MATH 345 Exam 3

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1 Problems

Theorem 6.24. The Intermediate Value Theorem (General Version): Let X be a connected topological space and $f : X \to \mathbb{R}$ be continuous. If $p, q \in f(X)$ and $p \le r \le q$, then $r \in f(X)$.

Proof. Suppose $f : X \to \mathbb{R}$ is continuous, $p, q \in f(X)$, and $p \le r \le q$. If $r = p$ or $r = q$, then we immediately have that $r \in f(X)$. Therefore we only need to consider the case where $p < r < q$.
Note that $f(X)$ is connected in $\mathbb{R}$ since X is connected and f is continuous [1].
BWOC, suppose that $r \notin f(X)$. Then $U = (-\infty, r)$ and $V = (r, \infty)$ are disjoint open subsets of $\mathbb{R}$ [2] whose union contains $f(X)$ [3]. Since $p \in U$ and $q \in V$, it follows that $f(X)$ intersects both U and V. Hence, U and V form a separation of $f(X)$ in $\mathbb{R}$ [4]. But this contradicts the fact that $f(X)$ is connected in $\mathbb{R}$ [5]. Therefore, $r \in f(X)$. □

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Theorem 6.25. The One-Dimensional Brouwer Fixed Point Theorem: Let $f : [-1, 1] \to [-1, 1]$ be continuous. There exists at least one $c \in [-1, 1]$ such that $f(c) = c$.

Proof. Let $f : [-1, 1] \to [-1, 1]$ be continuous. Define a function $g : [-1, 1] \to \mathbb{R}$ by $g(x) = f(x) - x$. The function g is continuous [6]. Note that $f(-1) \ge -1$, and therefore $g(-1) \ge 0$. Similarly $g(1) \le 0$ [7]. The Intermediate Value Theorem implies that there exists a value $c \in [-1, 1]$ such that $g(c) = 0$ (see Theorem 6.24 above). For such c it follows that $f(c) = c$ [8]. Therefore there exists at least one c in $[-1, 1]$ such that $f(c) = c$, as we wished to show. □

2 Supportive Results

[1] Theorem 6.6: If X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected in Y.

Proof. Suppose that $f(X)$ is not connected in Y. Then there exists open sets U and V that form a separation of $f(X)$ in Y. The function f is continuous, and therefore $f^{-1}(U)$ and $f^{-1}(V)$ are open in X [9]. Both U and V have nonempty intersections with $f(X)$; thus $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty [10]. Furthermore $f(X) \subset U \cup V$, implying that $X \subset f^{-1}(U) \cup f^{-1}(V)$ [11]. Finally, since $U \cap V \cap f(X) = \varnothing$, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint [12]. Therefore the pair of sets, $f^{-1}(U)$ and $f^{-1}(V)$, is a separation of X [13], contradicting the assumption that X is connected. Hence, $f(X)$ is connected in Y. □

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[2] Result 2: $U = (-\infty, r)$ and $V = (r, \infty)$ are disjoint open subsets of $\mathbb{R}$.

Proof. Since $U \cap V = \varnothing$, U and V are disjoint. We can see both U and V are both basis elements for $\mathbb{R}$ with the standard topology, and thus are both open sets. Therefore, U and V are disjoint open subsets of $\mathbb{R}$. □

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[3] Result 3: $f(X) \subset U \cup V$.

Proof. We know that $f(X) \in \mathbb{R}$, and $r \notin f(X)$. So $f(X) \notin \mathbb{R} - r = U \cup V$. Therefore, $f(X) \in U \cup V$. $\square$

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[4] **Result 4:** U and V form a separation of $f(X)$ in $\mathbb{R}$.

Proof. U and V are open sets in X. Also, $f(X) \subset U \cup V$, $U \cap f(X) \neq \varnothing$ ($p \in U$), and $V \cap f(X) \neq \varnothing$ ($q \in V$).
Since U and V are disjoint, $U \cap V \cap f(X) = \varnothing$.
Thus, by Definition 6.5, U and V form a separation of $f(X)$ in $\mathbb{R}$. $\square$

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Definition 6.5: Let A be a subspace of a topological space X. If U and V are open sets in X such that $A \subset U \cup V$, $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$, then we say that the pair of sets, U and V, is a separation of A in X. NOTE: This definition is built off of Theorem 6.4, which is proved below.

Theorem 6.4: A set A is disconnected in X if and only if there exist open sets U and V in X such that $A \subset U \cup V$, $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$.

Proof. Suppose that A is disconnected in X. Then there exist nonempty sets P and Q that are open in A, disjoint, and such that $P \cup Q = A$ [14]. Since P and Q are open in A there exist sets U and V that are open in X and such that $U \cap A = P$ and $V \cap A = Q$ [15]. Clearly, $A \subset U \cup V$, $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$.
Now suppose that U and V are open sets in X such that $A \subset U \cup V$, $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$. If we let $P = U \cap A$ and $Q = V \cap A$, then it follows that the pair of sets, P and Q, is a separation of A in the subspace topology [14], and therefore A is disconnected in X.
Therefore, a set A is disconnected in X if and only if there exist open sets U and V in X such that $A \subset U \cup V$, $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, and $U \cap V \cap A = \varnothing$. $\square$

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[5] **Result 5:** $f(X)$ has a separation but is connected, which is a contradiction.

Proof. By Definition 6.1, a connected space contains no separation, so this is a contradiction. $\square$

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[6] **Result 6:** Define a function $g : [-1,1] \to \mathbb{R}$ by $g(x) = f(x) - x$. The function g is continuous.

Proof. Let $h : [-1,1] \to [-1,1]$ given by $h(x) = -x$ be a function. Consider a basis element U in $[-1,1]$ of the form $(a,b) \cap [-1,1]$. $f^{-1}(U) = (-a,-b) \cap [-1,1]$ will be open in $[-1,1]$. $g(x)$ can be written as the sum of h and f, by $g(x) = f(x) + h(x)$. Since the sum of two continuous functions is continuous, it follows that g is continuous. $\square$

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[7] **Result 7:** Note that $f(-1) \geq -1$, and therefore $g(-1) \geq 0$. Similarly $g(1) \leq 0$.

$$g(-1) = f(-1) - (-1) \geq -1 + 1 = 0.\ g(-1) \geq 0.$$

$$g(1) = f(1) - 1 \leq 1 - 1 = 0.\ g(1) \leq 0.$$

[8] **Result 8:** For such c it follows that $f(c) = c$.

Proof. Note that $g(c) = 0$ as stated. Then

$$g(c) = f(c) - c = 0 \rightarrow f(c) = c.$$

So, $f(c) = c$. $\square$

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[9] **DEFINITION 4.2.** Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is continuous if $f^{-1}(V)$ is open in X for every open set V in Y.

[10] **Result 10:** Both U and V have nonempty intersections with $f(X)$; thus $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty.

Proof. Since U and V form a separation of $f(X)$ in Y, they both have nonempty intersections with $f(X)$. Because f is a continuous function, there exist sets in X that map into the parts U and V that intersect with $f(X)$. Thus, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. $\square$

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[11] **Result 11:** Furthermore $f(X) \subset U \cup V$, implying that $X \subset f^{-1}(U) \cup f^{-1}(V)$.

Proof. Since $f(X) \subset U \cup V$, $f^{-1}(f(X)) \subset f^{-1}(U \cup V)$ [16]. By [17], $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, so $X \subset f^{-1}(U) \cup f^{-1}(V)$. $\square$

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[12] **Result 12:** Since $U \cap V \cap f(X) = \varnothing$, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint.

Proof. Since $U \cap V \cap f(X) = \varnothing$, $f^{-1}(U \cap V \cap f(X) = f^{-1}(\varnothing) = \varnothing)$. Also, $f^{-1}(U \cap V \cap f(X)) = f^{-1}(U) \cap f^{-1}(V) \cap X$ [17]. Since X is our top space, it follows that $f^{-1}(U) \cap f^{-1}(V) = \varnothing$, and that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. $\square$

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[13] **DEFINITION 6.1.** Let X be a topological space.

1. We call X connected if there does not exist a pair of disjoint nonempty open sets whose union is X.
2. We call X disconnected if X is not connected.
3. If X is disconnected, then a pair of disjoint nonempty open sets whose union is X is called a separation of X.

[14] **DEFINITION 6.3.** A set A contained in a topological space X is said to be connected in X if A is connected in the subspace topology. If A is not connected in X, we say it is disconnected in X.

[15] **DEFINITION 3.1.** Let X be a topological space and let Y be a subset of X. Define $T_Y = \{U \cap Y | U \text{ is open in } X\}$. This is called the subspace topology on Y and, with this topology, Y is called a subspace of X. We say that $V \subset Y$ is open in Y if V is an open set in the subspace topology on Y.

[16] **DEFINITION 0.20** Given $f : X \rightarrow Y$ and a point $y \in Y$, define $f^{-1}(y)$, the preimage of y, to be the set $\{x \in X | f(x) = y\}$. Furthermore, given a subset W of Y, define $f^{-1}(W)$, the preimage of W, to be the set $\{x \in X | f(x) \in W\}$.

[17] **Theorem 0.22.** If $f : X \rightarrow Y$ is a function and V and W are subsets of Y, then

1. $f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$.
2. $f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W)$.