HW4

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Contents

1 3.3.12 **1**

2 3.3.13 **1**

3 3.3.15 **2**

1 3.3.12

1. $(i) \Rightarrow (iii)$
   Assume x is the projection of y on C and there exists some $c \in C\, s.t.\ \langle y - x, c - x \rangle > 0$. Now consider the point $x + \epsilon(c - x)$ where $\epsilon \in (0, 1)$ and thus it's in C. $||x + \epsilon(c - x) - y||^2 = ||x - y||^2 + \epsilon^2||c - x||^2 + 2\epsilon \langle x - y, c - x \rangle$. Its derivative w.r.t ϵ is $2(\epsilon||c - x||^2 + \langle x - y, c - x \rangle) < 0$ when ϵ is closed to 0. So there exists $epsilon\, s.t.\, x + \epsilon(c - x)$ is closer to y than x and is also in C. This contradicts our assumption. So (iii) must holds.

2. $(iii) \Rightarrow (i)$
   Assume (iii) holds.For any $c \in C$, we have $||c - y||^2 = ||c - x + x - y||^2 = ||c - x||^2 + ||y - x||^2 - 2 \langle y - x, c - x \rangle \geq ||x - y||^2$. So (i) holds.

3. $(ii) \Rightarrow (i)$
   Simply let τ be 1.

4. $(i) \Rightarrow (ii)$
   By $(i) \Leftrightarrow (iii)$, it's equivalent to show that $y - x \in N_C(x) \Rightarrow x + \tau(y - x) - x = \tau(y - x) \in N_C(x)$. This equation just follows by the definition of normal cone.

2 3.3.13

$N_K(k) \subset \{k^* \in K^o | \langle k^*, k \rangle = 0\}$

1. For any $k^* \in N_K(k)$, $\langle k^*, k\rangle = 0$.
   By definition, we have $\langle k^*, c-k\rangle \leq 0$, $\forall c \in K$. Since K is a cone, we can let $c = 2k \, and \, \frac{k}{2}$, then we have $\langle k^*, k\rangle = 0$.

2. For any $k^* \in N_K(k)$, $k^* \in K^o$.
   For any $c \in K$, we have $\langle k^*, c-k\rangle \leq 0$. We already known that $\langle k^*, k\rangle = 0$. So $\langle k^*, c\rangle \leq 0 \leq 1$. So $k^* \in K^o$.

Thus $N_K(k) \subset \{k^* \in K^o | \langle k^*, k\rangle = 0\}$.

$\{k^* \in K^o | \langle k^*, k\rangle = 0\} \subset N_K(k)$

For any $k^* \in \{k^* \in K^o | \langle k^*, k\rangle = 0\}$ and any $c \in K$, we have $\langle k^*, c\rangle \leq 0$ by Proposition 3.1.2. Then we have $\langle k^*, c-k\rangle = \langle k^*, c\rangle \leq 0$. Thus $\{k^* \in K^o | \langle k^*, k\rangle = 0\} \subset N_K(k)$.

3 3.3.15

It's sufficient to show that $N_C(x) \subset N_C(y)$. Then by symmetry, we also have $N_C(y) \subset N_C(x)$. So they are equvalent.
Assume there exists $z \in N_C(x)$ but $z \notin N_C(y)$. Then there is some $c \in C$ and $\langle z, c-y\rangle > 0$.

Now let's consider a point of the form $x + \epsilon(x-y) + \sigma(c-y)$, where ϵ, $\sigma > 0$.
Then $\langle z, (x + \epsilon(x-y) + \sigma(c-y)) - x\rangle$
$= \langle z, \epsilon(x-y) + \sigma(c-y)\rangle$
$= -\epsilon \langle z, y-x\rangle + \sigma \langle z, c-y\rangle > 0$.
Because $\langle z, y-x\rangle \leq 0$ by definition and $\langle z, c-y\rangle$ by assumption.

$x + \epsilon(x-y) + \sigma(c-y)$
$= (1-\sigma)x + (\epsilon+\sigma)(x-y) + \sigma c$
$= (1-\sigma)(x + \frac{\epsilon+\sigma}{1-\sigma}(x-y)) + \sigma c$

Because $x, y \in S$ is relatively open and $x - y \in par\, S$, if we choose ϵ and σ small enough, $x + \frac{\epsilon+\sigma}{1-\sigma}(x-y) \in S \subset C$. By the convexity of C, $(1-\sigma)(x + \frac{\epsilon+\sigma}{1-\sigma}(x-y)) + \sigma c \in C$.
This means that we found a point $c^{'} \in C$ satisfying $\left\langle z, c^{'} - x\right\rangle > 0$, which contradicts our assumption. So $N_C(x) \in N_C(y)$. We are done.