Homework 2. Solutions

1. Consider the upper half plane $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Use the definition of an open set to show that A is open in $\mathbb{R}^2$.

Let $(x, y) \in A$ be given. Then $y > 0$ and we claim that the open ball $B((x, y), y)$ is contained entirely within A. In fact, one has

$$\begin{aligned}
(a, b) \in B((x, y), y) \quad &\Longrightarrow \quad (x-a)^2 + (y-b)^2 < y^2 \\
&\Longrightarrow \quad (y-b)^2 < y^2 \\
&\Longrightarrow \quad b(b-2y) < 0 \\
&\Longrightarrow \quad 0 < b < 2y \\
&\Longrightarrow \quad (a, b) \in A.
\end{aligned}$$

This shows that $B((x, y), y) \subset A$, so the set A is open in $\mathbb{R}^2$.

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2. Show that the following set is open in $\mathbb{R}^2$.

$$B = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4x \text{ and } y > 0 \right\}.$$

The given set is the intersection $B = B_1 \cap B_2$, where

$$\begin{aligned} B_1 &= \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4x \right\} \text{ and} \\ B_2 &= \left\{ (x,y) \in \mathbb{R}^2 : y > 0 \right\}. \end{aligned}$$

Note that B_2 is the upper half plane and this is open in $\mathbb{R}^2$ by the previous problem. The set B_1 can be expressed in the form

$$B_1 = \left\{ (x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 4 \right\} = B((2,0),2),$$

so it is an open ball in $\mathbb{R}^2$ and thus open. Being the intersection of two open sets, the given set B is then open as well.

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3. Show that the following sets are open in $\mathbb{R}$.

$$A = \left\{x \in \mathbb{R} : x^3 > x\right\}, \qquad B = \left\{0 < x < 1 : \frac{1}{x} \notin \mathbb{Z}\right\}.$$

When it comes to the first set, one has

$$x^3 > x \iff x(x^2 - 1) > 0 \iff x(x+1)(x-1) > 0.$$

This implies that $A = (-1, 0) \cup (1, \infty)$ and so A is open in $\mathbb{R}$. The second set is the interval $(0, 1)$ with the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ removed. It is open because it is the union of open intervals, namely

$$B = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \cdots = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right).$$

4. Is the set $\mathbb{Q}$ of all rational numbers closed in $\mathbb{R}$? Why or why not?

We use the first part of Theorem 1.4. Were $\mathbb{Q}$ closed in $\mathbb{R}$, every convergent sequence of rational numbers would have to converge to a rational number. However, this is not really the case. As a simple example, consider a rational approximation of $\sqrt{2}$, say

$$\begin{aligned} x_1 &= 1.4, \\ x_2 &= 1.41, \\ x_3 &= 1.414, \\ x_4 &= 1.4142 \end{aligned}$$

and so on. This is a convergent sequence of rational numbers, but its limit $\sqrt{2}$ is not a rational number. Thus, $\mathbb{Q}$ is not closed in $\mathbb{R}$.