CS727: Convex Analysis **Scribe:** Tien-Lung Fu

Title: hw7

0.1 Exercise 5.2.9

Part 1: We want to show if each P and Q is polyhedral, then P × Q is polyhedral.

$$
\begin{aligned}
&P\ and\ Q\ are\ polyhedral,\ we\ have:\\
&P = \{x \in R^m | Ax \leqslant a\}\\
&Q = \{x \in R^n | By \leqslant b\}\\
&and\ we\ have\ P \times Q = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^{m+n} | \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leqslant \begin{bmatrix} a \\ b \end{bmatrix} \}\\
&Thus,\ P \times Q\ is\ polyhedral
\end{aligned}
$$

Part 2: We want to show if P × Q is polyhedral, then each P and Q is polyhedral

$$
Define\ a\ linear\ map\ L : R^{m+n} \to R^m\ as
$$

$$
L(\begin{bmatrix} x \\ y \end{bmatrix}) = [I, 0] \begin{bmatrix} x \\ y \end{bmatrix} = x
$$

We want to show $L(P \times Q) = P$

Part 2-1: We want to show $P \subseteq L(P \times Q)$

$$
\forall x \in P,\ \exists y \in Q\ s.t. \begin{bmatrix} x \\ y \end{bmatrix} \in P \times Q\ and\ L(\begin{bmatrix} x \\ y \end{bmatrix}) = x\ (Q.E.D)
$$

Part 2-2: We want to show $L(P \times Q) \subseteq P$

$$
\forall x \in L(P \times Q),\ we\ have\ x \in P\ (Q.E.D)
$$

So we have $L(P \times Q) = P$. Since $P \times Q$ is polyhedral, $L(P \times Q)$ is also polyhedral. Thus, P is polyhedral. We can use similar way to prove Q is also polyhedral.

0.2 Exercise 5.2.10

We know that for any $x_0 \in P$, there exists a face of P s.t. x_0 is in the relative interior of F. We can write F in the following form.

$$
F := \{x : A_I x = a_I, A_{cI} x \leqslant a_{cI}\}
$$

If $I = \phi$, then $x_0 \in int(P)$ thus $N_P(x_0) = \{0\}$ and $T_p(x_0) = R^n$. Then we can choose Q' to be a small neighborhood of x_0 s.t. $Q' \subseteq P$ and $Q := Q' - \{x_0\}$ will satisfy the result we want.

If $I \neq \phi$, we know that $N_P(x_0) = A_I^*(R^{|I|}) = \{w | w = \sum_{i=1}^{|I|} \lambda_i w^i, \lambda_i \geqslant 0\}$, where w^i is the i-th column of A_I^*. Then we know that

$$T_P(x_0) = N_P^\circ(x_0) = \{y| < y, w > \leqslant 0, \forall w \in N_P(x_0)\} = \{y| < y, w^i > \leqslant 0, i = 1, ..., |I|\} = \{A_I y \leqslant 0\}$$

Suppose $Ax_0 = a^0$, then we have

$$P - \{x_0\} = \{x : A_I x \leqslant 0, A_{cI} x \leqslant a_{cI} - a_{cI}^0\}$$

Since x_0 is in riF, we know that $A_{cI} x_0 = a_{cI}^0 < a_{cI}$, then we have $A_{cI} x_0 < a_{cI} - a_{cI}^0$. This implies that there exists a very small neighborhood Q of the origin s.t. $\forall x \in Q, A_{cI} x \leqslant a_{cI} - a_{cI}^0$. Thus, $Q \cap (P - \{x_0\}) = Q \cap T_P(x_0)$