MATH 200 Homework 1

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1 Preliminaries

Exercise 1.1 *Are there any graphs (not multi-graphs) with three vertices other than those shown above (G_0, G_1, G_2, G_3)? If so, provide sketches of any additional graphs. If not, explain why you are certain that you have accounted for all the possible graphs on three vertices.*

Observing Figure 1.1.2, there are no other possible graphs drawn with 0 or 3 edges as no other edge combinations exist. Figure 1.1.2 also shows that all other possible graphs with 1 edge are isomorphic to G_1, and all other graphs with 2 edges are isomorphic to G_2. Thus, there are no other graphs with three vertices other than those shown in Figure 1.1.1.

Exercise 1.2 *Suppose we ask the same question about multi-graphs. How many distinct multi-graphs are there on 3 vertices?*

In a multi-graph, an infinite number of edges can be added between two vertices. As depicted in Figure 1.2.1, every edge added between two specific vertices will create a unique multi-graph. Thus, there are infinite distinct multi-graphs on three vertices.

Exercise 1.3 *Sketch all of the possible distinct graphs on one vertex. Then sketch all of the distinct graphs on two vertices. Explain how you know that you have all of the possible graphs.*

As there exist no other possible edge combinations, all possible distinct graphs on one and two vertices are depicted in Figure 1.3.1 and Figure 1.3.2, respectively.

Exercise 1.4 *Sketch all of the possible distinct graphs on four vertices. Explain how you know that you have all of the possible graphs.*

In an **empty graph** on n vertices, there are no edges connecting any of the vertices. In a **complete graph** on n vertices, all vertices are connected to each

other. Both the empty graph and the complete graph on four vertices are shown in Figure 1.4.1 as G_0 and G_6, respectively.
Notice that the empty graph and the complete graph on four vertices contain no similar edges; no two vertices are adjacent in both graphs. Thus, the empty graph and the complete graphs are complementary. Since there are no edges in the empty graph and all possible edges exist on the complete graph, there exist no other possible graphs on four vertices with zero or six edges.
From adding an edge to G_0, we are able to create one unique graph. All graphs on four vertices with one edge are isomorphic. These graphs are shown in Figure 1.4.2 below. All graphs on four vertices with five edges are the complements of a graph on four vertices with one edge. Thus, there exists only one unique graph on four vertices with five edges, also shown in Figure 1.4.2
Likewise, the complements of graphs on four vertices with two edges are graphs with four edges. To obtain all of the unique graphs on four vertices with two edges, we can add an edge onto G_1. There are two unique graphs we can create with two edges, shown along with their complements in Figure 1.4.3.
All unique graphs on four vertices can be obtained using the same method. By adding an edge onto G_2 and G_2', we can obtain three unique graphs, shown in Figure 1.4.4. All other graphs on four vertices with three edges are isomorphic. The 11 unique graphs on four vertices are shown in Figure 1.4.5

Exercise 1.5 *If a graph has n vertices, is there a minimum number of edges necessary in order for the graph to be connected? Is there a maximum number of edges it may have and still not be connected?*

If a graph has n vertices, the minimum number of edges necessary in order for the graph to be connected is $n-1$.

In a **complete graph** on n vertices, all vertices are connected to each other. In this case, each vertex has a degree of $n-1$, which gives the complete graph a **total degree** of $(n-1)n$. Since each edge connects two vertices, the total number of edges on a graph is equal to half of its total degree. For a graph to not be connected, at least on vertex must have a degree of 0 (must not be connected to any other vertices). In this case, the maximum number of edges a graph with n vertices may have and still not be connected is equal to the complete graph with $n-1$ vertices. Thus, the maximum number of edges is equal to

$$\frac{((n-1)-1)(n-1)}{2} = \frac{(n-1)(n-2)}{2} \tag{1}$$

Exercise 1.6 *The graphs below show several examples of paths. Find the path which*

(a) is a path, but is neither an Euler path nor a circuit, **(ii)**

(b) is a circuit, but not an Euler circuit, **(iii)**
(c) is an Euler path, but not an Euler circuit, **(iv)**
(d) is an Euler circuit. **(i)**

Exercise 1.7 *In each of the graphs below, see if you can find an Euler circuit.*

None of the graphs below contain an Euler circuit, except for **(d)**, which is shown in Figure 1.7.1.