General Lagrange and Hermite Interpolation in R^n with Applications to Finite Element Methods

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1. Introduction and Preliminaries

Recently there has been considerable interest in approximation theory in several variables, one of the main reasons for this being that it can be applied to obtain error estimates for various finite element methods of numerical approximation (we have given some references to these methods and their use in the bibliography, but the list is far from exhaustive).

Basically, there are two different types of approach. The first one consists in prescribing a set Σ of points in R^n and then interpolating the values of a given function u at the points of Σ by a polynomial of a given degree. This is *Lagrange interpolation*. If, in addition, the values of some derivatives of the function u are also to be interpolated at some points of Σ, then we have an *Hermite interpolation* problem. Generally, the error bounds are given in term of the sup-norm of some high-order derivatives of the interpolated function; see, for example, the papers of Strang [29], Strang & Fix [30], Zlámal [35, 36], Ženíšek [34], Thacher [32], Thacher & Milne [33], Hall [20], Nicolaides [22, 23, 24], Frederickson [17], Salzer [27], Guenther & Roetman [18]. The work of Coatmélec [14] is also particularly important and relevant in this respect.

In most cases, a special role is played by the canonical Cartesian coordinates, and indeed, this limits in some ways the obtainable results as well as the practical examples which can be handled in this way. However, a more powerful coordinate-free approach can be used, such as in Ciarlet & Wagschal [13], and in some parts of Coatmélec [14]. §§2 and 3 of the present paper depend upon the same idea, and their results can be considered as a generalization of [13], which was itself a generalization of [35]. Our main result concerns the following situation: given a finite set $\Sigma \subset R^n$ and given a function u, the interpolation problem consists in finding a polynomial $\tilde{u}$ of degree $\leqq k$, in such a way that the values of u and $\tilde{u}$ and of some corresponding derivatives of u and $\tilde{u}$ are equal at some specified points of Σ. Under the sole assumption that this problem has a unique solution, we prove the error estimate (cf. Theorems 2 and 4)

$$(1.1)\quad \sup\{\|D^m u(x) - D^m \tilde{u}(x)\|;\, x\in K\} \leqq C M_{k+1} \frac{h^{k+1}}{\rho^m} \quad \text{for all } 0 \leqq m \leqq k$$

where

K = closed convex hull of Σ,

$M_{k+1} = \sup\{\|D^{k+1}u(x)\|; x \in K\}$,

h = diameter of K,

ρ = supremum of the diameters of the inscribed spheres of K,

C = numerical constant independent of Σ (in a sense to be made more precise in the sequel).

The inequality (1.1) is obtained from generalized Taylor expansions (cf. Theorems 1 and 3), which contain as special cases those given in [13].

The second approach, taken up in §4, is more abstract, and perhaps more elegant from a mathematical standpoint. The approximation scheme is defined as follows: given a bounded open subset Ω of R^n, we associate with any function u in the Sobolev space $W^{k+1,p}(\Omega)$ a unique "approximation" Πu. With the sole assumption that Π leaves invariant all polynomials of degree $\leqq k$ (apart from this condition, Πu can be any "reasonable" function), we obtain (cf. Theorem 5)

$$\|u - \Pi u\|_{W^{m,p}(\Omega)} \leqq C |u|_{k+1,p,\Omega} \frac{h^{k+1}}{\rho^m} \quad \text{for all } 0 \leqq m \leqq k+1 \tag{1.2}$$

where

$$|u|_{k+1,p,\Omega} = \|D^{k+1}u\|_{L^p(\Omega)},$$

h = diameter of Ω,

ρ = supremum of the diameters of the inscribed spheres of $\bar{\Omega}$,

C = numerical constant independent of Ω;

this general result is then applied to the Lagrange and Hermite interpolations described in §§2 and 3 (cf. Theorem 6).

This type of approximation is not necessarily defined as a Lagrange or Hermite interpolation problem, but of course it contains such problems as special cases. Our approach is not new, and in some instances, is very similar to that of BRAMBLE [7], BRAMBLE & HILBERT [8, 9], BRAMBLE & ZLÁMAL [10], for example. However our results seem to have a fairly new and more concise form, which generalizes in one way or another these previous results, whether it be that we consider Sobolev spaces $W^{m,p}(\Omega)$ for $p \neq 2$, or that the dimension n is arbitrary, or else that we can handle more practical examples, once the coordinate-free approach developed in §§2 and 3 is used.

For this second approach we note the following relevant papers: AUBIN [1, 2], BABUŠKA [3, 4], DI GUGLIELMO [19], FIX & STRANG [16], BIRKHOFF, SCHULTZ & VARGA [6], NITSCHE [25, 26], SCHULTZ [28].

Several examples are examined in the text, each stemming from some particular finite element method. It is in this sense that the "applications to finite element methods" mentioned in the title are to be understood. The associated error bounds obtained in the solution of elliptic problems are not explicitly mentioned, since their derivation is almost immediate. Basically, our examples involve "elements" which are n-simplices of R^n, because it is perhaps for those types of elements that our results appear to be the newest. We have not considered the multivariate

Hermite approximations of BIRKHOFF, SCHULTZ & VARGA [6] and SCHULTZ [28], but our results could also be applied to these cases.

The geometrical dependence of the error estimates upon the geometry of the domain is exhibited by the two parameters h and ρ of (1.1) or (1.2). Thus, the "smallest angle θ of the triangle" (cf. Remark 3) used by ZLÁMAL [35, 36], STRANG [29] and others in the case where $n=2$ as a measure of the "badness" of the approximation for derivatives, is here replaced by the ratio ρ/h, which is its natural generalization to the case where n is arbitrary.

In practice, one often thinks of a *family* $(\Sigma_i)_{i\in I}$ of sets, with associated parameters h_i and ρ_i. As in [13, condition (6.18)], it is then natural to define a *regular family* as a family for which one has

$$h_i \leqq \alpha \rho_i \quad \text{for all } i \in I, \tag{1.3}$$

for some constant $\alpha > 0$. For such regular families the error bounds of (1.1) can be converted immediately into the following form:

$$\sup\{\|D^m u(x) - D^m \tilde{u}(x)\|;\ x \in K\} = O(h^{k+1-m}) \quad \text{for all } 0 \leqq m \leqq k \tag{1.4}$$

(and similarly for (1.2)), and this is generally the form which is found in the literature.

As in [13], heavy use will be made of differential calculus with Fréchet derivatives, for which we refer to [11] or [15]. We now briefly explain some of our subsequent notations. Given two vector spaces over the same scalar field, we denote by $\mathscr{L}_k(X; Y)$ the space of k-linear mappings $A_k: X^k \to Y$, where X^k is the Cartesian product of X, k times with itself (we recall that a mapping is k-linear if and only if it is linear with respect to any one of the k variables when the other $(k-1)$ variables are kept fixed). When $k=1$, we shall write $\mathscr{L}(X; Y)$ in lieu of $\mathscr{L}_1(X; Y)$, and $\mathscr{L}(X)$ if $Y=X$.

Given a function $u: R^n \to R$, its k-th derivative $D^k u(a)$ at a point a of R^n (whenever it is defined) is an element of $\mathscr{L}_k(R^n; R)$ which is symmetric in the sense that

$$D^k u(a) \cdot (\xi_1, \xi_2, \ldots, \xi_k) = D^k u(a) \cdot (\xi_{\sigma_1}, \xi_{\sigma_2}, \ldots, \xi_{\sigma_n})$$

for any permutation $\sigma: i \to \sigma_i$ of the set $\{1, 2, \ldots, k\}$. In case $\xi_i = \xi$ for all $1 \leqq i \leqq k$, we shall simply write

$$D^k u(a) \cdot (\xi, \xi, \ldots, \xi) = D^k u(a) \cdot (\xi)^k,$$

and likewise, if $\xi_i = \xi$ for all $2 \leqq i \leqq k$,

$$D^k u(a) \cdot (\xi_1, \xi, \ldots, \xi) = D^k u(a) \cdot (\xi_1, (\xi)^{k-1}).$$

The norm of $D^k u(a)$ is defined as

$$\|D^k u(a)\| = \sup\{|D^k u(a) \cdot (\xi_1, \xi_2, \ldots, \xi_k)|;\ \|\xi_i\| \leqq 1, 1 \leqq i \leqq k\}$$

where $\|\cdot\|$ denotes the usual Euclidean norm over R^n. If R^n is equipped with its canonical basis $(e_1, e_2, \ldots, e_n)$, then the usual partial derivatives are given by

$$\frac{\partial u}{\partial x_i}(a) = D u(a) \cdot (e_i), \qquad \frac{\partial^2 u}{\partial x_i \partial x_j}(a) = D^2 u(a) \cdot (e_i, e_j), \quad \text{etc.} \ldots,$$

so that as a consequence of the definition of $\|D^k u(a)\|$, we have

$$\left|\frac{\partial u}{\partial x_i}(a)\right| \leqq \|Du(a)\| \leqq C_1(n) \max\left\{\left|\frac{\partial u}{\partial x_i}(a)\right|; 1\leqq i\leqq n\right\},$$

$$\left|\frac{\partial^2 u}{\partial x_i \partial x_j}(a)\right| \leqq \|D^2 u(a)\| \leqq C_2(n) \max\left\{\left|\frac{\partial^2 u}{\partial x_i \partial x_j}(a)\right|; 1\leqq i,j\leqq n\right\}, \quad \text{etc.} \ldots$$

for some constants $C_1(n)$, $C_2(n)$, etc. ..., which depend only upon n; this last observation shows that error estimates such as those of (1.1), (2.15), and (3.15) can immediately be converted into similar estimates involving the usual partial derivatives.

2. Lagrange Interpolation

Given an integer $k\geqq 1$, we let P_k denote the space of polynomials of degree $\leqq k$ defined over R^n, and we let

$$N=N(k)=\dim P_k.$$

We shall say that N distinct points a_i of R^n form a *k-unisolvent set*

$$\Sigma=\{a_i\}_{i=1}^N$$

provided that, given any real numbers α_i, $1\leqq i\leqq N$, there exists one and only one polynomial $p\in P_k$ such that

$$p(a_i)=\alpha_i, \quad 1\leqq i\leqq N.$$

Given a function u defined on a k-unisolvent set Σ, we say that $\tilde{u}$ is its *interpolating polynomial* if it is the unique polynomial of degree $\leqq k$ with the property that

$$\tilde{u}(a_i)=u(a_i), \quad 1\leqq i\leqq N.$$

Example 1. Let K be a nondegenerate n-simplex of R^n with vertices a_i, $1\leqq i\leqq n+1$. We denote by $a_{ij}=a_{ji}$ the mid-point of the edge joining the vertex a_i to the vertex a_j, $i\neq j$. Then the set

$$\Sigma^{\mathrm{I}}=\{a_i\}_{i=1}^{n+1}$$

is 1-unisolvent, and the set

$$\Sigma^{\mathrm{II}}=\{a_i\}_{i=1}^{n+1}\cup\{a_{ij}\}_{1\leqq i<j\leqq n+1}$$

is 2-unisolvent (cf. [13]). If $\tilde{u}$ is the interpolating polynomial of a given function u on Σ^{I} (resp. Σ^{II}), we shall say that $\tilde{u}$ is an *approximation of type* I (resp. *of type* II) of the function u.

Now let $\Sigma=\{a_i\}_{i=1}^N$ be a set of points of R^n. We say that a subset $K\subset R^n$ is *Σ-admissible* if and only if whenever it contains a point x, it also contains the closed segments joining the points x and a_i, for all $1\leqq i\leqq N$ (in other words, K is star-shaped with respect to any point of Σ). Given a function u: $K\to R$, we say that u belongs to *the class* $\mathscr{T}^{k+1}(K)$ if and only if the Taylor formula

$$(2.1)\qquad \begin{aligned} u(a_i)&=u(x)+Du(x)\cdot(a_i-x)+\cdots+\frac{1}{k!}D^k u(x)\cdot(a_i-x)^k\\ &+\frac{1}{(k+1)!}D^{k+1}u(\eta_i(x))\cdot(a_i-x)^{k+1}\end{aligned}$$

holds for all points $x\in K$ and all $1\leqq i\leqq N$, where

$$(2.2)\qquad \eta_i(x)=\theta_i x+(1-\theta_i)a_i \quad \text{for some } 0<\theta_i<1.$$

In particular, this will be the case if K is the closed convex hull of Σ, $u\in C^k(K)$, and $D^{k+1}u(x)$ exists for all points x in K.

Theorem 1. *Let k be a fixed integer $\geqq 1$, and let there be given a k-unisolvent set $\Sigma=\{a_i\}_{i=1}^N$ of points of R^n. Let $K\subset R^n$ be a Σ-admissible set, and let $u\in\mathscr{T}^{k+1}(K)$ be given. Then, for any point $x\in K$ and any integer m with $0\leqq m\leqq k$, one has*

$$(2.3)\qquad D^m\tilde{u}(x)=D^m u(x)+\frac{1}{(k+1)!}\sum_{i=1}^N\{D^{k+1}u(\eta_i(x))\cdot(a_i-x)^{k+1}\}D^m p_i(x)$$

where the p_i's are the unique polynomials of degree $\leqq k$ such that

$$(2.4)\qquad p_i(a_j)=\delta_{ij},\quad 1\leqq i,j\leqq N,$$

and $\eta_i(x)$ is of the form (2.2).

Proof. From the definition of the polynomials p_i it follows that $\tilde{u}=\sum_{i=1}^N u(a_i)\,p_i$, and thus $D^m\tilde{u}=\sum_{i=1}^N u(a_i)\,D^m p_i$ for any $m\geqq 1$. Let x be a point of K. Since $u\in\mathscr{T}^{k+1}(K)$, we can replace $u(a_i)$ by (2.5) in the above expressions. In this fashion, we obtain for any integer m with $0\leqq m\leqq k$,

$$(2.5)\qquad \begin{aligned} D^m\tilde{u}(x)&=\sum_{l=0}^k\frac{1}{l!}\sum_{i=1}^N\{D^l u(x)\cdot(a_i-x)^l\}D^m p_i(x)\\ &+\frac{1}{(k+1)!}\sum_{i=1}^N\{D^{k+1}u(\eta_i(x))\cdot(a_i-x)^{k+1}\}D^m p_i(x).\end{aligned}$$

We shall now use the fact that $\tilde{u}=u$ whenever $u\in P_k$. This will yield the (somehow unexpected) result that in the last expression,

$$(2.6)\qquad \frac{1}{l!}\sum_{i=1}^N\{D^l u(x)\cdot(a_i-x)^l\}D^m p_i(x)=\begin{cases}0 & \text{for } 0\leqq l\leqq m-1,\\ D^m u(x) & \text{for } l=m,\\ 0 & \text{for } m+1\leqq l\leqq k.\end{cases}$$

In [13], this fact was observed in some particular cases, but was left unexplained. We begin by proving that

$$\sum_{i=1}^N\{D^l u(x)\cdot(a_i-x)^l\}D^m p_i(x)=0$$

for the values of l belonging to the (possibly empty) set $\{0, 1, \dots, m-1\}$. If u is a constant A_0, then (2.5) reduces to

$$0=\sum_{i=1}^{N}\{A_0(a_i-x)\}\,D^m p_i(x), \qquad \text{for all } x\in R^n,\ A_0\in R.$$

In particular, this is true for any $x\in K$ and for $A_0=u(x)$, so that the term corresponding to $l=0$ in (2.5) is equal to zero. The proof of (2.6) will now be achieved by induction on l. If (2.6) holds for $l=0, 1, \dots, l_0$ for some $l_0\leqq m-2$, then (2.5) yields (since $l_0+1\leqq m-1$)

$$(2.7)\qquad 0=\sum_{i=1}^{N}\{A_{l_0}\cdot(a_i-x)^{l_0}\}\,D^m p_i(x), \qquad \text{for all } x\in R^n,\ A_{l_0}\in\mathscr{L}_{l_0}(R^n;R),$$

since to any element $A_{l_0}\in\mathscr{L}_{l_0}(R^n;R)$ we can associate a polynomial $p\in P_{l_0}$ such that $D^{l_0}p(x)=A_{l_0}$ for all $x\in R^n$ $\left(\text{for example, choose } x\in R^n\to p(x)=\frac{1}{l_0!}A_{l_0}\cdot(x)^{l_0}\right)$. Thus, we obtain

$$0=\sum_{i=1}^{N}\{D^{l_0}u(x)\cdot(a_i-x)^{l_0}\}\,D^m p_i(x), \qquad \text{for all } x\in K,$$

by letting $A_{l_0}=D^{l_0}u(x)$ in (2.7). Therefore, (2.6) is proved for $0\leqq l\leqq m-1$. If $l=m$, then by observing that $\tilde u=u$ if $u\in P_m$, we get in the same fashion,

$$A_m=\frac{1}{m!}\sum_{i=1}^{N}\{A_m\cdot(a_i-x)^m\}\,D^m p_i(x), \qquad \text{for all } x\in R^n,\ A_m\in\mathscr{L}_m(R^n;R),$$

so that in particular

$$D^m u(x)=\frac{1}{m!}\sum_{i=1}^{N}\{D^m u(x)\cdot(a_i-x)^m\}\,D^m p_i(x), \qquad \text{for all } x\in K.$$

The proof of (2.6) for $m+1\leqq l\leqq k$ is identical, and is omitted.

Remark 1. With the same assumptions as in Theorem 1, we can expand the value $u(x)$ for any $x\in K$ via the *multipoint Taylor formula*

$$(2.8)\qquad u(x)=\sum_{i=1}^{N}u(a_i)\,p_i(x)+\mathscr{R}(D^{k+1}u,x)$$

where the functions p_i do not depend upon the function u, and the remainder, denoted by $\mathscr{R}(D^{k+1}u,x)$, depends upon the $(k+1)$-st derivative of u evaluated at a finite number of points of K, but does not depend upon lower order derivatives of u. This is an immediate consequence of (2.3) with $m=0$.

Before proving error estimates (Theorem 2), we must examine some geometrical properties of point sets of R^n. Let $\Sigma=\{a_i\}_{i=1}^N$ and $\hat\Sigma=\{\hat a_i\}_{i=1}^N$ be two sets of N points of R^n. Then we say that the two sets are *equivalent* if and only if there exists an invertible element $B\in\mathscr{L}(R^n)$ and a vector $b\in R^n$ such that

$$(2.9)\qquad a_i=B\hat a_i+b, \qquad 1\leqq i\leqq N.$$

Lemma 1. *Let $\hat\Sigma=\{\hat a_i\}_{i=1}^N$ be a k-unisolvent set, and let $\Sigma=\{a_i\}_{i=1}^N$ be an equivalent set of points. Then Σ is k-unisolvent.*

Proof. Let $\hat{p}_i$, $1 \leqq i \leqq N$, be the unique polynomials of degree $\leqq k$ such that $\hat{p}_i(\hat{a}_j) = \delta_{ij}$, $1 \leqq i, j \leqq N$. The polynomials p_i defined by

$$x \in R^n \to p_i(x) = \hat{p}_i(B^{-1}(x-b)), \quad 1 \leqq i \leqq N, \tag{2.10}$$

are such that $p_i(a_j) = \delta_{ij}$, $1 \leqq i, j \leqq N$, which proves that Σ is k-unisolvent.

Example 2. Let K (resp. $\hat{K}$) be a nondegenerate n-simplex of R^n with vertices a_i (resp. $\hat{a}_i$) and with mid-points a_{ij} (resp. $\hat{a}_{ij}$) on the edges (cf. Example 1). Then the 1-unisolvent sets

$$\Sigma^{\mathrm{I}} = \{a_i\}_{i=1}^{n+1} \quad \text{and} \quad \hat{\Sigma}^{\mathrm{I}} = \{\hat{a}_i\}_{i=1}^{n+1}$$

are equivalent, and likewise, the 2-unisolvent sets

$$\Sigma^{\mathrm{II}} = \{a_i\}_{i=1}^{n+1} \cup \{a_{ij}\}_{1 \leqq i < j \leqq n+1} \quad \text{and} \quad \hat{\Sigma}^{\mathrm{II}} = \{\hat{a}_i\}_{i=1}^{n+1} \cup \{\hat{a}_{ij}\}_{1 \leqq i < j \leqq n+1}$$

are equivalent.

Given a set $\Sigma = \{a_i\}_{i=1}^N$ of N points of R^n, we henceforth denote by $K = K(\Sigma)$ the closed convex hull of Σ, and we associate with K the two following geometrical parameters.

$$h = h(\Sigma) = \text{diameter of } K, \tag{2.11}$$

$$\rho = \rho(\Sigma) = \sup\{\text{diameter of the spheres contained in } K\} \tag{2.12}$$

(in many applications, K is an n-simplex, so that ρ is the diameter of the inscribed sphere). If $\hat{\Sigma}$ is another set of points of R^n, we shall likewise denote by $\hat{K}$ its closed convex hull, and by $\hat{h}$ and $\hat{\rho}$ the two parameters defined as in (2.11)–(2.12).

If Σ is any k-unisolvent set ($k \geqq 1$), the interior $\mathring{K}$ of K is nonempty, so that ρ is >0.

The following simple result will be of crucial importance in proving the error estimates.

Lemma 2. *Let $\Sigma = \{a_i\}_{i=1}^N$ and $\hat{\Sigma} = \{\hat{a}_i\}_{i=1}^N$ be two equivalent k-unisolvent sets, with*

$$a_i = B\hat{a}_i + b, \quad 1 \leqq i \leqq N,$$

for some $B \in \mathscr{L}(R^n)$ and some $b \in R^n$. If $\|\cdot\|$ denotes the operator norm induced by the usual Euclidean vector norm of R^n, we have

$$\|B\| \leqq \frac{h}{\hat{\rho}} \quad \text{and} \quad \|B^{-1}\| \leqq \frac{\hat{h}}{\rho}. \tag{2.13}$$

Proof. Since $\hat{\rho} > 0$, we can write $\|B\|$ as

$$\|B\| = \sup\left\{\frac{\|B\xi\|}{\|\xi\|};\ \xi \in R^n \text{ with } \|\xi\| = \hat{\rho}\right\}.$$

Each vector $\xi \in R^n$ which satisfies $\|\xi\| = \hat{\rho}$ can be written as $\xi = \hat{y} - \hat{z}$ with $\hat{y}$, $\hat{z} \in \hat{K}$, by definition of $\hat{\rho}$. The vector

$$B\xi = (B\hat{y} + b) - (B\hat{z} + b)$$

is thus of the form $B\xi = y - z$ with $y, z \in K$; thus $\|B\xi\| \leqq h$ by definition of h, from which the first inequality of (2.14) follows at once. The proof of the second inequality of (2.14) is identical.

Theorem 2. *Let $\Sigma = \{a_i\}_{i=1}^N$ be a k-unisolvent set of points of R^n, and let h and ρ be defined as in* (2.11)–(2.12). *Let $u \in \mathscr{T}^{k+1}(K)$ be given with*

$$M_{k+1} = \sup\{\|D^{k+1} u(x)\|;\, x \in K\} < +\infty. \tag{2.14}$$

If u is the unique interpolating polynomial of degree $\leqq k$ of u, we have for any integer m with $0 \leqq m \leqq k$,

$$\sup\{\|D^m u(x) - D^m \tilde{u}(x)\|;\, x \in K\} \leqq C M_{k+1} \frac{h^{k+1}}{\rho^m}, \tag{2.15}$$

for some constants

$$C = C(n, k, m, \hat{\Sigma}) \tag{2.16}$$

which are the same for all equivalent k-unisolvent sets and which can be computed once and for all in a k-unisolvent set $\hat{\Sigma}$ equivalent to Σ.

Proof. Since the closed convex hull $K = K(\Sigma)$ is always Σ-admissible, the formula (2.3) holds for any point $x \in K$. Thus, for any such point,

$$\|D^m u(x) - D^m \tilde{u}(x)\| \leqq \frac{1}{(k+1)!} \sum_{i=1}^N \left|\{D^{k+1} u(\eta_i(x)) \cdot (a_i - x)^{k+1}\}\right| \|D^m p_i(x)\|. \tag{2.17}$$

By definition of h and M_{k+1}, we first have

$$\left|\{D^{k+1} u(\eta_i(x)) \cdot (a_i - x)^{k+1}\}\right| \leqq M_{k+1} h^{k+1}. \tag{2.18}$$

Next, let us choose once and for all a k-unisolvent set $\hat{\Sigma}$ equivalent to Σ. From the proof of Lemma 1 (cf. (2.10)), the associated polynomials p_i and $\hat{p}_i$ satisfy

$$p_i(x) = \hat{p}_i(B^{-1}(x-b)) \quad \text{for all } x \in R^n,$$

so that, for any vectors $\xi_\mu \in R^n$, $1 \leqq \mu \leqq m$,

$$D^m p_i(x) \cdot (\xi_1, \xi_2, \ldots, \xi_m) = D^m \hat{p}_i(B^{-1}(x-b)) \cdot (B^{-1}\xi_1, B^{-1}\xi_2, \ldots, B^{-1}\xi_m),$$

whence

$$\sup\{\|D^m p_i(x)\|;\, x \in K\} \leqq \sup\{\|D^m \hat{p}_i(B^{-1}(x-b))\|;\, x \in K\} \|B^{-1}\|^m.$$

Since the image of $\hat{K}$ under the mapping $\hat{x} \to x = B\hat{x} + b$ is precisely K, we can write

$$\sup\{\|D^m \hat{p}_i(B^{-1}(x-b))\|;\, x \in K\} = \sup\{\|D^m \hat{p}_i(\hat{x})\|;\, \hat{x} \in \hat{K}\}.$$

From Lemma 2, $\|B^{-1}\|^m \leqq \hat{h}^m/\rho^m$. Combining this with (2.18) in (2.17), we obtain (2.15) with

$$C(n, k, m, \hat{\Sigma}) = \frac{\hat{h}^m}{(k+1)!} \sum_{i=1}^N \sup\{\|D^m \hat{p}_i(\hat{x})\|;\, \hat{x} \in \hat{K}\},$$

which completes the proof.

Remark 2. The quantities $\sup\{\|D^m \hat{p}_i(\hat{x})\|;\, \hat{x} \in \hat{K}\}$ for $m \geqq 1$ can all be bounded above in terms of $\sup\{|\hat{p}_i(\hat{x})|;\, \hat{x} \in \hat{K}\}$, for one can prove that if $p \in P_k$ and K is any compact convex subset of R^n, then the following generalization of MARKOFF's inequality in one variable holds (cf. [14, §I.3]):

$$\sup\{\|Dp(x)\|;\, x\in K\}\leqq C\frac{k^2 n}{\sigma(K)}\sup\{|p(x)|;\, x\in K\}$$

where C is a numerical constant and $\sigma(K)$ is a geometrical parameter associated with K (note that $\sigma(K)$ reduces to the diameter of the inscribed sphere when K is an n-simplex).

Remark 3. It is clear that the estimates of (2.15) for $m\geqq 1$ are better when the ratio h/ρ is small. The intuitive significance of this is that one should not consider k-unisolvent sets Σ whose closed convex hull is "too flat", i.e., which is "almost" contained in an $(n-1)$-dimensional linear manifold of R^n. For example, if K is a 2-simplex (i.e., a triangle) in R^2, one has the estimate

$$\frac{1}{2\tan\frac{\theta}{2}}\leqq\frac{h}{\rho}\leqq\frac{2}{\sin\theta}$$

where θ is the smallest angle of K, which shows that the smaller θ is, the poorer the estimate is.

Example 3. Consider the approximations of types I and II described in Example 1. In both cases, ρ is equal to the diameter of the inscribed sphere of the n-simplex K. Let us denote by $\lambda_1(x), \lambda_2(x), \ldots, \lambda_{n+1}(x)$ the barycentric coordinates of a point x of R^n with respect to the vertices a_i, $1\leqq i\leqq n+1$, of K. Then approximation of type I is given by [13, Theorem 1]

$$\tilde{u}(x)=\sum_{i=1}^{n+1} u(a_i)\,\lambda_i(x),$$

so that (for $k=1$, $N=N(k)=n+1$)

$$C(n,1,0,\hat{\Sigma})=\frac{N}{2!}\sum_{i=1}^{N}\sup\{|\lambda_i(\hat{x})|;\,\hat{x}\in\hat{K}\}=\frac{n+1}{2},$$

$$C(n,1,1,\hat{\Sigma})=\frac{\hat{h}}{2!}\sum_{i=1}^{N}\sup\{\|D\lambda_i(\hat{x})\|;\,\hat{x}\in\hat{K}\}\leqq\frac{n+1}{2}\,\frac{\hat{h}}{\hat{\rho}},$$

the last inequality being a consequence of [13, Lemma 1]. To obtain the lowest possible ratio $\hat{h}/\hat{\rho}$, one should pick for $\hat{\Sigma}$ an *equilateral* n-simplex, for which [12, Theorem 3]

$$\frac{\hat{h}}{\hat{\rho}}=\left(\frac{n(n+1)}{2}\right)^{\frac{1}{2}}.$$

Similarly [13, Theorem 3], approximation of type II is given by

$$\tilde{u}(x)=\sum_{i=1}^{n+1} p_i(x)\,u(a_i)+\sum_{1\leqq i<j\leqq n+1} p_{ij}(x)\,u(a_{ij})$$

with

$$p_i(x)=2(\lambda_i(x))^2-\lambda_i(x),\qquad 1\leqq i\leqq n+1,$$

$$p_{ij}(x)=4\lambda_i(x)\,\lambda_j(x),\qquad 1\leqq i<j\leqq n+1,$$

so that we can again compute upper bounds for the constants $C(n, 2, m, \hat{\Sigma})$ for $m=0$, 1 and 2.

Thus the error estimates of Corollaries 2 and 4 of [13] are direct consequences of Theorem 2, with the new estimate corresponding to $m=2$ in the case of approximations of type II.

3. Hermite Interpolation

Let k be a given integer $\geqq 1$. Then the natural extension of Hermite interpolation in one variable should be as follows: we are given an integer s with $1 \leqq s \leqq k$, and $(s+1)$ sets of points

$$(3.1)\qquad \Sigma^0=\{a_i^0\}_{i=1}^{N_0},\ \Sigma^1=\{a_i^1\}_{i=1}^{N_1},\ \dots,\ \Sigma^s=\{a_i^s\}_{i=1}^{N_s}.$$

In a given set $\Sigma^r=\{a_i^r\}_{i=1}^{N_r}$, we assume that $a_i^r \neq a_j^r$ if $i \neq j$, but we do not exclude the possibility that $a_i^r=a_{i'}^{r'}$ for some $1 \leqq i \leqq N_r$ and some $1 \leqq i' \leqq N_{r'}$ if $r \neq r'$. Indeed, in practical applications one has

$$\Sigma^s \subset \Sigma^{s-1} \subset \cdots \subset \Sigma^1 \subset \Sigma^0,$$

although this assumption is not necessary for our theory. We say that

$$(3.2)\qquad \Sigma=\Sigma^0 \cup \Sigma^1 \cup \cdots \cup \Sigma^s$$

is a *k-unisolvent* set provided that, given any $A_i^r \in \mathscr{L}_r(R^n; R)$, $1 \leqq i \leqq N_r$, $0 \leqq r \leqq s$, there exists one and only one polynomial $p \in P_k$ such that

$$D^r p(a_i^r)=A_i^r, \quad 1 \leqq i \leqq N_r,\ 0 \leqq r \leqq s,$$

with the convention that $\mathscr{L}_0(R^n; R)=R$.

Example 4. Let K be a nondegenerate n-simplex of R^n with vertices a_i, $1 \leqq i \leqq n+1$. We denote by $a_{ijk}=a_{jik}=\cdots=a_{kji}$ the barycentre of each triangle with vertices a_i, a_j and a_k, with $i \neq j$, $j \neq k$, $k \neq i$. Then the set

$$\Sigma^{\mathrm{III}}=\Sigma^0 \cup \Sigma^1 \quad \text{with} \quad \begin{cases} \Sigma^0=\{a_i\}_{i=1}^{n+1} \cup \{a_{ijk}\}_{\substack{i,j,k=1\\ i\neq j,\, j\neq k,\, k\neq i}}^{n+1}, \\ \Sigma^1=\{a_i\}_{i=1}^{n+1}, \end{cases}$$

is 3-unisolvent [13, Theorem 5], and the associated interpolating function (as defined below) will be called an *approximation of type III.*

The above formulation is, however, not general enough: in many cases, all the conditions needed to define the derivative $D^r p(a_i^r)$ as an element of $\mathscr{L}_r(R^n; R)$ cannot be specified at all the points a_i^r, the reason being that we need to match the number of given parameters with the number of coefficients of a polynomial of degree k. Instead, here it is only required that some "partial" derivatives $D^r p(a_i^r) \cdot (\xi_1, \xi_2, \dots, \xi_r)$ are to be interpolated for $(\xi_1, \xi_2, \dots, \xi_r)$ belonging to some subset X_i^r of $(R^n)^r$. For example, if R^n is equipped with the canonical basis $(e_k)_{k=1}^n$, it can be required that $\dfrac{\partial^2 p}{\partial x_k^2}(a_i^2)$, $1 \leqq k \leqq n$, must be interpolated at some point a_i^2, but the cross-derivatives $\dfrac{\partial^2 p}{\partial x_k \partial x_l}(a_i^2)$, $k \neq l$, are not to be considered. In this case, $X_i^2 = \bigcup_{k=1}^{n} \{(e_k, e_k)\}$, whereas the full knowledge of the second derivative $D^2 p(a_i^2)$ would require that, for instance, $X_i^2 = \bigcup_{k,l=1}^{n} \{(e_k, e_l)\}$.

Accordingly, the *Hermite interpolation problem* is defined as follows: We are given a set Σ of the form (3.2), each Σ^r being as in (3.1); in addition, *to each point a_i^r, for $1 \leqq r \leqq s$, we associate a subset X_i^r of $(R^n)^r$* (X_i^r is of course assumed to be nonempty, and $(\xi_1, \xi_2, \ldots, \xi_r) \in X_i^r$ implies that all ξ_i's are $\neq 0$). Then we say that the set Σ is *k-unisolvent* if and only if, given any $A_i^r \in \mathscr{L}_r(R^n; R)$, $1 \leqq i \leqq N_r$, $0 \leqq r \leqq s$, there exists one and only one polynomial $p \in P_k$ such that, for all $1 \leqq i \leqq N_r$ and all $0 \leqq r \leqq s$,

$$(3.3)\ D^r p(a_i^r) \cdot (\xi_1, \xi_2, \ldots, \xi_r) = A_i^r \cdot (\xi_1, \xi_2, \ldots, \xi_r) \quad \text{for all} \quad (\xi_1, \xi_2, \ldots, \xi_r) \in X_i^r,$$

with the convention that (3.3) reduces to $p(a_i^0) = A_i^0$ when $r=0$. Likewise, given a function u such that $D^r u(a_i^r)$ is defined for $1 \leqq i \leqq N_r$ and $0 \leqq r \leqq s$, we say that $\tilde{u}$ is its *interpolating polynomial* if it is the unique polynomial of degree $\leqq k$ such that

$$(3.4) \qquad D^r \tilde{u}(a_i^r) \cdot (\xi_1, \xi_2, \ldots, \xi_r) = D^r u(a_i^r) \cdot (\xi_1, \xi_2, \ldots, \xi_r)$$

for all $(\xi_1, \xi_2, \ldots, \xi_r) \in X_i^r$, $1 \leqq i \leqq N_r$, $0 \leqq r \leqq s$, with the convention that this reduces to $\tilde{u}(a_i^0) = u(a_i^0)$ for $r=0$.

Example 5. Let K be a triangle in R^2 with vertices a_i, $1 \leqq i \leqq 3$, and let $a_{ij} = a_{ji}$ denote the mid-points of the edges joining a_i to a_j, for $i \neq j$. Then, given a function u, *the interpolation problem* consists in finding a polynomial $\tilde{u} \in P_5$ such that

$$(3.5) \qquad \begin{aligned} \tilde{u}(a_i) &= u(a_i), & 1 \leqq i \leqq 3, \\ D\tilde{u}(a_i) &= Du(a_i), & 1 \leqq i \leqq 3, \\ D^2 \tilde{u}(a_i) &= D^2 u(a_i), & 1 \leqq i \leqq 3, \\ \frac{\partial \tilde{u}}{\partial \nu}(a_{ij}) &= \frac{\partial u}{\partial \nu}(a_{ij}), & 1 \leqq i < j \leqq 3 \end{aligned}$$

where $\partial/\partial \nu$ denotes the (interior) normal derivative along the boundary of K. This type of approximation was considered by ZLÁMAL [35], who showed that the problem has a unique solution. In fact, the above set of conditions implies that $D\tilde{u}(a_{ij})$ is known, since $\tilde{u}$ is a polynomial of degree 5 in one variable along the edge joining a_i to a_j, with known values for itself and its first and second derivatives at the points a_i and a_j.

As will become apparent later on, the key observation is that we can replace conditions (3.5) by

$$(3.5') \qquad D\tilde{u}(a_{ij}) \cdot (a_k - a_{ij}) = Du(a_{ij}) \cdot (a_k - a_{ij}), \quad 1 \leqq i < j \leqq 3$$

where $k \neq i$ and $k \neq j$. The associated interpolation function will be called an *approximation of type V*.

The set

$$\Sigma^{\mathrm{V}}=\Sigma^0\cup\Sigma^1\cup\Sigma^2$$

where

$$\begin{aligned}&\Sigma^0=\{a_i\}_{i=1}^3,\\ &\Sigma^1=\{a_i\}_{i=1}^3\cup\{a_{ij}\}_{i=1}^3 \quad \text{with} \quad X_i^1=R^2, \quad X_{ij}^1=a_k-a_{ij},\\ &\Sigma^2=\{a_i\}_{i=1}^3 \quad \text{with} \quad X_i^2=(R^2)^2,\end{aligned}$$

is clearly 5-unisolvent. Let us notice that for X_i^1 (resp. X_i^2), we could as well take the sets $X_i^1=\{a_j-a_i,\ a_k-a_i\}$ (resp. $X_i^2=\{(a_j-a_i,\ a_j-a_i),\ (a_j-a_i,\ a_k-a_i),\ (a_k-a_i,\ a_k-a_i)\}$).

We shall now prove the analog of Theorem 1 for Hermite interpolation, but for the sake of simplicity we shall restrict ourselves to the case $s=1$. The extension to the case $s\geqq 2$ offers no difficulties, but it is long and cumbersome and for this reason will be omitted.

Theorem 3. *Let there be given a k-unisolvent set*

$$\Sigma=\Sigma^0\cup\Sigma^1, \quad \textit{with} \quad \Sigma^0=\{a_i^0\}_{i=1}^{N_0} \quad \textit{and} \quad \Sigma^1=\{a_i^1\}_{i=1}^{N_1},$$

such that with each point a_i^1, $1\leqq i\leqq N_1$, is associated a subset X_i of R^n, in each of which some vectors $\xi_{i1}^1, \xi_{i2}^1, \ldots, \xi_{1d_i}^1$ form a basis. Let $K\subset R^n$ be a Σ-admissible set, and let $u\in\mathscr{T}^{k+1}(K)$ be given. Then, for any point $x\in K$ and any integer m with $0\leqq m\leqq k$, one has

$$\begin{aligned}D^m\tilde{u}(x)=D^m u(x)&+\frac{1}{(k+1)!}\sum_{i=1}^{N_0}\{D^{k+1}u(\eta_i^0(x))\cdot(a_i^0-x)^{k+1}\}\,D^m p_i^0(x)\\ &+\frac{1}{k!}\sum_{i=1}^{N_0}\sum_{k=1}^{d_i}\{D^{k+1}u(\eta_{ik}^1(x))\cdot(\xi_{ik}^1,(a_i^1-x)^k)\}\,D^m p_{ik}^1(x)\end{aligned}\tag{3.6}$$

where the p_i^0's and p_{ik}^1's are the unique polynomials of degree $\leqq k$ such that

$$\begin{aligned}p_i^0(a_j^0)&=\delta_{ij}, && 1\leqq i,\ j\leqq N_0,\\ D p_i^0(a_j^1)\cdot(\xi_{jk}^1)&=0, && 1\leqq i\leqq N_0,\ 1\leqq j\leqq N_1,\ 1\leqq k\leqq d_j,\\ p_{ik}^1(a_j^0)&=0, && 1\leqq i\leqq N_1,\ 1\leqq j\leqq N_0,\ 1\leqq k\leqq d_i,\\ D p_{ik}^1(a_j^1)\cdot(\xi_{jl}^1)&=\delta_{ij}\delta_{kl}, && 1\leqq i,\ j\leqq N_1,\ 1\leqq k,\ l\leqq d_i,\end{aligned}\tag{3.7}$$

and, for some $0<\theta_i^0, \theta_{ik}^1<1$,

$$\eta_i^0(x)=\theta_i^0 x+(1-\theta_i^0)\,a_i^0, \qquad \eta_{ik}^1(x)=\theta_{ik}^1 x+(1-\theta_{ik}^1)\,a_i^1.\tag{3.8}$$

Proof. It is readily seen that the interpolating polynomial $\tilde{u}$ is given by

$$\tilde{u}=\sum_{i=1}^{N_0}u(a_i^0)\,p_i^0+\sum_{i=1}^{N_1}\sum_{k=1}^{d_i}\{Du(a_i^1)\cdot(\xi_{ik}^1)\}\,p_{ik}^1$$

so that, for any $0\leqq m\leqq k$,

$$D^m\tilde{u}=\sum_{i=1}^{N_0}u(a_i^0)\,D^m p_i^0+\sum_{i=1}^{N_1}\sum_{k=1}^{d_i}\{Du(a_i^1)\cdot(\xi_{ik}^1)\}\,D^m p_{ik}^1.$$

Let $x \in K$ be given. Since $u \in \mathcal{T}^{k+1}(K)$, we may write

$$u(a_i^0) = u(x) + Du(x) \cdot (a_i^0 - x) + \cdots + \frac{1}{k!} D^k u(x) \cdot (a_i^0 - x)^k + \frac{1}{(k+1)!} D^{k+1} u(\eta_i^0(x)) \cdot (a_i^0 - x)^{k+1}$$

for $1 \leqq i \leqq N_0$, and

$$Du(a_i^1) \cdot (\xi_{ik}^1) = Du(x) \cdot (\xi_{ik}^1) + D^2 u(x) \cdot (\xi_{ik}^1, a_i^1 - x) + \cdots + \frac{1}{(k-1)!} D^k u(x) \cdot (\xi_{ik}^1, (a_i^1 - x)^{k-1}) + \frac{1}{k!} D^{k+1} u(\eta_{ik}^1(x)) \cdot (\xi_{ik}^1, (a_i^1 - x)^k)$$

for $1 \leqq i \leqq N_1$, $1 \leqq k \leqq d_i$, where η_i^0 and η_{ik}^1 are of the form (3.8). Thus, we have

$$\begin{aligned} D^m \tilde{u}(x) = & \sum_{l=0}^{k} \frac{1}{l!} \sum_{i=1}^{N_0} \{D^l u(x) \cdot (a_i^0 - x)^l\} D^m p_i^0(x) \\ & + \sum_{l=1}^{k} \frac{1}{(l-1)!} \sum_{i=1}^{N_1} \sum_{k=1}^{d_i} \{D^l u(x) \cdot (\xi_{ik}^1, (a_i^1 - x)^{l-1})\} D^m p_{ik}^1(x) \\ & + \frac{1}{(k+1)!} \sum_{i=1}^{N_0} \{D^{k+1} u(\eta_i^0(x)) \cdot (a_i^0 - x)^{k+1}\} D^m p_i^0(x) \\ & + \frac{1}{k!} \sum_{i=1}^{N_1} \sum_{k=1}^{d_i} \{D^{k+1} u(\eta_{ik}^1(x)) \cdot (\xi_{ik}^1, (a_i^1 - x)^k)\} D^m p_i^1(x). \end{aligned}$$

Using the fact that $\tilde{u} = u$ whenever $u \in P_k$, it can now be shown, exactly as in the proof of Theorem 1, that

$$\begin{aligned} & \sum_{l=0}^{k} \frac{1}{l!} \sum_{i=1}^{N_0} \{D^l u(x) \cdot (a_i^0 - x)^l\} D^m p_i^0(x) \\ & \quad + \sum_{l=1}^{k} \frac{1}{(l-1)!} \sum_{i=1}^{N_1} \sum_{k=1}^{d_i} \{D^l u(x) \cdot (\xi_{ik}^1, (a_i^1 - x)^k)\} D^m p_i^1(x) = D^m u(x), \end{aligned}$$

which completes the proof.

As in §2, we must introduce the notion of equivalent k-unisolvent sets. Let

$$\Sigma = \{a_i^0\}_{i=1}^{N_0} \cup \{a_i^1\}_{i=1}^{N_1} \cup \cdots \cup \{a_i^s\}_{i=1}^{N_s}$$

and

$$\hat{\Sigma} = \{\hat{a}_i^0\}_{i=1}^{N_0} \cup \{\hat{a}_i^1\}_{i=1}^{N_1} \cup \cdots \cup \{\hat{a}_i^s\}_{i=1}^{N_s}$$

be two sets of points of R^n, a subset X_i^r (resp. $\hat{X}_i^r$) of $(R^n)^r$ being associated with each point a_i^r (resp. $\hat{a}_i^r$) for $1 \leqq i \leqq N_r$, $1 \leqq r \leqq s$. Then we say that the two sets are *equivalent* if and only if there exists an invertible element $B \in \mathcal{L}(R^n)$ and a vector $b \in R^n$ such that

(3.9) $$a_i^r = B \hat{a}_i^r + b, \quad 1 \leqq i \leqq N_r, \ 0 \leqq r \leqq s,$$

and

(3.10) $$X_i^r = \{(\xi_1, \xi_2, \ldots, \xi_r) \in (R^n)^r;\ \xi_i = B \hat{\xi}_i \text{ for all } (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_r) \in \hat{X}_i^r\},$$

$1 \leqq i \leqq N_r$, $1 \leqq r \leqq s$.

Example 6. It is clear that condition (3.10) is automatically satisfied at a point a_i^r if the derivative $D^r \tilde{u}(a_i^r)$ is specified there, in which case we may choose $X_i^r = \hat{X}_i^r = (R^n)^r$. The same holds if the elements $(\xi_1, \xi_2, \ldots, \xi_r)$ of X_i^r are such that each vector ξ_i, $1 \leqq i \leqq r$, is expressed as a linear combination of points of Σ. This is the case for approximation of type III (cf. Example 4) and for approximation of type V (cf. Example 5), once the interpolating conditions (3.5) are replaced by conditions (3.5′).

Lemma 3. *If two sets Σ and $\hat{\Sigma}$ are equivalent in the above sense, and if Σ is k-unisolvent, then Σ is k-unisolvent.*

Proof. We give the proof in the case $s=1$. Let $\hat{p}_i^0$ and $\hat{p}_{ik}^1$ be the polynomials associated with the k-unisolvent set $\hat{\Sigma}$ and which satisfy (3.7). It is readily seen that the polynomials p_i^0 and p_{ik}^1 defined by

$$(3.11)\qquad \begin{aligned} &x \in R^n \to p_i^0(x) = \hat{p}_i^0(B^{-1}(x-b)), \quad 1 \leqq i \leqq N_0, \\ &x \in R^n \to p_{ik}^1(x) = \hat{p}_{ik}^1(B^{-1}(x-b)), \quad 1 \leqq i \leqq N_1,\ 1 \leqq k \leqq d_i, \end{aligned}$$

satisfy (3.7), which shows that the set Σ is k-unisolvent.

We can now prove our main result.

Theorem 4. *Let there be given a k-unisolvent set of points of R^n*

$$\Sigma = \Sigma^0 \cup \Sigma^1 \cup \cdots \cup \Sigma^s, \quad \textit{with} \quad \Sigma^r = \{a_i^r\}_{i=1}^{N_r}, \quad 0 \leqq r \leqq s,$$

such that each point a_i^r, $1 \leqq i \leqq N_r$, $1 \leqq r \leqq s$, is associated with a subset X_i^r of $(R^n)^r$. Let $K \subset R^n$ be a Σ-admissible set, and let $u \in \mathscr{T}^{k+1}(K)$ be given, with

$$(3.12)\qquad M_{k+1} = \sup\{\|D^{k+1} u(x)\|;\ x \in K\} < +\infty.$$

If $\tilde{u}$ is the interpolating polyomial of u, i.e., $u \in P_k$ and

$$(3.13)\qquad \tilde{u}(a_i^0) = u(a_i^0), \quad 1 \leqq i \leqq N_0,$$

$$(3.14)\qquad D^r \tilde{u}(a_i^r) \cdot (\xi_1, \xi_2, \ldots, \xi_r) = D^r u(a_i^r) \cdot (\xi_1, \xi_2, \ldots, \xi_r)$$

for all $(\xi_1, \xi_2, \ldots, \xi_r) \in X_i^r$, $1 \leqq i \leqq N_r$, $1 \leqq r \leqq s$, we have for any integer m with $0 \leqq m \leqq k$,

$$(3.15)\qquad \sup\{\|D^m u(x) - D^m \tilde{u}(x)\|;\ x \in K\} \leqq C M_{k+1} \frac{h^{k+1}}{\rho^m},$$

for some constants

$$(3.16)\qquad C = C(n, k, m, \hat{\Sigma})$$

which are the same for all equivalent k-unisolvent sets and which can be computed once and for all in a k-unisolvent set $\hat{\Sigma}$ equivalent to Σ. The parameters h and ρ are defined as in (2.11)–(2.12).

Proof. For simplicity, we shall give the proof in the case $s=1$. From (3.6), we obtain for any point $x \in K$,

$$\|D^m \tilde{u}(x) - D^m u(x)\| \leqq \frac{1}{(k+1)!} \sum_{i=1}^{N_0} \left|\{D^{k+1} u(\eta_i^0(x)) \cdot (a_i^0 - x)^{k+1}\}\right| \, \|D^m p_i^0(x)\|$$

$$+ \frac{1}{k!} \sum_{i=1}^{N_1} \sum_{k=1}^{d_i} \left|\{D^{k+1} u(\eta_{ik}^1(x)) \cdot (\xi_{ik}^1, (a_i^1 - x)^k)\}\right| \, \|D^m p_{ik}^1(x)\|.$$

By definition of h and M_{k+1}, we have

$$\left|\{D^{k+1} u(\eta_i^0(x)) \cdot (a_i^0 - x)^{k+1}\}\right| \leqq M_{k+1} h^{k+1},$$
$$\left|\{D^{k+1} u(\eta_{ik}^1(x)) \cdot (\xi_{ik}^1, (a_i^1 - x)^k)\}\right| \leqq M_{k+1} \|\xi_{ik}^1\| h^k.$$

Now let $\hat{\Sigma}$ be a k-unisolvent set equivalent to Σ. From the definition of equivalence, it follows that the vectors ξ_{ik}^1 are related to the corresponding vectors $\hat{\xi}_{ik}^1$ by (cf. (3.10))

$$\xi_{ik}^1 = B \hat{\xi}_{ik}^1.$$

Thus

$$\|\xi_{ik}^1\| \leqq h \frac{\|\hat{\xi}_{ik}^1\|}{\hat{\rho}},$$

by Lemma 2, so that

$$\left|\{D^{k+1} u(\eta_{ik}^1(x)) \cdot (\xi_{ik}^1, (a_i^1 - x)^k)\}\right| \leqq M_{k+1} h^{k+1} \frac{\|\hat{\xi}_{ik}^1\|}{\hat{\rho}}.$$

Next, as in the proof of Theorem 2 we have

$$\sup\{\|D^m p_i^0(x)\|;\, x \in K\} \leqq \sup\{\|D^m \hat{p}_i(\hat{x})\|;\, \hat{x} \in \hat{K}\} \frac{\hat{h}^m}{\rho^m},$$

$$\sup\{\|D^m p_{ik}^1(x)\|;\, x \in K\} \leqq \sup\{\|D^m \hat{p}_{ik}^1(\hat{x})\|;\, \hat{x} \in \hat{K}\} \frac{\hat{h}^m}{\rho^m},$$

using (3.11). Hence, we obtain (3.15) with

$$C(n, k, m, \hat{\Sigma}) = \frac{\hat{h}^m}{(k+1)!} \sum_{i=1}^{N_0} \sup\{\|D^m \hat{p}_i^0(\hat{x})\|;\, \hat{x} \in \hat{K}\}$$

$$+ \frac{1}{k!} \frac{\hat{h}^m}{\hat{\rho}} \sum_{i=1}^{N_1} \sum_{k=1}^{d_i} \|\hat{\xi}_{ik}^1\| \sup\{\|D^m p_{ik}^1(\hat{x})\|;\, \hat{x} \in \hat{K}\}.$$

Example 7. If we consider approximations of type III, as defined in Example 4, we obtain

$$\sup\{\|D^m u(x) - D^m \tilde{u}(x)\|;\, x \in K\} \leqq C M_4 \frac{h^4}{\rho^m} \quad \text{for } m = 0, 1, 2 \text{ and } 3.$$

These bounds were announced in [13].

Similarly, we obtain for approximations of type V (cf. Example 5),

$$\sup\{\|D^m u(x) - D^m \tilde{u}(x)\|;\, x \in K\} \leqq C M_6 \frac{h^6}{\rho^m} \quad \text{for } 0 \leqq m \leqq 5.$$

These bounds were proved by ZLÁMAL [35, Theorem 3] for $0 \leqq m \leqq 4$. Again, we emphasize that central in our approach is the fact that the interpolating condition (3.5) must be written in a form which is the same for all equivalent 5-unisolvent sets (triangles in the present case). This requirement is not satisfied by (3.5) but is satisfied by (3.5′).

Finally, we want to mention a development of the preceeding approach which is a generalization of the Hermite interpolation problem in the following sense. We are still given a set Σ of the form (3.2), and, given a function u, it is again required to find an interpolate $\tilde{u}$ such that the interpolating conditions (3.4) are satisfied. However the condition that $\tilde{u}$ be a polynomial of degree $\leqq k$ will now be replaced by the condition that $\tilde{u}$ belong to some finite-dimensional vector space P such that

(*a*) the set Σ is *P-unisolvent*, i.e., there exists one and only one $\tilde{u} \in P$ which satisfies conditions (3.4) for all $1 \leqq i \leqq N_r$, and all $0 \leqq r \leqq s$,

(3.17) (*b*) $P_k \subset P$ for some fixed k, and thus

(3.18) (*c*) $\tilde{u} = u$ whenever $u \in P_k$.

In all the cases previously considered we had $P = P_k$, but the necessity for considering such more general problems will be made clearer in Example 8 below. Let us just mention that the exact analog of Theorem 4 holds without modification. It suffices to observe that Lemma 3 is now replaced by

Lemma 4. *If two sets Σ and $\hat{\Sigma}$ are equivalent in the sense of* (3.9)–(3.10) *and if Σ is P-unisolvent, then $\hat{\Sigma}$ is $\hat{P}$-unisolvent, if we let*

$$\hat{P} = \{\hat{p}: \hat{K} \to R;\ \hat{p}(\hat{x}) = p(B\hat{x} + b), \forall x \in K, \forall p \in P\}. \tag{3.19}$$

Remark 4. As a space of functions, $\hat{P}$ might in general be different from P (except if $P = P_k$ for some k), but this is irrelevant.

Example 8. The set Σ^{III} described in Example 4 is 3-unisolvent. We shall define an *approximation of type* III′ by specifying that the values of the interpolating polynomial $\tilde{u}$ at the points a_{ijk} will no longer be considered as free parameters (and which could thus be equated to the values $u(a_{ijk})$ in approximation of type III), but instead that these values $\tilde{u}(a_{ijk})$ will be replaced by the following linear combination of the values of the approximated function u, together with its first derivative, at the vertices a_i, a_j and a_k:

$$\begin{aligned} \tilde{u}(a_{ijk}) &= \tfrac{1}{3}\{u(a_i) + u(a_j) + u(a_k)\} \\ &\quad - \tfrac{1}{6}\{Du(a_i)\cdot(a_i - a_{ijk}) + Du(a_j)\cdot(a_j - a_{ijk}) + Du(a_k)\cdot(a_k - a_{ijk})\}. \end{aligned} \tag{3.20}$$

This is a generalization of the approximation considered in [36] (for an engineering source, see [5]). Although the order of accuracy obtained is one less than the corresponding order of accuracy for approximations of type III (as will be shown below), this method has the practical advantage that the computing time for the associated finite-element method is smaller than for approximations of type III, or even for approximations of type II (cf. [36]). To study this type of approximation, we prove

Lemma 5. *Let T be a triangle with vertices a_i, $1 \leqq i \leqq 3$, in R^n $(n \geqq 2)$, and let a be its barycentre. Then, if $u \in P_2$, one has*

$$u(a) = \tfrac{1}{3} \sum_{i=1}^{3} u(a_i) - \tfrac{1}{6} \sum_{i=1}^{3} Du(a_i) \cdot (a_i - a). \tag{3.21}$$

Proof. Denoting by $K \in \mathscr{L}_2(R^n, R)$ the constant second order derivative of u, one has

$$u(a_i) = u(a) + Du(a) \cdot (a_i - a) + \tfrac{1}{2} K \cdot (a_i - a)^2, \quad 1 \leqq i \leqq 3;$$

hence, since $\sum_{i=1}^{3} (a_i - a) = 0$ by definition of a, we obtain

$$\sum_{i=1}^{3} u(a_i) = 3u(a) + \tfrac{1}{2} \sum_{i=1}^{3} K \cdot (a_i - a)^2. \tag{3.22}$$

On the other hand, we have

$$Du(a_i) = Du(a) + K \cdot (a_i - a), \quad 1 \leqq i \leqq 3,$$

whence, again using the relation $\sum_{i=1}^{3} (a_i - a) = 0$, there results

$$\sum_{i=1}^{3} Du(a_i) \cdot (a_i - a) = \sum_{i=1}^{3} K \cdot (a_i - a)^2 \tag{3.23}$$

and (3.21) follows by eliminating $\sum_{i=1}^{3} K \cdot (a_i - a)^2$ between (3.22) and (3.23).

From the preceeding lemma it follows that $u = \tilde{u}$ whenever $u \in P_2$ (instead of P_3 for approximations of type III). We can then apply the theory developed in §§2 and 3: the Taylor series will then be written with a remainder involving the third derivative of u instead of the fourth; hence we obtain for approximations of type III′

$$\sup\{\|D^m u(x) - D^m \tilde{u}(x)\|;\, x \in K\} \leqq C M_3 \frac{h^3}{\rho^m} \quad \text{for } m = 0, 1,$$

which is the generalization of the bounds obtained in [36] for the case $n = 2$.

In approximations of type III′, the space P consists of the subspace of P_3 for which the homogeneous relation (3.20) holds for all $i \neq j$, $j \neq k$, $k \neq i$.

4. Interpolation in Sobolev Spaces

For any integer $m \geqq 1$ and any $1 \leqq p \leqq \infty$, we denote by $W^{m,p}(\Omega)$ the Sobolev space of (classes of) real-valued functions which, together with all their partial distributional derivatives of order $\leqq m$, belong to $L^p(\Omega)$. It will be convenient here to use the norms and semi-norms respectively given by*

$$\|u\|_{m,p,\Omega} = \left(\sum_{l=0}^{m} \|D^l u\|_{p,\Omega}^p \right)^{1/p}, \quad |u|_{m,p,\Omega} = \|D^m u\|_{p,\Omega}$$

* Here, for example, $\|D^l u\|_{p,\Omega}$ means $(\int_\Omega \|D^l u(x)\|^p \, dx)^{1/p}$, etc.

for $1 \leqq p < \infty$, and by

$$\|u\|_{m,\infty,\Omega} = \max\{\|D^l u\|_{\infty,\Omega};\ 0 \leqq l \leqq m\}, \qquad |u|_{m,\infty,\Omega} = \|D^m u\|_{\infty,\Omega}$$

for $p=\infty$. Those norms and semi-norms are readily seen to be equivalent to the usual ones.

We denote by $(W^{m,p}(\Omega))'$ the strong dual space of $W^{m,p}(\Omega)$, by (f, u) the pairing between an element $f \in (W^{m,p}(\Omega))'$ and an element $u \in W^{m,p}(\Omega)$, and by

$$\|f\|^*_{m,p,\Omega} = \sup\left\{\frac{|(f,v)|}{\|v\|_{m,p,\Omega}};\ v \in W^{m,p}(\Omega),\ v \neq 0\right\}$$

the dual norm.

We begin by proving a preliminary result, which is essentially Theorem 2 of [8] (see also [7]). We give the proof for the sake of completeness.

Lemma 6. *Let Ω be a bounded open subset of R^n with a continuous boundary (in the sense of [21]), let p be given with $1 \leqq p \leqq \infty$, let $k \geqq 0$ be a fixed integer, and let $f \in (W^{k+1,p}(\Omega))'$ be such that*

$$(f, u) = 0 \quad \textit{for all } u \in P_k. \tag{4.1}$$

Then there exists a constant

$$C = C(n, k, p, \Omega) \tag{4.2}$$

such that

$$|(f,u)| \leqq C \|f\|^*_{k+1,p,\Omega} |u|_{k+1,p,\Omega} \quad \textit{for all } u \in W^{k+1,p}(\Omega). \tag{4.3}$$

Proof. From a result given in [21, Theorem 7.2, page 112] for the case $1 \leqq p < \infty$ and which can be easily extended to the case $p=\infty$, it follows that the semi-norm $|\cdot|_{k+1,p,\Omega}$ is a norm on the quotient space $W^{k+1,p}(\Omega)/P_k$ equivalent to the quotient norm

$$\|[u]\|_{W^{k+1,p}(\Omega)/P_k} = \inf\{\|u+v\|_{k+1,p,\Omega};\ v \in P_k\},$$

where $[u]$ denotes the equivalence class of any of its elements u. Thus there exists a constant $C = C(n, k, p, \Omega)$ such that

$$\|[u]\|_{W^{k+1,p}(\Omega)/P_k} \leqq C |u|_{k+1,p,\Omega} \quad \text{for all } u \in W^{k+1,p}(\Omega).$$

Given an $f \in (W^{k+1,p}(\Omega))'$ which satisfies (4.1), we have

$$(f,u) = (f, u+v) \quad \text{for all } u \in W^{k+1,p}(\Omega) \text{ and all } v \in P_k,$$

so that

$$\begin{aligned} |(f,u)| &\leqq \|f\|^*_{k+1,p,\Omega} \inf\{\|u+v\|_{k+1,p,\Omega};\ v \in P_k\} \\ &\leqq C \|f\|^*_{k+1,p,\Omega} |u|_{k+1,p,\Omega}, \end{aligned}$$

which completes the proof.

We need another technical result:

Lemma 7. *Let Ω be a bounded open subset of R^n with a continuous boundary, let p be given with $1 \leqq p \leqq \infty$, let $k \geqq 0$ be a fixed integer, and let m be an integer* *with* $0 \leq m \leq k+1$. *Let* $\Pi \in \mathscr{L}(W^{k+1,p}(\Omega);\ W^{m,p}(\Omega))$ *be such that*

$$\Pi u = u \quad \text{for all } u \in P_k. \tag{4.4}$$

Then there exists a constant

$$C = C(n, k, p, \Omega) \tag{4.5}$$

(*the same as in Lemma* 5) *such that*

$$\|u - \Pi u\|_{m,p,\Omega} \leq C \|I - \Pi\|_{\mathscr{L}(W^{k+1,p}(\Omega);\ W^{m,p}(\Omega))} |u|_{k+1,p,\Omega} \tag{4.6}$$

for all $u \in W^{k+1,p}(\Omega)$.

Proof. Given $g \in (W^{m,p}(\Omega))'$, the linear form defined by

$$f: u \to (f, u) = (g, u - \Pi u)$$

is continuous over $W^{k+1,p}(\Omega)$ (which is contained in $W^{m,p}(\Omega)$) since

$$\begin{aligned}
\|f\|^*_{k+1,p,\Omega} &= \sup\left\{\frac{|(g, u-\Pi u)|}{\|u\|_{k+1,p,\Omega}};\ u \in W^{k+1,p}(\Omega),\ u \neq 0\right\} \\
&\leq \|g\|^*_{m,p,\Omega} \sup\left\{\frac{\|u-\Pi u\|_{m,p,\Omega}}{\|u\|_{k+1,p,\Omega}};\ u \in W^{k+1,p}(\Omega),\ u \neq 0\right\} \\
&\leq \|g\|^*_{m,p,\Omega} \|I - \Pi\|_{\mathscr{L}(W^{k+1,p}(\Omega);\ W^{m,p}(\Omega))}.
\end{aligned}$$

Since $(f, u) = 0$ for any $u \in P_k$, the conclusion follows by applying Lemma 5 combined with the fact that [31, Theorem 4.3-B, page 186]

$$\|u - \Pi u\|_{m,p,\Omega} = \sup\left\{\frac{|(g, u-\Pi u)|}{\|g\|^*_{m,p,\Omega}};\ g \in (W^{m,p}(\Omega))',\ g \neq 0\right\}.$$

This completes the proof.

The proof of the fundamental error estimates (4.13) of Theorem 5 will depend upon the notion of equivalent domains, as in the preceeding sections. Let Ω (resp. $\hat{\Omega}$) be a bounded open subset of R^n. Then we say that Ω and $\hat{\Omega}$ are *equivalent* if and only if there exists an invertible element $B \in \mathscr{L}(R^n)$ and a vector $b \in R^n$ such that

$$\Omega = \{x \in R^n;\ x = B\hat{x} + b \quad \text{for each } \hat{x} \in \hat{\Omega}\}. \tag{4.7}$$

With each function u defined over Ω, we associate a function $\hat{u}$ defined over $\hat{\Omega}$ by letting

$$\hat{u}(\hat{x}) = u(B\hat{x} + b) \quad \text{for each } \hat{x} \in \hat{\Omega}. \tag{4.8}$$

As will become clear from the proof of Theorem 5, the correspondence $u \to \hat{u}$ is an isomorphism between $W^{m,p}(\Omega)$ and $W^{m,p}(\hat{\Omega})$ for each m and p.

Likewise, if Π is an element of $\mathscr{L}(W^{k+1,p}(\Omega);\ W^{m,p}(\Omega))$, we associate with Π an element $\hat{\Pi} \in \mathscr{L}(W^{k+1,p}(\hat{\Omega});\ W^{m,p}(\hat{\Omega}))$ by letting

$$\hat{\Pi}\hat{u} = \widehat{\Pi u} \quad \text{for each } u \in W^{k+1,p}(\Omega). \tag{4.9}$$

It can easily be seen that P_k is left invariant by Π if and only if it is left invariant by $\hat{\Pi}$.

Finally, with each bounded open subset Ω of R^n, we associate as in §2 the two geometrical parameters:

$$h = \text{diameter of } \Omega, \tag{4.10}$$

$$\rho = \sup\{\text{diameter of the spheres contained in } \bar{\Omega}\}; \tag{4.11}$$

we leave it to the reader to verify that Lemma 2 holds without modification.

Theorem 5. *Let Ω be a bounded open subset of R^n with a continuous boundary, let p be given with $1 \leqq p \leqq \infty$, let $k \geqq 0$ be a fixed integer, and let m be an integer with $0 \leqq m \leqq k+1$. Let $\Pi \in \mathscr{L}(W^{k+1,p}(\Omega);\, W^{m,p}(\Omega))$ be such that*

$$\Pi u = u \quad \text{for all } u \in P_k. \tag{4.12}$$

Then for any $u \in W^{k+1,p}(\Omega)$ (and for h small enough if $p<\infty$; cf. (4.18)),

$$\|u - \Pi u\|_{m,p,\Omega} \leqq \mathscr{C}\, |u|_{k+1,p,\Omega} \frac{h^{k+1}}{\rho^m}, \tag{4.13}$$

for some constants

$$\mathscr{C} = \mathscr{C}(n, k, p, \hat{\Omega}, \hat{\Pi}) \tag{4.14}$$

(*where $\hat{\Pi}$ is defined as in* (4.9)) *which are the same for all equivalent domains Ω and which can be computed once and for all in a domain $\hat{\Omega}$ equivalent to Ω.*

Proof. If $D^l u$ is any distributional derivative of u, then $D^l \hat{u}$ is defined for each $\hat{x} \in \hat{\Omega}$ by

$$D^l \hat{u}(\hat{x}) \cdot (\xi_1, \xi_2, \ldots, \xi_l) = D^l u(B\hat{x}+b) \cdot (B\xi_1, B\xi_2, \ldots, B\xi_l),$$

so that for $1 \leqq p < \infty$

$$\begin{aligned} \int_{\hat{\Omega}} \|D^l \hat{u}(\hat{x})\|^p \, d\hat{x} &\leqq \|B\|^{lp} \int_{\hat{\Omega}} \|D^l u(B\hat{x}+b)\|^p \, d\hat{x} \\ &= \|B\|^{lp} |\det(B)|^{-1} \int_{\Omega} \|D^l u(x)\|^p \, dx, \end{aligned}$$

that is,

$$|\hat{u}|_{l,p,\hat{\Omega}} \leqq \|B\|^l |\det(B)|^{-1/p} |u|_{l,p,\Omega}, \tag{4.15}$$

for all $1 \leqq p \leqq \infty$, if we understand that $1/p = 0$ if $p = \infty$. Similarly, we obtain

$$|u|_{l,p,\Omega} \leqq \|B^{-1}\|^l |\det(B)|^{1/p} |\hat{u}|_{l,p,\hat{\Omega}}. \tag{4.16}$$

It follows from (4.16) that

$$\|u\|_{m,p,\Omega}^p \leqq |\det(B)| \sum_{l=0}^{m} \|B^{-1}\|^{lp} |\hat{u}|_{l,p,\hat{\Omega}}^p \leqq |\det(B)|\, \|B^{-1}\|^{mp} \|\hat{u}\|_{m,p,\hat{\Omega}}^p \tag{4.17}$$

under the assumption that $1 \leqq \|B^{-1}\|$; this in turn is certainly satisfied for

$$h \leqq \hat{\rho}, \tag{4.18}$$

i.e., for h small enough (once the domain $\hat{\Omega}$ is chosen) since we then have from Lemma 2

$$1 \leqq \frac{\hat{\rho}}{h} \leqq \frac{1}{\|B\|} \leqq \|B^{-1}\|.$$

Thus for h satisfying (4.18), we obtain from (4.17)

$$\|u - \Pi u\|_{m,p,\Omega} \leqq |\det(B)|^{1/p} \|B^{-1}\|^m \|\hat{u} - \hat{\Pi}\hat{u}\|_{m,p,\hat{\Omega}}. \tag{4.19}$$

From Lemma 6, we have

$$\|\hat{u} - \hat{\Pi}\hat{u}\|_{m,p,\hat{\Omega}} \leqq C(n,k,p,\hat{\Omega}) \|I - \hat{\Pi}\|_{\mathscr{L}(W^{k+1,p}(\hat{\Omega});\, W^{m,p}(\hat{\Omega}))} |\hat{u}|_{k+1,p,\hat{\Omega}}. \tag{4.20}$$

Therefore, by using the inequality (4.15) for $l = k+1$ and the estimates (2.13) of Lemma 2, we finally obtain the inequality (4.13) with

$$\mathscr{C} = \mathscr{C}(n,k,p,\hat{\Omega},\hat{\Pi}) = \frac{\hat{h}^m}{\hat{\rho}^{k+1}} C(n,k,p,\hat{\Omega}) \|I - \hat{\Pi}\|_{\mathscr{L}(W^{k+1,p}(\hat{\Omega});\, W^{m,p}(\hat{\Omega}))} \tag{4.21}$$

where $C(n,k,p,\hat{\Omega})$ is the constant (4.2) of Lemma 5. This completes the proof.

We shall now apply the above theorem to the generalized Hermite interpolation defined at the end of §3. In so doing, we generalize similar results obtained in [10] for the case $p=2$, $n=2$, and obtained in [34] for a restricted class of polynomial approximations. Likewise, we generalize some of the results of [6], [8] and [9] which were obtained for spline and Hermite interpolation over "rectangles" of R^n.

Theorem 6. *Let there be given a P-unisolvent set Σ of the form* (3.2). *If $u \in C^s(K)$ (we recall that K is the closed convex hull of Σ), we let $\tilde{u}$ denote (as before) its Hermite interpolation, in the sense that $\tilde{u}$ is the unique element of P such that*

$$D^r \tilde{u}(a_i^r) \cdot (\xi_1, \xi_2, \ldots, \xi_r) = D^r u(a_i^r) \cdot (\xi_1, \xi_2, \ldots, \xi_r) \tag{4.22}$$

for all $(\xi_1, \xi_2, \ldots, \xi_r) \in X_i^r$, $1 \leqq i \leqq N_r$, $0 \leqq r \leqq s$. We also assume that

$$P_k \subset P \subset W^{k+1,p}(K) \quad \textit{for some fixed integer } k \geqq 0 \tag{4.23}$$

so that

$$\tilde{u} = u \quad \textit{whenever} \quad u \in P_k. \tag{4.24}$$

Then if $u \in W^{k+1,p}(K)$, with

$$s < k+1 - \frac{n}{p} \quad \textit{if } p < \infty \textit{ or } s \leqq k \textit{ if } p = \infty, \tag{4.25}$$

we have (for h small enough if $p < \infty$)

$$\|u - \tilde{u}\|_{m,p,K} \leqq \mathscr{C} |u|_{k+1,p,K} \frac{h^{k+1}}{\rho^m}, \quad \textit{for any } 0 \leqq m \leqq k+1, \tag{4.26}$$

for some constants

$$\mathscr{C} = \mathscr{C}(n,k,m,\hat{\Sigma},\hat{P}) \tag{4.27}$$

which are the same for all equivalent sets and which can be computed once and for all in a $\hat{P}$-unisolvent set $\hat{\Sigma}$ equivalent to Σ. The parameters h and ρ are defined as n (2.11)–(2.12).

Proof. The condition (4.25) guarantees that $W^{k+1,p}(K) \subset C^s(K)$, so that the Hermite interpolation $\tilde{u}$ is well-defined. For any $u \in W^{k+1,p}(K)$, let $\Pi u = \tilde{u}$. This defines an element which is in $W^{m,p}(K)$ for all $0 \leqq m \leqq k+1$, and moreover $\Pi \in \mathscr{L}(W^{k+1,p}(K);\, W^{m,p}(K))$ since the injection $W^{k+1,p}(K) \to C^s(K)$ is continuous (cf. [21, §3.5]) and P is finite-dimensional. Thus the inequality (4.26) is an immediate consequence of (4.13), the constant $\mathscr{C}$ of (4.27) being given by (4.21) with $\hat{\Pi}$ defined by $\hat{\Pi}\hat{u} = \tilde{\hat{u}}$.

We leave it to reader to apply the above theorem to the various types of approximations (I, II, III, III′ and V) considered in Examples 1 to 8.

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