MATH 345 Homework 6

T.J. Liggett

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1 Problems

2.24 Determine δA in each case.

(a) $A = (0, 1]$ in the lower limit topology on $\mathbb{R}$. $\delta A = \{0, 1\}$.

(b) $A = \{a\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$. $\delta A = \{b, c\}$.

(c) $A = \{a, c\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$. $\delta A = \{b, c\}$.

(d) $A = \{b\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$. $\delta A = \{b, c\}$.

(e) $A = (-1, 1) \cup \{2\}$ in the standard topology on $\mathbb{R}$. $\delta A = \{-1, 1, 2\}$.

(f) $A = (-1, 1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$. $\delta A = \{-1, 2\}$.

(g) $A = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology. $\delta A = A$

2.26 Determine the boundary of each of the following subsets of $\mathbb{R}^2$ in the standard topology:

(a) $A = \{(x, x) \in \mathbb{R}^2 | x \in \mathbb{R}\}$. $\delta A = A$.

(b) $B = \{(x, y) \in \mathbb{R}^2 | x > 0, y \neq 0\}$. $\delta B = \{(0, y) | y \in \mathbb{R}\} \cup \{(x, 0) | x \geq 0\}$.

(c) $C = \{(\frac{1}{n}, 0) \in \mathbb{R}^2 | n \in \mathbb{Z}_+\}$. $\delta C = C$

(d) $D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x^2 - y^2 < 1\}$ $\delta D = \{(x, y) \in \mathbb{R}^2 | x^2 - y^2 = 1\}$.

2.28 Prove Theorem 2.15: Let A be a subset of a topological space X.

(a) δA is closed.

Proof. From part b below, we can see that δA is an intersection of closed sets. Since we know that the intersection of closed sets is closed, we know δA is closed. □

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(b) $\delta A = Cl(A) \cap Cl(X - A)$

Proof. First, let $x \in \delta A$. Then by definition, $x \in Cl(A)$, and $x \notin Int(A)$. So, $x \in X - Int(A)$. By Theorem 2.6, $X - Int(A) = Cl(X - A), x \in Cl(X - A)$. Thus, $x \in Cl(A) \cap Cl(X - A)$, and $\delta A \subset Cl(A) \cap Cl(X - A)$.
Next, let $x \in Cl(A) \cap Cl(X - A)$ We know that $x \in Cl(A)$, $x \in Cl(X - A)$. So $x \in X - Int(X)$ by Theorem 2.6, and thus $x \notin Int(A)$, $x \in Cl(A) - Int(A)$. Hence, $Cl(A) \cap Cl(X - A) \subset \delta A$.
Therefore, $\delta A = Cl(A) \cap Cl(X - A)$. □

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(g) $\delta A = \varnothing$ if and only if A is both open and closed.

Proof. ($\rightarrow$) Let $\delta A = \varnothing$. Since $Cl(A) - Int(A) = \varnothing$, it follows that $Cl(A) = Int(A) = A$. Since A is equal to both its interior and closure, by Theorem 2.2 A is both open and closed.
($\leftarrow$) Let A be both closed and open. Then by Theorem 2.2, $A = Int(A)$ and $A = Cl(A)$, and hence $Cl(A) = Int(A)$. It follows that $\delta A = Cl(A) - Int(A) = \varnothing$.
Therefore, $\delta A = \varnothing$ if and only if A is both open and closed. □

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3.3 Prove Theorem 3.4: Let X be a topological space, and let $Y \subset X$ have the subspace topology. Then $C \subset Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X.

Proof. ($\rightarrow$) Let $C \subset Y$ be closed in Y. Then $C = Y - V$ for some open set V in Y, where $V = U \cap Y$ and U is an open set in X. Consider $D = X - U$, so D is a closed set in X. Then

$$D \cap Y = (X - U) \cap Y = (X \cap Y) - (U \cap Y) = Y - (U \cap Y) = Y - V = C,$$

So $C = D \cap Y$. Thus, if $C \subset Y$ is closed in Y, then $C = D \cap Y$ for some closed set D in X.

($\leftarrow$) Let $C = D \cap Y$, where D is closed in X. Consider $U = X - D$, an open set in X.

$$U \cap Y = (X - D) \cap Y = (X \cap Y) - (D \cap Y) = Y - (D \cap Y) = Y - C.$$

Since $Y - C = U \cap Y$, by the definition of the subspace topology the complement of C is an open set in the subspace topology, and thus C is closed in Y. Hence, if $C = D \cap Y$, then $C \subset Y$ is closed in Y.

Therefore, $C \subset Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X. □

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3.4 Let $Y = (0,5]$ inherit the standard topology:

(a) $(0,1)$ open

(b) $(0,1]$ closed

(c) $\{1\}$ closed

(d) $(0,5]$ both

(e) $(1,2)$ open

(f) $[1,2)$ neither

(g) $(1,2]$ neither

(h) $[1,2]$ closed

(i) $(4,5]$ neither

(j) $[4,5]$ closed

3.5 Let $Y = (0,5]$ inherit the lower limit topology:

(a) $(0,1)$ both

(b) $(0,1]$ neither

(c) $\{1\}$ neither

(d) $(0,5]$ both

(e) $(1,2)$ closed

(f) $[1,2)$ both

(g) $(1,2]$ neither

(h) $[1,2]$ neither

(i) $(4,5]$ neither

(j) $[4,5]$ both

3.7 Let X be a Hausdorff topological space, and Y be a subset of X. Prove that the subspace topology on Y is Hausdorff.

Proof. Let X be a Hausdorff topological space, and Y be a subset of X. Consider $x,y \in X$, where $x,y \in Y$ as well. There exist disjoint neighborhoods U and V for x and y in X because it is Hausdorff. Consider the open sets in the subspace topology on Y, by definition, $U \cap Y$ and $V \cap Y$. Then

$$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \varnothing \cap Y = \varnothing$$

So, since U and V are disjoint, $U \cap Y$ and $V \cap Y$ are disjoint as well. Thus, there exist disjoint neighborhoods $x \in U \cap Y$ and $y \in V \cap Y$ in the subspace topology, for arbitrary elements in Y.
Therefore, if X is a Hausdorff topological space, and Y is a subset of X, then the subspace topology on Y is Hausdorff. □

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3.8 **(a)** Let X be a topological space, and let $Y \subset X$ have the subspace topology. If A is open in Y, and Y is open in X, show that A is open in X.

Proof. Let A be open in the subspace topology Y, and Y be open in X. By definition of the subspace topology, $A = U \cap Y$ for some open set U in X. Since U and Y are both open, A is a finite intersection of open sets in X, and is therefore an open set in X by the definition of a topology.
Therefore, if A is open in Y, and Y is open in X, then A is open in X. □

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3.9

(a) Let $K = \{\frac{1}{n} \in \mathbb{R} | n \in \mathbb{Z}_+\}$. Show that the standard topology on K is the discrete topology.

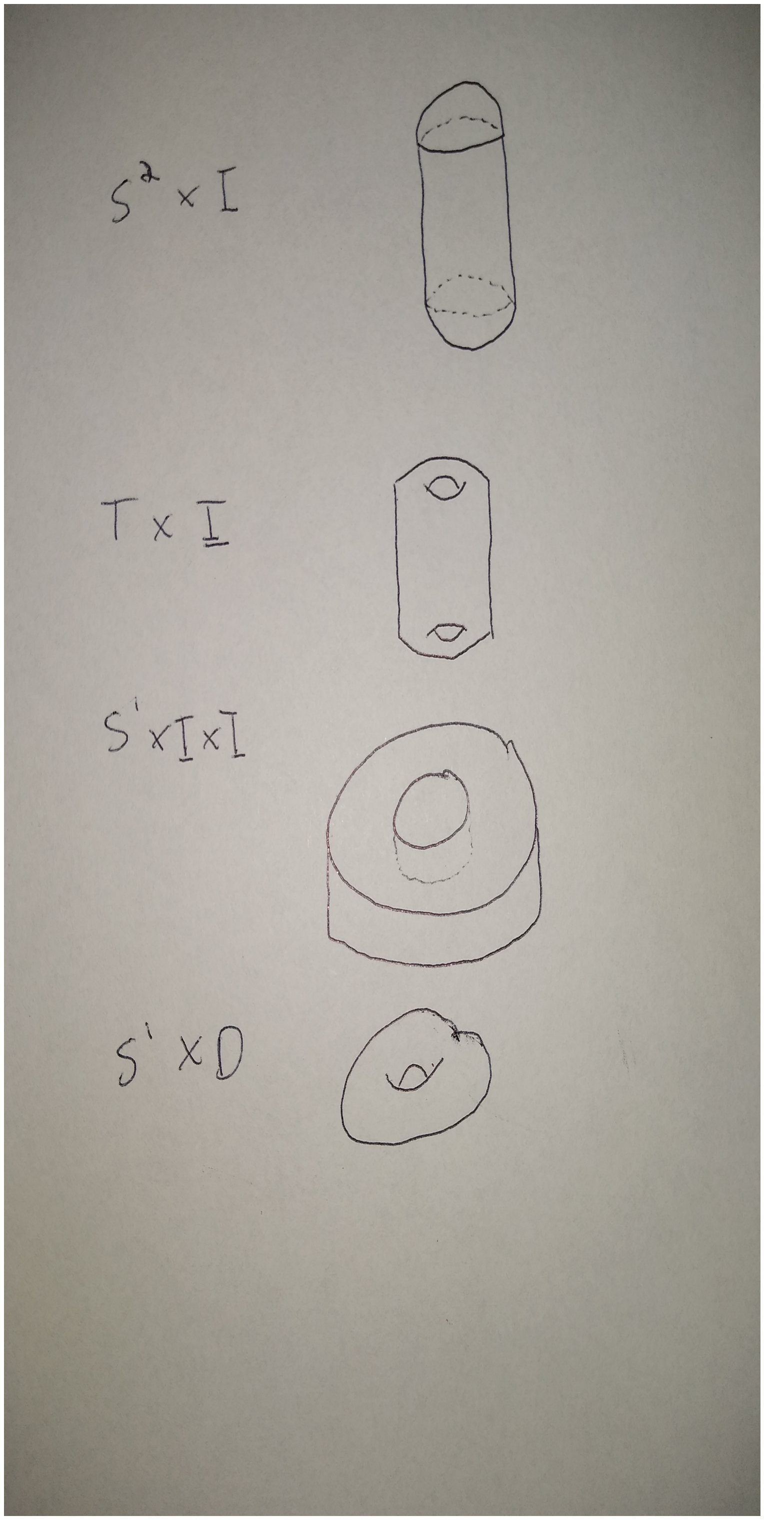
(b) Let $K^* = K \cup \{0\}$. Show that the standard topology on K^* is not the discrete topology.

a

Proof. For any element $k \in K$, $k = \frac{1}{n} | n \in \mathbb{Z}_+$, the open set in the standard topology $(\frac{1}{n+1}, \frac{1}{n+1})$ generates the singleton set $\{k\}$ in the subspace topology for K. Using unions of these open sets, we can generate any subset of K, making the standard topology on K the discrete topology. □

b

Proof. BWOC, assume the standard topology on K^* is the discrete topology. Consider the singleton set $\{0\} \in K^*$. Let (a,b) be an open set in the standard topology on $\mathbb{R}$ that contains $\mathbb{R}$. For any value of $b > 0$, there exists an element in $k = \frac{1}{n} | n \in \mathbb{Z}_+$ where $b < k$. Thus, this set will not generate the singleton set in the subspace topology. This is a contraction.
Therefore, the standard topology on K^* is not the discrete topology. □



3.16 Pictures!

3.18 Show that if X and Y are Hausdorff spaces, then so is the product space $X \times Y$.

Proof. Let X and Y be Hausdorff spaces, consider the disjoint open sets $u_x \in U_x$ and $v_x \in V_x$ in X and $u_y \in U_y$ and $v_y \in V_y$ in Y. Consider $(u_x, u_y) \in U_x \times U_y$ and $(v_x, v_y) \in V_x \times V_y$. By definition of the product topology, $(U_x \times U_y)$ and $(V_x \times V_y)$ are open sets in $X \times Y$, and

$$(U_x \times U_y) \cap (V_x \times V_y) = (U_x \cap V_x) \times (U_y \cap V_y) = \varnothing \times \varnothing = \varnothing.$$

So, $(U_x \times U_y)$ and $(V_x \times V_y)$ are disjoint neighborhoods of (u_x, u_y) and (v_x, v_y) in $X \times Y$, respectively. Therefore, the product space is a Hausdorff space. $\square$

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3.19 Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Let A be closed in X and B be closed in Y. Then $X - A$ is open in X and $Y - B$ is open in Y. Consider $(X - A) \times (Y - B) = (X \times Y) - (A \times B)$ is an open set in $X \times Y$. Since the complement of $(A \times B)$ is open in the product topology, it follows that $A \times B$ is closed.
Therefore, if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$. $\square$

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3.21 Determine whether or not the sets in Figure 3.13 are open, closed, both, or neither in the product topologies on the plane given by $\mathbb{R} \times \mathbb{R}$, $\mathbb{R}_l \times \mathbb{R}$, and $\mathbb{R}_l \times \mathbb{R}_l$, where $\mathbb{R}_l$ is the real line in the lower limit topology. 1 $\mathbb{R} \times \mathbb{R}$, 2 $\mathbb{R}_l \times \mathbb{R}$, and 3 $\mathbb{R}_l \times \mathbb{R}_l$ (a) Closed, Closed, Closed (b) Neither, Both, Closed (c) Neither, Neither, Both