Foundations Homework 4

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Chapter Two

Exercise 1 *If possible, give an example of a collection of objects which satisfies*

(a) both the well-ordering property and the trichotomy law,

(b) the well-ordering property, but not the trichotomy law,

(c) the trichotomy law, but not the well-ordering property,

(d) neither the well-ordering property nor the trichotomy law.

(a) All NFL linebackers ordered by career tackles.

(b) You cannot have this; If two objects in the collection cannot be compared, then there exists a set of those two objects which does not satisfy the well-ordering law.

(c) The set of all real integers

(d) The set of all functions on the interval 0 to 1. For functions f and g, $f > g$ if $f(x) > g(x)$ for the entire interval 0 to 1.

Assignment 2 *Use mathematical induction to prove that for any natural number* n

(a) $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, *and*

(b) $1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

PROOF (A) For the base case, notice that

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \tag{1}$$

$$\frac{n(n+1)}{2} = \frac{2(2+1)}{2} = \frac{6}{2} = 3 = 1 + 2 \tag{2}$$

$$\frac{n(n+1)}{2} = \frac{3(3+1)}{2} = \frac{12}{2} = 6 = 1+2+3 \tag{3}$$

Inductive Hypothesis: For some natural number k,

$$1+2+3+\ldots+k = \frac{k(k+1)}{2} \tag{4}$$

Consider the $k+1$ case.

$$\frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{k^2+3k+2}{2} \tag{5}$$

$$= \frac{k^2+k+2k+2}{2} = \frac{k(k+1)}{2} + (k+1) \tag{6}$$

By the inductive hypothesis,

$$\frac{k(k+1)}{2} + (k+1) = 1+2+3+\ldots+k+(k+1) \tag{7}$$

Therefore, for any natural number n,

$$1+2+3+\ldots+n = \frac{n(n+1)}{2} \tag{8}$$

PROOF (B) For the base case, notice that

$$1^2 = 1 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1 \tag{9}$$

$$1^2+2^2 = 5 = \frac{2(2+1)(2(2)+1)}{6} = \frac{30}{6} = 5 \tag{10}$$

$$1^2+2^2+3^2 = 14 = \frac{3(3+1)(2(3)+1)}{6} = \frac{84}{6} = 14 \tag{11}$$

Inductive Hypothesis: For some natural number k,

$$1+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6} \tag{12}$$

Consider the $k+1$ case.

$$\frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \tag{13}$$

$$= \frac{2k^3 + 6k^2 + 4k + 3k^2 + 9k + 6}{6} \tag{14}$$

$$= \frac{2k^3 + 3k^2 + k}{6} + \frac{6k^2 + 12k + 6}{6} \tag{15}$$

$$= \frac{k(2k^2 + 3k + 1)}{6} + k^2 + k + 1 \tag{16}$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \tag{17}$$

By the inductive hypothesis,

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = 1 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 \tag{18}$$

Therefore, by induction, for any natural number n

$$1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{19}$$

Assignment 3 *State and prove, using induction, a theorem about the sum* $2^0 + 2^1 + 2^2 + \cdots + 2^n$*, where* n *is a natural number.*

We will attempt to prove that for any natural number n, $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

Base case: $2^0 = 1 = 2^1 - 1$

Inductive Hypothesis: Assume that for some natural number k,

$$2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 \tag{20}$$

Consider the $k+1$ case:

$$2^(k+1) + 1 - 1 = 2^{k+1} * 2 - 1 = 2(2^{k+1} - 1 + \frac{1}{2}) \tag{21}$$

By the Inductive Hypothesis,

$$2(2^k + 1 - 1 + \frac{1}{2}) = 2(2^0 + 2^1 + 2^2 + \cdots + 2^k + \frac{1}{2}) \tag{22}$$

$$= 2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} \tag{23}$$

Therefore, by induction, for any natural number n,

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \tag{24}$$

Assignment 4 *Prove Theorem 2.1*
For natural numbers $n, m, and M$

1. $n|m$ *implies* $n|km$ *for any natural number* k
2. $n|m$ *and* $m|M$ *implies* $n|M$
3. $n|m$ *and* $n|M$ *implies* $n|(sm + tM)$ *for any natural numbers* s *and* t
4. $n|m$ *implies* $n \leq m$
5. $n|m$ *and* $m|n$ *implies* $n = m$

PROOF (i)

Let n, m, and k be natural numbers, where $n|m$. Since $n|m$, it follows that $m = an$ for some integer a. Thus, $km = k(an) = (ka)n$. Since k,a are integer numbers closed under multiplication, $ka = l$ for some integer l, and $n|lm$. Therefore, $n|m$ implies $n|km$ for any natural number k.

PROOF (ii)

Let n, m and M be natural numbers, where $n|m$ and $m|M$. Then, $m = an$ and $M = bm$ for some integers a and b. Thus, $M = b(an) = (ba)n$. Since b and a are integers, their product is an integer since integers are a closed set. Because of this, $n|M$ by some integer ba. Therefore, $n|m$ and $m|M$ implies $n|M$

PROOF (iii)

Let n, m, M, s, and t be natural numbers where $n|m$ and $n|M$. Then $m = an$ and $M = bn$ for some natural numbers a, b. Using substitution and the distributive property, $sm + tM = san + tbn = (sa + tb)n$. Since natural numbers are closed under addition and subtraction, it follows that $sa + tb$ is a natural number, and that $n|(sm + tM)$ under the definition of divides. Therefore, $n|m$ and $n|M$ implies $n|(sm + tM)$ for any natural numbers s and t.

PROOF (iv)

Let n, m be natural numbers where $n|m$. Assume, BWOC, that $n > m$. By the definition of divides, $m = kn$ for some natural number k. However, if $n > m$, then k must be less than one. But k is a natural number! Therefore, BWOC, $n|m$

implies $n \leq m$.

PROOF (v)
(v) $n|m$ and $m|n$ implies $n = m$

Let n, m be natural numbers where $n|m$ and $m|n$. By the definition of divides, $m = kn$ and $n = jm$ for some natural numbers k, j. Then $n = jm = jkn$

Assignment 5 *Prove that there are infinite primes.*

Assume, BWOC, that there exist a finite number of primes,

$$2, 3, 5, 7, \cdots, p_n \tag{25}$$

where p_n is the largest prime number. Consider the number p where,

$$p = p_1 * p_2 * p_3 * \cdots * p_n + 1 \tag{26}$$

Since p is the product of all primes, it is not prime. Thus, there exists a prime p_k where $p_k|p$, $1 \leq k \leq n$. However, since p_k is in our list of primes, $p_k|p_1*p_2*p_3*\cdots*p_n$, and cannot divide p. This is a contradiction.
Therefore, BWOC, there are infinitely many primes.

Exercise 7 *If the exponent vectors for a and b are $(a_1, a_2, ..., a_k)$ and $(b_1, b_2, ..., b_k)$, what is the exponent vector for ab? For a^2? For (a, b), the greatest common divisor of a and b? For $[a, b]$, the least common multiple of a and b?*

Exponent vector for ab: $(a_1 + b_1, a_2 + b_2, ..., a_k + b_k)$
Exponent vector for a^2: $(2a_1, 2a_2, ..., 2a_k)$
The exponent vector for (a, b) will be the lower value for each component of the vector. Likewise, the exponent vector for $[a, b]$ will be the greater value for each component of the vector.