MATH 345 Homework 5

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March 2021

1 Problems

2.13 Determine the set of limit points of A in each case.

(a) $A = (0,1]$ in the lower limit topology on $\mathbb{R}$. $A' = [0,1)$

(b) $A = \{a\}$ in $X = \{a,b,c\}$ with topology $\{X, \varnothing, \{a\}, \{a,b\}\}$. $A' = \{b,c\}$

(c) $A = \{a,c\}$ in $X = \{a,b,c\}$ with topology $\{X, \varnothing, \{a\}, \{a,b\}\}$ $A' = \{b,c\}$

(d) $A = \{b\}$ in $X = \{a,b,c\}$ with topology $\{X, \varnothing, \{a\}, \{a,b\}\}$. $A' = \{c\}$

(e) $A = (-1,1) \cup \{2\}$ in the standard topology on $\mathbb{R}$. $A' = [-1,1]$

(f) $A = (-1,1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$. $A' = [-1,1)$

(g) $A = \{(x,0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology. $A' = A$

2.14 For each $n \in \mathbb{Z}_+$, let $B_n = \{n, n+1, n+2, ...\}$, and consider the collection $B = \{B_n | n \in \mathbb{Z}_+\}$.

(a) Show that B is basis for a topology on $\mathbb{Z}_+$.

(b) Show that the topology on X generated by B is not Hausdorff.

(c) Show that the sequence $(2,4,6,8,...)$ converges to every point in $\mathbb{Z}_+$ with the topology generated by B.

(a)

Proof. Let $x \in \mathbb{Z}_+$. Consider the set $B_1 \in B$, where $B_1 = \{1,2,3,...\} = \mathbb{Z}_+$. $x \in B_1$, therefore for every $x \in \mathbb{Z}_+$, there exists a basis element $U \in B$ where $x \in U$.
For the second condition, let $x \in \mathbb{Z}_+$ and $B_m \in B$, $B_n \in B$, $x \in B_m \cap B_n$. WLOG, assume $m \leq n$. Then

$$B_m \cap B_n = \{m, m+1, m+2, ...\} \cap \{n, n+1, n+2, ...\} = \{m, m+1, m+2, ...\} = B_m$$

Since $B_m \in B$, there exists a third basis element, $B_s = B_m$, where $x \in B_s \subset B_m \cap B_n$. Thus, the second condition of a basis is satisfied.
Therefore, B is a basis for a topology on $\mathbb{Z}_+$. $\square$

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(b)

Proof. BWOC, assume the topology on X generated by B is Hausdorff. Then for $x, y \in X$, there exist open sets $x \in U$, $y \in V$, where $U \cap V = \varnothing$. Since U and V are open sets, they are each a union of basis elements and contain at least one basis element of the form $B_n = \{n, n+1, n+2, ...\}$. Let $B_m \subset U$ and $B_n \subset V$ be basis elements. But there exists one $s \in \mathbb{Z}_+$ where $m < s$ and $n < s$, and thus $s \in B_m, B_n$, and more importantly $s \in U$ and $s \in V$. Thus, $U \cap V \neq \varnothing$, which is a contradiction!
Therefore, the topology on X generated by B is not Hausdorff. $\square$

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(c)

Proof. Let $z \in \mathbb{Z}_+$, where U is a neighborhood of z in the topology generated by B. Then U is a union of basis elements in B of the form $B_n = \{n, n+1, n+2, ...\}$, so $U = \{u, u+1, u+2, ...\}$, where $u \in \mathbb{Z}_+$ and B_u is the largest of these basis elements. Consider $x_e \in (2,4,6,8,...)$, where $u \leq e$. Then for every element $x_f \in (2,4,6,8,...)$ where $e < f$, $f \in u$, and thus there is a positive integer e such that $x_f \in U$ for all $f \geq e$. Since this is true for every neighborhood U of z, it follows that $(2,4,6,8,...)$ converges to z.
Therefore, the sequence $(2,4,6,8,...)$ converges to every point in $\mathbb{Z}_+$ with the topology generated by B. ☐

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2.15 Determine the set of limit points of $[0,1]$ in the finite complement topology on $\mathbb{R}$.

The set of limit points is $\mathbb{R}$.

2.19 Show that if (x_n) is an injective sequence in $\mathbb{R}$, then (x_n) converges to every point in $\mathbb{R}$ with the finite complement topology on $\mathbb{R}$.

Proof. Let (x_n) be an injective sequence in $\mathbb{R}$, $y \in \mathbb{R}$. Then (x_n) has an infinite range. Consider U, a neighborhood of y in the finite complement topology, where $U = \mathbb{R} - F$ and F is a finite set in $\mathbb{R}$ where $F \cap (x_n) = \varnothing$. Since (x_n) is infinite and F is finite, there exists some N where $x_n \notin F$ for $n \geq N$, and it follows that (x_n) converges to y. Therefore, if (x_n) is an injective sequence in $\mathbb{R}$, then (x_n) converges to every point in $\mathbb{R}$ with the finite complement topology on $\mathbb{R}$. ☐

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2.20 Prove Theorem 2.11: Let A be a subset of $\mathbb{R}^n$ in the standard topology. If x is a limit point of A, then there is a sequence of points in A that converges to x.

Proof. Let A be a subset of $\mathbb{R}^n$ and x be a limit point of A. Then every neighborhood of x intersects A at a point other than x. Let U be a neighborhood of x, by definition of the standard topology an open n-ball, $B(y,)$. Let $x_n \in B(y, \epsilon - \frac{1}{n}) \cap A$, where $n \in \mathbb{N}$, $x \in B(y, \frac{1}{n})$, and $y \in \mathbb{R}$. The sequence (x_n) generated from this converges to x, as all of the elements of this sequence where $n \geq m$ are in $U = B(y, \epsilon)$, an arbitrary neighborhood of x. Therefore, If x is a limit point of A, then there is a sequence of points in A that converges to x. ☐

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2.21 Determine the set of limit points of the set

$$S = \{(x, sin(\frac{1}{x})) \in \mathbb{R}^2 | 0 < x \leq 1\}$$

as a subset of $\mathbb{R}^2$ in the standard topology.

$S \cup \{(0,y) | y \in [-1,1]\}$. The S, because obviousness (this is a determine not a show). The $\{(0,y) | y \in [-1,1]\}$ because any neighborhood of this will intersect with a point on the sinusoid.

2.23 Let T be the collection of subsets of $\mathbb{R}$ consisting of the empty set and every set whose complement is countable.

(a) Donut have to do this one.

(b) Show that the point 0 is a limit point of the set $A = \mathbb{R} - \{0\}$ in the countable complement topology.

(c) Show that in $A = \mathbb{R} - \{0\}$ there is no sequence converging to 0 in the countable complement topology.

(b)

Proof. Let U be an open set containing 0 in the countable complement topology. Then $U = \mathbb{R} - C$ where $C \in \mathbb{R}$ is countable, and $0 \in U$. Since C is countable, there exists an $x \in U$ where $x \neq 0$, so $x \in \mathbb{R} - \{0\}$. Thus, $U \cap A \neq 0$, and every neighborhood of 0 intersects A, and since $0 \notin A$, the intersection contains points other than 0. Therefore, the point 0 is a limit point of the set $A = \mathbb{R} - \{0\}$ in the countable complement topology. □

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(c)

Proof. Consider a sequence $S = (x_1, x_2, x_3, ...)$ in $\mathbb{R} - \{0\}$ that converges to 0 in the countable complement topology. Consider the open set $O = \mathbb{R} - S$, where $0 \in O$. There are no elements of S that are in O, this is a neighborhood of 0, so S does not converge to 0. Therefore, in $A = \mathbb{R} - \{0\}$ there is no sequence converging to 0 in the countable complement topology. □