# Foundations Homework 11

TJ Liggett

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## Chapter Four

**Assignment 6** *Show that, if* $[(m',n')] = [(m,n)]$ *and* $[(p',q')] = [(p,q)]$*, then* $[(m',n')] + [(p',q')] = [(m,n)] + [(p,q)]$*. Explain how this proves that the addition of* $\sim$ *equivalence classes is a "well-defined" operation.*

Let $[(m',n')], [(m,n)], [(p',q')], [(p,q)] \varepsilon \mathbb{N}^2$, where $[(m',n')] = [(m,n)]$ and $[(p',q')] = [(p,q)]$. Then,

$$m' + n = n' + m, p' + q = q' + p, \text{ so that } n - m = n' - m', q - p = q' - p'.$$

$$m' + n + p' + q = m + n' + p + q'$$

$$\text{By the definition of } \sim\text{: } [(m' + p', n' + q')] = [(m + p, n + q)]$$

$$\text{By the definition of } \sim \text{ addition: } [(m',n')] + [(p',q')] = [(m,n)] + [(p,q)]$$

Therefore, if $[(m',n')] = [(m,n)]$ and $[(p',q')] = [(p,q)]$, then $[(m',n')] + [(p',q')] = [(m,n)] + [(p,q)]$. Because the output of this operation does not change regardless of which values of each equivalence class are inputed, this operation is well-defined. That is, the addition of two values will be the same as the addition of any two values in their respective equivalence classes.

**Assignment 7** *Verify that the set of all* $\sim$ *equivalence classes forms an abelian group under the addition defined above. Which equivalence class serves as e? How is* $[(m,n)]^{-1}$ *situated in* $\mathbb{N}^2$ *relative to* $[(m,n)]$*?*

Let $[(m,n)], [(p,q)], [(s,t)] \varepsilon \mathbb{N}^2$. Consider that

$$[(m,n)] + ([(p,q)] + [(s,t)]) = [(m,n)] + [(p+s, q+t)] = [(m+p+s, n+q+t)]$$

$$= [(m+p, n+q)] + [(s,t)] = ([(m,n)] + [p,q]) + [(s,t)]$$

Thus, $\forall [(m,n)], [(p,q)], [(s,t)] \varepsilon \mathbb{N}^2, [(m,n)] + ([(p,q)] + [(s,t)]) = ([(m,n)] + [p,q]) + [(s,t)]$.

Let $[(m,n)]\varepsilon\mathbb{N}^2$. Observe that for the equivalence class $[(s,s)]$ related to the integer 0,

$$[(m,n)]+[(s,s)] = [(m+s,n+s)] = [(s+m,s+n)] = [(s,s)]+[(m,n)]$$

Since $[(m+s,n+s)]$ corresponds to the integer $(n+s)-(m+s) = m-n$, it follows that $[(m,n)]+[(s,s)] = [(m,n)]$. Thus, for $[(s,s)]\varepsilon\mathbb{N}^2, [(m,n)]+[(s,s)] = [(s,s)]+[(m,n)] = [(m,n)]$. $(s,s)$, the equivalence class corresponding to the integer 0, serves as $e$.
Let $[(m,n)]\varepsilon\mathbb{N}^2$. Observe that for $[(n,m)]$,

$$[(m,n)]+[(n,m)] = [(m+n,n+m)] = [(s,s)] = e.$$

Thus, $\forall[(m,n)]\varepsilon\mathbb{N}^2, [(m,n)]^{-1} = [(n,m)]$.
Let $[(m,n)],[(p,q)]\varepsilon\mathbb{N}^2$. Consider that,

$$[(m,n)]+[(p,q)] = [(m+p,n+q)] = [(p+m,q+n)] = [(p,q)]+[(m,n)]$$

Hence, $\forall[(m,n)],[(p,q)]\varepsilon\mathbb{N}^2, [(m,n)]+[(p,q)] = [(p,q)]+[(m,n)]$.
Therefore, the set of all $\sim$ equivalence classes forms an abelian group under the addition defined above.

**Assignment 8** *Verify that the set of all non-zero $\sim$ equivalence classes forms an abelian semi-group under the multiplication defined above.*

Let $[(m,n)],[(p,q)],[(s,t)]\varepsilon\mathbb{N}^2$. Consider that

$$[(m,n)]\times([(p,q)]\times[(s,t)]) = [(m,n)]\times[(qs+pt,qt+ps)]$$
$$= [(n(qs+pt)+m(qt+ps), n(qt+ps)+m(qs+pt))]$$
$$= [(nqs+npt+mqt+mps, nqt+nps+mqs+mpt)]$$
$$= [(t(np+mq)+s(nq+mp), t(nq+mp)+s(np+mq))]$$
$$= [(nq+mp,np+mq)]\times[(s,t)] = ([(m,n)]\times[(p,q)])\times[(s,t)]$$

Thus, $\forall[(m,n)],[(p,q)],[(s,t)]\varepsilon\mathbb{N}^2, [(m,n)]\times([(p,q)]\times[(s,t)]) = ([(m,n)]\times[p,q])\times[(s,t)]$.

Let $[(m,n)]\varepsilon\mathbb{N}^2$. Observe that for the equivalence class $[(s,s+1)]$ related to the integer 1,

$$[(m,n)]\times[(s,s+1)] = [(ns+m(s+1), n(s+1)+ms)] = [((s+1)m+sn, (s+1)n+sm)] = [(s,s+1$$

Since $[(ns+ms+m, ns+n+ms)]$ corresponds to the integer $(n+ms+ns)-(m+ms+ns) = m-n$, it follows that $[(m,n)]\times[(s,s+1)] = [(m,n)]$. Thus, for

$[(s,s+1)]\varepsilon\mathbb{N}^2, [(m,n)] \times [(s,s+1)] = [(s,s+1)] \times [(m,n)] = [(m,n)]$. $(s,s+1)$, the equivalence class corresponding to the integer 1, serves as $e$.

Let $[(m,n)], [(p,q)]\varepsilon\mathbb{N}^2$. Consider that,

$[(m,n)]+[(p,q)] = [(np+mq, nq+mp)] = [(qm+pn, qn+pm)] = [(p,q)]\times[(m,n)]$

Hence, $\forall[(m,n)], [(p,q)]\varepsilon\mathbb{N}^2, [(m,n)] \times [(p,q)] = [(p,q)] \times [(m,n)]$.

**Assignment 9** *Verify the right and left distributive laws*

(a) $[(m,n)] \times ([(p,q)] + [(s,t)]) = [(m,n)] \times [(p,q)] + [(m,n)] \times [(s,t)]$

$$\begin{gathered}[(m,n)] \times ([(p,q)] + [(s,t)]) \\ = [(m,n)] \times [(p+s, q+t)] \text{ (definition of} \sim \text{addition)} \\ = [(n(p+s) + m(q+t), n(q+t) + m(p+s))] \text{ (definition of} \sim \text{multiplication)} \\ = [(np+ns+mq+mt, nq+nt+mp+ms))] \text{ (distributive property)} \\ = [(np+mq, nq+mp)] + [(ns+mt, nt+ms)] \text{ (definition of} \sim \text{addition)} \\ = [(m,n)] \times [(p,q)] + [(m,n)] \times [(s,t)] \text{ (definition of} \sim \text{multiplication)}\end{gathered}$$

(b) $([(m,n)] + [(p,q)]) \times [s,t] = [(m,n)] \times [(s,t)] + [(p,q)] \times [(s,t)]$

$$\begin{gathered}([(m,n)] + [(p,q)]) \times [(s,t)] \\ = ([(m+p, n+q)]) \times [(s,t)] \text{ (definition of} \sim \text{addition)} \\ = [((n+q)s + (m+p)t, (n+q)t + (m+p)s)] \text{ (definition of} \sim \text{multiplication)} \\ = [(ns+qs+mt+pt, nt+qt+ms+ps)] \text{ (distributive property)} \\ = [(mt+ns, ms+nt)] + [(pt+qs, ps+qt)] \text{ (definition of} \sim \text{addition)} \\ = [(m,n)] \times [(s,t)] + [(p,q)] \times [(s,t)] \text{ (definition of} \sim \text{multiplication)}\end{gathered}$$

(c) $[(m,n)] \times [(p,q)] = 0$ iff $[(m,n)] = 0$ or $[(p,q)] = 0$

Let $[(m,n)] \times [(p,q)] = 0$. Then,

$$(np+mq) - (nq+mp) = 0 \text{ (definition of} \sim \text{multiplication)}$$

$$\longrightarrow (np-nq) + (mq-mp) = 0$$

$$\longrightarrow n(p-q) - m(p-q) = 0$$

$$\longrightarrow (n-m)(p-q)=0$$

Observe that since $n-m, p-q \varepsilon \mathbb{Z}$, by the multiplication property of zero $(n-m)$ or $(p-q)$ must be 0. Therefore, if $[(m,n)] \times [(p,q)] = 0$ then $[(m,n)] = 0$ or $[(p,q)] = 0$.
For the other side, consider that the integer corresponding to $[(m,n)] \times [(p,q)]$ is equivalent to $(n-m)(p-q)$. Since the multiplication of integers is commutative, without loss of generality let $[(m,n)] = 0$. Translating this to a subtraction of integers, we see that $(m-n) = 0$. By the multiplication property of zero, $(n-m)(p-q) = 0$, and thus $[(m,n)] \times [(p,q)] = 0$. Thus, if $[(m,n)] = 0$ or $[(p,q)] = 0$, then $[(m,n)] \times [(p,q)] = 0$. Therefore, $[(m,n)] \times [(p,q)] = 0$ iff $[(m,n)] = 0$ or $[(p,q)] = 0$.

**Exercise 10** *Complete each of the following:*

(a) $[(2,5)] < [(3,17)]$

(b) $[(5,2)] > [(17,3)]$

(c) $[(m,m+1)] < [(n,n+3)]$

(d) $[(a,2a)] < [(b,b+2a)]$

(e) $[(a,a+4)] > [(b,b)]$

(f) $[(a+4,a)] = [(7,3)]$