Math 345 Homework 1

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1 Problems

1.1 Let A, B, and C be sets. Prove that if $A \subset B$ and $B \subset C$, then $A \subset C$.

Let A, B, and C be sets, and let $a \in A$. Since $A \subset B$, it follows that $a \in B$. And likewise since $B \subset C$ and $a \in B$, then $a \in C$. Therefore, if $A \subset B$ and $B \subset C$, then $A \subset C$.

1.2 Let $\{U_d\}_{d \in D}$ be an indexed family of subsets of a set S. Let $B \subset S$. Prove that $B \subset \bigcap_{d \in D} U_d$ if and only if for each $d \in D$, $B \subset U_d$.

proof:
First, let's prove if $B \subset \bigcap_{d \in D} U_d$ then for each $d \in D$, $B \subset U_d$. Let $b \in B$, where $B \subset \bigcap_{d \in D} U_d$. Then by definition of a subset, $b \in \bigcap_{d \in D} U_d$. By the definition of an intersection, it follows that for each $d \in D$, $b \in U_d$. Thus, if $B \subset \bigcap_{d \in D} U_d$ then for each $d \in D$, $B \subset U_d$.

Next, we prove if for each $d \in D$, $B \subset U_d$, then $B \subset \bigcap_{d \in D} U_d$. Let $b \in B$. By the defn. of a subset, for each $d \in D$, $b \in U_d$. By the defn. of an intersection, it follows that if for each $d \in D$, $b \in U_d$, b is an element of the intersection of these sets, $b \in \bigcap_{d \in D} U_d$. Thus, if for each $d \in D$, $B \subset U_d$, then $B \subset \bigcap_{d \in D} U_d$.

Therefore, $B \subset \bigcap_{d \in D} U_d$ if and only if for each $d \in D$, $B \subset U_d$.

1.3 Let A and B be sets, both of which have at least two distinct elements. Prove that there is a subset $W \subset A \times B$ that is not the product of a subset of A with a subset of B. [Thus, not every subset of a product is the product of a pair of subsets.]

BWOC, let $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$ be two sets with at least two distinct elements, where every $W \subset A \times B$, is the product of a subset of A

with a subset of B. Consider the set $W' = \{(a_1, b_2), (a_2, b_1)\}$ This would have to be the product of a subset of A containing a_1 and a subset of B containing b_1. However, a product of these two sets would have to contain the element $\{a_1, b_1\}$, which is a contradiction. Therefore, there is a subset $W \subset A \times B$ that is not the product of a subset of A with a subset of B.

1.4 Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be two sets, each having precisely two distinct elements. Let $f : A \to B$ be the constant function such that $f(a) = b_1$ for each $a \in A$.

- Prove that $f^{-1}(f(\{a_1\}) \neq \{a_1\}$. [Thus it is usually that case that $f^{-1}(f(X))$ and X are not equal for a set X.]
  proof: Observe that
  $$f^{-1}(f(\{a_1\}) = f^{-1}(\{b_1\}) = \{a_1, a_2\} \neq \{a_1\} \tag{1}$$
  Therefore $f^{-1}(f(\{a_1\}) \neq \{a_1\}$.
- Prove that $f(f^{-1}(B)) \neq B$. [Thus it is usually the case that $f(f^{-1}(Y))$ and Y are not equal for a set Y.]
  proof: Observe that
  $$f(f^{-1}(B)) = f(\{a_1, a_2\}) = \{b_1\} \neq B \tag{2}$$
  Therefore $f(f^{-1}(B)) \neq B$.
- Prove that $f(\{a_1\} \cap \{a_2\}) \neq f(\{a_1\}) \cap f(\{a_2\})$. [Thus it is usually the case that $f(X \cap X')$ and $f(X) \cap f(X')$ are not equal for sets X and X'.]
  proof: Observe that
  $$f(\{a_1\} \cap \{a_2\}) = f(\emptyset) = \emptyset \tag{3}$$
  but
  $$f(\{a_1\}) \cap f(\{a_2\}) = \{b_1\} \cap \{b_1\} = \{b_1\} \neq \emptyset \tag{4}$$
  Therefore $f(\{a_1\} \cap \{a_2\}) \neq f(\{a_1\}) \cap f(\{a_2\})$.

1.5 Let $f : A \to B$ and $g : B \to C$ be functions between sets. Prove that for $Z \subset C$, $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$.

proof:
Let $f : A \to B$ and $g : B \to C$ be functions between sets, and $h = g \circ f$, and $Z \subset C$. Consider
$$h(f^{-1}(g^{-1}(Z))) = (g \circ f)(f^{-1}(g^{-1}(Z))) = g(f(f^{-1}(g^{-1}(Z)))) \tag{5}$$
Which, by the definition of an inverse
$$= g(g^{-1}(Z)) = Z \tag{6}$$
We can thus say that $f^{-1}(g^{-1}(Z))$ is the inverse of h, or $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$. Therefore, for $Z \subset C$, $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$.

1.6 Prove that the function $f : [3, \infty) \to [-9, \infty)$ defined by $f(x) = x^2 - 6x$ is a bijection.

proof:
First, we must prove that f is injective. Let $x_1, x_2 \in [3, \infty)$, where $f(x_1) = f(x_2)$. Then

$$f(x_1) = x_1^2 - 6x_1 = f(x_2) = x_2^2 - 6x_2 \tag{7}$$

$$x_1^2 - 6x_1 = x_2^2 - 6x_2 \tag{8}$$

$$x_1^2 - x_2^2 = 6x_1 - 6x_2 \tag{9}$$

$$(x_1 + x_2)(x_1 - x_2) = 6(x_1 - x_2) \tag{10}$$

$$x_1 + x_2 = 6 \tag{11}$$

Inside the domain $[3, \infty)$, the only elements that add up to 6 are $3 + 3$, so $x_1 = x_2 = 3$. Therefore, f is injective.
Next, we prove that f is surjective. Let $y \in [-9, \infty)$.

$$y = f(x) = x^2 - 6x \tag{12}$$

$$y + 9 = x^2 - 6x + 9 \tag{13}$$

$$y + 9 = (x - 3)^2 \tag{14}$$

$$\sqrt{y + 9} = \sqrt{(x - 3)^2} \tag{15}$$

$$3 + \sqrt{y + 9} = x \tag{16}$$

Since $y \in [-9, \infty)$, it follows that $x \in [3, \infty)$, and thus f is surjective. Therefore, the function $f : [3, \infty) \to [-9, \infty)$ defined by $f(x) = x^2 - 6x$ is a bijection.