MATH 345 Homework 7

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1 Problems

1. Let X and Y be sets. Let A be a subset of X and B be a subset of Y. For each element $\alpha \in J$ in an indexing set J, let U_α be a subset of X and V_α be a subset of Y. Prove that

$$(\bigcup_{\alpha \in J}(U_\alpha \times V_\alpha)) \cap (A \times B) = \bigcup_{\alpha \in J}((U_\alpha \cap A) \times (V_\alpha \cap B)).$$

First, consider **Lemma 420**.

Lemma 420: Let X, Y be sets. Let A, C be subsets of X and B, D be subsets of Y. Then

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

Proof. Let X, Y be sets. Let A, C be subsets of X and B, D be subsets of Y.

$(\subset) Consider$(x,y) $\in (A \times B) \cap (C \times D)$. Then $(x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$. So, $x \in A, C$ and $y \in B, D$ by defn. of the Cartesian product of sets. Thus, $x \in (A \cap C)$ and $y \in (B \cap D)$. Hence, $(x, y) \in (A \cap C) \times (B \cap D)$.

$(\supset) Consider$(x,y) $\in (A \cap C) \times (B \cap D)$. Then by defn. of the Cartesian product of sets, $x \in (A \cap C)$ and $y \in (B \cap D)$. By defn. of intersections, $x \in A, C$ and $y \in B, D$. Then $(x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$. Thus, $(x, y) \in (A \times B) \cap (C \times D)$.

Therefore, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. ☐

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Now, consider the proof at hand:

Proof. Let X and Y be sets. Let A be a subset of X and B be a subset of Y. For each element $\alpha \in J$ in an indexing set J, let U_α be a subset of X and V_α be a subset of Y. Consider

$$(\bigcup_{\alpha \in J}(U_\alpha \times V_\alpha)) \cap (A \times B)$$

Writing it out and letting $|J| = j + 1$, we get this

$$= ((U_0 \times V_0) \cup (U_1 \times V_1) \cup ... \cup (U_j \times V_j)) \cap (A \times B)$$

By the distributive property of sets,

$$= ((U_0 \times V_0) \cap (A \times B)) \cup ((U_1 \times V_1) \cap (A \times B)) \cup ... \cup ((U_j \times V_j) \cap (A \times B))$$

By Lemma 420,

$$= ((U_0 \cap A) \times (V_0 \cap B)) \cup ((U_1 \cap A) \times (V_1 \cap B)) \cup ... \cup ((U_j \cap A) \times (V_j \cap B)) = \bigcup_{\alpha \in J}((U_\alpha \cap A) \times (V_\alpha \cap B)).$$

Therefore,

$$(\bigcup_{\alpha\in J}(U_\alpha \times V_\alpha)) \cap (A \times B) = \bigcup_{\alpha\in J}((U_\alpha \cap A) \times (V_\alpha \cap B)).$$

□

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2. Prove Theorem 3.9. Include the portion we proved in class, that $T_{prod} \subset T_{sub}$, in your proof.

Proof. ($\subset$) We must first show $T_{prod} \subset T_{sub}$. Let T_A be the subspace topology on A and T_B be the subspace topology on B. Consider the basis for T_{prod},

$$B_{prod} = \{(U \cap A) \times (V \cap B) | U \text{ is open in } T_A, V \text{ is open in } T_B\}$$

Let $S \in B_{prod}$. Then $S = (U \cap A) \times (V \cap B)$ for some open sets U and V in T_A and T_B, respectively. By Theorem 420:

$$(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B)$$

Since $U \in X$ and $V \in Y$, it follows that $(U \times V) \in (X \times Y)$. Thus, by the defn. of T_{sub} $S \in T_{prod}$, and basis elements of T_{prod} are in T_{sub}. Consider $W \in T_{prod}$. We know that W is a union of basis elements of B_{prod}, and since unions of open sets in T_{sub} are in T_{sub}, it follows that $W \in T_{sub}$. Thus, $T_{prod} \subset T_{sub}$.

($\supset$) We must now show that $T_{sub} \subset T_{prod}$. Consider the basis for T_{sub}

$$B_{sub} = \{(U \times V) \cap (A \times B) | U \times V \in B_{X \times Y}\}$$

Let $S \in B_{sub}$. Then $S = (U \times V) \cap (A \times B)$ for some $U \times V \in B_{X \times Y}$. By Theorem 420:

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

Since U is open in X, $U \cap A$ is in the subspace topology on A, and since V is open in Y, $V \cap B$ is in the subspace topology on B. With $U \cap A \in T_A$ and $V \cap B \in T_B$, it follows that $(U \cap A) \times (V \cap B) \in T_A \times T_B$, so $S \in T_{prod}$. So basis elements in B_{sub} are in T_{prod}. Consider $W \in T_{sub}$. We know that W is a union of basis elements of B_{sub}, and since unions of open sets in T_{prod} are in T_{prod}, it follows that $W \in T_{prod}$. Thus, $T_{sub} \subset T_{prod}$. □

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3.24 Let $X = \mathbb{R}$ in the standard topology. Take the partition

$$X^* = \{..., (-1, 0], (0, 1], (1, 2], ...\}.$$

Describe the open sets in the resulting quotient topology on X^*.

The quotient topology on X^* is homomorphic to the trivial topology on $\mathbb{Z}$, $T_{\mathbb{Z}} = \{\varnothing, \mathbb{Z}\}$. Any union of sets in X^*, except for the union of ALL open sets, will be a union of sets of the form $(n-1, n]$, where $n \in \mathbb{Z}$. The preimage of these sets is not open in the standard topology on $\mathbb{R}$.

3.25 Define a partition of $X = \mathbb{R}^2 - \{O\}$ by taking each ray emanating from the origin as an element in the partition. Which topological space that we have previously encountered appears to be topologically equivalent to the quotient space that results from this partition?

This space appears to be topologically equivalent to S_1. Each ray can be compacted down to a single point on the circle.

3.27 Provide an example showing that a quotient space of a Hausdorff space need not be a Hausdorff space.

Consider **Example 3.14** and the digital line topology. Let X be the topological space of $\mathbb{R}$ with the standard topology. X is a Hausdorff space. Define $p : \mathbb{R} \to \mathbb{Z}$ by $p(x) = x$ if x is an integer, and $p(x) = n$ if $x \in (n-1, n+1)$ and n is an odd integer. So p is the identity on the integers, and p maps non integer values to the nearest odd integer. In previous work, we proved that the digital line topology is not Hausdorff, as open sets with even integers must contain adjacent odd numbers in them. Thus, this quotient space of a Hausdorff space is not a Hausdorff space.

3.28 Consider the equivalence relation on $\mathbb{R}$ defined by $x\ y$ if $x - y \in \mathbb{Z}$. Describe the quotient space that results from the partition on $\mathbb{R}$ into the equivalence classes in the equivalence relation.

The resulting quotient space is topologically equivalent to S^1.

3.30 Consider the equivalence relation on $\mathbb{R}^2$ defined by $(x_1, x_2)\ (w_1, w_2)$ if $x_1^2 + x_2^2 = w_1^2 + w_2^2$. Describe the quotient space that results from the partition of $\mathbb{R}^2$ into the equivalence classes in this equivalence relation.

Equivalence classes are circles of a radius from the origin. The resulting quotient space is topologically equivalent to $\mathbb{R}_+$, or $[0, \infty)$

3.33 In each of the following cases, describe or draw a picture of the resulting quotient space. Assume that points are identified only with themselves unless they are explicitly said to be identified with other points.

(a) The disk with its boundary points identified with each other to form a single point.

(b) The circle S^1 with each pair of antipodal points identified with each other.

(c) The interval $[0, 4]$, as a subspace of $\mathbb{R}$, with integer points identified with each other.

(d) The interval $[0, 9]$, as a subspace of $\mathbb{R}$, with even integer points identified with each other to form a point and with odd integer points identified with each other to form a different point.

(e) The real line $\mathbb{R}$ with $[-1, 1]$ collapsed to a point.

(f) The real line $\mathbb{R}$ with $[-2, -1] \cup [1, 2]$ collapsed to a point.

(g) The real line $\mathbb{R}$ with $(-1, 1)$ collapsed to a point.

(h) The plane $\mathbb{R}^2$ with the circle S^1 collapsed to a point.

(i) The plane $\mathbb{R}^2$ with the circle S^1 and the origin collapsed to a point.

(j) The sphere with the north and south pole identified with each other.

(k) The sphere with the equator collapsed to a point.

More Pictures!