02157 Functional Programming

Finite Trees (I)

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Overview

Finite Trees

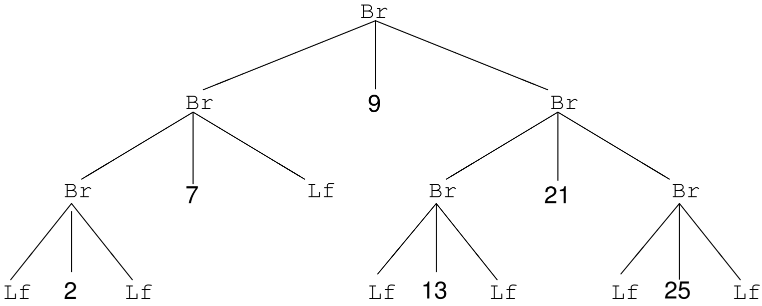
- recursive declarations of algebraic types
- meaning of type declarations: rules generating values
- typical recursions following the structure of trees
- trees with a fixed branching structure
- trees with a variable number of sub-trees
- illustrative examples

Mutually recursive type and function declarations

Finite trees

A *finite tree* is a value containing subcomponents of the same type

Example: A *binary tree*



A tree is a connected, acyclic, undirected graph, where

- the top node (carrying value 9) is called the root
- a branch node has two children — constructor Br
- a node without children is called a leaf — constructor Lf

Example: Binary Trees

A *recursive datatype* is used to represent values that are trees.

```
type Tree = | Lf
            | Br of Tree*int*Tree;;
```

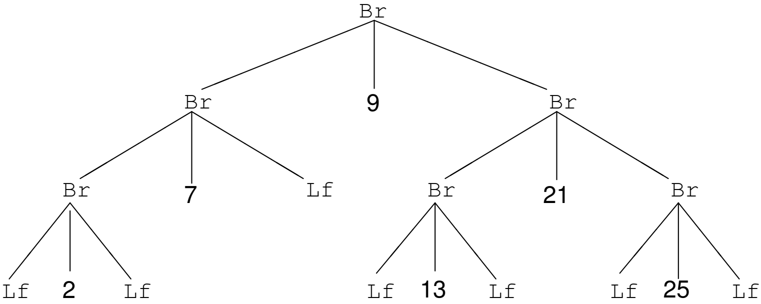
The declaration provides rules for generating trees:

1. `Lf` is a tree
2. if t_1, t_2 are trees and n is an integer, then $\mathtt{Br}(t_1, n, t_2)$ is a tree.
3. the type Tree contains no other values than those generated by repeated use of Rules 1. and 2.

The tags Lf and Br are called constructors:

$$\begin{array}{lcl} \mathtt{Lf} & : & \mathtt{Tree} \\ \mathtt{Br} & : & \mathtt{Tree} * \mathtt{int} * \mathtt{Tree} \rightarrow \mathtt{Tree} \end{array}$$

Example: Binary Trees



Corresponding F#-value:

```
Br(Br(Br(Lf,2,Lf),7,Lf),
   9,
   Br(Br(Lf,13,Lf),21,Br(Lf,25,Lf)))
```

Traversals of binary trees

- Pre-order traversal: First visit the root node, then traverse the left sub-tree in pre-order and finally traverse the right sub-tree in pre-order.
- In-order traversal: First traverse the left sub-tree in in-order, then visit the root node and finally traverse the right sub-tree in in-order.
- Post-order traversal: First traverse the left sub-tree in post-order, then traverse the right sub-tree in post-order and finally visit the root node.

In-order traversal

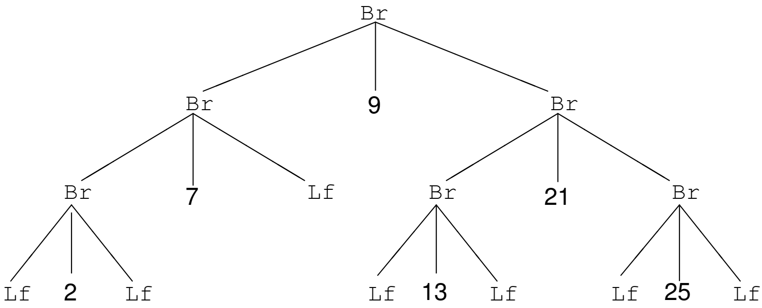
```
let rec inOrder =
   function
   | Lf          -> []
   | Br(t1,j,t2) -> inOrder t1 @ [j] @ inOrder t2;;
val toList : Tree -> int list

inOrder(Br(Br(Lf,1,Lf), 3, Br(Br(Lf,4,Lf), 5, Lf)));;
val it : int list = [1; 3; 4; 5]
```

Binary search tree

Condition: for every node containing the value x: every value in the left subtree is smaller then x, and every value in the right subtree is greater than x.

Example: A *binary search tree*



Binary search trees: Insertion

- Recursion following the structure of trees
- Constructors `Lf` and `Br` are used in patterns to decompose a tree into its parts
- Constructors `Lf` and `Br` are used in expressions to construct a tree from its parts
- The search tree condition is an invariant for `insert`

```
let rec insert i =
   function
   | Lf                -> Br(Lf,i,Lf)
   | Br(t1,j,t2) as tr -> // Layered pattern
         match compare i j with
         | 0           -> tr
         | n when n<0  -> Br(insert i t1 , j, t2)
         | _           -> Br(t1,j, insert i t2);;
val insert : int -> Tree -> Tree
```

Example:

```
let t1 = Br(Lf, 3, Br(Lf, 5, Lf));;
let t2 = insert 4 t1;;
val t2 : Tree = Br (Lf,3,Br (Br (Lf,4,Lf),5,Lf))
```

Binary search trees: contains

```
let rec contains  i =
   function
   | Lf                    -> false
   | Br(_,j,_)  when i=j -> true
   | Br(t1,j,_) when i<j -> contains i t1
   | Br(_,j,t2)            -> contains i t2;;
val contains : int -> Tree -> bool

let t = Br(Br(Br(Lf,2,Lf),7,Lf),
           9,
           Br(Br(Lf,13,Lf),21,Br(Lf,25,Lf)));;

contains 21 t;;
val it : bool = true

contains 4 t;;
val it : bool = false
```

Parameterize type declarations

The programs on search trees require only an ordering on elements
– they no not need to be integers.

A polymorphic tree type is declared as follows:

```
type Tree<'a> = | Lf | Br of Tree<'a> * 'a * Tree<'a>;;
```

Program text is unchanged (though polymorphic now), for example

```
let rec insert i = function
       ....
  | Br(t1,j,t2) as tr -> match compare i j with
       .... ;;
val insert: 'a -> Tree<'a> -> Tree<'a> when 'a: comparison

let ti = insert 4  (Br(Lf, 3, Br(Lf, 5, Lf)));;
val ti : Tree<int> = Br (Lf,3,Br (Br (Lf,4,Lf),5,Lf))

let ts = insert "4" (Br(Lf, "3", Br(Lf, "5", Lf)));;
val ts : Tree<string>
       = Br (Lf,"3",Br (Br (Lf,"4",Lf),"5",Lf))
```

So far

- Declaration of a recursive algebraic data type, that is, a type for a finite tree
- Meaning of the type declaration in the form of rules for generating values
- typical functions on trees:
  - gathering information from a tree — Example: inOrder
  - inspecting a tree — Example: contains
  - constructs of a new tree — Example: insert

Manipulation of arithmetical expressions

Consider $f(x)$:

$$3 \cdot (x - 1) - 2 \cdot x$$

We may be interested in

- computation of values, e.g. $f(2)$
- differentiation, e.g. $f'(x) = (3 \cdot 1 + 0 \cdot (x - 1)) - (2 \cdot 1 + 0 \cdot x)$
- simplification of the expressions, e.g. $f'(x) = 1$
- .....

We would like a suitable representation of such arithmetical expressions that supports the above manipulations

How would you visualize the expressions as a tree?

- root?
- leaves?
- branches?

Example: Expression Trees

```
type Fexpr =
   | Const of float
   | X
   | Add of Fexpr * Fexpr
   | Sub of Fexpr * Fexpr
   | Mul of Fexpr * Fexpr
   | Div of Fexpr * Fexpr;;
```

Defines 6 constructors:

- `Const: float -> Fexpr`
- `X : Fexpr`
- `Add: Fexpr * Fexpr -> Fexpr`
- `Sub: Fexpr * Fexpr -> Fexpr`
- `Mul: Fexpr * Fexpr -> Fexpr`
- `Div: Fexpr * Fexpr -> Fexpr`

- Can you write 3 values of type Fexpr?
- Drawings of trees?

Expressions: Computation of values

Given a value (a float) for `x`, then every expression denote a float.

```
compute : float -> Fexpr -> float
```

```
let rec compute x =
   function
   | Const r        -> r
   | X              -> x
   | Add(fe1,fe2)   -> compute x fe1 + compute x fe2
   | Sub(fe1,fe2)   -> compute x fe1 - compute x fe2
   | Mul(fe1,fe2)   -> compute x fe1 * compute x fe2
   | Div(fe1,fe2)   -> compute x fe1 / compute x fe2;;
```

Example:

```
compute 4.0 (Mul(X, Add(Const 2.0, X)));;
val it : float = 24.0
```

Blackboard exercise: Substitution

```
type Fexpr = | Const of float
             | X
             | Add of Fexpr * Fexpr
             | Sub of Fexpr * Fexpr
             | Mul of Fexpr * Fexpr
             | Div of Fexpr * Fexpr;;
```

Declare a function

```
substX: Fexpr -> Fexpr -> Fexpr
```

so that `substX` e' e is the expression obtained from e by substituting every occurrence of `X` with e'

For example:

```
let ex = Add(Sub(X, Const 2.0), Mul(Const 4.0, X));;

substX (Div(X,X)) ex;;
val it : Fexpr =
   Add(Sub(Div(X,X), Const 2.0), Mul(Const 4.0, Div(X,X)))
```

Symbolic Differentiation `D: Fexpr -> Fexpr`

A classic example in functional programming:

```
let rec D = function
  | Const _       -> Const 0.0
  | X             -> Const 1.0
  | Add(fe1,fe2) -> Add(D fe1,D fe2)
  | Sub(fe1,fe2) -> Sub(D fe1,D fe2)
  | Mul(fe1,fe2) -> Add(Mul(D fe1,fe2),Mul(fe1,D fe2))
  | Div(fe1,fe2) -> Div(
                      Sub(Mul(D fe1,fe2),Mul(fe1,D fe2)),
                      Mul(fe2,fe2));;
```

Notice the direct correspondence with the rules of differentiation.

Can be tried out directly, as tree are "just" values, for example:

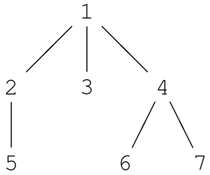
```
D(Add(Mul(Const 3.0, X), Mul(X, X)));;
val it : Fexpr =
  Add
    (Add (Mul (Const 0.0,X),Mul (Const 3.0,Const 1.0)),
     Add (Mul (Const 1.0,X),Mul (X,Const 1.0)))
```

Trees with a variable number of sub-trees

An archetypical declaration:

```
type ListTree<'a> = Node of 'a * (ListTree<'a> list)
```

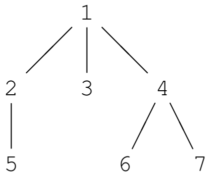
- `Node(`x`,[])` represents a leaf tree containing the value x
- `Node(`x`,[`t_0`;...;`t_{n-1}`])` represents a tree with value x in the root and with n sub-trees represented by the values $t_0, \ldots, t_{n-1}$



It is represented by the value `t1` where

```
let t7 = Node(7,[]);;      let t6 = Node(6,[]);;
let t5 = Node(5,[]);;      let t3 = Node(3,[]);;
let t2 = Node(2,[t5]);;    let t4 = Node(4,[t6; t7]);;
let t1 = Node(1,[t2; t3; t4]);;
```

Depth-first traversal of a ListTree



Corresponds to the following order of the elements: 1, 2, 5, 3, 4, 6, 7

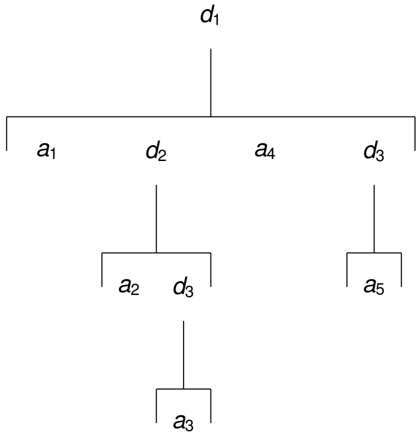
Invent a more general function traversing a list of List trees:

```
let rec depthFirstList =
   function
   | [] -> []
   | Node(n,ts)::trest -> n::depthFirstList(ts @ trest)
depthFirstList : ListTree<'a> list -> 'a list

let depthFirst t = depthFirstList [t]
 depthFirst1 : t:ListTree<'a> -> 'a list

depthFirst t1;;
val it : int list = [1; 2; 5; 3; 4; 6; 7]
```

Mutual recursion. Example: File system



- A file system is a list of elements
- an element is a file or a directory, which is a named file system

We focus on structure now – not on file content

Mutually recursive type declarations

- are combined using and

```
type FileSys = Element list
and Element =
   | File of string
   | Dir of string * FileSys
```

```
let d1 =
  Dir("d1",[File "a1";
            Dir("d2", [File "a2";
                       Dir("d3", [File "a3"])]);
            File "a4";
            Dir("d3", [File "a5"])
           ])
```

The type of d1 is ?

Mutually recursive function declarations

- are combined using and

Example: extract the names occurring in file systems and elements.

```
let rec namesFileSys =
  function
  | []     -> []
  | e::es -> (namesElement e) @ (namesFileSys es)
and namesElement =
  function
  | File s     -> [s]
  | Dir(s,fs) -> s :: (namesFileSys fs) ;;
val namesFileSys : Element list -> string list
val namesElement : Element -> string list


namesElement d1 ;;
val it : string list = ["d1"; "a1"; "d2"; "a2";
                        "d3"; "a3"; "a4"; "d3"; "a5"]
```

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Mutually recursive type and function declarations