ISYE 727 Homework 7

Mingchen Ma

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1 5.2.9

We first show that if P and Q each is polyhedral, then $P \times Q$ is polyhedral.

Since P and Q are polyhedral, we write

$$P = \{x \in R^m | Ax \leq b\}$$
$$Q = \{y \in R^n | By \leq d\}.$$

Then we have

$$P \times Q = \{\binom{x}{y} \in R^{m+n} | \begin{pmatrix} A & O \\ O & B \end{pmatrix} \leq \binom{b}{d}\}$$

Thus, $P \times Q$ is also polyhedral.

Then we show that if $P \times Q$ is polyhedral then P, Q each is polyhedral. We can just show that P is polyhedral and Q is polyhedral symmetrically. Define a linear map $L : R^{m+n} \to R^m$ as following.

$$L\binom{x}{y} = (I, O)\binom{x}{y} = x$$

Then we show that $L(P \times Q) = P$. In fact, for any $x \in P$ there is an $y \in Q$ such that $\binom{x}{y} \in P \times Q$. And $L\binom{x}{y} = x$. So $P \subseteq L(P \times Q)$. On the other hand, for any $x \in L(P \times Q)$, $x \in P$. So $L(P \times Q) \subseteq P$. And we can conclude that $L(P \times Q) = P$. Since $P \times Q$ is polyhedral, we know that $L(P \times Q)$ is also polyhedral. i.e. P is polyhedral.

2 5.2.10

First, we know that for any $x_0 \in P$, there exists a of F of P such that x_0 is in the relative interior of F. And we can write F in the following form.

$$F := \{x : A_I x = a_I, A_{cI} x \leq a_{cI}\}.$$

If $I = \emptyset$, then $x_0 \in \text{int}(P)$ thus $N_P(x_0) = \{0\}$ and $T_P(x_0) = R^n$. Then result is obviously true, because we can choose Q' to be a small neighborhood of x_0 such that $Q' \subseteq P$ and in this case, $Q := Q' - \{x_0\}$ satisfies the result we want.

From now on, we assume that $I \neq \emptyset$. Then we know that $N_P(x_0) = A_I^*(R^{|I|}) = \{w | w = \sum_{i=1}^{|I|} \lambda_i w^i, \lambda_i \geq 0\}$, where w_i is the ith column of A_I^*. Then we know that

$$T_P(x_0) = N_P^\circ(x_0) = \{y | < y, w > \leq 0, \forall w \in N_P(x_0)\} = \{y | < y, w^i > \leq 0, i = 1, \ldots, |I|\} = \{y | A_I y \leq 0\}.$$

Now we focus on the set $P - \{x_0\}$. Suppose that $Ax_0 = a^0$. Then we have

$$P - \{x_0\} = \{x : A_I x \leq 0, A_{cI} x \leq a_{cI} - a_{cI}^0\}.$$

Since x_0 is in ri$F$, we know that $A_{cI} x_0 = a_{cI}^0 < a_{cI}$, which means that $A_{cI} 0 < a_{cI} - a_{cI}^0$. This implies that there exists a very small neighborhood Q of the origin such that $\forall x \in Q$, $A_{cI} x \leq a_{cI} - a_{cI}^0$. Thus $Q \cap (P) - \{x_0\} = Q \cap T_P(x_0)$.