MATH 200 Homework 2

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1 Preliminaries

Assignment 1.8 *Prove Theorem 1.1 of Euler*

konig.png

The Konigsberg Bridges
Image found at 'https://www.researchgate.net/figure/The-Koenigsberg-bridge-problem-a-seven-bridges-of-Koenigsberg-b-graph-representation$_{f}ig3_265219734'$

Observe the figure above, in which (a) is a drawing of the Konigsberg bridges, and (b) is a graph in which the edges represent the seven bridges and the vertices represent the surfaces connected by these bridges.

For any given vertex, consider an **arriving edge** to be any edge which the Euler circuit arrives, or travels to, the given vertex. Consider a **departing edge** to be the opposite; any edge which the Euler circuit departs the vertex.

In the graph above, all of the vertices contain an odd degree. Thus, each vertex must have an odd number of arriving and departing edges. Because of this, no circuit can start and end at the same vertex while visiting every edge. Therefore, the Konigsberg graph contains no Euler circuit.

Exercise 1.9 *Draw a graph which remains connected after the removal of any one edge, but which has no Hamiltonian circuit.*

ex1_12.png

The graph above remains connected after the removal of any one edge, but has no Hamiltonian circuit.

Exercise 1.10 *Suppose that a salesman lives in city A and must visit cities B, C, and D then return home. List all of the possible Hamiltonian circuits. Verify that your list confirms the formula above.*

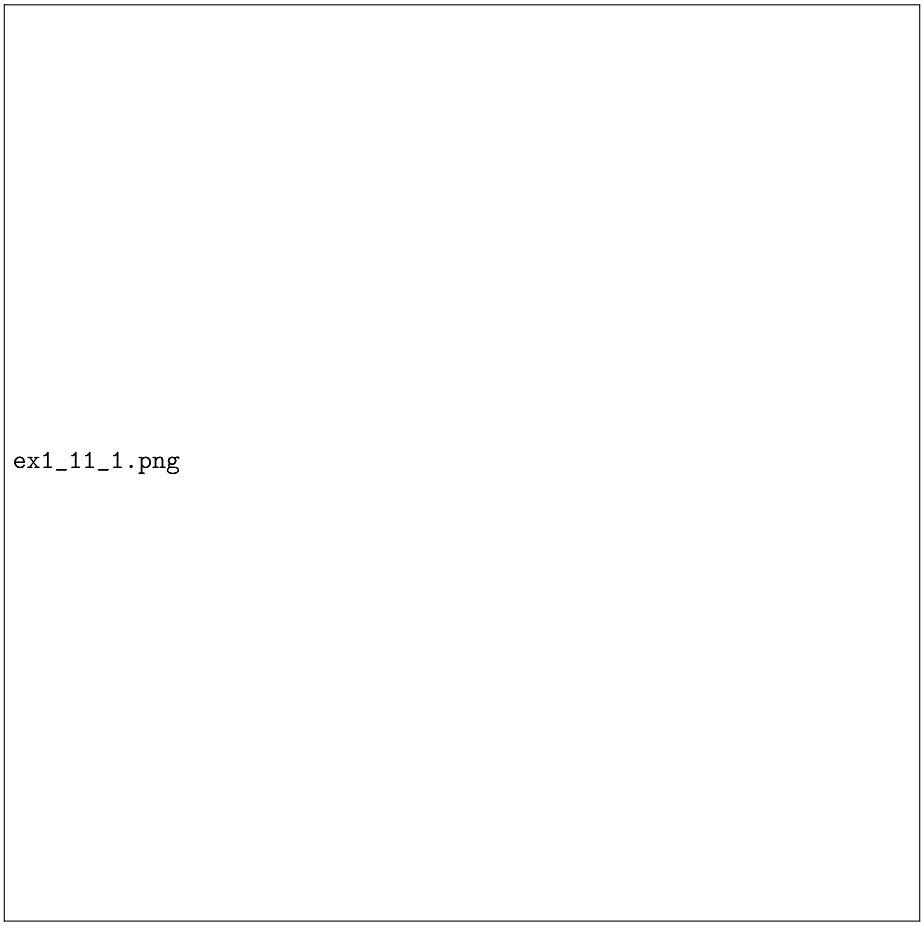
All of the possible Hamiltonian Circuits are listed below:

- $ABCDA$
- $ABDCA$
- $ACBDA$
- $ACDBA$
- $ADBCA$
- $ADCBA$

Observe that $(n-1)! = (4-1)! = 3! = 6$, which is exactly the number of possible Hamiltonian Circuits listed above.

Assignment 1.11 *Prove the Reduction Theorem.*

Consider a planar graph G that is non-trivalent. That is, G contains at least one point where more than three countries' boundary arcs intersect. Observe the figure below:



Notice that graph (b) is almost equivalent to graph (a), except at the point E where (a) is non-trivalent, a region is inserted and the graph becomes trivalent. If this process is repeated at every point in G where G is non-trivalent, a trivalent graph G' can be created. As G' is trivalent, we can assume that G' can be four-colored. By removing the interior region we just created, we see that no two countries of the same color touch at point E. This is shown in the figure below:

ex1_11_2.png

With this, we can infer that by reducing a graph G to its trivalent counterpart G' as done above, we are able to four-color the countries of G that intersect where G is not trivalent. Therefore, if it were possible to four-color any trivalent map on the plane, then it would be possible to four-color any planar map.

Exercise 1.12 *Make a copy of the planar map below, and carry out the process described above: place a vertex in the interior of each "country," and connect a pair of vertices with an edge if the two "countries" share a boundary arc. Now use four colors to color the vertices in such a way that no two adjacent vertices are the same color.*

ex1.png

Exercise 1.13 *In each of the maps depicted below find the number V of vertices, the number E of edges, and the number R of regions where the exterior region is always included as a region. What relationship exists between V, E, and R? Draw three additional maps to see if your conjecture holds in those graphs.*

ex1_13_1.jpg

- Graph (a) has 9 vertices, 12 edges, and 5 regions.
- Graph (b) has 10 vertices, 13 edges, and 5 regions.

- Graph (c) has 12 vertices, 18 edges, and 8 regions.

Conjecture: For all connected planar graphs, $V - E + R = 2$, where V, E, and R are the number of vertices, edges, and regions, respectively. Observe graphs (d), (e), and (f) below:

ex1_13_2.png

- Graph (d) has 6 vertices, 7 edges, and 3 regions.
- Graph (e) has 4 vertices, 4 edges, and 2 regions.
- Graph (f) has 5 vertices, 6 edges, and 3 regions.

Thus, the conjecture holds true for these graphs.

Assignment 1.14 *The fact that $V - E + R = 2$ for these examples of connected planar graphs with no loops suggests that the same is true of any such* *graph. Prove or disprove the foregoing conjecture.*

Consider an arbitrary, connected, planar graph G with V vertices, E edges, and R regions, such that $V - E + R = T$. To remove an edge e_1 from G, we must also remove either a region or a vertex.
In the case that the one of the vertices connected by e_1 has a degree of 1, that vertex must be removed from the graph as well. Otherwise, the vertex will have a degree of 0 and G will no longer be connected. Since the vertex has a degree of 1, it cannot enclose a region on the graph, and no regions would be removed. Thus, the relationship between vertices, edges, and regions will remain $V - E + R = T$, as V and E will both decrease by 1.
In the case that both vertices connected by e_1 have a degree greater than 1, the graph will still be connected if e_1 is removed and both vertices are kept. However, an edge in this case either divides an interior region from the exterior region or it divides two interior regions. By removing e_1, a region on the graph will be removed, and both E and R will decrease by one. Thus, the relationship between vertices, edges, and regions will remain $V - E + R = T$.
To remove a vertex v_1 from graph G, all edges connected to v_1 must be removed. For a vertex of degree 1, one edge will be removed as argued above. For a vertex with degree greater than 1, each additional edge removed will remove a region from the graph, also argued above. Likewise, a region can only be removed by removing an edge without removing a vertex.
If vertices, edges, and regions of G are removed methodically by removing one edge at a time, the relationship will be preserved as either a region or vertex will be removed each time. When G has zero edges remaining, it will be a connected, planar graph where $V + R = T$. Since a connected graph must contain $V - 1$ edges, G will be isomorphic to the connected graph on one vertex, which contains one region. Notice that $V - E + R = 2$ for G. Since this relationship between vertices, edges, and regions has stayed consistent as edges have been removed, $V - E + R = 2$ for the arbitrary graph G.
Therefore, for all connected planar graphs, $V - E + R = 2$, where V, E, and R are the number of vertices, edges, and regions, respectively.