MATH 345 Homework 3

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1 Problems

1 Let (X, T_x) and (Y, T_y) be topological spaces. Consider the set $X \times Y$. Prove that the collection

$$B = \{U \times V \subset X \times Y | U \in T_x, V \in T_y\}$$

is a basis for a topology on $X \times Y$.

Proof. Consider that since $X \in T_x$ and $Y \in T_y$, $X \times Y \in B$. Then for $x \in X \times Y$ there exists the basis element $X \times Y \in B$.
Now, let $(x, y) \in U_1 \times V_1$ and $(x, y) \in U_2 \times V_2$ where $U_1 \times V_1, U_2 \times V_2 \in B$. Then $(x, y) \in U_1 \times V_1 \cap U_2 \times V_2$. Consider $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Because U_3 and V_3 are finite intersections of open sets in T_x and T_y, respectively, it follows that they are open sets in their relative topologies. It also holds that

$$U_3 \times V_3 = (U_1 \cap U_2) \times (V_1 \times V_2) = (U_1 \times V_1) \cap (U_2 \times V_2)$$

Thus, $(x, y) \in U_3 \times V_3 \subset (U_1 \times V_1) \cap (U_2 \times V_2)$, and the second condition for a basis holds.
Therefore B is a basis for a topology on $X \times Y$. $\square$

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1.39 For each point (m, n) in the digital plane, determine the smallest closed set containing (m, n).
m, n are even: $\{(m, n)\}$
m, n are odd: $\{(m-1, n-1), (m, n-1), (m+1, n-1), (m-1, n), (m, n), (m+1, n), (m-1, n+1), (m, n+1), (m+1, n+1)\}$
m is even, n is odd: $\{(m, n-1), (m, n), (m, n+1)\}$
m is odd, n is even: $\{(m-1, n), (m, n), (m+, n)\}$

2.1 (a-g) Determine $Int(A)$ and $Cl(A)$ in each case.

(a) $A = (0, 1]$ in the lower limit topology on $\mathbb{R}$. $Int(A) = (0, 1)$, $Cl(A) = [0, 1]$.

(b) $A = \{a\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$. $Int(A) = \{a\}$, $Cl(A) = X$

(c) $A = \{a, c\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$. $Int(A) = \{a\}$, $Cl(A) = X$

(d) $A = \{b\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$. $Int(A) = \varnothing$, $Cl(A) = \{b, c\}$

(e) $A = (-1, 1) \cup \{2\}$ in the standard topology on $\mathbb{R}$. $Int(A) = (-1, 1)$, $Cl(A) = [-1, 1] \cup \{2\}$

(f) $A = (-1, 1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$. $Int(A) = (-1, 1)$, $Cl(A) = [-1, 1) \cup \{2\}$

(g) $A = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology. $Int(A) = \varnothing$, $Cl(A)$

2.2 Prove Theorem 2.2, parts 2, 4, 6.

If C is a closed set in X and $A \subset C$, then $Cl(A) \subset C$.

Proof. Let C be a closed set in X and $A \subset C$. By definition of the closure of a set,

$$Cl(A) = C \cap U_1 \cap U_2 \cap U_n$$

where $U_1, U_2, ..., U_n$ are all the closed sets containing A besides C. By definition of an intersection, it follows that $Cl(A) \subset C$. Therefore, if C is a closed set in X and $A \subset C$, then $Cl(A) \subset C$. □

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If $A \subset B$ then $Cl(A) \subset Cl(B)$.

Proof. Consider that by definition, $B \subset Cl(B)$ so $A \subset B \subset Cl(B)$. Because $Cl(B)$ is a closed set in and $A \subset Cl(B)$, by the proof above it follows that $Cl(A) \subset Cl(B)$. Therefore, if $A \subset B$ then $Cl(A) \subset Cl(B)$. □

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A is closed if and only if $A = Cl(A)$.

Proof. First, Let A be a closed set. Then A is a closed set where $A \subset A$. Then by the first theorem $Cl(A) \subset A$, and by definition of the closure $A \subset Cl(A)$, and $A = Cl(A)$. Thus if A is closed then $A = Cl(A)$.
Second, let $A = Cl(A)$. Because the closure is a closed set, A is closed. Thus, If $A = Cl(A)$, then A is closed. Therefore, A is closed if and only if $A = Cl(A)$. □

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2.4 Consider the particular point topology PPX_p on a set X. Determine $Int(A)$ and $Cl(A)$ for sets A containing p and for sets A not containing p.
For sets A containing p, if $p \in A$ then p is open, and thus $Int(A) = A$. $Cl(A) = X$.
For sets A not containing p, $Int(A) = \varnothing$ and $Cl(A) = A$.

2.6 Prove that $Cl(\mathbb{Q}) = \mathbb{R}$ in the standard topology on $\mathbb{R}$.

Proof. Let $A = \mathbb{R} - \mathbb{Q}$. We aim to prove $Int(A) = \varnothing$, BWOC assume $Int(A) \neq$. Then there exists an interval $(x - \epsilon, x + \epsilon)$ where $x \in A$, $\epsilon < 0$. Let $n \in \mathbb{N}$, where $\frac{1}{10^n} < \epsilon$. Consider a rational number $r \in \mathbb{Q}$, where r is x rounded to the nth decimal place. Then $|x - r| \leq \frac{1}{10^n} < \epsilon$, so $r \in (x - \epsilon, x + \epsilon)$ and $r \in \mathbb{Q}$, which is a contradiction! Thus, $Int(A) = \varnothing$. By Theorem 2.6,

$$Cl(\mathbb{Q}) = Cl(\mathbb{R} - A) = \mathbb{R} - Int(A) = \mathbb{R} - \varnothing = \mathbb{R}$$

Therefore, $Cl(\mathbb{Q}) = \mathbb{R}$. □

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2.8 (a) Show that the set of odd integers is dense in the digital line topology on $\mathbb{Z}$. Is the same true for the set of even integers?
Denote the set of odd numbers O and the set of even numbers E in the digital line topology on $\mathbb{Z}$. It follows that $\mathbb{Z} - O = E$. We aim to prove $Int(E) = \varnothing$, BWOC assume $Int(E)$ is nonempty, and there exists an open set A containing only even numbers. Consider $e \in A$, so e is an even number. Since A is open, it can

be generated by a union of basis elements. Only one basis element, $B(e) = \{e-1, e, e+1\}$, contains e, so $B(e) \subset A$. But $e-1$ is odd, which is a contradiction! So $Int(E) = \varnothing$. By Theorem 2.6,

$$Cl(O) = Cl(\mathbb{Z} - E) = \mathbb{Z} - Int(E) = \mathbb{Z} - \varnothing = \mathbb{Z}$$

Thus, the set of odd integers is dense in the digital line topology on $\mathbb{Z}$.
However, it is NOT the case that the even integers are dense. Because O is the union of all singleton sets of odd numbers, which are all basis elements, it follows that O is open, which makes E closed and $Cl(E) = E$. So the even integers are not dense.

(b) Which subsets of $\mathbb{Z}$ are dense in the discrete topology on $\mathbb{Z}$??
All subsets of $\mathbb{Z}$ besides $\varnothing$ are dense in the discrete topology on $\mathbb{Z}$, as the only closed sets in this are $\varnothing$ and $\mathbb{Z}$. So $\mathbb{Z}$ is the smallest closed set that contains any set besides $\varnothing$.

2.10 Prove Theorem 2.5: Let X be a topological space, A be a subset of X, and y be an element of X. Then $y \in Cl(A)$ if and only if every open set containing y intersects A.

Proof. (=¿) If $y \in Cl(A)$ then every open set containing y intersects A.
Let $y \in Cl(A)$, BWOC assume there exists an open set Y containing y that does not intersect A. By definition, $X - Y$ is a closed set containing A that does not contain y. But then y would not be in $Cl(A)$, which is a contradiction. Thus, If $y \in Cl(A)$ then every open set containing y intersects A.
(¡=) If every open set containing y intersects A, then $y \in Cl(A)$.
Let every open set containing y intersect A. BWOC, consider $y \notin Cl(A)$, and then there exists a closed set C where $A \subset C$, $y \notin C$. Consider the complement of this set $X - C$, which by definition is an open set that contains y and does not intersect A, which is a contradiction. Thus, If every open set containing y intersects A, then $y \in Cl(A)$.
Therefore, $y \in Cl(A)$ if and only if every open set containing y intersects A. □