

## Complexity and global rates of trust-region methods based on probabilistic models

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Trust-region algorithms have been proved to globally converge with probability 1 when the accuracy of the trust-region models is imposed with a certain probability conditioning on the iteration history. In this article, we study the complexity of such methods, providing global rates and worst-case complexity bounds on the number of iterations (with overwhelmingly high probability), for both first- and second-order measures of optimality. Such results are essentially the same as the ones known for trust-region methods based on deterministic models. The derivation of the global rates and worst-case complexity bounds follows closely from a study of direct search methods based on the companion notion of probabilistic descent.

**Keywords:** trust-region methods; worst-case complexity; probabilistic models.

### 1. Introduction

Trust-region methods form a well-established and understood class of methods for the minimization of a nonlinear (possibly nonsmooth) function subject or not to constraints on its variables (see the book by Conn *et al.*, 2000 and the recent survey by Yuan, 2015). They have also been comprehensively studied in the context of derivative-free optimization (DFO), where the derivatives of the objective or constraint functions cannot be computed or approximated (see the book by Conn *et al.*, 2009b and the recent survey by Custódio *et al.*, 2017). In this article, we focus on the unconstrained minimization of a smooth objective function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  without using its derivatives.

In the derivative-free setting, trust-region algorithms use sampled points to build a model of the objective function around the current iterate, typically by quadratic interpolation. The quality of these models is measured by the accuracy they provide relatively to a Taylor expansion in a ball  $B(x, \delta)$  of center  $x$

and radius  $\delta$ . Models that are as accurate as first-order Taylor expansions are called fully linear (Conn *et al.*, 2008a, 2009b).

**DEFINITION 1.1** Given a function  $f \in \mathcal{C}^1$  and constants  $\kappa_{ef}, \kappa_{eg} > 0$ , a  $\mathcal{C}^1$  function  $m : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a  $(\kappa_{eg}, \kappa_{ef})$ -fully linear model of  $f$  on  $B(x, \delta)$  if, for all  $s \in B(0, \delta)$ ,

$$\begin{aligned} |m(s) - f(x + s)| &\leq \kappa_{ef} \delta^2, \\ \|\nabla m(s) - \nabla f(x + s)\| &\leq \kappa_{eg} \delta. \end{aligned}$$

Fully linear models are not necessarily linear or affine functions. Models that are as accurate as second-order Taylor expansions are called fully quadratic (Conn *et al.*, 2008a, 2009b). Similarly, such models are not necessarily quadratic functions.

**DEFINITION 1.2** Given a function  $f \in \mathcal{C}^2$  and constants  $\kappa_{ef}, \kappa_{eg}, \kappa_{eh} > 0$ , a  $\mathcal{C}^2$  function  $m : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic model of  $f$  on  $B(x, \delta)$  if, for all  $s \in B(0, \delta)$ ,

$$\begin{aligned} |m(s) - f(x + s)| &\leq \kappa_{ef} \delta^3, \\ \|\nabla m(s) - \nabla f(x + s)\| &\leq \kappa_{eg} \delta^2, \\ \|\nabla^2 m(s) - \nabla^2 f(x + s)\| &\leq \kappa_{eh} \delta. \end{aligned}$$

The construction of fully linear/quadratic models based on the sampled sets raises a number of geometrical questions. Conn *et al.* (2008a,b, 2009b) provided a systematic approach to the subject of deterministic sampling geometry in DFO, establishing error bounds for polynomial interpolation and regression models in terms of a constant measuring the quality or well-poisedness of the corresponding sample set (ensuring then the fully linear/quadratic properties). They also showed how to deterministically build or update such sets to ensure that such a constant remains moderate in size. Some numerical studies pioneered by Fasano *et al.* (2009) have, however, shown that trust-region methods can tolerate the use of models updated without strict geometry requirements, although it is also known (Scheinberg & Toint, 2010) that convergence cannot be ensured to first-order critical points without appealing to fully linear models when the size of the model gradient becomes small (a procedure known as the criticality step).

A DFO context of expensive function evaluations often makes it unaffordable to construct a deterministic model that is guaranteed to be fully quadratic, as such a process requires  $(n + 1)(n + 2)/2$  function evaluations. Practical approaches rely on considerably fewer points (but at least  $n + 1$  to preserve full linearity) and use the remaining degrees of freedom to minimize the norm of the model Hessian or its distance to the previous one. The most studied examples use minimum Frobenius-type norms (Conn *et al.*, 1998; Powell, 2004), yet in Bandeira *et al.* (2012), it was proposed to apply the theory of sparse  $\ell_1$ -recovery to build quadratic models based on random sampling. Such models were proved to be fully quadratic with high probability even when considerably less than  $(n + 1)(n + 2)/2$  points were used, depending on the sparsity of the Hessian of the objective.

Such findings have then called for a probabilistic analysis of derivative-free trust-region algorithms (Bandeira *et al.*, 2014), where the accuracy of the models is guaranteed only with a certain probability. It was shown in Bandeira *et al.* (2014) that the resulting trust-region methods converge with probability 1 to first- and second-order critical points. The main purpose of this article is to establish (with overwhelmingly high probability) the rate under which these methods drive to zero the corresponding criticality measures: the norm of the gradient  $\|\nabla f(x_k)\|$  (in the first-order case) and the maximum

$\sigma(x_k) = \max\{\|\nabla f(x_k)\|, -\lambda_{\min}(\nabla^2 f(x_k))\}$  between the gradient and the negative of the minimum Hessian eigenvalue (in the second-order case). The proofs rely heavily on the technique developed in [Gratton et al. \(2015\)](#) for establishing global rates and worst-case complexity bounds for randomized algorithms in which the new iterate depends on some object (directions in [Gratton et al., 2015](#); models here) and the quality of the object is favorable with a certain probability. The technique is based on counting the number of iterations for which the quality is favorable and examining the probabilistic behavior of this number. Although the road map for our article was described in [Gratton et al. \(2015, Section 6\)](#), its actual concretization poses a few delicate issues. In addition, this article goes beyond [Gratton et al. \(2015, Section 6\)](#) by establishing results in expectation and addressing the second-order case.

To place our work in the existing literature, we briefly review what is known about the complexity of trust-region methods in the deterministic unconstrained case. Using first-order Taylor expansions as models, trust-region methods are known ([Gratton et al., 2008](#)) to take at most  $\mathcal{O}(\epsilon^{-2})$  iterations<sup>1</sup> to reduce the measure  $\|\nabla f(\cdot)\|$  below  $\epsilon \in (0, 1)$ . This first-order iteration complexity bound can be improved to  $\mathcal{O}(\epsilon^{-1})$  and  $\mathcal{O}(-\log(\epsilon))$ , respectively, under convexity and strong convexity assumptions ([Grapiglia et al., 2016b](#)). A recent article by [Curtis et al. \(2017\)](#) also suggested a modification to the classical trust-region approach to improve the nonconvex case to  $\mathcal{O}(\epsilon^{-1.5})$ , a bound typical of cubic regularization methods. Using second-order Taylor expansions as models, it has been proved ([Curtis et al., 2012](#); [Grapiglia et al., 2016a](#)) that at most  $\mathcal{O}(\epsilon^{-3})$  iterations are needed to reduce the second-order criticality measure  $\sigma(\cdot)$  below  $\epsilon \in (0, 1)$ . For trust-region schemes exploiting derivatives, we point out that the above bounds are essentially valid for both iteration and function evaluation counts, as those methods typically compute one additional function value per iteration.

In the derivative-free case, we are particularly interested in counting function evaluations and in understanding the dependence of the complexity bounds in terms of the dimension  $n$  of the problem. Using fully linear models, the first-order iteration complexity bound is of the form  $\mathcal{O}(\kappa_d^{-2}\epsilon^{-2})$  ([Garmanjani et al., 2016](#)), where  $\kappa_d = 1/\max\{\kappa_{ef}, \kappa_{eg}\}$  and  $\kappa_{ef}, \kappa_{eg}$  are the constants arising in Definition 1.1. Since interpolation techniques can ensure  $\kappa_d^{-1} = \mathcal{O}(\sqrt{n})$  with at most  $\mathcal{O}(n)$  evaluations per iteration, one recovers the bounds  $\mathcal{O}(n\epsilon^{-2})$  for iterations and  $\mathcal{O}(n^2\epsilon^{-2})$  for function evaluations, a result also derived for direct search ([Vicente, 2013](#)). Using fully quadratic models, the second-order iteration complexity bound is of the form  $\mathcal{O}(\kappa_d^{-3}\epsilon^{-3})$  ([Júdice, 2015](#); [Garmanjani et al., 2016](#)), with  $\kappa_d = 1/\max\{\kappa_{ef}, \kappa_{eg}, \kappa_{eh}\}$  and the  $\kappa_{ef}, \kappa_{eg}, \kappa_{eh}$  constants defined as in Definition 1.2. Fully quadratic models constructed through interpolation satisfy  $\kappa_d^{-1} = \mathcal{O}(n)$  and require  $\mathcal{O}(n^2)$  evaluations per iteration, leading to a bound in  $\mathcal{O}(n^3\epsilon^{-3})$  (respectively,  $\mathcal{O}(n^5\epsilon^{-3})$ ) for measuring the number of iterations (respectively, function evaluations) needed to drive  $\sigma(\cdot)$  below  $\epsilon$ . We will see that the bounds developed in this article are comparable to the above-mentioned bounds known for deterministic derivative-free trust-region algorithms.

The remainder of the article is organized as follows. Section 2 presents the results related to first-order stationarity, whereas Section 3 is concerned with the second-order counterpart. Finally, in Section 4, we comment on the extension of our work to other settings.

## 2. Complexity of first-order trust-region methods based on probabilistic models

We consider now the scenario analysed in [Bandeira et al. \(2014\)](#), where the models used in a trust-region method are randomly generated at each iteration. As a result, the iterates and trust-region radii produced

<sup>1</sup> The notation  $\mathcal{O}(A)$  stands for a scalar times  $A$ , with the scalar depending solely on the problem considered or constants from the algorithm (with no further dependence on  $A$ ). The dependence on the problem dimension  $n$  will explicitly appear in  $A$  when considered appropriate.

by the algorithm will also be random. Upper-case letters will be then used to designate random variables and lower case their realizations. Hence,  $m_k, x_k, \delta_k$  will denote, respectively, the realizations of the random model, iterate and trust-region radius  $M_k, X_k, \Delta_k$  at iteration  $k$ . The random models are then required to be fully linear with a certain favorable property regardless of the past iteration history. The following definition was proposed in [Bandeira et al. \(2014\)](#) to analyse global convergence of the corresponding trust-region methods to first-order critical points.

**DEFINITION 2.1** A sequence of random models  $\{M_k\}$  is said to be  $(p)$ -probabilistically  $(\kappa_{eg}, \kappa_{ef})$ -fully linear for a corresponding ball sequence  $\{B(X_k, \Delta_k)\}$ , if the events

$$S_k = \{M_k \text{ is a } (\kappa_{eg}, \kappa_{ef})\text{-fully linear model of } f \text{ on } B(X_k, \Delta_k)\}$$

satisfy  $\mathbb{P}(S_0) \geq p$  and, for each  $k \geq 1$ , the following submartingale-like condition:

$$\mathbb{P}(S_k \mid M_0, \dots, M_{k-1}) \geq p.$$

An example is given in [Bandeira et al. \(2014\)](#), where, using random matrix theory, it was shown that linear interpolation based on Gaussian sample sets of cardinality  $n + 1$  (with a fixed point and the remaining  $n$  points being generated randomly from a standard Gaussian distribution) gives rise to fully linear models with a favorable probability (say  $p > 1/2$ ). Using increasingly more sample points and building the models by linear regression, it is possible to reach a probability as high as desired ([Conn et al., 2009b](#), Chapter 4; see also [Larson & Billups, 2016](#)).

## 2.1 Algorithm and assumptions

To simplify the presentation, we describe the trust-region methods under consideration (later given in Algorithm 2.4) for each realization of the model randomness. A few components of these methods are classical, with or without derivatives. At each iteration  $k$ , one minimizes a quadratic model

$$m_k(x_k + s) = f(x_k) + g_k^T s + \frac{1}{2} s^T H_k s$$

in a trust region of the form  $B(x_k, \delta_k)$ . For global convergence to first-order criticality, the Hessian models are assumed to be uniformly bounded and the step  $s_k$  is asked to satisfy a fraction of the model decrease given by the negative model gradient within the trust region. These two assumptions are formalized below.

**ASSUMPTION 2.2** There exists a positive constant  $\kappa_{bhm}$  such that, for every  $k$ , the Hessians  $H_k$  of all realizations  $m_k$  of  $M_k$  satisfy

$$\|H_k\| \leq \kappa_{bhm}.$$

**ASSUMPTION 2.3** For every  $k$ , and for all realizations  $m_k$  of  $M_k$  (and of  $X_k$  and  $\Delta_k$ ), we are able to compute a step  $s_k$  so that it satisfies a *fraction of Cauchy decrease*, i.e.,

$$m(x_k) - m(x_k + s_k) \geq \frac{\kappa_{fcd}}{2} \|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \delta_k \right\} \quad (2.1)$$

for some constant  $\kappa_{fcd} \in (0, 1]$ , and with the convention that  $\frac{\|g_k\|}{\|H_k\|} = \infty$  if  $\|H_k\| = 0$ .

Finally, the step acceptance and trust-region radius update are based on the ratio between the actual decrease in the objective function and the one predicted by the model, namely,

$$\rho_k = \frac{f(x_k) - f(x_k + s_k)}{m_k(x_k) - m_k(x_k + s_k)}.$$

What is different now from the classical derivative-based case is that some form of criticality step has to be taken into account, where models are recomputed in regions small enough compared with the size of the model gradient. Following [Bandeira et al. \(2014\)](#), the presentation and the analysis can be significantly simplified if this requirement is mitigated at each iteration. So, in Algorithm 2.4, the trust-region radius is reduced (at iterations where  $\rho_k$  is large enough and the step is taken), provided  $\delta_k$  is too large compared with  $\|g_k\|$ .

ALGORITHM 2.4 (A simple first-order derivative-free trust-region framework).

Fix parameters  $\eta_1, \eta_2, \delta_{\max} > 0$  and  $0 < \gamma_1 < 1 < \gamma_2$ . Select initial  $x_0$  and  $\delta_0 \in (0, \delta_{\max})$ .

For  $k = 0, 1, 2, \dots$ ,

Build a quadratic model  $m_k(x_k + s)$  of  $f$ , and compute  $s_k$  by approximately minimizing  $m_k$  in  $B(x_k, \delta_k)$  so that it satisfies (2.1).

If  $\rho_k \geq \eta_1$ , set  $x_{k+1} = x_k + s_k$  and

$$\delta_{k+1} = \begin{cases} \min\{\gamma_2 \delta_k, \delta_{\max}\} & \text{if } \|g_k\| \geq \eta_2 \delta_k, \\ \gamma_1 \delta_k & \text{otherwise.} \end{cases}$$

Otherwise, set  $x_{k+1} = x_k$  and  $\delta_{k+1} = \gamma_1 \delta_k$ .

End For

Note that we have slightly extended the framework in [Bandeira et al. \(2014\)](#) using two different parameters (namely,  $\gamma_1$  and  $\gamma_2$ ) to update the trust-region radius, instead of using a single one and its inverse. As we will see, these parameters are intimately connected to the minimum probability with which the models are required to be fully linear. Also, the safeguard  $\delta_{\max}$  is not used in the analysis of the first-order methods and is only there for coherence with the second-order case (where it appears in the analysis) as well as with [Bandeira et al. \(2014\)](#).

The algorithm will be analysed under the following two assumptions on  $f$ . As in [Bandeira et al. \(2014\)](#), it would be enough to assume continuous differentiability in an enlarged initial level set, but we skip this detail for keeping the presentation simple.

ASSUMPTION 2.5 The function  $f$  is continuously differentiable on  $\mathbb{R}^n$ , and its gradient is Lipschitz continuous.

ASSUMPTION 2.6 The objective function  $f$  is bounded from below on  $\mathbb{R}^n$ , and we denote by  $f_{\text{low}}$  a lower bound.

At this point, we can state a fundamental result for establishing the complexity of trust-region methods based on probabilistic models. As in the classical setting of trust-region methods, it ensures that the step

is taken if the trust-region radius is small enough compared with the size of the true gradient. There are two differences between Lemma 2.7 and [Bandeira et al. \(2014, Lemma 3.2\)](#): first, the result is stated for the true gradient, in alignment with what is needed to establish complexity bounds; second, it is inferred additionally that the trust-region radius is increased under the same condition.

LEMMA 2.7 Let Assumptions 2.2, 2.3 and 2.5 hold. If  $m_k$  is  $(\kappa_{eg}, \kappa_{ef})$ -fully linear on  $B(x_k, \delta_k)$  and

$$\delta_k < \kappa \|\nabla f(x_k)\|, \quad (2.2)$$

where

$$\kappa = \left( \kappa_{eg} + \max \left\{ \eta_2, \kappa_{bhm}, \frac{4\kappa_{ef}}{\kappa_{fcd}(1 - \eta_1)} \right\} \right)^{-1},$$

then at the  $k$ th iteration, the step is taken ( $x_{k+1} = x_k + s_k$ ) and  $\delta_k$  is increased.

*Proof.* From (2.2), one has

$$\kappa_{eg}\delta_k + \max \left\{ \eta_2, \kappa_{bhm}, \frac{4\kappa_{ef}}{\kappa_{fcd}(1 - \eta_1)} \right\} \delta_k < \|\nabla f(x_k)\|,$$

and from this and Definition 1.1,

$$\max \left\{ \eta_2, \kappa_{bhm}, \frac{4\kappa_{ef}}{\kappa_{fcd}(1 - \eta_1)} \right\} \delta_k < \|\nabla f(x_k)\| - \|g_k - \nabla f(x_k)\| \leq \|g_k\|.$$

Hence,

$$\delta_k < \min \left\{ \frac{1}{\eta_2}, \frac{1}{\kappa_{bhm}}, \frac{\kappa_{fcd}(1 - \eta_1)}{4\kappa_{ef}} \right\} \|g_k\|,$$

and from [Conn et al. \(2009b, Proof of Lemma 10.6\)](#), we obtain  $\rho_k \geq \eta_1$ . Because the first term in the minimum gives  $\eta_2\delta_k < \|g_k\|$ , the trust-region radius is increased.  $\square$

## 2.2 Behavior of the trust-region radius

We will now prove that the sequence of the trust-region radii is square summable and establish an explicit upper bound for the sum. Such a property is not standard in derivative-based trust-region algorithms, as the trust-region radius is not expected to go to zero. Yet the summability holds for Algorithm 2.4, because the algorithm does not increase the trust-region radius unless it is bounded by  $\eta_2^{-1}\|g_k\|$  (and the reduction ratio  $\rho_k$  is at least  $\eta_1$ ). The proof makes use of the set of indices corresponding to iterations where the trust-region radius is increased, that is,

$$\mathcal{H} = \{k \in \mathbb{N} : \rho_k \geq \eta_1 \text{ and } \|g_k\| \geq \eta_2\delta_k\}. \quad (2.3)$$

REMARK 2.8 In the context of direct search based on probabilistic descent ([Gratton et al., 2015, Lemma 4.1](#)), a similar result was proved for the sequence of the step size  $\alpha_k$ . There an iteration attains a

decrease of the order of  $\alpha_k^2$  whenever the step is taken, in which case such direct search algorithms always increase the step size. In the trust-region context, as detailed in the proof below, a decrease of the order of  $\delta_k^2$  corresponds only to the iterations where the trust-region radius is increased. There are iterations where the step is taken and the trust region is decreased, in which case the decrease obtained is not necessarily of this order.

LEMMA 2.9 Under Assumptions 2.2, 2.3 and 2.6, for any realization of Algorithm 2.4,

$$\sum_{k=0}^{\infty} \delta_k^2 \leq \beta := \frac{\gamma_2^2}{1 - \gamma_1^2} \left[ \frac{\delta_0^2}{\gamma_2^2} + \frac{f_0 - f_{\text{low}}}{\eta} \right],$$

where  $f_0 = f(x_0)$  and

$$\eta = \eta_1 \eta_2 \frac{\kappa_{fcd}}{2} \min \left\{ \frac{\eta_2}{\kappa_{bhm}}, 1 \right\}.$$

*Proof.* For any  $k \in \mathcal{K}$ , from (2.1),

$$\begin{aligned} f(x_k) - f(x_k + s_k) &\geq \eta_1 (m_k(x_k) - m_k(x_k + s_k)) \\ &\geq \eta_1 \frac{\kappa_{fcd}}{2} \eta_2 \min \left\{ \frac{\eta_2}{\kappa_{bhm}}, 1 \right\} \delta_k^2 = \eta \delta_k^2. \end{aligned}$$

Consequently, if we sum over all iteration indices in  $\mathcal{K}$  less than a given integer  $k$ , we obtain

$$\eta \sum_{\substack{j \in \mathcal{K} \\ j \leq k}} \delta_j^2 \leq \sum_{\substack{j \in \mathcal{K} \\ j \leq k}} [f(x_j) - f(x_{j+1})] \leq \sum_{j \leq k} [f(x_j) - f(x_{j+1})] = f_0 - f(x_{k+1}) \leq f_0 - f_{\text{low}},$$

leading to

$$\sum_{k \in \mathcal{K}} \delta_k^2 \leq \frac{f_0 - f_{\text{low}}}{\eta}.$$

From now on, the proof is as in the proof of Gratton *et al.* (2015, Lemma 4.1). Suppose first that  $\mathcal{K}$  has infinite cardinality, and let  $\mathcal{K} = \{k_1, k_2, \dots\}$  and, for auxiliary reasons,  $k_0 = -1$  and  $\delta_{-1} = \delta_0/\gamma_2$ . The sum  $\sum_{k=0}^{\infty} \delta_k^2$  can thus be rewritten as

$$\sum_{k=0}^{\infty} \delta_k^2 = \sum_{i=0}^{\infty} \sum_{k=k_i+1}^{k_{i+1}} \delta_k^2.$$

Additionally, one has for each index  $i$ ,  $\delta_k \leq \gamma_2(\gamma_1)^{k-k_i-1} \delta_{k_i}$  for  $k = k_i + 1, \dots, k_{i+1}$ . Hence,

$$\sum_{k=k_i+1}^{k_{i+1}} \delta_k^2 \leq \frac{\gamma_2^2}{1 - \gamma_1^2} \delta_{k_i}^2,$$

and we finally obtain the desired result:

$$\sum_{k=0}^{\infty} \delta_k^2 \leq \frac{\gamma_2^2}{1 - \gamma_1^2} \sum_{i=0}^{\infty} \delta_{k_i}^2 = \frac{\gamma_2^2}{1 - \gamma_1^2} \left[ \delta_{-1}^2 + \sum_{k \in \mathcal{K}} \delta_k^2 \right] \leq \frac{\gamma_2^2}{1 - \gamma_1^2} \left[ \frac{\delta_0^2}{\gamma_2^2} + \frac{f_0 - f_{\text{low}}}{\eta} \right].$$

In the case of a finite set  $\mathcal{K} = \{k_1, k_2, \dots, k_M\}$ , defining  $k_{M+1} = \infty$  allows us to reproduce the same reasoning as above, leading to an identical bound.  $\square$

This is a stronger version of the result  $\lim_{k \rightarrow \infty} \delta_k = 0$  proved in [Bandeira et al. \(2014\)](#) for the purpose of global convergence.

### 2.3 Number of iterations with fully linear models

Recall the set of indices (2.3) corresponding to iterations where the step is taken and the trust-region radius is increased, and let  $y_k$  denote the corresponding indicator ( $y_k = 1$  if  $k \in \mathcal{K}$ ,  $y_k = 0$  otherwise). Such choices will allow us to exploit the analysis of direct search based on probabilistic descent ([Gratton et al., 2015](#)), with  $\delta_k$  and  $\mathcal{K}$ , respectively, playing the roles of the step size and the set of successful iterations in [Gratton et al. \(2015\)](#).

Now let  $\nabla f(\tilde{x}_k)$ , with  $\tilde{x}_k \in \{x_0, \dots, x_k\}$ , represent a minimum norm gradient among  $\nabla f(x_0), \dots, \nabla f(x_k)$ . In addition, we define

$$p_0 = \frac{\ln(\gamma_1)}{\ln(\gamma_1/\gamma_2)}. \quad (2.4)$$

The quantity  $p_0 \in (0, 1)$  will emerge naturally in our upcoming analysis, as we will need the probability  $p$  in Definition 2.1 to be higher than  $p_0$  to render the complexity results. In practice, how to fulfill this requirement depends upon the model-generation scenario. If the probability under which these models are sufficiently accurate can be adjusted in advance, then for any particular choices of  $\gamma_1$  and  $\gamma_2$  ( $0 < \gamma_1 < 1 < \gamma_2$ ), we can adjust the probability so that it is higher than  $p_0$ . There may also exist scenarios where the model generation has a given nonadjustable probability of being sufficiently accurate, in which case one can then set  $\gamma_1, \gamma_2$  properly, so that the quantity  $p_0$  is smaller than this probability.

The next step in the analysis is to show, similarly to [Gratton et al. \(2015, Lemma 4.2\)](#), that the number of iterations that benefited from a fully linear model depends on the inverse of the square of  $\|\nabla f(\tilde{x}_k)\|$ , thus indicating that such a number is not necessarily low when the gradient is large.<sup>2</sup> The binary variable  $z_k$  below indicates whether  $m_k$  is  $(\kappa_{eg}, \kappa_{ef})$ -fully linear or not.

**LEMMA 2.10** Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. Given a realization of Algorithm 2.4 and a positive integer  $k$ ,

$$\sum_{l=0}^{k-1} z_l \leq \frac{\beta}{(\min\{\delta_0/\gamma_2, \kappa \|\nabla f(\tilde{x}_k)\|\})^2} + p_0 k.$$

<sup>2</sup> As opposed to Lemma 2.9, this result follows more directly from the theory in [Gratton et al. \(2015\)](#), given that the discrepancy mentioned in Remark 2.8 does not need special treatment.



*Proof.* For each  $l \in \{0, 1, \dots, k-1\}$ , define

$$v_l = \begin{cases} 1 & \text{if } \delta_l < \min\{\gamma_2^{-1}\delta_0, \kappa \|\nabla f(\tilde{x}_k)\|\}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

The proof relies then on  $z_l \leq (1 - v_l) + v_l y_l$ , which is true because, when  $v_l = 1$ , Lemma 2.7 implies that  $y_l \geq z_l$  (since  $\|\nabla f(\tilde{x}_k)\| \leq \|\nabla f(\tilde{x}_{k-1})\| \leq \|\nabla f(\tilde{x}_l)\|$ ). It suffices then to separately prove

$$\sum_{l=0}^{k-1} (1 - v_l) \leq \frac{\beta}{(\min\{\delta_0/\gamma_2, \kappa \|\nabla f(\tilde{x}_k)\|\})^2} \quad (2.6)$$

and

$$\sum_{l=0}^{k-1} v_l y_l \leq p_0 k. \quad (2.7)$$

Inequality (2.6) is justified by Lemma 2.9 and the fact that (2.5) implies

$$1 - v_l \leq \frac{\delta_l^2}{(\min\{\delta_0/\gamma_2, \kappa \|\nabla f(\tilde{x}_k)\|\})^2}.$$

The proof of inequality (2.7) is verbatim as in the proof of Gratton *et al.* (2015, Lemma 4.2), thanks to the definition of  $p_0$  given by (2.4).  $\square$

#### 2.4 Worst-case complexity and global rate

We will now consider events belonging to the  $\sigma$ -algebra generated by the random model sequence  $\{M_k\}$ . The probabilities will then be expressed with respect to the corresponding probability space (see Royer, 2016, Appendix A for more details).

From Lemma 2.10, one then has the following inclusion of events

$$\left\{ \|\nabla f(\tilde{X}_k)\| \geq \epsilon \right\} \subset \left\{ \sum_{l=0}^{k-1} Z_l \leq \frac{\beta}{\kappa^2 \epsilon^2} + p_0 k \right\}, \quad (2.8)$$

for any  $\epsilon$  satisfying

$$0 < \epsilon < \frac{\delta_0}{\kappa \gamma_2}. \quad (2.9)$$

On the other hand, the probabilistic behavior of the event on the right-hand side is known from the application of the Chernoff bound to the lower tail of  $\sum_{l=0}^{k-1} Z_l$  (see, e.g., Gratton *et al.*, 2015, Lemma 4.4). Note that the conditioning on the past from Assumption 2.1 appears in the proof of the result below.

LEMMA 2.11 Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. Suppose that  $\{M_k\}$  is  $(p)$ -probabilistically  $(\kappa_{eg}, \kappa_{ef})$ -fully linear and  $\lambda \in (0, p)$ . Then,

$$\pi_k(\lambda) := \mathbb{P}\left(\sum_{l=0}^{k-1} Z_l \leq \lambda k\right) \leq \exp\left[-\frac{(p-\lambda)^2}{2p}k\right].$$

Thus, when (2.9) holds and

$$k \geq \frac{2\beta}{(p-p_0)\kappa^2\epsilon^2}, \quad (2.10)$$

the inclusion (2.8) and the monotonicity of  $\pi_k(\cdot)$  will give us (setting  $\lambda = (p+p_0)/2$  in Lemma 2.11)

$$\begin{aligned} \mathbb{P}\left(\|\nabla f(\tilde{X}_k)\| \leq \epsilon\right) &\geq \mathbb{P}\left(\|\nabla f(\tilde{X}_k)\| < \epsilon\right) \geq 1 - \mathbb{P}\left(\sum_{l=0}^{k-1} Z_l \leq \frac{\beta}{\kappa^2\epsilon^2} + p_0 k\right) \\ &\geq 1 - \pi_k\left(\frac{\beta}{\kappa^2\epsilon^2} + p_0\right) \\ &\geq 1 - \pi_k\left(\frac{p-p_0}{2} + p_0\right) \\ &\geq 1 - \exp\left[-\frac{(p-p_0)^2}{8p}k\right], \end{aligned} \quad (2.11)$$

leading to the following global rate result.

THEOREM 2.12 Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. Suppose that  $\{M_k\}$  is  $(p)$ -probabilistically  $(\kappa_{eg}, \kappa_{ef})$ -fully linear with  $p > p_0$  and

$$k \geq \frac{2\beta\gamma_2^2}{(p-p_0)\delta_0^2}. \quad (2.12)$$

Then, the minimum gradient norm  $\|\nabla f(\tilde{X}_k)\|$  satisfies

$$\mathbb{P}\left(\|\nabla f(\tilde{X}_k)\| \leq \frac{\sqrt{2}\beta^{1/2}}{\kappa(p-p_0)^{1/2}} \frac{1}{\sqrt{k}}\right) \geq 1 - \exp\left[-\frac{(p-p_0)^2}{8p}k\right].$$

*Proof.* Let

$$\epsilon = \frac{\sqrt{2}\beta^{1/2}}{\kappa(p-p_0)^{1/2}} \frac{1}{\sqrt{k}}.$$

Then, (2.10) holds with equality, and  $\epsilon < \delta_0/(\kappa\gamma_2)$  is guaranteed by (2.12). Hence, (2.11) gives us the bound.  $\square$

We have thus proved a global rate of  $1/\sqrt{k}$  for the norm of the gradient with overwhelmingly high probability.

Similarly, one can prove a worst-case bound of the order of  $\mathcal{O}(\epsilon^{-2})$  for the first iteration index  $K_\epsilon$ , for which  $\|\nabla f(\tilde{X}_{K_\epsilon})\| \leq \epsilon$ , also with overwhelmingly high probability (and we note that  $K_\epsilon$  is a random variable due to the randomness of the models). The proof relies again on applying (2.11), on the observation that  $\mathbb{P}(K_\epsilon \leq k) = \mathbb{P}(\|\nabla f(\tilde{X}_k)\| \leq \epsilon)$ , and on taking  $k$  as

$$k = \left\lceil \frac{2\beta}{(p-p_0)\kappa^2\epsilon^2} \right\rceil.$$

**THEOREM 2.13** Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. Suppose that  $\{M_k\}$  is  $(p)$ -probabilistically  $(\kappa_{eg}, \kappa_{ef})$ -fully linear with  $p > p_0$  and that  $\epsilon$  satisfies (2.9). Then,  $K_\epsilon$  satisfies

$$\mathbb{P}\left(K_\epsilon \leq \left\lceil \frac{2\beta}{(p-p_0)\kappa^2\epsilon^2} \right\rceil\right) \geq 1 - \exp\left[-\frac{\beta(p-p_0)}{4p\kappa^2\epsilon^2}\right].$$

Results in expectation are a natural by-product of our analysis. It can be shown that  $\mathbb{E}(\|\nabla f(\tilde{X}_k)\|)$  is bounded by a function of the order of  $k^{-1/2} + \exp(-k)$  up to multiplicative constants (see Gratton *et al.*, 2015, Proposition 5.2). It is also possible to bound the expected value of  $K_\epsilon$  (Royer, 2016).

**THEOREM 2.14** Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. Suppose that  $\{M_k\}$  is  $(p)$ -probabilistically  $(\kappa_{eg}, \kappa_{ef})$ -fully linear with  $p > p_0$  and that  $\epsilon$  satisfies (2.9). Then,

$$\mathbb{E}(K_\epsilon) \leq c_1\epsilon^{-2} + \frac{1}{1 - \exp(-c_2)},$$

where

$$c_1 = \frac{2\beta}{(p-p_0)\kappa^2}, \quad c_2 = \frac{(p-p_0)^2}{8p}.$$

*Proof.* Since  $K_\epsilon$  takes only non-negative integer values, the expected value of  $K_\epsilon$  satisfies

$$\mathbb{E}(K_\epsilon) = \sum_{j=0}^{\infty} j\mathbb{P}(K_\epsilon = j) = \sum_{j \geq 0} \sum_{\substack{k \in \mathbb{N} \\ k < j}} \mathbb{P}(K_\epsilon = j) = \sum_{k \in \mathbb{N}} \sum_{\substack{j \geq 0 \\ k < j}} \mathbb{P}(K_\epsilon = j) = \sum_{k \in \mathbb{N}} \mathbb{P}(K_\epsilon > k).$$

Hence,

$$\begin{aligned} \mathbb{E}(K_\epsilon) &= \sum_{\substack{0 \leq k < c_1\epsilon^{-2} \\ k \in \mathbb{N}}} \mathbb{P}(K_\epsilon > k) + \sum_{\substack{k \geq c_1\epsilon^{-2} \\ k \in \mathbb{N}}} \mathbb{P}(K_\epsilon > k) \\ &\leq c_1\epsilon^{-2} + 1 + \sum_{\substack{k \geq c_1\epsilon^{-2} \\ k \in \mathbb{N}}} \mathbb{P}(K_\epsilon > k) \\ &= c_1\epsilon^{-2} + 1 + \sum_{\substack{k \geq c_1\epsilon^{-2} \\ k \in \mathbb{N}}} \mathbb{P}\left(\|\nabla f(\tilde{X}_k)\| > \epsilon\right). \end{aligned}$$

For any index  $k \geq c_1 \epsilon^{-2}$ , similar to (2.11), we have

$$\mathbb{P} \left( \|\nabla f(\tilde{X}_k)\| > \epsilon \right) \leq \exp(-c_2 k).$$

As a result,

$$\mathbb{E}(K_\epsilon) \leq c_1 \epsilon^{-2} + 1 + \sum_{\substack{k \geq c_1 \epsilon^{-2} \\ k \in \mathbb{N}}} \exp(-c_2 k) \leq c_1 \epsilon^{-2} + \sum_{k \in \mathbb{N}} \exp(-c_2 k) \leq c_1 \epsilon^{-2} + \frac{1}{1 - \exp(-c_2)},$$

which proves the desired result.  $\square$

The obtained bound is thus

$$\mathcal{O} \left( \frac{\kappa^{-2} \epsilon^{-2}}{p - p_0} \right) + \mathcal{O}(1),$$

which matches the results of [Cartis & Scheinberg \(2017\)](#) for line search methods based on probabilistic models (where  $p_0$  is taken as  $1/2$ ).

More generally, one observes that for both Theorems 2.13 and 2.14, the dominant term in iteration effort is

$$\mathcal{O} \left( (p - p_0)^{-1} \kappa^{-2} \epsilon^{-2} \right), \quad (2.13)$$

where  $\kappa = \mathcal{O}(1/\max\{\kappa_{ef}, \kappa_{eg}\})$  according to Lemma 2.7. As mentioned in the introductory section, one can ensure  $\max\{\kappa_{ef}, \kappa_{eg}\} = \mathcal{O}(\sqrt{n})$  in a deterministic setting with  $\mathcal{O}(n)$  points. Hence, in a simple scenario, where a random model is selected among a finite number of fixed fully linear models, the models are then almost always  $(\kappa_{eg}, \kappa_{ef})$ -fully linear, namely  $(p)$ -probabilistically  $(\kappa_{eg}, \kappa_{ef})$ -fully linear with  $p = 1$ , and one recovers from (2.13) the same iteration and function evaluation complexity bounds mentioned in the introduction for the deterministic setting, up to a constant factor  $1 - p_0$  that involves only algorithmic parameters. In a more general setting of probabilistic models, and when building linear models by randomly generating  $n$  sampling points, one can obtain  $(p)$ -probabilistically  $(\kappa_{eg}, \kappa_{ef})$ -fully linear models with  $\max\{\kappa_{ef}, \kappa_{eg}\} = \mathcal{O}(\sqrt{n})$  while  $p > p_0$  independent of  $n$  (see [Bandeira et al., 2014](#), Section 6.2, ‘Analysis of poisedness of random sample sets’). Hence, from (2.13), one again recovers the same iteration and function evaluation complexity bounds known for the deterministic setting.

## 2.5 A note on global convergence

Our complexity theory implies (see [Gratton et al., 2015](#), Proposition 5.1)

$$\mathbb{P} \left( \inf_{k \geq 0} \|\nabla f(X_k)\| = 0 \right) = 1.$$

If we assume for all realizations of Algorithm 2.4 that the iterates never arrive at a stationary point in a finite number of iterations, then the events  $\{\liminf_{k \rightarrow \infty} \|\nabla f(X_k)\| = 0\}$  and  $\{\inf_{k \geq 0} \|\nabla f(X_k)\| = 0\}$  are identical, and we recover the  $\liminf$  result with probability 1 of [Bandeira et al. \(2014, Theorem 4.2\)](#). Note

also that such a result could be derived even more directly using the argument of Gratton *et al.* (2015, Lemma 3.2 and Theorem 3.1). The article by Bandeira *et al.* (2014) goes further into the analysis and also establishes a lim-type result.

### 3. Complexity of second-order trust-region methods based on probabilistic models

The proof technology of Section 2 enables us to derive a similar complexity study for the class of trust-region algorithms under consideration but now with the goal of approaching or converging to second-order critical points. To do so, additional assumptions need to be enforced regarding both the quality of the models and the properties of the step resulting from the approximate solution of the trust-region subproblem. Moreover, instead of Assumption 2.5, the objective function will now be required to satisfy the following assumption.

**ASSUMPTION 3.1** The function  $f$  is twice continuously differentiable on  $\mathbb{R}^n$ , and its Hessian is Lipschitz continuous.

The probabilistic counterpart to the concept of fully quadratic models is given by Definition 1.2.

**DEFINITION 3.2** A sequence of random models  $\{M_k\}$  is said to be  $(p)$ -probabilistically  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic for a corresponding ball sequence  $\{B(X_k, \Delta_k)\}$  if the events

$$S_k = \{M_k \text{ is a } (\kappa_{eh}, \kappa_{eg}, \kappa_{ef})\text{-fully quadratic model of } f \text{ on } B(X_k, \Delta_k)\}$$

satisfy  $\mathbb{P}(S_0) \geq p$  and, for each  $k \geq 1$ , the following submartingale-like condition

$$\mathbb{P}(S_k \mid M_0, \dots, M_{k-1}) \geq p.$$

The authors of Bandeira *et al.* (2012) show how to build fully quadratic models with high probability from quadratic interpolation and uniformly generated sample sets. It was also proved there that such a procedure may recover such models with considerably less than  $(n+1)(n+2)/2$  function evaluations when the Hessian of the function is sparse.

#### 3.1 Algorithm and assumptions

As before, we consider quadratic models around the iterate  $x_k$ , with the same definitions for  $m_k$ ,  $g_k$  and  $H_k$  (and the imposition of Assumption 2.2). As curvature is now present in our analysis, we will make use of the notation  $\tau_k = \lambda_{\min}(H_k)$ . The solution of the trust-region subproblem has now to be second-order accurate, in the sense of Assumption 3.3 below (which replaces Assumption 2.3).

**ASSUMPTION 3.3** For every  $k$ , and for all realizations  $m_k$  of  $M_k$  (and of  $X_k$  and  $\Delta_k$ ), we are able to compute a step  $s_k$  so that it satisfies both a *fraction of Cauchy decrease* and a *fraction of eigendecrease*, i.e.,

$$m(x_k) - m(x_k + s_k) \geq \frac{\kappa_{fod}}{2} \max \left\{ \|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \delta_k \right\}, -\tau_k \delta_k^2 \right\}. \quad (3.1)$$

for some constant  $\kappa_{fod} \in (0, 1]$ .

The first part of (3.1) can be satisfied by a Cauchy step, while considering a step of norm  $\delta_k$  along an eigenvector of the model Hessian associated with the eigenvalue  $\tau_k$  yields a second-order decrease in  $\delta_k^2$ . A step satisfying (3.1) can be obtained by taking the one corresponding to the largest decrease caused in the model value.

Such considerations lead us from Algorithm 2.4 to Algorithm 3.5, preserving the overall structure of the method (and, in particular, the updating rules for the trust-region radius). As mentioned in the introduction, we make use of the second-order criticality measure

$$\sigma(x) = \max \{ \|\nabla f(x)\|, -\lambda_{\min}(\nabla^2 f(x)) \},$$

for which a natural estimator at  $x_k$  is

$$\sigma^m(x_k) := \max \{ \|g_k\|, -\tau_k \}.$$

In the case of a fully quadratic model, the two quantities are related as follows.

LEMMA 3.4 (Conn et al., 2009b, Lemma 10.15) Let Assumption 3.1 hold. If  $m_k$  is  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic on  $B(x_k, \delta_k)$ , then

$$|\sigma(x_k) - \sigma^m(x_k)| \leq \kappa_\sigma \delta_k, \quad (3.2)$$

where  $\kappa_\sigma = \max \{ \kappa_{eg} \delta_{\max}, \kappa_{eh} \}$ .

ALGORITHM 3.5 (A simple second-order derivative-free trust-region framework).

Fix parameters  $\eta_1, \eta_2, \delta_{\max} > 0$  and  $0 < \gamma_1 < 1 < \gamma_2$ . Select initial  $x_0$  and  $\delta_0 \in (0, \delta_{\max})$ .

For  $k = 0, 1, 2, \dots$ ,

Build a quadratic model  $m_k(x_k + s)$  of  $f$ , and compute  $s_k$  by approximately minimizing  $m_k$  in  $B(x_k, \delta_k)$  so that it satisfies (3.1).

If  $\rho_k \geq \eta_1$ , set  $x_{k+1} = x_k + s_k$  and

$$\delta_{k+1} = \begin{cases} \min \{ \gamma_2 \delta_k, \delta_{\max} \} & \text{if } \sigma^m(x_k) \geq \eta_2 \delta_k, \\ \gamma_1 \delta_k & \text{otherwise.} \end{cases}$$

Otherwise, set  $x_{k+1} = x_k$  and  $\delta_{k+1} = \gamma_1 \delta_k$ .

End For

LEMMA 3.6 Let Assumptions 2.2, 3.1 and 3.3 hold. If  $m_k$  is  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic on  $B(x_k, \delta_k)$  and

$$\delta_k < \kappa_{2\text{nd}} \sigma(x_k),$$

where

$$\kappa_{2\text{nd}} = \left( \kappa_\sigma + \max \left\{ \eta_2, \kappa_{bhm}, \frac{4\kappa_{ef}\delta_{\max}}{\kappa_{fod}(1-\eta_1)}, \frac{4\kappa_{ef}}{\kappa_{fod}(1-\eta_1)} \right\} \right)^{-1},$$

then at the  $k$ th iteration the step is taken ( $x_{k+1} = x_k + s_k$ ) and  $\delta_k$  is increased.

*Proof.* The proof is similar to the one of Lemma 2.7. Combining (3.2) with the error bound of Lemma 3.4 yields

$$\delta_k < \min \left\{ \frac{1}{\eta_2}, \frac{1}{\kappa_{bhm}}, \frac{\kappa_{fod}(1-\eta_1)}{4\kappa_{ef}}, \frac{\kappa_{fod}(1-\eta_1)}{4\kappa_{ef}\delta_{\max}} \right\} \sigma^m(x_k).$$

This allows us to directly conclude that  $\rho_k \geq \eta_1$  (see Conn *et al.*, 2009b, Lemma 10.17). Besides, since the first term in the minimum gives  $\eta_2\delta_k \leq \sigma^m(x_k)$ , the trust-region radius is increased.  $\square$

### 3.2 Behavior of the trust-region radius

Given the decrease properties now enforced by the algorithm, similar to Lemma 2.9, we can prove that the sequence of the trust-region radii is cube summable. Let  $\mathcal{K}_{2nd}$  be the set of indices corresponding to iterations where the step is taken and the trust-region radius is increased, i.e.,

$$\mathcal{K}_{2nd} = \{k \in \mathbb{N} : \rho_k \geq \eta_1 \text{ and } \sigma^m(x_k) \geq \eta_2\delta_k\}.$$

LEMMA 3.7 Under Assumptions 2.2, 2.6 and 3.3, for any realization of Algorithm 3.5,

$$\sum_{k=0}^{\infty} \delta_k^3 \leq \beta_{2nd} := \frac{\gamma_2^3}{1-\gamma_1^3} \left[ \frac{\delta_0^3}{\gamma_2^3} + \frac{f_0 - f_{\text{low}}}{\eta_{2nd}} \right],$$

where

$$\eta_{2nd} = \eta_1\eta_2 \frac{\kappa_{fod}}{2} \min \left\{ \min \left\{ \frac{\eta_2}{\kappa_{bhm}}, 1 \right\} \frac{1}{\delta_{\max}}, 1 \right\}.$$

*Proof.* Similar to the proof of Lemma 2.9, we need to consider only the case where  $|\mathcal{K}_{2nd}| = \infty$ . For any  $k \in \mathcal{K}_{2nd}$ , if  $\sigma^m(x_k) = \|g_k\|$ , we have by Assumption 3.3 that

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq \eta_1 [m_k(x_k) - m_k(x_k + s_k)] \\ &= \eta_1 \frac{\kappa_{fod}}{2} \|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \delta_k \right\} \\ &\geq \eta_1\eta_2 \frac{\kappa_{fod}}{2} \min \left\{ \frac{\eta_2}{\kappa_{bhm}}, 1 \right\} \delta_k^2 \\ &\geq \eta_1\eta_2 \frac{\kappa_{fod}}{2} \min \left\{ \frac{\eta_2}{\kappa_{bhm}}, 1 \right\} \frac{1}{\delta_{\max}} \delta_k^3. \end{aligned}$$

Meanwhile, if  $\sigma^m(x_k) = -\tau_k$ , similar reasoning leads to

$$f(x_k) - f(x_{k+1}) \geq \eta_1\eta_2 \frac{\kappa_{fod}}{2} \delta_k^3.$$

As a result, for any index  $k \in \mathcal{K}_{2nd}$ , one has

$$f(x_k) - f(x_{k+1}) \geq \eta_1\eta_2 \frac{\kappa_{fod}}{2} \min \left\{ \min \left\{ \frac{\eta_2}{\kappa_{bhm}}, 1 \right\} \frac{1}{\delta_{\max}}, 1 \right\} \delta_k^3 = \eta_{2nd} \delta_k^3.$$

The rest of the proof follows the lines of Lemma 2.9 replacing the squares of the trust-region radii by cubes.  $\square$

### 3.3 Number of iterations with fully quadratic models

An upper bound on the number of iterations where the models are fully quadratic is derived similarly to the first-order case. The main changes consist in defining  $\tilde{x}_k \in \{x_0, \dots, x_k\}$ , such that  $\sigma(\tilde{x}_k)$  is a minimum value among  $\{\sigma(x_0), \dots, \sigma(x_k)\}$  and  $z_k$  as the binary variable indicating whether  $m_k$  is  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic or not.

LEMMA 3.8 Let Assumptions 2.2, 2.6, 3.1 and 3.3 hold. Given a realization of Algorithm 3.5 and a positive integer  $k$ ,

$$\sum_{l=0}^{k-1} z_l \leq \frac{\beta_{2nd}}{(\min\{\delta_0/\gamma_2, \kappa_{2nd}\sigma(\tilde{x}_k)\})^3} + p_0 k.$$

### 3.4 Worst-case complexity and global rate

The derivation of the second-order complexity theory is based on observing that Lemma 3.8 now implies

$$\left\{ \|\sigma(\tilde{X}_k)\| \geq \epsilon \right\} \subset \left\{ \sum_{l=0}^{k-1} Z_l \leq \frac{\beta_{2nd}}{\kappa_{2nd}^3 \epsilon^3} + p_0 k \right\},$$

for any  $\epsilon$  satisfying

$$0 < \epsilon < \frac{\delta_0}{\kappa_{2nd} \gamma_2}. \quad (3.3)$$

Then, a result identical to Lemma 2.11 can be ensured considering the definition of  $Z_l$  based on fully quadratic models and replacing the probabilistic fully linear assumption by the probabilistic fully quadratic one. The rest of the analysis proceeds similarly with minor changes in constants. The global rate is now  $1/\sqrt[3]{k}$ , as shown below.

THEOREM 3.9 Let Assumptions 2.2, 2.6, 3.1 and 3.3 hold. Suppose that  $\{M_k\}$  is  $(p)$ -probabilistically  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic with  $p > p_0$  and

$$k \geq \frac{2\beta_{2nd}\gamma_2^3}{(p - p_0)\delta_0^3}.$$

Then, the minimum second-order criticality measure  $\sigma(\tilde{X}_k)$  satisfies

$$\mathbb{P}\left(\sigma(\tilde{X}_k) \leq \frac{\sqrt[3]{2}\beta_{2nd}^{1/3}}{\kappa_{2nd}(p - p_0)^{1/3}} \frac{1}{\sqrt[3]{k}}\right) \geq 1 - \exp\left[-\frac{(p - p_0)^2}{8p}k\right].$$

Similar conclusions to those of Section 2.5 can be drawn here regarding global convergence results of the liminf type that can be deduced from the above theorem and regarding their interplay with the



second-order convergence theory of [Bandeira \*et al.\* \(2014\)](#). However, a significant difference from the first-order case is the absence of a rigorous proof for deriving a lim-type result ([Bandeira \*et al.\*, 2014](#)).

As in the first-order case, we can establish a worst-case complexity bound with overwhelmingly high probability, but now of the order of  $\epsilon^{-3}$ .

**THEOREM 3.10** Let Assumptions 2.2, 2.6, 3.1 and 3.3 hold. Suppose that  $\{M_k\}$  is  $(p)$ -probabilistically  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic with  $p > p_0$  and that  $\epsilon$  satisfies (3.3). Let  $K_\epsilon$  be the first iteration index, for which  $\sigma(\bar{X}_{K_\epsilon}) \leq \epsilon$ . Then,  $K_\epsilon$  satisfies

$$\mathbb{P}\left(K_\epsilon \leq \left\lceil \frac{2\beta_{2nd}}{(p-p_0)\kappa_{2nd}^3\epsilon^3} \right\rceil\right) \geq 1 - \exp\left[-\frac{\beta_{2nd}(p-p_0)}{4p\kappa_{2nd}^3\epsilon^3}\right].$$

As in Theorem 2.14, we can also bound the expected number of iterations needed to reach the desired accuracy.

**THEOREM 3.11** Let Assumptions 2.2, 2.6, 3.1 and 3.3 hold. Suppose that  $\{M_k\}$  is  $(p)$ -probabilistically  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic with  $p > p_0$  and that  $\epsilon$  satisfies (3.3). Then,

$$\mathbb{E}(K_\epsilon) \leq c_3\epsilon^{-3} + \frac{1}{1 - \exp(-c_2)},$$

where

$$c_3 = \frac{2\beta_{2nd}}{(p-p_0)\kappa_{2nd}^3},$$

and  $c_2$  is defined as in Theorem 2.14.

The iteration complexity bound is thus essentially  $\mathcal{O}((p-p_0)^{-1}\kappa_{2nd}^{-3}\epsilon^{-3})$ , where we observe from Lemma 3.6 that  $\kappa_{2nd} = \mathcal{O}(1/\max\{\kappa_{ef}, \kappa_{eg}, \kappa_{eh}\})$ . Similarly to the first-order case, we can recover the deterministic bounds for iteration and function evaluation complexity mentioned in the introduction when the models are almost always  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic. Such models are  $(p)$ -probabilistically  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic with  $p = 1$  and  $\max\{\kappa_{ef}, \kappa_{eh}, \kappa_{eg}\} = \mathcal{O}(n)$  and can be built selecting among a finite number of fixed fully quadratic models using  $\mathcal{O}(n^2)$  points.

The implication for Theorems 3.10 and 3.11 on the function evaluation complexity becomes more interesting in a broader probabilistic setting. It was proved in [Bandeira \*et al.\* \(2012\)](#) that  $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ -fully quadratic models with  $\max\{\kappa_{eh}, \kappa_{eg}, \kappa_{ef}\} = \mathcal{O}(n)$  can be obtained with high probability (yet less than 1) using fewer than  $\mathcal{O}(n^2)$  sample points if the objective function has a sparse Hessian. For instance, when the Hessian is tridiagonal, only  $\mathcal{O}(n(\log^4 n))$  points are needed, and  $p$  can be chosen so that  $p - p_0$  is positive and independent of  $n$ . Hence, Theorems 3.10 and 3.11 lead to a function evaluation complexity bound of

$$\mathcal{O}(n(\log^4 n) \max\{\kappa_{ef}^3, \kappa_{eg}^3, \kappa_{eh}^3\}\epsilon^{-3}) = \mathcal{O}((n \log n)^4 \epsilon^{-3}),$$

which compares favorably in terms of  $n$  with the bound mentioned in the introduction for the deterministic case.

#### 4. Remarks on open questions

Recently, a number of articles have appeared proposing and analysing derivative-free trust-region methods for the unconstrained optimization of a stochastic function. In this setting, what is observable is  $\tilde{f}(x, \varepsilon(\omega))$ , where  $\varepsilon$  is a random variable. The objective function  $f(x)$  may be given by  $\mathbb{E}(\tilde{f}(x, \varepsilon))$ . One approach (Shashaani *et al.*, 2016) extended the framework (Conn *et al.*, 2009a) using sample average approximation (SAA). The number of observations in each Monte Carlo oracle may be up to  $\mathcal{O}(\delta_k^{-4})$ . First-order global convergence was proved with probability 1 but for algorithmic parameters that depend on unknown problem constants. Another approach (Chen *et al.*, 2017) extended trust-region methods based on probabilistic models (Bandeira *et al.*, 2014) to cover also probabilistic estimates of the objective function. In the nonbiased case with  $f(x) = \mathbb{E}(\tilde{f}(x, \varepsilon))$ , the probabilistic assumptions can be ensured by SAA within  $\mathcal{O}(\delta_k^{-4})$  observations. This approach can also handle biased cases like failures in function evaluations or even processor failures (thus accommodating gradient failures when using finite differences). First-order global convergence was also proved with probability 1 but again for algorithmic parameters that depend on unknown problem constants. A similar approach by Larson & Billups (2016) led to first-order global convergence in probability (weaker than with probability 1) but under more practical assumptions. Very recently, an article by Blanchet *et al.* (2016) developed a complexity analysis for the approach in Chen *et al.* (2017) showing a complexity bound of  $\mathcal{O}(\epsilon^{-2})$  on the expected number of iterations needed to satisfy  $\|\nabla f(X_k)\| \leq \epsilon$  (again under unverifiable assumptions on algorithmic parameters).

It is an open question whether our proof technology can improve upon these stochastic optimization approaches in the sense of establishing global convergence with probability 1 and global rates and complexity bounds with overwhelmingly high probability without unverifiable assumptions on algorithmic parameters. Another challenging prospect for future work is to develop better rates and bounds for the convex and strongly convex cases for either deterministic or stochastic functions.

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