More Undecidable Problems

Rice's Theorem

Post's Correspondence Problem

Some Real Problems

Properties of Languages

- Any set of languages is a *property* of languages.
- Example: The infiniteness property is the set of infinite languages.
- In what follows, we'll focus on properties of RE languages, because we can't represent other languages by TM's.

Properties of Langauges – (2)

- Thus, we shall think of a property as a problem about Turing machines.
- Let L_P be the set of binary TM codes for TM's M such that L(M) has property P.

Trivial Properties

- There are two (*trivial*) properties P for which L_P is decidable.
  1. The *always-false property*, which contains no RE languages.
  2. The *always-true property*, which contains every RE language.
- Rice's Theorem: For every other property P, L_P is undecidable.

Reductions

- A *reduction* from language L to language L′ is an algorithm (TM that always halts) that takes a string w and converts it to a string x, with the property that:

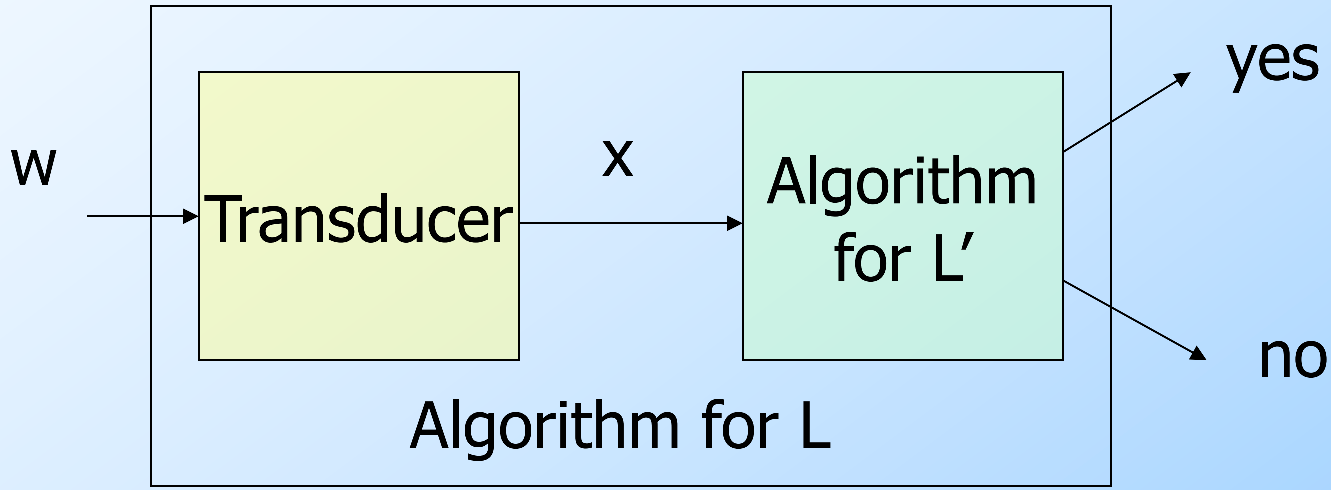
    x is in L′ if and only if w is in L.

TM's as *Transducers*

- We have regarded TM's as acceptors of strings.
- But we could just as well visualize TM's as having an *output tape*, where a string is written prior to the TM halting.

Reductions – (2)

- If we reduce L to L′, and L′ is decidable, then the algorithm for L′ + the algorithm of the reduction shows that L is also decidable.



Reductions – (3)

- Normally used in the contrapositive.
- If we know L is not decidable, then L' cannot be decidable.

Reductions – Aside

- This form of reduction is not the most general.
- Example: We "reduced" L_d to L_u, but in doing so we had to complement answers.
- More in NP-completeness discussion on Karp vs. Cook reductions.

Proof of Rice's Theorem

- We shall show that for every nontrivial property P of the RE languages, L_P is undecidable.
- We show how to reduce L_u to L_P.
- Since we know L_u is undecidable, it follows that L_P is also undecidable.

The Reduction

- Our reduction algorithm must take M and w and produce a TM M′.
- L(M′) has property P if and only if M accepts w.
- M′ has two tapes, used for:
  1. Simulates another TM M_L on the input to M′.
  2. Simulates M on w.
     - Note: neither M, M_L, nor w is input to M′.

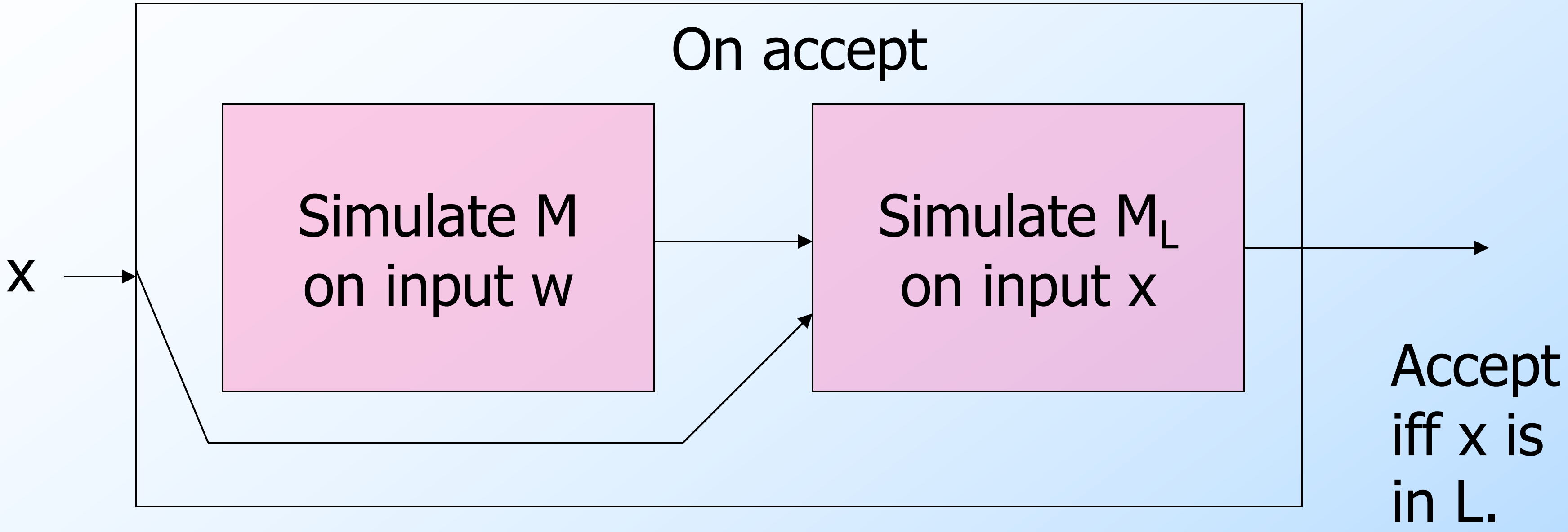
The Reduction – (2)

- Assume that ∅ does not have property P.
  - If it does, consider the complement of P, which would also be decidable if P were, because the recursive languages are closed under complementation.
- Let L be any language with property P, and let M_L be a TM that accepts L.

Design of M’

1. On the second tape, write w and then simulate M on w.
2. If M accepts w, then simulate M_L on the input x to M’, which appears initially on the first tape.
3. M’ accepts its input x if and only if M_L accepts x.

Action of M' if M Accepts w



Design of M’ – (2)

- Suppose M accepts w.
- Then M’ simulates M_L and therefore accepts x if and only if x is in L.
- That is, L(M’) = L, L(M’) has property P, and M’ is in L_P.

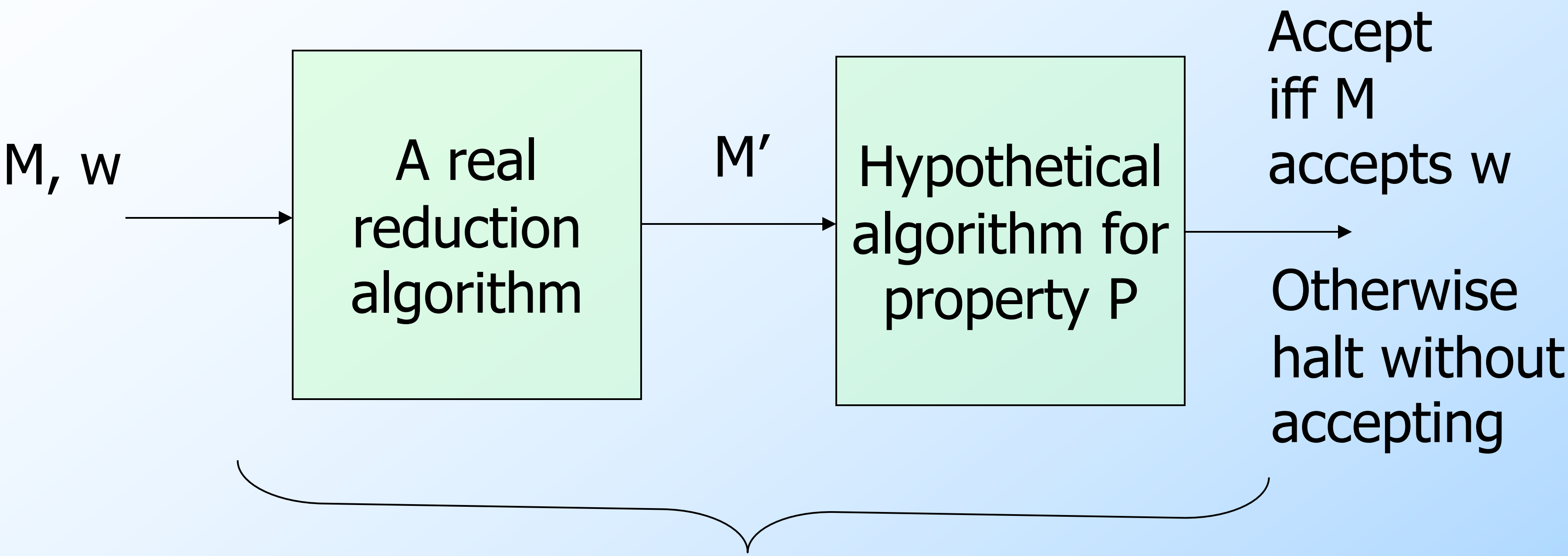
Design of M' – (3)

- Suppose M does not accept w.
- Then M' never starts the simulation of M_L, and never accepts its input x.
- Thus, $L(M') = \varnothing$, and L(M') does not have property P.
- That is, M' is not in L_P.

Design of M' – Conclusion

- Thus, the algorithm that converts M and w to M' is a reduction of L_u to L_P.
- Thus, L_P is undecidable.

Picture of the Reduction



This would be an algorithm for L_u, which doesn’t exist

Applications of Rice's Theorem

- We now have any number of undecidable questions about TM's:
  - Is L(M) a regular language?
  - Is L(M) a CFL?
  - Does L(M) include any palindromes?
  - Is L(M) empty?
  - Does L(M) contain more than 1000 strings?
  - Etc., etc.

Post's Correspondence Problem

- *Post's Correspondence Problem* (PCP) is an example of a problem that does not mention TM's in its statement, yet is undecidable.
- From PCP, we can prove many other non-TM problems undecidable.

PCP Instances

- An instance of PCP is a list of pairs of nonempty strings over some alphabet Σ.
  - Say $(w_1, x_1), (w_2, x_2), \ldots, (w_n, x_n)$.
- The answer to this instance of PCP is "yes" if and only if there exists a nonempty sequence of indices $i_1,\ldots,i_k$, such that $w_{i1}\ldots w_{in} = x_{i1}\ldots x_{in}$.

Example: PCP

- Let the alphabet be {0, 1}.
- Let the PCP instance consist of the two pairs (0, 01) and (100, 001).
- We claim there is no solution.
- You can't start with (100, 001), because the first characters don't match.

Example: PCP – (2)

Recall: pairs are (0, 01) and (100, 001)

0100100
01001001

But we can never make the first string as long as the second.

Must start with first pair

Can add the second pair for a match

As many times as we like

Example: PCP – (3)

- Suppose we add a third pair, so the instance becomes: 1 = (0, 01); 2 = (100, 001); 3 = (110, 10).
- Now 1,3 is a solution; both strings are 0110.
- In fact, any sequence of indexes in **12***3** is a solution.

Proving PCP is Undecidable

- We'll introduce the *modified* PCP (MPCP) problem.
  - Same as PCP, but the solution must start with the first pair in the list.
- We reduce L_u to MPCP.
- But first, we'll reduce MPCP to PCP.

Example: MPCP

- The list of pairs (0, 01), (100, 001), (110, 10), as an instance of MPCP, has a solution as we saw.
- However, if we reorder the pairs, say (110, 10), (0, 01), (100, 001) there is no solution.
  - No string 110… can ever equal a string 10… .

Representing PCP or MPCP Instances

- Since the alphabet can be arbitrarily large, we need to code symbols.
- Say the i-th symbol will be coded by “a” followed by i in binary.
- Commas and parentheses can represent themselves.

Representing Instances – (2)

- Thus, we have a finite alphabet in which all instances of PCP or MPCP can be represented.
- Let L_{PCP} and L_{MPCP} be the languages of coded instances of PCP or MPCP, respectively, that have a solution.

Reducing L_{MPCP} to L_{PCP}

- Take an instance of L_{MPCP} and do the following, using new symbols * and $.
  1. For the first string of each pair, add * after every character.
  2. For the second string of each pair, add * before every character.
  3. Add pair ($, *$).
  4. Make another copy of the first pair, with *'s and an extra * prepended to the first string.

Example: L_{MPCP} to L_{PCP}

MPCP instance, in order:	PCP instance:
(110, 10)	(1*1*0*, *1*0)
(0, 01)	(0*, *0*1)
(100, 001)	(1*0*0*, *0*0*1)
	($, *$)
	(*1*1*0*, *1*0)

Ender

Special pair version of first MPCP choice – only possible start for a PCP solution.

L_{MPCP} to L_{PCP} – (2)

- If the MPCP instance has a solution string w, then padding with stars fore and aft, followed by a $ is a solution string for the PCP instance.
  - Use same sequence of indexes, but the special pair to start.
  - Add ender pair as the last index.

L_{MPCP} to L_{PCP} – (3)

- Conversely, the indexes of a PCP solution give us a MPCP solution.
  1. First index must be special pair – replace by first pair.
  2. Remove ender.

Reducing L_u to L_{MPCP}

- We use MPCP to simulate the sequence of ID's that M executes with input w.
- Suppose $q_0w \vdash I_1 \vdash I_2 \vdash \ldots$ is the sequence of ID's of M with input w.
- Then any solution to the MPCP instance we can construct will begin with this sequence of ID's, separated by #'s.

Reducing L_u to L_{MPCP} – (2)

- But until M reaches an accepting state, the string formed by concatenating the second components of the chosen pairs will always be a full ID ahead of the string from the first pairs.
- If M accepts, we can even out the difference and solve the MPCP instance.

Reducing L_u to L_{MPCP} – (3)

- Key assumption: M has a semi-infinite tape; it never moves left from its initial head position.
- Alphabet of MPCP instance: state and tape symbols of M (assumed disjoint) plus special symbol # (assumed not a state or tape symbol).

Reducing L_u to L_{MPCP} – (4)

- First MPCP pair: (#, #q_0w#).
  - We start out with the second string having the initial ID and a full ID ahead of the first.
- (#, #).
  - We can add ID-enders to both strings.
- (X, X) for all tape symbols X of M.
  - We can copy a tape symbol from one ID to the next.

Example: Copying Symbols

- Suppose we have chosen MPCP pairs to simulate some number of steps of M, and the partial strings from these pairs look like:

. . . #AB

. . . #ABqCD#A B

Reducing L_u to L_{MPCP} – (5)

- For every state q of M and tape symbol X, there are pairs:
  1. (qX, Yp) if δ(q, X) = (p, Y, R).
  2. (ZqX, pZY) if δ(q, X) = (p, Y, L) [any Z].
- Also, if X is the blank, # can substitute.
  1. (q#, Yp#) if δ(q, B) = (p, Y, R).
  2. (Zq#, pZY#) if δ(q, X) = (p, Y, L) [any Z].

Example: Copying Symbols – (2)

- Continuing the previous example, if $\delta(q, C) = (p, E, R)$, then:

. . . #ABqCD#

. . . #ABqCD#ABEpD#

- If M moves left, we should not have copied B if we wanted a solution.

Reducing L_u to L_{MPCP} – (6)

- If M reaches final state f, then f "eats" the neighboring tape symbols, one or two at a time, to enable M to reach an "ID" that is essentially empty.
- The MPCP instance has pairs (XfY, f), (fY, f), and (Xf, f) for all tape symbols X and Y.
- To even up the strings and solve: (f##, #).

Example: Cleaning Up After Acceptance

… #ABfCDE#AfD E # f E #f##

… #ABfCDE#A fDE # f E # f##

CFG's from PCP

- We are going to prove that the *ambiguity problem* (is a given CFG ambiguous?) is undecidable.
- As with PCP instances, CFG instances must be coded to have a finite alphabet.
- Let *a* followed by a binary integer i represent the i-th terminal.

CFG’s from PCP – (2)

- Let A followed by a binary integer i represent the i-th variable.
- Let A1 be the start symbol.
- Symbols ->, comma, and ϵ represent themselves.
- Example: S -> 0S1 | A, A -> c is represented by A1->a1A1a10,A1->A10,A10->a11

CFG's from PCP – (3)

- Consider a PCP instance with k pairs.
  - i-th pair is (w_i, x_i).
- Assume *index symbols* $a_1, \ldots, a_k$ are not in the alphabet of the PCP instance.
- The *list language* for $w_1, \ldots, w_k$ has a CFG with productions $A \to w_iAa_i$ and $A \to w_ia_i$ for all $i = 1, 2, \ldots, k$.

List Languages

- Similarly, from the second components of each pair, we can construct a list language with productions B -> x_iBa_i and B -> x_ia_i for all i = 1, 2,…, k.
- These languages each consist of the concatenation of strings from the first or second components of pairs, followed by the reverse of their indexes.

Example: List Languages

- Consider PCP instance (a,ab), (baa,aab), (bba,ba).
- Use 1, 2, 3 as the index symbols for these pairs in order.

A -> aA1 | baaA2 | bbaA3 | a1 | baa2 | bba3

B -> abB1 | aabB2 | baB3 | ab1 | aab2 | ba3

Reduction of PCP to the Ambiguity Problem

- Given a PCP instance, construct grammars for the two list languages, with variables A and B.
- Add productions S -> A | B.
- The resulting grammar is ambiguous if and only if there is a solution to the PCP instance.

Example: Reduction to Ambiguity

A -> aA1 | baaA2 | bbaA3 | a1 | baa2 | bba3

B -> abB1 | aabB2 | baB3 | ab1 | aab2 | ba3

S -> A | B

- There is a solution 1, 3.
- Note abba31 has leftmost derivations:

S => A => aA1 => abba31

S => B => abB1 => abba31

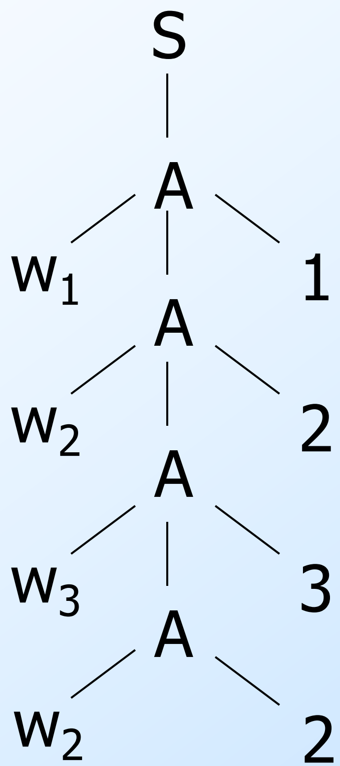
Proof the Reduction Works

- In one direction, if $a_1,\ldots, a_k$ is a solution, then $w_1\ldots w_k a_k\ldots a_1$ equals $x_1\ldots x_k a_k\ldots a_1$ and has two derivations, one starting S -> A, the other starting S -> B.
- Conversely, there can only be two leftmost derivations of the same terminal string if they begin with different first productions. Why? Next slide.

Proof – Continued

- If the two derivations begin with the same first step, say S -> A, then the sequence of index symbols uniquely determines which productions are used.
  - Each except the last would be the one with A in the middle and that index symbol at the end.
  - The last is the same, but no A in the middle.

Example: S =>A=>*…2321



More “Real” Undecidable Problems

- To show things like CFL-equivalence to be undecidable, it helps to know that the complement of a list language is also a CFL.
- We’ll construct a deterministic PDA for the complement langauge.

DPDA for the Complement of a List Language

- Start with a bottom-of-stack marker.
- While PCP symbols arrive at the input, push them onto the stack.
- After the first index symbol arrives, start checking the stack for the reverse of the corresponding string.

Complement DPDA – (2)

- The DPDA accepts after every input, with one exception.
- If the input has consisted so far of only PCP symbols and then index symbols, and the bottom-of-stack marker is exposed after reading an index symbol, do not accept.

Using the Complements

- For a given PCP instance, let L_A and L_B be the list languages for the first and second components of pairs.
- Let L_A^c and L_B^c be their complements.
- All these languages are CFL's.

Using the Complements

- Consider $L_A^c \cup L_B^c$.
- Also a CFL.
- $= \Sigma^*$ if and only if the PCP instance has no solution.
- Why? a solution $a_1, \ldots, a_n$ implies $w_1 \ldots w_n a_n \ldots a_1$ is not in L_A^c, and the equal $x_1 \ldots x_n a_n \ldots a_1$ is not in L_B^c.
- Conversely, anything missing is a solution.

Undecidability of "= Σ*"

- We have reduced PCP to the problem is a given CFL equal to all strings over its terminal alphabet?

Undecidablility of “CFL is Regular”

- Also undecidable: is a CFL a regular language?
- Same reduction from PCP.
- Proof: One direction: If $L_A^c \cup L_B^c = \Sigma^*$, then it surely is regular.

"= Regular" – (2)

- Conversely, we can show that if L = $L_A^c \cup L_B^c$ is not Σ^*, then it can't be regular.
- Proof: Suppose wx is a solution to PCP, where x is the indices.
- Define homomorphism h(0) = w and h(1) = x.

"= Regular" – (3)

- $h(0^n1^n)$ is not in L, because the repetition of any solution is also a solution.
- However, $h(y)$ is in L for any other y in $\{0,1\}^*$.
- If L were regular, so would be $h^{-1}(L)$, and so would be its complement = $\{0^n1^n \mid n \geq 1\}$.