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On the complexity of approximating the maximal inscribed ellipsoid for a polytope

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We give a new polynomial bound on the complexity of approximating the maximal inscribed ellipsoid for a polytope.

Key words: Maximal inscribed ellipsoid, maximal inscribed paraboloid, path-following Newton's method, computational complexity.

1. Introduction

Let Q be a full-dimensional polytope in $\mathbb{R}^n$ defined by m linear inequalities

$$Q = \{x \in \mathbb{R}^n \mid c_i^{\mathrm{T}} x \leqslant 1, i = 1, \ldots, m\}. \tag{1.1}$$

In this paper, we shall study the complexity of the following extremal geometric problem:

Problem I. Given a polytope (1.1) and a relative accuracy $\gamma \in (0, 1)$ in the volume, find an ellipsoid E, contained in the polytope, such that

$$\operatorname{vol} E / \operatorname{vol} E^* \geqslant \gamma,$$

where E^* is the ellipsoid of maximum volume inscribed in Q.

One of the motivations for studying the complexity of Problem I is that it appears as a basic subroutine at each iteration of the method of inscribed ellipsoids [11],

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which achieves relative error ε of minimization of an arbitrary nondifferentiable convex function F on Q in at most

$$6.64n \ln(1/\varepsilon)$$

iterations. At each iteration of this method it is required to solve Problem I with accuracy $\gamma = 0.99$ for a current polytope localizing the set of extrema, and to evaluate F and the subgradient of F at the center of E. Note that, similar to the method of volumetric centers by Vaidya [13], the method of inscribed ellipsoids is an optimal method for convex programming in terms of the order of the number of iterations. In particular, good algorithms for solving Problem I may prove useful for developing efficient methods for decomposition and nondifferential convex optimization (for a similar motivation see also [5, 9, 14]).

Another useful application of γ-maximal ellipsoids is related to the fact that they give "well rounding" affine transformations for convex bodies [6]. More precisely, it is known [11] that

$$E \subseteq Q \subseteq n\left(\frac{1+3\sqrt{1-\gamma}}{\gamma}\right)E \tag{1.2}$$

for an arbitrary n-dimensional body Q, where λE stands for the ellipsoid obtained by the homothetic dilatation of E by a factor of λ.

Extremal inscribed and circumscribed ellipsoids are also used for approximating reachability regions for linear control problems, in optimal design [12], and in some other applications. We also find Problem I interesting in itself.

In this paper we show that a γ-maximal ellipsoid for a polytope (1.1) can be computed in at most

$$\mathrm{O}\left(m^{3.5} \ln\left[\frac{mR}{\ln(1/\gamma)}\right] \ln\left[\frac{n \ln R}{\ln(1/\gamma)}\right]\right), \tag{1.3}$$

arithmetical operations, where R is an a priori known ratio of the radii of two Euclidean balls, the first of which is circumscribed about Q and the second inscribed in Q. This improves by a factor of m the best previously known complexity bound for the problem due to Nesterov and Nemirovsky [7]. We also show that the computational cost of the mth iteration of the method of inscribed ellipsoids can be bounded by $\mathrm{O}(m^{3.5} \ln m \ln \ln n)$ operations. Note that for the method of inscribed ellipsoids one can assume without loss of generality that $\gamma = 0.99$, $R = 3n$, $m = \mathrm{O}(n \ln n)$.

The paper is organized as follows: In Section 2 we consider four computational problems of finding extremal ellipsoids for convex polytopes and show that these problems can be reduced in linear time to Problem I. In Section 3 Problem I is formulated as a convex program with nonlinear constraints. In Section 4 we describe an algorithm ElliP that reduces Problem I to a small number of special convex programming problems P with linear constraints. The number k of subproblems P

is indeed very small: $k \leqslant 12$ for the method of inscribed ellipsoids with $n \leqslant 10^6$ variables. In Section 5 Algorithm ElliP is viewed in geometrical terms. It turns out that each problem P can be interpreted as the problem of inscribing the maximal paraboloid in a polyhedral cone defined by the pair (Q, b), where b is an interior point of the polytope Q. In Section 6 we bound the complexity of Problem P by using the general path-following Newton's method stated in [7]. The number of Newton iterations of the method does not exceed $O(m^{1/2}\ln[mR/\ln(1/\gamma)])$ and, though the number of unknowns in the problem grows as $\frac{1}{2}n(n+1)+n \approx \frac{1}{2}n^2$, the computational cost of one iteration can be bounded by $O(m^3)$ arithmetical operations. To prove the latter, we develop a system of $n+m$ linear equations with $n+m$ unknowns to compute the Newton direction, which is similar to but simpler than the system suggested in [7] for the case of $\frac{1}{2}n(n+1)$ unknowns. In Section 7 we obtain the upper bound (1.3) on the complexity of Problem I. Section 8 of the paper contains some concluding remarks and open questions.

2. Extremal ellipsoids

Let Q be a convex body in $\mathbb{R}^n$. It is known that

- among the ellipsoids E, centred at a given point $a \in \operatorname{int} Q$ and inscribed in Q, there exists a unique ellipsoid $E^*(a)$ of maximum volume;
- there exists a unique maximal ellipsoid

$$E^* = \operatorname{argmax}\{\operatorname{vol} E \mid E \subseteq Q\}$$

for Q [3].

Let $\gamma \in (0, 1]$. An ellipsoid E, inscribed in Q, is called γ-maximal for Q, if $\operatorname{vol} E \geqslant \gamma \cdot \operatorname{vol} E^*$. We say that E is (γ, a)-maximal for Q, if E is centered at a, $E \subseteq Q$, and $\operatorname{vol} E \geqslant \gamma \cdot \operatorname{vol} E^*(a)$.

Similarly,

- among the ellipsoids, centred at a and circumscribed about Q, there exists a unique ellipsoid $E_*(a)$ of minimum volume;
- there exists a unique minimal ellipsoid

$$E_* = \operatorname{argmin}\{\operatorname{vol} E \mid Q \subseteq E\}.$$

for any arbitrary convex body Q. Moreover, the center of E_* is an interior point of Q [3].

Again, let $\gamma \in (0, 1]$ be a given relative accuracy in the volume. An ellipsoid E, containing Q, is said to be γ-minimal for Q if $\gamma \cdot \operatorname{vol} E \leqslant \operatorname{vol} E_*$. Next, E is said to be (γ, a)-minimal for Q if E is centered at a, $E \supseteq Q$, and $\gamma \cdot \operatorname{vol} E \leqslant \operatorname{vol} E_*(a)$.

Suppose without loss of generality that Q contains the origin $a = 0$ as an interior point and consider the following four computational problems.

Problem I. Given $\gamma \in (0, 1)$ and a full-dimensional polytope Q in $\mathbb{R}^n$ defined by m linear inequalities

$$Q = \{x \in \mathbb{R}^n \mid c_i^{\mathrm{T}} x \leq 1, i = 1, \ldots, m\}, \tag{2.1}$$

find a γ-maximal ellipsoid for Q.

Problem $\mathbf{I_0}$ ("centered" version of I). Find a $(\gamma, 0)$-maximal ellipsoid for (2.1).

Problem C. Given $\gamma \in (0, 1)$ and a full-dimensional polytope Q defined as the convex hull of m points in $\mathbb{R}^n$,

$$Q = \text{conv.hull}\{d_1, \ldots, d_m\}, \tag{2.2}$$

find a γ-minimal ellipsoid for Q.

Problem $\mathbf{C_0}$ ("centered" version of C). Find a $(\gamma, 0)$-minimal ellipsoid for (2.2).

In this section we describe five geometric transformations

$$\mathrm{C} \rightleftarrows \mathrm{C}_0 \rightleftarrows \mathrm{I}_0 \rightarrow \mathrm{I}. \tag{2.3}$$

yielding "linear-time" reductions among the above listed computational problems. We begin with the reduction $\mathrm{C}(n, m, \gamma) \rightarrow \mathrm{C}_0(n+1, 2m, \gamma)$, suggested for the case $\gamma = 1$ by Titterington [12].

Suppose we wish to compute a γ-minimal ellipsoid for a given n-dimensional polytope (2.2), containing the origin as an interior point. Let us introduce a new "vertical" coordinate x_{n+1}, and consider the $(n+1)$-dimensional polytope

$$Q' = \text{conv.hull}\{\pm(d_1, 1), \ldots, \pm(d_m, 1)\}, \tag{2.4}$$

still containing $0 \in \mathbb{R}^{n+1}$ as an interior point. Let E' be a $(\gamma, 0)$-minimal ellipsoid for Q'. Then the intersection of E' with the hyperplane $\Pi = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$ gives an n-dimensional ellipsoid E which is γ-minimal for Q. Indeed, $E' \supseteq Q'$ if and only if $E' \cap \Pi \supseteq Q$. Moreover,

$$\mathrm{vol}_{n+1} E' = \mathrm{const}(n) \cdot \mathrm{vol}_n[E' \cap \Pi] \cdot v(h)$$

where $h > 1$ is the "height" of E', and

$$v(h) = h^{n+1}(h^2 - 1)^{-n/2} \geq v(\sqrt{n+1}).$$

In particular, if E'_* is the $(1, 0)$-minimal ellipsoid for Q', then the height of E'_* equals $\sqrt{n+1}$, and $E'_* \cap \Pi$ is the minimal ellipsoid for Q. Furthermore,

$$\frac{\mathrm{vol}_n(E'_* \cap \Pi)}{\mathrm{vol}_n(E' \cap \Pi)} = \frac{\mathrm{vol}_{n+1} E'_*}{\mathrm{vol}_{n+1} E'} \cdot \frac{v(h)}{v(\sqrt{n+1})} \geq \gamma,$$

i.e., $E' \cap \Pi$ is γ-minimal for Q.

The reverse reduction $C_0(n, m, \gamma) \to C(n, 2m, \gamma)$ is simpler. In order to determine a $(\gamma, 0)$-minimal ellipsoid for (2.2), it suffices to find a γ-minimal ellipsoid E for the polytope

$$Q_{\pm} = \text{conv.hull}\{\pm d_1, \ldots, \pm d_m\},$$

and shift E to the origin. It is easy to see that the shifted ellipsoid $\frac{1}{2}[E + (-E)]$ still contains the centrally symmetric polytope $Q_{\pm}$, and by definition this ellipsoid is $(\gamma, 0)$-minimal for both $Q_{\pm}$ and Q.

Thus, Problem C is equivalent to its "centered" version.

The equivalence $I_0(n, m, \gamma) \rightleftarrows C_0(n, m, \gamma)$ follows by standard polarity arguments: An ellipsoid E is inscribed in

$$Q = \{x \in \mathbb{R}^n \mid c_i^T x \leq 1, i, \ldots, m\} \tag{2.1}$$

if and only if its polar

$$E^\circ = \{y \in \mathbb{R}^n \mid x^T y \leq 1 \text{ for all } x \in E\}$$

contains the polar

$$Q^\circ = \text{conv.hull}\{c_1, \ldots, c_m\} \tag{2.5}$$

of the polytope Q. If E is centered at the origin, then E° is centered at the origin as well, and

$$\text{vol}\, E \cdot \text{vol}\, E^\circ = \mu_n^2,$$

where μ_n is the volume of the unit n-dimensional Euclidean ball. Therefore E is $(\gamma, 0)$-maximal for (2.1) if and only if E° is $(\gamma, 0)$-minimal for (2.5).

Now for the last reduction $I_0(n, m, \gamma) \to I(n, 2m, \gamma)$: to find a $(\gamma, 0)$-maximal ellipsoid E for (2.1), one can compute a γ-maximal ellipsoid E for the centrally symmetric polytope

$$Q^{\pm} = \{x \in \mathbb{R}^n \mid \pm c_i^T x \leq 1, i = 1, \ldots, m\},$$

and translate E to the origin.

We do not know whether there exists a reduction $I \to I_0$, similar to the geometric reductions (2.3); see also Question 2 in Section 8. Henceforth we focus on the computational complexity of Problem I, the most difficult among our four computational problems. In Sections 4 and 5 we will reduce Problem I to a small number of subproblems P, each of which can be interpreted as a problem of the approximate computation of the maximal paraboloid inscribed in a polyhedral cone.

In most of the paper, we need the following technical assumption: Q contains the unit Euclidean ball $B_1 = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ and is contained in the Euclidean ball $B_R = \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$ of a given radius R,

$$B_1 \subseteq Q \subseteq B_R. \tag{2.6}$$

Note that by means of dilatations we can keep R constant in the reductions $C \leftarrow C_0 \rightleftarrows I_0 \to I$. In the reduction $C \to C_0$ we have $R' = \sqrt{R^2 + 1} < 2R$ for the polytope Q', see (2.4).

3. The problem of finding the maximal inscribed ellipsoid as a convex programming problem

Problem I can be reformulated as a convex program [9, 10]. This can be done as follows.

An arbitrary ellipsoid E in $\mathbb{R}^n$ can be given in the form $E=\{x \mid x=a+Bz, \|z\| \leqslant 1\}$ where $a \in \mathbb{R}^n$ is the center of E and B is an $n \times n$ symmetric positive definite matrix. Thus E is the image of the Euclidean unit ball $\{z \mid \|z\| \leqslant 1\}$ shifted to the point a after the linear transformation B. In particular, in the above representation the support function of the ellipsoid has the form

$$\phi_E(c)=\max\{c^{\mathrm{T}}x \mid x \in E\}=c^{\mathrm{T}}a+\|c^{\mathrm{T}}B\|,$$

and its volume is given by vol $E=\mu_n \det B$. Hence, in order to find a γ-maximal ellipsoid E for (2.1) it suffices to solve the convex program

$$\begin{aligned} &f(B)=-\ln \det B \to \min, \\ &c_i^{\mathrm{T}}a+\|c_i^{\mathrm{T}}B\| \leqslant 1, \quad i=1,\ldots,m, \end{aligned} \tag{3.1}$$

with the unknowns $a \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n(n+1)/2}$, to an absolute accuracy of $\ln(1/\gamma)$ in the functional. Here B is a symmetric positive definite matrix of order n. The convexity of $f(B)$ on the set of positive definite matrices is well known [2].

Letting $A=B^2$, (3.1) can be rewritten as

$$\begin{aligned} &f(A)=-\ln \det A \to \min, \\ &c_i^{\mathrm{T}}Ac_i \leqslant (1-c_i^{\mathrm{T}}a)^2, \quad i=1,\ldots,m. \end{aligned} \tag{3.2}$$

Problem (3.3) is also formulated in [5]. Note that unlike those of (3.1), the constraints of (3.2) are not convex. However, for a fixed a (say $a=0$) these constraints are linear in A. In particular, Problems C, C_0 and I_0 can be reduced to the problem of minimizing $f(A)$ with A subject to linear constraints (see [1] for the case I_0).

The convex programming problem (3.1) can be solved by the ellipsoid method, which yields a polynomial (but poor) bound

$$n^6(n^2+m)\ln[Rn/\ln(1/\gamma)]$$

on the number of arithmetical operations sufficient to solve Problems I and C [10, 11]. For reasonable m, this result was substantially improved by Nesterov and Nemirovsky [7]. Using a path-following Newton's method for minimizing the function

$$F_t=-\ln \det A-t\sum_{i=1}^{m}\ln[(1-c_i^{\mathrm{T}}a)^2-c_i^{\mathrm{T}}Ac_i]$$

with penalty parameter $t \downarrow 0$, see (3.2), they reduced the arithmetical complexity of computing a γ-maximal ellipsoid to the bound

$$\mathrm{O}(m^{4.5}\ln[Rm/\ln(1/\gamma)]). \tag{3.3}$$

Applying the same approach to the centered version of the problem ($a=0$), Nesterov and Nemirovsky also obtained a better bound

$$\mathrm{O}(m^{3.5}\ln[Rm/\ln(1/\gamma)]) \tag{3.4}$$

on the complexity of finding a γ-minimal ellipsoid for a polytope. In both cases the bound $\mathrm{O}(m^{0.5}\ln[Rm/\ln(1/\gamma)])$ on the number of Newton steps is "standard" [8]. However, in the case $a=0$, where the constraints of (3.2) are linear, the corresponding linear system for computing Newton's direction is simpler. In time $\mathrm{O}(m^3)$ this system can be rewritten as a linear system in m unknowns, and consequently, it can be solved in $\mathrm{O}(m^3)$ operations. For the general case, where a is not fixed, Nesterov and Nemirovsky described a more complicated method, which requires $\mathrm{O}(m^4)$ operations per Newton iteration [7, pp. 163–188].

In this paper we reduce (3.2) to a small number of subproblems

$$\begin{aligned} \mathrm{P}(b):\quad & f(A)=-\ln\det A\to\min,\\ & c_i^{\mathrm{T}}Ac_i\leqslant(1-c_i^{\mathrm{T}}a)(1-c_i^{\mathrm{T}}b),\quad i=1,\ldots,m, \end{aligned} \tag{3.5}$$

with fixed values of $b\in\mathbb{R}^n$. So each $\mathrm{P}(b)$ is a problem of minimizing $f(A)$ with A and a subject to linear constraints. This allows us to bring down the complexity of finding a γ-maximal ellipsoid for a polytope to a bound close to (3.4).

4. Reduction I→P

Consider the following algorithm ElliP for computing a γ-maximal ellipsoid for a polytope Q.

Step 0. Set

$$\delta:=\tfrac{1}{3}\ln(1/\gamma),$$

$$k:=0,$$

$$b_k:=b_0:=\text{an arbitrary interior point in } Q.$$

Step 1. Find an approximate solution $a_k=a(b_k)$ and $A_k=A(b_k)$ to Problem $\mathrm{P}(b_k)$, see (3.5), with absolute error δ in the functional.

Step 2. Update

$$b_{k+1}:=\tfrac{1}{2}(b_k+a_k),$$

$$k:=k+1.$$

Go to Step 1 and start a new iteration.

Let

$$\gamma(b)=\operatorname{vol}E^*(b)/\operatorname{vol}E^* \tag{4.1}$$

be the ratio of the volumes of the $(1, b)$-maximal and the maximal ellipsoids for Q. So $\gamma(b) \in (0,1]$ for all $b \in \operatorname{int} Q$, and $\gamma(b)=1$ if and only if $b=a^*$, where a^* is the center of the maximal ellipsoid E^* for Q.

Theorem 1. *Algorithm ElliP converges in at most $\kappa+1$ iterations, where*

$$\kappa = \lceil \log(\ln(1/\gamma(b_0))/\ln(1/\gamma)) \rceil + 1, \tag{4.2}$$

with the ellipsoid

$$E_\kappa = \{x \in \mathbb{R}^n \mid x = b_{\kappa+1} + A_\kappa^{1/2} z,\ \|z\| \leq 1\}$$

γ-maximal for Q.

Proof. Let us first show that $E_\kappa \subseteq Q$. By the definition of A_κ (see Step 1 in the description of the algorithm) we have

$$c_i^{\mathrm{T}} A_\kappa c_i \leq (1 - c_i^{\mathrm{T}} a_\kappa)(1 - c_i^{\mathrm{T}} b_\kappa), \quad i = 1, \ldots, m.$$

Since

$$(1 - c_i^{\mathrm{T}} a)(1 - c_i^{\mathrm{T}} b) \leq (1 - c_i^{\mathrm{T}}[\tfrac{1}{2}(a+b)])^2 \tag{4.3}$$

for all $a, b \in Q$, we conclude that

$$c_i^{\mathrm{T}} A_\kappa c_i \leq (1 - c_i^{\mathrm{T}}[\tfrac{1}{2}(a_\kappa + b_\kappa)])^2 = (1 - c_i^{\mathrm{T}} b_{\kappa+1})^2,$$

see Step 2. Therefore

$$\|c_i^{\mathrm{T}} A_\kappa^{1/2}\| \leq 1 - c_i^{\mathrm{T}} b_{\kappa+1}, \quad i = 1, \ldots, m.$$

This proves the inclusion $E_\kappa \subseteq Q$, see (3.1).

To prove the γ-maximality of E_κ, consider the function $\phi : Q \times Q \to \mathbb{R}$ defined as

$$\phi(a, b) = \min\{f(A) \mid c_i^{\mathrm{T}} A c_i \leq (1 - c_i^{\mathrm{T}} a)(1 - c_i^{\mathrm{T}} b),\ i = 1, \ldots, m\}.$$

Here $f(A) = -\ln \det A$ and A is a symmetric positive definite matrix of order n. We need the following property of ϕ: For all $a, b \in Q$,

$$\phi(\tfrac{1}{2}(a+b), \tfrac{1}{2}(a+b)) \leq \phi(a, b) \leq \tfrac{1}{2}(\phi(a, a) + \phi(b, b)). \tag{4.4}$$

The first inequality of (4.4) follows from (4.3). To prove the second inequality suppose that A and B are the optimal matrices for (a, a) and (b, b):

$$\begin{aligned} f(A) &= \phi(a, a), \qquad c_i^{\mathrm{T}} A c_i \leq (1 - c_i^{\mathrm{T}} a)^2, \\ f(B) &= \phi(b, b), \qquad c_i^{\mathrm{T}} B c_i \leq (1 - c_i^{\mathrm{T}} b)^2. \end{aligned} \tag{4.5}$$

We can assume without loss of generality that A and B are diagonal matrices. Multiplying the inequalities (4.5) for each $i = 1, \ldots, m$, we get

$$c_i^{\mathrm{T}} (AB)^{1/2} c_i \leq [(c_i^{\mathrm{T}} A c_i)(c_i^{\mathrm{T}} B c_i)]^{1/2} \leq (1 - c_i^{\mathrm{T}} a)(1 - c_i^{\mathrm{T}} b).$$

Hence,

$$\phi(a, b) \leq f((AB)^{1/2}) = \tfrac{1}{2}[\phi(a, a) + \phi(b, b)].$$

Observe that the first inequality of (4.4) implies that the minimum of $\phi(a, b)$ on $Q \times Q$ is attained at (a^*, a^*):

$$\begin{aligned}\min\{\phi(a, b) \mid a, b \in Q\} &= \min\{\phi(a, a) \mid a \in Q\} \\ &= \phi(a^*, a^*) = f(A^*) = -2 \ln(\text{vol } E^*/\mu_n).\end{aligned}$$

Here $E^* = \{x \in \mathbb{R}^n \mid x = a^* + (A^*)^{1/2} z, \|z\| \leq 1\}$ is the maximal ellipsoid for Q.

Now we can prove the inequality

$$\ln \frac{\text{vol } E^*}{\text{vol } E_\kappa} = \tfrac{1}{2}[f(A_\kappa) - f(A^*)] \leq \ln(1/\gamma), \tag{4.6}$$

equivalent to the γ-maximality of E_κ. From the description of the algorithm we know that

$$f(A_\kappa) \leq \min\{\phi(a, b_\kappa) \mid a \in Q\} + \delta \leq \phi(b_\kappa, b_\kappa) + \delta,$$

see Step 1. Hence

$$\ln \frac{\text{vol } E^*}{\text{vol } E_\kappa} \leq \tfrac{1}{2}[\phi(b_\kappa, b_\kappa) - \phi(a^*, a^*)] + \tfrac{1}{2}\delta. \tag{4.7}$$

Let

$$\xi_\kappa = \phi(b_\kappa, b_\kappa) - \phi(a^*, a^*) = 2 \ln(1/\gamma(b_\kappa)),$$

see (4.1). From the description of ElliP and (4.4) we have

$$\begin{aligned}\xi_\kappa &= \phi(\tfrac{1}{2}(a_{\kappa-1} + b_{\kappa-1}), \tfrac{1}{2}(a_{\kappa-1} + b_{\kappa-1})) - \phi(a^*, a^*) \\ &\leq \phi(a_{\kappa-1}, b_{\kappa-1}) - \phi(a^*, a^*) \\ &\leq \min\{\phi(a, b_{\kappa-1}) \mid a \in Q\} + \delta - \phi(a^*, a^*) \\ &\leq \min\{\tfrac{1}{2}(\phi(a, a) + \phi(b_{\kappa-1}, b_{\kappa-1})) \mid a \in Q\} + \delta - \phi(a^*, a^*) \\ &= \tfrac{1}{2}(\phi(b_{\kappa-1}, b_{\kappa-1}) - \phi(a^*, a^*)) + \delta \\ &= \tfrac{1}{2}\xi_{\kappa-1} + \delta.\end{aligned}$$

The latter recurrence implies

$$\xi_\kappa \leq 2^{-\kappa}\xi_0 + \delta(1 + 2^{-1} + \cdots + 2^{-\kappa+1}) < 2^{-\kappa}\xi_0 + 2\delta.$$

Now from (4.7) and (4.1) it follows that

$$\ln \frac{\text{vol } E^*}{\text{vol } E_\kappa} \leq 2^{-\kappa} \ln(1/\gamma(b_0)) + \tfrac{3}{2}\delta.$$

Since $\delta = \frac{1}{3} \ln(1/\gamma)$, we obtain (4.6) from (4.2). This completes the proof of the theorem. $\square$

Selecting $b_0 = 0$ as the starting point for ElliP, we obtain from (2.6) and (4.2),

$$\kappa \leq \left\lceil \log\left[\frac{2n \ln R}{\ln(1/\gamma)}\right]\right\rceil. \tag{4.8}$$

This already low upper bound on the number of iterations of the algorithm ElliP can be still lowered in the case where the algorithm is applied as a subroutine in the method of inscribed ellipsoids [8]. At the sth step of this method we have a γ-maximal ellipsoid E^s for a polytope Q^s. Next we pass a halfspace $\pi^s = \{x \in \mathbb{R}^n \mid g_s^{\mathrm{T}}(x - a^s) \geq 0\}$ through the center a^s of E^s, and compute a new γ-maximal ellipsoid E^{s+1} for the polytope $Q^{s+1} = Q^s \cap \pi^s$. Since Q^{s+1} contains the half ellipsoid $E^s \cap \pi^s$, we can start ElliP for Q^{s+1} at the center b_0^s of the maximal ellipsoid inscribed in the half ellipsoid $E^s \cap \pi^s$. Clearly, the point b_0^s can be computed in $O(n^2)$ operations, and

$$\gamma(b_0^s) \geq 0.5\gamma n^{-1/2}$$

as one can inscribe an ellipsoid E of volume $0.5\mu_n n^{-1/2}$ in the halfball

$$\{x \in \mathbb{R}^n \mid \|x\| \leq 1, x_1 \geq 0\}$$

(place the center of E at $(n^{-1/2}, 0, \ldots, 0)$). In the method of inscribed ellipsoids we also fix $\gamma = 0.99$, see [11]. This yields the following bound

$$\kappa + 1 \leq 9 + \log(1 + \ln\sqrt{n}) \tag{4.9}$$

on the number of subproblems $P(b_0), \ldots, P(b_\kappa)$ that are to be solved at each step of the method inscribed ellipsoids. In particular, $\kappa + 1 \leq 12$ for $n \leq 10^6$.

We show in Sections 6 and 7 that the arithmetical complexity of each problem $\mathrm{P}(b)$ does not exceed (3.4). Before that, however, we shall describe a geometric interpretation of these problems.

5. Geometric interpretation

5.1. Paraboloids

Let $\mathcal{P}$ be the set of vertical paraboloids in $\mathbb{R}^{n+1}$ tangent to the hyperplane

$$\Pi = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}.$$

An arbitrary paraboloid $P \in \mathcal{P}$ can be represented in the form

$$P = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \geq \tfrac{1}{4}(x - a)^{\mathrm{T}} A^{-1}(x - a) + 1\}, \tag{5.1}$$

where A is an $n \times n$ symmetric positive definite matrix and $a \in \mathbb{R}^n$ is the "center" of the paraboloid. We call the quantity $V(P) = \mu_n(\det A)^{1/2}$ the "volume" of P. Geometrically, $V(P)$ is the n-volume of the ellipsoid obtained by intersecting P with the hyperplane $x_{n+1} = \frac{5}{4}$.

Let $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Consider the halfspace π in $\mathbb{R}^{n+1}$ of the form

$$\pi = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \geqslant c^{\mathrm{T}}x + d\}.$$

Clearly,

$$P \subseteq \pi \quad \text{if and only if} \quad c^{\mathrm{T}}Ac \leqslant 1 - d - c^{\mathrm{T}}a. \tag{5.2}$$

Note that the constraints (5.2) are linear in A and a.

5.2. Shadows of paraboloids

For a point $b \in \mathbb{R}^n$ denote by $S = S(b)$ the projection of the paraboloid P from the point $(b, 0)$ onto the hyperplane Π,

$$S = \text{conv.hull}\{P \cup (b, 0)\} \cap \Pi.$$

We call S the "b-shadow" of P, see Figure 1. It is easy to see that

$$S \text{ is a } n\text{-dimensional ellipsoid centered at the point } \tfrac{1}{2}(a+b), \tag{5.3}$$

$$\text{vol } S = \mu_n([1 + \tfrac{1}{4}(a-b)^{\mathrm{T}}A^{-1}(a-b)] \det A)^{1/2}. \tag{5.4}$$

Indeed, $x \in S$ if and only if the ray

$$(b, 0) + t \cdot (x - b, 1), \quad t \in [0, \infty), \tag{5.5}$$

meets P, or equivalently the quadratic inequality

$$t \geqslant \tfrac{1}{4}[(x-b)t - (a-b)]^{\mathrm{T}}A^{-1}[(x-b)t - (a-b)] + 1$$

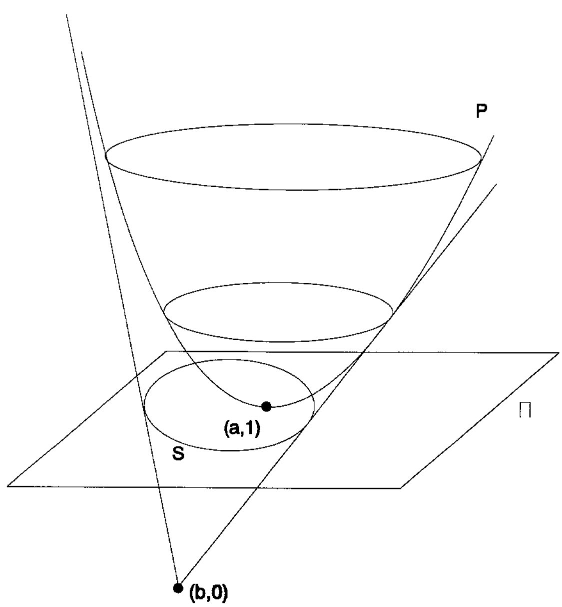

Fig. 1.

has real roots t. Therefore $x \in S$ is equivalent to

$$y^{\mathrm{T}}A^{-1}y(1+\xi^{\mathrm{T}}A^{-1}\xi) \leqslant (1+\xi^{\mathrm{T}}A^{-1}y)^2$$

with $y = x - b$ and $\xi = \frac{1}{2}(a-b)$. The latter inequality can be written as

$$y^{\mathrm{T}}[A^{-1}(1+\xi^{\mathrm{T}}A^{-1}\xi) - A^{-1}\xi\xi^{\mathrm{T}}A^{-1}]y \leqslant 1 + 2\xi^{\mathrm{T}}A^{-1}y.$$

Since

$$A^{-1} - \frac{A^{-1}\xi\xi^{\mathrm{T}}A^{-1}}{1+\xi^{\mathrm{T}}A^{-1}\xi} = (A+\xi\xi^{\mathrm{T}})^{-1},$$

we see that $x \in S$ if and only if

$$(x - \tfrac{1}{2}(a+b))^{\mathrm{T}}[A + \tfrac{1}{2}(a-b)\tfrac{1}{2}(a-b)^{\mathrm{T}}]^{-1}(x - \tfrac{1}{2}(a+b)) \leqslant 1.$$

This proves (5.3); (5.4) follows from

$$\det(A+\xi\xi^{\mathrm{T}}) \equiv (1+\xi^{\mathrm{T}}A^{-1}\xi) \cdot \det A.$$

Note that (5.4) shows that the volume of any b-shadow of P exceeds the "volume" $\mu_n(\det A)^{1/2}$ of P, and that the latter quantity is the linear approximation to vol S for a close to b.

5.3. *Geometric interpretation of the problem* P(b)

Let Q be a given polytope in $\mathbb{R}^n$. Suppose that $b \in \operatorname{int} Q$, and consider the polyhedral cone $K(b) \subset \mathbb{R}^{n+1}$ such that it has the vertex $(b, 0)$ and $K(b) \cap \Pi = Q$. In other words, $K(b)$ is the union of all rays (5.5) which intersect a copy of Q placed in the hyperplane Π. If Q is given by (1.1), then $K(b)$ is defined by the following system of linear inequalities

$$x_{n+1} \geqslant \frac{c_i^{\mathrm{T}}(x-b)}{1-c_i^{\mathrm{T}}b}, \quad i = 1, \ldots, m. \tag{5.6}$$

Now from (5.2) it follows that a paraboloid P of the form (5.1) is contained in the cone (5.6) if and only if

$$c_i^{\mathrm{T}}Ac_i \leqslant (1-c_i^{\mathrm{T}}a)(1-c_i^{\mathrm{T}}b), \quad i = 1, \ldots, m.$$

Therefore Problem P(b), see (2.5), can be interpreted as the problem of finding the maximal paraboloid $P \in \mathcal{P}$ inscribed in the cone $K(b)$.

Remark. It can be shown that for an arbitrary $b \in \operatorname{int} Q$ such a maximal paraboloid is unique. Note also that $P \subseteq K(b)$ if and only if the b-shadow of P, the ellipsoid S, is inscribed in the polytope Q.

5.4. *Geometric interpretation of the reduction* I→P

The iterative procedure ElliP can be interpreted as follows. We select an interior point b_0 in Q and inscribe a $\gamma^{1/3}$-maximal paraboloid P_0 in the cone $K(b_0)$. Maximizing the "volume" of $P \subseteq K(b_0)$, we maximize the linear approximation (5.4) to the volume of its b_0-shadow in Q. Next we move to the center $b_1 = \frac{1}{2}(b_0 + a_0)$ of the b_0-shadow of P_0 (see (5.3) and Step 2 of ElliP), and start the procedure anew. "Looking at Q from b_1," we inscribe a $\gamma^{1/3}$-maximal paraboloid P_1 in the cone $K(b_1)$ and so on. As we know from Theorem 1, the shadows $S(b_0), S(b_1), \ldots$ of the paraboloids $P_0, P_1, \ldots$ converge in a small number of iterations to a γ-maximal ellipsoid for Q. (In fact, instead of the shadows

$$S(b_k) = \{x \in \mathbb{R}^n \mid (x - b_{k+1})^{\mathrm{T}}[A_k + \tfrac{1}{2}(a_k - b_k)\tfrac{1}{2}(a_k - b_k)^{\mathrm{T}}]^{-1}(x - b_{k+1}) \leq 1\},$$

we used in the proof of Theorem 1 the smaller ellipsoids

$$E_k = \{x \in \mathbb{R}^n \mid (x - b_{k+1})^{\mathrm{T}} A_k^{-1}(x - b_{k+1}) \leq 1\} \subset S(b_k),$$

which also converge to a γ-maximal ellipsoid for Q. This observation can be used to improve the convergence of ElliP.)

6. The complexity of finding a γ-maximal paraboloid for a polyhedral cone

The change of variables

$$\begin{aligned} &A \to A, \\ &a \to a - b, \qquad\qquad (6.1)\\ &c_i \to \frac{c_i}{1 - c_i^{\mathrm{T}} b}, \quad i = 1, \ldots, m \end{aligned}$$

transforms Problem P(b), see (3.5), into the following standard problem P = P(0):

$$\begin{aligned} &-\ln \det A \to \min, \\ &c_i^{\mathrm{T}} A c_i + c_i^{\mathrm{T}} a \leq 1, \quad i = 1, \ldots, m. \end{aligned} \qquad (6.2)$$

The latter problem can be solved by the barrier method, stated in Section 3 of [7] for the general convex programming problem

$$f(x) \to \min, \quad x \in G \subset \mathbb{R}^N, \qquad (6.3)$$

with a thrice differentiable convex objective function f. The method is a special Newton procedure that follows the central path of the minimizers $x(t)$ of the function

$$F_t(x) = f(x) + t g(x), \quad t \downarrow 0.$$

Here g is a barrier function for the convex feasible region G.

For the applicability of the method it suffices to check the following three conditions:

(C1) *g is strongly self-concordant on* int $G \neq \emptyset$. By definition this means that
 (a) $g : \text{int } G \to \mathbb{R}$ is convex and thrice differentiable;
 (b) the level sets $\{x \in \text{int } G \mid g(x) \leq l\}$ of g are closed in $\mathbb{R}^N$ for each $l \in \mathbb{R}$;
 (c) for any $x \in \text{int } G$ and $h \in \mathbb{R}^N$,

$$|D^3 g(x)[h, h, h]| \leq 2(D^2 g(x)[h, h])^{3/2}. \tag{6.4}$$

(C2) *g is a σ-self-concordant barrier for G with some* $\sigma \geq 1$, i.e.,

$$\lambda(x) \leq \sigma^{1/2} \tag{6.5}$$

for all $x \in \text{int } G$, where by definition

$$\lambda(x) = \min\{\lambda \geq 0 \mid \forall h \in \mathbb{R}^N : |Dg(x)[h]| \leq \lambda (D^2 g(x)[h, h])^{1/2}\}. \tag{6.6}$$

(C3) *f is β-compatible with g for some* $\beta \geq 0$. This condition means that:

 (a) f is lower semicontinuous and convex on G, and finite and thrice differentiable on int G;
 (b) for all $x \in \text{int } G$ and $h \in \mathbb{R}^N$,

$$|D^3 f(x)[h, h, h]| \leq \beta(3D^2 f(x)[h, h])(3D^2 g(x)[h, h])^{1/2}. \tag{6.7}$$

Suppose that the conditions (C1)-(C3) are satisfied. Then, given an initial point $x_0 \in \text{int } G$ and an absolute error $\delta > 0$, the barrier method can produce an approximate solution

$$x^\delta \in \text{int } G, \quad f(x^\delta) \leq \min\{f(x) \mid x \in G\} + \delta,$$

to the problem (6.3) in at most

$$\mathrm{O}\left(\sigma^{1/2} \ln\left[\frac{\sigma V_g(f)}{\delta(1 - \pi_{x_g}(x_0))}\right]\right) \tag{6.8}$$

Newton iterations applied to convex combinations of f and g (or of g and some linear form). Here

(i) $\mathrm{O}(\cdot)$ depends on β;

(ii) $x_g \in \text{int } G$ is the (unique) minimizer of g;

(iii) $$V_g(f) = \sup\{f(x) \mid x \in W_{1/2}(x_g)\} - \inf\{f(x) \mid x \in W_{1/2}(x_g)\} \tag{6.9}$$

is the variation of f on the ellipsoid $W_{1/2}(X_g)$, where

$$W_r(x) = \{y \in \mathbb{R}^n \mid D^2 g(x)[y - x, y - x] < r^2\}; \tag{6.10}$$

and

(iv) $$\pi_x(y) = \inf\{t \geq 0 \mid x + t^{-1}(y - x) \in G\} \tag{6.11}$$

is the Minkowski function of G with the pole at x.

Moreover, it is shown in [7] that under the assumption (C1),

$$W_1(x) \subset G \tag{6.12}$$

for all $x \in \operatorname{int} G$.

To apply these results to our problem (6.2) we set

$$\begin{aligned} &x = (A, a), \\ &G = \{(A, a) \,|\, (A, a) \text{ satisfies the constraints (6.2) and} \\ &\qquad\qquad A \text{ is positive definite and symmetric}\}, \\ &f = -\ln \det A, \\ &g = f - \sum_{i=1}^{m} \ln(1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i). \end{aligned} \tag{6.13}$$

Let us first check the conditions (C1)-(C3).

(C1) Clearly, the conditions (a) and (b) are satisfied. The Taylor expansion of g has the form

$$\begin{aligned} g(A+H, a+h) = g(A, a) - \operatorname{tr}\{A^{-1}H\} &+ \sum_{i=1}^{m} \left[\frac{c_i^{\mathrm{T}} h + c_i^{\mathrm{T}} H c_i}{1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i}\right] \\ +\tfrac{1}{2} \operatorname{tr}\{(A^{-1}H)^2\} &+ \tfrac{1}{2} \sum_{i=1}^{m} \left[\frac{c_i^{\mathrm{T}} h + c_i^{\mathrm{T}} H c_i}{1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i}\right]^2 \\ -\tfrac{1}{3} \operatorname{tr}\{(A^{-1}H)^3\} &+ \tfrac{1}{3} \sum_{i=1}^{m} \left[\frac{c_i^{\mathrm{T}} h + c_i^{\mathrm{T}} H c_i}{1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i}\right]^3 + \cdots. \end{aligned} \tag{6.14}$$

Hence we get (6.4) and (c):

$$\begin{aligned} |D^3 g(A, a)[(H, h), (H, h), (H, h)]| &= \left| -2 \operatorname{tr}\{(A^{-1}H)^3\} + 2 \sum_{i=1}^{m} \left[\frac{c_i^{\mathrm{T}} h + c_i^{\mathrm{T}} H c_i}{1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i}\right]^3 \right| \\ &= \left| -2 \operatorname{tr}\{X^3\} + 2 \sum_{i=1}^{m} \psi_i^3 \right| \\ &\leq 2 |\operatorname{tr}\{X^3\}| + 2 \sum_{i=1}^{m} |\psi_i|^3 \\ &\leq 2 (\operatorname{tr}\{X^2\})^{3/2} + 2 \left(\sum_{i=1}^{m} \psi_i^2 \right)^{3/2} \\ &\leq 2 \left(\operatorname{tr}\{X^2\} + \sum_{i=1}^{m} \psi_i^2 \right)^{3/2} \\ &= 2 (\|X\|^2 + \|\psi\|^2)^{3/2} \\ &= 2 (D^2 g(A, a)[(H, h), (H, h)])^{3/2}. \end{aligned}$$

Here

$$X = A^{-1/2} H A^{-1/2},$$

$$\psi_i = \frac{c_i^{\mathrm{T}} h + c_i^{\mathrm{T}} H c_i}{1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i}, \quad i = 1, \ldots m,$$

$$\psi = (\psi_1, \ldots, \psi_m),$$

and $\|\cdot\|$ stands for the ℓ_2-norm.

(C2) Let us prove (6.5) for $\sigma = 2m$. By the definition (6.6) of the Newton decrement $\lambda(A, a)$ of g at (A, a) we have

$$\begin{aligned}
\lambda(A, a) &= \max_{H,h} \frac{|Dg(A, a)[(H, h)]|}{(D^2 g(A, a)[(H, h), (H, h)])^{1/2}} \\
&\leqslant \max_{X,\psi} \frac{|-\mathrm{tr}\{X\} + \psi_1 + \cdots + \psi_m|}{(\|X\|^2 + \|\psi\|^2)^{1/2}} \\
&\leqslant (n+m)^{1/2} < (2m)^{1/2}.
\end{aligned}$$

The last inequality follows from the fact that the polytope (2.1) is bounded and $n + 1 \leqslant m$.

(C3) f is $(2 \cdot 3^{-3/2})$-compatible with g:

$$\begin{aligned}
&|D^3 f(A, a)[(H, h), (H, h), (H, h)]| \\
&\quad = 2|\mathrm{tr}\{(A^{-1}H)^3\}| \\
&\quad = 2|\mathrm{tr}\{X^3\}| \leqslant 2 \cdot 3^{-3/2}(3\|X\|^2)(3\|X\|^2)^{1/2} \\
&\quad < 2 \cdot 3^{-3/2}(3\|X\|^2)(3\|X\|^2 + 3\|\psi\|^2)^{1/2} \\
&\quad = 2 \cdot 3^{-3/2}(3D^2 f(A, a)[(H, h), (H, h)])(3D^2 g(A, a)[(H, h), (H, h)])^{1/2}.
\end{aligned}$$

To use the bound (6.8) on the number of Newton iterations we have to obtain upper bounds on the quantities $V_g(f)$ and $\pi_{x_g}(x_0)$. We first prove the following two lemmas.

Lemma 6.1. *Let $x_g = (A_g, a_g)$ be the minimizer of the function g defined in* (6.13). *Then*

$$A_g^{-1} = \sum_{i=1}^{m} c_i c_i^{\mathrm{T}} / (1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i), \tag{6.15}$$

$$0 = \sum_{i=1}^{m} c_i / (1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i), \tag{6.16}$$

$$n + m = \sum_{i=1}^{m} 1 / (1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i), \tag{6.17}$$

$$\omega_*(A_g) > \left(2m \cdot \max_{i=1,\ldots,m} \|c_i\|^2\right)^{-1}, \tag{6.18}$$

where $\omega_*(\cdot)$ is the minimal eigenvalue of $(\cdot)$.

Proof. The equations (6.15) and (6.16) are the first-order optimality conditions $\partial g/\partial H = 0$ and $\partial g/\partial h = 0$, see (6.14). Multiplying (6.15) by A_g and (6.16) by a_g^{T}, we get (6.17):

$$n = \mathrm{tr}(A_g^{-1}A_g) = \mathrm{tr}\left\{\sum_{i=1}^{m} \frac{c_i c_i^{\mathrm{T}} A_g}{1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i}\right\} = \mathrm{tr}\left\{\sum_{i=1}^{m} \frac{c_i a_g^{\mathrm{T}} + c_i c_i^{\mathrm{T}} A_g}{1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i}\right\}$$

$$= \sum_{i=1}^{m} \frac{c_i^{\mathrm{T}} a_g + c_i^{\mathrm{T}} A_g c_i}{1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i} = -m + \sum_{i=1}^{m} 1/(1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i).$$

The last inequality (6.18) can be obtained as follows:

$$[\omega_*(A_g)]^{-1} = \max\{x^{\mathrm{T}} A_g^{-1} x \mid x \in \mathbb{R}^n,\ \|x\| = 1\}$$

$$= \max\left\{\sum_{i=1}^{m} \frac{(c_i^{\mathrm{T}} x)^2}{1 - c_i^{\mathrm{T}} a_g - c_i^{\mathrm{T}} A_g c_i} \;\middle|\; x \in \mathbb{R}^n,\ \|x\| = 1\right\} \quad \text{(by (6.15))}$$

$$\leqslant (n+m) \max_i \max\{(c_i^{\mathrm{T}} x)^2 \mid x \in \mathbb{R}^n,\ \|x\| = 1\} \quad \text{(by (6.17))}$$

$$= (n+m) \max_i \|c_i\|^2 < 2m \cdot \max_i \|c_i\|^2. \qquad \square$$

Lemma 6.2. *Suppose that Q is contained in some Euclidean ball of radius R. Then*

$$\omega^*(A) \leqslant \tfrac{25}{16} R^2 \tag{6.19}$$

for all feasible points $(A, a) \in G$, where $\omega^*(\cdot)$ is the maximal eigenvalue of $(\cdot)$.

Proof. From the definition of paraboloids (5.1) it follows that for any feasible point $(A, a) \in G$ the ellipsoid $E = \{x \in \mathbb{R}^n \mid (x-a)^{\mathrm{T}} A^{-1} (x-a) \leqslant 1\}$ is contained in the intersection of the cone $K = K(0)$ with the hyperplane $x_{n+1} = \frac{5}{4}$. Hence E can be covered by a copy of the polytope $\frac{5}{4}Q$, and consequently by an Euclidean ball of radius $\frac{5}{4}R$. $\square$

Now we can prove that

$$V_g(f) \leqslant 2n \ln(2mR \max\|c_i\|); \tag{6.20}$$

see (6.9) and (6.10) for the definition of $V_g(f)$. Indeed, (6.12) implies that $\omega_*(A) \geqslant \frac{1}{2}\omega_*(A_g)$ for all $(A, a) \in W_{1/2}(A_g, a_g)$. Since

$$-n \ln \omega^*(A) \leqslant f = -\ln \det A \leqslant -n \ln \omega_*(A),$$

(6.20) follows from (6.18) and (6.19).

Let us select the pair

$$A_0 = 0.5I/\max\|c_i\|^2,$$

$$a_0 = 0,$$

as the starting point x_0 for the barrier method (I is the identity matrix of order n). Since

$$c_i^{\mathrm{T}} A_0 c_i + c_i^{\mathrm{T}} a_0 \leq \tfrac{1}{2}, \quad i = 1, \ldots, m,$$

and

$$\omega_*(A_0) = 0.5/\max\|c_i\|^2,$$
$$\omega^*(A_g) \leq \tfrac{25}{16} R^2,$$

we get from the definition (6.11) of the Minkowski function

$$\frac{1}{1 - \pi_{x_g}(x_0)} \leq 4R^2 \max\|c_i\|^2. \tag{6.21}$$

Letting $\delta = \ln(1/\gamma)$ and substituting (6.20) and (6.21) in (6.8), we see that the number of Newton iterations of the barrier method for inscribing a γ-maximal paraboloid in the cone $K(0)$ does not exceed

$$\mathrm{O}(m^{1/2} \ln[mR \max\|c_i\|/\ln(1/\gamma)]). \tag{6.22}$$

The arithmetical cost of one Newton iteration can be bounded by $\mathrm{O}(m^3)$ operations, as in the case $a \equiv 0$, $h \equiv 0$ considered in [7]. Indeed, let F be a convex combination of g, f and some linear form of the variables, say

$$F = -\ln \det A - \tau \sum_{i=1}^{m} \ln(1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i) + \langle L, A\rangle + \langle l, a\rangle.$$

Here L is a given symmetric matrix of order n, l is a given n-dimensional vector, and

$$\langle L, A\rangle = \mathrm{tr}(LA),$$
$$\langle l, a\rangle = l^{\mathrm{T}} a,$$

stands for the inner product in the Euclidean space (A, a). Then

$$\begin{aligned} DF(A, a)[H, h] &= -\mathrm{tr}(A^{-1}H) + \tau \sum_{i=1}^{m} \frac{c_i^{\mathrm{T}} h + c_i^{\mathrm{T}} H c_i}{1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i} + \langle L, H\rangle + \langle l, h\rangle \\ &= \left\langle -A^{-1} + L + \tau \sum_{i=1}^{m} C_i \Delta_i^{-1}, H \right\rangle + \left\langle l + \tau \sum_{i=1}^{m} c_i \Delta_i^{-1}, h \right\rangle, \end{aligned}$$

where

$$C_i = c_i c_i^{\mathrm{T}} \quad \text{and} \quad \Delta_i = 1 - c_i^{\mathrm{T}} a - c_i^{\mathrm{T}} A c_i, \qquad i = 1, \ldots, m.$$

Therefore both the A-component

$$\nabla_A = -A^{-1} + L + \tau \sum_{i=1}^{m} \Delta_i^{-1} C_i$$

and the a-component

$$\nabla_a = l + \tau \sum_{i=1}^{m} \Delta_i^{-1} c_i$$

of the gradient of F can be computed at any point (A, a) in $O(mn^2)$ arithmetical operations.

Furthermore,

$$D^2F(A, a)[(H, h), (H, h)] = \langle A^{-1}HA^{-1}, H\rangle + \tau \sum_{i=1}^{m} \Delta_i^{-2}(\langle C_i, H\rangle + \langle c_i, h\rangle)^2,$$

and to find the Newton direction (H°, h°) of F at (A, a) one has to solve the following system of linear equations:

$$\begin{aligned} &\tfrac{1}{2}\frac{\partial D^2F(A, a)[(H, h), (H, h)]}{\partial H} \\ &\quad = A^{-1}HA^{-1} + \tau \sum_{i=1}^{m} \Delta_i^{-2}\{\langle C_i, H\rangle + \langle c_i, h\rangle\}C_i = -\nabla_A, \\ &\tfrac{1}{2}\frac{\partial D^2F(A, a)[(H, h), (H, h)]}{\partial h} = \tau \sum_{i=1}^{m} \Delta_i^{-2}\{\langle C_i, H\rangle + \langle c_i, h\rangle\}c_i = -\nabla_a. \end{aligned} \tag{6.23}$$

Let

$$\lambda_i = \tau\Delta_i^{-2}\{\langle C_i, H\rangle + \langle c_i, h\rangle\}, \quad i = 1, \ldots, m. \tag{6.24}$$

From the matrix equation (6.23) it follows that H can be represented in the form

$$H = -A\nabla_A A - \sum_{j=1}^{m} \lambda_j AC_jA. \tag{6.25}$$

Treating $\lambda_1, \ldots, \lambda_m$ as unknowns, we can substitute (6.25) in (6.24) and obtain the following system of $m+n$ linear equations in $m+n$ unknowns λ, h,

$$\begin{aligned} &\lambda_i = -\tau\Delta_i^{-2}\left[\sum_{j=1}^{m} \langle C_i, AC_jA\rangle\lambda_j - \langle c_i, h\rangle + \langle C_i, A\nabla_A A\rangle\right], \quad i = 1, \ldots, m, \\ &\sum_{j=1}^{m} c_j\lambda_j = -\nabla_a, \end{aligned} \tag{6.26}$$

equivalent to the system (6.23). Since A is symmetric,

$$\langle C_i, A\nabla_A A\rangle = \operatorname{tr}(c_ic_i^{\mathrm{T}}A\nabla_A A) = \langle Ac_i, \nabla_A Ac_i\rangle$$

and

$$\langle C_i, AC_jA\rangle = \operatorname{tr}(c_ic_i^{\mathrm{T}}Ac_jc_j^{\mathrm{T}}A) = \langle c_i, Ac_j\rangle^2.$$

Thus, letting

$$f_i = Ac_i, \quad i = 1, \ldots, m, \tag{6.27}$$

(6.26) can be rewritten as

$$\begin{aligned} &\lambda_i = -\tau\Delta_i^{-2}\left[\sum_{j=1}^{m} \langle c_i, f_j\rangle^2\lambda_j - \langle c_i, h\rangle + \langle f_i, \nabla_A f_i\rangle\right], \quad i = 1, \ldots, m, \\ &\sum_{j=1}^{m} c_j\lambda_j = -\nabla_a. \end{aligned} \tag{6.28}$$

Now it is easy to see that the vectors f_i, $i=1,\ldots,m$, and all the coefficients of the system (6.28) can be computed in $O(m^3)$ arithmetical operations. As $n<m$, the system itself can also be solved in $O(m^3)$ operations. Having found the solution (λ^0, h^0) to the system (6.28), one can determine the A-component of the Newton direction by the formula (6.25),

$$H^0=-A\nabla_A A-\sum_{j=1}^{m}\lambda_j^0 f_j f_j^{\mathrm{T}}$$

in $O(m^3)$ operations as well.

Remark. For the case $a\equiv 0$, $h\equiv 0$, considered in [7], the system (6.28) turns into

$$\lambda_i=-\frac{\tau}{(1-\langle c_i,f_i\rangle)^2}\left[\sum_{j=1}^{m}\langle c_i,f_j\rangle^2\lambda_j+\langle f_i,\nabla_A f_i\rangle\right],\quad i=1,\ldots,m,$$

which is simpler than the system derived in [7], see p. 175–177.

We have thus shown that the computational cost of one Newton iteration is $O(m^3)$ arithmetical operations. Combining this fact with the bound (6.22) on the number of Newton iterations and with (6.1), we obtain the following result.

Theorem 2. *Let $Q=\{x\in\mathbb{R}^n\mid c_i^{\mathrm{T}}x\leq 1, i=1,\ldots,m\}$ be a polytope that can be covered by an Euclidean ball of radius R, and let $b\in\operatorname{int} Q$. To inscribe a γ-maximal paraboloid in the polyhedral cone $K(b)$ defined by* (5.6), *it suffices to perform*

$$O\left(m^{3.5}\ln\left[\frac{mR}{r(b)\ln(1/\gamma)}\right]\right)\tag{6.29}$$

arithmetical operations, where

$$\frac{1}{r(b)}=\max_{i=1,\ldots,m}\frac{\|c_i\|}{1-c_i^{\mathrm{T}}b}.\quad\square\tag{6.30}$$

Note that $r(b)$ is the Euclidean distance from b to the boundary of Q. Observe also that in Theorem 2 we can replace the adjective "γ-maximal" by "$\gamma^{1/3}$-maximal".

7. Bounding the complexity of determining a γ-maximal ellipsoid for a polytope

Theorem 3. *Let $Q=\{x\in\mathbb{R}^n\mid c_ix\leq 1\}$ be a polytope satisfying* (2.6). *To find a γ-maximal ellipsoid for Q, it suffices to perform*

$$O\left(m^{3.5}\cdot\ln\left[\frac{mR}{\ln(1/\gamma)}\right]\cdot\ln\left[\frac{n\ln R}{\ln(1/\gamma)}\right]\right)\tag{7.1}$$

arithmetical operations.

Proof. Applying ElliP, we reduce the problem of finding a γ-maximal ellipsoid for Q to κ subproblems $\mathrm{P}(b_0), \mathrm{P}(b_1), \ldots, \mathrm{P}(b_\kappa)$, each of which is the problem of finding a $\gamma^{1/3}$-maximal paraboloid inscribed in the corresponding cone $K(b_s)$, $s = 0, 1, \ldots, \kappa$. Here

$$b_0 = 0, \qquad b_1, \ldots, b_\kappa \in \operatorname{int} Q, \quad \kappa \leqslant \log[2n \ln R/\ln(1/\gamma)]$$

is the sequence of points generated by ElliP. The condition (1.6) implies $r(b_0) \geqslant 1$, see (6.30), and by (6.29) the complexity of $P(b_0)$ does not exceed

$$\mathrm{O}(m^{3.5} \ln[mR/\ln(1/\gamma)]) \tag{7.2}$$

arithmetical operations. Now we have to show that the points $b_1, \ldots, b_\kappa$ do not come too close to the boundary of Q, i.e. to bound the quantities $r(b_1), \ldots, r(b_\kappa)$ from below. Since from the description of ElliP we know that $b_{s+1} = \frac{1}{2}(b_s + a_s)$, $a_s \in \operatorname{int} Q$, and the distance function $r: \operatorname{int} Q \to \mathbb{R}$ is concave and positive, we have the following trivial recurrence:

$$r(b_0) \geqslant 1, \qquad r(b_{s+1}) > \tfrac{1}{2} r(b_s).$$

The number $\kappa + 1$ of iterations of ElliP is so small that even this recurrence yields an estimate

$$\min\{r(b_1), \ldots, r(b_\kappa)\} > 2^{-\kappa} \geqslant \frac{\ln(1/\gamma)}{2n \ln R}, \tag{7.3}$$

sufficient for our needs. Indeed, substituting (7.3) in (6.29), we see that the upper bound (7.2) still applies to each of the problems $\mathrm{P}(b_1), \ldots, \mathrm{P}(b_\kappa)$. Multiplying (7.2) by (4.8), we get (7.1). □

Applying ElliP in the method of inscribed ellipsoids, we also obtain from (4.9) and (1.2) the following:

Corollary. *The complexity of the mth iteration of the method of inscribed ellipsoids does not exceed* $\mathrm{P}(m^{3.5} \cdot \ln m \cdot \ln \ln n)$ *arithmetical operations.* □

Note that in the latter method one may assume without loss of generality that $m = \mathrm{O}(n \ln n)$. This yields the bound $\mathrm{O}(n^{3.5+\varepsilon})$ arithmetical operations per iteration of the method of inscribed ellipsoids and improves the bound $\mathrm{O}(n^{4.5+\varepsilon})$ in [7].

8. Concluding remarks and questions

(1) We believe that the Newton system (6.28) needs further investigation. In particular, is it possible to use "Karmarkar's speed-up" to reduce the average iteration cost to $\mathrm{O}(m^{2.5})$ arithmetical operations?

(2) Consider the following algorithm IC:

Step 0. Set

$k := 0,$

$b := b_0 :=$ an arbitrary interior point of Q.

Step 1. Find the minimal covering ellipsoid E_k for the set of points

$c_i/(1 - c_i^{\mathrm{T}} b_k), \quad i = 1, \ldots, m.$

Step 2. Update:

$b_{k+1} :=$ the center of the polar of E_k,

$k := k + 1.$

Go to Step 1 and start a new iteration.

Does the sequence of the polars $E_0^0, E_1^0, \ldots, E_k^0 \subseteq Q$ converge to the maximal inscribed ellipsoid for Q?

(3) We conjecture that the following "polar" versions of Problems I and C are NP-hard:

- find a γ-minimal covering ellipsoid for a given polytope (2.1);
- find a γ-maximal inscribed ellipsoid for a given polytope (2.2).

It is known [4] that these problems are NP-hard in the case where we consider balls instead of ellipsoids.

(4) Suppose that $Q = \{x \in \mathbb{R}^n \mid c_i^{\mathrm{T}} x \leqslant 1, i = 1, \ldots, m\}$ is a rational polytope: $c_1, \ldots, c_m \in \mathbb{Z}^n$. Can one obtain non-trivial bounds on the algebraic degrees of the entries of the maximal inscribed ellipsoids $E^* = E^*(Q)$ as functions of n?

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