CS727: Convex Analysis	**Scribe:** Tien-Lung Fu
Title: hw4	

0.1 Exercise 3.3.12

1. $(i) \Rightarrow (iii)$

$$\begin{aligned}
&x \textit{ is the closest point in } C \textit{ to } y \\
&\Rightarrow \forall c \in C, < y - x, c - x > \leqslant 0 \ (\textit{By Lemma } 2.2.1) \\
&\Rightarrow y - x \in N_c(x)
\end{aligned}$$

2. $(iii) \Rightarrow (i)$

$$\begin{aligned}
&If\ y - x \in N_C(x),\ then\ \forall c \in C,\ (y-x)^T(c-x) \leqslant 0 \\
&\Rightarrow (y-x)^T(y-x) + (y-x)^T(c-y) \leqslant 0 \\
&\Rightarrow ||y-x||^2 \leqslant (y-x)^T(y-c) \leqslant ||y-x||||y-c||\ (Cauchy - Schwarz\ inequality) \\
&\Rightarrow ||y-x|| \leqslant ||y-c|| \\
&\Rightarrow x\ is\ the\ closest\ point\ in\ C\ to\ y \\
&\Rightarrow x\ is\ the\ projection\ of\ y\ on\ C
\end{aligned}$$

3. $(ii) \Rightarrow (i)$

$$If\ we\ let\ \tau := 1,\ QED$$

4. $(i) \Rightarrow (ii)$

$$\begin{aligned}
&By\ (i) \Leftrightarrow (iii),\ y - x \in N_C(x) \Leftrightarrow x\ is\ the\ projection\ of\ y\ on\ C \\
&If\ y - x \in N_C(x),\ implying\ \tau(y-x) \in N_C(x)\ since\ \tau > 0 \\
&\Rightarrow \tau(y-x) = (x + \tau(y-x)) - x \in N_C(x) \\
&\Rightarrow x\ is\ the\ projection\ of\ x + \tau(y-x)\ on\ C
\end{aligned}$$

0.2 Exercise 3.3.13

We want to show $N_K(k) \subset \{k^* \in K^0| < k^*, k >= 0\}$

$$\begin{aligned}
&\forall k^* \in N_K(k),\ we\ have\ \ < k^*, c-k >\leqslant 0,\ \forall c \in K \\
&If\ we\ let\ c := 2k,\ < k^*, k >\leqslant 0 \\
&If\ we\ let\ c := \frac{k}{2},\ \frac{-1}{2} < k^*, k >\leqslant 0 \\
&\Rightarrow < k^*, k >= 0 \\
&\Rightarrow < k^*, c-k >=< k^*, c > - < k^*, k >=< k^*, c >\leqslant 0 \\
&\Rightarrow k^* \in K^0 \\
&\Rightarrow Thus,\ k^* \in \{k^* \in K^0| < k^*, k >= 0\}\ (QED)
\end{aligned}$$

We want to show $\{k^* \in K^0| < k^*, k >= 0\} \subset N_K(k)$

$$\begin{aligned}
&\forall k^* \in \{k^* \in K^0| < k^*, k >= 0\},\ we\ have\ \ < k^*, c >\leqslant 0,\ \forall c \in K \\
&\Rightarrow < k^*, c >=< k^*, c > -0 =< k^*, c > - < k^*, k >=< k^*, c-k >\leqslant 0 \\
&\Rightarrow k^* \in N_K(k)\ (QED)
\end{aligned}$$

0.3 Exercise 3.3.15

$$\begin{aligned}
&Let\ x, y\ \in S \\
&\Rightarrow x, y \in riC \\
&\Rightarrow N_C(x) = N_C(y) = (parC)^{\perp}\ (By\ Theorem\ 3.3.7b)
\end{aligned}$$