MATH 345 Homework 2

T.J. Liggett

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1 Problems

1.1 Determine all of the possible topologies on $X = \{a, b\}$.

$$T_1 = \{\emptyset, X\}$$

$$T_2 = \{\emptyset, \{a\}, X\}$$

$$T_3 = \{\emptyset, \{b\}, X\}$$

$$T_4 = \{\emptyset, \{a\}, \{b\}, X\}$$

1.4

1. Give an example of a space where the discrete topology is the same as the finite complement topology.

$$X = \{a, b, c\}$$

2. Make and prove (sike) a conjecture indicating for what class of sets the discrete and finite complement topologies coincide.

The discrete and finite complement topologies coincide for all finite sets.

1.6 Define a topology on \mathbb{R} (by listing the open sets within it) that contains the open sets (0,2) and (1,3) and that contains as few open sets as possible.

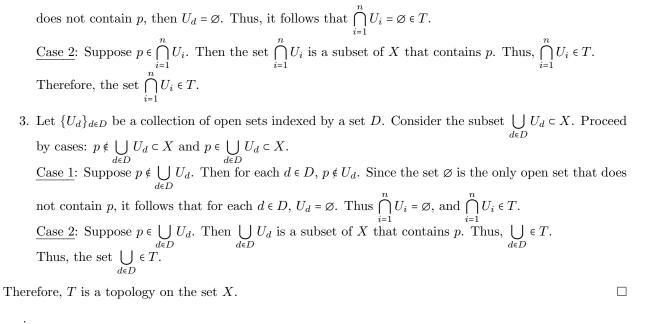
$$T = \{\emptyset, (0, 2), (1, 3), (1, 2), (0, 3), \mathbb{R}\}\$$

1.7 Let X be a set and assume $p \in X$. Show that the collection T, consisting of \emptyset and all subsets of X containing p, is a topology on X. This topology is called the **particular point topology** on X, and we denote it by PPX_p

Proof. Let X be a set and assume $p \in X$. Let T consist of \emptyset and all subsets of X containing p.

- 1. By definition, $\emptyset \in X$. Because $X \subset X$ and $p \in X$, it follows that $X \in T$.
- 2. Let $U_1, \dots U_n \in \text{be}$ elements of T. Consider the subset $\bigcap_{i=1}^n U_i \subset X$. Proceed by cases: $p \notin \bigcap_{i=1}^n U_i$ or $p \in \bigcap_{i=1}^n U_i$.

Case 1: Suppose $p \notin \bigcap_{i=1}^{n} U_i$. Then, there exists $d \in D$ such that $p \notin U_d$. Since U_d is an element of T that



1.9 Let T consist of \emptyset , \mathbb{R} , and all intervals $(-\infty, p)$ for $p \in \mathbb{R}$. Prove that T is a topology on \mathbb{R} . *Proof.* Let T consist of \emptyset , \mathbb{R} , and all intervals $(-\infty, p)$ for $p \in \mathbb{R}$.

- 1. By definition, \emptyset , \mathbb{R} are elements of T.
- 2. Let $U_1, \ldots U_n \in T$ be elements of T. Consider the subset $\bigcap_{i=1}^n U_i \subset \mathbb{R}$. If any U_i is empty, then the intersection is empty and thus an open set. So assume each U_i is not empty, and each U_i is an interval $(-\infty, p)$ where $p \in \mathbb{R}$ or $p = \mathbb{R}$. Then $\bigcap_{i=1}^n U_i$ is of the form $(-\infty, s)$ where s is the upper bound of the smallest U_i . Since $s \in \mathbb{R}$ or $s = \infty$, it follows that $\bigcap_{i=1}^n U_i \in T$.
- 3. Let $\{U_d\}_{d\in D}$ be a collection of open sets indexed by a set D. Consider the subset $\bigcup_{d\in D} U_d \subset \mathbb{R}$. Suppose $\bigcup_{d\in D} U_d = \emptyset$. Then for each $d\in D, U_d = \emptyset$, and thus $\bigcup_{d\in D} U_d \in T$. So suppose $\bigcup_{d\in D} U_d \neq \emptyset$. Then at least one U_d is nonempty, and either of the form $(-\infty, p), p\in \mathbb{R}$ or \mathbb{R} . The union of these sets would be of the form $(-\infty, s), s\in \mathbb{R}$ or \mathbb{R} , where s is the upper bound of the largest U_i . Since $s\in \mathbb{R}$ or $\bigcup_{d\in D} U_d \in T$. it follows that $\bigcup_{d\in D} U_d \in T$.

Therefore, T is a topology of the set \mathbb{R} .