Foundations Homework 10

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18 November 2019

Chapter Three

Exercise 22 *For each sequence, write out the first few terms of each sequence, and determine which of these sequences is a subsequence of another.*

- $\{a_n\} = \{\frac{n}{n+1}\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$
- $\{b_n\} = \{\frac{2n}{2n+1}\} = \{\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \dots\}$
- $\{c_n\} = \{(-1)^n \frac{n}{n+1}\} = \{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \dots\}$
- $\{d_n\} = \{\frac{n^2}{n^2+1}\} = \{\frac{1}{2}, \frac{4}{5}, \frac{9}{10}, \frac{16}{17}, \dots\}$
- $\{e_n\} = \{\frac{n-(-1)^n}{n+1-(-1)^n}\} = \{\frac{2}{3}, \frac{1}{2}, \frac{4}{5}, \frac{3}{4}, \dots\}$
- $\{f_n\} = \{\frac{4n}{4n+1}\} = \{\frac{4}{5}, \frac{8}{9}, \frac{12}{13}, \frac{16}{17}, \dots\}$

$\{b_n\}$ is a subsequence of $\{a_n\}$, $\{c_n\}$, and $\{e_n\}$. $\{d_n\}$ is a subsequence of $\{a_n\}$, $\{e_n\}$. $\{f_n\}$ is a subsequence of $\{a_n\}$, $\{b_n\}$, $\{e_n\}$.

Exercise 23 *Notice that in Exercise 3.22, $\{b_n\} = \{a_{2n}\}$. Is $\{a_{kn}\}$ always a subsequence of $\{a_n\}$? What conditions must a function $f(n)$ satisfy so that $\{a_{f(n)}\}$ is a subsequence of $\{a_n\}$?*

$\{a_{kn}\}$ is always a subsequence of $\{a_n\}$. A function $f(n)$ must be continuous so that $\{a_{f(n)}\}$ is a subsequence of $\{a_n\}$.

Exercise 24 *Write out Σ_1,Σ_2, and Σ_3. How many elements are in Σ_4? In Σ_{10}?*

$\Sigma_1 = \{\{0\}, \{1\}\}$
$\Sigma_2 = \{\{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}\}$
$\Sigma_3 = \{\{0,0,0\}, \{0,0,1\}, \{0,1,0\}, \{0,1,1\}, \{1,0,0\}, \{1,0,1\}, \{1,1,0\}, \{1,1,1\}\}$

Σ_4 has $2^4 = 16$ elements. $\Sigma_1 0$ has $2^{10} = 1024$ elements.

Exercise 25 *Let $\mathbb{N}_k = \{n \varepsilon \mathbb{N} | n \leq k\} = \{1, 2, \dots, k\}$. List the elements in $P(\mathbb{N}_1)$, $P(\mathbb{N}_2)$, and $P(\mathbb{N}_3)$. How many elements are in $P(\mathbb{N}_4)$? In $P(\mathbb{N}_{10})$?*

$P(\mathbb{N}_1) = \{\emptyset, \{1\}\}$
$P(\mathbb{N}_2) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
$P(\mathbb{N}_3) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$P(\mathbb{N}_4)$ has $2^4 = 16$ elements. $P(\mathbb{N}_{10})$ has $2^{10} = 1024$ elements.

Assignment 26 *Prove that $P(\mathbb{N}) \sim \Sigma$, where Σ is the set of all infinite sequences whose range is contained in $\{0, 1\}$.*

Observe the function,

$F : P(\mathbb{N}) \to \Sigma$ given by $F = \{(S, \{a_1, a_2, a_3, \dots\}) | S \varepsilon P(\mathbb{N}), a_n = 1 \text{ if } n \varepsilon S, \text{ 0 otherwise}\}$

Let $S, T \varepsilon P(\mathbb{N})$ where $F(S) = F(T)$, $F(S) = s_1, s_2, s_3, \dots$, $F(T) = t_1, t_2, t_3, \dots$. Then $s_1 = t_1, s_2 = t_2, \dots$, and so if a natural number $n \varepsilon S$, then $n \varepsilon T$, and vice versa. Therefore, $S \equiv T$, and F is one-to-one.
Let a_i be a sequence in Σ. Observe that a corresponding set $S \varepsilon P(\mathbb{N})$ can be obtained from the items in a_i equal to one. This can be done for any such sequence in Σ, and thus F is onto. Since F is onto and one-to-one, it follows that the domain and range of F have the same cardinality. Therefore, $P(\mathbb{N}) \sim \Sigma$, where Σ is the set of all infinite sequences whose range is contained in $\{0, 1\}$.

Assignment 27 *Prove (by contradiction) that Σ is not countably infinite.*

Assume, by way of contradiction, that Σ is countably infinite. Then then elements of Σ can be arranged in an infinite sequence, in example $\{s_1, s_2, s_3, s_4, \dots\}$, where:
$s_1 = \{0, 1, 0, 1, 0, \dots\}$
$s_2 = \{0, 0, 0, 1, 0, \dots\}$
$s_3 = \{0, 1, 1, 1, 0, \dots\}$
$s_4 = \{1, 1, 1, 1, 0, \dots\}$
$\dots$
Observe that we can obtain the sequence t_k, where $t_k^j = s_j^j + 1 \pmod 2$ with subscripts representing an item of the sequence and superscripts representing an element of said subsequence. for any arbitrary infinite sequence of Σ. However, since t_k has a uniquely different element from $s_1, s_2, s_3, \dots$, it follows that t_k is not in this sequence – and hence not in Σ – which is a contradiction. Therefore, Σ is not countably infinite.

Assignment 28 *Prove that if $A_1, A_2, A_3, \dots$ is a sequence of countably infinite sets, then $A = \cup_{n=1}^{\infty} A_n$ is also countably infinite.*

Let $F : \mathbb{N} \times \mathbb{N} \to A$ given by $F = \{((n, m), a_n^m) | (n, m) \varepsilon \mathbb{N} \times \mathbb{N}, a_n^m \varepsilon A\}$, and a_n^m is the mth item in A_n.
Consider $F(n_1, m_1) = F(n_2, m_2)$. Then $a_{n_1}^{m_1} = a_{n_2}^{m_2}$. While the values of these numbers may exist at other points in sequences, each value of F maps to a distinct element of a distinct sequence A_n. Thus, $|A| \leq |\mathbb{N} \times \mathbb{N}|$.
Observe that for a given element a_n^m of A, the corresponding element in $\mathbb{N} \times \mathbb{N}|$ can be found as (n, m). Thus, F is onto.
Since $|A| \leq |\mathbb{N} \times \mathbb{N}|$ and F is onto, it follows that A and $\mathbb{N} \times \mathbb{N}$ have the same cardinality. Therefore, A is countably infinite.

Exercise 1 *Give three elements* (a, b) *of* $\mathbb{N}^2$ *which satisfy* $(a, b) \sim (2, 5)$. *Determine whether* $(12, 26)$ *and* $(57, 71)$ *are equivalent.*

$(2, 5) \sim (4, 7) \sim (1, 4) \sim (10, 13)$
$12 + 71 = 83 = 26 + 57$. By definition, $(12, 26) \sim (57, 71)$

Assignment 2 *Verify that* $\sim$ *is an equivalence relation on* $\mathbb{N}^2$.

For $\sim$ to be an equivalence relation, it must be reflexive, symmetric, and transitive.
Let $(a, b) \varepsilon \mathbb{N}^2$. Since $a+b = a+b$, it follows that $(a, b) \sim (a, b)$. Thus, $\sim$ is reflexive.
Let $(a, b) \sim (c, d)$, $(a, b), (c, d) \varepsilon \mathbb{N}^2$. Then $a + d = b + c$. Since addition is commutative, $c + b = d + a$, and it follows that $(c, d) \sim (a, b)$. Hence, $\sim$ is symmetric.
Let $(a, b), (c, d), (e, f) \varepsilon \mathbb{N}^2$, with $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then

$$a + d = b + c, c + f = d + e$$

$$a - b = c - d, c - d = e - f$$

$$a - b = e - f \to a + f = b + e$$

Since $a + f = b + e$, $(a, b) \sim (e, f)$. Thus, $\sim$ is transitive.
$\sim$ is reflexive, symmetric, and transitive. Therefore, $\sim$ is an equivalence relation.

Exercise 3 *The relation* S *on* $\mathbb{N}^2$ *given by* $(m, n)S(m', n')$ *if and only if* $gcd(m, n) = gcd(m', n')$ *is also an equivalence relation.*

(a) Give three elements of the $\mathbb{N}^2$ which are equivalent to $(2, 5)$.
$(3, 4)$, $(5, 6)$, $(5, 19)$.

(b) Give three elements of the $\mathbb{N}^2$ which are equivalent to $(15, 9)$
$(3, 18)$,$(6, 21)$,$(9, 12)$.

Assignment 4 *Show that*

(a) $(m+1, n+1) \sim (m, n)$.
Let $(m, n) \varepsilon \mathbb{N}^2$. Observe that $m+1+n = m+n+1$, and thus $(m+1, n+1) \sim (m, n)$.

(b) $(m-1, n-1) \sim (m, n)$ unless $m = 1$ and/or $n = 1$.
Let $(m, n) \varepsilon \mathbb{N}^2$, $m > 1$, $n > 1$. Since $m - 1 + n = m + n - 1$, it follows that $(m-1, n-1) \sim (m, n)$.

Exercise 5 *Using the above definitions for addition and subtraction, complete the following:*

(a) Compute $[(2, 5)] + [(4, 9)]$ and $[(12, 7)] + [(3, 8)]$

- $[(2, 5)] + [(4, 9)] = [(6, 14)]$
- $[(12, 7)] + [(3, 8)] = [15, 15]$

(b) Verify that $(2, 5) \sim (10, 13) \sim (4, 7)$ and that $(4, 9) \sim (2, 7) \sim (12, 17)$. Then compute $[(10, 13)] + [(2, 7)]$ and $[(4, 7)] + [(12, 17)]$, and check that the sum is the same $\sim$ equivalence class computed in part (a).

- $2 + 13 = 15 = 5 + 10$, and $10 + 7 = 17 = 13 + 4$. Thus, $(2, 5) \sim (10, 13) \sim (4, 7)$. Likewise, since $4 + 7 = 11 = 9 + 2$ and $2 + 17 = 19 = 7 + 12$, it follows that $(4, 9) \sim (2, 7) \sim (12, 17)$.
- $[(10, 13)] + [(2, 7)] = [(12, 20)]$. $12 + 14 = 26 = 20 + 6$, so $(12, 20) \sim (6, 14)$.
- $[(4, 7)] + [(12, 17)] = [(16, 24)]$. $16 + 14 = 30 = 24 + 6$, so $(16, 24) \sim (6, 14)$.

(c) We identify the integer 3 with $\sim$ equivalence class $[(2, 5)]$. What integers are identified with the equivalence classes $\sim$ equivalence classes $[(4, 9)]$, $[(12, 7)]$, and $[(3, 8)]$? Translate the two additions of $\sim$ equivalence classes from part(a) into additions of integers.

- $[(4, 9)] \to 5$
- $[(12, 7)] \to -5$
- $[(2, 5)] + [(4, 9)] = [(6, 14)] \to 3 + 5 = 8$
- $[(12, 7)] + [(3, 8)] = [15, 15] \to -5 + 5 = 0$

(d) Compute $[(12, 7)] - [(3, 8)]$. Translate to a subtraction of integers.

- $[(12, 7)] - [(3, 8)] = [(12, 7)] + [(8, 3)] = [(20, 10)]$
- $-5 - 5 = -10$