Deterministic Finite Automata

Alphabets, Strings, and Languages

Transition Graphs and Tables

Some Proof Techniques

Alphabets

- An *alphabet* is any finite set of symbols.
- Examples:

  ASCII, Unicode,

  {0,1} (*binary alphabet*),

  {a,b,c}, {s,o},

  set of signals used by a protocol.

Strings

- A *string* over an alphabet Σ is a list, each element of which is a member of Σ.
  - Strings shown with no commas or quotes, e.g., abc or 01101.
- Σ^* = set of all strings over alphabet Σ.
- The *length* of a string is its number of positions.
- ϵ stands for the *empty string* (string of length 0).

Example: Strings

- $\{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$
- Subtlety: 0 as a string, 0 as a symbol look the same.
  - Context determines the type.

Languages

- A *language* is a subset of Σ^* for some alphabet Σ.
- Example: The set of strings of 0's and 1's with no two consecutive 1's.
- L = {ϵ, 0, 1, 00, 01, 10, 000, 001, 010, 100, 101, 0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010, . . . }

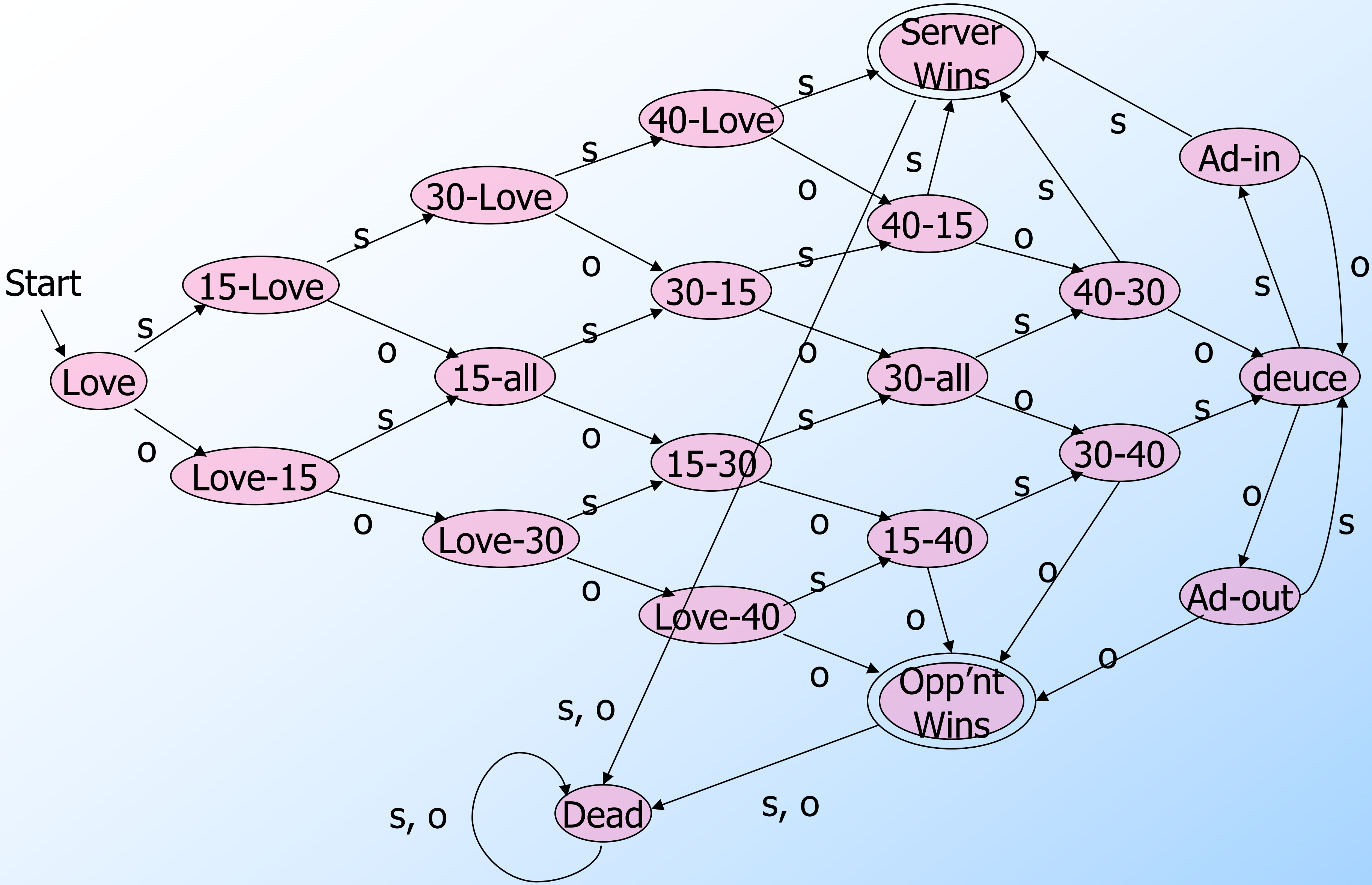
Hmm… 1 of length 0, 2 of length 1, 3, of length 2, 5 of length 3, 8 of length 4. I wonder how many of length 5?

Deterministic Finite Automata

- A formalism for defining languages, consisting of:
  1. A finite set of *states* (Q, typically).
  2. An *input alphabet* (Σ, typically).
  3. A *transition function* (δ, typically).
  4. A *start state* (q_0, in Q, typically).
  5. A set of *final states* ($F \subseteq Q$, typically).
     - "Final" and "accepting" are synonyms.

The Transition Function

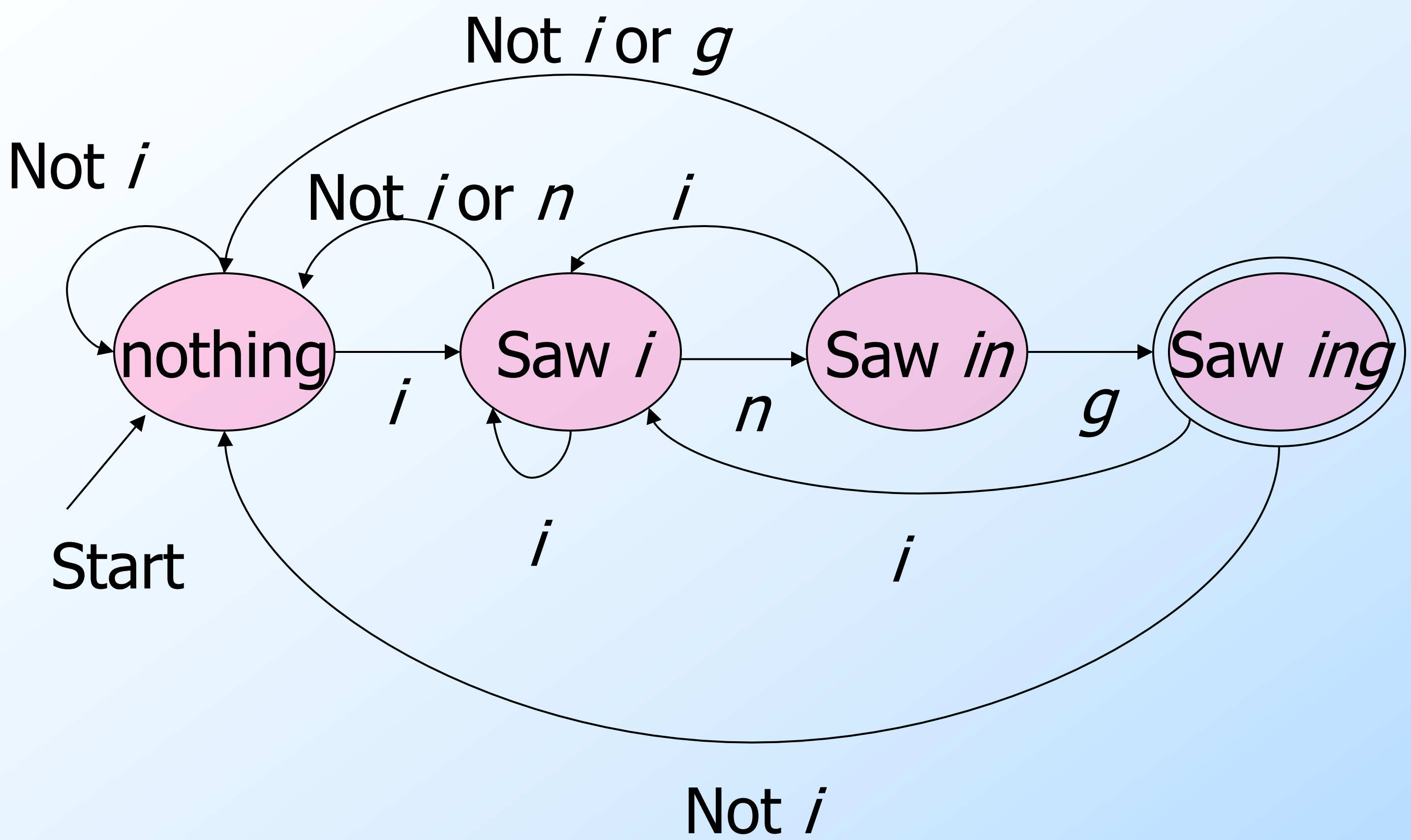
- Takes two arguments: a state and an input symbol.
- $\delta(q, a)$ = the state that the DFA goes to when it is in state *q* and input *a* is received.
- Note: always a next state – add a *dead state* if no transition (Example on next slide).


Start
Love
15-Love
Love-15
30-Love
15-all
Love-30
40-Love
30-15
15-30
Love-40
Server Wins
40-15
30-all
15-40
Opp'nt Wins
40-30
30-40
Ad-in
deuce
Ad-out
Dead
s, o
s, o
s, o

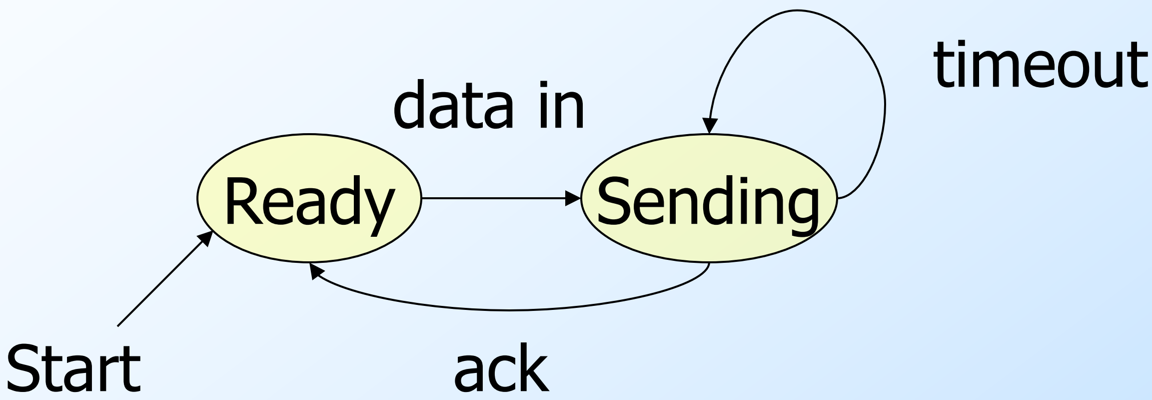
Graph Representation of DFA’s

- Nodes = states.
- Arcs represent transition function.
  - Arc from state p to state q labeled by all those input symbols that have transitions from p to q.
- Arrow labeled “Start” to the start state.
- Final states indicated by double circles.

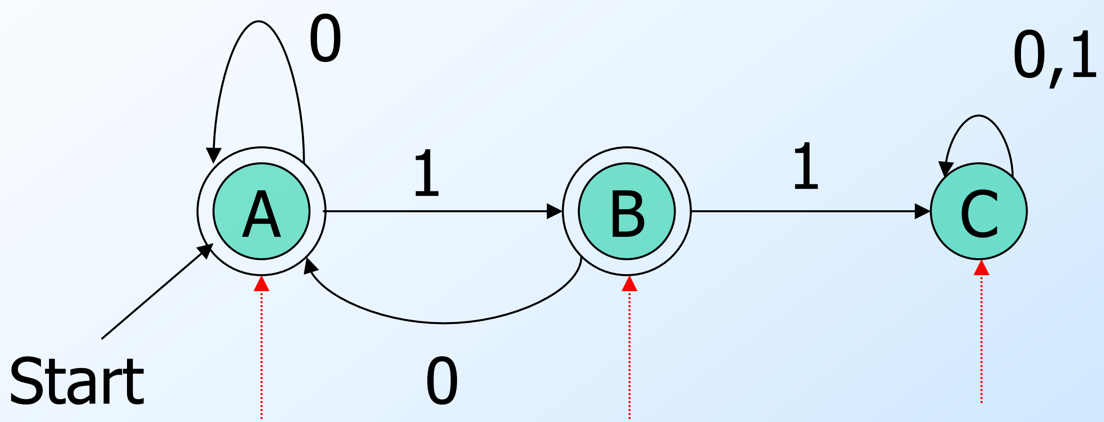
Example: Recognizing Strings Ending in "ing"



Example: Protocol for Sending Data



Example: Strings With No 11



String so far has no 11, does not end in 1.	String so far has no 11, but ends in a single 1.	Consecutive 1's have been seen.

Alternative Representation: Transition Table

Final states starred

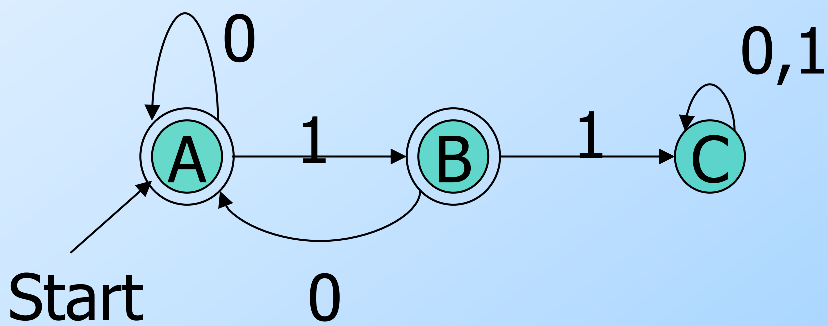
Columns = input symbols

Arrow for start state

	0	1
→ * A	A	B
* B	A	C
C	C	C

Rows = states

Each entry is δ of the row and column.



Convention: Strings and Symbols

- … w, x, y, z are strings.
- a, b, c,… are single input symbols.

Extended Transition Function

- We describe the effect of a string of inputs on a DFA by extending δ to a state and a string.
- Intuition: Extended δ is computed for state q and inputs $a_1a_2...a_n$ by following a path in the transition graph, starting at q and selecting the arcs with labels a_1, a_2,…, a_n in turn.

Inductive Definition of Extended δ

- Induction on length of string.
- Basis: $\delta(q, \epsilon) = q$
- Induction: $\delta(q,wa) = \delta(\delta(q,w),a)$
  - Remember: w is a string; a is an input symbol, by convention.

Example: Extended Delta

	0	1
A	A	B
B	A	C
C	C	C

$\delta(B,011) = \delta(\delta(B,01),1) = \delta(\delta(\delta(B,0),1),1) =$

$\delta(\delta(A,1),1) = \delta(B,1) = C$

Delta-hat

- We don't distinguish between the given delta and the extended delta or delta-hat.
- The reason:
- $\hat{\delta}(q, a) = \delta(\hat{\delta}(q, \epsilon), a) = \delta(q, a)$

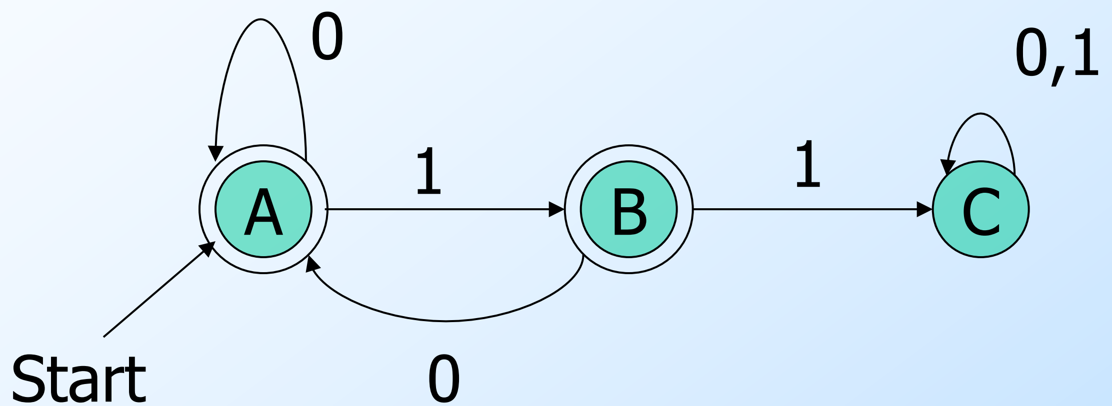
Extended deltas

Language of a DFA

- Automata of all kinds define languages.
- If A is an automaton, L(A) is its language.
- For a DFA A, L(A) is the set of strings labeling paths from the start state to a final state.
- Formally: L(A) = the set of strings w such that $\delta(q_0, w)$ is in F.

Example: String in a Language

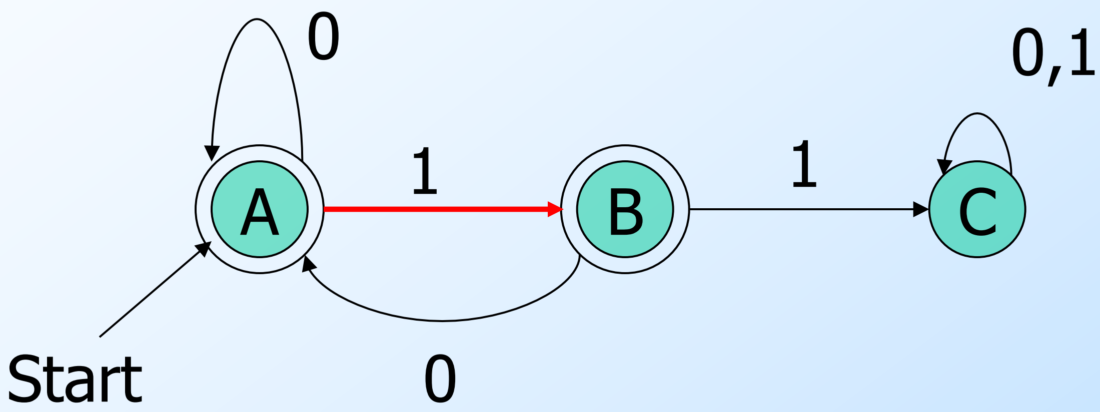
String 101 is in the language of the DFA below.
Start at A.



Example: String in a Language

String 101 is in the language of the DFA below.

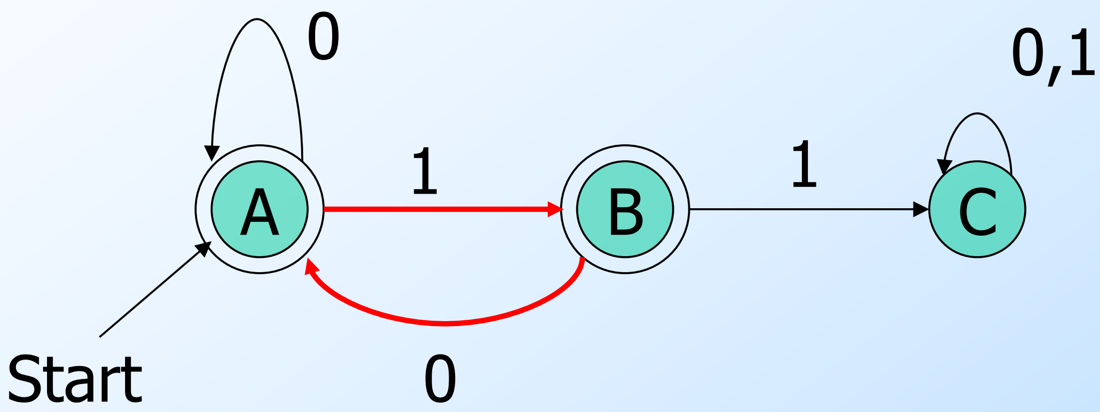
Follow arc labeled 1.



Example: String in a Language

String 101 is in the language of the DFA below.

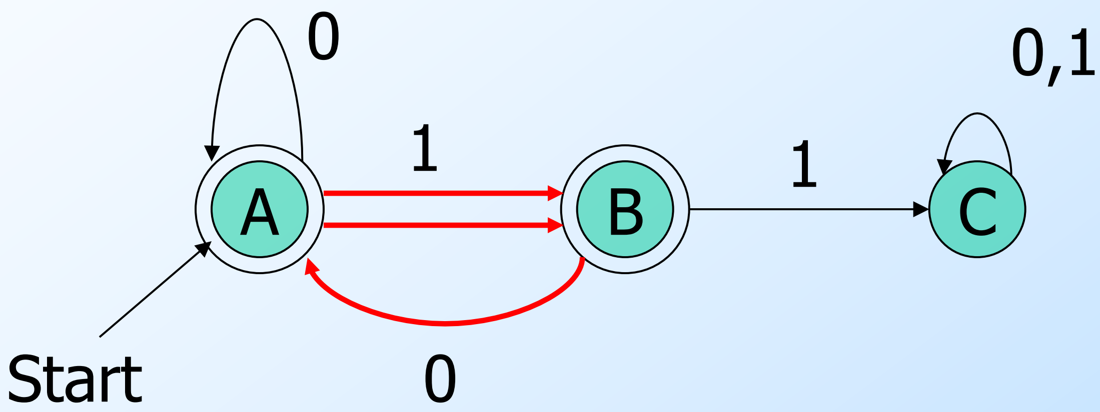
Then arc labeled 0 from current state B.



Example: String in a Language

String 101 is in the language of the DFA below.

Finally arc labeled 1 from current state A. Result is an accepting state, so 101 is in the language.



Example – Concluded

- The language of our example DFA is:

{w | w is in {0,1}* and w does not have two consecutive 1's}

Such that…

These conditions about w are true.

Read a *set former* as "The set of strings w…

Proofs of Set Equivalence

- Often, we need to prove that two descriptions of sets are in fact the same set.
- Here, one set is "the language of this DFA," and the other is "the set of strings of 0's and 1's with no consecutive 1's."

Proofs – (2)

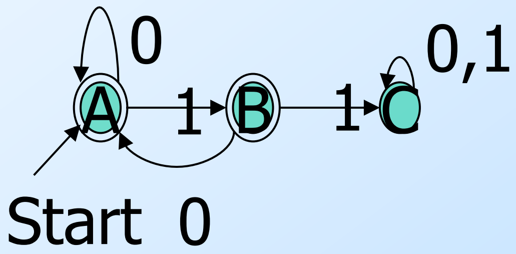
- In general, to prove S = T, we need to prove two parts: $S \subseteq T$ and $T \subseteq S$.

  That is:

  1. If w is in S, then w is in T.
  2. If w is in T, then w is in S.

- Here, S = the language of our running DFA, and T = "no consecutive 1's."

Part 1: $S \subseteq T$



- To prove: if w is accepted by
then w has no consecutive 1's.
- Proof is an induction on length of w.
- Important trick: Expand the inductive hypothesis to be more detailed than the statement you are trying to prove.

The Inductive Hypothesis

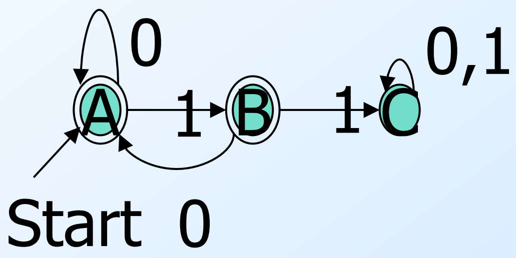
1. If $\delta(A, w) = A$, then w has no consecutive 1's and does not end in 1.
2. If $\delta(A, w) = B$, then w has no consecutive 1's and ends in a single 1.

- Basis: $|w| = 0$; i.e., $w = \epsilon$.
  - (1) holds since ϵ has no 1's at all.
  - (2) holds *vacuously*, since $\delta(A, \epsilon)$ is not B.

"length of"

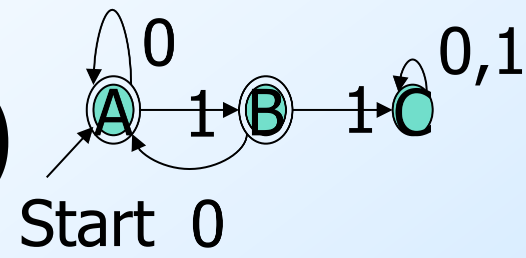
Important concept:
If the "if" part of "if..then" is false, the statement is true.

Inductive Step



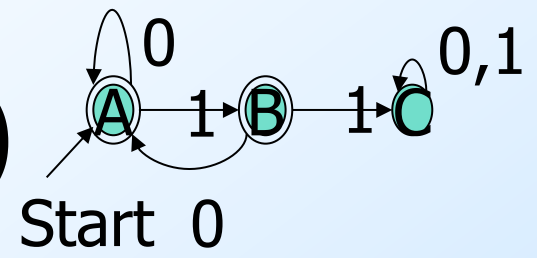
- Assume (1) and (2) are true for strings shorter than w, where |w| is at least 1.
- Because w is not empty, we can write w = xa, where *a* is the last symbol of w, and x is the string that precedes.
- IH is true for x.

Inductive Step – (2)



- Need to prove (1) and (2) for w = xa.
- (1) for w is: If δ(A, w) = A, then w has no consecutive 1's and does not end in 1.
- Since δ(A, w) = A, δ(A, x) must be A or B, and *a* must be 0 (look at the DFA).
- By the IH, x has no 11's.
- Thus, w has no 11's and does not end in 1.

Inductive Step – (3)



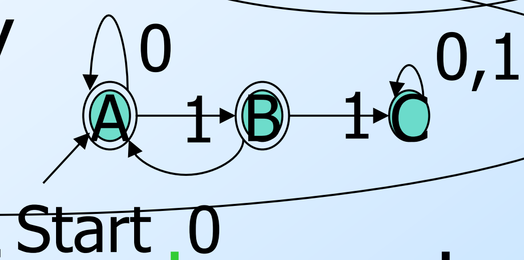
- Now, prove (2) for w = xa: If δ(A, w) = B, then w has no 11's and ends in 1.
- Since δ(A, w) = B, δ(A, x) must be A, and *a* must be 1 (look at the DFA).
- By the IH, x has no 11's and does not end in 1.
- Thus, w has no 11's and ends in 1.

Part 2: $T \subseteq S$

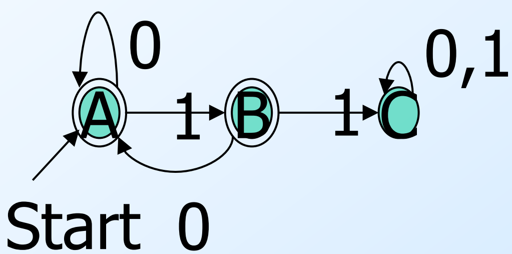
X

- Now, we must prove: if w has no 11's, then w is accepted by

Y



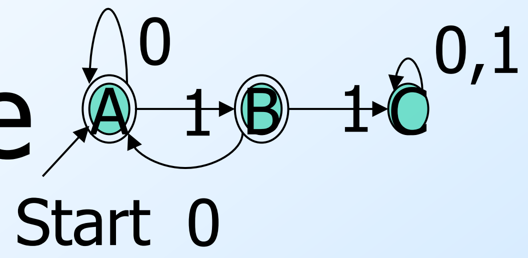
- *Contrapositive* : If w is not accepted by



then w has 11.

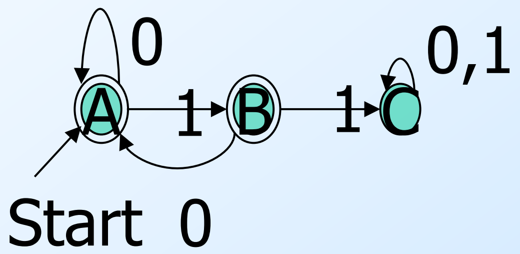
Key idea: contrapositive of "if X then Y" is the equivalent statement "if not Y then not X."

Using the Contrapositive



- Because there is a unique transition from every state on every input symbol, each w gets the DFA to exactly one state.
- The only way w is not accepted is if it gets to C.

Using the Contrapositive – (2)



- The only way to get to C [formally: $\delta(A,w) = C$] is if $w = x1y$, x gets to B, and y is the tail of w that follows what gets to C for the first time.
- If $\delta(A,x) = B$ then surely $x = z1$ for some z.
- Thus, $w = z11y$ and has 11.

Regular Languages

- A language L is *regular* if it is the language accepted by some DFA.
  - Note: the DFA must accept only the strings in L, no others.
- Some languages are not regular.
  - Intuitively, regular languages "cannot count" to arbitrarily high integers.

Example: A Nonregular Language

$L_1 = \{0^n1^n \mid n \geq 1\}$

- Note: a^i is conventional for i a's.
  - Thus, $0^4 = 0000$, e.g.
- Read: "The set of strings consisting of n 0's followed by n 1's, such that n is at least 1.
- Thus, $L_1 = \{01, 0011, 000111, \ldots\}$

Another Example

L_2 = {w | w in {(,)}* and w is *balanced* }

- Balanced parentheses are those sequences of parentheses that can appear in an arithmetic expression.
- E.g.: (), ()(), (()), (()()),…

But Many Languages are Regular

- They appear in many contexts and have many useful properties.
- Example: the strings that represent floating point numbers in your favorite language is a regular language.

Example: A Regular Language

L_3 = { w | w in {0,1}* and w, viewed as a binary integer is divisible by 23}

- The DFA:
  - 23 states, named 0, 1,…,22.
  - Correspond to the 23 remainders of an integer divided by 23.
  - Start and only final state is 0.

Transitions of the DFA for L_3

- If string w represents integer i, then assume $\delta(0, w) = i\%23$.
- Then w0 represents integer 2i, so we want $\delta(i\%23, 0) = (2i)\%23$.
- Similarly: w1 represents 2i+1, so we want $\delta(i\%23, 1) = (2i+1)\%23$.
- Example: $\delta(15,0) = 30\%23 = 7$; $\delta(11,1) = 23\%23 = 0$.

Another Example

L_4 = { w | w in {0,1}* and w, viewed as the reverse of a binary integer is divisible by 23}

- Example: 01110100 is in L_4, because its reverse, 00101110 is 46 in binary.
- Hard to construct the DFA.
- But there is a theorem that says the reverse of a regular language is also regular.