MATH 345 Exam 2

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1 Problems

Let A be a subset of a topological space X. Prove that $\partial A = \varnothing$ iff A is open and closed.

Proof. ($\rightarrow$) First, we must show that if $\partial A = \varnothing$, then A is open and closed.
Let A be a subset of a topological space X, where $\partial A = \varnothing$. By the definition of boundary, $Cl(A) - Int(A) = \varnothing$, so $Cl(A) = Int(A)$.
Consider that the interior of A is the union of all open sets contained in A, and thus $Int(A) \subset A$. Also, the closure of A is the intersection of all closed sets that contain A, and $A \subset Cl(A)$.
Since $Cl(A) = Int(A) \subset A$ and $A \subset Cl(A)$, it follows that $Cl(A) = A$ and $Int(A) = Cl(A) = A$. By Theorem 2.2, A is open as $Int(A) = A$ and A is closed as $Cl(A) = A$. Thus, if $\partial A = \varnothing$, then A is open and closed.

($\leftarrow$) Next, we show that if A is open and closed, then $\partial A = \varnothing$.
Let A be both open and closed. By Theorem 2.2, since A is closed $Cl(A) = A$ and since A is open $Int(A) = A$. Thus,

$$\partial A = Cl(A) - Int(A) = A - A = \varnothing.$$

Thus, if A is open and closed, then $\partial A = \varnothing$.

Therefore, $\partial A = \varnothing$ iff A is open and closed.

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