MATH 345 Homework 3

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March 2021

1 Problems

1.10 Show that $B = \{[a,b) \subset \mathbb{R} | a < b\}$ is a basis for a topology on $\mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. Then there exists a basis element $B_x = [x, x+1) \in B$ $\square$

1.11 Determine which of the following collections of subsets of $\mathbb{R}$ are bases:

1. $C_1 = \{(n, n+2) \subset \mathbb{R} | n \in \mathbb{Z}\}$ No
2. $C_2 = \{[a,b] \subset \mathbb{R} | a < b\}$ Yes
3. $C_3 = \{[a,b] \subset \mathbb{R} | a \leq b\}$ Yes
4. $C_4 = \{(-x,x) \subset \mathbb{R} | x \in \mathbb{R}\}$ Yes
5. $C_5 = \{(a,b) \cup \{b+1\} \subset \mathbb{R} | a < b\}$ No

1.12 Determine which of the following are open sets in $\mathbb{R}_l$. In each case, prove your assertion.

$$A = [4,5), B = \{3\}, C = [1,2], D = (7,8)$$

- Set A: $[4,5)$ is a basis element in the basis that generates $\mathbb{R}_l$, so it is also an open set.
- Set B: BWOC, assume $\{3\}$ is an open set. Then there is a union of basis elements $\bigcup_{n=1}^{t}[a,b)$ which generates $\{3\}$. However, a singleton set in this form would have to be $[a,a]$, with a closed upper bound. A union of these basis elements will never have a closed upper bound, and thus never generate a singleton set, which is a contradiction. Therefore, the set $\{3\}$ is not open.
- Set C: Similarly to above, there is no union of basis elements of the form $[a,b)$ which will generate a closed upper bound. Thus, the set $[1,2]$ is not open.
- Set D: $(7,8)$ is not a basis element, but can be generated by the union of basis elements $\bigcup_{n=1}^{\infty}[7+\frac{1}{n}, 8)$, so it is an open set.

1.14 Let B be the collection of subsets of $\mathbb{Z}$ used in defining the digital line in Example 1.10. Show that B is a basis for a topology on $\mathbb{Z}$

Proof. Let $x \in \mathbb{Z}$. Consider x is odd. Then $B(x) = \{x\}$. If x is even, then $B(x) = \{n-1, n, n+1\}$. Thus, for every $x \in \mathbb{Z}$, there exists a basis element which contains x.
For the second condition of a basis, consider the case where x is even. Let $B_1, B_2 \in B$ be basis elements where $x \in B_1, B_2$. For an even element of B, there exists only one set $B(x) = \{x-1, x, x+1\}$ which contains x. Thus, it follows that $B_1 = B_2 = B(x)$, and $B_1 \cap B_2 = B(x) \in B$. So if x is even, for two basis elements where $x \in B_1 \cap B_2$, there exists a third basis element where $x \in B_3 \subset B_1 \cap B_2$.
Now consider the case where x is odd. Let $B_1, B_2 \in B$ be basis elements where $x \in B_1, B_2$. Then either

1. B_1 and B_2 are both generated by odd integers.

2. B_1 and B_2 are both generated by even integers.

3. WLOG, B_1 is generated by an even integer and B_2 is generate by an odd integer.

Case 1: Assume B_1 and B_2 are both generated by odd integers. Then $B_1 = \{a\}$, $B_2 = \{b\}$ where $a, b \in \mathbb{Z}$. Since $x \in B_1$ and $x \in B_2$, it follows that $a = b = x$, and $B_1 \cap B_2 = \{x\}$, which is a basis element.
Case 2: Assume B_1 and B_2 are both generated by even integers. Since x is in both, B_1 and B_2 can either be $\{x-2, x-1, x\}$ or $\{x, x+1, x+2\}$, and thus $B_1 \cap B_2$ can either be $\{x\}$, $\{x-2, x-1, x\}$, or $\{x, x+1, x+2\}$. In each case, it is true that for the basis element $B(x) = \{x\}$, $x \in B(x) \subset B_1 \cap B_2$.
Case 3: WLOG, assume B_1 is generated by an even integer and B_2 is generate by an odd integer. Since $x \in B_2$, it follows that $B_2 = \{x\}$. $x \in B_1$, which makes B_1 equal to either $\{x-2, x-1, x\}$ or $\{x, x+1, x+2\}$. In either case, $B_1 \cap B_2 = \{x\} = B(x)$, hence $x \in B(x) \subset B_1 \cap B_2$.
Therefore, B is a basis for a topology on $\mathbb{Z}$. □

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1.16

1. Show that B is a basis for a topology on $\mathbb{R}^2$.
   First, let $x = (x, y) \in \mathbb{R}^2$. Then $x \in B_1 = \{(a,b) \times (c,d) | a < x < b, c < y < d\}$. So for every $x \in \mathbb{R}^2$, there exists a basis element that contains x.
   Now let $x = (x, y) \in \mathbb{R}^2$ where $x \in B_1 \cap B_2$, $B_1 = \{(a,b) \times (c,d) | a < b, c < d, a, b, c, d \in \mathbb{R}\}$ and $B_2 = \{(e,f) \times (g,h) | e < f, g < h, e, f, g, h \in \mathbb{R}\}$. Then $B_1 \cap B_2 = \{(max(a,e), min(b,f)) \times (max(c,g), min(d,h))\}$. Since $x \in B_1 \cap B_2$, we can assume this intersection is non-empty, and this rectangle B_3 is an element of B. Thus $x \in B_3 \subset B_1 \cap B_2$.
   Therefore, B is a basis for a topology on $\mathbb{R}^2$.

2. Show that the topology T', generated by B is the standard topology on $\mathbb{R}^2$ (T_{std}).
   First, consider an element $S \in T'$, where $S = \{(a,b) \times (c,d) | a < b, c < d\}$. For a point $x \in S$, there exists an open ball $B(x, \epsilon) = \{y \in \mathbb{R}^2 | d(x,y) < \epsilon\}$ where $\epsilon < |x_x - a|$, $\epsilon < |x_x - b|$, $\epsilon < |x_y - c|$, $\epsilon < |x_y - d|$. Hence, this open ball is contained in S and is a basis element of the standard topology on $\mathbb{R}^2$, so for every basis element $B_t \in T'$, any point $p \in B_t$ is contained in a basis element of T_{std} that is contained in the B_t. Thus, $T' \subset T_{std}$.
   Next, consider an element $B(x, \epsilon) = \{y \in \mathbb{R}^2 | d(x,y) < \epsilon\} \in T_{std}$. For a point $r \in B(x, \epsilon)$, there exists an open rectangle R where $r \in R$, R $\{(r_x - s, r_x + s) \times (r_y - s, r_y + s)\}$, and $2s\sqrt{2} + d(x,r) < \epsilon$. Hence, this open rectangle is contained in $B(x, \epsilon)$ and is a basis element of T' on $\mathbb{R}^2$, so for every basis element $B_t \in T_{std}$, any point $p \in B_t$ is contained in a basis element of T' that is contained in the B_t. Thus, $T_{std} \subset T'$.
   Therefore the topology T', generated by B is the standard topology on $\mathbb{R}^2$ (T_{std}).

Exercises for Section 1.3

1.26 Prove that closed balls are closed sets in the standard topology on $\mathbb{R}^2$.

Proof. Let $B(x,) = \{y \in \mathbb{R}^2 | d(x,y) \leq \epsilon\}$ be a closed ball on $\mathbb{R}^2$. Then the complement of $B(x, \epsilon)$ in $\mathbb{R}^2$ is

$$\mathbb{R}^2 - B(x, \epsilon) = \{y \in \mathbb{R}^2 | d(x,y) > \epsilon\}.$$

Consider an arbitrary point $z \in \mathbb{R}^2 - B(x, \epsilon)$. There exists an open ball $B(z, r) = \{a \in \mathbb{R}^2 | d(a, z) < r\}$ where $r < d(x, z) - \epsilon$, so $B(z, r) \subset \mathbb{R}^2 - B(x, \epsilon)$. Hence, for any point $z \in \mathbb{R}^2 - B(x, \epsilon)$, there is a basis element contained in $B(x, \epsilon)$, and thus $\mathbb{R}^2 - B(x, \epsilon)$ is an open set. Since the complement of $B(x, \epsilon)$ is open, $B(x, \epsilon)$ is closed. Therefore, closed balls are closed sets in the standard topology on $\mathbb{R}^2$. □

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1.27 The infinity comb C is the subset of the plane illustrated in Figure 1.17 and defined by

$$C = \{(x,0)|0 \le x \le 1\} \cup \{(\frac{1}{2^n}, y)|n = 0,1,2,...and 0 \le y \le 1\}$$

Prove that C is not closed in the standard topology on $\mathbb{R}^2$.

Proof. Consider the complement of C on $\mathbb{R}^2$, $C' = C - \mathbb{R}^2$. Consider $p = (0,1) \in C'$, and an open ball $B(b,\epsilon)$, $\epsilon > 0$, $b \in \mathbb{R}^2$, where $p \in B(b,\epsilon)$. Thus $d(b,p) < \epsilon$. Then there exists a point $x = (\frac{1}{2^n}, 1)$ where $\frac{1}{2^n} < \epsilon - d(b,p), n \in \{0,1,2,...\}$. It follows that $d(b,x) \le d(b,p) + d(p,x) = d(b,p) + \frac{1}{2^n} < \epsilon$, so $x \in B(b,\epsilon)$. So there exists $p = (0,1) \in C'$ such that for every $B \in \{B(x,\epsilon)|x \in \mathbb{R}^2, \epsilon > 0\}$, either $p \notin B$ or $B \not\subset C'$. Thus C' is not open, and C is not closed. Therefore, C is not closed in the standard topology on $\mathbb{R}^2$. □

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1.28 Which sets are closed sets in the finite complement topology on a topological space X?

The closed sets in the finite complement topology on X are X and all finite subsets of X. If $A \subset X$ is finite, then its complement $X - A$ is open in the finite complement topology, making A closed.

1.30 Which sets are closed sets in the particular point topology PPX_p on a set X

All sets in $A \subset X$ that don't contain p, and X are closed sets in PPX_p.

1.35 Show that $\mathbb{R}$ in the lower limit topology is Hausdorff.

Consider $a, b \in \mathbb{R}$. WLOG, assume $a < b$. Observe that two open sets in the lower limit topology, $U = [a,b)$ and $V = [b, b+1)$ are disjoint sets, where $a \in U$ and $b \in V$. So there exist disjoint neighborhoods for a and b. Thus, for any two elements of $\mathbb{R}$, there exist disjoint neighborhoods for those elements in $\mathbb{R}_l$, and therefore $\mathbb{R}$ in the lower limit topology is Hausdorff.

1.36 Show that $\mathbb{R}$ in the finite complement topology is not Hausdorff.

Let S_1 and S_2 be open sets in the finite complement topology on $\mathbb{R}$, where $S_1 = \mathbb{R} - F_1$ and $S_2 = \mathbb{R} - F_2$ for some finite sets, $F_1, F_2 \in \mathbb{R}$. Then:

$$S_1 \cap S_2 = (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2)$$

By DeMorgan's Law,

$$(\mathbb{R} - F_1) \cap (\mathbb{R} - F_2) = \mathbb{R} - (F_1 \cup F_2)$$

Since $F_1 \cup F_2$ is a finite set, it follows that $S_1 \cap S_2$ is non-empty. This means that there are no two disjoint open sets in the topology, and thus no two points have disjoint neighborhoods. Therefore, $\mathbb{R}$ in the finite complement topology is not Hausdorff.