CS787: Advanced Algorithms	**Scribe:** Tien-Lung Fu
Lecture 25: Probabilistic Method, Method of Conditional Expectation, Lovasz Local Lemma **Date:** 04/30/2019	

25.1 Probabilistic Method

In this section, several theorems are listed as examples of probabilistic method.

Theorem 25.1.1 *For any $n \in Z^+$ and $\epsilon > 0$, $\exists$ $k = \exp(\text{const.}\,\epsilon^2 n)$ unit vectors $x_1, \ldots, x_k$ in $\mathbb{R}^n$ with $|x_i \cdot x_j| \leq \epsilon$, $\forall i, j \in [k]$.*

Proof: Assign $+1$ or -1 to each coordinate of x_i uniformly at random. Denote $Z_l = \mathbb{1}(x_{il} = x_{jl})$ as an indicator for the l^{th} entry of vector x_i and x_j to be equal. Also we let $Z = \sum_l Z_l$.

Then we have

$$\begin{aligned} |x_i \cdot x_j| &= |\sum_l Z_l - n + \sum_l Z_l| = 2|\frac{n}{2} - Z|, \\ \mathbb{E}[Z] &= \frac{n}{2}. \end{aligned}$$

By Chernoff bound

$$Pr[|x_i \cdot x_j| > \epsilon n] = Pr[|\frac{n}{2} - z| > \frac{\epsilon}{2} n] \leq \exp\left(-\frac{\epsilon^2}{3} \cdot \frac{n}{2}\right).$$

Choosing $k = \exp(\frac{\epsilon^2 n}{12})$, we have

$$Pr[\exists i, j \in [k] \; with \; |x_i \cdot x_j| > \epsilon n] \leq \frac{k^2}{2} \exp(-\frac{\epsilon^2 n}{6}) = \frac{1}{2}.$$

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Theorem 25.1.2 *In any undirected graph, G=(V,E), $\exists$ an independent set of size $\leq \sum_{v \in V} \frac{1}{degree(v)+1}$.*

Proof: The algorithm:

- Assign independently random number drawn from Uniform$[0, 1]$ to every vertex v.
- Output v if it is a local minimum.

Notice that the algorithm above always returns an independent set and the expected size of this independent set is $\sum_v \frac{1}{degree(v)+1}$.

This holds by linearity of expectation and these two observations:

1. If $Pr[E] > 0$, then there exists a state of the world where E happens.

2. For any random variable X, there exists an instantiation with $X \geq \mathbb{E}[X]$

$\blacksquare$

Theorem 25.1.3 *In any undirected graph G=(V,E), there exists a cut of size* $\geq \frac{|E|}{2}$

Proof: The algorithm:

- Place every vertex independently in left part L with probability $\frac{1}{2}$ and right part R with probability $\frac{1}{2}$.
- Return partition $(L; R)$.

$\blacksquare$

Theorem 25.1.4 *For any k-CNF formula ϕ, there exists an assignment that satisfies at least $(1 - 2^{-k})$ fraction of the clauses in ϕ*

Proof: MaxSat: What is the most number of clauses we can satisfy in ϕ?

Set every variable to True of False with equal probability independently, then

$Pr[a\ clause\ is\ not\ satisfied] = \frac{1}{2^{size\ of\ clauses}}$. $\blacksquare$

25.2 Method of Conditional Expectation

Here is an example of satisfactory problem:

$$(x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\vee x_1 \wedge \neg x_3 \vee \neg x_4).$$

Then there exists an assignment that satisfies $\frac{3}{4} + \frac{15}{16} + \frac{7}{8}$ fraction
If we randomly assign x_1 first, the expected number of fraction is

$$\begin{aligned}
&Pr[x_1 = True] * \mathbb{E}[fraction\ |x_1 = True] + Pr[x_1 = False] * \mathbb{E}[fraction\ |x_1 = False] \\
&= \frac{1}{2} * \frac{11}{4} + \frac{1}{2} * \frac{19}{8}.
\end{aligned}$$

We should assign x_1 = True since $\mathbb{E}[fraction\ |x_1 = True]$ is greater. We can do similar analysis on $x_2,\ x_3,\ x_4$

25.3 Lovasz Local Lemma

Theorem 25.3.1 *Given a k-CNF formula ϕ with number of clauses $< 2^k$, ϕ is always satisfied.*

Lemma 25.3.2 (Symmetric Lovasz Local Lemma (SLL)) *Let $E_1, E_2, ..., E_n$ be n random events and let Γ_i denote the dependency neighborhood of event E_i. Suppose that $Pr[E_i] \leq p\ \forall i$, and $|\Gamma_i| < d\ \forall i$ and $e \cdot p \cdot (d + 1) \leq 1$, where e is the base of natural logarithms. Then $Pr[\cup_{i \in [n]} E_i] < 1$*

Note: The condition of this lemma is saying every event is independent of most of the events. By independency, we mean $\forall S \subseteq [n] \setminus \{\Gamma_i \cup i\},\ Pr[E_i|S] = Pr[E_i]$.

Theorem 25.3.3 *Let ϕ be a k-CNF formula where each variable appears in at most $\frac{2^k}{e \cdot k}$ clauses. Then ϕ is satisfied.*

Proof: Denote E_i as clause i is not satisfied. If we set $p = \frac{1}{2^k}$, we have

$$d \leq k\frac{2^k}{ek} - 1 = \frac{2^k}{e} - 1.$$

Conditions for SLL are satisfied. ■