CS787: Advanced Algorithms	Scribe: Vishnu Lokhande
Lecture 15: OBM (contd.) and Stochastic Online Problem	Date: March 26, 2019

15.1 Online Bipartite Matching contd.

We will analyze $\mathbb{E}[\alpha_u] + \mathbb{E}[\beta_u]$ for any edge (u, v). We follow the following procedure

- Fix an edge (u, v).
- Imagine taking u out of the graph.
- Run the whole matching process.
- β gets assigned β_v. Reintroduce α_u. Note, in this procedure, β_v only increases.

In $G\backslash u$, v gets matched to u'. Now, y_u is the only undetermined random variable. This gives us a lower bound on β_v.

$$\mathbb{E}[\alpha_u] + \mathbb{E}[\beta_v] = \int_0^{y'_u} \frac{g(y_u)}{F} dy + \frac{1 - g(y_{u'})}{F}$$

$$\int_0^z \frac{g(y_u)}{F} dy + \frac{1 - g(z)}{F} \rightarrow \text{ let } z = y_{u'}, \text{ want this to be greater than 1 for all } z.$$

$$\frac{G(z) - G(0)}{F} + \frac{1 - G'(z)}{F} = 1$$

Using the boundary condition that $g(1) = 1$, we get the solution as

$$g(z) = \frac{e^z}{e}$$
$$F = 1 - \frac{1}{e}$$

15.2 Stochastic Online Problems

The problem that we will study in this lecture is called the Secratary Problem. As per this problem, there are candidates that are being interviewed by us (the secratary), in an online fashion. It is online because the hiring decision is made immediately after interviewing a candidate. If the secratary hires someone, he doesn't get to see the future candidates.

There are two kinds of objectives that we seek to solve,

Cardinal Objective We have some measure of quality of a person. Our goal, as a secratary, is to maximize the quality of the person hired.

Ordinal Objective We do not have an objective measure of quality. We get to observe a relative ranking of everyone we interview. The goal here is to maximize the likelihood of picking the best candidate.

Without any new information, no randomized algorithm can do better than $\frac{1}{n}$ C.R., where n is the number of candidates.

There are two kinds of models, each providing us with additional infomation to solve the problem and gain better C.R.

- Candidates arrive in random order. We will discuss this problem with ordinal objective in mind. These are typically called the secretary problems.
- For each candidate, we know some distribution D_i such the $v_i \sim D_i$. Here, v_i is some measure of quality of the candidate. We have the stochastic information about the candidates, however, the order of arrival is still adversarial, i.e, it's decided by the adversary and unknown to us apriori. Problems in this model are called prophet inequality problems.

There are many intermediate models, one example is, say we have the stochastic information about the candidate and the candidates arrive in uniformly random order.

Let's study two strategies to solve the secretary problem. The candidates are arriving in some order. When we see the i^{th} candidate, we get to know about the ranking of first i candidates. Our goal is to pick the best the candidate or maximize the likelihood of picking the best candidate.

15.2.1 Strategy 1

- Don't hire first r candidates

- After r candidates, hire the first candidate that is better than all seen before.

Let Q_r denote the probability that the above strategy hires the best candidate.
For example, when $r = n/2$, $Q_{n/2} \geq Pr($ Best in second half & second best in first half$)$. This implies, $Q_{n/2} \geq 1/4$. Let's now analyze for a general case,

$$\begin{aligned}
Q_r &= \sum_{i=r+1}^{n} Pr(\text{Cand } i \text{ is the best}) Pr(\text{Alg hires } i \mid i \text{ is the best}) \\
&= \sum_{i=r+1}^{n} \frac{1}{n} Pr(\text{best out of } 1^{st}\ i-1 \text{ appear in the } 1^{st}\ r) \\
&= \sum_{i=r+1}^{n} \frac{1}{n} \frac{r}{i-1} \\
&= \frac{r}{n}(H_{n-1} - H_{r-1}) \quad \text{ where } H_k \text{ denotes the } k_{th} \text{ harmonic sum} \\
&\approxeq \frac{r}{n} \ln(\frac{n}{r})
\end{aligned}$$

Optimal Choice of r: $r = n/e$ and $Q_{n/e} = 1/e$.

15.2.2 Strategy 2

We adopt a gentler strategy here where we have a non-zero "probability of hiring" for every candidate instead of just not hiring some fixed r candidate.
Let p_i denote the probability of hiring the i^{th} candidate given it is the best so far.
Let x_i denote the probability of hiring the i^{th} candidate. So, $x_i = \frac{1}{i} p_i$

The sum of the probability of hiring a candidate from among the first $i-1$ candidates and p_i cannot be more than one. This gives us a constraint that, $p_i \leq 1 - \sum_{j=1}^{i-1} x_j$.
The objective for us is to maximize $\sum_i \frac{1}{n} p_i$.
Thus, the linear program is,

$$\max \sum_{i=1}^{n} \frac{i}{n} x_i$$

$$\text{s.t. } ix_i + \sum_{j=1}^{i-1} x_j \leq 1 \forall i$$

$$x_i \geq 0 \forall i$$

Observation 1 Any algorithm for this problem gives a feasible solution to this LP.

Observation 2 Any feasible solution to this LP can be achieved algorithmically.
Proof: Given a feasible solution x_i. For $i = 1$ to n, if i is the best so far, hire i with probability

$$\frac{ix_i}{1 - \sum_{j=1}^{i-1} x_j}$$

We can prove that the probability we hire the i^{th} candidate is x_i. This is because, it is the product of probability that we haven't picked anybody so far $(1 - \sum_{j=1}^{i-1} x_j)$, the probability that i is the best so far (i/n) and the probability that we hire i, $(\frac{ix_i}{1-\sum_{j=1}^{i-1} x_j})$, which comes out as x_i. ∎

Claim 15.2.1 *The optimal value of this L.P. is no more than* $1/e$.

In the next lecture , we will discuss profit inequalities.