Foundations Homework 8

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Chapter Three

Exercise 4 *Let $A = 1, 2, 3$ and $B = a, b$. Find $A \times B$, $B \times A$, and $A \times A$*

$A \times B = (1, a), (1, b), (2, a), (2, b), (3, a), (3, b)$
$B \times A = (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)$
$A \times A = (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)$

Exercise 5 *The three curves in the figure (3.2) represent the relations $R_1 = (m, n)|m = n$, $R_2 = (m, n)|mn = 60$, $R_3 = (m, n)|m^2 + n^2 = 169$ in $\mathbb{N}_0 \times \mathbb{N}_0$*

(a) Identify the domains and ranges of R_1, R_2, and R_3.
R_1: $D = \{m|m\varepsilon\mathbb{N}_0\}$, $R = \{n|n\varepsilon\mathbb{N}_0\}$
R_2: $D = \{m|m|60, m\varepsilon\mathbb{N}\}$, $R = \{n|n \leq 60n\varepsilon\mathbb{N}\}$
R_3: $D = \{m|m \leq 13, m\varepsilon\mathbb{N}_0\}$, $R = \{n|n \leq 13, n\varepsilon\mathbb{N}_0\}$

(b) How many points of $\mathbb{N}_0 \times \mathbb{N}_0$ do each of the foregoing relations contain?
$R_1 : \infty$, $R_2 : 12$, $R_3 : 14$

(c) How would your answers to parts (a) and (b) change if "=" were replaced by "<" in the definitions of R_1, R_2, and R_3? R_1: $D = \{m|m\varepsilon\mathbb{N}_0\}$, $R = \{n|n\varepsilon\mathbb{N}\}$
R_2: $D = \{m|m < 60, m\varepsilon\mathbb{N}\}$, $R = \{n|n < 60n\varepsilon\mathbb{N}\}$
R_3: $D = \{m|m < 13, m\varepsilon\mathbb{N}_0\}$, $R = \{n|n < 13, n\varepsilon\mathbb{N}_0\}$ $R_1 : \infty$, $R_2 : 248$, $R_3 : 144$

Exercise 6 *Let $\mathcal{P}$ be the set of all people. Determine which of the following relations are equivalence relations. If a relation is not an equivalence relation, indicate which property or properties fail.*

(a) For p and $q\ \varepsilon\ \mathcal{P}$ say $p\ q$ if p and q were born in the same year.
This is an equivalence relation.

(b) For p and $q\ \varepsilon\ \mathcal{P}$ say $p\ q$ if p is younger that q.
This relation is not reflexive, as one cannot be younger than themselves. It is also not symmetric, as if one person is younger than the other then the other cannot be younger.

(c) For p and $q \varepsilon \mathcal{P}$ say $p\ q$ if p and q have the same number of children.
This is an equivalence relation.

(d) For p and $q \varepsilon \mathcal{P}$ say $p\ q$ if p has seen every movie that q has seen.
This relation is not symmetric or transitive.

(e) For p and $q \varepsilon \mathcal{P}$ say $p\ q$ if p and q both speak the same language.
This relation is not transitive.

Exercise 7 *Let $\mathcal{T}$ be the set of all triangles.*

(a) For two triangles $\triangle ABC$ and $\triangle XYZ \varepsilon \mathcal{T}$, write $\triangle ABC \cong \triangle XYZ$ if the triangles are congruent. Is $\cong$ an equivalence relation? Give a brief explanation.
Yes, this is an equivalence relation. It is reflexive because a triangle is congruent to itself. It is symmetric because if two triangles are congruent, it follows that both $\triangle ABC \cong \triangle XYZ$ and $\triangle XYZ \cong \triangle ABC$. And they are transitive because congruence among triangles is transitive.

(b) For two triangles $\triangle ABC$ and $\triangle XYZ \varepsilon \mathcal{T}$, write $\triangle ABC \asymp \triangle XYZ$ if one side of $\triangle ABC$ is the same length as a side of $\triangle XYZ$. Is $\asymp$ and equivalence relation? Give a brief explanation.
This is not an equivalence relation because it is not transitive. Let $\triangle ABC \asymp \triangle DEF$ and $\triangle DEF \asymp \triangle XYZ$. $\triangle DEF$ has one side in common with both other triangles, but that could be a different side of each triangle.

Exercise 8 *Choose a set S and give*

(a) an example of an equivalence relation on S, and

(b) a relation on S which fails to be an equivalence relation. Specify which of the three properties (reflexivity, symmetricity, and transitivity) fails.

The set of all linebackers on the Augustana football team is S. The equivalence relation on S is year of school. An example of a relation which fails is positions played. This is reflexive and symmetric, but not transitive. For example. Wick and I can both play Will, so we are related, and Wick and Seth can both play Sam, so they are related. However, Seth can only play Sam, and I can't play Sam, so we are not related, and the relation is not transitive.

Assignment 9 *Define a relation $\smile$ between matrices by stipulating that $A \smile B$ if and only if it is possible to obtain B from A by a sequence of interchanges of rows and/or interchanges of columns. Show that $\smile$ is an equivalence relation.*

For $\smile$ to be an equivalence relation, it must be shown that $\smile$ is reflexive, symmetric, and transitive. Let $R_1, R_2, \ldots, R_m$ and $C_1, C_2, \ldots, C_n$ denote row and column interchanges.

1. Observe that for any matrix A, it is possible to obtain A from itself without interchanging any rows or columns, and thus $A \smile A$. Hence, $\smile$ is reflexive.

2. Let R_{n-} be the inverse of a given row interchange R_n. Let A, B be matrices where $A \smile B$. Then B can be obtained from A by a sequence of interchanges of rows and/or columns $R_1, R_2, \ldots, R_m$ and $C_1, C_2, \ldots, C_n$. Observe that A can likewise be obtained from B through the inverse of these interchanges, $R_{1-}, R_{2-}, \ldots, R_{m-}$ and $C_{1-}, C_{2-}, \ldots, C_{n-}$. Thus, $B \smile A$ and $\smile$ is symmetric.

3. Let A, B, C be matrices where $A \smile B$ and $B \smile C$. Then B can be obtained from A by a sequence of interchanges of rows and/or columns $R_1, R_2, \ldots, R_m$ and $C_1, C_2, \ldots, C_n$, and C can be obtained from B by a separate sequence of interchanges of rows and/or columns $R_{m+1}, R_{m+2}, \ldots, R_p$ and $C_{n+1}, C_{n+2}, \ldots, C_q$. Observe that C can then be obtained from A by the sequence of interchanges of rows and/or columns $R_1, R_2, \ldots, R_p$ and $C_1, C_2, \ldots, C_q$. Thus, $A \smile C$ and so $\smile$ is transitive.

Since $\smile$ is reflexive, symmetric, and transitive, it follows that $\smile$ is an equivalence relation.

Exercise 10 *What are the equivalence classes induced by the relations in Exercises 3.6, 3.7, and 3.8?*

- 3.6a: Year born (. . . , 1999, 2000, 2001, . . .)
- 3.6c: Number of children (0, 1, 2, . . .)
- 3.7: All triangles congruent to each other represent an equivalence class
- 3.8: Year of school (True Freshman, Redshirt Freshman, Sophomore, Junior, Senior)