CS727 HW5

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1. (5.4.19)

   (a) Let $P = \{x : Dx \geq d; Fx = f; x \geq 0\}$. We have P is closed. Note that P contains no line since any x with negative entry is not in P. Then by theorem 4.3.4, $P = \text{conv}(X) + \text{pos}(Y)$, where X is the set of extreme points of P, and Y is a set of points of $\mathbb{R}^n$ such that $\text{pos}(Y)$ is the set of extreme rays of P.

   Now by Minkowski-Weyl, since P is polyhedral, for any $x \in P$, we can write $x = \sum_{i=1}^p \lambda_i x_i + \sum_{i=1}^q \mu_i y_i$ such that $\lambda_i \geq 0, \sum_i^p \lambda_i = 1, \mu_i \geq 0$, and $x_i \in X, y_i \in Y$.

   Note that we must have $\langle c^*, y_i \rangle \geq 0$ for all $i = 1, \ldots, q$. If there exists $\langle c^*, y_i \rangle < 0$, then we can make the corresponding μ_i approaching ∞ and $\inf\{\langle c^*, x\rangle, x \in P\}$ will be $-\infty$. Hence:

   $$\forall x \in P : \langle c^*, x \rangle \geq \left\langle c^*, \sum_{i=1}^p \lambda_i x_i \right\rangle$$

   note $\sum_{i=1}^p \lambda_i x_i \in P$ since P is convex.

   Since X is finite, we can find $x^* \in X$ with the smallest $\langle c^*, x^* \rangle$, i.e. $\langle c^*, x^* \rangle \leq \langle c^*, x_i \rangle \forall x_i \in X$. Then we have

   $$\langle c^*, x^* \rangle \leq \left\langle c^*, \sum_{i=1}^p \lambda_i x_i \right\rangle, \text{ for all suitable } \lambda_i$$

   Hence this x^* satisfies the condition that $\forall x \in P, \langle c^*, x^* \rangle \leq \langle c^*, x \rangle$. By definition, x^* attains the infimum that we want.

   (b) First let us rewrite the constraints in P in terms of $Ax \leq a$:

   $$Dx \geq d; Fx = f; x \geq 0 \iff \underbrace{\begin{bmatrix} -D \\ F \\ -F \\ -I \end{bmatrix}}_{A} x \leq \underbrace{\begin{bmatrix} -d \\ f \\ -f \\ 0 \end{bmatrix}}_{a}$$

   where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

   Let $x_0 \in P$ be the feasible solution that satisfies base demand d_0. Use the Hoffman's theorem, there exists $\gamma_A > 0$ such that given x_0, we can find $x \in P$ (equivalently, $Ax \leq a$) such that

   $$\|x - x_0\| \leq \gamma_A \|(Ax_0 - a)_+\|$$

   Note that for x_0:

   $$Ax_0 = \begin{bmatrix} -D \\ F \\ -F \\ -I \end{bmatrix} x \leq \begin{bmatrix} -d_0 \\ f \\ -f \\ 0 \end{bmatrix}$$

therefore:

$$(Ax_0 - a)_+ \leq \left(\begin{bmatrix} -d_0 \\ f \\ -f \\ 0 \end{bmatrix} - \begin{bmatrix} -d \\ f \\ -f \\ 0 \end{bmatrix} \right)_+ = \left(\begin{bmatrix} d - d_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)_+$$

Let the solution we get from (a) be x^*, we have:

$$\begin{aligned} & \langle c^*, x^* \rangle \\ \leq & \langle c^*, x \rangle \\ = & \langle c^*, x_0 \rangle + \langle c^*, x - x_0 \rangle \\ \leq & c_0 + \|c^*\| \|x - x_0\| \\ \leq & c_0 + \|c^*\| \gamma_A \|(Ax_0 - a)_+\| \\ \leq & c_0 + \|c^*\| \gamma_A \|(d - d_0)_+\| \end{aligned}$$

Let $\alpha = \gamma_A \|c^*\|$, and we finish the proof.