MATH 345 Exam 1

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1 Problems

Consider the set of integers $\mathbb{Z}$ with the digital line topology. Prove that the topological space is not Hausdorff and that there exists a singleton set that is not closed.

Proof. Consider the set of integers $\mathbb{Z}$ with the digital line topology T_{DL}. By way of contradiction, assume the topological space is Hausdorff. Then $\forall x, y \in \mathbb{Z}$, there exist disjoint neighborhoods $x \in X$, $y \in Y$.
Consider the case where $x = 0, y = 1$. Note that 0 is an element of only one basis element in the digital line topology,

$$B(0) = \{-1, 0, 1\}$$

Since the neighborhood X is an open set in the digital line topology and $0 \in X$, it is either a basis element, $B(0)$, or a union of basis elements

$$X = B(0) \cup B(x_1), ..., B(x_n)$$

where $0 \leq n < \infty$. In either case, it follows that $1 \in X$. However, X and Y are disjoint, but $1 \in X$ and $1 \in Y$, which is a contradiction.
Therefore, this topological space is NOT Hausdorff.

Similarly, by way of contradiction assume the singleton set $S = \{1\} \in T_{DL}$ is closed. Then the complement of this set is $S' = \mathbb{Z} - \{1\}$ is open. Since $0 \in S'$, and 0 is an element of only one basis element in the digital line topology, it follows that $B(0) = \{-1, 0, 1\} \subset S'$. But then $1 \in S'$, which is a contradiction.
Thus, the singleton set containing 1 is not closed.

Therefore, for the set of integers $\mathbb{Z}$ with the digital line topology, the topological space is not Hausdorff and there exists a singleton set that is not closed. $\square$