MATH 345 Homework 9

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1 Problems

1. Let $f : X \to Y$ be a bijection between sets X and Y. Let U and V be subsets of Y. Prove that

$$f(U) \cap f(V) = f(U \cap V)$$

Proof. ($\subset$) Let $y \in f(U) \cap f(V)$. Because y is in the intersection of $f(U)$ and $f(V)$, y is in both $f(U)$ and $f(V)$. $f^{-1}(y)$ exists as f is a bijection. Then $f^{1}(y) \in U$ and $f^{-1}(y) \in V$, and so $f^{-1}(y) \in U \cap V$. It follows that $y \in f(U \cap V)$. Since an arbitrary element in $f(U) \cap f(V)$ is in $f(U \cap V)$, $f(U) \cap f(V) \subset f(U \cap V)$.
($\supset$) Next, let $y \in f(U \cap V)$. Then $f^{-1}(y) \in U \cap V$. Since $f^{-1}(y)$ is in the intersection of U and V, it is in both sets. Since the inverse of y is in both U and V, $y \in f(U)$ and $y \in f(V)$. It follows that $y \in f(U) \cap f(V)$. Since an arbitrary element in $f(U \cap V)$ is in $f(U) \cap f(V)$, $f(U \cap V) \subset f(U) \cap f(V)$.
Therefore, $f(U) \cap f(V) = f(U \cap V)$. □

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4.24 Prove that a bijection $f : X \to Y$ is a homeomorphism if and only if f and f^{-1} map closed sets to closed sets.

Proof. ($\to$) Let f be a homeomorphism. Consider a closed set $C \subset X$. Then $X - C$ is open in X. Because f is a homeomorphism, $f(X - C)$ is open in Y. As f is a bijection, it follows by Theorem 0.22 that

$$f(X - C) = f^{-1-1}(X - C) = f^{-1-1}(X) - f^{-1-1}(C) = f(X) - f(C) = Y - f(C).$$

Since $Y - f(C)$ is open in Y, $f(C)$ is closed in Y, and f maps closed sets to closed sets. Likewise, Let $D \subset Y$ be closed. $Y - D$ is open in Y, $f^{-1}(Y - D)$ is open in X, and by Theorem 0.22

$$f^{-1}(Y - D) = f^{-1}(Y) - f^{-1}(D) = X - f^{-1}(D).$$

As $X - f^{-1}(D)$ is open, $f^{-1}(D)$ is closed, and f^{-1} maps closed sets to closed sets. Therefore, If a bijection $f : X \to Y$ is a homeomorphism then f and f^{-1} map closed sets to closed sets.
($\leftarrow$) Let f be a bijection where f and f^{-1} map closed sets to closed sets. Consider closed sets C and $f(C)$. Consider the open set in Y, $Y - f(C)$. By Theorem 0.22,

$$f^{-1}(Y - f(C)) = f^{-1}(Y) - f^{-1}(f(C)) = X - C.$$

Since the preimage of an open set in Y is open in X, f is continuous. Now, consider $X - C$ is open in X. By Theorem 0.22,

$$f(X - C) = f^{-1-1}(X - C) = f(X) - f(C) = Y - f(C).$$

Since the preimage of an open set in X is open in Y, f^{-1} is continuous. Thus, f is a homeomorphism, and if f and f^{-1} map closed sets to closed sets, then f is a homeomorphism.
Therefore, a bijection $f : X \to Y$ is a homeomorphism if and only if f and f^{-1} map closed sets to closed sets. □

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4.26

(a) Provide a formula for a homeomorphism between the intervals $[0,\infty)$ and $[a,b)$, with $a<b$.

$$f:[0,\infty)\to[a,b),\text{ given by } f(x)=\frac{a+bx}{x+1}$$

(b) Provide a formula for a homeomorphism between the intervals $(-\infty,0]$ and $(a,b]$, with $a<b$.

$$g:(-\infty,0]\to(a,b],\text{ given by } g(x)=\frac{b-ax}{1-x}$$

(c) Given the homeomorphisms in Example 4.12 and the first two parts of this exercise, prove that if I_1 and I_2 are intervals in the collection in Example 4.12, then I_1 and I_2 are topologically equivalent.

Proof. Consider h given in Example 4.12, a homeomorphism from $[a,\infty)$ to $(-\infty,a]$, so those spaces are homeomorphic. We have homeomorphisms $f,g,$ and h which map between all intervals in collection (iii). Thus, all of these intervals are topologically equivalent. □

4.28 Prove each of the following statements, and then use them to show that topological equivalence is an equivalence relation on the collection of all topological spaces.

(a) The function $id:X\to X$, defined $id(x)=x$, is a homeomorphism.

Proof. Consider an open set $U\subset X$. The preimage of U, $id^{-1}(U)=U$. We know that U is open in X, and thus the preimage of open sets in the range is open, making id continuous. Now consider the inverse of id, which is just id. Since id is continuous, its equivalent inverse is continuous, making id a homeomorphism. □

(b) If $f:X\to Y$ is a homeomorphism, then so is $f^{-1}:Y\to X$.

Proof. Since f is a homeomorphism, it follows that f and f^{-1} are both continuous functions. Because both f^{-1} and its inverse f are continuous functions, it follows that f^{-1} is a homeomorphism. □

(c) if $f:X\to Y$ and $g:Y\to Z$ are homeomorphisms, then so is the composition $gof:X\to Z$.

Proof. First, since f and g are both homeomorphisms and thus continuous, by Theorem 4.9 the composition gof is also continuous. Now, consider $gof^{-1}=f^{-1}(g^{-1}(x))$, by the first homework. Since f and g are homeomorphisms, it follows that their inverses are continuous, and thus by Theorem 4.9, the composition of f^1 and g^1, $gof^{-1}=f^{-1}(g^{-1}(x))$, is continuous as well.
Because both gof and gof^{-1} are continuous, gof is a homeomorphism. □

4.29 Show that $\mathbb{R}^2-\{0\}$ in the standard topology is homeomorphic to $S^1\times\mathbb{R}$.

Proof. Consider the functions

$$f:\mathbb{R}^2-\{O\}\to\mathbb{R}^+\times[0,2\pi],\text{ given by } f(x,y)=(\sqrt{x^2+y^2},tan^{-1}(\frac{y}{x}))$$

$$g:\mathbb{R}^+\times[0,2\pi]\to S^1\times\mathbb{R},\text{ given by } g(r,\theta)=((cos(\theta),sin(\theta)),log(r))$$

We will show that both f and g are homeomorphisms, and thus $gof:\mathbb{R}^2-\{0\}\to S^1\times\mathbb{R}$ is a homeomorphism.
First, consider f, which is a mapping of x,y coordinates to polar coordinates. We know that coordinate mappings are homeomorphic, and both sets do not include the origin.
Next, consider g. Basis elements in $\mathbb{R}^+\times[0,2\pi]$ are of the form $U\times V$, where U is open in $\mathbb{R}^+$ and V is open in $[0,2\pi]$. Consider $log:\mathbb{R}^+\to\mathbb{R}$ is a homeomorphism, and $[0,2\pi]\to S^1$ is a homeomorphism in g. Thus, g is a homeomorphism.
Since f and g are homeomorphisms, it follows that $gof:\mathbb{R}^2-\{0\}\to[0,2\pi]\to S^1$ is a homeomorphism, and the spaces are homeomorphic. □

4.31

(a) Show that having nonempty open sets that contain finitely many points is a topological property.

Proof. Let X and Y be topological spaces where $f : X \to Y$ is a homeomorphism and X contains nonempty, finite open sets. Consider a nonempty, finite open set $U \subset X$. Because f is a homeomorphism, f^{-1} is a continuous function, and the preimage $f^{-1}(U) = V$ is open in Y. Since f, as a homeomorphism, is also bijective, then finite sets map to finite sets, and thus if U is open, then V is open as well, and Y contains nonempty open sets with finitely many points.
Therefore, having nonempty open sets that contain finitely many points is a topological property. □

(b) Prove that the digital line topology is not homeomorphic to $\mathbb{Z}$ with the finite complement topology.

Proof. BWOC, assume these topological spaces are homeomorphic. Consider the digital line topology. There exist open sets in this topology with finitely many points. However, all sets in $\mathbb{Z}$ with a finite complement F, are of the form $\mathbb{Z} - F$, and contain infinitely many elements. The only open set in the finite complement topology on $\mathbb{Z}$ is $\varnothing$, and so the topological property proved in part a fails.
Therefore, the digital line topology is not homeomorphic to $\mathbb{Z}$ with the finite complement topology. □

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4.32 (a) If $f : X \to Y$ is a homeomorphism, then $f(Int(A)) = Int(f(A))$ for every $A \subset X$.

Proof. First, let $A \subset X$ and consider $x \in f(Int(A))$. Then $f^{-1}(x) \in Int(A)$. Because $f^{-1}(x)$ is in the interior of A, there exists a neighborhood U in X where $f^{-1}(x) \in U \subset A$. Then $x \in f(U) \subset f(A)$. Because f is a homeomorphism, the open set U maps to open set $f(U)$. So x is in an open neighborhood contained in $f(A)$, and it follows $x \in Int(f(A))$, and $f(Int(A)) \subset Int(f(A))$
Next, let $x \in Int(f(A))$. As x is in the interior of $f(A)$, there exists an open set V where $x \in V \in f(A)$. Then $f^{-1}(x) \in f^{-1}(V) \subset A$. Because f is a homeomorphism and thus continuous, the preimage of open set V is open in X. It follows that $f^{-1}(x) \in Int(A)$, as $f^{-1}(x)$ is contained in an open set $f^{-1}(V)$ contained in A. Thus, $x \in f(Int(A))$, and $Int(f(A)) \subset f(Int(A))$.
Therefore, if $f : X \to Y$ is a homeomorphism, then $f(Int(A)) = Int(f(A))$ for every $A \subset X$. □

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4.33 Let $X \times Y$ be partitioned into subsets of the form $X \times \{y\}$ for all y in Y. If we let $(X \times Y)^*$ denote the collection of sets in the partition, show that $(X \times Y)^*$ with the resulting quotient topology is homeomorphic to Y.

Proof. Consider a subset U of the quotient topology on $(X \times Y)^*$. These subsets are of the form

$$U = \{X \times \{y\} : y \in V \subset Y\},$$

Where V is open in Y. Consider the function

$$\delta : Y \to (X \times Y)^* \text{ given by } \delta(y) = X \times \{y\}.$$

Consider an open set $V \subset Y$. $\delta(V) = \{X \times \{y\} : y \in V\}$, which is open in $(X \times Y)^*$ by definition. Likewise, for an open set $U \subset (X \times Y)^*$ $\delta^{-1}(U) = V$, where V is an open set in Y. So both δ and δ^{-1} are continuous.
To show that f is a bijection, it must be surjective and injective. Consider $X \times y \in (X \times Y)^*$. We know $\delta(y) = X \times y$, so δ is surjective. Also, consider $y, z \in Y$ where $y \neq z$. We can see that $\delta(y) = \{X \times y\}$, and $\delta(z) = \{X \times z\}$, so $\delta(y) \neq \delta(z)$. Thus, δ is injective and a bijection.
All of this shows that δ is a homeomorphism from Y to $(X \times Y)^*$, and thus the spaces are topologically equivalent. □