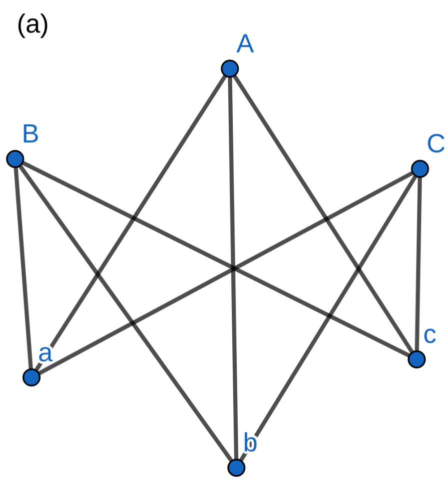
Foundations Homework 3

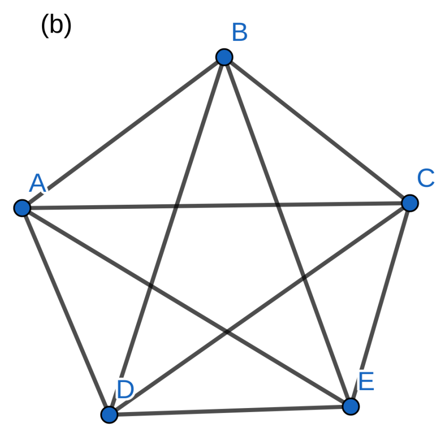
TJ Liggett

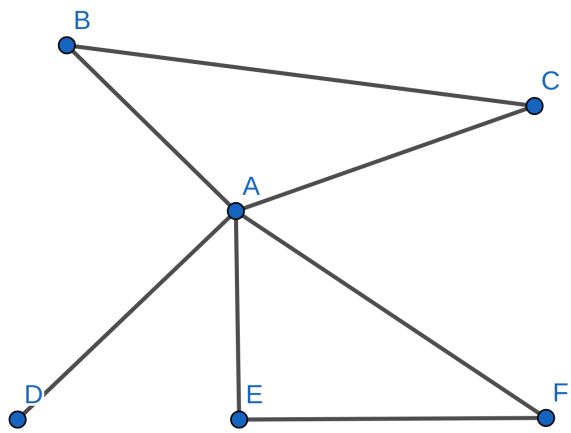
16 September 2019

Chapter One

Exercise 15 *Draw the two graphs represented by incidence matrices (a) and (b). Then give the incidence matrix for the graph (c) shown below.*







$$(c) = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Exercise 16 *How can we tell if two incidence matrices represent the same graph? How can we tell if two graphs are isomorphic?*

Two incidence matrices A and B represent the same graph and are isomorphic if we can rearrange the rows and columns of B to match those of A.

Assignment 17 *Verify that in any connected planar graph with no loops and at least two (straight) edges, it must be the case that $\frac{3}{2}R \leq E \leq 3V - 6$.*

If edges are to be straight-line segments, every region in a connected planar graph must have at least three edges (given). Because edges of a graph are shared by two regions, it follows that $\frac{3}{2}R \leq E$. Observe that $V - E + R = 2$ for any connected planar graph. It follows that:

$$\frac{3}{2}(V - E + R) = \frac{3}{2}(2) \tag{1}$$

$$\frac{3}{2}V - \frac{3}{2}E + \frac{3}{2}R = 3 \tag{2}$$

$$\frac{3}{2}R = 3 - \frac{3}{2}V + \frac{3}{2}E \tag{3}$$

Through substitution to the found inequality, we observe that:

$$3 - \frac{3}{2}V + \frac{3}{2}E \leq E \tag{4}$$

$$\frac{E}{2} \leq \frac{3}{2}V - 3 \tag{5}$$

$$E \leq 3V - 6 \tag{6}$$

Therefore, we can conclude that $\frac{3}{2}R \leq E \leq 3V - 6$

Assignment 18 *Prove that the graphs corresponding to incidence matrices (a) and (b) above are non-planar.*

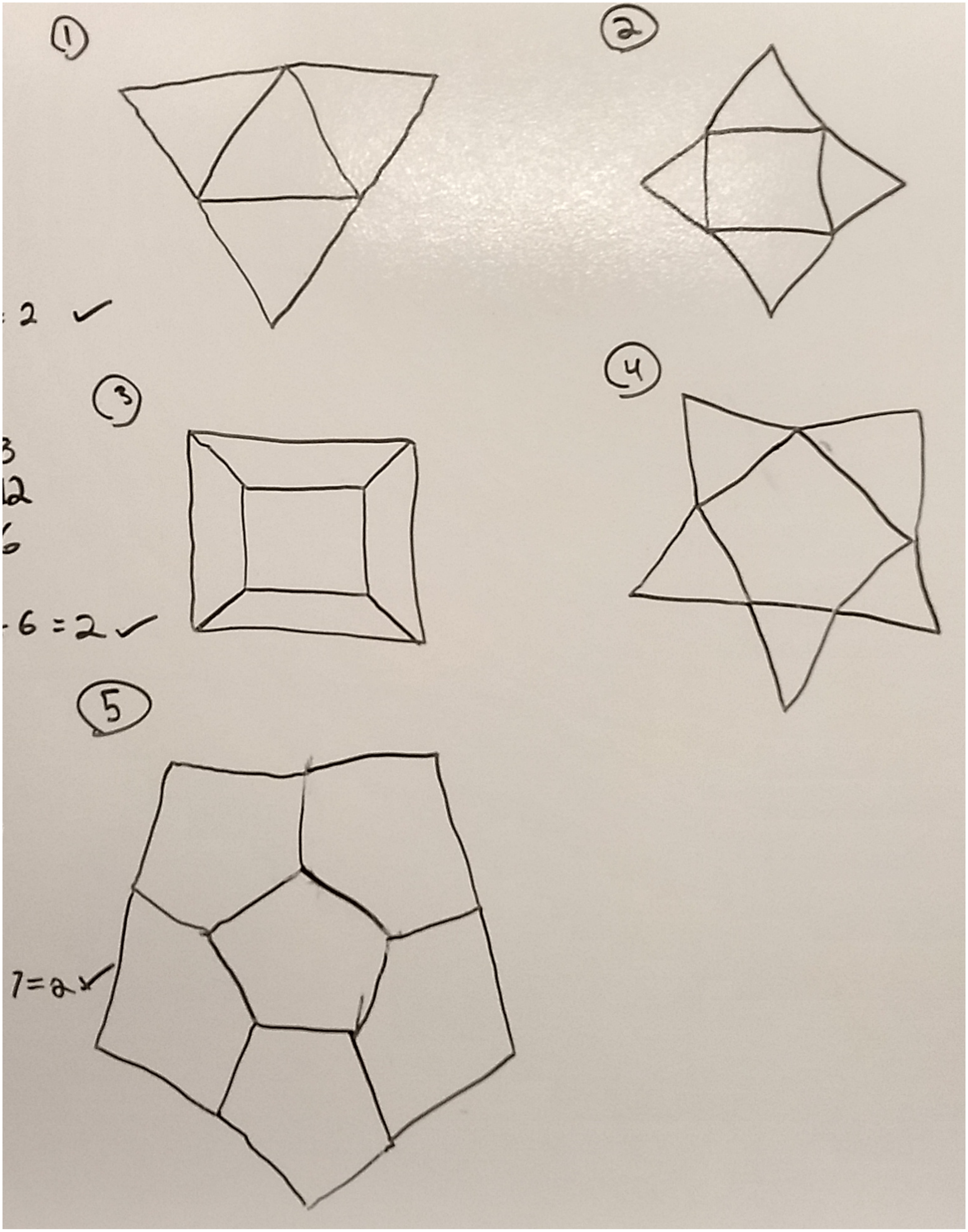
For the graph corresponding to incidence matrix (a), observe that the graph contains 6 vertices and 9 edges. By Euler's theorem, $R = 2 - V + E = 2 - 6 + 9 = 5$, *so there must be five regions in the graph if the graph is a connected planar graph. Observe that in the incidence matrix (a), there exist no circuits between less than four vertices. Because of this, each region must be bound by at least four edges. Since each edge is shared by two regions, there must exist at least* $\frac{4}{2}$ *edges per region, for a total of 10 edges for the 5 regions of the graph. However, there exist only 9 edges in the graph. Therefore, the graph corresponding to incidence matrix (a) is non-planar.*

For the graph corresponding to incidence matrix (b), observe that the (b) contains 10 edges and 5 vertices. However, $3(5) - 6 < 10$, *and does not satisfy the inequality* *proved above. Therefore, the graph corresponding to incidence matrix (b) is non-planar.*

Assignment 19 *Prove that a connected planar graph with at least two straight edges and no loops must have a vertex of degree $\delta < 6$.*

By way of contradiction, assume that a connected planar graph with at least two straight edges and no loops has no vertex of degree $\delta < 6$. In other words, all vertices of this graph have a degree of 6 or higher, and the total degree $T \geq 6V$. We know the total degree of a graph to be equal to twice the number of edges. From this, we can obtain that $6V \leq 2E$, or that $3V \leq E$. However, Theorem 1.4 states that $E \leq 3V - 6$, contradicting the previous statement. Therefore, by contradiction, a connected planar graph with at least two straight edges and no loops must have a vertex of degree $\delta < 6$.

Assignment 20 *Prove that there are only five Platonic solids.*



The five Platonic solids' projections onto a plane are shown above. In a Platonic solid, the same number of faces meet at each vertex. Because of this, the degree of *each vertex of the projection of each solid must be equal. In a three dimensional solid, each vertex must have a degree of at least three. By the theorem proved in Assignment 1.19, we know this degree must be less than six. Thus, the degree* δ *of each vertex in a Platonic solid must be* $3 \leq \delta \leq 5$.

Observe that any regular polygon with six edges (or more) has a degree of [120] or greater. If three of these polygons meet at a vertex, their degree will be greater than or equal to [360], which is impossible. For this reason, no Platonic solids can exist with faces of polygons with more than 5 edges.

The shape with the least amount of edges is a triangle, let us start there. Observe that no matter how many faces meet at each vertex, each edge will be shared by two faces, and $E = \frac{3}{2}R$.

Consider a Platonic solid where 3 triangles meet at each vertex. If $\delta = 3$, *then each vertex is shared by 3 regions, and* $V = \frac{3}{3}R = R$. *Substituting for the equation* $V - E + R = 2$, *we see that* $R - \frac{3}{2}R + R = 2$, *and* $R = 4$. *Hence, there exists a Platonic solid with 4 triangle faces, where 3 faces meet at each vertex.*

Consider a Platonic solid where 4 triangles meet at each vertex. If $\delta = 4$, *then each vertex is shared by 4 regions, and* $V = \frac{3}{4}R$. *Substituting for the equation* $V - E + R = 2$, *we see that* $\frac{3}{4} - \frac{3}{2}R + R = 2$, *and* $R = 8$. *Hence, there exists a Platonic solid with 8 triangle faces, where 4 faces meet at each vertex.*

Consider a Platonic solid where 5 triangles meet at each vertex. If $\delta = 5$, *then each vertex is shared by 5 regions, and* $V = \frac{3}{5}R$. *Substituting for the equation* $V - E + R = 2$, *we see that* $\frac{3}{5} - \frac{3}{2}R + R = 2$, *and* $R = 20$. *Hence, there exists a Platonic solid with 20 triangle faces, where 5 faces meet at each vertex.*

In a Platonic Solid created with squares, observe that no matter how many faces meet at each vertex, each edge will be shared by two faces, and $E = \frac{4}{2}R = 2R$. *Consider a Platonic solid where 3 squares meet at each vertex. If* $\delta = 3$, *then each vertex is shared by 3 regions, and* $V = \frac{4}{3}R$. *Substituting for the equation* $V - E + R = 2$, *we see that* $\frac{4}{3} - \frac{4}{2}R + R = 2$, *and* $R = 6$. *Hence, there exists a Platonic solid with 6 square faces, where 3 faces meet at each vertex.*

Consider a Platonic solid where 4 squares meet at each vertex. If $\delta = 4$, *then each vertex is shared by 4 regions, and* $V = \frac{4}{4}R$. *Substituting for the equation* $V - E + R = 2$, *we see that* $\frac{4}{4} - \frac{4}{2}R + R = 2$, *and* $R = 0$. *Since the number of regions cannot be zero, this Platonic solid does not exist. Likewise, a Platonic solid of squares with* $\delta = 5$ *have less than zero regions by the method above, and does not exist. In a Platonic Solid created with pentagons, observe that no matter how many faces meet at each vertex, each edge will be shared by two faces, and* $E = \frac{5}{2}R = 2R$. *Consider a Platonic solid where 3 Pentagons meet at each vertex. If* $\delta = 3$, *then each vertex is shared by 3 regions, and* $V = \frac{5}{3}R$. *Substituting for the*

equation $V - E + R = 2$*, we see that* $\frac{5}{3} - \frac{5}{2}R + R = 2$*, and* $R = 12$*. Hence, there exists a Platonic solid with 6 pentagon faces, where 3 faces meet at each vertex.*
Consider a Platonic solid where 4 pentagons meet at each vertex. If $\delta = 4$*, then each vertex is shared by 4 regions, and* $V = \frac{5}{4}R$*. Substituting for the equation* $V - E + R = 2$*, we see that* $\frac{5}{4} - \frac{5}{2}R + R = 2$*, and* $R = -2$*. Since the number of regions cannot be negative, this Platonic solid does not exist. Likewise, a Platonic solid of pentagons with* $\delta = 5$ *have less than -2 regions by the method above, and does not exist.*
Therefore, there exist only five Platonic solids.