MATH 345 Homework 2

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1 Problems

1.1 Determine all of the possible topologies on $X = \{a, b\}$.

$$T_1 = \{\varnothing, X\}$$
$$T_2 = \{\varnothing, \{a\}, X\}$$
$$T_3 = \{\varnothing, \{b\}, X\}$$
$$T_4 = \{\varnothing, \{a\}, \{b\}, X\}$$

1.4

1. Give an example of a space where the discrete topology is the same as the finite complement topology.

$$X = \{a, b, c\}$$

2. Make and prove (sike) a conjecture indicating for what class of sets the discrete and finite complement topologies coincide.

   The discrete and finite complement topologies coincide for all finite sets.

1.6 Define a topology on $\mathbb{R}$ (by listing the open sets within it) that contains the open sets (0,2) and (1,3) and that contains as few open sets as possible.

$$T = \{\varnothing, (0,2), (1,3), (1,2), (0,3), \mathbb{R}\}$$

1.7 Let X be a set and assume $p \in X$. Show that the collection T, consisting of $\varnothing$ and all subsets of X containing p, is a topology on X. This topology is called the **particular point topology** on X, and we denote it by PPX_p

Proof. Let X be a set and assume $p \in X$. Let T consist of $\varnothing$ and all subsets of X containing p.

1. By definition, $\varnothing \in X$. Because $X \subset X$ and $p \in X$, it follows that $X \in T$.

2. Let $U_1, \ldots U_n \in$ be elements of T. Consider the subset $\bigcap_{i=1}^{n} U_i \subset X$. Proceed by cases: $p \notin \bigcap_{i=1}^{n} U_i$ or $p \in \bigcap_{i=1}^{n} U_i$.

   <u>Case 1</u>: Suppose $p \notin \bigcap_{i=1}^{n} U_i$. Then, there exists $d \in D$ such that $p \notin U_d$. Since U_d is an element of T that

does not contain p, then $U_d = \varnothing$. Thus, it follows that $\bigcap_{i=1}^{n} U_i = \varnothing \in T$.

Case 2: Suppose $p \in \bigcap_{i=1}^{n} U_i$. Then the set $\bigcap_{i=1}^{n} U_i$ is a subset of X that contains p. Thus, $\bigcap_{i=1}^{n} U_i \in T$.

Therefore, the set $\bigcap_{i=1}^{n} U_i \in T$.

3. Let $\{U_d\}_{d\in D}$ be a collection of open sets indexed by a set D. Consider the subset $\bigcup_{d\in D} U_d \subset X$. Proceed by cases: $p \notin \bigcup_{d\in D} U_d \subset X$ and $p \in \bigcup_{d\in D} U_d \subset X$.

   Case 1: Suppose $p \notin \bigcup_{d\in D} U_d$. Then for each $d \in D$, $p \notin U_d$. Since the set $\varnothing$ is the only open set that does not contain p, it follows that for each $d \in D$, $U_d = \varnothing$. Thus $\bigcap_{i=1}^{n} U_i = \varnothing$, and $\bigcap_{i=1}^{n} U_i \in T$.

   Case 2: Suppose $p \in \bigcup_{d\in D} U_d$. Then $\bigcup_{d\in D} U_d$ is a subset of X that contains p. Thus, $\bigcup_{d\in D} \in T$.

   Thus, the set $\bigcup_{d\in D} \in T$.

Therefore, T is a topology on the set X. $\square$

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1.9 Let T consist of $\varnothing$, $\mathbb{R}$, and all intervals $(-\infty, p)$ for $p \in \mathbb{R}$. Prove that T is a topology on $\mathbb{R}$.

Proof. Let T consist of $\varnothing$, $\mathbb{R}$, and all intervals $(-\infty, p)$ for $p \in \mathbb{R}$.

1. By definition, $\varnothing$, $\mathbb{R}$ are elements of T.

2. Let $U_1, \ldots U_n \in T$ be elements of T. Consider the subset $\bigcap_{i=1}^{n} U_i \subset \mathbb{R}$. If any U_i is empty, then the intersection is empty and thus an open set. So assume each U_i is not empty, and each U_i is an interval $(-\infty, p)$ where $p \in \mathbb{R}$ or $p = \mathbb{R}$. Then $\bigcap_{i=1}^{n} U_i$ is of the form $(-\infty, s)$ where s is the upper bound of the smallest U_i. Since $s \in \mathbb{R}$ or $s = \infty$, it follows that $\bigcap_{i=1}^{n} U_i \in T$.

3. Let $\{U_d\}_{d\in D}$ be a collection of open sets indexed by a set D. Consider the subset $\bigcup_{d\in D} U_d \subset \mathbb{R}$. Suppose $\bigcup_{d\in D} U_d = \varnothing$. Then for each $d \in D, U_d = \varnothing$, and thus $\bigcup_{d\in D} U_d \in T$. So suppose $\bigcup_{d\in D} U_d \neq \varnothing$. Then at least one U_d is nonempty, and either of the form $(-\infty, p), p \in \mathbb{R}$ or $\mathbb{R}$. The union of these sets would be of the form $(-\infty, s), s \in \mathbb{R}$ or $\mathbb{R}$, where s is the upper bound of the largest U_i. Since $s \in \mathbb{R}$ or $\bigcup_{d\in D} U_d = \mathbb{R}$, it follows that $\bigcup_{d\in D} U_d \in T$.

Therefore, T is a topology of the set $\mathbb{R}$. $\square$