SIAM J. NUMER. ANAL.
Vol. 10, No. 2, April 1973

ON PERTURBATIONS IN SYSTEMS OF LINEAR INEQUALITIES*

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Abstract. We consider what happens to sets defined by systems of linear inequalities when elements of the system are perturbed. If $S = \{x|Gx \leqq g, Dx = d\}$, and if S' and S'' are defined in the obvious manner by perturbed matrices $G', G'', g', g'', D', D'', d', d''$, we show that, under certain hypotheses, to each element x' in S' there corresponds x'' in S'' with $\|x' - x''\| \leqq c\{\|G' - G''\| + \|g' - g''\| + \|D' - D''\| + \|d' - d''\|\}(1 + \|x'\|)$ for some constant c depending on S.

1. Preliminaries. Consider a set S defined by a system of linear equalities and inequalities $S = \{x|Gx \leqq g, Dx = d\}$, where G is an $m' \times n'$ matrix, D is $r' \times n'$, g is $m' \times 1$, and d is $r' \times 1$; all numbers are assumed to be real. We shall use the symbols $\leqq, \leq, <$, and their analogues $\geqq, \geq, >$, in the sense of [Mangasarian (1969)]. That is, $b \leqq 0$ if and only if each component of b is nonpositive, $b < 0$ if and only if each component of b is negative, and $b \leq 0$ if and only if $b \leqq 0$ but $b \neq 0$. We wish to consider in what way the set S is perturbed when the matrices G, g, D and d are perturbed to G', g', D' and d', respectively, of compatible dimensions, yielding a new set S'. The earliest result of this type known to the author is that of [Hoffman (1952)]; since we shall use this result often, we state it precisely here without proof. For any vector x, we denote by x^+ that vector, of the same dimension, whose ith component equals the maximum of zero and the ith component of x. For convenience, throughout this paper we use a single norm $\|\cdot\|$ always assumed to be the ℓ_∞-norm: $\|x\| = \max_i\{|x_i|\}$.

HOFFMAN'S THEOREM. *Under the above hypotheses, let S be nonempty. Then there exist constants c_0 and c, depending only on G and D, such that, if s' is any point in S', then there exists s in S satisfying*

$$\begin{aligned}\|s - s'\| &\leqq c_0\{\|(Gs' - g)^+\| + \|Ds' - d\|\}\\ &\leqq c\{\|[(G - G')s' - (g - g')]^+\| + \|(D - D')s' - (d - d')\|\}.\end{aligned}$$

Unfortunately this theorem does not go in the opposite direction; that is, given s in S, we do not know that there is an s' in S' whose distance away can be measured via the *same* constants c and c_0. The problem here is that we do not know how c and c_0 depend on G and D. For many applications it is precisely this kind of result that is needed, saying that S and S' are close to each other essentially in the Hausdorff metric; this is important, for example, in estimating by how much the minimum values of a function over S and S' can differ.

Stronger results of the type we seek were developed in [Robinson (1972b)]; as in Hoffman's results, a constant c is found, depending only on G and D, so that

* Received by the editors July 28, 1972, and in revised form September 26, 1972.

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both forms of the fundamental estimate are true with the same constant c. Unfortunately, this requires a very strong hypothesis,[1] namely that, for all vectors g and d, there exist at least one x satisfying $Gx \leqq g$, $Dx = d$. This makes the result inapplicable for many simple polyhedral sets S, such as bounded polyhedra. The most general results in [Robinson (1972b)], specialized to this case, are actually somewhat stronger than that described above, but are still much more restrictive than the result we shall present here; they do not, for example, allow for arbitrary perturbations in G when S is bounded and super-consistent. We are not able to obtain results as general as we would like, however. In great generality we can estimate the distances between a perturbed set S' and a fixed set S; in order to estimate sharply distances between two perturbations S' and S'' of the same set S we must, however, significantly restrict ourselves. This is a drawback we would like to see eliminated.[1] The interested reader can find more abstract results on perturbations of convex sets in [Joly (1970)], [Laurent (1972)]; estimates of the magnitude of the perturbations are not included, however.

Throughout this paper we shall use c as a generic constant, whose value *is seldom the same* from place to place; it will always depend on fixed objects, such as G, g, D, d, et cetera. To avoid confusion with the notation x^+ previously introduced, we shall denote the Moore–Penrose [Moore (1919)], [Penrose (1955)] pseudo-inverse of a matrix A by the symbol $A^{\#}$. For brevity, we shall no longer inform the reader of the precise dimensions of matrices and vectors appearing in this paper; the dimensions are always such that all indicated multiplications, order relations, et cetera, are well-defined.

In the next section we shall discuss the structure of the set S (whether it has a nonempty interior, et cetera), since this has great impact on the behavior of S under perturbations. In § 3 we shall consider some examples and discuss the need for restricting the form of the perturbations we can allow; in § 4 we give our main results.

2. On the structure of S. In § 4 we shall discover that, if S is super-consistent, i.e., if D is vacuous and there exists x with $Gx < g$, then S is very well-behaved under perturbations; if S is not super-consistent, then S can behave wildly unless the perturbations take special form. For S to be super-consistent clearly D needs to be vacuous (or identically zero) so that we have no explicit equality constraints; this is not sufficient, however, since $Gx \leqq g$ can contain implicit equalities. For example, one might have $G = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. We need to study these hidden equalities separately. By interchanging rows in G if necessary, we can partition G into the form $G = \begin{pmatrix} A \\ B \end{pmatrix}$, and g into $\begin{pmatrix} a \\ b \end{pmatrix}$, where $Bx = b$ for all x in S, and, for any row, say the ith, A_i^T of A, there exists an x_i in S with $A_i^T x_i$ less than the ith component of a. Any of A, B and D may, of course, be vacuous. The following proposition is then obvious [Dantzig, et al. (1967)].

[1] The referee has kindly observed that one can probably apply the results of [Robinson (1972b)] to a carefully chosen subset of active constraints, thereby extending those results and yielding a different approach to the results of this present paper; it appears that this approach could also be used to extend our results on the distance between two perturbations S' and S''.

PROPOSITION 2.1. *Under the above hypotheses, either A is vacuous or there exists $\hat{x}$ in S with $A\hat{x} < a$.*

Thus we can now describe the set S precisely as follows.

(2.2) $S = \{x \mid Ax \leqq a, Bx \leqq b, Dx = d\}$ is not empty; for all x in S we have $Bx = b$, and either A is vacuous or there exists $\hat{x}$ in S with $A\hat{x} \leqq a - h$, where $h > 0$.

Our analysis of the behavior of S under perturbations will depend on A, a, B, b, D, d, $\hat{x}$ and h.

The fact that x in S implies $Bx = b$ says that B and D are strongly related. First note that if $By \leqq 0$ and $Dy = 0$, then for small positive r we have $\hat{x} + ry$ (or $x' + ry$ for some x' in S if A is vacuous) also in S, and hence, $b = B(\hat{x} + ry) = B\hat{x} + rBy = b + rBy$; thus $By \leqq 0$ and $Dy = 0$ implies $By = 0$. That is to say, the system $By \leq 0$ and $Dy = 0$ has no solution. It follows from Tucker's theorem of the alternative [Tucker (1956)], [Mangasarian (1969)] that there exist vectors $p > 0$ and q such that $p^TB + q^TD = 0$. We shall need this relationship in Lemma 3.5.

We are now ready to consider some examples and to see what types of perturbations we can hope to analyze.

3. Introduction to perturbation results. Suppose for the moment that A and B are vacuous so that we have only equality constraints $Dx = d$ to be perturbed to $D'x = d'$. If we consider the example $D = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, d = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = d', D' = \begin{pmatrix} 0 & 1 \\ \varepsilon & 2 \end{pmatrix}$ for small nonzero ε, we see that the perturbed set S' is radically different and quite distant from S for all $\varepsilon \neq 0$. The problem here is that the ranks of D and D' are unequal; one possibility is to assume that D and D' have equal rank so that this phenomenon cannot occur.

Having seen the type of perturbation we can hope to allow in the equality constraints, we now turn to the question of the inequality constraints. First, suppose that A and D are vacuous, so that we have entirely hidden equality constraints $Bx \leqq b$ which are to be perturbed to $B'x \leqq b'$. As an example, consider $B = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, B' = \begin{pmatrix} -\varepsilon & 1 \\ 0 & -1 \end{pmatrix}, b' = \begin{pmatrix} 1 - \varepsilon \\ -1 \end{pmatrix}$ for small positive ε; again we see that S and S' are radically different and distant for all $\varepsilon > 0$. Again, it turns out that the change in the rank of B to B' is the culprit; one possibility then is to assume that $\operatorname{rank}(B) = \operatorname{rank}(B')$.

Unfortunately, we still have not eliminated all the difficulties; if D and B are suitably related, we still can get into trouble even though we keep $\operatorname{rank}(D) = \operatorname{rank}(D')$ and $\operatorname{rank}(B) = \operatorname{rank}(B')$. For example, let $B = (0 \quad 1)$, $b = (1)$, $D = (0 \quad 1)$, $d = (1)$, $B' = (-\varepsilon \quad 1)$, $b' = (1 - \varepsilon)$, $D' = (0 \quad 1)$, $d' = (1)$; again S and S' are radically different and distant for all $\varepsilon \neq 0$. It turns out that this can happen only if the rank of the matrix

$$N = \begin{pmatrix} B \\ D \end{pmatrix} \tag{3.1}$$

is changed by the perturbation. In the other two examples considered either B or D was vacuous, so that this latest possible assumption subsumes the two suggested earlier.

We shall henceforth assume

$$\text{rank}\,(N) = \text{rank}\,(N') = r \quad \text{for some integer } r, \quad \text{where } N' = \begin{pmatrix} B' \\ D' \end{pmatrix}. \tag{3.2}$$

We now want to discover what this assumption means when N' is near N. To give us a convenient notation for use here and later, let us define

$$\varepsilon' = \max\{\|A' - A\|, \|B' - B\|, \|D' - D\|, \|(a - a')^+\|, \|b' - b\|, \|d' - d\|\}. \tag{3.3}$$

PROPOSITION 3.4. *Under the hypothesis of equation* (3.2), *for ε' small enough we can write $PN = \begin{pmatrix} N_r \\ UN_r \end{pmatrix}$ and $PN' = \begin{pmatrix} N'_r \\ U'N'_r \end{pmatrix}$, where P is a permutation matrix, N_r and N'_r have full row rank r, $N'_r = N_r(I + H')$ with $\|H'\| \leqq c\varepsilon'$, and $\|U - U'\| \leqq c\varepsilon'$.*

Proof. There exists a permutation matrix P such that $PN = \begin{pmatrix} N_r \\ L \end{pmatrix}$ where N_r has full row rank r, that is $x^T N_r = 0$ only if $x = 0$, and each row of L can be written as a linear combination of the rows of N_r, that is, $L = UN_r$. In fact, we may take $U = LN_r^{\#}$. It follows easily that for small ε' the same r rows of N' are independent, so that $PN' = \begin{pmatrix} N'_r \\ L' \end{pmatrix}$ with $L' = U'N'_r$ and $U' = L'N_r'^{\#}$. Since N_r and N'_r both have full row rank r, since $\|N_r - N'_r\| \leqq \varepsilon'$, and since $\|L - L'\| \leqq \varepsilon'$, by the continuity of the pseudo-inverse operation in this case in which $N_r^{\#} = N_r^T(N_r N_r^T)^{-1}$ we conclude that $\|U' - U\| \leqq c\varepsilon'$. Also, we can write $N'_r = N_r(I + H')$ with $\|H'\| \leqq c\varepsilon'$ by similarly using the pseudo-inverse. This completes the proof.

Before we proceed to the main results of the next section, we derive a useful result describing a strong relationship between B' and D'.

LEMMA 3.5. *Suppose that the general hypotheses of equations* (2.2), (3.1), *and* (3.2) *are valid. Then there exists a constant c_1 independent of B' and D' for small ε' such that $\|B'x\| \leqq c_1\{\|[B'x]^+\| + \|D'x\|\}$ for all x.*

Proof. From the remarks before Proposition 2.3 we know that we have vectors $p > 0$ and q such that $p^T B + q^T D = 0$; we rewrite this as

$$0 = (p^T \quad q^T)N = (p^T \quad q^T)P^{-1}PN = (p^T \quad q^T)P^{-1}\begin{pmatrix} N_r \\ UN_r \end{pmatrix}$$

$$= (p^T \quad q^T)P^{-1}\begin{pmatrix} I \\ U \end{pmatrix}N_r = (u^T \quad v^T)\begin{pmatrix} I \\ U \end{pmatrix}N_r,$$

where $(u^T \quad v^T) = (p^T \quad q^T)P^{-1}$. Since N_r has full rank, this implies

$$(u^T \quad v^T)\begin{pmatrix} I \\ U \end{pmatrix} = 0,$$

that is, $u^T + v^T U = 0$. Define $\bar{v} = v$ and $\bar{u}^T = -\bar{v}^T U'$; clearly $\|\bar{u} - u\| \leqq c\varepsilon'$ by Proposition 3.4. We have $0 = (\bar{u}^T \quad \bar{v}^T)\begin{pmatrix} I \\ U' \end{pmatrix}$, so that

$$0 = (\bar{u}^T \quad \bar{v}^T)\begin{pmatrix} I \\ U' \end{pmatrix} N'_r = (\bar{u}^T \quad \bar{v}^T) P N' = (\bar{p}^T \quad \bar{q}^T) N',$$

where $(\bar{p}^T \quad \bar{q}^T) = (\bar{u}^T \quad \bar{v}^T)P$. Clearly $\|(\bar{p}^T \quad \bar{q}^T) - (p^T \quad q^T)\| \leqq c\varepsilon'$ and $\bar{p}^T B' + \bar{q}^T D' = 0$; since $p > 0$ we have $\bar{p} > 0$ for small ε'. Let $B_i'^T$ and $D_i'^T$ denote the rows of B' and D'. For a given x, let $B_M'^T x$ maximize and $B_m'^T x$ minimize $B_i'^T x$ over i; then either $\|B'x\| = B_M'^T x = \|(B'x)^+\|$ or $\|B'x\| = -B_m'^T x$. In the former case, the desired inequality of the lemma follows trivially, so we suppose instead that $\|B'x\|$ equals $-B_m'^T x$ which in turn equals

$$\sum_{i \neq m} \frac{\bar{p}_i}{\bar{p}_m} B_i'^T x + \sum_j \frac{\bar{q}_j}{\bar{p}_m} D_j'^T x \leqq \sum_{i \neq m} \frac{\bar{p}_i}{\bar{p}_m} (B_M'^T x)^+ + \|D'x\| \sum_j \frac{|\bar{q}_j|}{\bar{p}_m}$$

$$\leqq \|(B'x)^+\| \sum_{i \neq m} \frac{\bar{p}_i}{\bar{p}_m} + \|D'x\| \sum_j \frac{|\bar{q}_j|}{\bar{p}_m}.$$

Since $(\bar{p} \quad \bar{q})$ is near $(p \quad q)$, the sums in the above inequality can be replaced by uniform constants defined via p and q for small ε'.

4. The main results. To prove our main results, we shall construct solutions to a sequence of perturbed systems of equalities and inequalities for which only the right-hand sides are perturbed; before we can do this, we need a simple result on the solvability of such systems.

LEMMA 4.1. *Suppose that the hypotheses of equations* (2.2), (3.1), *and* (3.2) *are valid; define* $\hat{f}_r = \left[P \begin{pmatrix} b \\ d \end{pmatrix} \right]_r$, *where P is as in Proposition* 3.4. *Then there exists a constant c_2 such that the system $Ax \leqq a''$ and $N_r x = f_r$ is simultaneously solvable whenever $\|f_r - \hat{f}_r\| \leqq \delta$, $\|(a - a'')^+\| \leqq \delta$, and $\delta \leqq c_2 \min_i h_i$, where the h_i are the components of h in equation* (2.2).

Proof. If A is vacuous, there is of course nothing to prove; if A is not vacuous, then consider $\hat{x}$ from equation (2.2), so that $A\hat{x} \leqq a - h$ and $N_r\hat{x} = \hat{f}_r$. Now define $\bar{x} = \hat{x} + N_r^{\#}(f_r - \hat{f}_r)$; it follows that $\|\bar{x} - \hat{x}\| \leqq \|N_r^{\#}\|\delta$. Since N_r has full row rank, it maps onto its column space, and hence it follows easily that $N_r\bar{x} = f_r$. Thus $A\bar{x} - a'' = A\hat{x} - a + A(\bar{x} - \hat{x}) + a - a'' \leqq -h + A(\bar{x} - \hat{x}) + (a - a'')^+$; it is clear then that for some c_2 and for $\delta \leqq c_2 \min_i h_i$ this vector on the right will be negative and, in fact, less than or equal to $-\frac{1}{2}h$. Thus $\bar{x}$ solves $A\bar{x} \leqq a'$ and $N_r\bar{x} = f_r$ as required. This completes the proof.

We are finally prepared for our main results.

THEOREM 4.2. *Let nonempty sets S and S' be defined from $\{A, B, D, a, b, d\}$ and $\{A', B', D', a', b', d'\}$, respectively, and define ε' by equation* (3.3). *Suppose that the hypotheses in equations* (2.2), (3.1), *and* (3.2) *hold. Then there exist positive constants c and ε_0 depending on S such that to every s in S satisfying $\varepsilon'(1 + \|s\|) \leqq \varepsilon_0$ there corresponds an s' in S' satisfying $\|s - s'\| \leqq c\varepsilon'(1 + \|s\|)$.*

Proof. We have s given in S, so s solves $As \leqq a$, $Bs = b$, $Ds = d$. We also know that S' is not empty, so we choose $\bar{s}$ in S' thus satisfying $A'\bar{s} \leqq a'$, $B'\bar{s} \leqq b'$, $D'\bar{s} = d'$. Defining $\bar{b} = B'\bar{s} \leqq b'$, we have, by using Lemma 3.5, that

$$\begin{aligned}\|\bar{b} - B's\| = \|B'(\bar{s} - s)\| &\leqq c_1\|[B'(\bar{s} - s)]^+\| + c_1\|D'(\bar{s} - s)\| \\ &\leqq c_1\|[\bar{b} - Bs + Bs - B's]^+\| + c_1\|d' - Ds + Ds - D's\| \\ &\leqq c_1\|[\bar{b} - b]^+\| + c_1\|s\|\varepsilon' + c_1\varepsilon' + c_1\varepsilon'\|s\| \\ &\leqq c_1\|[b' - b]^+\| + c_1\varepsilon' + 2c_1\varepsilon'\|s\| \leqq 2c_1\varepsilon'(1 + \|s\|);\end{aligned}$$

since also $\|\bar{b} - b\| \leqq \|\bar{b} - B's\| + \|B's - Bs\|$, we are able to conclude that

$$\|\bar{b} - b\| \leqq 2c_1\varepsilon'(1 + \|s\|) + \varepsilon'\|s\|. \tag{4.3}$$

We remark that the inequality in equation (4.3) holds simultaneously for all s in S and $\bar{s}$ in S'.

We define $x_1 = s$. We know that x_1 solves

$$Ax_1 \leqq a, \qquad N_r x_1 = f_r = \left[P\begin{pmatrix} b \\ d \end{pmatrix}\right]_r. \tag{4.4}$$

Consider the question of finding an x_{n+1} for $n \geqq 1$ to solve

$$\begin{aligned} Ax_{n+1} &\leqq a' + (A - A')x_n, \\ N_r x_{n+1} &= \bar{f}_r + (N_r - N_r')x_n, \end{aligned} \tag{4.5}$$

where we define

$$\bar{f}_r = \left[P\begin{pmatrix} \bar{b} \\ d' \end{pmatrix}\right]_r = N_r'\bar{s}. \tag{4.6}$$

We can conclude from Lemma 4.1 that equation (4.5) is solvable for all n if both $\|[a - a' - (A - A')x_n]^+\|$ and $\|\bar{f}_r + (N_r - N_r')x_n - \hat{f}_r\|$ are less than or equal to $c_2 \min_i h_i$. The first of these terms is bounded by $\varepsilon'(1 + \|x_n\|)$, while the second is bounded by

$$\begin{aligned}\varepsilon'\|x_n\| + \|\bar{f}_r - \hat{f}_r\| \leqq \varepsilon'\|x_n\| + \left\|\begin{pmatrix} \bar{b} \\ d' \end{pmatrix} - \begin{pmatrix} b \\ d \end{pmatrix}\right\| &\leqq \varepsilon'\|x_n\| + \varepsilon' + \|\bar{b} - b\| \\ &\leqq \varepsilon'\|x_n\| + \varepsilon' + 2c_1\varepsilon'(1 + \|s\|) + \varepsilon'\|s\|,\end{aligned}$$

where we have used equation (4.3).

Thus, in order to have the terms we need be bounded by $c_2 \min_i h_i$ so that we can solve equation (4.5), it is clear that we merely require a uniform bound on $\|x_n\|$ and a suitably small ε'. For $n = 1$, of course, we have $x_n = s$; by Hoffman's theorem there is a constant c_0 depending only on A, B and D such that we can find x_2 solving equation (4.5) whose distance from s, the x_1 solving equation (4.4), is given by

$$\|x_2 - x_1\| \leqq c_0\|[Ax_1 - a' - (A - A')s]^+\| + c_0\|N_r x_1 - \bar{f}_r - (N_r - N_r')s\|. \tag{4.7}$$

For the first term we have $\|[Ax_1 - a' - (A - A')s]^+\| \leqq \|[a - a' - (A - A')s]^+\| \leqq \varepsilon'(1 + \|s\|)$. The second term is just $\|\hat{f}_r - \bar{f}_r - (N_r - N_r')s\|$ which we just bounded in the preceding paragraph by $\varepsilon'\|s\| + \varepsilon' + 2c_1\varepsilon'(1 + \|s\|) + \varepsilon'\|s\|$.

Putting this together yields

$$\|x_2 - x_1\| \leqq c_0\varepsilon'[2 + 3\|s\| + 2c_1(1 + \|s\|)], \tag{4.8}$$

and hence,

$$\|x_2\| \leqq \|s\| + c_0\varepsilon'[2 + 3\|s\| + 2c_1(1 + \|s\|)]. \tag{4.9}$$

Now suppose that x_k solves equation (4.5) for $n = k - 1$ and we seek an x_{k+1} for $n = k$; assuming a uniform bound on $\|x_n\|$ and a small enough ε', we find again from Hoffman's theorem that we can choose x_{k+1} with

$$\|x_{k+1} - x_k\| \leqq c_0\|[Ax_k - a' - (A - A')x_k]^+\| + c_0\|N_r x_k - \bar{f}_r - (N_r - N_r')x_k\|. \tag{4.10}$$

The first of these terms is less than or equal to

$$c_0\|(A - A')(x_{k-1} - x_k)\| \leqq c_0\varepsilon'\|x_{k-1} - x_k\|,$$

while the second is bounded by $c_0\|(N_r - N_r')(x_{k-1} - x_k)\| \leqq c_0\varepsilon'\|x_{k-1} - x_k\|$. Thus we conclude that

$$\|x_{k+1} - x_k\| \leqq 2c_0\varepsilon'\|x_k - x_{k-1}\| \quad \text{for } k \geqq 2. \tag{4.11}$$

The usual induction argument then gives us the uniform bound on $\|x_n\|$ as

$$\|x_{n+1}\| \leqq \|s\| + \|x_2 - x_1\|\frac{1 - (2c_0\varepsilon')^n}{1 - 2c_0\varepsilon'} \leqq \|s\| + \frac{c_0\varepsilon'[2 + 3\|s\| + 2c_1(1 + \|s\|)]}{1 - 2c_0\varepsilon'} \tag{4.12}$$

and the estimate

$$\|x_{n+1} - x_n\| \leqq (2c_0\varepsilon')^{n-1}\|x_2 - x_1\|, \tag{4.13}$$

from which we conclude that $\{x_n\}$ is a Cauchy sequence converging to some point, say s', satisfying

$$\|s' - x_1\| \leqq \frac{1}{1 - 2c_0\varepsilon'}\|x_2 - x_1\| \leqq \varepsilon'\frac{c_0[2 + 3\|s\| + 2c_1(1 + \|s\|)]}{1 - 2c_0\varepsilon'}. \tag{4.14}$$

The induction argument yielding these bounds is valid so long as ε' is small enough for the inequalities mentioned in the paragraph following equation (4.6) to hold with $\|x_n\|$ replaced by its bound in the first part of equation (4.12) with n replaced by $n - 1$; it is this requirement that determines ε_0.

Since x_{n+1} solves equation (4.5), it follows that s' solves $As' \leqq a' + (A - A')s'$, $N_r s' = \bar{f}_r + (N_r - N_r')s'$, from which we see that

$$A's' \leqq a', \qquad N_r's' = \bar{f}_r. \tag{4.15}$$

Now recall that the system $N'x = \begin{pmatrix} \bar{b} \\ d \end{pmatrix}$ is consistent since it is solved by $x = \bar{s}$. From the definition of N_r' as giving a basis for the row space of N' it follows immediately that $N'x = \begin{pmatrix} \bar{b} \\ d' \end{pmatrix}$ for any x solving $N_r'x = \bar{f}_r$. Since s' is such an x according to equation (4.15), we see that $A's' \leqq a'$, $B's' = \bar{b} \leqq b'$, $D's' = d'$; thus s' is in S', and equation (4.14) gives our desired estimate since $x_1 = s$. This completes the proof.

As we remarked in § 1, this result is somewhat disappointing since it serves only to relate perturbations S' to a fixed set S; it does not allow us to relate S' to S'' in some neighborhood of S. Although such a result appears to be true, we have been unable to prove it; if one tries to duplicate the proof of Theorem 4.2, the difficulty arises in finding a result analogous to that in equation (4.3). However, if we assume that B is vacuous, this difficulty disappears and we can prove the following result.

THEOREM 4.16. *Let the general hypothesis of equation* (2.2) *hold for* S, *with* B *vacuous so that there is an* $\hat{x}$ *with* $A\hat{x} \leqq a - h$, $h > 0$, $D\hat{x} = d$, *and* $N = D$. *Let nonempty sets* S' *and* S'' *be defined in the obvious way by two sets of perturbations. As in equation* (3.3), *let* ε', ε'' *and* ε *measure the perturbations in the parameters between* S *and* S', S *and* S'', *and* S'' *and* S', *respectively. Suppose that* $\operatorname{rank}(D) = \operatorname{rank}(D') = \operatorname{rank}(D'')$. *Then there exist positive constants* c *and* ε_0 *depending on* S *such that to any* s'' *in* S'' *satisfying* $\varepsilon'(1 + \|s''\|) \leqq \varepsilon_0$ *there corresponds an* s' *in* S' *satisfying* $\|s' - s''\| \leqq c\varepsilon(1 + \|s''\|)$.

Proof. We merely indicate the crucial changes from the proof of Theorem 4.2; essentially we simply replace s and S by s'' and S''. As before, take s'' in S'', and let $x_1 = s''$; x_1 solves

$$Ax_1 \leqq a'' + (A - A'')s'', \qquad N_r x_1 = d_r'' + (N_r - N_r'')s''.$$

Define x_{n+1} via

$$Ax_{n+1} \leqq a' + (A - A')x_n, \quad N_r x_{n+1} = d_r' + (N_r - N_r')x_n$$

as before. By Proposition 4.1 this is solvable if, as before, we get a uniform bound on $\|x_n\|$ and if ε' is small enough. Starting at $n = 1$, $\|x_1\| = \|s''\|$. Using Hoffman's theorem we get x_2 with

$$\begin{aligned} \|x_2 - x_1\| &\leqq c_0\|[Ax_1 - a' - (A - A')s'']^+\| + c_0\|N_r x_1 - d_r' - (N_r - N_r')s''\| \\ &\leqq c_0\|[a'' + (A - A'')s'' - a' - (A - A')s'']^+\| \\ &\quad + c_0\|d_r'' + (N_r - N_r'')s'' - d_r' - (N_r - N_r')s''\| \\ &\leqq c_0\|[a'' - a' + (A' - A'')s]^+\| + c_0\|d_r'' - d_r' + (N_r' - N_r'')s''\| \\ &\leqq c_0(1 + \|s''\|)\varepsilon + c_0(1 + \|s''\|)\varepsilon, \end{aligned} \tag{4.7'}$$

so

$$\|x_2 - x_1\| \leqq 2c_0(1 + \|s''\|)\varepsilon, \tag{4.8'}$$

$$\|x_2\| \leqq \|s''\| + 2c_0(1 + \|s''\|)\varepsilon. \tag{4.9'}$$

Proceeding precisely as earlier through (4.10) and (4.11), we obtain

$$\|x_{n+1}\| \leqq \|s''\| + \|x_2 - x_1\|\frac{1 - (2c_0\varepsilon')^n}{1 - 2c_0\varepsilon'} \leqq \|s''\| + \frac{2c_0(1 + \|s''\|)\varepsilon}{1 - 2c_0\varepsilon'} \tag{4.12'}$$

as our uniform bound. Therefore, $\{x_n\}$ exists if $\varepsilon'(\|s''\| + 1)$ is small enough. Then we get

$$\|s' - x_1\| \leqq \frac{1}{1 - 2c_0\varepsilon'}\|x_2 - x_1\| \leqq \frac{2c_0(1 + \|s''\|)}{1 - 2c_0\varepsilon'}\varepsilon. \tag{4.14'}$$

Acknowledgment. We wish to thank Dr. Stephen M. Robinson for making available to us very valuable preprints of several of his works in this area [Robinson (1972a,b,c)].

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