {0,1} then {
                tmp <- not tmp
                if j != 0 then {
                        // There should write judgement statement,
                        // But for convenience, write like this.
                        tmp_lable <- stack_lable[j]
                        expr <- stack_ex[j]
                        i <- stack_i[j]
                        paren <- stack_paren[j]
                        j <- j-1
                        got tmp_lable
                    }
            }

            else {
                stack_i[j] <- i
                stack_paren[j] <- parent
                stack_ex[j] <- expr[i:end] //Worst situation,t=O(n)
```

```
            stack_label[j] <- 'lab_not'
            j <- j+1
            goto label_loop
            lable_not:
                if tmp == 0 then tmp <- 1
                else tmp <- 0
                if j != 0 then {
                    tmp_lable <- stack_lable[j]
                    expr <- stack_ex[j]
                    i <- stack_i[j]
                    paren <- stack_paren[j]
                    j <- j-1
                    got tmp_lable
                }
        }
    }

    else if expr[i] == '&' then {
        if expr[i-1] in {0,1} then {
            stack_left <- expr[i-1]
            k_left <- k_left+1
        }

        else {
            stack_i[j] <- i
            stack_paren[j] <- parent
            stack_ex[j] <- expr[star:i]
            stack_label[j] <- 'lab_land'
            j <- j+1
            goto label_loop
            label_land:
                stack_left[k_left] <- tmp
                k_left <- k_left+1
                if j != 0 then {
                    tmp_lable <- stack_lable[j]
                    expr <- stack_ex[j]
                    i <- stack_i[j]
                    paren = stack_paren[j]
                    j <- j-1
                    got tmp_lable
        }

        if expr[i+1] in {0,1} then {
            stack_right <- expr[i+1]
            k_right <- k_right+1
        }
```

```
            else {
                stack_i[j] <- i
                stack_paren[j] <- parent
                stack_ex[j] <- expr[i:end]
                stack_label[j] <- 'lab_rand'
                j <- j+1
                goto label_loop
                label_rand:
                    stack_left[k_right] <- tmp
                    k_right <- k_right+1
                    if j != 0 then {
                        tmp_lable <- stack_lable[j]
                        expr <- stack_ex[j]
                        i <- stack_i[j]
                        paren = stack_paren[j]
                        j <- j-1
                        got tmp_lable
            }

            k_left < k_left-1
            k_right < k_right -1
            tmp <- stack_left[k_left] and stack[k_right]
            if j != 0 then {
                tmp_lable <- stack_lable[j]
                expr <- stack_ex[j]
                i <- stack_i[j]
                paren = stack_paren[j]
                j <- j-1
                got tmp_lable
        }

        else if expr[i] == '|' then
            ...
            // It is very similar to the previous case.
    }

print tmp
```

The main idea I used here is Expression Tree. For this purpose, our input must be processed applicable form. For example, if our input has form

$$x_1 \wedge x_2 \wedge x_3 \vee x_4 \vee x_5 \wedge x_6 \vee \neg x_7$$

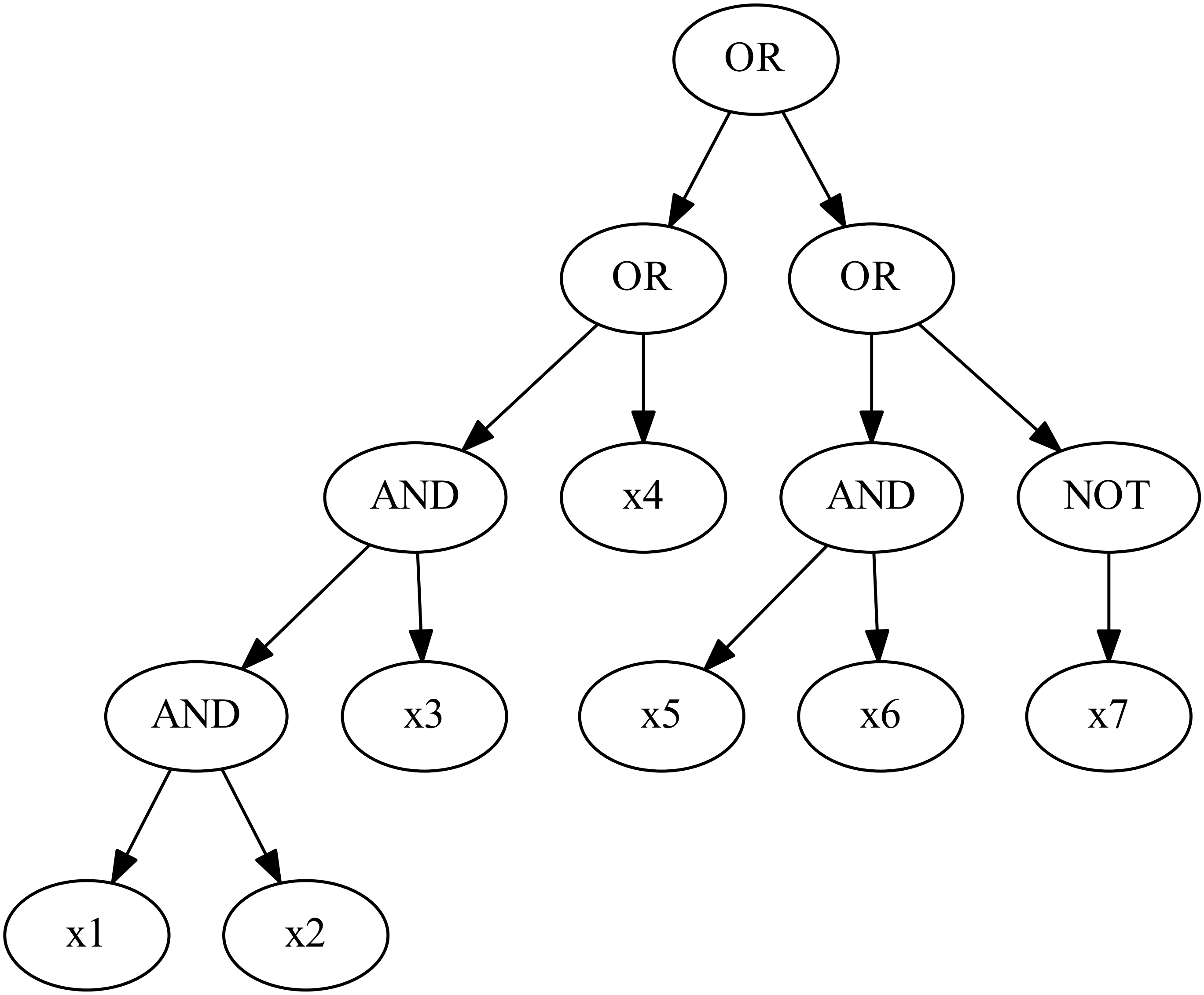
According to the associative rules of logical symbols, we well know it is equivalent to

$$(x_1 \wedge x_2 \wedge x_3) \vee x_4 \vee (x_5 \wedge x_6) \vee (\neg x_7)$$

But it is not enough for my algorithm, because I find the outmost logical symbol based on the parenthesis. So I need add more parenthesis

$$(((x_1 \wedge x_2) \wedge x_3) \vee x_4) \vee ((x_5 \wedge x_6) \vee (\neg x_7))$$

All the three formulas are the same, they have the same expression tree


OR
OR
OR
AND
x4
AND
NOT
AND
x3
x5
x6
x7
x1
x2

In my program, it counts the parentheses as number n_{pa}, if it meets (then let increase n_{pa} by 1, if it meets) then let reduce n_{pa} by 1; if it meets a logical symbol and $n_{pa} = 0$ at the same time, then it is the root node of expression(subexpression). Then evaluate the value of the expression(subexpression). It is a recursive process, but I translate it to a loop with many stacks which can store these statuses.

Now I analyze complexity of this algorithm. First, if the input string of expression has n symbols (including parentheses), so we will use at most $a \cdot n$ registers, so $S = O(n)$, and every register at most $b \cdot n$ bits, so $l = O(n)$, where a, b are some constants. And the program evaluates does not exceed n times, because there are less than n logical symbols and every logical symbol only need evaluate once. So $t = O(n)$.

I have to explain that, in my algorithm I didn't perform actions that receive a formula, and the values of arguments of the formula. But it is obvious that it can be reached in $O(n)$ time and only use $O(n)$ spaces. So the complexity will not change.

Hence, I built an algorithm with $l = O(n), S = O(n), t = O(n)$ by RAM, and according to the theorem, there will be a TM with $t = O(tsl^2 + n) = O(n^4)$ which can execute this algorithm. So this problem belongs to the class FP.

The second solution:

I also consider this problem and try to build a Turing Machine to answer the question. I find that is can be achieved in $O(n^3)$ time. The main idea is similar to the first one. In this TM T, there are some statuses include the most important "reset" status, if there are n logical symbols, T will evaluate each logical symbol with values of its arguments including removing parentheses in $O(n)$ time, and then reset the head to the initial position and reset status to initial status, then there will be at most $n - 1$ logical symbols which are not evaluated. So,

$$t = O(n \cdot \sum_{i=1}^{n} i) = O(n^3)$$

It indicates that this problem belongs to the class FP.