Foundations Homework 9

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Chapter Three

Exercise 11 *Define a relation on* $\mathbb{N}^2$ *as follows:* $(m, n)\ (p, q)$ *if and only if* $m + q = n + p$*. Draw the equivalence classes on the lattice, as is done in Figure 3 for the relations* R_1*,* R_2*, and* R_3 *of Exercise 3.5.*

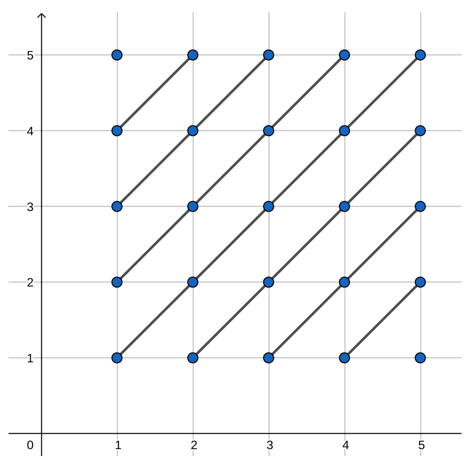

Figure 11: Equivalence Classes

Each equivalence class consists of points $(m, n)\ (p, q)$, where $m + q = n + p$.

Exercise 12 *Define a relation* $\approx$ *on* $\mathbb{N}^2$ *as follows:* $(m, n) \approx (p, q)$ *if and only if* $m + n = p + q$*. As in Exercise 3.11, draw the equivalence classes.*

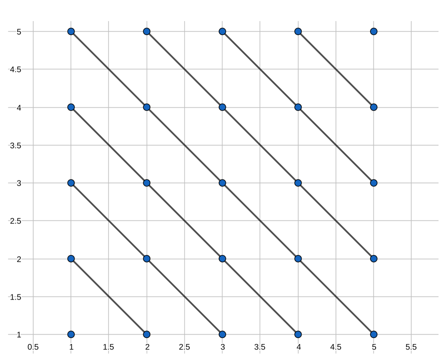

Figure 12: Equivalence Classes

Each equivalence class consists of points $(m, n) \approx (p, q)$, where $m + n = p + q$.

Exercise 13 *Give two examples of relations which are functions, and two which are not.*

Functions:

- $F_1 = \{(1, 3), (2, 2), (3, 1)\}$
- $F_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Non-Functions:

- $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
- $R_2 = \{(1, 1), (2, 2), (3, 3), (3, 4), (1, 5)\}$

Exercise 14 *Give examples of two functions which are one-to-one, and two which are not.*

One-to-one Functions:

- $F_1 = \{(1, 3), (2, 2), (3, 1)\}$
- $F_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Non-one-to-one Functions:

- $F_3 = \{(1, 3), (2, 2), (3, 3)\}$
- $F_4 = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$

Exercise 15 *Give the converse of each function from Exercise 3.14. Which of these are inverses?*

- $F_1^t = \{(1, 3), (2, 2), (3, 1)\}$
- $F_2^t = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$
- $F_3^t = \{(3, 1), (2, 2), (3, 3)\}$
- $F_4^t = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5)\}$

F_1^t, F_2^t are inverses

Assignment 16 *Show that if a function F is one-to-one on its domain, then F^t is also a function. Thus, if a function is one-to-one, it has an inverse.*

Let F be a function which is one-to-one on its domain. So each value x in the domain of F maps to a unique value y in its codomain, such that $F(x) = y$. It follows that for F^t, $F^t(y) = x$ for a unique value x. By definition, F^t is a function. Therefore, if a function is one-to-one, it has an inverse.

Exercise 17 *Find $P(S)$ if $S = \{a, b, c\}$. How many elements are in $P(S)$ if $S = \{a, b, c, d\}$?*

$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
If S contains 4 elements, then $|P(S)| = 16$

Assignment 18 *Show by induction that, if $|S| = n \varepsilon \mathbb{N}$, then $|P(S)| = 2^n = 2^{|S|}$.*

Base case $n = 1$: $|S| = 1$, $P(S) = \{\emptyset, S\}$, $|P(S)| = 2 = 2^{|S|}$
Inductive Hypothesis: For some set S where $|S| = k, k \varepsilon \mathbb{N}$, $|P(S)| = 2^k = 2^{|S|}$. Consider the $n + 1$ case, where $S_{n+1} = S_n + T, S_n \cap T = \emptyset$, and $|S| = n, |T| = 1$. For each set in $P(S_n)$, there exists an additional set in $P(S_{n+1})$ which contains the item in T. Thus, $|P(S_{n+1})| = 2|P(S_n)|$. By induction, $P(S_n) = 2^n$, and thus $|P(S_{n+1})| = 2(2^n) = 2^{n+1}$. Therefore, by induction, if $|S| = n \varepsilon \mathbb{N}$, then $|P(S)| = 2^n = 2^{|S|}$.

Exercise 19 *Is the function $F : \mathbb{N} \to \mathbb{N}$ given by $F = \{(n, n + 3) | n \varepsilon \mathbb{N})\}$ onto? What about the function $G : \mathbb{Q}' \to \mathbb{Q}'$ given by $G = \{(q, \frac{1}{q}) | q \varepsilon \mathbb{Q}'\}$?*

F is not onto because 0,1, and 2 cannot be accessed in the codomain. G is onto.

Exercise 20 *Using each pair of sets D and C below as the domain and codomain, give an example or explain why it is impossible to do so:*

1. *a relation which is not a function*
2. *a function which is one-to-one, but not onto*
3. *a function which is onto, but not one-to-one*
4. *a function which is both one-to-one and onto*
5. *a function which is neither one-to-one or onto*

What generalizations can you state about the number of elements in the domain and range of a function which is one-to-one? onto?

(a) $D = \{1, 2, 3\}$, $C = \{a, b, c\}$

   (a) $R = \{(1, a), (1, b)\}$
   (b) Not possible. In order to use the whole domain, the whole codomain must be used.
   (c) Not possible. In order to use the whole codomain, the whole domain must be used.
   (d) $F_3 = \{(1, a), (2, b), (3, c)\}$

(e) $F_4 = \{(1,b),(2,a),(3,a)\}$

(b) $D = \{1,2,3,4\}$, $C = \{a,b,c\}$

(a) $R = \{(1,a),(1,b),(1,c)\}$

(b) Not possible. In order to use the whole domain, the whole codomain must be used.

(c) $F_2 = \{(1,a),(2,a),(3,b),(4,c)\}$

(d) Not possible. In order to use the whole domain, the whole codomain must be used.

(e) $F_4 = \{(1,b),(2,b),(3,b),(4,b)\}$

(c) $D = \{1,2,3\}$, $C = \{a,b,c,d\}$

(a) $R = \{(1,a),(1,b),(1,c)\}$

(b) $F_1 = \{(1,a)(2,b)(3,c)\}$

(c) Not possible. The codomain is greater than the domain, and cannot be fully covered.

(d) Not possible. The codomain is greater than the domain, and cannot be fully covered.

(e) $F_4 = \{(1,a)(2,a)(3,a)\}$

For a function to be one-to-one, the domain must be greater than or equal to the range. For a function to be onto, the range must be greater than or equal to the domain.

Assignment 21 *Show that* $\mathbb{N}$ *and* $2\mathbb{N} = \{2n|n\varepsilon\mathbb{N}\} = \{2,4,6,\dots\}$ *have the same cardinality.*

Observe that the function F exists where $F : \mathbb{N} \to 2\mathbb{N}$ given by $F = \{(n,2n)|n\varepsilon\mathbb{N}\}$. Since F is onto and one-to-one, it follows that the domain and range of F have the same cardinality. Therefore, $\mathbb{N}$ and $2\mathbb{N} = \{2n|n\varepsilon\mathbb{N}\} = \{2,4,6,\dots\}$ have the same cardinality.