MATH 345 Homework 8

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1 Problems

1. Let $f : X \to Y$ be function between sets X and Y. Let U and V be subsets of Y. Prove that

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V).$$

Proof. ($\subset$) Consider $x \in f^{-1}(U \cup V)$. Then $f(x) = y \in U \cup V$. Then $y \in U$ or $y \in V$. In any case, the preimage $f^{-1}(y) \subset f^{-1}(U) \cup f^{-1}(V)$. Since $f(x) = y$, $x \in f^{-1}(y)$ and thus $x \in f^{-1}(U) \cup f^{-1}(V)$. Hence, $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.
($\supset$) Now let $x \in f^{-1}(U) \cup f^{-1}(V)$. Then $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$, and $f(x) = y \in U$ or $f(x) = y \in V$, respectively. In either case, it follows that $y \in U \cup V$. Thus, the preimage $f^{-1}(y) \subset f^{-1}(U \cup V)$. Since $f(x) = y$ and $x \in f^{-1}(y)$, $x \in f^{-1}(U \cup V)$. Hence, $f^{-1}(U \cup V) \supset f^{-1}(U) \cup f^{-1}(V)$.
Therefore, $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. □

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4.2 Prove Theorem 4.8: Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.

Proof. ($\to$) Let $f : X \to Y$ be a continuous function. Consider the closed set $C \subset Y$. Its complement $Y - C$ is open in Y. By the definition of a continuous function (4.2), the preimage of the open set $Y - C$, $f^{-1}(Y - C)$, is open in X. By Theorem 0.22,

$$f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C) = X - f^{-1}(C).$$

Since the complement $X - f^{-1}(C)$ of the preimage if C is open in X, it follows that $f^{-1}(C)$ is closed in X. Thus, if a function $f : X \to Y$ is continuous then $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$.
($\leftarrow$) Now let $f^{-1}(C)$ be closed in X for every closed set $C \subset Y$. Consider an open set V in Y. We know that $f^{-1}(Y - V)$, the preimage of a closed set, is closed in X. Then by Theorem 0.22,

$$f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V).$$

Since the complement of the preimage of V is closed in X, it follows that $f^{-1}(V)$ is open in X. Thus, if $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$, then the function $f : X \to Y$ is continuous.
Therefore, a function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subset Y$. □

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4.4 Prove whether or not each given function is continuous.

1. The function $f : \mathbb{R}_l \to \mathbb{R}$, defined by $f(x) = 3x - 5$, where $\mathbb{R}_l$ is the real line with the lower limit topology. Consider a basis element (a, b) where $a < b$ of $\mathbb{R}$. Its preimage $f^{-1}((a, b)) = (\frac{a+5}{3}, \frac{b+5}{3})$ is a union of basis elements $\bigcup_{i=1}^{\infty}[\frac{a+5}{3} + \frac{1}{i}, \frac{b+5}{3})$ in the lower limit topology, so it is open in $\mathbb{R}_l$. By Theorem 4.3, since the preimage of basis elements in $\mathbb{R}$ is open in $\mathbb{R}_l$, f is continuous.

2. The function $g : \mathbb{R}_{fc} \to \mathbb{R}$, defined by $g(x) = 3x - 5$, where $\mathbb{R}_{fc}$ is the real line with the finite complement topology. Consider a basis element (a,b) where $a < b$ of $\mathbb{R}$. Its preimage $f^{-1}((a,b)) = (\frac{a+5}{3}, \frac{b+5}{3})$ has the complement $(-\infty, \frac{a+5}{3}] \cup [\frac{b+5}{3}, \infty)$ in $\mathbb{R}_f c$, which is not finite. Thus, this function is NOT continuous.

4.7 Suppose X is a space with topologies T_1 and T_2. Let $id : X \to X$ be the identity, $id(x) = x$, and assume that the domain X has the topology T_1 and that the range X has topology T_2. Show that id is continuous if and only if T_1 is finer than T_2.

Proof. ($\to$) Let $id : X_{T_1} \to X_{T_2}$, $id(x) = x$ be a continuous function. Consider and open set $U \in T_2$. Since id is continuous, $id^{-1}(U)$ is open in the domain X with T_1. $id^{-1}(U) = U$, so U is open in the domain X with T_1. Because any open set $U \in T_2$ is an open set $U \in T_1$, $T_2 \subset T_1$. Thus, T_1 is finer than T_2.
($\leftarrow$) Let T_1 be finer than T_2, and so $T_2 \subset T_1$. Consider an open set $U \in T_2$. Then $id^{-1}(U) = U$. Since $U \in T_2 \subset T_1$, the preimage of an open set U in X with T_2 is open in the domain X with T_1. Thus, id is a continous function.
Therefore, id is continuous if and only if T_1 is finer than T_2. $\square$

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4.8 Let $f : X \to Y$ be a continuous function. If x is a limit point of a subset A of X, is it true that $f(x)$ is a limit point of $f(A)$ in Y? Prove this or find a counterexample.

Consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ where $fx = 2$, and the set $A = (0,1)$. The limit points $A' = \{0,1\}$, and $f(A) = \{2\}$ and $f(0) = 2$. However, for 2 to be a limit point of $f(A)$, all open sets containing 2 must intersect $f(A)$ at a point other than 2, which is impossible. This is a contradiction, and thus if x is a limit point of a subset A of X, then it is not necessarily true that $f(x)$ is a limit point of $f(A)$ in Y.

4.10 Let $f : X \to Y$ be a function. The graph of f is the subset of $X \times Y$ given by $G = \{(x, f(x)) | x \in X\}$. Show that, if f is continuous and Y is Hausdorff, then G is closed in $X \times Y$.

Proof. Let $(x,y) \in (X \times Y) - G$. Since $y \notin G$, $y \neq f(x)$. Because Y is a Hausdorff space, there exist disjoint neighborhoods U and V where $y \in U$, $f(x) \in V$. And because f is continuous, there exists a neighborhood W of x where $f(W) \subset V$.
From this, we can see the open set $W \times U$ in $X \times Y$, where $(x,y) \in W \times U$. Let $(z, f(z)) \in G$. Then either:

1. $z \notin W$, so $(z, f(z)) \notin W \times U$
2. $z \in W$, but then $f(z) \in V$, so $f(z) \notin U$ and thus $(z, f(z)) \notin W \times U$.

So any point in G is not in $W \times U$, and thus $(W \times U) \cap G = \varnothing$ and $(W \times U) \subset (X \times Y) - G$. Hence, for any point $(x,y) \in (X \times Y) - G$, there exists a neighborhood $W \times U$ of (x,y) contained in $(X \times Y) - G$ disjoint from G. Thus, $(X \times Y) - G$ is open, and G is closed in $(X \times Y)$.
Therefore, if f is continuous and Y is Hausdorff, then G is closed in $X \times Y$. $\square$

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4.11 Let $f : X \to Y$ be continuous, and let A be a subspace of X. Prove that $f|A : A \to Y$, the restriction of f to A, is continuous.

Proof. Consider an open set V in Y. The preimage of this set

$$f|A^{-1}(V) = f^{-1}(V) \cap A,$$

By the definition of the restriction of f to A. Since f is continuous, $f^{-1}(V)$ is open in X. It follows that $f^{-1}(V) \cap A$ is open in A by the definition of the subspace topology. Thus, the preimage of an open set V in Y is open in A, so $f|A$ is continuous.
Therefore, $f|A : A \to Y$, the restriction of f to A, is continuous. $\square$

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4.13a Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be continuous functions. Show that $h : X \to Y_1 \times Y_2$, defined by $h(x) = (f_1(x), f_2(x))$, is continuous as well.

Proof. Let $U \times V$ be an arbitrary basis element in $Y_1 \times Y_2$, so U is open in Y_1 and V is open in Y_2. Consider $x \in h^{-1}(U \times V)$. Observe that

$$\begin{gathered} x \in h^{-1}(U \times V) \\ \to h(x) \in U \times V \\ \to (f_1^{-1}(x), f_2^{-1}(x)) \in (U \times V) \\ \to f_1(x) \in U \text{ and } f_2(x) \in V \\ \to x \in f_1^{-1}(U) \text{ and } x \in f_2^{-1}(V) \\ \to x \in f_1^{-1}(U) \cap f_2^{-1}(V), \end{gathered}$$

So $h^{-1}(U \times V) \subset f_1^{-1}(U) \cap f_2^{-1}(V)$. Likewise, we can see with $x \in f_1^{-1}(U) \cap f_2^{-1}(V)$ that

$$\begin{gathered} x \in f_1^{-1}(U) \cap f_2^{-1}(V) \\ \to x \in f_1^{-1}(U) \text{ and } x \in f_2^{-1}(V) \\ \to f_1(x) \in U \text{ and } f_2(x) \in V \\ \to (f_1^{-1}(x), f_2^{-1}(x)) \in (U \times V) \\ \to h(x) \in U \times V \\ \to x \in h^{-1}(U \times V) \end{gathered}$$

So $h^{-1}(U \times V) \supset f_1^{-1}(U) \cap f_2^{-1}(V)$, and $h^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. Because f_1 and f_2 are continuous functions, it follows that the preimages $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open sets in X. This makes $h^{-1}(U \times V)$ a finite intersection of open sets, and it follows that preimages of basis elements in $Y_1 \times Y_2$ in h are open in X. Thus, by Theorem 4.3, h is a continuous function. Therefore, $h : X \to Y_1 \times Y_2$, defined by $h(x) = (f_1(x), f_2(x))$, is continuous. □

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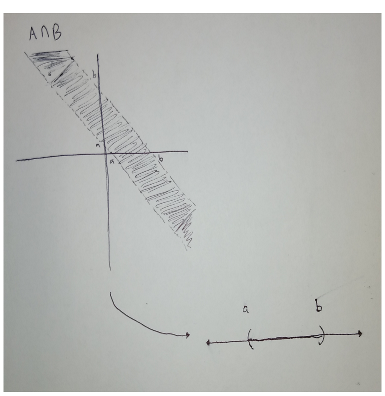
4.14 Show that the addition function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = x + y$, is a continuous function.

Proof. Let (a, b) with $a < b$ be an arbitrary basis element of $\mathbb{R}$. Consider the preimage,

$$f^{-1}((a, b)) = A \cap B, \text{ where } A = \{(x, y) | x + y > a\} \text{ and } \{(x, y) | x + y < b\}.$$



Let (p, q) be a point in $A \cap B$. There exists a circular open set in $\mathbb{R}^2$ centered at (p, q) with radius δ where

$x + y - a > \delta$ and $b - x + y > \delta$, so the open set is contained in $A \cap B$. Thus, every point in $A \cap B$ has a neighborhood contained in $A \cap B$, and so $A \cap B$ is open. Since $f^{-1}((a,b))$ is open, the preimage of an arbitrary basis element in $\mathbb{R}$, it follows that f is continuous by Theorem 4.3.

Therefore, the addition function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x,y) = x + y$, is a continuous function. $\square$