CS787: Advanced Algorithms	**Scribe:** Group 4
Title: hw4	

0.1 Limited Interview Slots

(A) let $D_i(X)$ = P(observed candidate i's quality = X)
LP:
$max \sum_u y_{iu} * u$

s.t. $\sum_u y_{iu} D_i(u) <= x_i$, for all i

$\sum_{i,u} y_{iu} <= 1$

$\sum_i x_i <= k$

0.2 Adaptive Learning

The denotations are as below:
t_1: The begin time stamp
t_2: The end time stamp after a block of some steps
B: The expert with the minimum mistakes during the block
m: The amount of minimum mistakes made by B during the block
M: The amount of mistakes made by ALG during the block

First, we derive the lower bound of w_{B,t_1}
If $w_{B,t_1-1} \geq \frac{W_{t_1-1}}{4n}$ and B made a mistake between $t_1 - 1$ and t_1
Then we have:

$$w_{B,t_1} \geq \frac{1}{2}\frac{W_{t_1-1}}{4n} \geq \frac{W_{t_1}}{8n}$$

Second, we derive the relation between W_t and W_{t+1} whenever ALG makes a mistake

$$\begin{aligned}
W_{t+1} &= \sum_{i:\ correct} w_{i,t} + \sum_{i:\ wrong,\ w_{i,t} < \frac{W_t}{4n}} w_{i,t} + \frac{1}{2} \sum_{i:\ wrong,\ w_{i,t} \geq \frac{W_t}{4n}} w_{i,t} \\
&= W_t - \frac{1}{2} \sum_{i:\ wrong,\ w_{i,t} \geq \frac{W_t}{4n}} w_{i,t} \\
&\leq W_t - \frac{1}{2}\frac{W_t}{4n} \\
&= W_t(1 - \frac{1}{8n})
\end{aligned}$$

Finally, combining above we get:

$$\begin{aligned}
\frac{W_{t_1}}{8n}(\frac{1}{2})^m \le w_{B,t_1}(\frac{1}{2})^m \le w_{B,t_2} \le W_{t_2} \le W_{t_1}(1-\frac{1}{8n})^M \\
(\frac{1}{2})^m \le 8n(1-\frac{1}{8n})^M \\
(Apply\ \log_2\ operation) \\
m * \log_2 \frac{1}{2} \le 3 + \log_2 n + M * \log_2(1-\frac{1}{8n}) \\
M \le \frac{m + \log_2 n + 3}{\log_2(1-\frac{1}{8n})}\ (Since\ \log_2(1-\frac{1}{8n})\ is\ negative) \\
= O(m + \log_2 n)
\end{aligned}$$

0.3 Sleeping Experts

(A) Step1, when t = 0, $W_0 = \sum_{i=1}^n w_{i,0}$ = n.
Step2, denote W_t' the summation of all weights at timestamp t. Assuming $W_t' \le n$
Step3, by definition, we can deduct as below:

$$\begin{aligned}
W_{t+1} &= \sum_{i=1}^n (1+\epsilon)^{R_{i,t}} w_{i,t} \le \sum_{i=1}^n (1+R_{i,t}\epsilon) w_{i,t} = W_t + \epsilon \sum_{i=1}^n R_{i,t} w_{i,t} \\
&= W_t + \epsilon[\frac{\sum_j p_{j,t} c_{j,t}}{1+\epsilon} W_t - \sum_{i=1}^n c_{i,t} w_{i,t}] \\
&= W_t + \frac{\epsilon}{1+\epsilon} \sum_{i=1}^n c_{i,t} w_{i,t} - \epsilon \sum_{i=1}^n c_{i,t} w_{i,t} \\
&= W_t + \epsilon \sum_{i=1}^n c_{i,t} w_{i,t} (\frac{1}{1+\epsilon} - 1)\ \ (Since\ \frac{1}{1+\epsilon} - 1\ is\ negative) \\
&\le W_t
\end{aligned}$$

Since both the awake experts' weights summation and sleeping experts' weights' summation do not increase during update, we concludes: $W'_{t+1} \le n$
Hence, by induction we complete the proof.

(B) Let $\beta = \frac{1}{1+\epsilon}$. Then, $W_{i,t+1} = W_{i,t} \beta^{c_{i,t} - \beta E[c_t]}$

And the weight of expert i at time T is: $w_{i,T} = \prod_{t>=0, t \in T_i} \beta^{c_{i,t} - \beta E[c_t]}$

From part A), $w_{i,T} <= n$.
$n <= w_{i,T} = \beta^{\sum_{t>=0, t\in T_i} (c_{i,t} - \beta E[c_t])}$

If we log both sides, we can get: $log_\beta n >= \sum_{t>=0, t\in T_i} c_{i,t} - \beta \sum_{t>=0, t \in T_i} E[c_t]$

$log_\beta n >= cost_i(i) - \beta cost_i(ALG)$

Then, we change the base of log, we can get
$cost_i(ALG) <= (1+\epsilon)cost_i(i) + \frac{1+\epsilon}{log(1+\epsilon)} logn$

When ϵ is very small, $\frac{1+\epsilon}{log(1+\epsilon)} \approx \frac{1}{\epsilon}$. Thus we can prove that, $cost_i(ALG) <= (1+\epsilon)cost_i(i) + O(\frac{1}{\epsilon} logn)$