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Multipoint Taylor Formulas and Applications to the Finite Element Method

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§ 1. Introduction

In a recent paper, Zlámal [13] has proved the following type of result: let $\mathscr{S}$ be a triangle in the plane R^2 with vertices A_i, $1 \leqq i \leqq 3$, and let B_i, $1 \leqq i \leqq 3$, be the mid points of the sides of $\mathscr{S}$. Then, given a function ϕ defined at the points A_i and B_i, $1 \leqq i \leqq 3$, there exists a unique polynomial of degree 2:

$$\tilde{\phi}(x_1, x_2) = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 x_1^2 + \alpha_5 x_1 x_2 + \alpha_6 x_2^2,$$

with the following interpolation properties:

$$\tilde{\phi}(A_i) = \phi(A_i), \quad \text{and} \quad \tilde{\phi}(B_i) = \phi(B_i), \quad 1 \leqq i \leqq 3. \tag{1.1}$$

Moreover, if the function ϕ is defined over all of $\mathscr{S}$ and has bounded third derivatives over $\mathscr{S}$, then

$$\sup\{|\tilde{\phi}(P) - \phi(P)|;\ P \in \mathscr{S}\} \leqq D M_3 h^3, \tag{1.2}$$

$$\sup\left\{\left|\frac{\partial \tilde{\phi}}{\partial x_i}(P) - \frac{\partial \phi}{\partial x_i}(P)\right|;\ P \in \mathscr{S}\right\} \leqq D' M_3 \frac{h^2}{\sin\theta}, \quad i = 1, 2, \tag{1.3}$$

where

$$M_3 = \sup\left\{\left|\frac{\partial^3 \phi}{\partial x_i\, \partial x_j\, \partial x_k}(P)\right|;\ 1 \leqq i, j, k \leqq 2; P \in \mathscr{S},\right\},$$

$h =$ greatest length of the sides of $\mathscr{S}$,

$\theta =$ smallest angle of $\mathscr{S}$,

$D, D' =$ numerical constants independent of the geometry of the triangle.

The purpose of this paper is to generalize Zlámal's results to the n-dimensional case, so that, in particular, we will derive upper bounds such as those of (1.2)–(1.3). But, instead of deriving those bounds directly as is done in [13], it turns out that they will be obtained as immediate consequences of what might be called *multipoint Taylor formulas* (see (1.4) below). To be more specific, let there be given an n-simplex $\mathscr{S}$ in R^n, and let there be given a function $\phi \in C^3(\mathscr{S})$. If we denote by A_i, $1 \leqq i \leqq n+1$, the vertices of $\mathscr{S}$, and by A_{ij}, $1 \leqq i, j \leqq n+1$; $i \neq j$, the mid points of the edges of $\mathscr{S}$, we will show that we can expand the value $\phi(P)$ of the function ϕ at any point $P \in \mathscr{S}$ in terms of the values $\phi(A_i)$ and $\phi(A_{ij})$. This will be achieved through the *multipoint Taylor formula*

$$\phi(P) = \sum_{i=1}^{n+1} \alpha_i(P)\, \phi(A_i) + \sum_{i,j=1}^{n+1} \alpha_{ij}(P)\, \phi(A_{ij}) + \mathscr{R}(P, D^3\phi), \tag{1.4}$$

where the functions $\alpha_i(P)$ and $\alpha_{ij}(P)$ are independent of the particular function ϕ, and the *remainder*, denoted symbolically by $\mathscr{R}(P, D^3\phi)$, depends *only* on the third-order derivatives of the function ϕ evaluated at a finite number of points of $\mathscr{S}$, in such a way that if all the third-order derivatives of ϕ are uniformly bounded on $\mathscr{S}$ by a constant M_3, then

$$\sup\{|\mathscr{R}(P, D^3\phi)|;\ P\in\mathscr{S}\}\leqq CM_3 h^3, \tag{1.5}$$

for some constant C.

Moreover the function

$$\tilde{\phi}(P) = \sum_{i=1}^{n+1} \alpha_i(P)\,\phi(A_i) + \sum_{i,j=1}^{n+1} \alpha_{ij}(P)\,\phi(A_{ij}),$$

which turns out to be a polynomial of degree 2 in n variables, will be shown to be the unique such polynomial with the interpolation properties:

$$\tilde{\phi}(A_i) = \phi(A_i),\quad 1\leqq i\leqq n+1, \quad\text{and}\quad \tilde{\phi}(A_{ij}) = \phi(A_{ij}),\quad 1\leqq i,\ j\leqq n+1,\ i\neq j,$$

which are of course the n-dimensional analogs of (1.1).

As an obvious consequence of (1.5), we thus obtain the error bound

$$\sup\{|\tilde{\phi}(P) - \phi(P)|;\ P\in\mathscr{S}\}\leqq C M_3 h^3$$

for the difference between the function and its interpolating polynomial. Let us mention that we will likewise derive error bounds for the quantities

$$\sup\left\{\left|\frac{\partial\tilde{\phi}}{\partial x_i}(P) - \frac{\partial\phi}{\partial x_i}(P)\right|;\ P\in\mathscr{S}\right\}$$

for all $1\leqq i\leqq n$.

It is worth mentioning here the work of Coatmélec [3] where, in particular, multipoint Taylor formulas are also derived. However, his approach and methods are of a different type, since in particular, the degree of the approximating polynomials that he uses depends on the dimension n of the space, for a given type of approximation. Similarly, Di Guglielmo [6] has considered approximations on n-simplices but, again, in a different setting.

Since the coefficient functions $\alpha_i(P)$ and $\alpha_{ij}(P)$ which appear in (1.4) will be expressed in terms of the barycentric coordinates of P, we first briefly mention in §2 some relevant results about n-simplices and barycentric coordinates, as well as some results on differential calculus which we shall need in the sequel.

We examine in §3 a first type of approximation, called *approximation of type 1*, in which the interpolating polynomial $\tilde{\phi}$ is linear and is only required to satisfy the conditions

$$\tilde{\phi}(A_i) = \phi(A_i),\quad 1\leqq i\leqq n+1.$$

The *approximation of type 2*, which we described at the beginning of this section, is then examined in §4, and we also very briefly mention in §5 a *third type of approximation* which similarly generalizes a special case considered by Zlámal in R^2.

Finally, in § 6, we consider the *application of the above results to the finite element method* in R^n for solving (non necessarily linear) second-order boundary value problems over domains $\Omega \subset R^n$ which can be covered by generalized triangulations consisting of a finite number of n-simplices, over each of which the trial functions are linear or polynomials of degree 2. In so doing, we generalize to the n-dimensional case the classical Ritz method using piecewise linear functions over a triangulation of the plane introduced by Courant [4], and further studied by Friedrichs and Keller [8], Oganesjan [9], Tong and Pian [10], among others. We likewise generalize to the n-dimensional case the case where the trial functions are piecewise polynomials of degree 2 over triangles [13], or over tetrahedra as is the case for the finite element method in R^3 (cf. Chapter 9 of Fraeijs de Veubeke [7], or Zienkiewicz [12]). In both cases, we derive a priori bounds for the error of the type

$$\|\phi - \hat{\phi}\|_{W_0^{1,2}(\Omega)} = O(h^\alpha), \tag{1.6}$$

where

ϕ is the solution of the boundary value problem,

$\hat{\phi}$ is the approximation found by using the finite element method over a generalized triangulation of the domain Ω,

$\|\cdot\|_{W_0^{1,2}(\Omega)}$ is the usual Sobolev norm,

$h =$ greatest diameter of the n-simplices of the triangulation,

and

$\alpha = 1$ or 2 depending on whether the trial functions are piecewise linear or piecewise polynomials of degree 2.

§ 2. Background: n-Simplices and Differential Calculus

In the n-dimensional Euclidean space $E = R^n$, let there be given $(n+1)$ points A_i with coordinates $a_{1i}, a_{2i}, \ldots, a_{ni}$, $1 \leq i \leq n+1$, such that the matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,n+1} \\ a_{21} & a_{22} & \cdots & a_{2\,n+1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,n+1} \\ 1 & 1 & \cdots & 1 \end{vmatrix} \tag{2.1}$$

is regular. Given any point $P \in E$ with coordinates $x_1, x_2, \ldots, x_n$, there exist $(n+1)$ real numbers

$$\lambda_i = \lambda_i(P), \quad 1 \leq i \leq n+1,$$

such that

$$OP = \sum_{i=1}^{n+1} \lambda_i OA_i, \tag{2.2}$$

with

$$\sum_{i=1}^{n+1} \lambda_i = 1, \tag{2.3}$$

where O is the origin of E. Those quantities λ_i are called the *barycentric coordinates of* P (with respect to the $(n+1)$ points A_i), and they are uniquely determined. To see this, it suffices to solve the linear system

$$\sum_{j=1}^{n+1} a_{ij}\lambda_j = x_i, \quad 1 \leqq i \leqq n, \quad \text{and} \quad \sum_{j=1}^{n+1} \lambda_j = 1,$$

which has a unique solution since the matrix A of (2.1) is regular. At the same time, we see that the barycentric coordinates of P are linear (but generally non-homogeneous) functions of the coordinates $x_1, x_2, \ldots, x_n$ of P, since

$$\lambda_i = \sum_{j=1}^{n} b_{ij} x_j + b_{in+1}, \quad 1 \leqq i \leqq n+1, \tag{2.4}$$

where the matrix $B = (b_{ij})$ is the inverse of the matrix A. The barycentric coordinates are independent of the basis in E, and as a consequence,

$$\sum_{i=1}^{n+1} \lambda_i PA_i = 0. \tag{2.5}$$

The convex hull $\mathscr{S}$ of the $(n+1)$ points A_i, i.e., the set of those points of E whose barycentric coordinates satisfy the conditions

$$0 \leqq \lambda_i \leqq 1, \quad 1 \leqq i \leqq n+1, \tag{2.6}$$

is called the *n-simplex* generated by the points A_i, and the points A_i are called the *vertices* of the n-simplex. The *barycentre* G of $\mathscr{S}$ is the point of $\mathscr{S}$ whose barycentric coordinates are all equal (to $1/(n+1)$). An *m-dimensional face* of $\mathscr{S}$ is any m-simplex $(1 \leqq m \leqq n)$ generated by $(m+1)$ of the vertices of $\mathscr{S}$. A 1-dimensional face is an *edge*. For details about n-simplices, see for example [11, p. 95].

Let us now recall some results from differential calculus. For a detailed treatment, see for example [1] or [5, Chapter 8].

Given two vector spaces E and F over the same scalar field, and a positive integer k, a mapping $L\colon E^k \to F$ (where $E^k = E \times E \times \cdots \times E$ denotes the Cartesian product of E, k times with itself) is *k-linear* iff, whenever $(k-1)$ variables are fixed, it is a linear mapping with respect to the last variable. A k-linear mapping $L\colon E^k \to F$ is *symmetric* iff, for any permutation $\sigma\colon l \to \sigma_l$ of the set $\{1, 2, \ldots, k\}$ and for any collection of k vectors X_l, $1 \leqq l \leqq k$, in E, one has

$$L \cdot (X_1, X_2, \ldots, X_k) = L \cdot (X_{\sigma_1}, X_{\sigma_2}, \ldots, X_{\sigma_k}).$$

A k-linear mapping $L\colon E^k \to F$, where E and F are normed vector spaces is *continuous* iff

$$\|L\| \equiv \sup\{\|L \cdot (X_1, X_2, \ldots, X_k)\|_F;\ \|X_l\|_E = 1,\ 1 \leqq l \leqq k\} < +\infty.$$

This will be the case in particular if E and F are both finite-dimensional.

Let K be some subset of E. Given a real-valued function $\phi\colon K \to R$, the *k-th derivative* $D^k\phi(P)$ $(k \geqq 1)$ of the function ϕ at a point $P \in K$ is (assuming its existence) a k-linear and symmetric mapping from E^k into R. Following the above notations, the result of the application of $D^k\phi(P)$ on the element

$(X_1, X_2, \ldots, X_k) \in E^k$, i.e., a real number, will be denoted

$$D^k \phi(P) \cdot (X_1, X_2, \ldots, X_k).$$

In case where $X_l = X$, $1 \leqq l \leqq k$, we shall use for brevity the notations

$$D^k \phi(P) \cdot (X^k),$$

and, in case $k=1$,

$$D\phi(P) \cdot (X).$$

In case where $E = R^n$, if we denote by $e_1, e_2, \ldots, e_n$ the basis of E, then for sufficiently differentiable functions, we have

$$\frac{\partial \phi}{\partial x_i}(P) = D\phi(P) \cdot (e_i), \quad \frac{\partial^2 \phi}{\partial x_i \, \partial x_j}(P) = D^2 \phi(P) \cdot (e_i, e_j), \quad \text{etc.}$$

As a consequence, if the norm $\|\cdot\|_E$ in E is such that $\|e_i\|_E = 1$, $1 \leqq i \leqq n$, then,

$$\left|\frac{\partial \phi}{\partial x_i}(P)\right| \leqq \|D\phi(P)\|, \quad \left|\frac{\partial^2 \phi}{\partial x_i \, \partial x_j}(P)\right| \leqq \|D^2 \phi(P)\|, \quad \text{etc.}$$

Given a function $\phi: K \to R$ and an open segment $]A, P[$ whose closure $[A, P]$ is contained in K, we say that we can apply *Taylor formula of the* $(k+1)$*-st order between* P *and* A if we can write

$$\phi(A) = \phi(P) + \sum_{l=1}^{k} \frac{1}{l!} D^l \phi(P) \cdot (PA^l) + \frac{1}{(k+1)!} D^{k+1} \phi\big(\Pi(P)\big) \cdot (PA^{k+1}), \tag{2.7}$$

for some point $\Pi(P)$ of the open segment $]A, P[$.

In what follows, we shall frequently encounter the situation where the above Taylor formula must be applied between all points P of a given subset $K \subset E$ and a finite number of points $\{A_1, \ldots, A_{n+1}, A_{n+2}, \ldots, A_N\}$ which are all contained in an n-simplex $\mathscr{S}$ of vertices $A_1, \ldots, A_{n+1}$. For this reason, we say that a subset $K \subset E$ is $\mathscr{S}$*-admissible* iff whenever K contains a point P, it also contains the closed segments $[P, A_i]$ for all $1 \leqq i \leqq n+1$ (and hence K also contains all the closed segments $[P, A_i]$ for all $n+2 \leqq i \leqq N$). This is a generalization of the notion of a star-shaped domain, since it could be equivalently said that K is star-shaped with respect to any point of the n-simplex $\mathscr{S}$. Notice that $\mathscr{S}$ is $\mathscr{S}$-admissible.

We say that a function ϕ belongs to the class $\mathscr{T}^{k+1}(K)$ iff (a): the domain of definition of ϕ contains K, and (b): between any point $P \in K$ and any point $A \in \mathscr{S}$, the above Taylor formula (2.7) holds.

In particular, a function $\phi \in C^k(K) \cap C^{k+1}(\mathring{K})$ is in the class $\mathscr{T}^{k+1}(K)$ if K is $\mathscr{S}$-admissible ($\mathring{K}$ denotes the interior of K); cf. [1].

Let us also note the following relations that we will use repeatedly in the sequel:

$$\sum_{i=1}^{n+1} D\lambda_i(P) = 0, \tag{2.8}$$

$$D\lambda_i(P) \cdot (PA_j) = \delta_{ij} - \lambda_i(P), \quad 1 \leqq i, \; j \leqq n+1, \tag{2.9}$$

where δ_{ij} is the symbol of Kronecker. The relation (2.8) follows from (2.3). From (2.4), we have $\frac{\partial \lambda_i}{\partial x_k} = b_{ik}$, $1 \leqq i \leqq n+1$, $1 \leqq k \leqq n$, so that

$$D\lambda_i(P) \cdot (PA_j) = \sum_{k=1}^{n} b_{ik} a_{kj} - \sum_{k=1}^{n} b_{ik} x_k = \delta_{ij} - \lambda_i(P),$$

again by (2.4), which proves (2.9).

Observe that the linear mappings $D\lambda_i(P)$: $E \to R$, $1 \leqq i \leqq n+1$, are independent of P. For this reason, we shall henceforth denote them by $D\lambda_i$. We will need an estimate for $\|D\lambda_i\|$ which we now compute (the vector norm $\|\cdot\|_E$ in E is the usual Euclidean norm).

Lemma 1. Given an n-simplex $\mathscr{S}$, let

(2.10) $\qquad h' = \text{diameter of the inscribed sphere of } \mathscr{S}.$

Then,

(2.11) $\qquad \|D\lambda_i\| \leqq \frac{1}{h'}, \quad \text{for all} \quad 1 \leqq i \leqq n+1.$

Proof. By definition, we have

$$\|D\lambda_i\| = \sup\{|D\lambda_i \cdot (X)|\,;\ X \in E,\ \|X\|_E = 1\}.$$

Given any vector $X \in E$ which satisfies $\|X\|_E = 1$, we can write it as

$$X = \frac{1}{h'} PQ, \quad \text{with} \quad P, Q \in \mathscr{S},$$

by definition of h'. Since $Q \in \mathscr{S}$, there exist $(n+1)$ real numbers μ_j such that

$$PQ = \sum_{j=1}^{n+1} \mu_j PA_j, \quad \text{with} \quad 0 \leqq \mu_j \leqq 1,\ 1 \leqq j \leqq n+1, \quad \text{and} \quad \sum_{j=1}^{n+1} \mu_j = 1.$$

Thus, it follows, by (2.9),

$$D\lambda_i \cdot (X) = \frac{1}{h'} \sum_{j=1}^{n+1} \mu_j D\lambda_i \cdot (PA)_j = \frac{1}{h'} (\mu_i - \lambda_i).$$

Since $P \in \mathscr{S}$, the quantity λ_i also satisfies $0 \leqq \lambda_i \leqq 1$, so that finally

$$|D\lambda_i \cdot (X)| \leqq \frac{1}{h'},$$

which proves (2.11). Q.E.D.

Let us finally remark that in the case $n=2$, then $\mathscr{S}$ is a triangle, and if we denote by h the greatest side of $\mathscr{S}$ and by θ the smallest angle of $\mathscr{S}$, the following inequality can be easily established:

(2.12) $\qquad \frac{1}{h'} \leqq \frac{2}{h \sin\theta}.$

§3. Approximation of Type 1

Corresponding to each type of approximation, we will have the same sequence of results. First (Theorems 1, 3 and 5), the existence and uniqueness of the approximation will be proved by merely exhibiting the approximation in question, thereby proving its existence. The uniqueness of this approximation will then be a consequence of the following simple observation: in each case, finding the approximating polynomial amounts to solving a linear system whose number of unknowns (which are the coefficients of the polynomial) is equal to the number of equations. Since this system always has a solution, no matter what the right-hand sides of the linear equations are, uniqueness is then guaranteed.

Then (Theorems 2 and 4), we give an expression for the difference between the function and its approximation, and also for the difference between the first derivative of the function and the first derivative of the approximation. As a first consequence (Corollaries 1 and 3) we obtain a *multipoint Taylor formula*, and as a second consequence (Corollaries 2 and 4), we obtain an upper bound for the sup-norm of the difference between the function and its approximation, as well as an upper bound for the sup-norm of the difference between the first derivative of the function and the first derivative of the approximation.

Let there be given an n-simplex $\mathscr{S}\subset E$ with vertices A_i, $1\leqq i\leqq n+1$. Given a real-valued function ϕ defined at the points A_i, $1\leqq i\leqq n+1$, we say that $\tilde{\phi}$: $E\to R$ (observe that $\tilde{\phi}$ is defined for all points $P=(x_1, x_2, \ldots, x_n)$ of E) is an *approximation of type 1 of* ϕ iff

(3.1) $\tilde{\phi}$ is a linear function (i.e., a polynomial of degree 1 in the n variables $x_1, x_2, \ldots, x_n$),

(3.2) $\tilde{\phi}(A_i)=\phi(A_i), \quad 1\leqq i\leqq n+1.$

Observe that the number $N_1=n+1$ of conditions contained in (3.2) is precisely equal to the number of coefficients of a linear function of n variables.

Theorem 1. The unique approximation of type 1 is given by

$$\tilde{\phi}(P)=\sum_{i=1}^{n+1}\lambda_i(P)\,\phi(A_i), \qquad \text{for all} \quad P\in E. \tag{3.3}$$

Proof. Condition (3.1) is satisfied since the λ_i's are linear functions of $x_1, x_2, \ldots, x_n$, and condition (3.2) is also satisfied since $\lambda_i(A_j)=\delta_{ij}$, $1\leqq i$, $j\leqq n+1$. Q.E.D.

Theorem 2. Let $K\subset E$ be $\mathscr{S}$-admissible, and let $\phi\in\mathscr{T}^2(K)$. Then, for all $P\in K$, $P\neq A_i$, $1\leqq i\leqq n+1$,

$$\tilde{\phi}(P)-\phi(P)=\tfrac{1}{2}\sum_{i=1}^{n+1}\{D^2\phi(\Pi_i(P))\cdot(PA_i^2)\}\,\lambda_i(P), \tag{3.4}$$

$$D\tilde{\phi}(P)-D\phi(P)=\tfrac{1}{2}\sum_{i=1}^{n+1}\{D^2\phi(\Pi_i(P))\cdot(PA_i^2)\}\,D\lambda_i, \tag{3.5}$$

where the points $\Pi_i(P)$ are some points on the open segments $]P, A_i[$, $1\leqq i\leqq n+1$.

Proof. For brevity, we will use the notations $h_i=PA_i$, $1\leqq i\leqq n+1$. We have any $1\leqq i\leqq n+1$,

$$\phi(A_i)=\phi(P)+D\phi(P)\cdot(h_i)+\tfrac{1}{2}D^2\phi(\Pi_i(P))\cdot(h_i^2), \tag{3.6}$$

so that by replacing $\phi(A_i)$ by the above expression in (3.3), we obtain (3.4) as a direct consequence of (2.3) and (2.5).

Similarly, we obtain from (3.3),

$$D\tilde{\phi}(P) = \sum_{i=1}^{n+1} \phi(A_i)\, D\lambda_i,$$

so that by (3.6),

$$D\tilde{\phi}(P) = \phi(P)\sum_{i=1}^{n+1} D\lambda_i + \sum_{i=1}^{n+1}\{D\phi(P)\cdot(h_i)\}\, D\lambda_i + \tfrac{1}{2}\sum_{i=1}^{n+1}\{D^2\phi(\Pi_i(P))\cdot(h_i^2)\}\, D\lambda_i.$$

The factor of $\phi(P)$ is zero by (2.8). To prove that

$$\sum_{i=1}^{n+1}\{D\phi(P)\cdot(h_i)\}\, D\lambda_i \quad \text{and} \quad D\phi(P)$$

are equal, it suffices to apply those two linear mappings to the $(n+1)$ vectors h_j, $1 \leqq j \leqq n+1$, and to use the relations (2.9). Then,

$$\begin{aligned}\sum_{i=1}^{n+1}\{D\phi(P)\cdot(h_i)\}\, D\lambda_i\cdot(h_j) &= D\phi(P)\cdot(h_j) - D\phi(P)\cdot\left(\sum_{i=1}^{n+1}\lambda_i h_i\right)\\ &= D\phi(P)\cdot(h_j),\end{aligned}$$

since $\sum_{i=1}^{n+1}\lambda_i h_i = 0$ by (2.5). Q.E.D.

Corollary 1. Let $K \subset E$ be $\mathscr{S}$-admissible, and let $\phi \in \mathscr{T}^2(K)$. Then, for all $P \in K$, $\phi(P)$ can be expanded in terms of its values $\phi(A_i)$, $1 \leqq i \leqq n+1$, as

$$\phi(P) = \sum_{i=1}^{n+1}\lambda_i(P)\,\phi(A_i) - \tfrac{1}{2}\sum_{i=1}^{n=1}\{D^2\phi(\Pi_i(P))\cdot(PA_i^2)\}\,\lambda_i(P), \tag{3.7}$$

for some points $\Pi_i(P)$ on the open segments $]P, A_i[$, $1 \leqq i \leqq n+1$.

Proof. This is an immediate consequence of (3.3) and (3.4). Q.E.D.

Corollary 2. Let $\phi \in \mathscr{T}^2(\mathscr{S})$ be given, and assume that

$$M_2 = \sup\left\{\left|\frac{\partial^2\phi}{\partial x_i\,\partial x_j}(Q)\right|;\ 1 \leqq i, j \leqq n,\ Q \in \mathscr{S}\right\} < +\infty. \tag{3.8}$$

Then, denoting by $\tilde{\phi}$ the unique approximation of type 1 of ϕ, we have

$$\sup\{|\tilde{\phi}(P) - \phi(P)|;\ P \in \mathscr{S}\} \leqq C M_2 h^2, \tag{3.9}$$

and for any $1 \leqq i \leqq n$,

$$\sup\left\{\left|\frac{\partial\tilde{\phi}}{\partial x_i}(P) - \frac{\partial\phi}{\partial x_i}(P)\right|;\ P \in \mathscr{S}\right\} \leqq C' M_2 \frac{h^2}{h'}, \tag{3.10}$$

where

(3.11) $h =$ diameter of $\mathscr{S}$,

(3.12) $h' =$ diameter of the inscribed sphere of $\mathscr{S}$,

$C, C' =$ numerical constants which are independent of the geometry of $\mathscr{S}$.

Proof. Condition (3.8) implies that

$$M_2' = \sup\{\|D^2\phi(Q)\|;\ Q \in \mathscr{S}\} < +\infty.$$

For any point $P \in \mathscr{S}$, we have by (3.4),

$$|\tilde{\phi}(P) - \phi(P)| \leqq \tfrac{1}{2} \sum_{i=1}^{n+1} \lambda_i(P)\, |D^2\phi(\Pi_i(P)) \cdot (PA_i^2)|.$$

Since

$$|D^2\phi(\Pi_i(P)) \cdot (PA_i^2)| \leqq M_2' h^2, \quad \text{and} \quad 0 \leqq \lambda_i(P) \leqq 1, \text{ for all } 1 \leqq i \leqq n+1,$$

we thus obtain (3.9). Similarly, the bound (2.11) of Lemma 1 yields (3.10). Q.E.D.

§4. Approximation of Type 2

Let there be given an n-simplex $\mathscr{S} \subset E$ with vertices A_i, $1 \leqq i \leqq n+1$. On each edge generated by the vertices A_i and A_j, let $A_{ij} = A_{ji}$ denote the mid-point. Given a real-valued function ϕ defined at the points A_i, $1 \leqq i \leqq n+1$, and A_{ij}, $1 \leqq i, j \leqq n+1$, we say that $\tilde{\phi}: E \to R$ is an *approximation of type 2 of* ϕ iff

(4.1) $\tilde{\phi}$ is a polynomial of degree 2 in the n variables $x_1, x_2, \ldots, x_n$,

$$\begin{aligned} \tilde{\phi}(A_i) &= \phi(A_i), \qquad 1 \leqq i \leqq n+1, \\ \tilde{\phi}(A_{ij}) &= \phi(A_{ij}), \qquad 1 \leqq i,\ j \leqq n+1,\ i \neq j. \end{aligned} \tag{4.2}$$

Observe that the number N_2 of conditions expressed in (4.2) is

$$N_2 = \binom{n}{0} + 2\binom{n}{1} + \binom{n}{2} = \frac{(n+1)(n+2)}{2}.$$

It is also the number of coefficients of a polynomial of degree 2 in n variables.

Theorem 3. The unique approximation of type 2 is given by

$$\tilde{\phi}(P) = \sum_{i=1}^{n+1} \big(2(\lambda_i(P))^2 - \lambda_i(P)\big)\, \phi(A_i) + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n+1} \lambda_i(P)\, \lambda_j(P)\, \phi(A_{ij}), \quad \text{for all} \quad P \in E. \tag{4.3}$$

Proof. From (2.4), it is clear that $\tilde{\phi}$ satisfies (4.1). Next, if $P = A_k$ for some k, then $\lambda_i = \delta_{ik}$, so that $\tilde{\phi}(A_k) = \phi(A_k)$, and finally, if $P = A_{kl}$ for some $k \neq l$, then $\lambda_i = (\delta_{ik} + \delta_{il})/2$, so that $\tilde{\phi}(A_{kl}) = \phi(A_{kl})$, proving that (4.2) is satisfied. Q.E.D.

Theorem 4. Let $K \subset E$ be $\mathscr{S}$-admissible, and let $\phi \in \mathscr{T}^3(K)$. Then, for all $P \in K$, $P \neq A_i$, $1 \leqq i \leqq n+1$, $P \neq A_{ij}$, $1 \leqq i, j \leqq n+1$, $i \neq j$,

$$\begin{aligned}\tilde{\phi}(P) - \phi(P) = \tfrac{1}{6}\Big\{&\sum_{i=1}^{n+1}\{D^3\phi(\Pi_i(P)) \cdot (PA_i^3)\}\{2(\lambda_i(P))^2 - \lambda_i(P)\}\\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\{D^3\phi(\Pi_{ij}(P)) \cdot (PA_{ij}^3)\}\,\lambda_i(P)\,\lambda_j(P)\Big\},\end{aligned} \tag{4.4}$$

$$\begin{aligned}D\tilde{\phi}(P) - D\phi(P) = \tfrac{1}{6}\Big\{&\sum_{i=1}^{n+1}\{D^3\phi(\Pi_i(P)) \cdot (PA_i^3)\}\{4\lambda_i(P) - 1\}\,D\lambda_i\\ &+ 4\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\{D^3\phi(\Pi_{ij}(P)) \cdot (PA_{ij}^3)\}\,\lambda_i(P)\,D\lambda_j\Big\},\end{aligned} \tag{4.5}$$

where the points $\Pi_i(P)$ and $\Pi_{ij}(P)$ are some points on the open segments $]P, A_i[$ and $]P, A_{ij}[$ respectively.

Proof. For brevity, we will use the notations $\lambda_i = \lambda_i(P)$, $h_i = PA_i$, and $h_{ij} = PA_{ij}$. We have

$$\phi(A_i) = \phi(P) + D\phi(P) \cdot (h_i) + \tfrac{1}{2} D^2\phi(P) \cdot (h_i^2) + \tfrac{1}{6} D^3\phi(\Pi_i(P)) \cdot (h_i^3), \quad \text{for all } 1 \leqq i \leqq n+1, \tag{4.6}$$

and

$$\phi(A_{ij}) = \phi(P) + D\phi(P) \cdot (h_{ij}) + \tfrac{1}{2} D^2\phi(P) \cdot (h_{ij}^2) + \tfrac{1}{6} D^3\phi(\Pi_{ij}(P)) \cdot (h_{ij}^3), \quad \text{for all } 1 \leqq i, j \leqq n+1,\ i \neq j. \tag{4.7}$$

We then replace in (4.3) the quantities $\phi(A_i)$ and $\phi(A_{ij})$ by (4.6) and (4.7) respectively. In the resulting expression, the factor of $\phi(P)$ is

$$\sum_{i=1}^{n+1}(2\lambda_i^2 - \lambda_i) + 2\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\lambda_i\lambda_j = 1, \quad \text{by (2.3)}.$$

Using

$$h_{ij} = (h_i + h_j)/2,$$

it follows that the vector to which the operator $D\phi(P)$ is applied is

$$\sum_{i=1}^{n+1}(2\lambda_i^2 - \lambda_i)\,h_i + 2\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\lambda_i\lambda_j h_{ij} = 2\left(\sum_{j=1}^{n+1}\lambda_j\right)\sum_{i=1}^{n+1}\lambda_i h_i - \sum_{i=1}^{n+1}\lambda_i h_i = 0,$$

by (2.5). The term involving the second derivative is

$$\tfrac{1}{2}\sum_{i=1}^{n+1}(2\lambda_i^2 - \lambda_i)\,D^2\phi(P) \cdot (h_i^2) + \sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\lambda_i\lambda_j D^2\phi(P) \cdot (h_{ij}^2).$$

Using the 2-linearity and the symmetry of the operator $D^2\phi(P)$, and using again (4.8), this term can be rewritten as

$$\tfrac{1}{2} D^2\phi(P) \cdot \left(\left\{\sum_{i=1}^{n+1}\lambda_i h_i\right\}^2\right) + \tfrac{1}{2}\sum_{i=1}^{n+1} D^2\phi(P) \cdot \left(\lambda_i h_i, \left(\sum_{j=1}^{n}\lambda_j - 1\right) h_i\right) = 0,$$

by (2.3) and (2.5). Thus, (4.4) is proved.

Let us now prove (4.5). From (4.3), we have

$$D\tilde{\phi}(P)=\sum_{i=1}^{n+1}\phi(A_i)(4\lambda_i-1)D\lambda_i+4\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\phi(A_{ij})\lambda_i D\lambda_j.$$

Using (4.6) and (4.7) in this expression, we find that the resulting factor of $\phi(P)$ is

$$4\sum_{i=1}^{n+1}\lambda_i\left(\sum_{j=1}^{n+1}D\lambda_j\right)-\sum_{i=1}^{n+1}D\lambda_i=0 \quad \text{by (2.8)},$$

and the term involving $D\phi(P)$ is, using (4.8),

$$\sum_{i=1}^{n+1}\{D\phi(P)\cdot(h_i)\}\{4\lambda_i-1\}D\lambda_i+2\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\{D\phi(P)\cdot(h_i+h_j)\}\lambda_i D\lambda_j.$$

To show that this is again equal to $D\phi(P)$, it suffices to apply the above operator to the $(n+1)$ vectors h_k, $1\leqq k\leqq n+1$, and to use the relations (2.5) and (2.9).

Finally, a similar analysis shows that the term involving the second derivative, i.e.,

$$\frac{1}{2}\left\{\sum_{i=1}^{n+1}\{D^2\phi(P)\cdot(h_i^2)\}\{4\lambda_i-1\}D\lambda_i+4\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\{D^2\phi(P)\cdot(h_{ij}^2)\}\lambda_i D\lambda_j\right\}$$

is equal to 0, which achieves the proof. Q.E.D.

Corollary 3. Let $K\subset E$ be $\mathscr{S}$-admissible, and let $\phi\in\mathscr{T}^3(K)$. Then, for all $P\in K$, $\phi(P)$ can be expanded in terms of its values $\phi(A_i)$, $1\leqq i\leqq n+1$, and $\phi(A_{ij})$ $1\leqq i,\ j\leqq n+1$, $i\neq j$, as

$$\begin{aligned}\phi(P)=&\sum_{i=1}^{n+1}\big(2(\lambda_i(P))^2-\lambda_i(P)\big)\phi(A_i)+2\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\lambda_i(P)\lambda_j(P)\phi(A_{ij})\\ &-\frac{1}{6}\left\{\sum_{i=1}^{n+1}\{D^3\phi(\Pi_i(P))\cdot(PA_i^3)\}\{2(\lambda_i(P))^2-\lambda_i(P)\}\right.\\ &\left.+\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\{D^3\phi(\Pi_{ij}(P))\cdot(PA_{ij}^3)\}\lambda_i(P)\lambda_j(P)\right\},\end{aligned}\tag{4.9}$$

for some points $\Pi_i(P)$ on $]P, A_i[$, $1\leqq i\leqq n+1$, and for some points $\Pi_{ij}(P)$ on $]P, A_{ij}[$, $1\leqq i,\ j\leqq n+1$, $i\neq j$.

Proof. This is an obvious consequence of (4.3) and (4.4). Q.E.D.

Corollary 4. Let $\phi\in\mathscr{T}^3(\mathscr{S})$ be given, and assume that

$$M_3=\sup\left\{\left|\frac{\partial^3\phi}{\partial x_i\,\partial x_j\,\partial x_k}(Q)\right|;\ 1\leqq i,j,k\leqq n,\ Q\in\mathscr{S}\right\}<+\infty.\tag{4.10}$$

Then, denoting by $\tilde{\phi}$ the unique approximation of type 2 of ϕ, we have

$$\sup\{|\tilde{\phi}(P)-\phi(P)|;\ P\in\mathscr{S}\}\leqq D M_3 h^3,\tag{4.11}$$

and for any $1 \leqq i \leqq n$,

$$\sup\left\{\left|\frac{\partial \tilde{\phi}}{\partial x_i}(P) - \frac{\partial \phi}{\partial x_i}(P)\right|;\ P \in \mathscr{S}\right\} \leqq D' M_3 \frac{h^3}{h'}, \tag{4.12}$$

where h and h' have been defined in (3.11) and (3.12) respectively, and D and D' are numerical constants which are independent of the geometry of $\mathscr{S}$.

Proof. The proof is very similar to that of Corollary 2 and is left to the reader. Q.E.D.

Remark. In case $n=2$, (4.12) reduces to

$$\sup\left\{\left|\frac{\partial \tilde{\phi}}{\partial x_i}(P) - \frac{\partial \phi}{\partial x_i}(P)\right|;\ P \in \mathscr{S}\right\} \leqq D' M_3 \frac{h^2}{\sin\theta} \tag{4.12'}$$

by virtue of (2.12), which is precisely the bound obtained by Zlámal [13, Theorem 1].

Remark. It is clear that the bounds derived in Corollaries 2 and 4 could be extended without modifying the exponent of h so as to be valid at all points P of a given $\mathscr{S}$-admissible set $K \subset E$, provided the diameter of K is still of order h.

§ 5. Approximation of Type 3

We now define a third type of approximation and we give its explicit form in Theorem 5 below. An analysis similar to that made in the two preceeding paragraphs holds, but for the sake of brevity (the statements and their proofs are long and cumbersome), we just mention that such an analysis can indeed be made. This type of approximation generalizes to the n-dimensional case the second type of approximated first considered by Zlámal in [13] and subsequently in [14]. In particular, it is worth mentionning that the error bounds of Zlámal [13, inequalities (12) and (13)] are also valid in the n-dimensional case.

Let there be given an n-simplex $\mathscr{S} \subset E$ with vertices A_i, $1 \leqq i \leqq n+1$, and let $A_{ijk} = A_{jik} = \cdots = A_{kji}$ denote the barycentre on each 2-face generated by the vertices A_i, A_j, A_k (such a 2-face is an ordinary triangle). Given a real-valued function ϕ defined at the $(n+1)$ points A_i and at the $\binom{n+1}{3}$ points A_{ijk}, as well as its first order derivatives at the $(n+1)$ points A_i, we say that $\tilde{\phi}\colon E \to R$ is an *approximation of type 3 of* ϕ iff

(5.1) $\tilde{\phi}$ is a polynomial of degree 3 in the n variables $x_1, x_2, \ldots, x_n$,

$$\begin{aligned} &\tilde{\phi}(A_i) = \phi(A_i), \quad 1 \leqq i \leqq n+1,\\ &\tilde{\phi}(A_{ijk}) = \phi(A_{ijk}) \quad \text{for all pairwise distinct } i, j, k, \text{ with } 1 \leqq i, j, k \leqq n+1\\ &D\tilde{\phi}(A_i) = D\phi(A_i),\ 1 \leqq i \leqq n+1. \end{aligned} \tag{5.2}$$

Observe that the number of conditions expressed in (5.2) is

$$N_3 = \binom{n}{0} + 3\binom{n}{1} + 3\binom{n}{2} + \binom{n}{3}.$$

It is also the number of coefficients of a polynomial of degree 3 in n variables.

Theorem 5. The unique approximation of type 3 is given by

$$
\begin{aligned}
\tilde{\phi}(P) &= \sum_{i=1}^{n+1} \left(-2(\lambda_i(P))^3 + 3(\lambda_i(P))^2\right)\phi(A_i) \\
&+ \tfrac{1}{6} \sum_{i,j,k}{}' \lambda_i(P)\,\lambda_j(P)\,\lambda_k(P)\left(27\phi(A_{ijk}) - 7\{\phi(A_i) + \phi(A_j) + \phi(A_k)\}\right) \\
&+ \sum_{\substack{i,j=1 \\ i\neq j}}^{n+1} (\lambda_i(P))^2 \lambda_j(P)\{D\phi(A_i)\cdot(A_iA_j)\} \\
&- \sum_{i,j,k}{}' \lambda_i(P)\,\lambda_j(P)\,\lambda_k(P)\{D\phi(A_i)\cdot(A_iA_j)\},
\end{aligned} \tag{5.3}
$$

where $\sum'_{i,j,k}$ denotes the summation over all $1 \leq i, j, k \leq n+1$ for which $i \neq j$, $j \neq k$, $k \neq i$.

Proof. It is clear that $\tilde{\phi}(A_i) = \phi(A_i)$, $1 \leq i \leq n+1$, and that $\tilde{\phi}(A_{ijk}) = \phi(A_{ijk})$ for all pairwise distinct i, j, k. Let us check the third condition. By differentiating (5.3), we obtain:

$$D\tilde{\phi}(P)|_{P=A_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \{D\phi(A_i)\cdot(A_iA_j)\}\, D\lambda_j.$$

To show that the above expression is equal to $D\phi(A_i)$, it suffices to apply it to the n vectors A_iA_k, $k \neq i$, and to use the relations

$$D\lambda_j \cdot (A_iA_k) = \delta_{jk} - \lambda_j(A_i),$$

which are consequences of the relations (2.9). Q.E.D.

§6. Applications to the Numerical Solution of Boundary Value Problems by the Finite Element Method

Consider the nonlinear second order boundary value problem

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{a_i\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)\right\} + a\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0, \quad x \in \Omega, \tag{6.1}$$

$$u(x) = 0, \quad x \in \partial\Omega, \tag{6.2}$$

where Ω is a bounded domain in R^n, whose boundary $\partial\Omega$ is a simplicial complex (a simplicial complex is the generalization to R^n of a polygon in R^2; for details, cf. [11, Chapter IX]).

We assume that the monotone operator theory, as developed in [2], applies to this problem. For example, given a uniformly elliptic operator of the form

$$\mathscr{L}u(x) \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left\{a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right\} + a(x)\,u(x), \qquad x \in \Omega,$$

this will be the case for the nonlinear problem

$$
\begin{aligned}
\mathscr{L}u(x) + f(x, u) &= 0, \qquad x \in \Omega, \\
u(x) &= 0, \qquad x \in \partial\Omega,
\end{aligned}
$$

for which there exists a constant γ such that

$$\frac{\partial f}{\partial u}(x, u) \geqq \gamma > -\Lambda, \qquad \text{for all} \quad x \in \Omega \quad \text{and all} \quad u,$$

where Λ is the smallest eigenvalue of the operator $\mathscr{L}$ over Ω, and for which there exists an a priori bound for the solution in the uniform norm over Ω.

Such problems can be approximated by a variational approximation procedure. Given any finite dimensional subspace S^M of the Sobolev space $S = W_0^{1,2}(\Omega)$, an approximation $\hat{\phi}^M$ to the solution ϕ of (6.1)–(6.2) can be found in S^M, and moreover, there exists a constant K independent of the subspace S^M such that (cf. [2, Theorem 3.3])

$$\|\phi - \hat{\phi}^M\|_S \leqq K \inf\{\|\phi - w\|_S;\ w \in S^M\}, \tag{6.3}$$

where

$$\|w\|_S = \left\{\int_\Omega \sum_{i=1}^n \left(\frac{\partial w}{\partial x_i}\right)^2 dx\right\}^{\frac{1}{2}} \tag{6.4}$$

denotes the Sobolev norm over Ω. For further details, cf. [2].

We now establish a generalized *triangulation* T over $\bar{\Omega}$, as follows. The set $\bar{\Omega}$ is expressed as the set-theoretic union of a finite number of n-simplices $\mathscr{S}_i$, $1 \leqq i \leqq N$, whose interior are pairwise disjoint, and such that, given any n-simplex of the triangulation, each one of its $(n-1)$-face is either a portion of the boundary $\partial\Omega$, or else is also an $(n-1)$-face of another n-simplex of the triangulation. For the actual construction of such a triangulation, see [3, § III.1]).

To each such triangulation, we associate the two parameters

$$h = \max\{\text{diameter of } \mathscr{S}_i;\ 1 \leqq i \leqq N\}, \tag{6.5}$$

$$h' = \min\{\text{diameter of the inscribed sphere of } \mathscr{S}_i;\ 1 \leqq i \leqq N\}. \tag{6.6}$$

We then define the space $S_1(T)$ as follows: a function w defined over $\bar{\Omega}$ belongs to $S_1(T)$ iff

$$w \text{ is a first degree polynomial over each } \mathscr{S}_i,\ 1 \leqq i \leqq N, \tag{6.7}$$

$$w \in C^0(\bar{\Omega}), \tag{6.8}$$

$$w = 0 \quad \text{on } \partial\Omega. \tag{6.9}$$

Thus, the space $S_1(T)$ is a finite-dimensional subspace of S, and $w \in S_1(T)$ is determined in a one-to-one fashion by its values at all the vertices of all the n-simplices of the triangulation T.

Likewise, the space $S_2(T)$ will consist of functions w defined over $\bar{\Omega}$ which satisfy the following properties:

$$w \text{ is a second degree polynomial over each } \mathscr{S}_i,\ 1 \leqq i \leqq N, \tag{6.10}$$

$$w \in C_0(\bar{\Omega}), \tag{6.11}$$

$$w = 0 \quad \text{on } \partial\Omega. \tag{6.12}$$

The space $S_2(T)$ is again a finite-dimensional subspace of S and a function $w \in S_2(T)$ is determined in a one-to-one fashion by its values at all the vertices and all the mid-points of the edges of all the n-simplices of the triangulation T.

It is worth mentionning here that the explicit formula of (3.3) (resp. (4.3)) allows to construct an *explicit basis* for expanding the functions of $S_1(T)$ (resp. $S_2(T)$), and this observation might be helpful from a numerical standpoint.

Theorem 6. Assume that the monotone operator theory applies to the boundary value problem (6.1)–(6.2), so that the inequality of (6.3) holds, and let the solution ϕ of (6.1)–(6.2) be of class $C^1(\bar{\Omega}) \cap C^2(\Omega)$ with bounded second derivatives, i.e.,

$$M_2 = \sup\left\{\left|\frac{\partial^2 \phi}{\partial x_i \, \partial x_j}(Q)\right|;\ 1 \leqq i, j \leqq n,\ Q \in \Omega\right\} < +\infty. \tag{6.13}$$

Then the unique approximation $\hat{\phi}_1$ obtained in the subspace $S_1(T)$ satisfies

$$\|\phi - \hat{\phi}_1\|_S \leqq \mathscr{C} M_2 \frac{h^2}{h'}, \tag{6.14}$$

where h and h' have been defined in (6.5)–(6.6), and $\mathscr{C}$ is a numerical constant independent of the triangulation.

Similarly, if the solution ϕ is of class $C^2(\bar{\Omega}) \cap C^3(\Omega)$ with

$$M_3 = \sup\left\{\left|\frac{\partial^3 \phi}{\partial x_i \, \partial x_j \, \partial x_k}(Q)\right|;\ 1 \leqq i, j, k \leqq n,\ Q \in \Omega\right\} < +\infty, \tag{6.15}$$

then the unique approximation $\hat{\phi}_2$ obtained in the subspace $S_2(T)$ satisfies

$$\|\phi - \hat{\phi}_2\|_S \leqq \mathscr{D} M_3 \frac{h^3}{h'}, \tag{6.16}$$

where, again, $\mathscr{D}$ is a numerical constant independent of the triangulation.

Proof. Since the proofs of the two parts of the theorem are very similar, we just prove the first one.

Let $\tilde{\phi}_1$ be the unique function in $S_1(T)$ which interpolates the solution ϕ in the sense that its values at all the vertices of the n-simplices of the triangulation T coincide with the values of ϕ. Since $\phi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ it is clear that $\phi \in \mathscr{T}^2(\mathscr{S}_i)$ for each n-simplex $\mathscr{S}_i$, and hence we can apply the bounds (3.10) of Corollary 2 which, coupled with the definition (6.4) of the Sobolev norm $\|\cdot\|_S$, gives us that, for some constant C'',

$$\|\phi - \tilde{\phi}_1\|_S \leqq C'' M_2 \frac{h^2}{h'},$$

from which the bound of (6.14) follows by applying (6.3). Q.E.D.

Next, we have a *convergence criterion for a sequence of subspaces.*

Corollary 5. With the same assumptions as in Theorem 6, let there be given a sequence $(S_1(T_i))_{i=1}^{\infty}$ (resp. $(S_2(T_i))_{i=1}^{\infty}$) of subspaces of the above type whose associated parameters h_i and h_i' (as defined in (6.5)–(6.6)) satisfy

$$\lim_{i \to \infty} \frac{h_i^2}{h_i'} = 0 \quad \left(\text{resp. } \lim_{i \to \infty} \frac{h_i^3}{h_i'} = 0\right).$$

Then if we denote by $\hat{\phi}_{1i}$ (resp. $\hat{\phi}_{2i}$) the approximation found in $S_1(T_i)$ (resp. $S_2(T_i)$), we have

$$\lim_{i\to\infty} \|\hat{\phi}_{1i} - \phi\|_S = 0 \quad \left(\text{resp. } \lim_{i\to\infty} \|\hat{\phi}_{2i} - \phi\|_S = 0\right). \tag{6.17}$$

Proof. This is an obvious consequence of (6.14) (resp. (6.16)). Q.E.D.

For practical purposes, we say that a sequence of triangulations $(T_i)_{i=1}^{\infty}$ is *regular* iff there exists a constant $\alpha > 0$ such that

$$h_i \leqq \alpha h_i' \quad \text{for all} \quad i \geqq 1. \tag{6.18}$$

In the particular case $n = 2$, condition (6.18) is satisfied if there exists an angle $\theta_0 > 0$ such that the smallest angle θ_i found in all the triangles of the triangulation T_i satisfies

$$\theta_i \geqq \theta_0 \quad \text{for all} \quad i \geqq 1.$$

This special case thus shows that the idea of a regular sequence of triangulations is indeed a natural one. Also, with this assumption, we can now improve the asymptotic estimates of (6.17).

Corollary 6. With the same assumptions as in Theorem 6, let there be given a sequence $\left(S_1(T_i)\right)_{i=1}^{\infty}$ (resp. $\left(S_2(T_i)\right)_{i=1}^{\infty}$) of subspaces of the above type whose associated sequence $(T_i)_{i=1}^{\infty}$ of triangulations is regular. Then

$$\|\hat{\phi}_{1i} - \phi\|_S = O(h), \tag{6.19}$$

and

$$\|\hat{\phi}_{2i} - \phi\|_S = O(h^2). \tag{6.20}$$

Proof. The above bounds are obvious consequences of the bounds of (6.14)-(6.15) coupled with condition (6.18). Q.E.D.

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