More NP-Complete Problems

NP-Hard Problems

Tautology Problem

Node Cover

Knapsack

Next Steps

- We can now reduce 3-SAT to a large number of problems, either directly or indirectly.
- Each reduction must be polytime.
- Usually we focus on length of the output from the transducer, because the construction is easy.

Next Steps – (2)

- Another essential part of an NP-completeness proof is showing the problem is in **NP**.
- Sometimes, we can only show a problem *NP-hard* = "if the problem is in **P**, then **P** = **NP**," but the problem may not be in **NP**.

Example: NP-Hard Problem

- The *Tautology Problem* is: given a Boolean expression, is it satisfied by all truth assignments?
  - Example: x + -x + yz
- Not obviously in **NP**, but it's complement is.
  - Guess a truth assignment; accept if that assignment doesn't satisfy the expression.

Co-**NP**

- A problem/language whose complement is in **NP** is said to be in ***Co-NP***.
- Note: **P** is closed under complementation.
- Thus, **P** $\subseteq$ Co-**NP**.
- Also, if **P** = **NP**, then **P** = **NP** = Co-**NP**.

Tautology is NP-Hard

- While we can't prove Tautology is in **NP**, we can prove it is NP-hard.
- Suppose we had a polytime algorithm for Tautology.
- Take any Boolean expression E and convert it to NOT(E).
  - Obviously linear time.

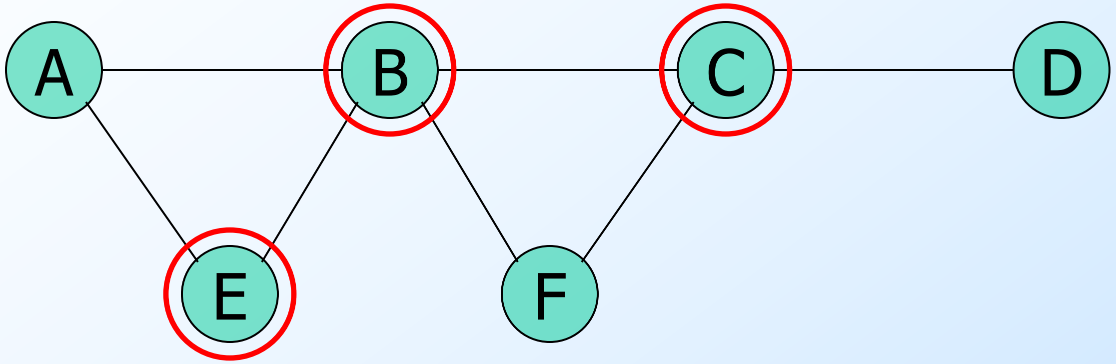
Tautology is NP-Hard – (2)

- E is satisfiable if and only NOT(E) is not a tautology.
- Use the hypothetical polytime algorithm for Tautology to test if NOT(E) is a tautology.
- Say "yes, E is in SAT" if NOT(E) is not a tautology and say "no" otherwise.
- Then SAT would be in **P**, and **P** = **NP**.

The Node Cover Problem

- Given a graph G, we say N is a *node cover* for G if every edge of G has at least one end in N.
- The problem Node Cover is: given a graph G and a "budget" k, does G have a node cover of k or fewer nodes?

Example: Node Cover



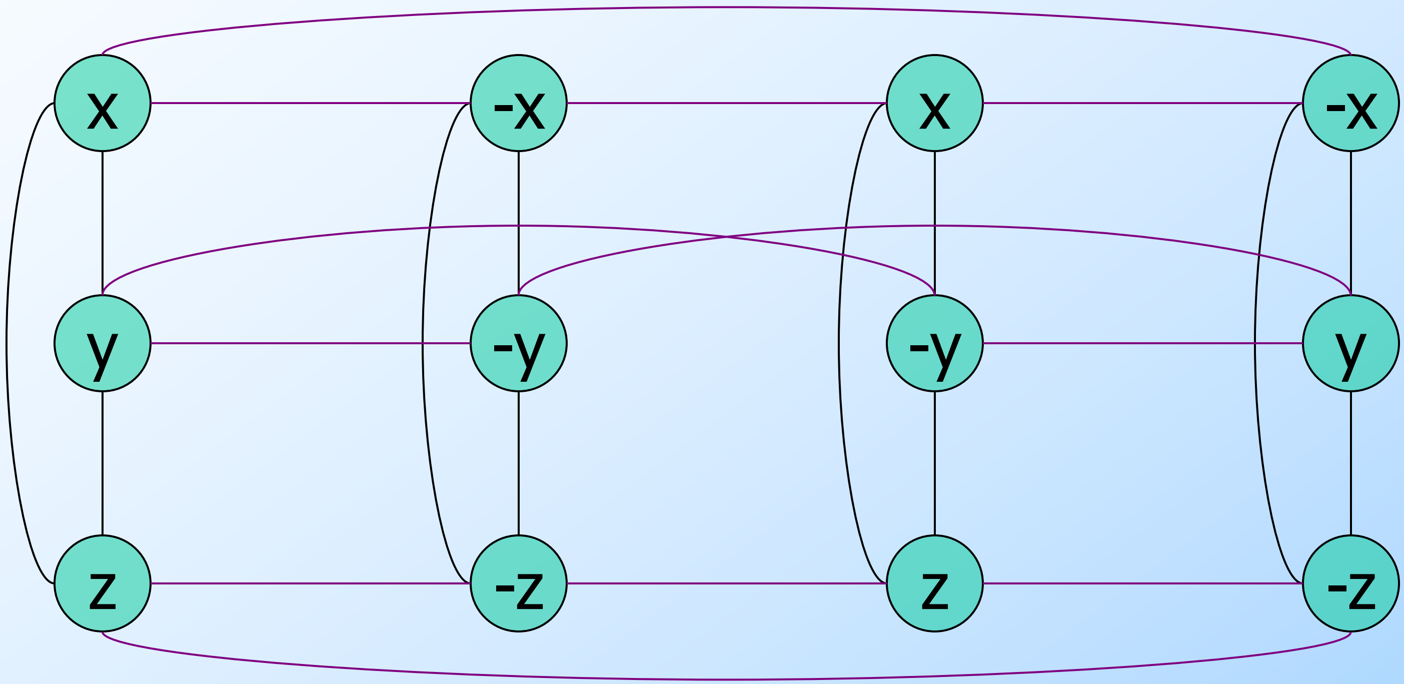
One possible node cover of size 3: {B, C, E}

NP-Completeness of Node Cover

- Reduction from 3-SAT.
- For each clause (X+Y+Z) construct a "column" of three nodes, all connected by *vertical* edges.
- Add a *horizontal* edge between nodes that represent any variable and its negation.
- Budget = twice the number of clauses.

Example: The Reduction to Node Cover

(x + y + z)(-x + -y + -z)(x + -y +z)(-x + y + -z)



Budget = 8

Example: Reduction – (2)

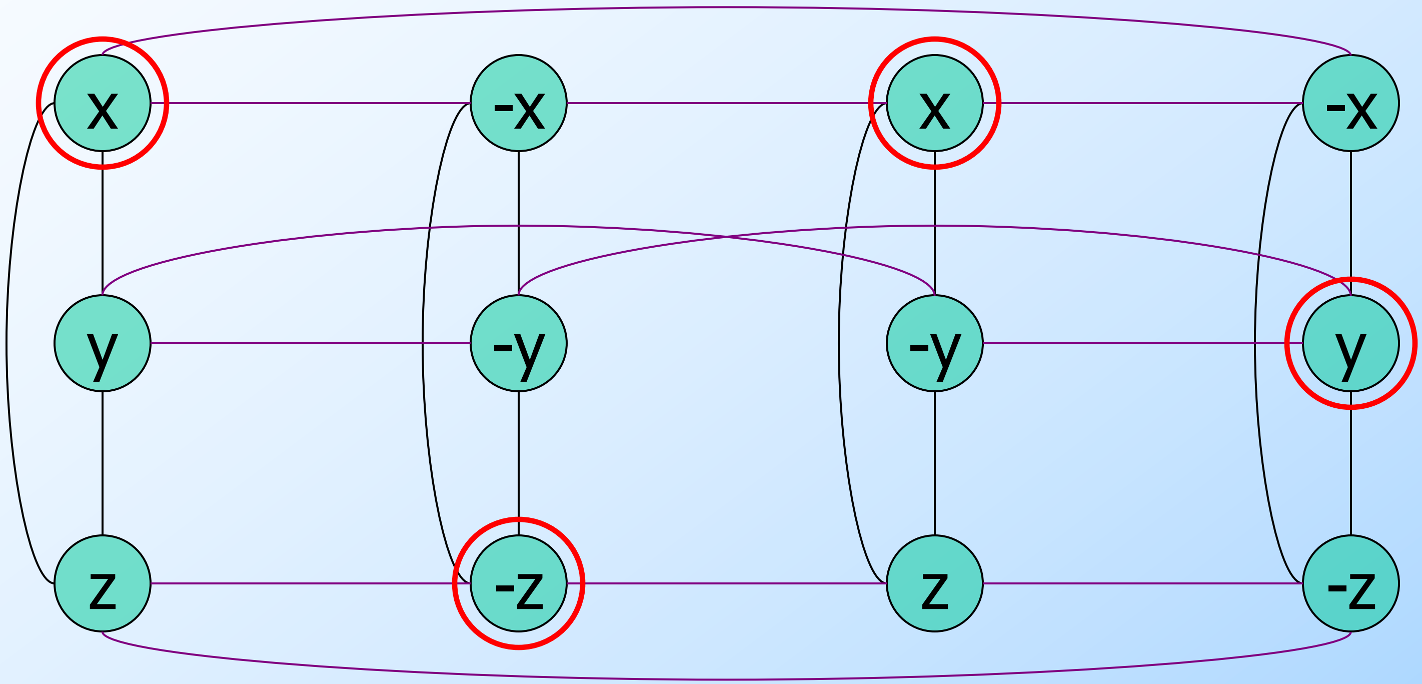
- A node cover must have at least two nodes from every column, or some vertical edge is not covered.
- Since the budget is twice the number of columns, there must be exactly two nodes in the cover from each column.
- Satisfying assignment corresponds to the node in each column not selected.

Example: Reduction – (3)

(x + y + z)(-x + -y + -z)(x + -y +z)(-x + y + -z)

Truth assignment: x = y = T; z = F

Pick a true node in each column

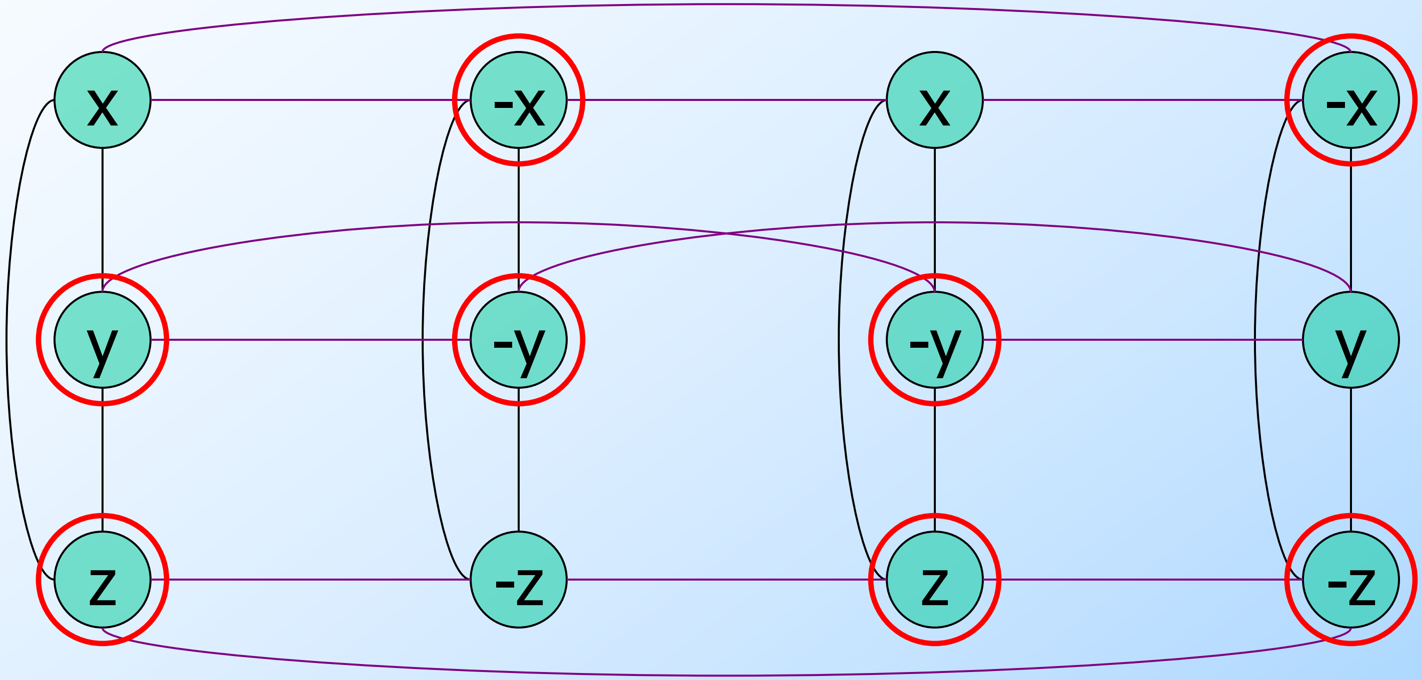


Example: Reduction – (4)

(x + y + z)(-x + -y + -z)(x + -y +z)(-x + y + -z)

Truth assignment: x = y = T; z = F

The other nodes form a node cover



Proof That the Reduction Works

- The reduction is clearly polytime.
- Need to show: If we construct from 3-SAT instance E a graph G and a budget k, then G has a node cover of size $\leq$ k if and only if E is satisfiable.

Proof: If

- Suppose we have a satisfying assignment A for E.
- For each clause of E, pick one of its three literals that A makes true.
- Put in the node cover the two nodes for that clause that do not correspond to the selected literal.
- Total = k nodes – meets budget.

Proof: If – (2)

- The selected nodes cover all vertical edges.
  - Why? Any two nodes for a clause cover the triangle of vertical edges for that clause.
- Horizontal edges are also covered.
  - A horizontal edge connects nodes for some x and -x.
  - One is false in A and therefore its node must be selected for the node cover.

Proof: Only If

- Suppose G has a node cover with at most k nodes.
- One node cannot cover the vertical edges of any column, so each column has exactly 2 nodes in the cover.
- Construct a satisfying assignment for E by making true the literal for any node not in the node cover.

Proof: Only If – (2)

- Worry: What if there are unselected nodes corresponding to both x and -x?
  - Then we would not have a truth assignment.
- But there is a horizontal edge between these nodes.
- Thus, at least one is in the node cover.

The Knapsack Problem

- We shall prove NP-complete a version of Knapsack with a target:
  - Given a list L of integers and a target k, is there a subset of L whose sum is exactly k?
- Later, we'll reduce this version of Knapsack to our earlier one: given an integer list L, can we divide it into two equal parts?

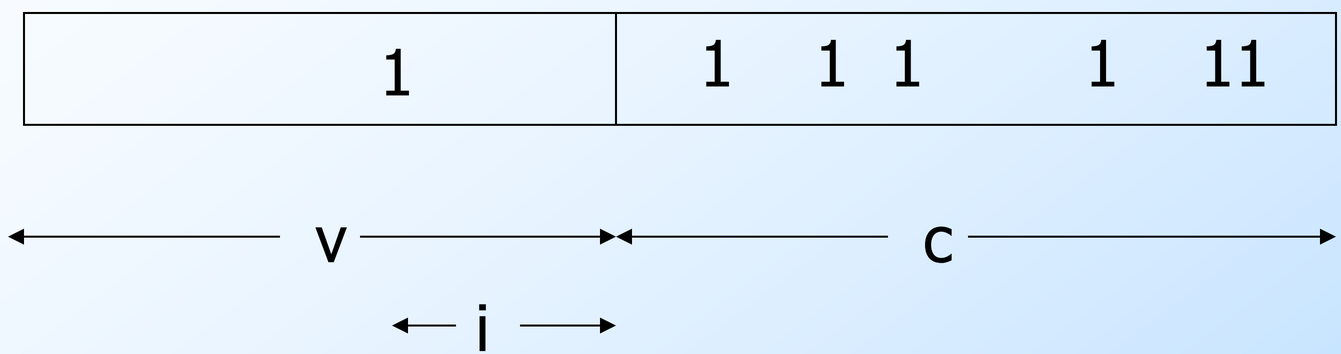
Knapsack-With-Target is in **NP**

- Guess a subset of the list L.
- Add ‘em up.
- Accept if the sum is k.

Polytime Reduction of 3-SAT to Knapsack-With-Target

- Given 3-SAT instance E, we need to construct a list L and a target k.
- Suppose E has c clauses and v variables.
- L will have base-32 integers of length c+v, and there will be 3c+2v of them.

Picture of Integers for Literals

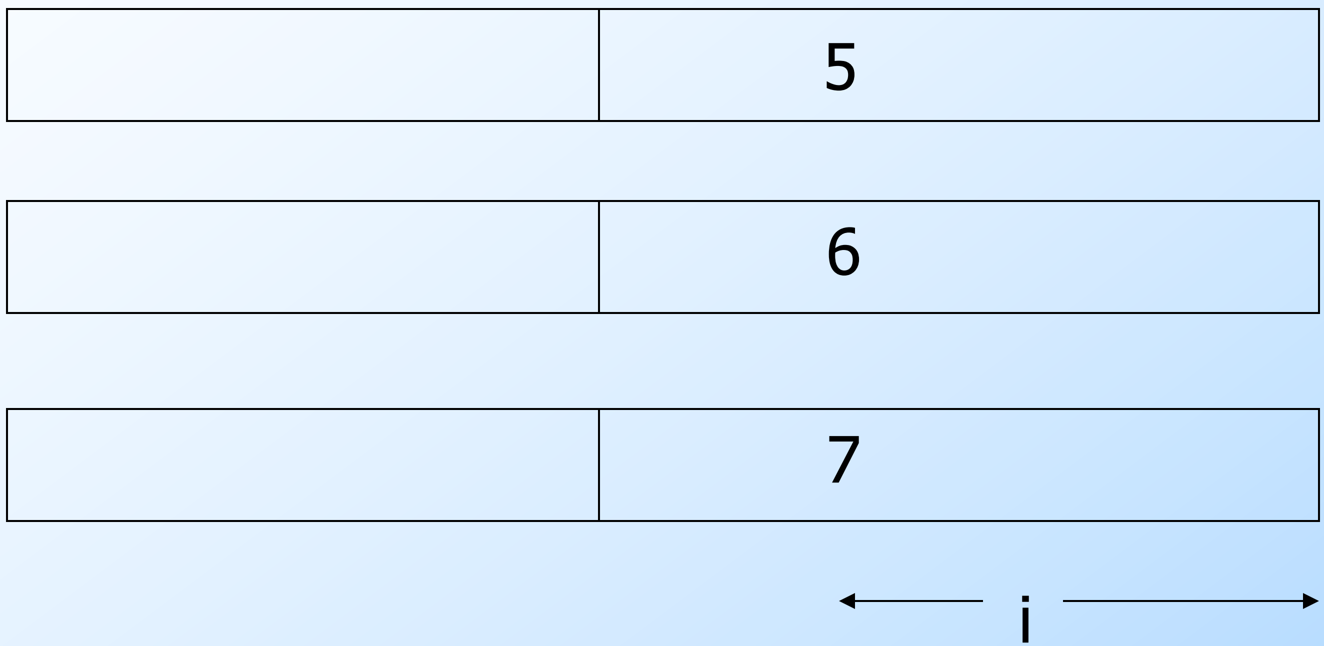


1 in i-th position if this integer is for x_i or $-x_i$.

1's in all positions such that this literal makes the clause true.

All other positions are 0.

Pictures of Integers for Clauses



For the i-th clause

Example: Base-32 Integers

$(x + y + z)(x + -y + -z)$

- $c = 2$; $v = 3$.
- Assume x, y, z are variables 1, 2, 3, respectively.
- Clauses are 1, 2 in order given.

Example: (x + y + z)(x + -y + -z)

- For x: 00111
- For -x: 00100
- For y: 01001
- For -y: 01010
- For z: 10001
- For -z: 10010

- For first clause: 00005, 00006, 00007
- For second clause: 00050, 00060, 00070

The Target

- $k = 8(1+32+32^2+\ldots+32^{c-1}) + 32^c(1+32+32^2+\ldots+32^{v-1})$
- That is, 8 for the position of each clause and 1 for the position of each variable.
  - $k = (11\ldots188\ldots8)_{32}$.
- Key Point: Base-32 is high enough that there can be no carries between positions.

Key Point: Details

- Among all the integers, the sum of digits in the position for a variable is 2.
- And for a clause, it is 1+1+1+5+6+7 = 21.
  - 1's for the three literals in the clause; 5, 6, and 7 for the integers for that clause.
- Thus, the target must be met on a position-by-position basis.

Key Point: Concluded

- Thus, if a set of integers matches the target, it must include exactly one of the integers for x and -x.
- For each clause, at least one of the integers for literals must have a 1 there, so we can choose either 5, 6, or 7 to make 8 in that position.

Proof the Reduction Works

- Each integer can be constructed from the 3-SAT instance E in time proportional to its length.
  - Thus, reduction is $O(n^2)$.
- If E is satisfiable, take a satisfying assignment A.
- Pick integers for those literals that A makes true.

Proof the Reduction Works – (2)

- The selected integers sum to between 1 and 3 in the digit for each clause.
- For each clause, choose the integer with 5, 6, or 7 in that digit to make a sum of 8.
- These selected integers sum to exactly the target.

Proof: Converse

- We must also show that a sum of integers equal to the target k implies E is satisfiable.
- In each digit for a variable x, either the integer for x or the digit for -x, but not both is selected.
- Let truth assignment A make this literal true.

Proof: Converse – (2)

- In the digits for the clauses, a sum of 8 can only be achieved if among the integers for the variables, there is at least one 1 in that digit.
- Thus, truth assignment A makes each clause true, so it satisfies E.

The *Partition-Knapsack Problem*

- This problem is what we originally referred to as "knapsack."
- Given a list of integers L, can we partition it into two disjoint sets whose sums are equal?
- Partition-Knapsack is NP-complete; reduction from Knapsack-With-Target.

Reduction of Knapsack-With-Target to Partition-Knapsack

- Given instance (L, k) of Knapsack-With-Target, compute the sum s of all the integers in L.
  - Linear in input size.
- Output is L followed by two integers: 2k and s.
- Example: L = 3, 4, 5, 6; k = 7.
  - Partition-Knapsack instance = 3, 4, 5, 6, 14, 18.

Solution

Solution

Proof That Reduction Works

- The sum of all integers in the output instance is 2(s+k).
  - Thus, the two partitions must each sum to s+k.
- If the input instance has a subset of L that sums to k, then pick it plus the integer s to solve the output instance.

Proof: Converse

- Suppose the output instance of Partition-Knapsack has a solution.
- The integers s and 2k cannot be in the same partition.
  - Because their sum is more than half 2(s+k).
- Thus, the subset of L that is in the partition with s sums to k.
  - Thus, it solves the input instance.