Searching and Sorting

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Today's Plan

Recap

Searching algorithms and their analysis

Sorting algorithms and their analysis

Announcements

Questions?

Searching

Looking for something!

In this discussion we will assume searching for an element in an array

Linear search

Most intuitive

Start at first position and keep looking until you find it

```
int linearSearch(int a[], int size, int value)
{

    for (int i = 0; i < size; i++)
    {
        if (a[i] == value) {
            return i;
        }
    }
    return-1;
}
```

How long does linear search take?

If you assume value is in the array and probability of finding it at any location is uniform, on **average n/2**

If value is not in the array (worst case) **n**

Either way it's $O(n)$

What if you know **array is sorted**?
Can you do better than linear search?

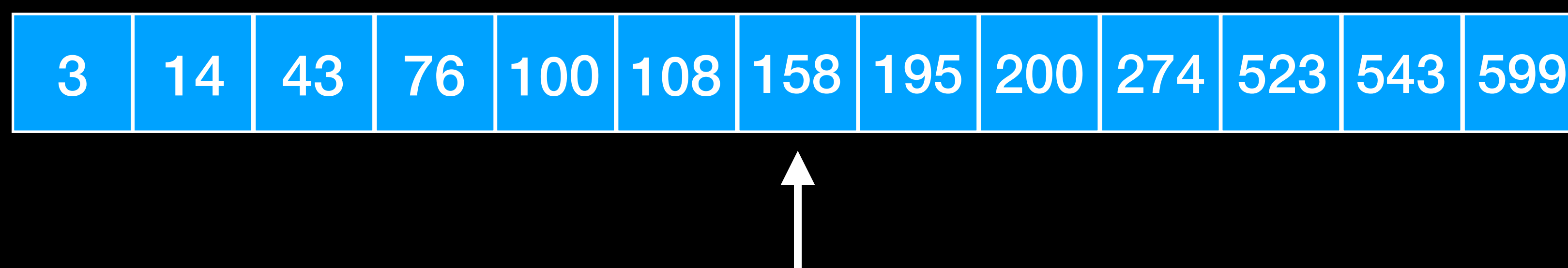
Lecture Activity

You are given a sorted array of integers.

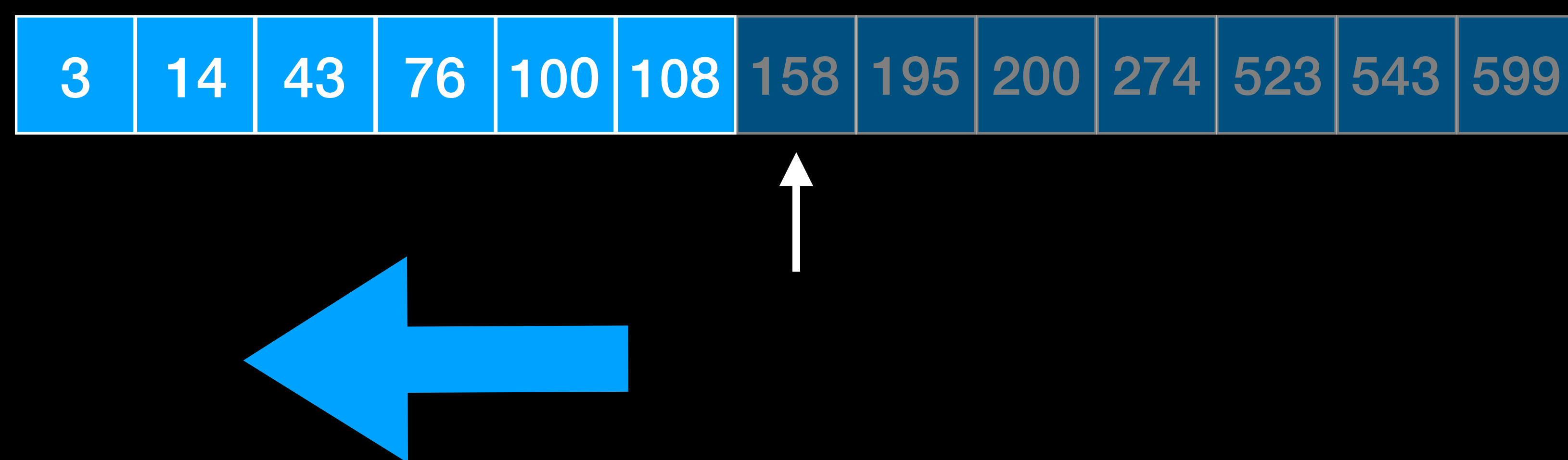
How would you search for 115? (try to do it in fewer than n steps: don’t search sequentially)

You can write pseudocode or succinctly explain your algorithm

Binary Search



Binary Search



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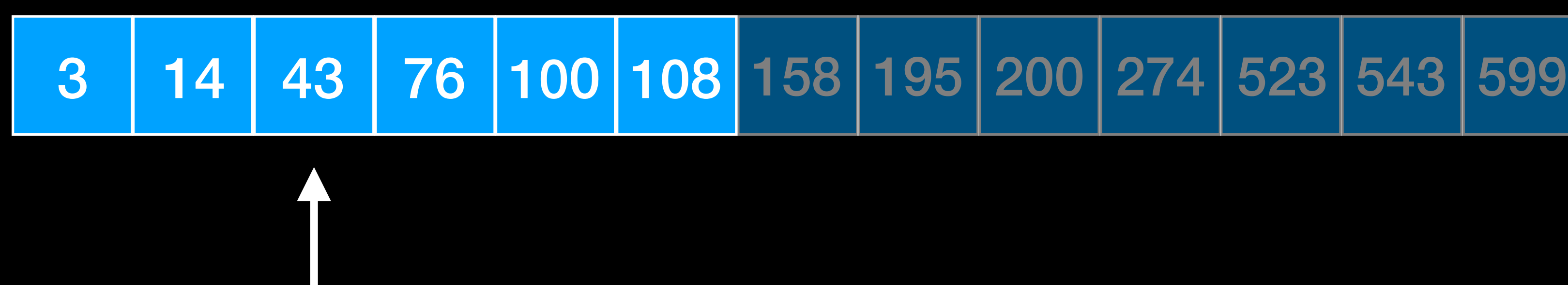

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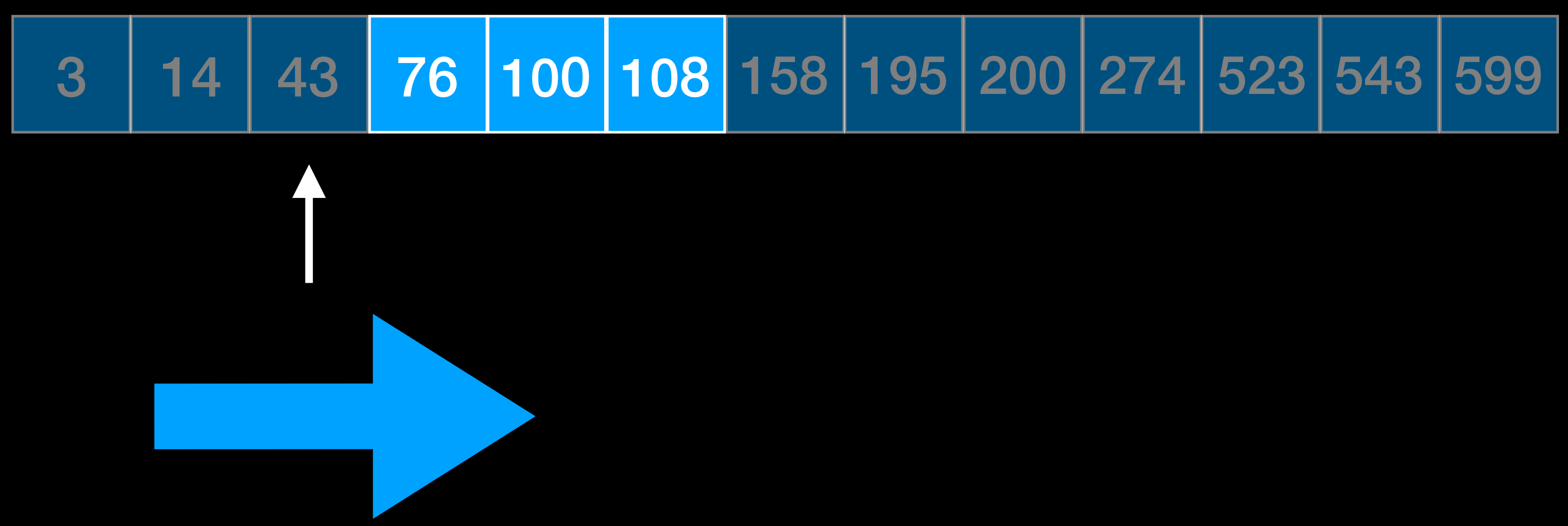
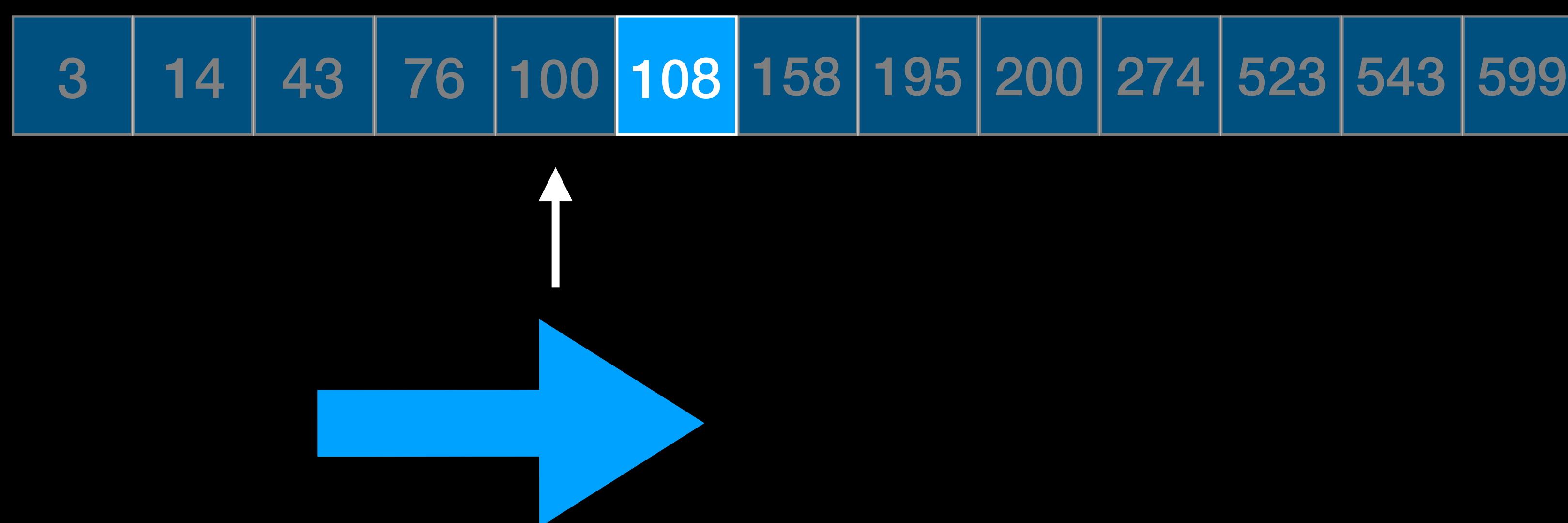


Binary Search

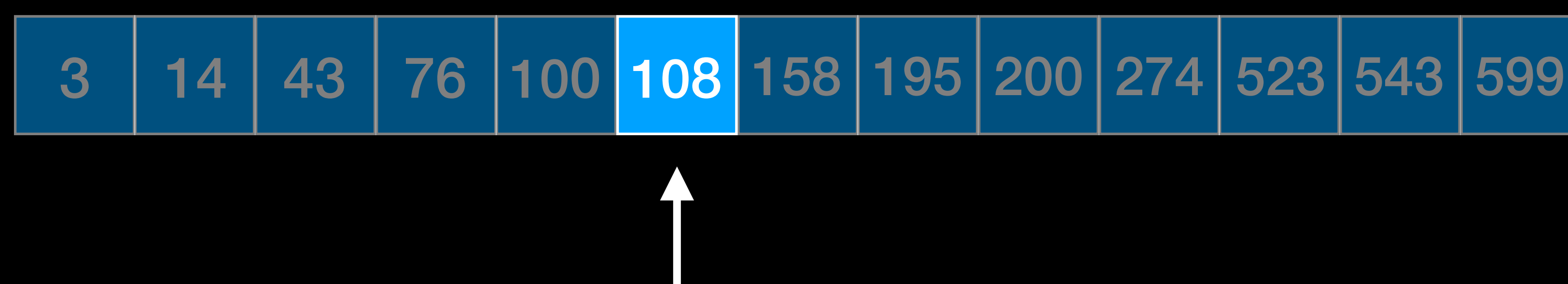
Binary Search



Binary Search

3	14	43	76	100	108	158	195	200	274	523	543	599

Binary Search



We have seen this before… where?

Binary Search

What is happening here?

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Simplification: assume n is a power of 2 so it can be evenly divided in two parts

The running time

Let $T(n)$ be the running time and **assume $n = 2^k$**

$T(n) = T(n/2) + 1$

One comparison

Search lower OR upper half

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Let $T(n)$ be the running time and **assume $n = 2^k$**

$T(n) = T(n/2) + 1$

$$\underbrace{T(n/2) = T(n/4)}_{\text{Search lower OR upper half of n/2}} + \underbrace{1}_{\text{One comparison}}$$

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Let $T(n)$ be the running time and **assume $n = 2^k$**

$T(n) = \boxed{T(n/2)} + 1$

$T(n/2) = T(n/4) + 1$

$T(n) = \boxed{T(n/4) + 1} + 1$

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Let $T(n)$ be the running time and **assume $n = 2^k$**

$T(n) = T(n/2) + 1$

2^1 1

$T(n) = T(n/4) + 2$

. . .

2^2 2

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Let $T(n)$ be the running time and **assume $n = 2^k$**

$T(n) = T(n/2) + 1$

$T(n) = T(n/4) + 2$

. . .

$T(n) = T(n/2^k) + k$

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Let $T(n)$ be the running time and **assume $n = 2^k$**

$T(n) = T(n/2) + 1$

The number to which I need to raise 2 to get n
And we said $n = 2^k$

$T(n) = T(n/4) + 2$

. . .

$T(n) = T(n/2^k) + k$

$T(n) = T(1) + \log_2(n)$

$n/n = 1$

Binary Search

What is happening here?

Size of search is **cut in half** at each step

Let $T(n)$ be the running time and **assume $n = 2^k$**

$T(n) = T(n/2) + 1$

$T(n) = T(n/4) + 2$

. . .

$T(n) = T(n/2^k) + k$

$T(n) = T(1) + \log_2(n)$

Binary search is $O(\log(n))$

Sorting

Rearranging a sequence into increasing (decreasing) order!

Several approaches

Can do it in may ways

What is the best way?

Let's find out using Big-O

Lecture Activity

Write **pseudocode** to sort an array.

543	3	523	76	200	158	195	108	43	274	100	14	599

There are many approaches to sorting
We will look at some comparison-based approaches here

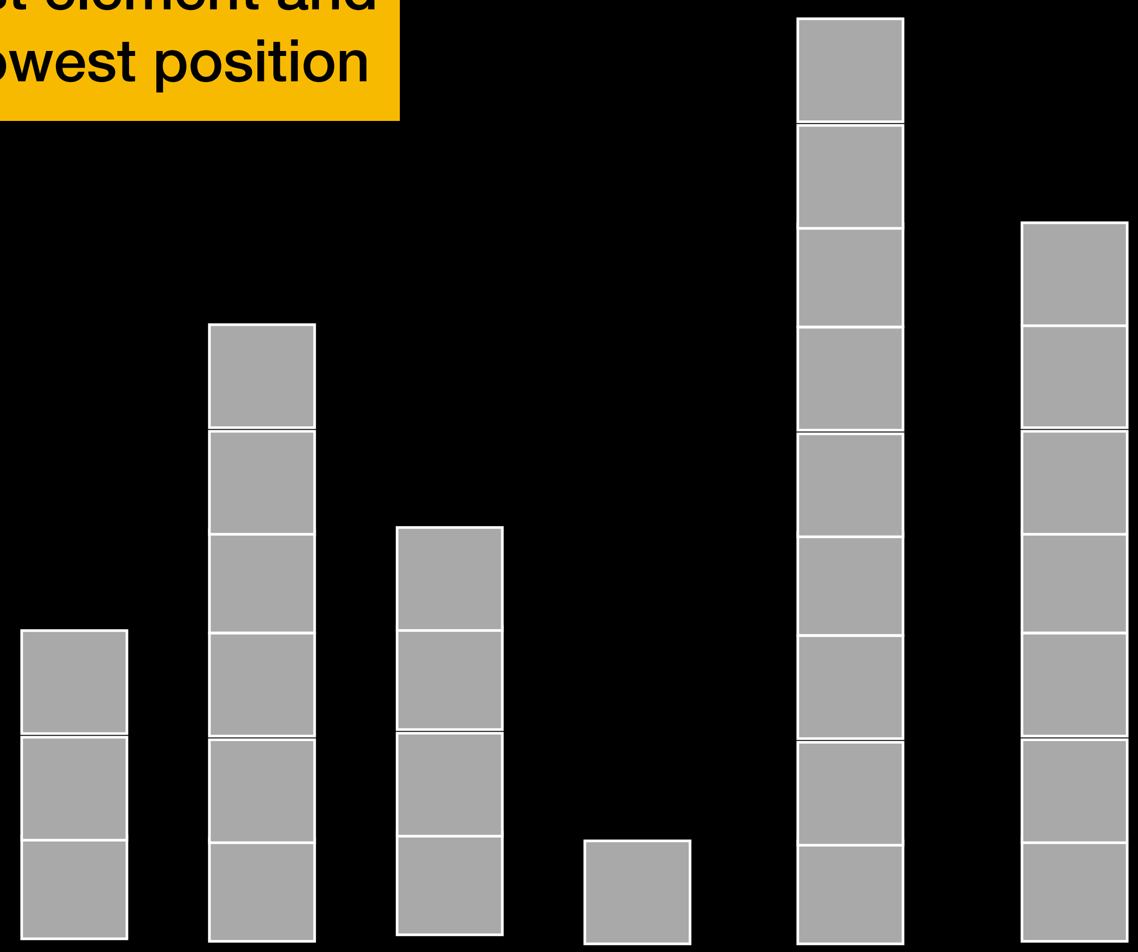
Selection Sort

Selection Sort

Unsorted

Sorted

Find smallest element and move it at lowest position



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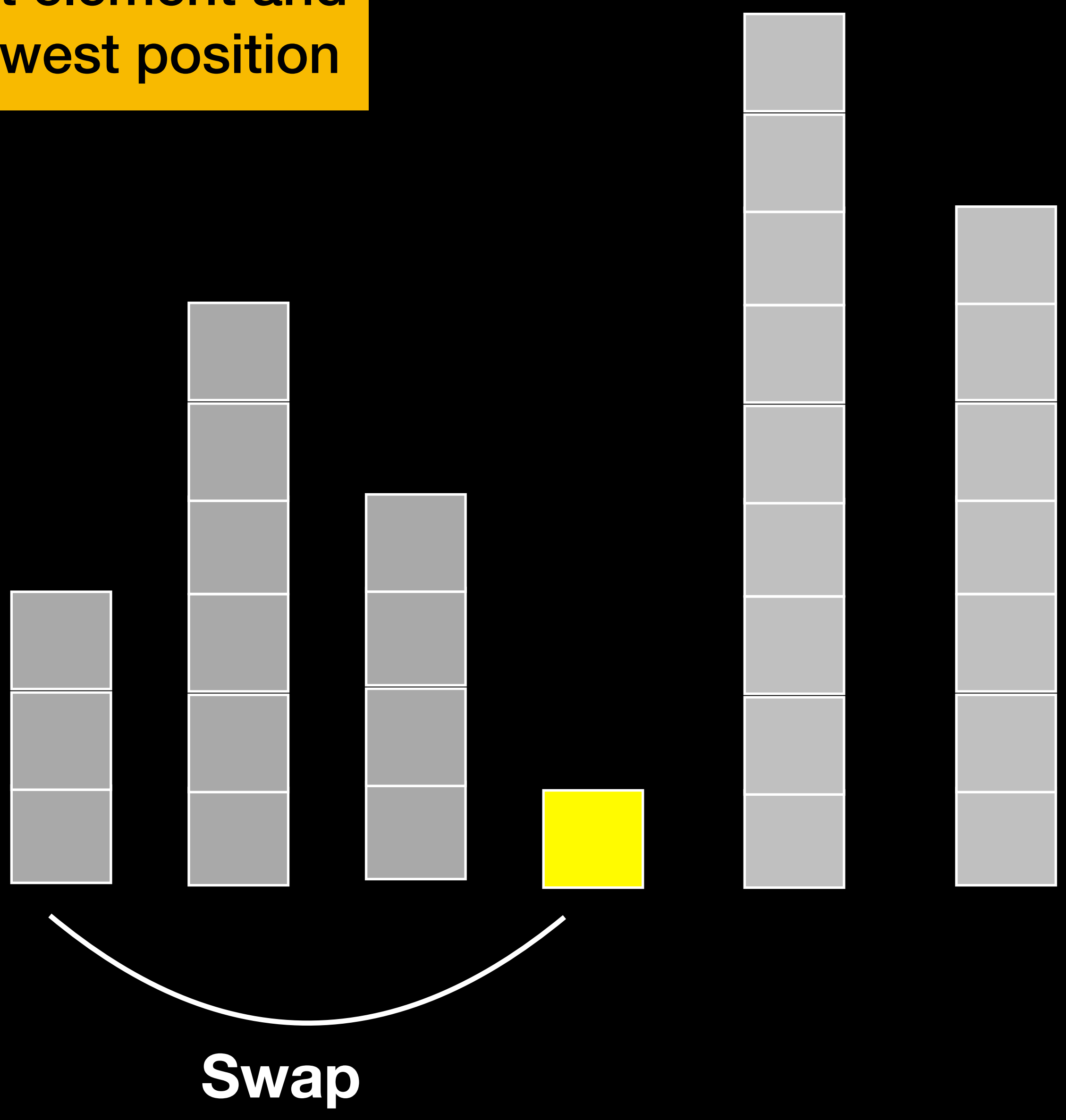

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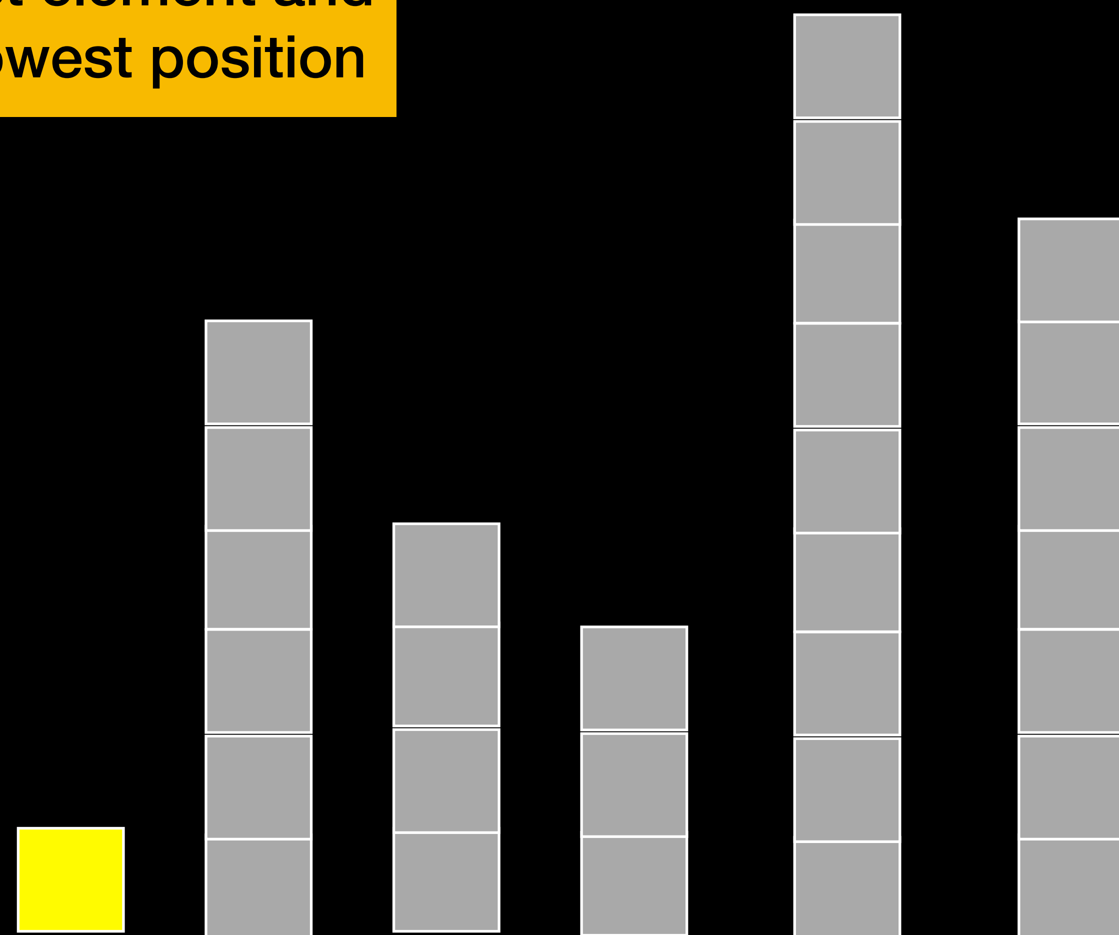

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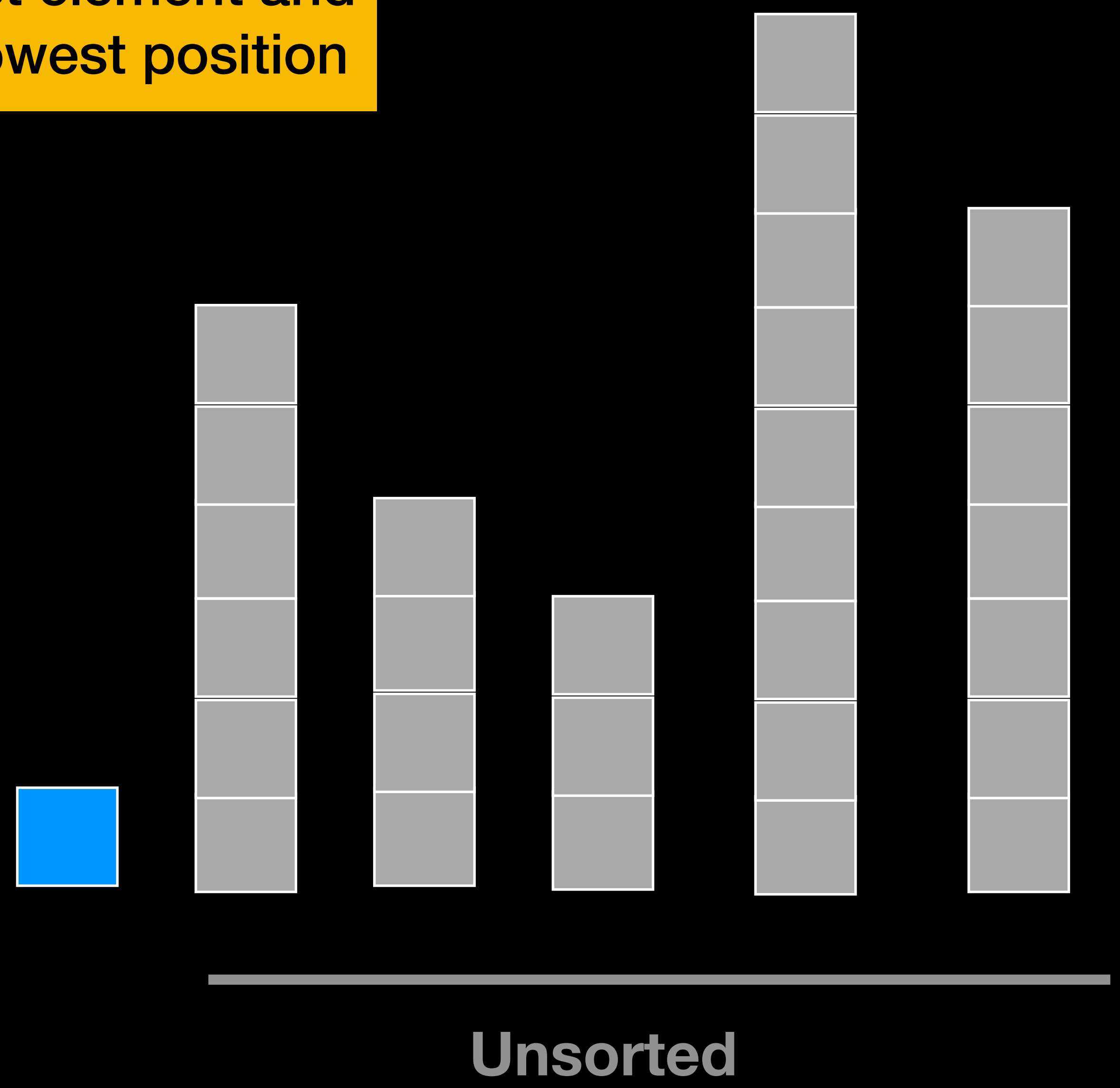
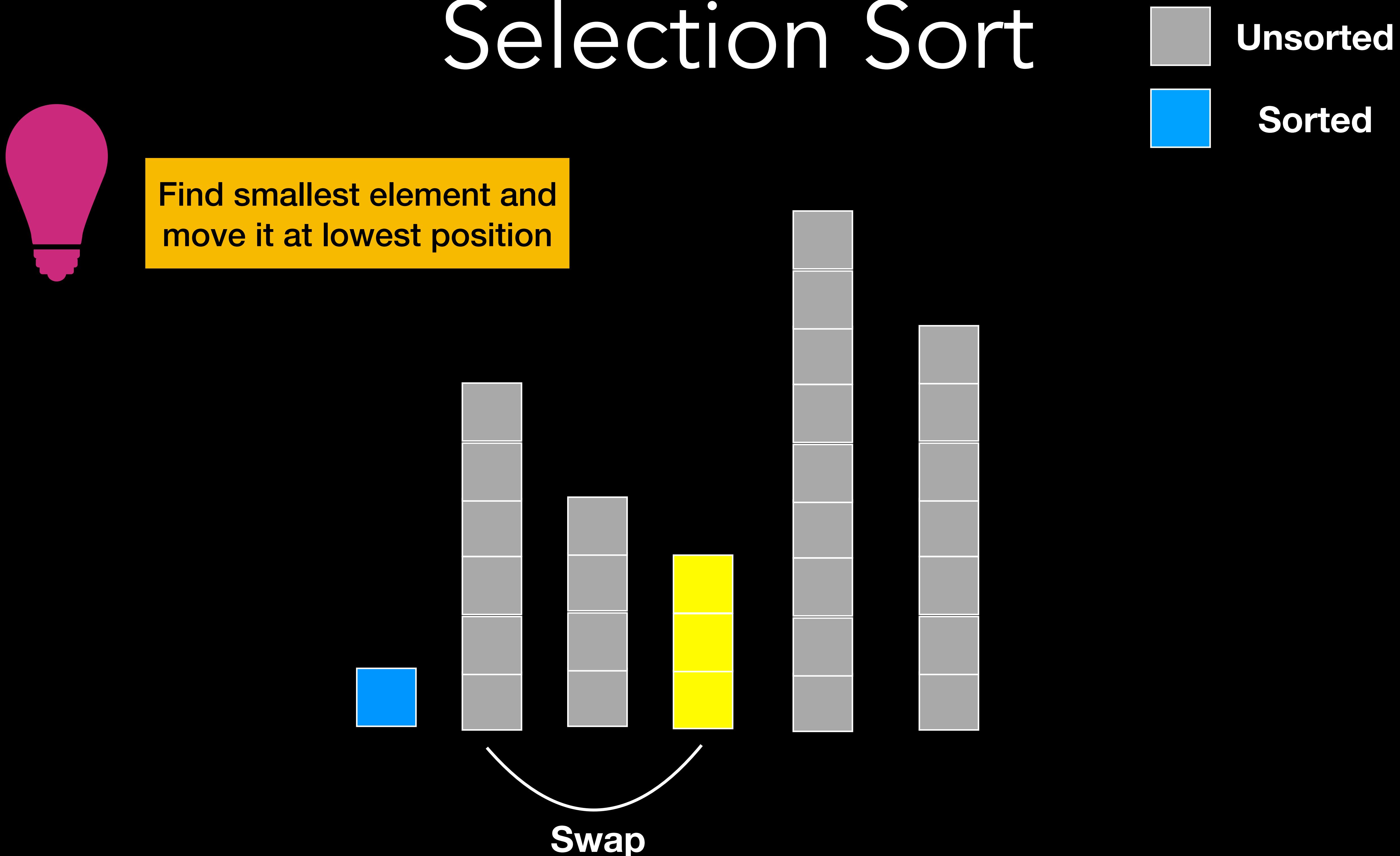


Selection Sort



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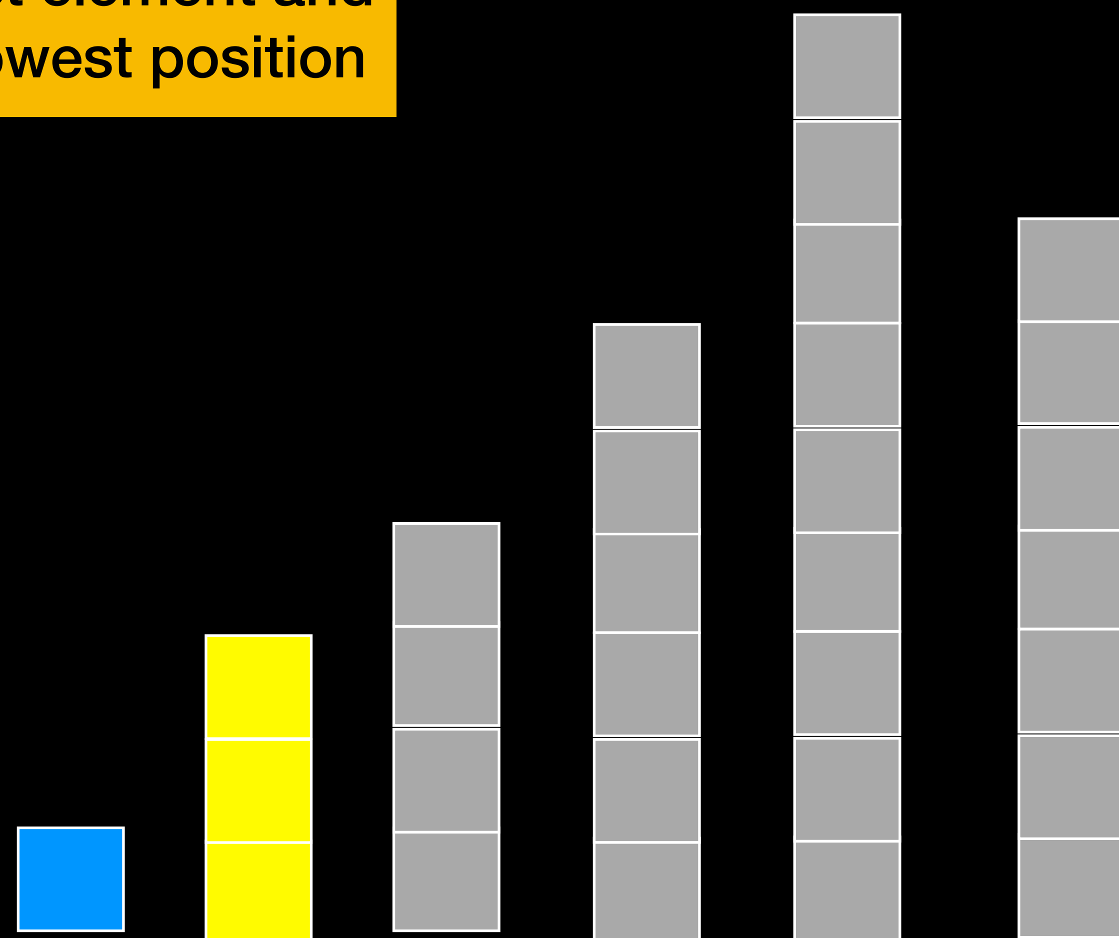

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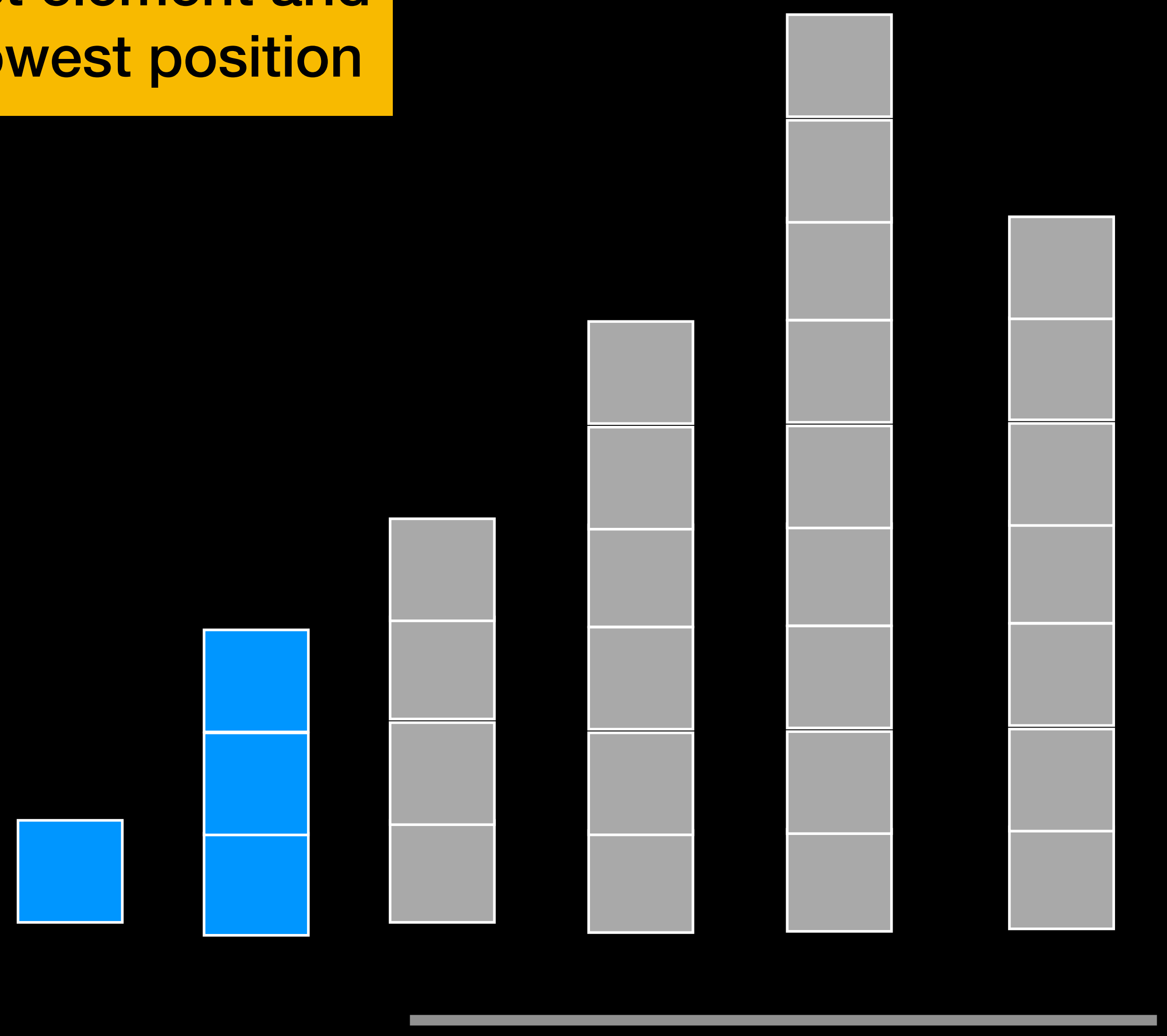

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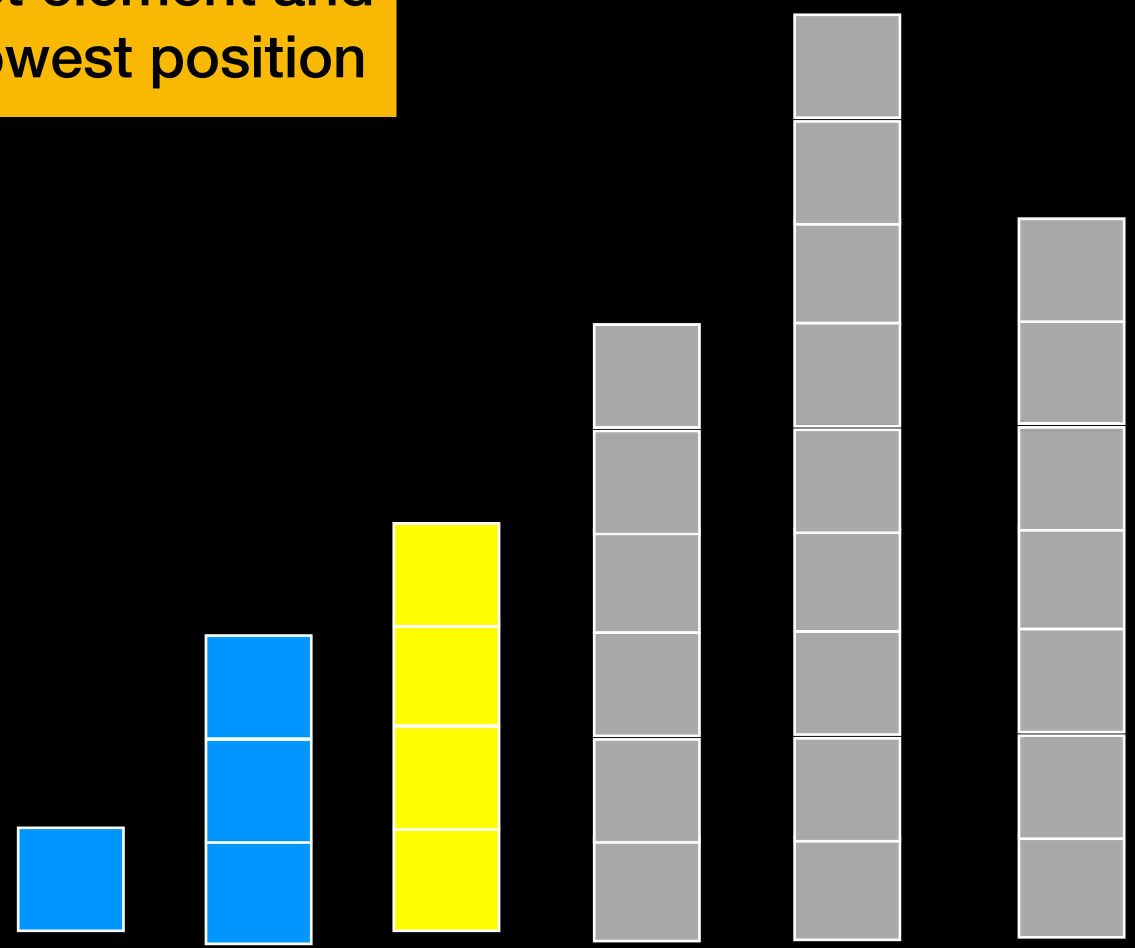

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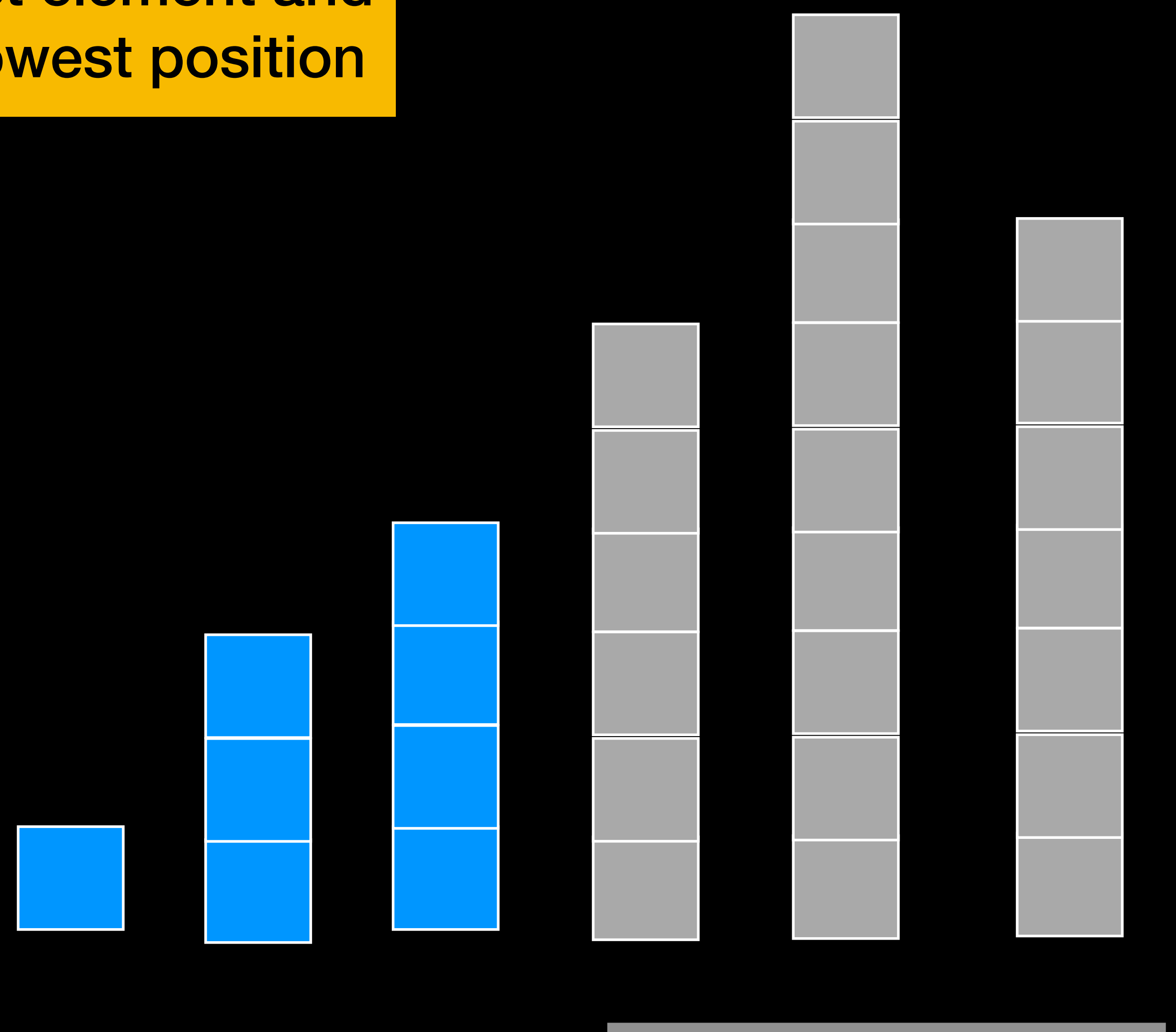
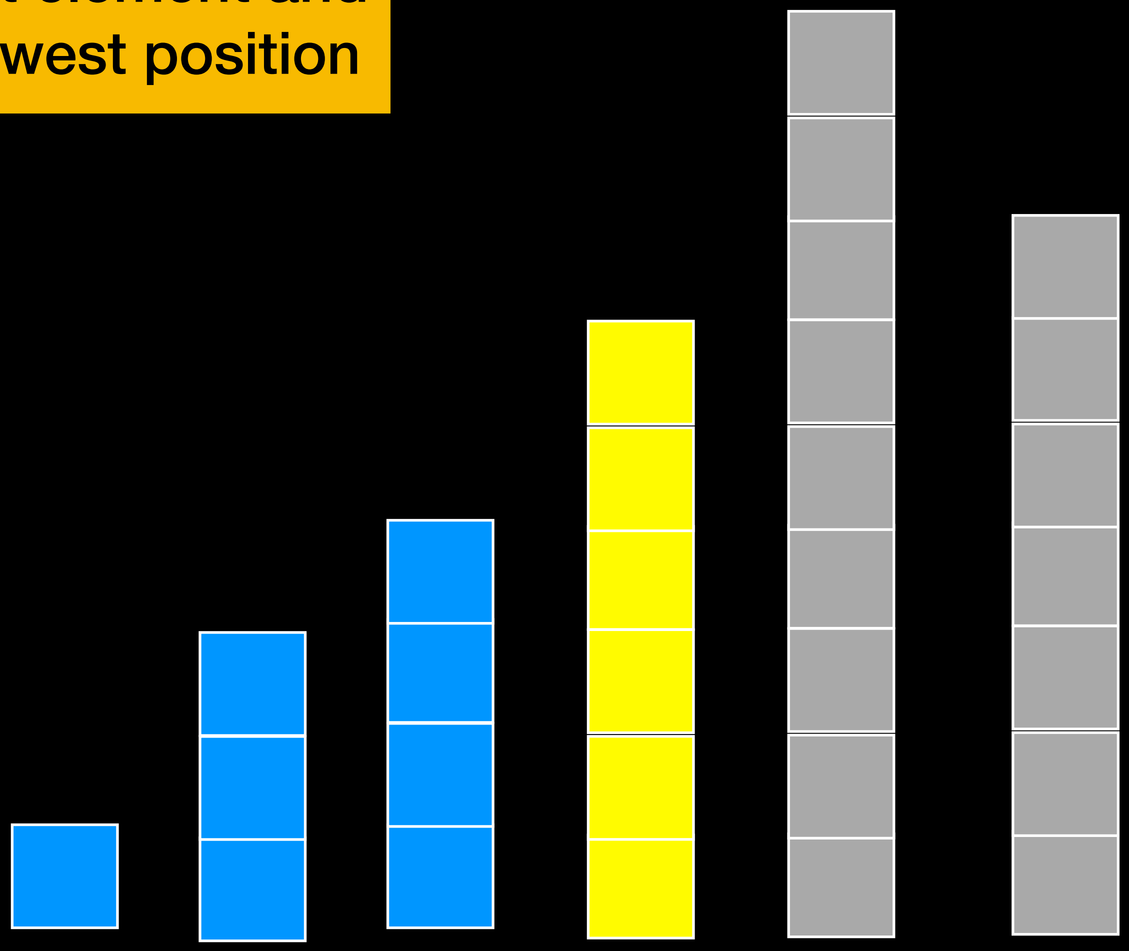


Selection Sort

Find smallest element and move it at lowest position



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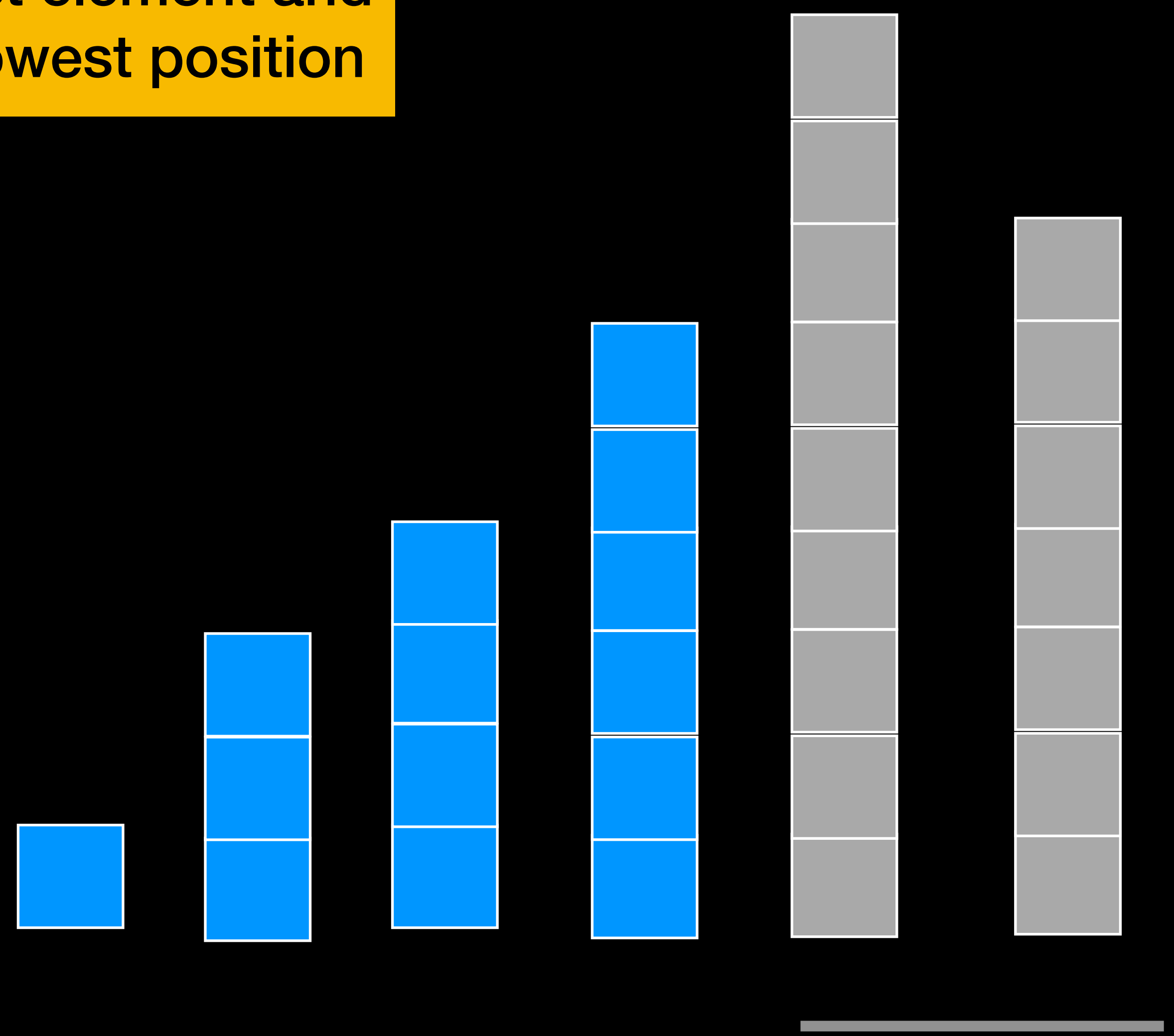

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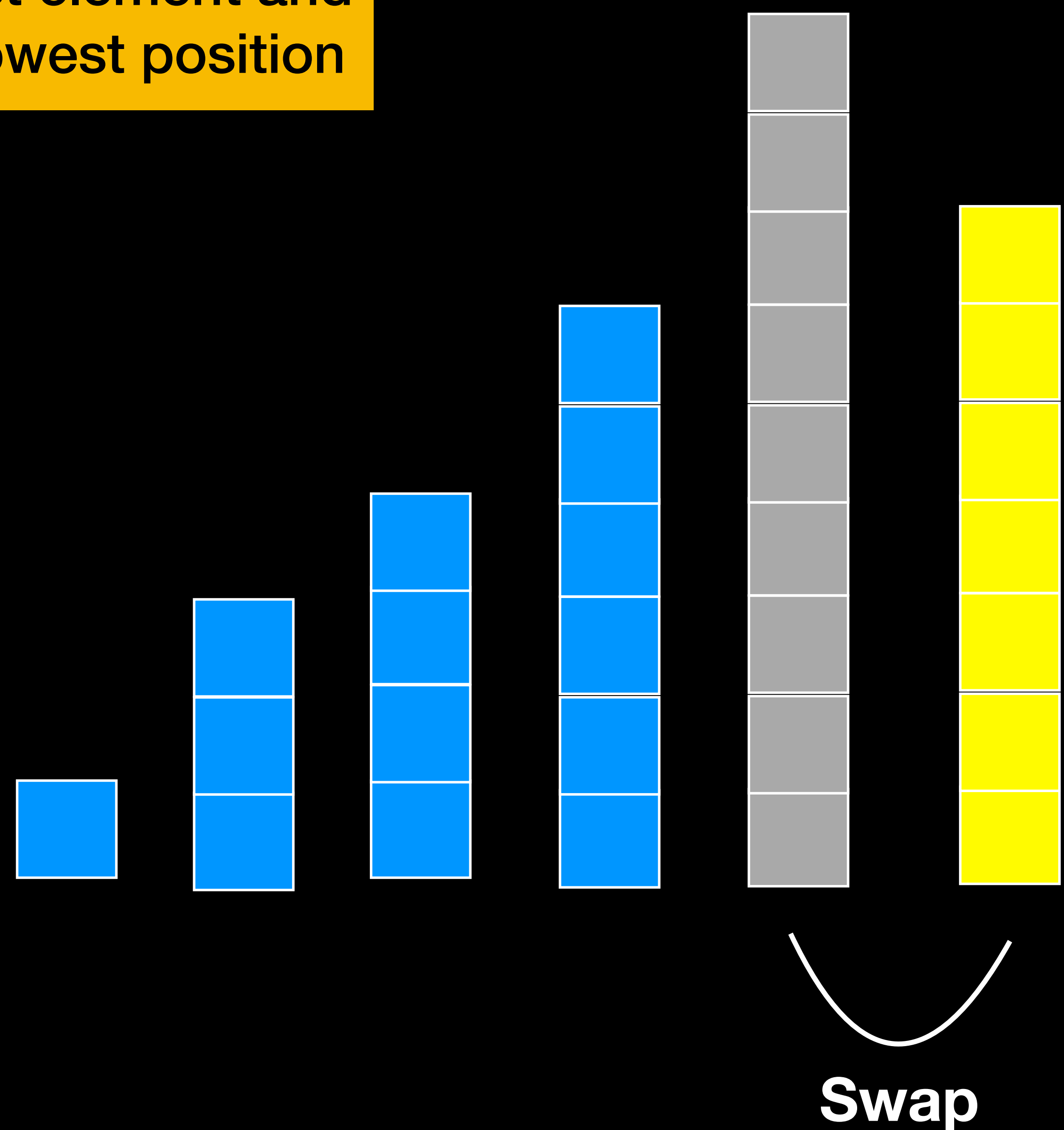
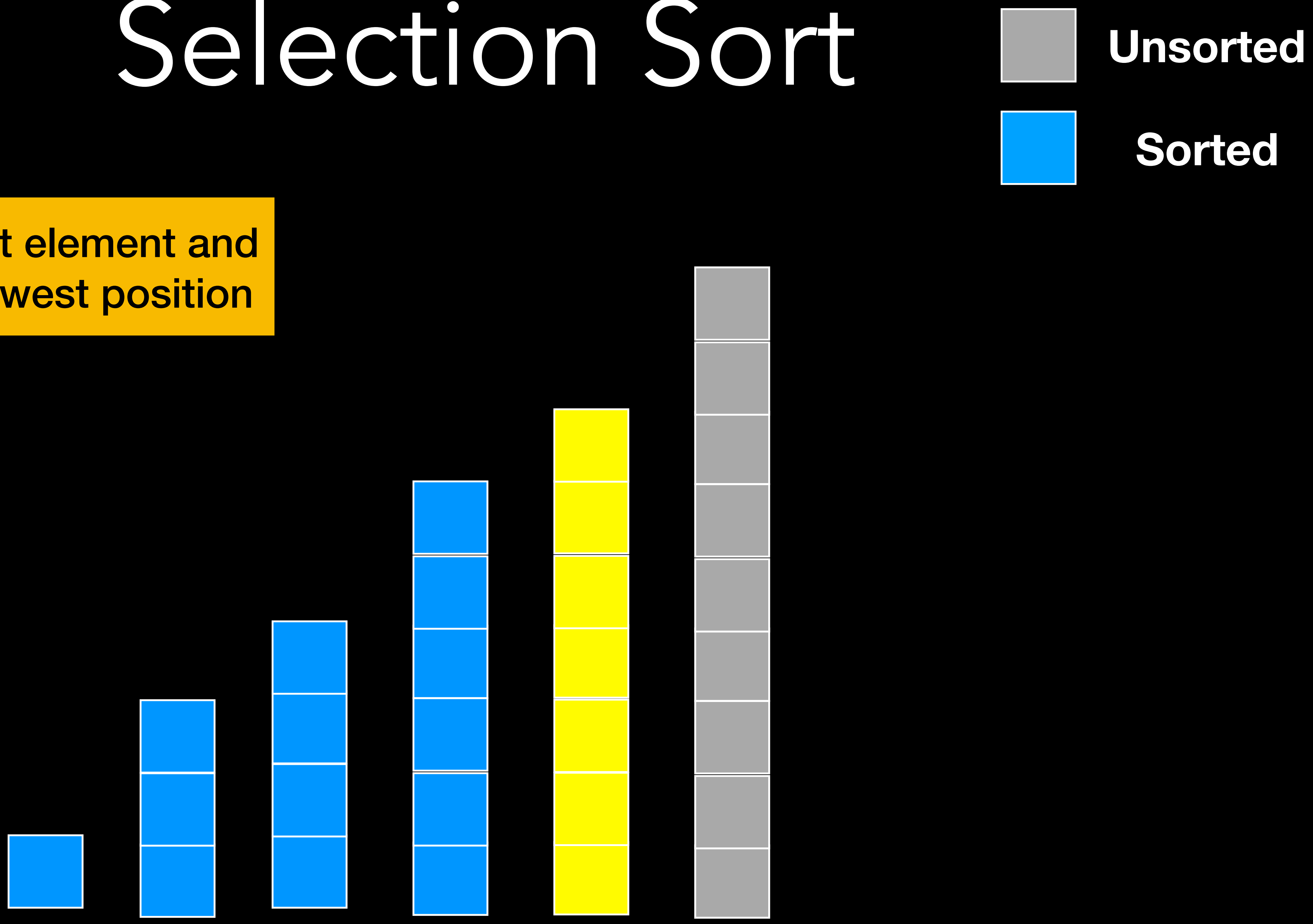


Selection Sort

Find smallest element and move it at lowest position



Selection Sort

Unsorted

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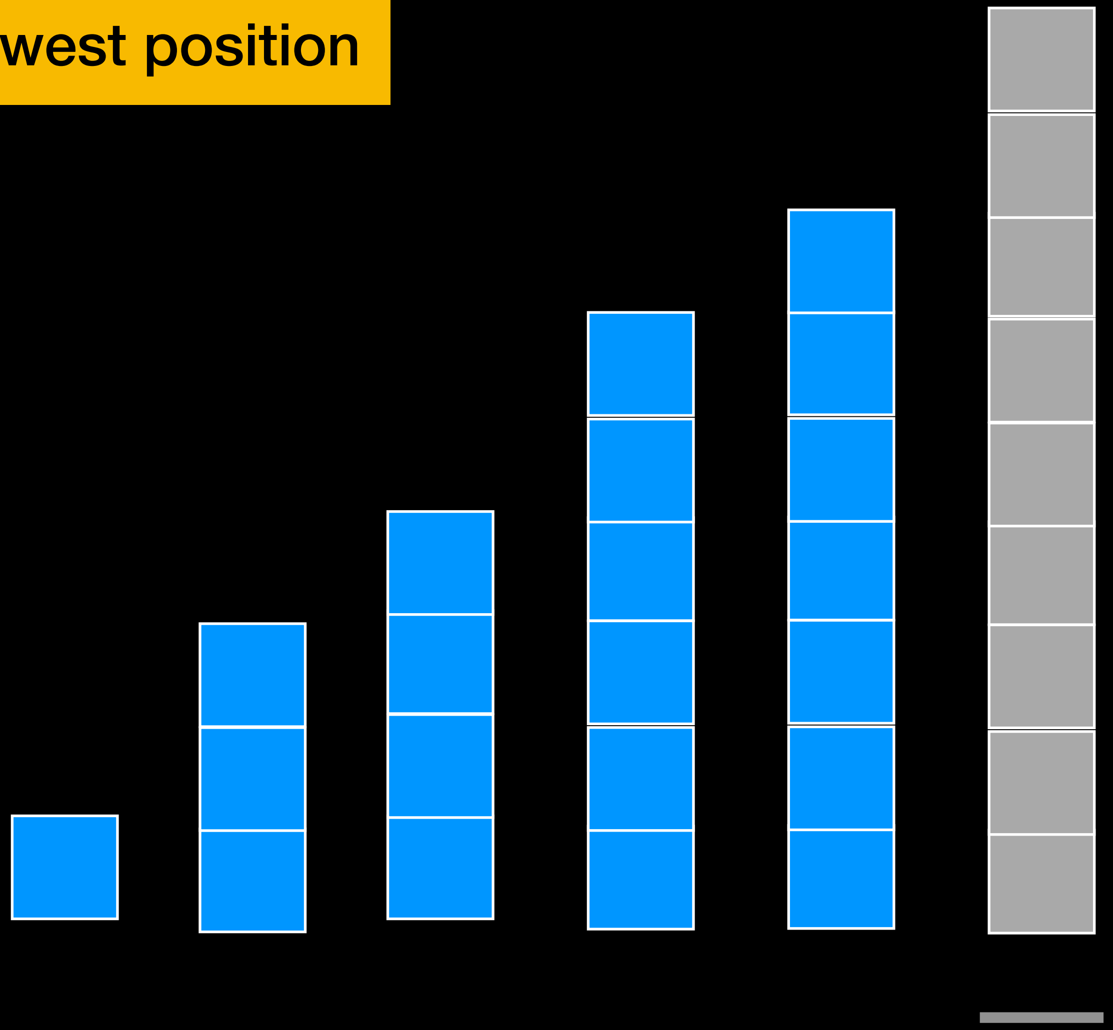
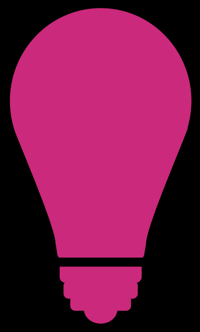
Find smallest element and move it at lowest position



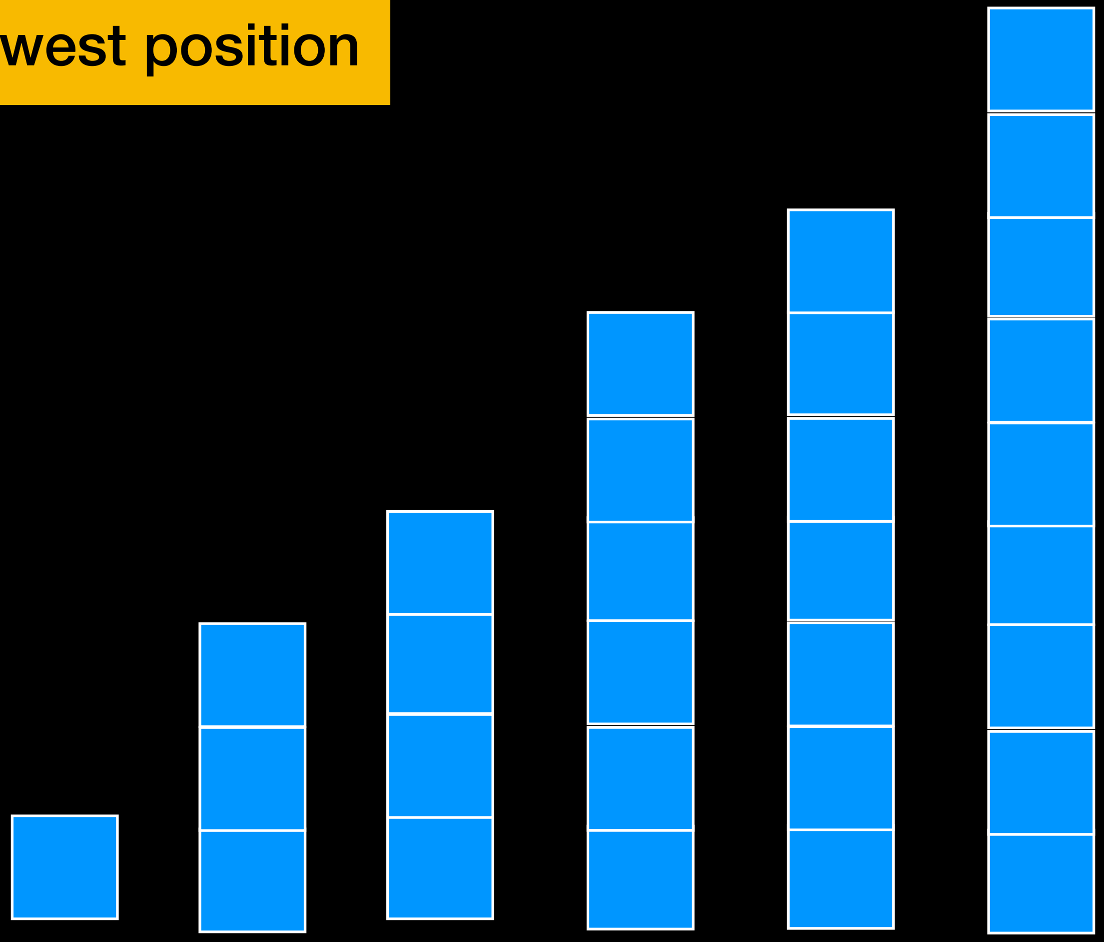
Selection Sort

Unsorted

Sorted



Find smallest element and move it at lowest position



Selection Sort

Find the smallest item and move it at position 1

Find the next-smallest item and move it at position 2

. . .

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

Find second smallest: look at **n-1** elements

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

Find second smallest: look at **n-1** elements

Find third smallest: look at **n-2** elements

. . .

Selection Sort Analysis

How much work?

Find smallest: look at **n** elements

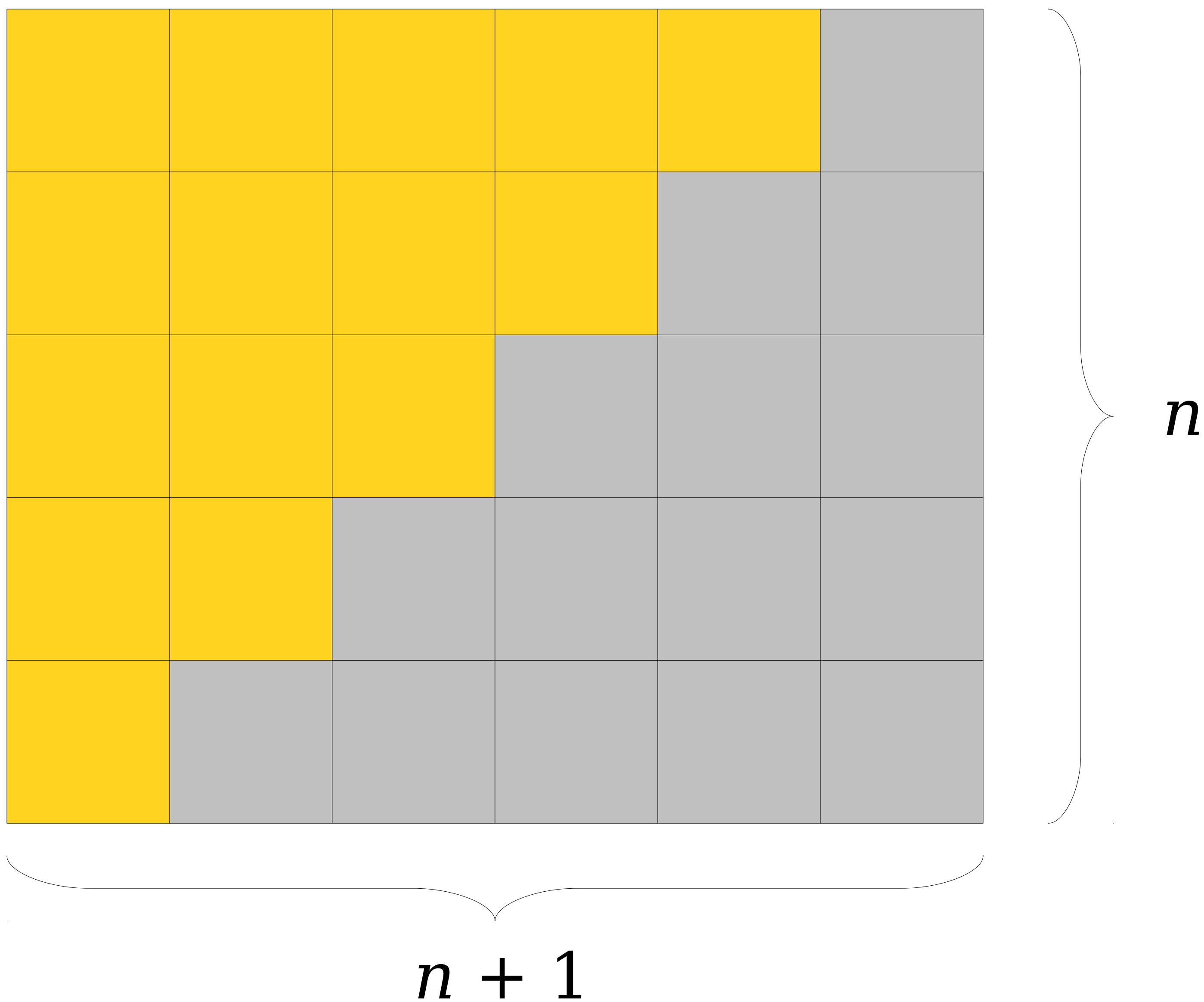
Find second smallest: look at **n-1** elements

Find third smallest: look at **n-2** elements

. . .

Total work: **n + (n-1) + (n-2) + . . . +1**

$$n + (n\text{-}1) + \ldots + 2 + 1 = n(n+1) / 2$$



Selection Sort Analysis

T(n) = n(n+1) / 2 comparisons + n data moves = O( )?

Selection Sort Analysis

$T(n) = n(n+1) / 2$ comparisons $+ n$ data moves $= O(\ )$?

$T(n) = (n^2+n) / 2 + n = O(\ )$?

Selection Sort Analysis

$T(n) = n(n+1) / 2$ comparisons $+ n$ data moves $= O(\ )$?

$T(n) = (n^2+n) / 2 + n = O(\ )$?

Ignore constant

Ignore non-dominant terms

Selection Sort Analysis

$T(n) = n(n+1) / 2$ comparisons $+ n$ data moves $= O(\ )$?

$T(n) = (n^2+n) / 2 + n = O(\mathbf{n^2})$

Ignore constant

Ignore non-dominant terms

Selection Sort Analysis

$T(n) = n(n+1) / 2$ comparisons + n data moves = $O(\ )$?

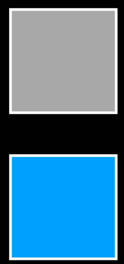
$T(n) = (n^2+n) / 2 + n = O(n^2)$

Selection Sort run time is **$O(n^2)$**

Stability

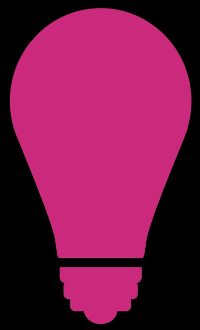
A sorting algorithm is Stable if elements that are equal remain is same order relative to each other after sorting

Selection Sort

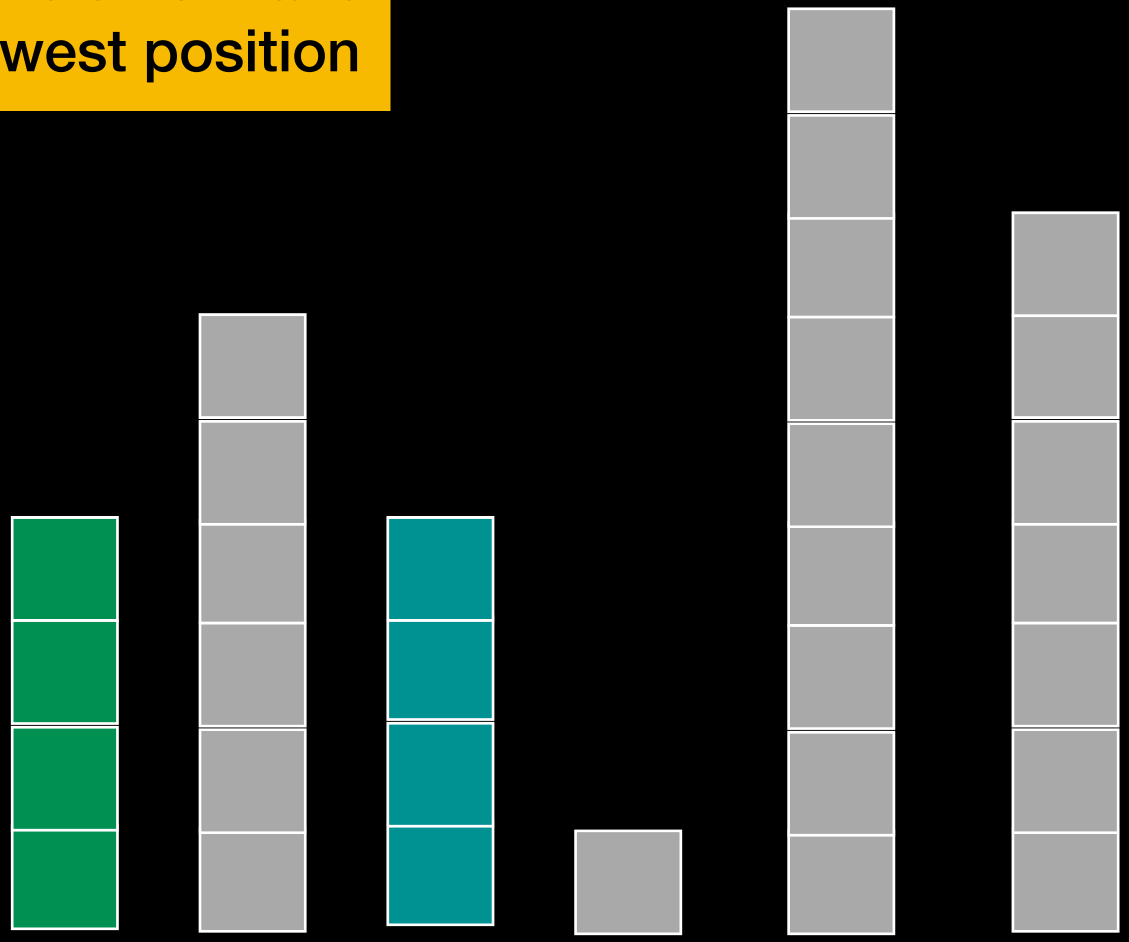


Unsorted

Sorted



Find smallest element and move it at lowest position



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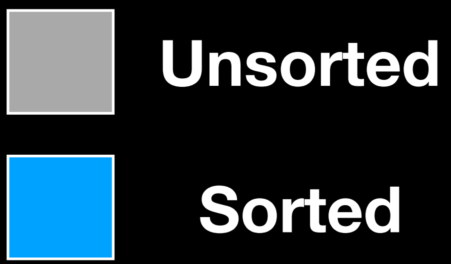

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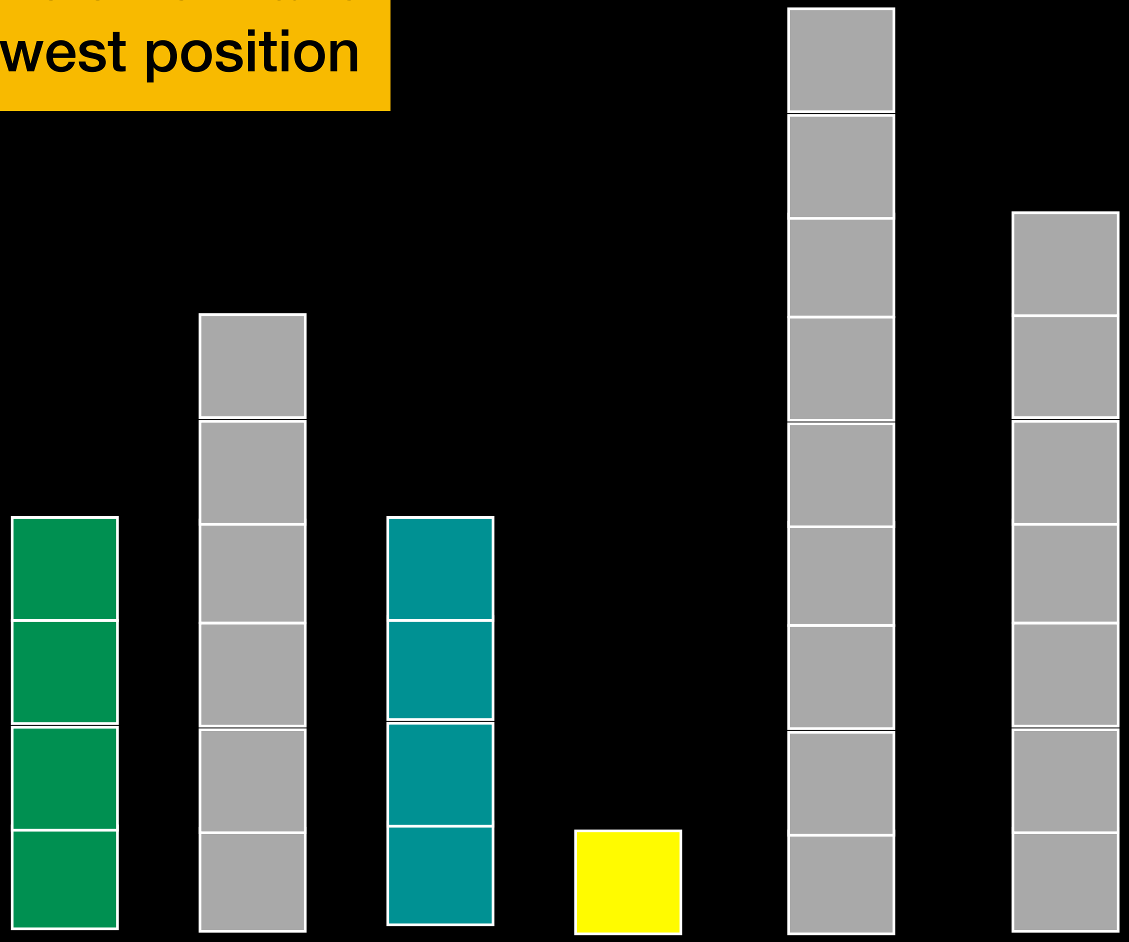

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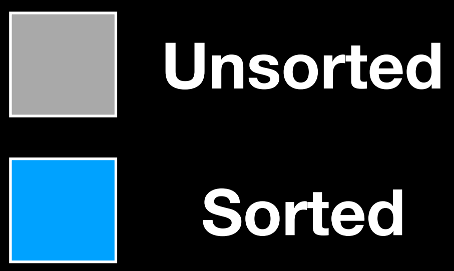

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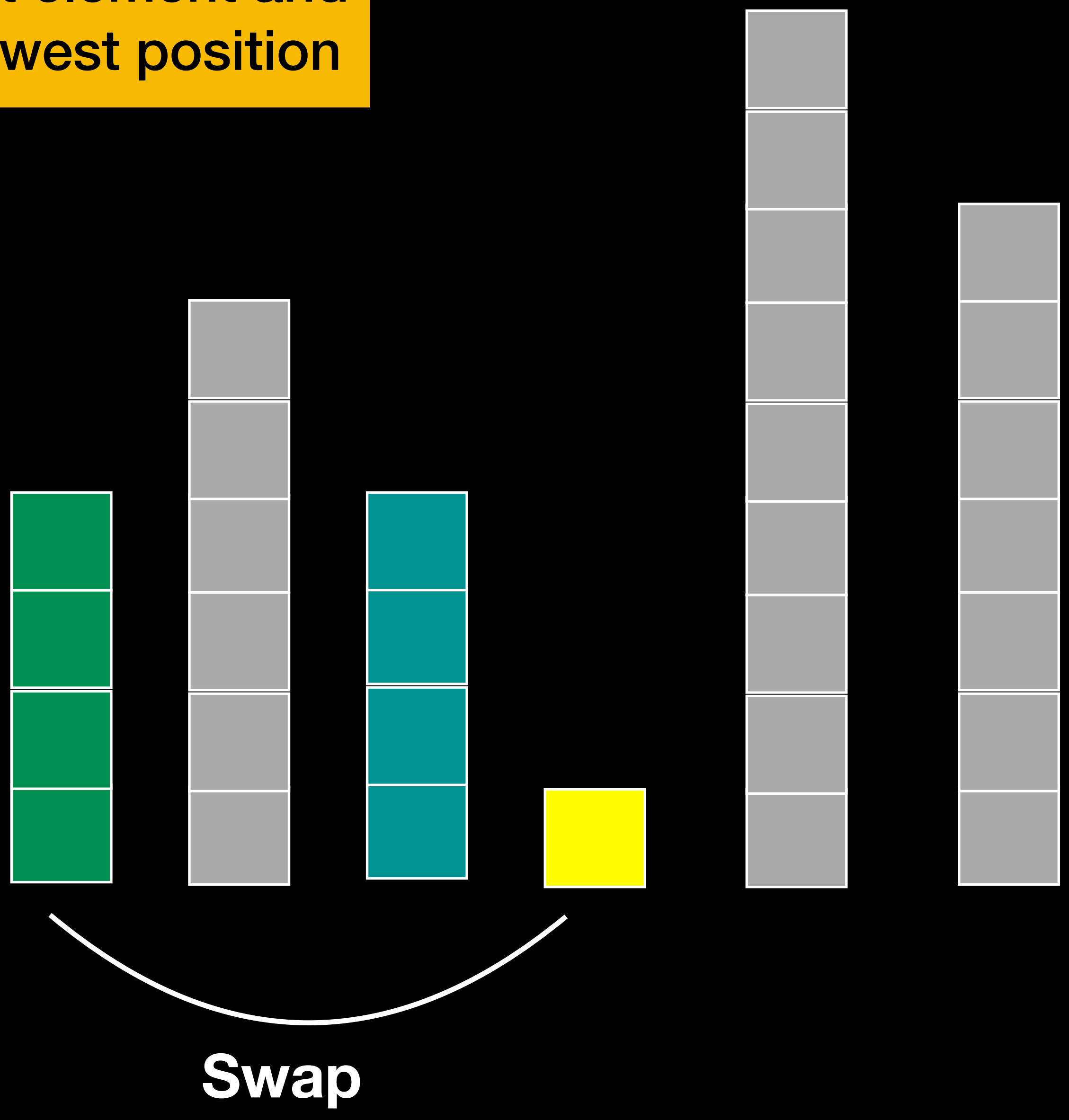

Selection Sort

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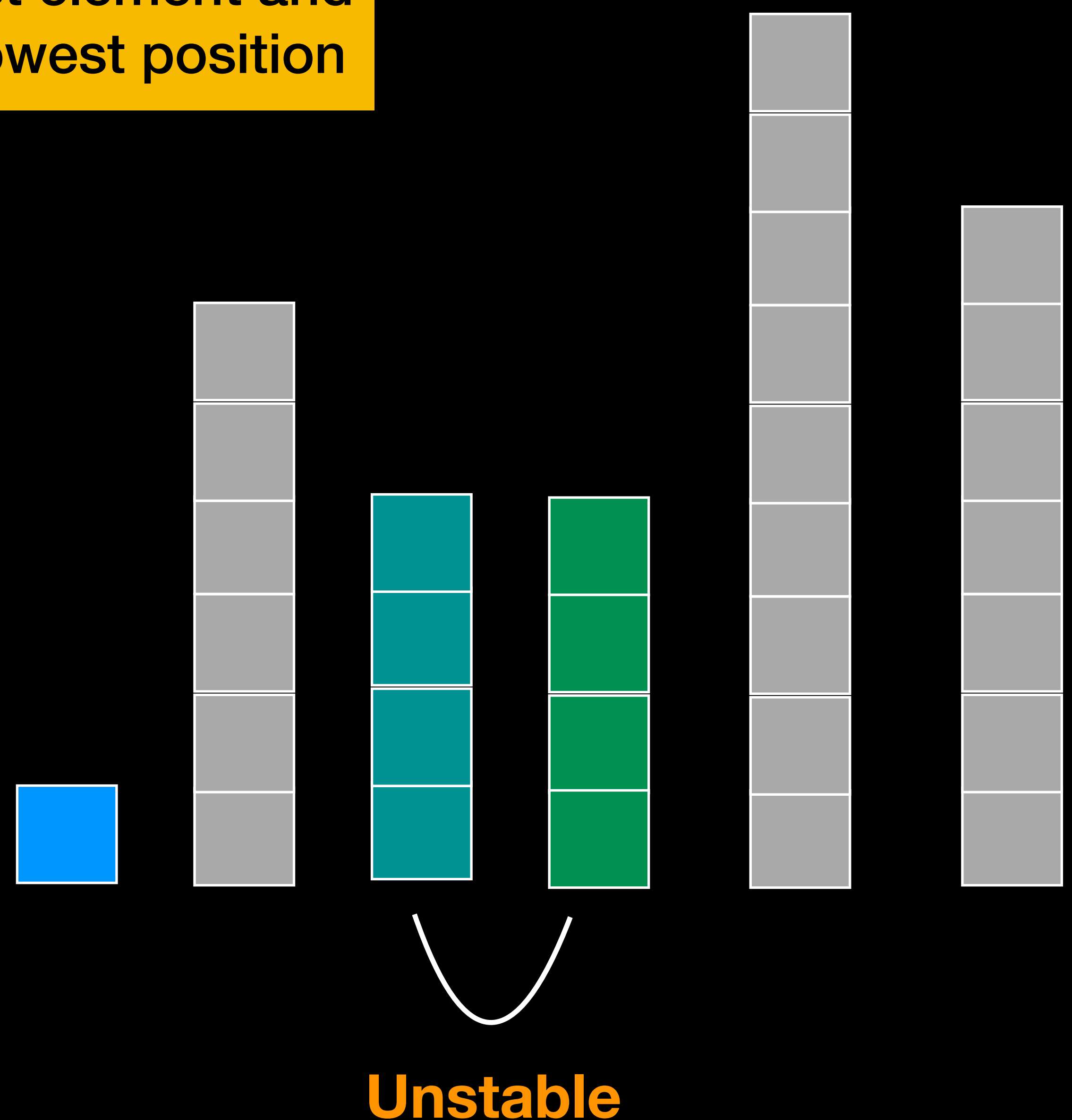
Find smallest element and move it at lowest position



Selection Sort Analysis

Execution time DOES NOT depend on initial arrangement of data => ALWAYS O(n^2)

O(n^2) comparisons

Good choice for small **n** and/or data moves are costly (O(n) data moves)

Unstable

Understanding $O(n^2)$

100	14	3	43	200	274

T(n)

Understanding $O(n^2)$

100	14	3	43	200	274

$T(n)$

100	14	3	43	200	274	523	108	76	195	599	158

$T(2n) \approx 4T(n)$

$(2n)^2 = 4n^2$

Understanding O(n^2)

100	14	3	43	200	274

$T(n)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(3n) \approx 9T(n)$

$(3n)^2 = 9n^2$

Understanding $O(n^2)$ on large input

If size of input increases by factor of 100

Execution time increases by factor of 10,000

$T(100n) = 10{,}000T(n)$

!

Understanding $O(n^2)$ on large input

If size of input increases by factor of 100

Execution time increases by factor of 10,000

$T(100n) = 10{,}000T(n)$

Assume $n = 100{,}000$ and $T(n) = 17$ seconds

Sorting 10,000,000 takes 10,000 longer

!

Understanding $O(n^2)$ on large input

If size of input increases by factor of 100
Execution time increases by factor of 10,000
$T(100n) = 10{,}000T(n)$

Assume $n = 100{,}000$ and $T(n) = 17$ seconds
Sorting 10,000,000 takes 10,000 longer

Sorting 10,000,000 entries takes ≈ 2 days

Multiplying input by 100 to go from 17sec to 2 days!!!

Raise your hand if you had Selection Sort

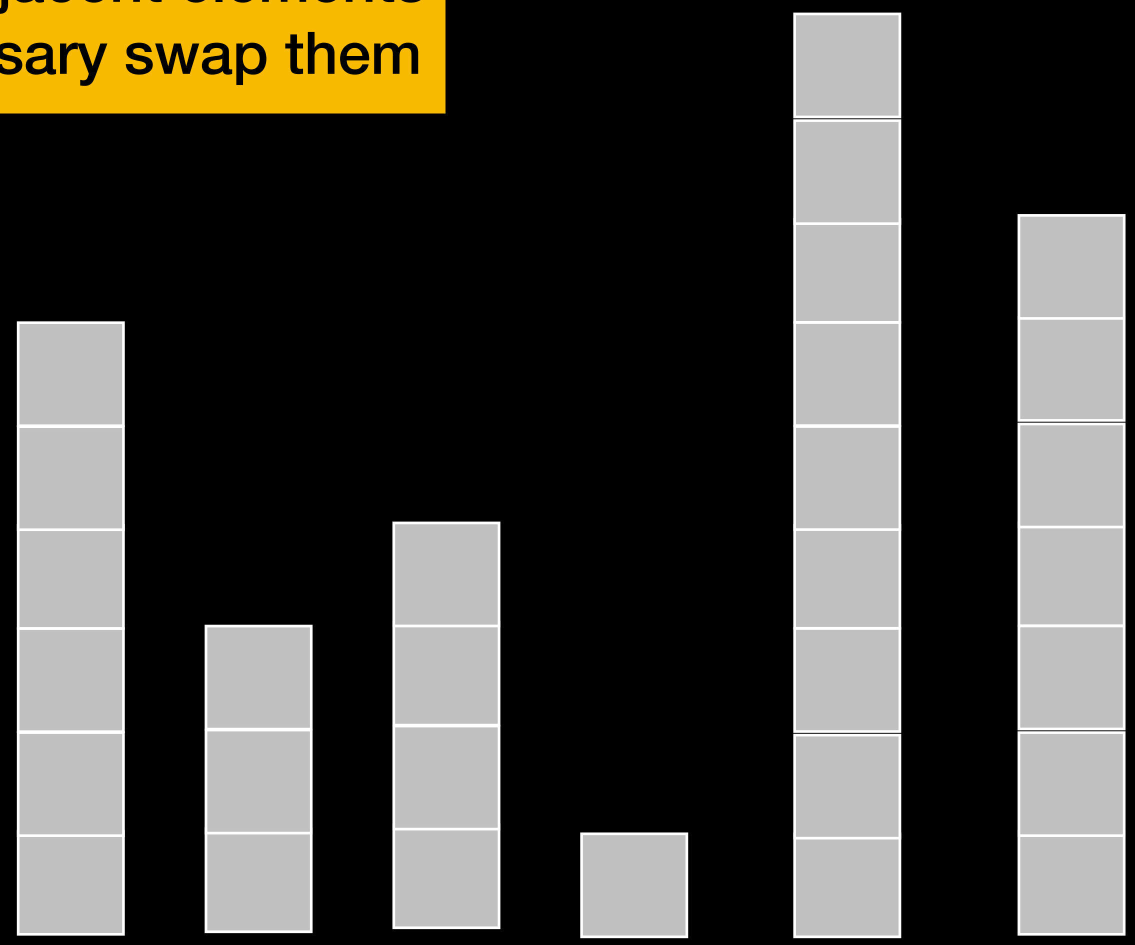
Bubble Sort

Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them



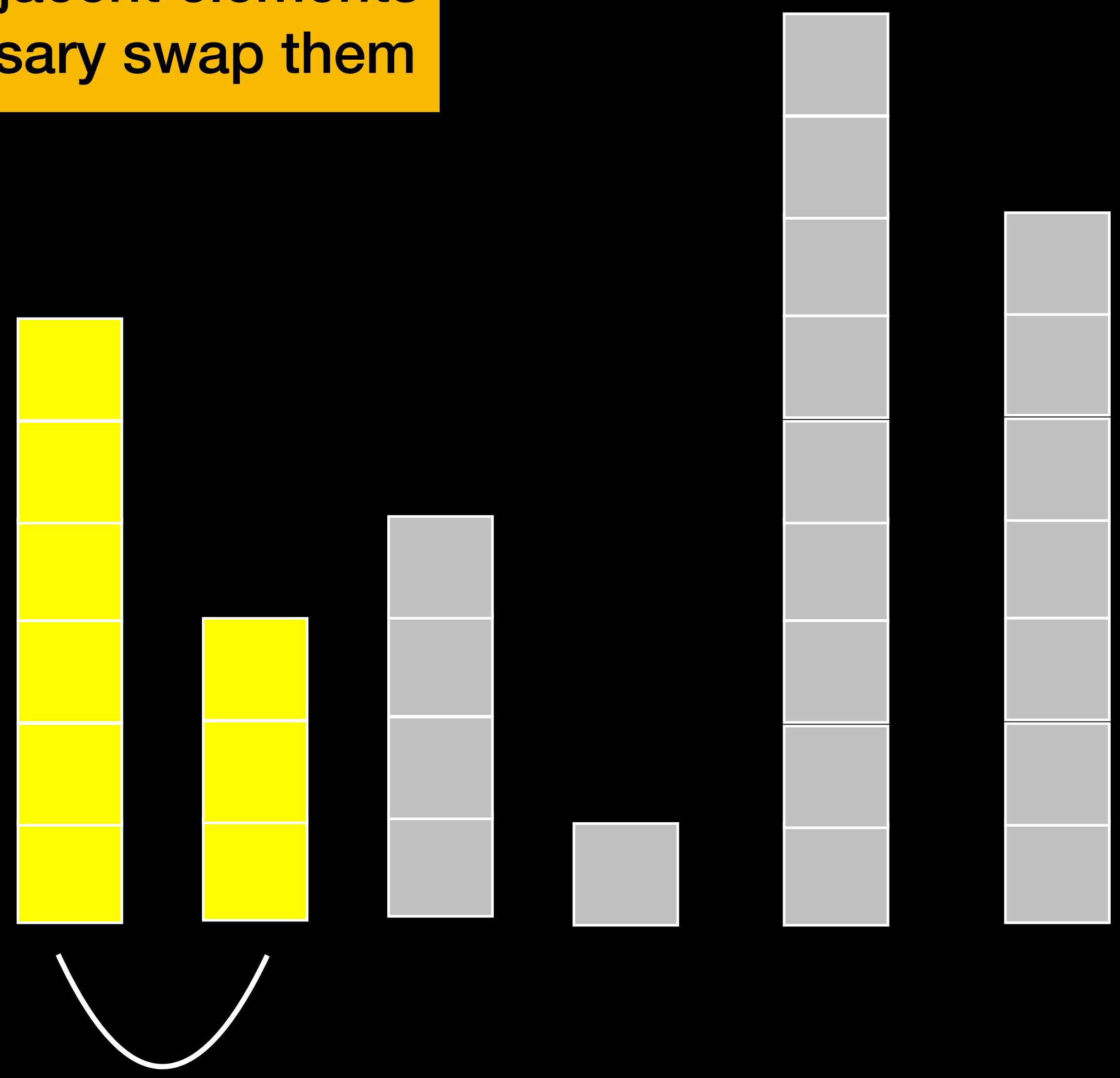
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

1st Pass



Bubble Sort

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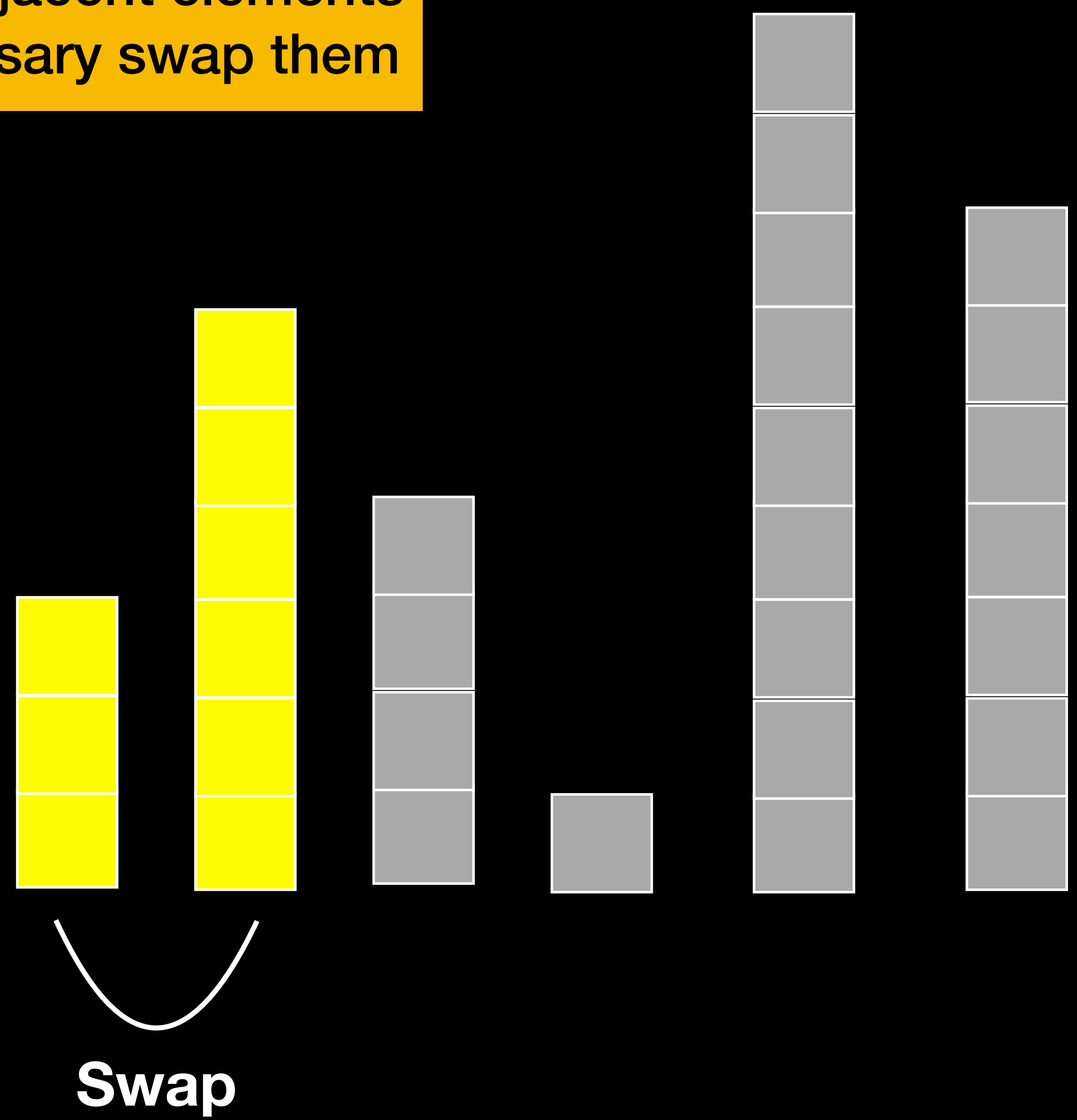

Bubble Sort

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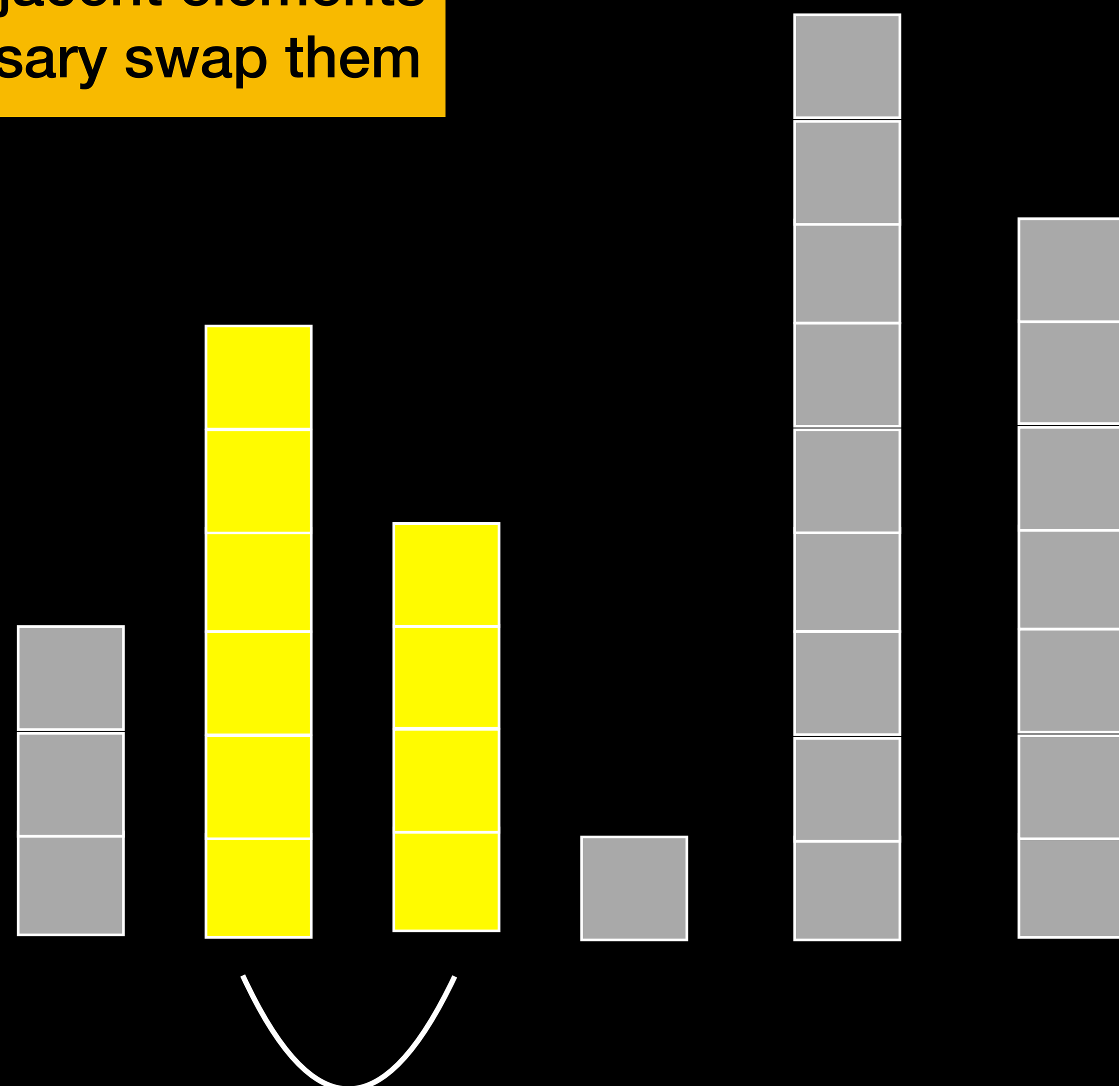

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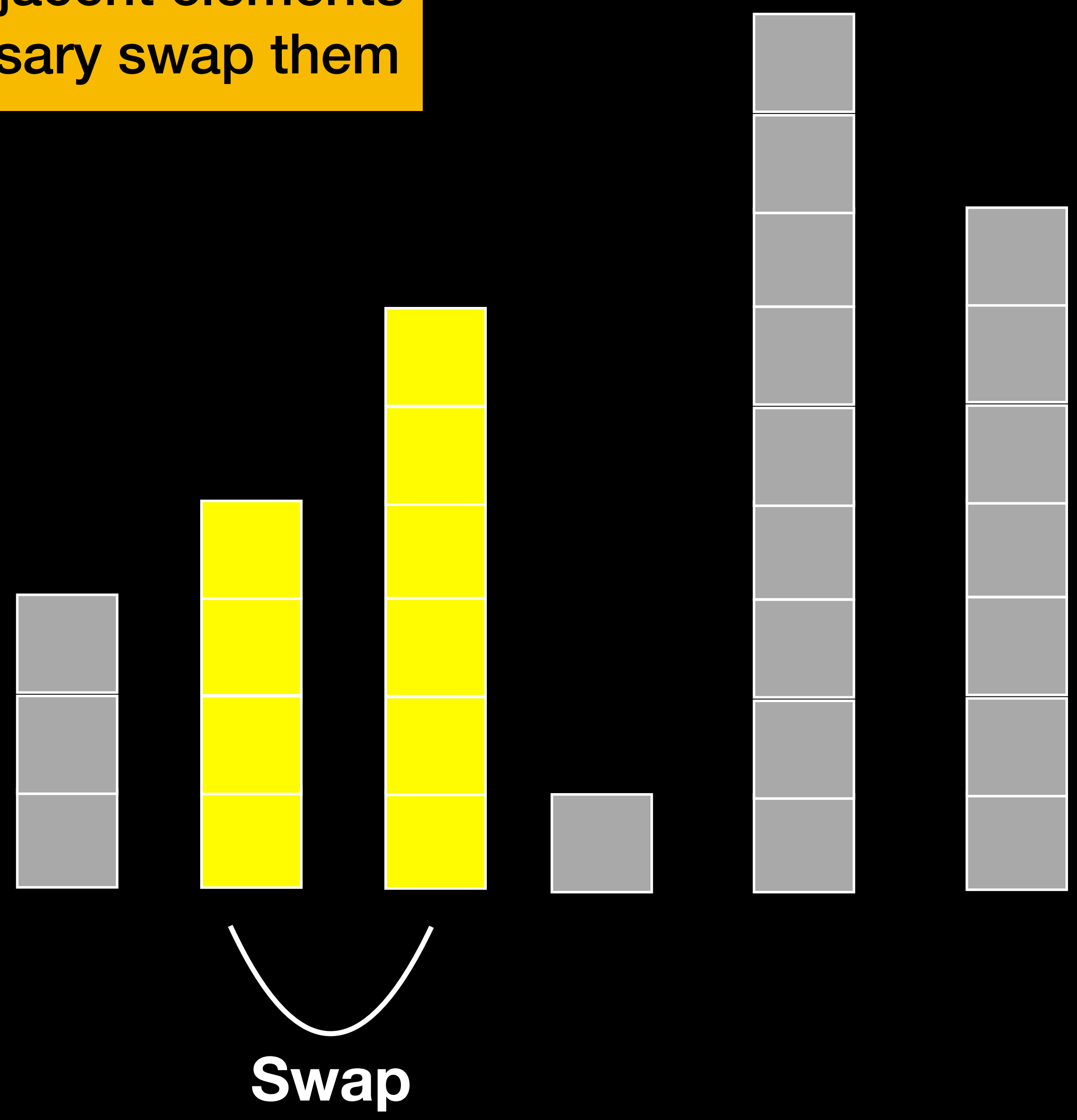

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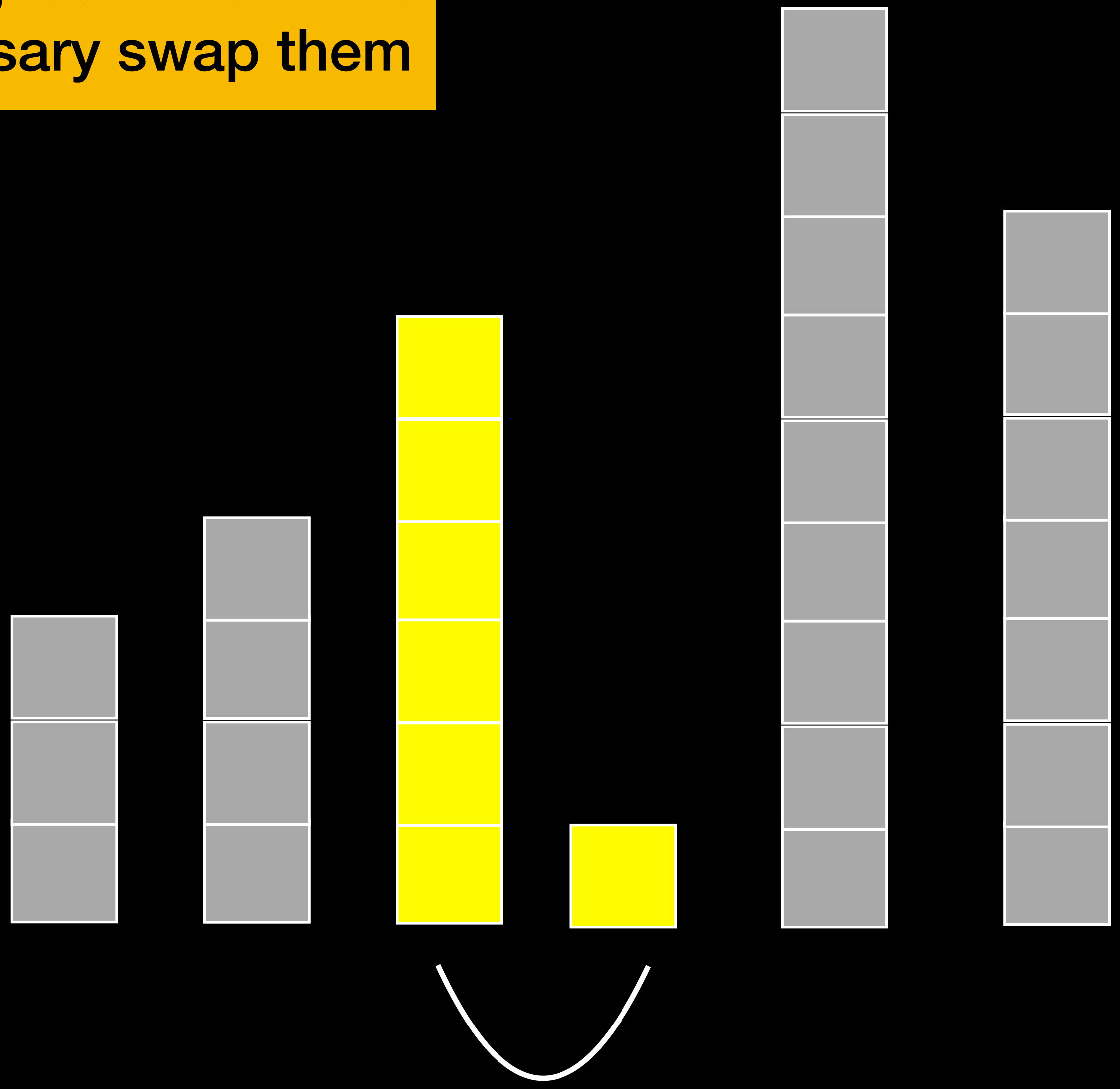

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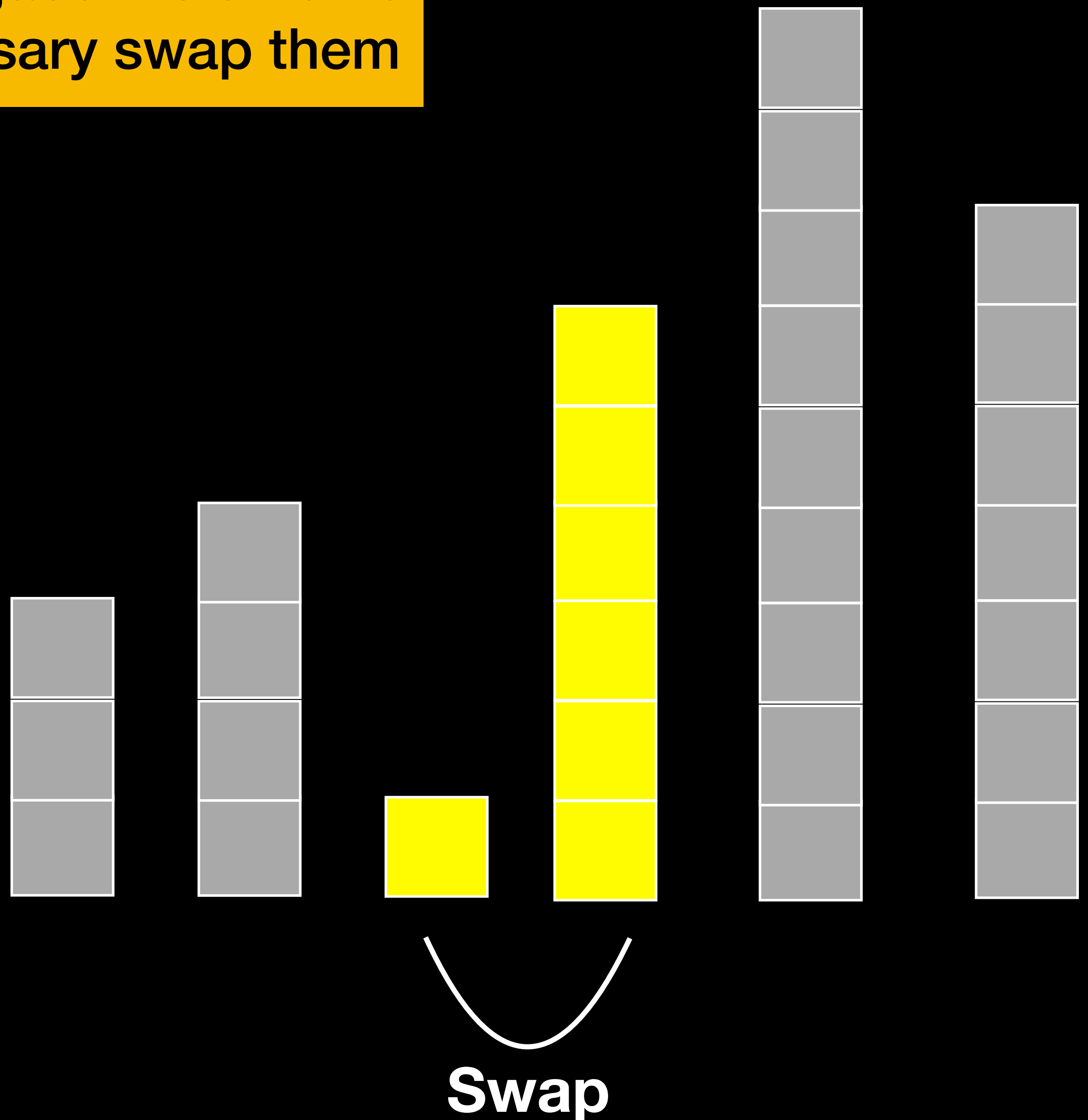
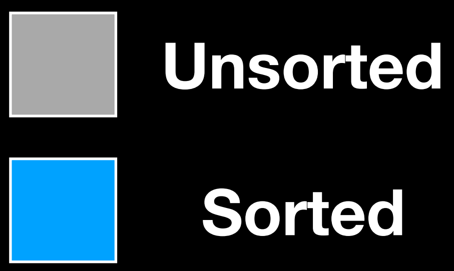


Bubble Sort



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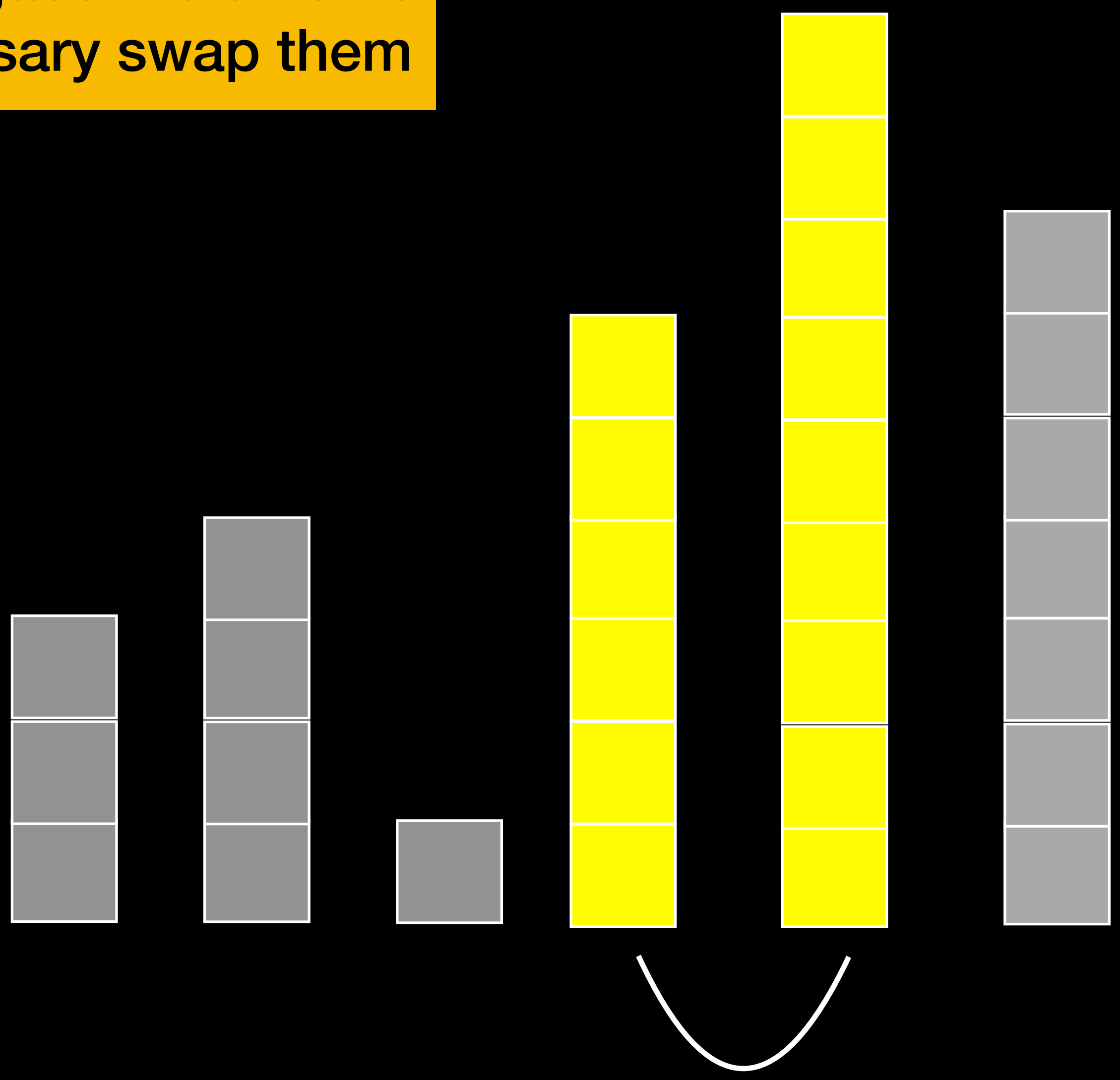

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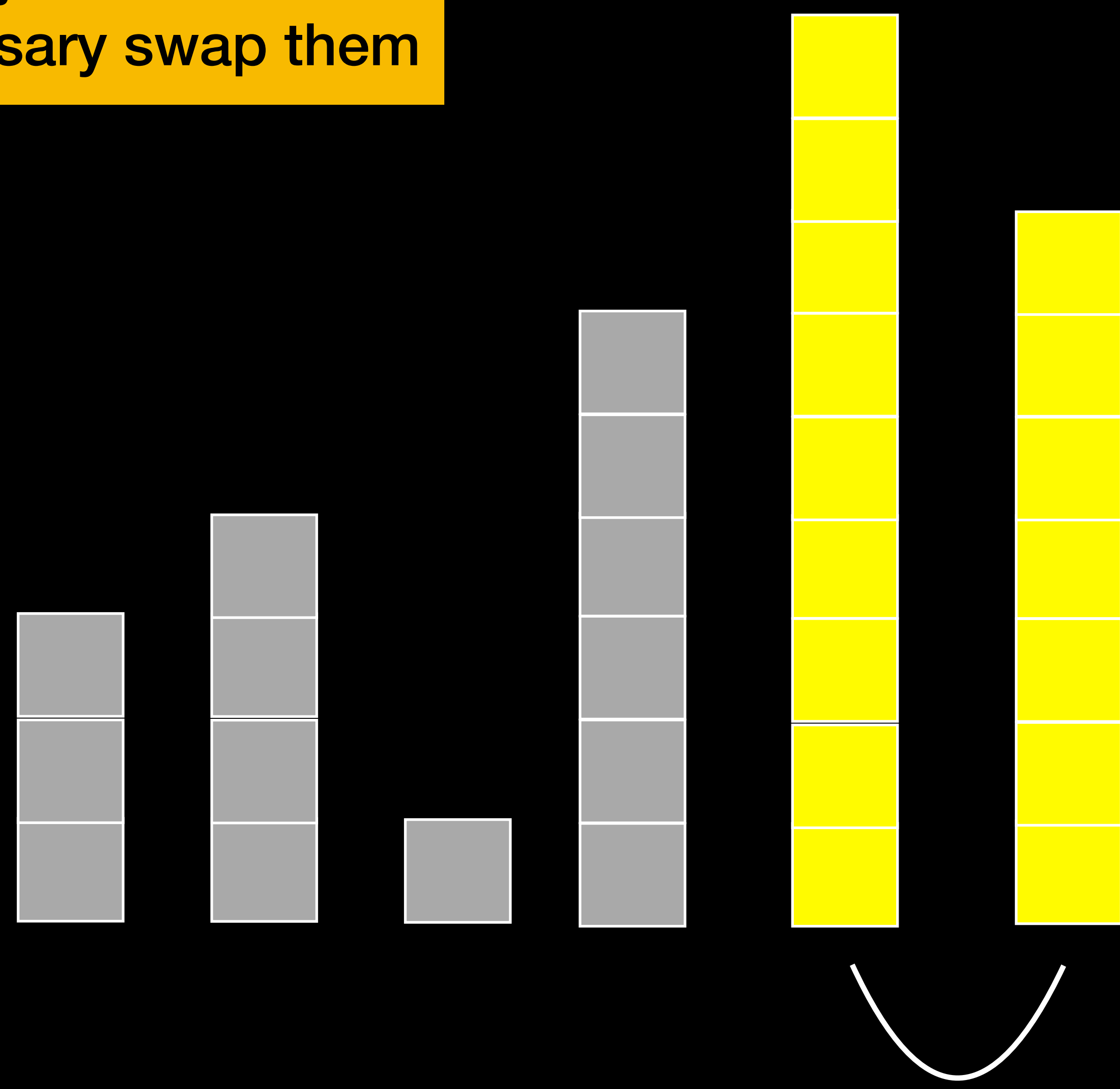

Bubble Sort

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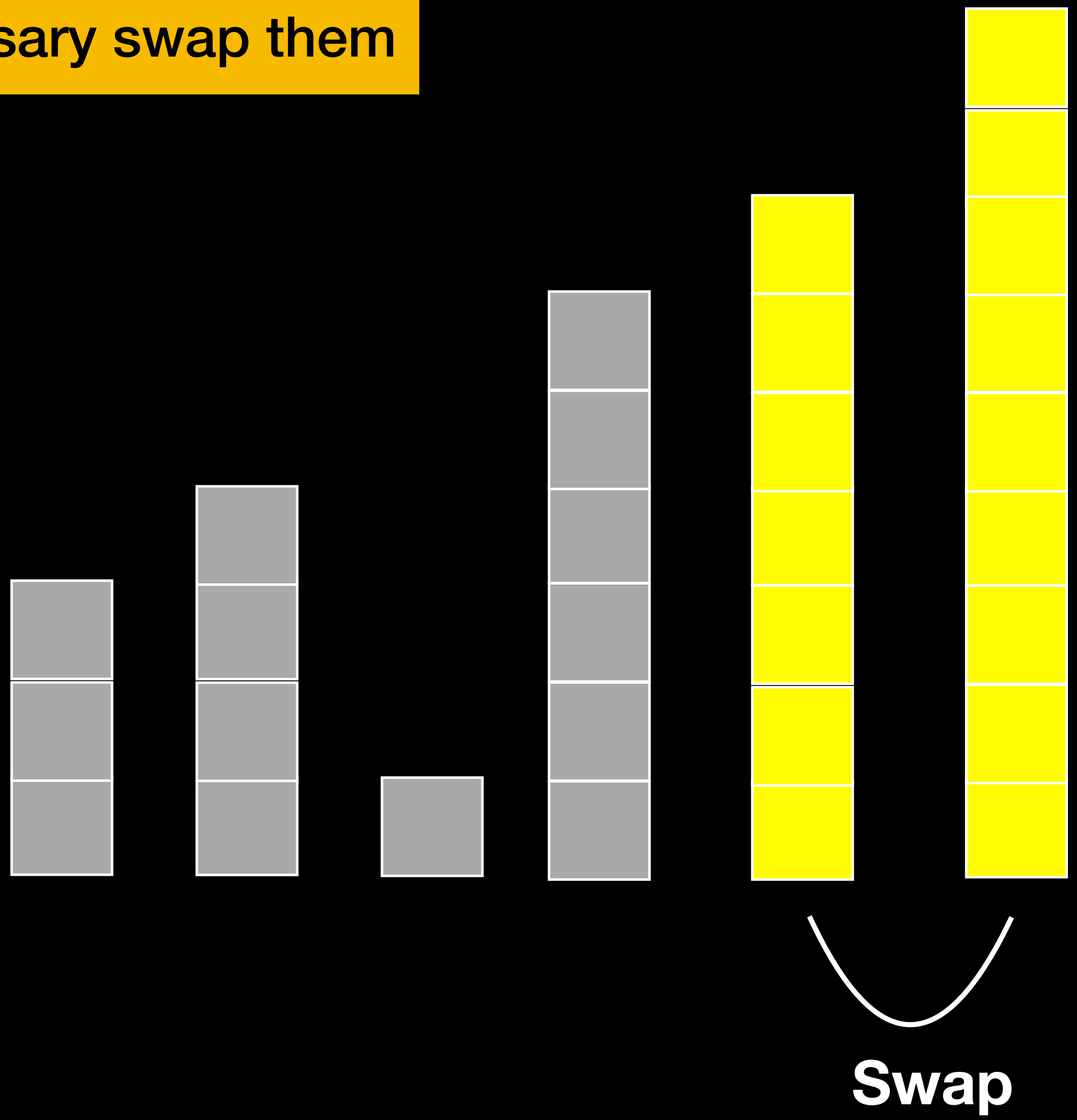
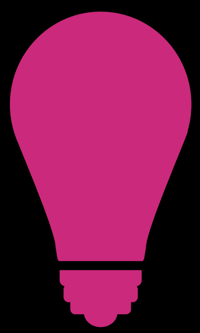
1st Pass

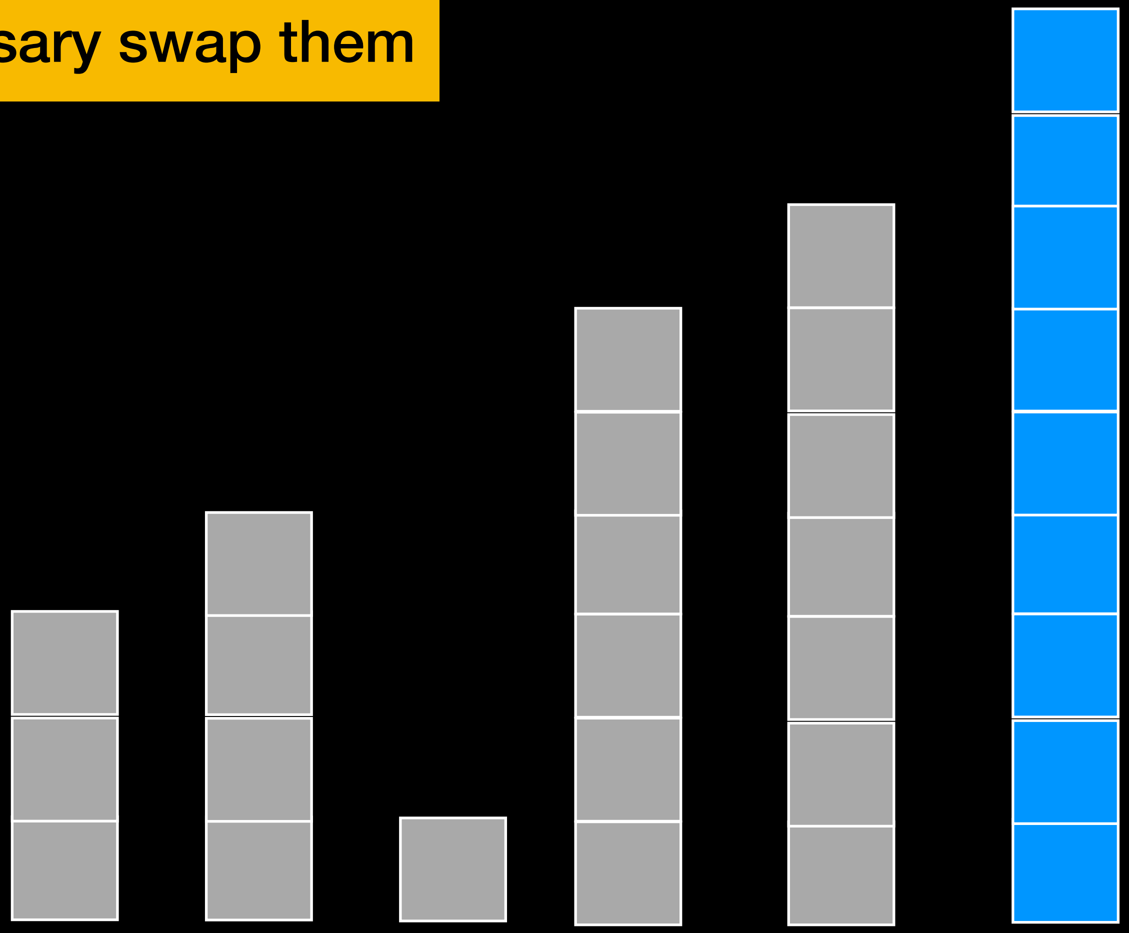


Bubble Sort



Compare adjacent elements and if necessary swap them

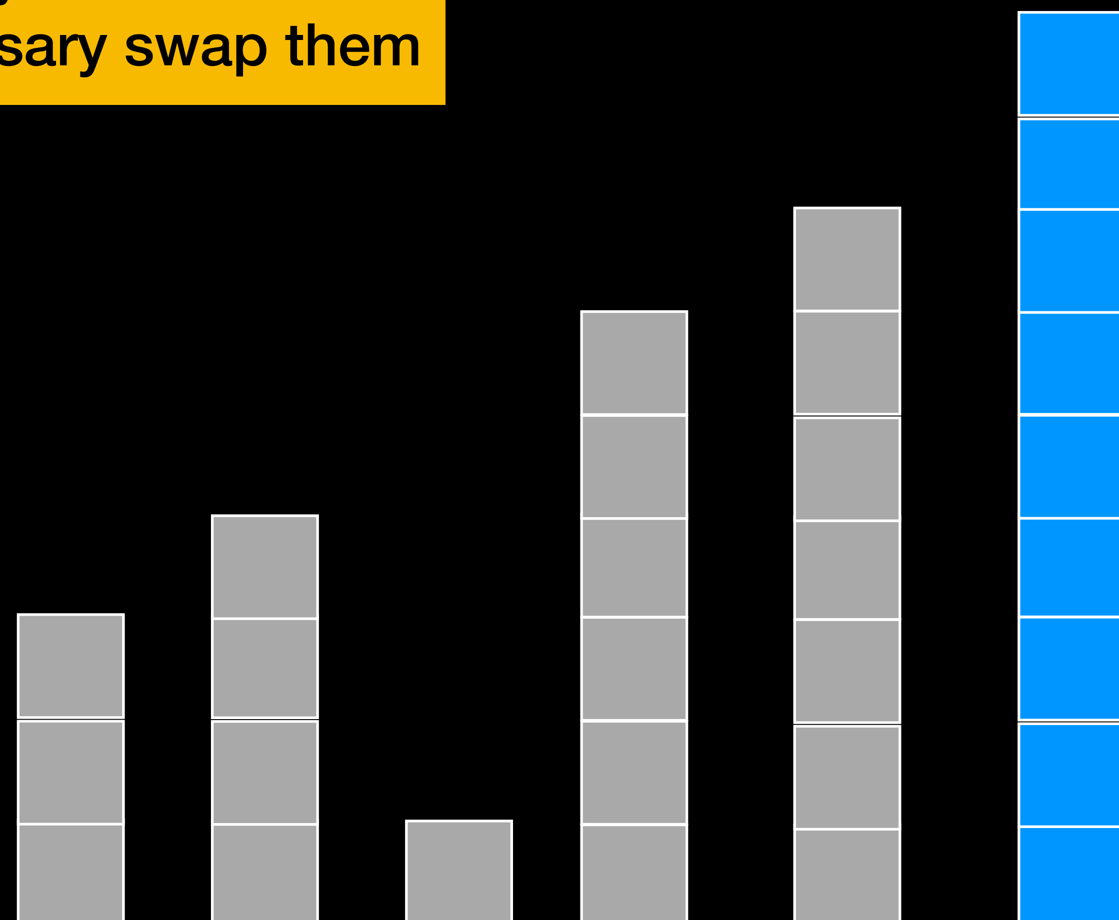
End of1st Pass:
Not sorted, but largest has ***"bubbled up"*** to its proper position



Bubble Sort

Compare adjacent elements and if necessary swap them

2nd Pass:
Sort **n-1**



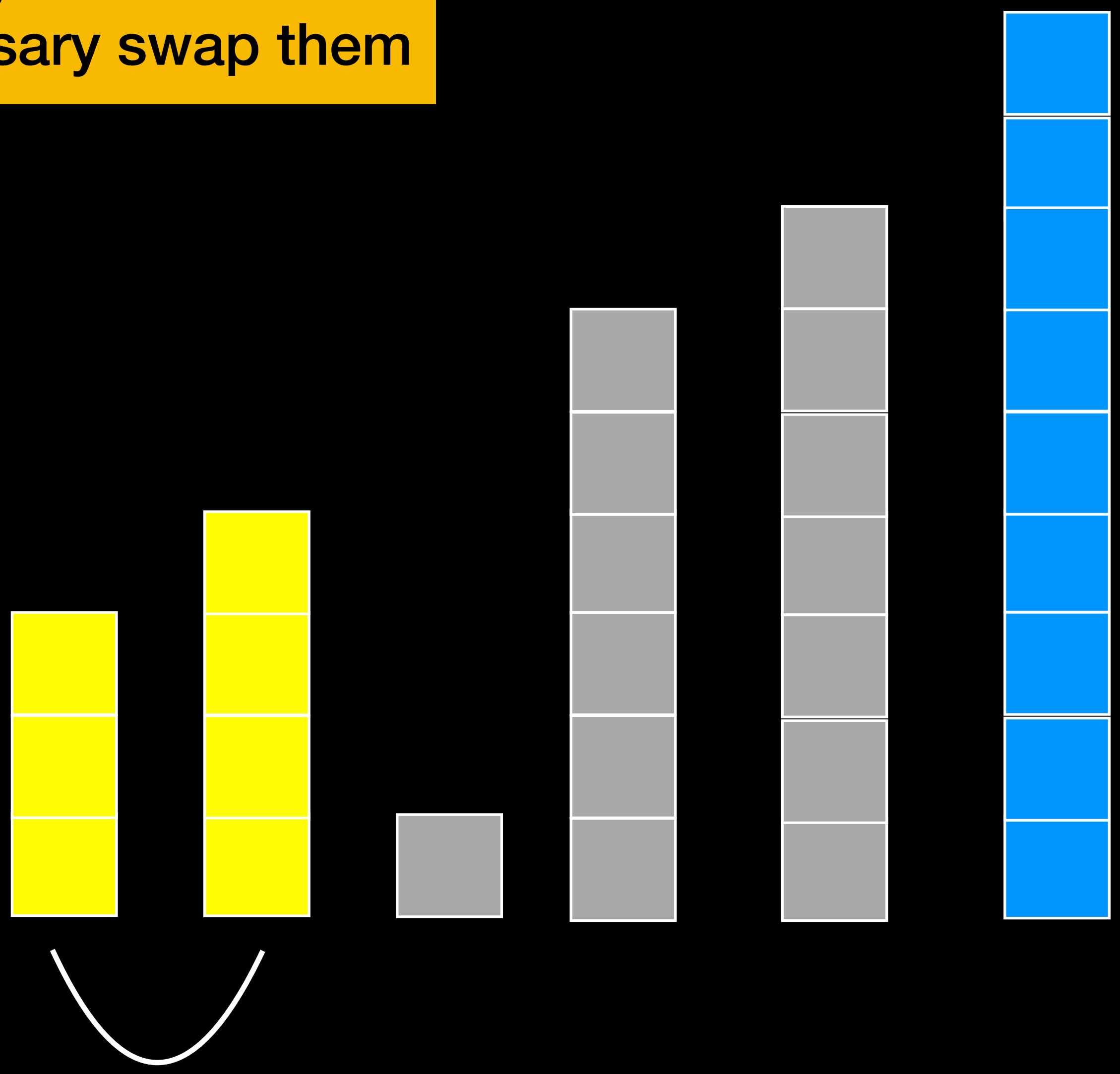
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

2nd Pass



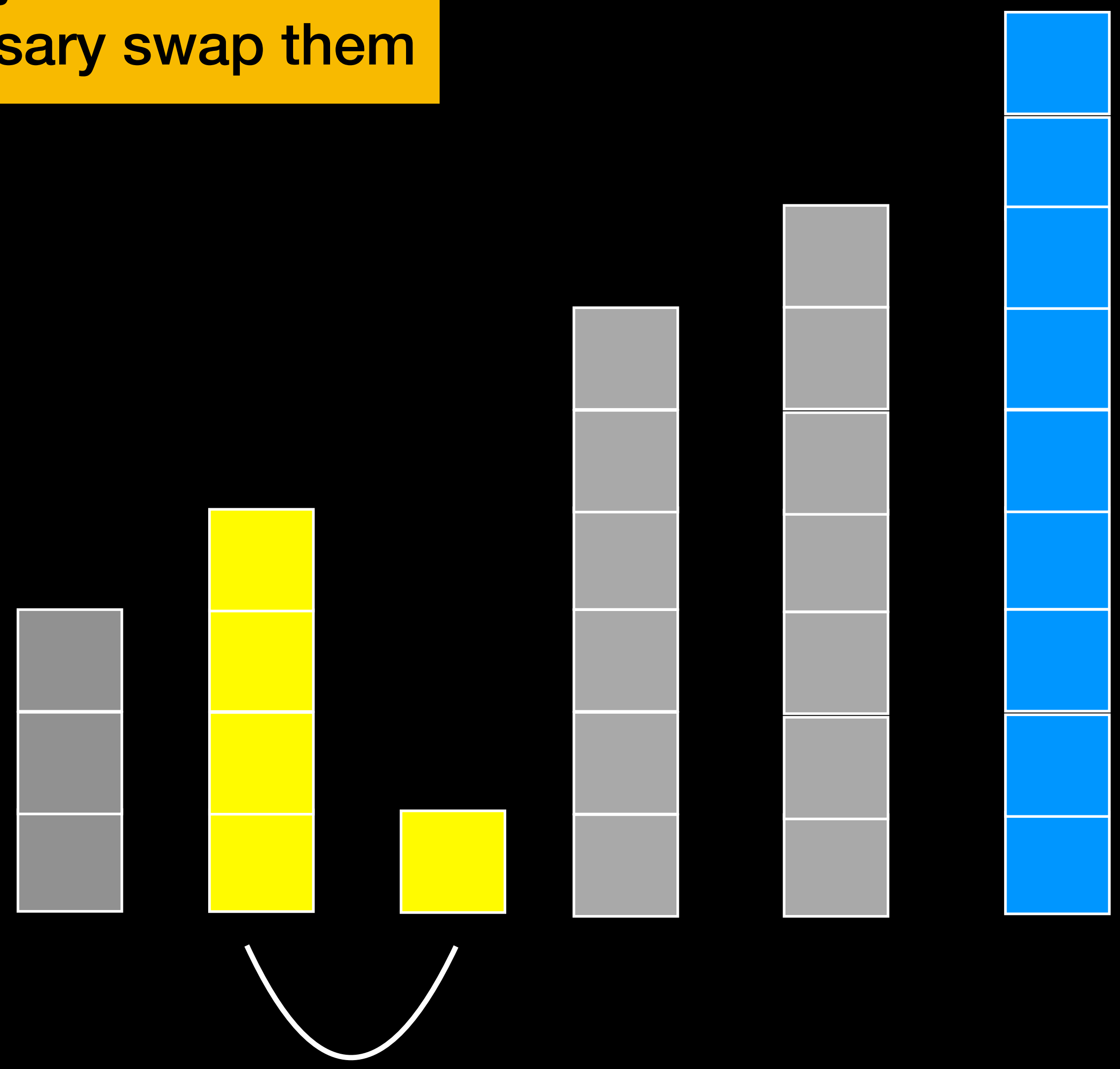
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

2nd Pass



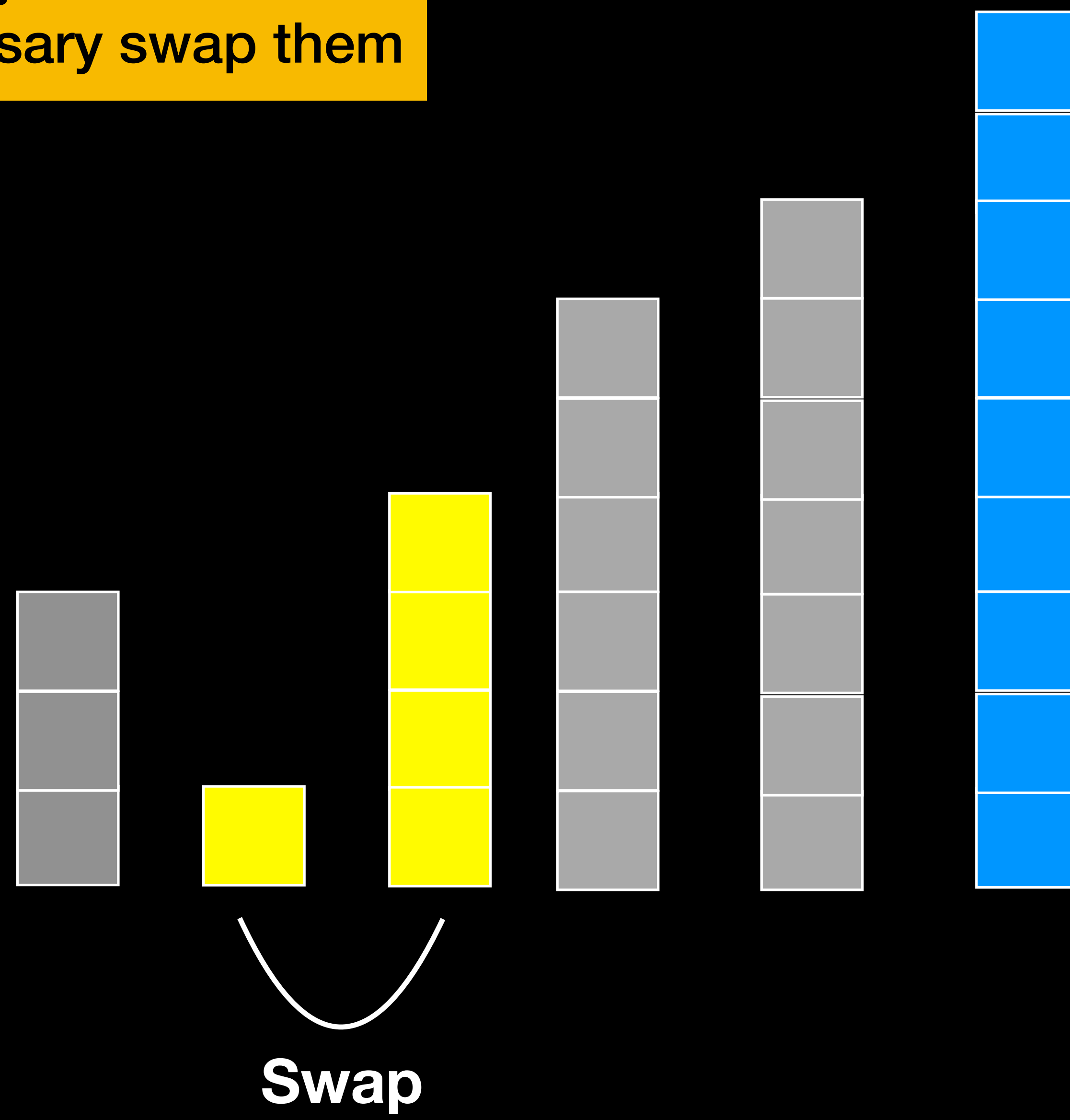
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

2nd Pass



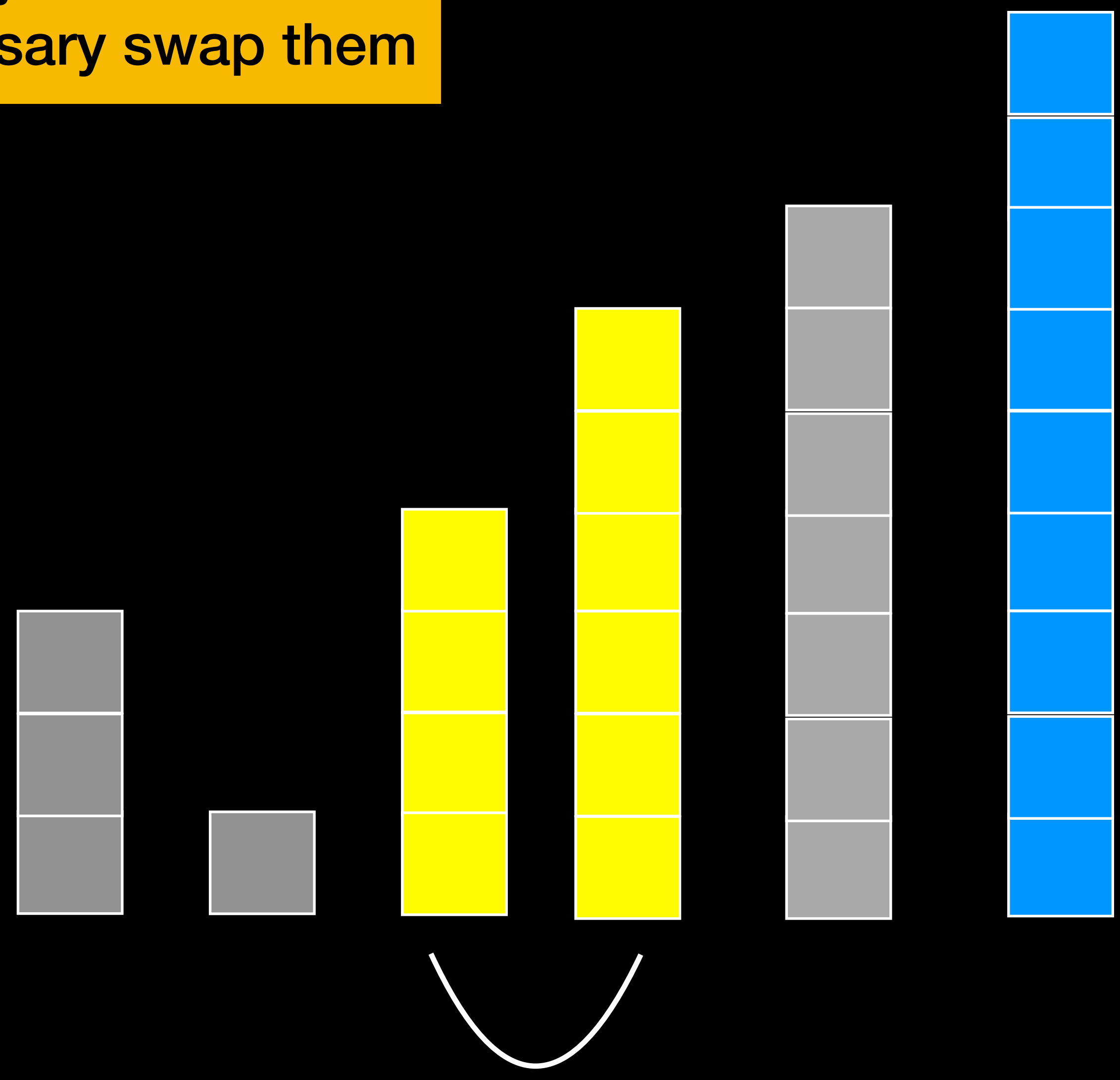
Bubble Sort

Unsorted

Sorted

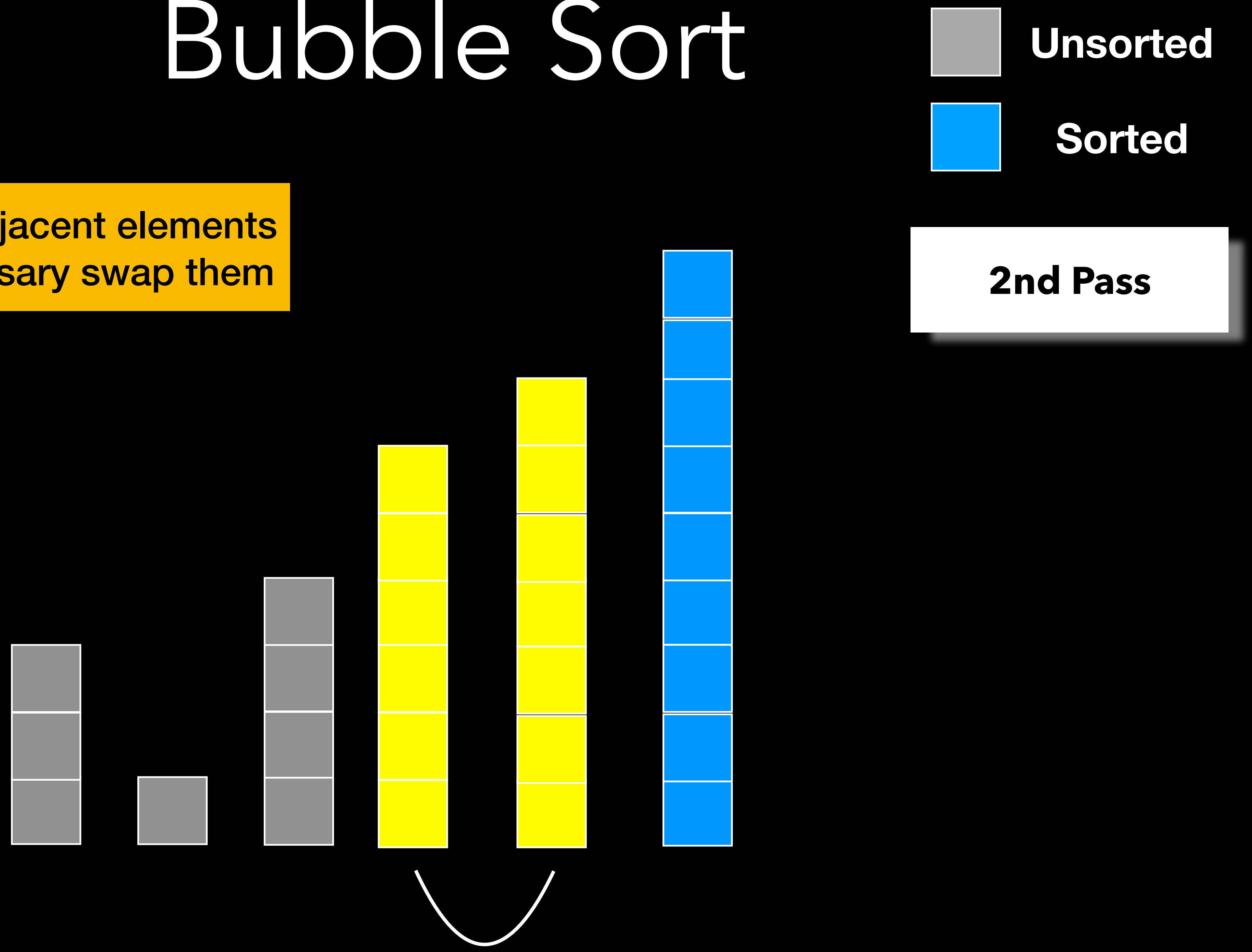
Compare adjacent elements and if necessary swap them

2nd Pass

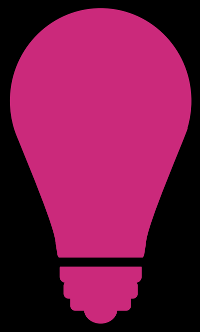


Bubble Sort

Compare adjacent elements and if necessary swap them

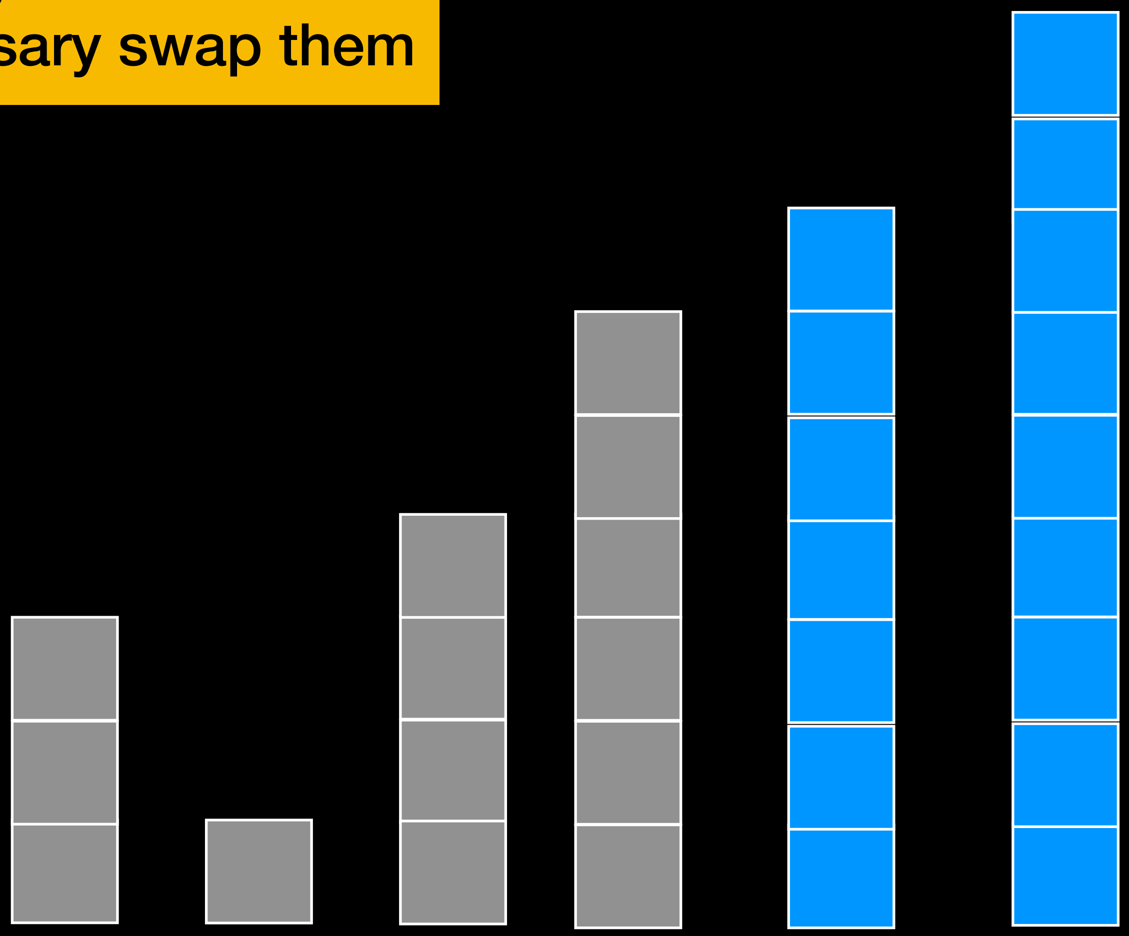


Bubble Sort



Compare adjacent elements and if necessary swap them

3rd Pass:
Sort **n-2**



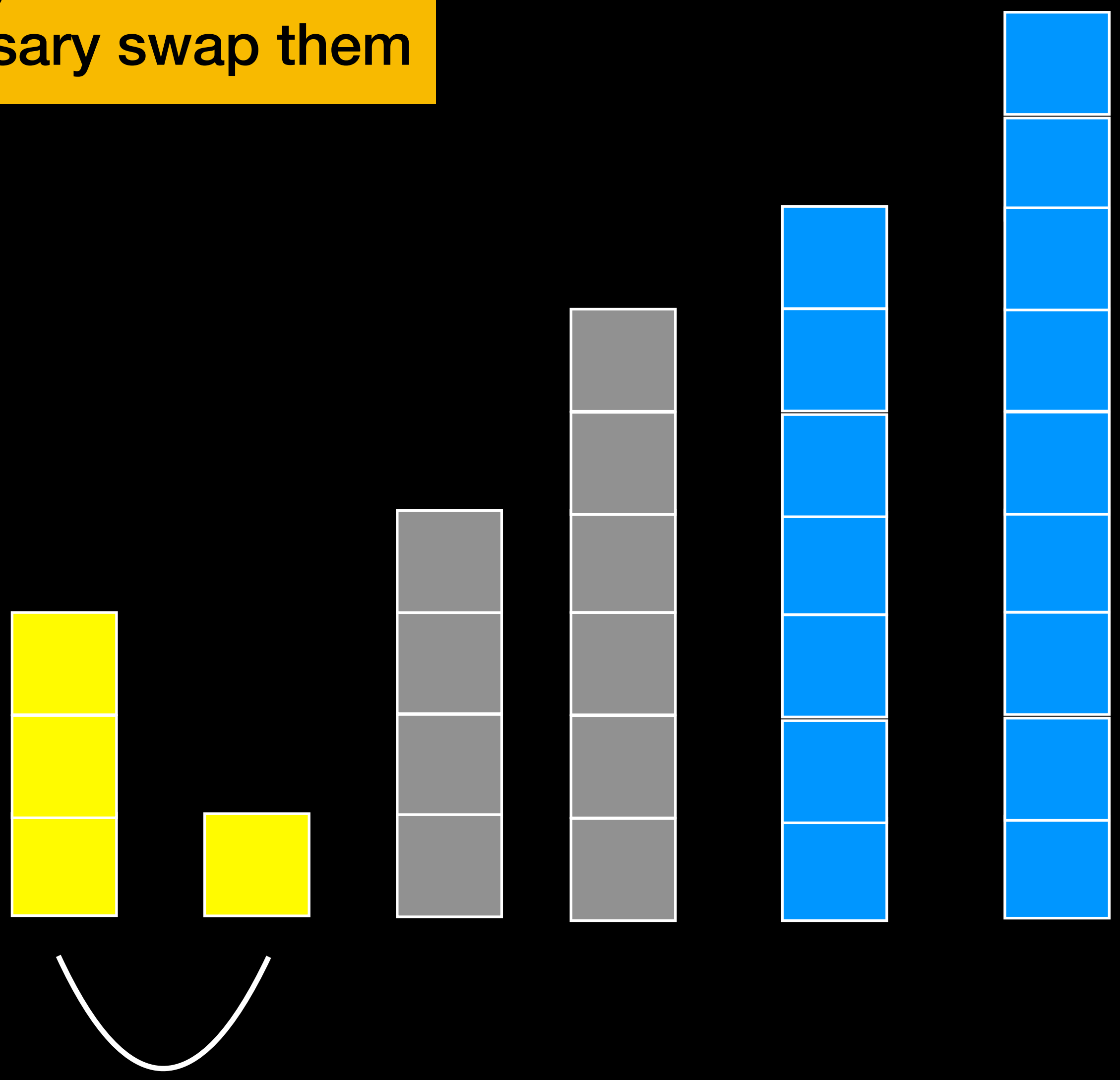
Bubble Sort

Unsorted

Sorted

3rd Pass

Compare adjacent elements and if necessary swap them



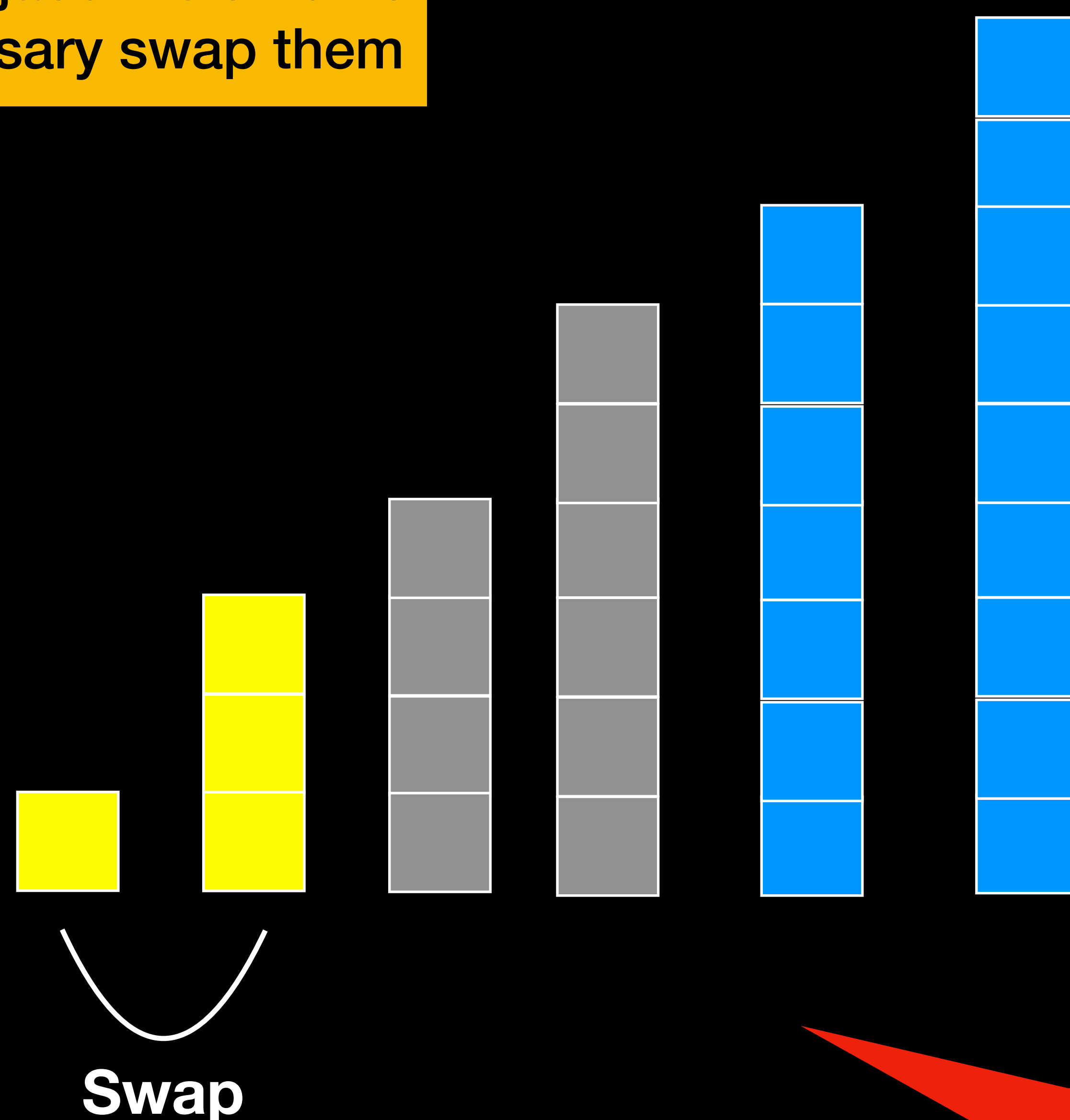
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

3rd Pass



Array is sorted
But our algorithm doesn't know
It keeps on going

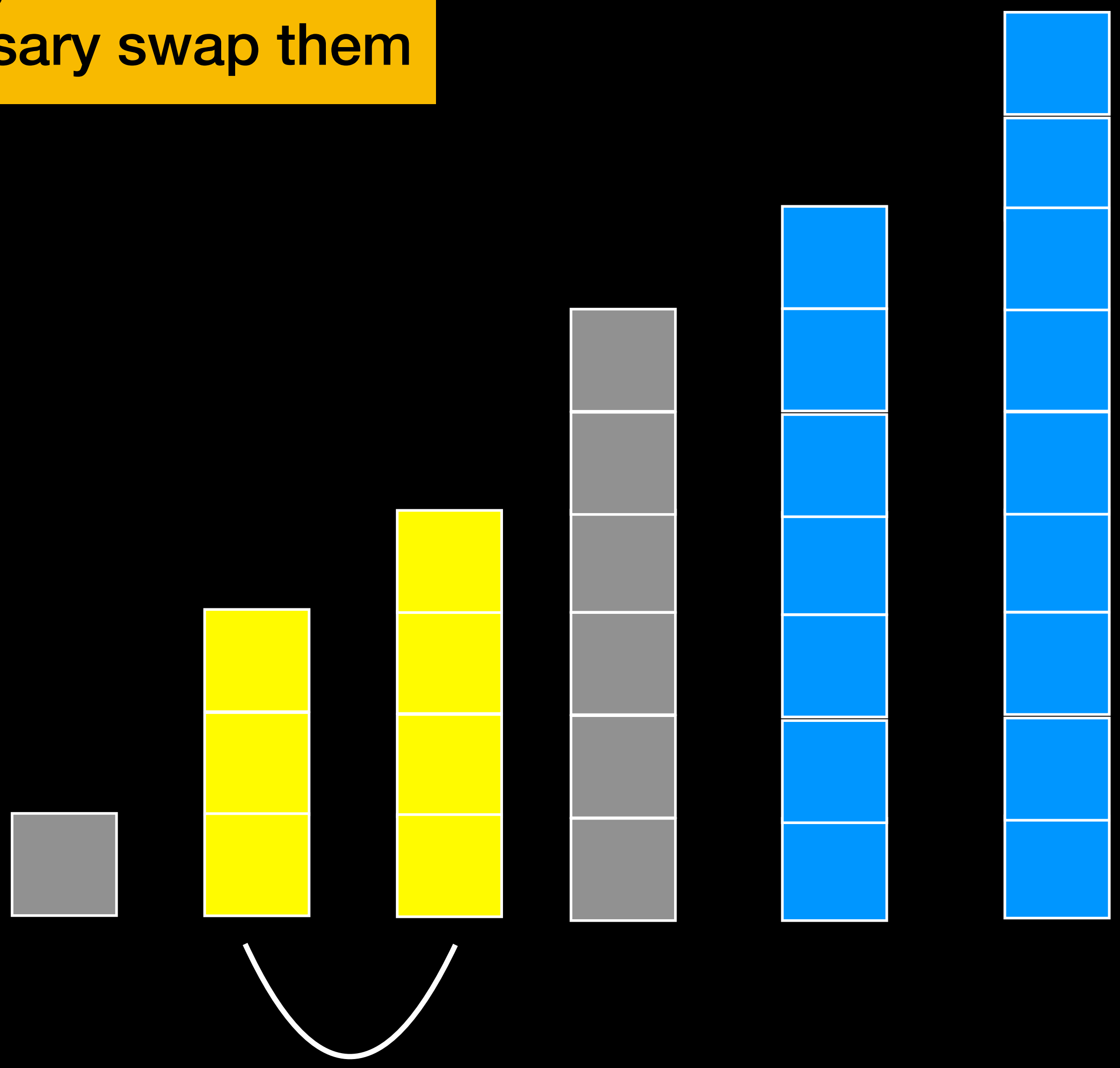
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

3rd Pass



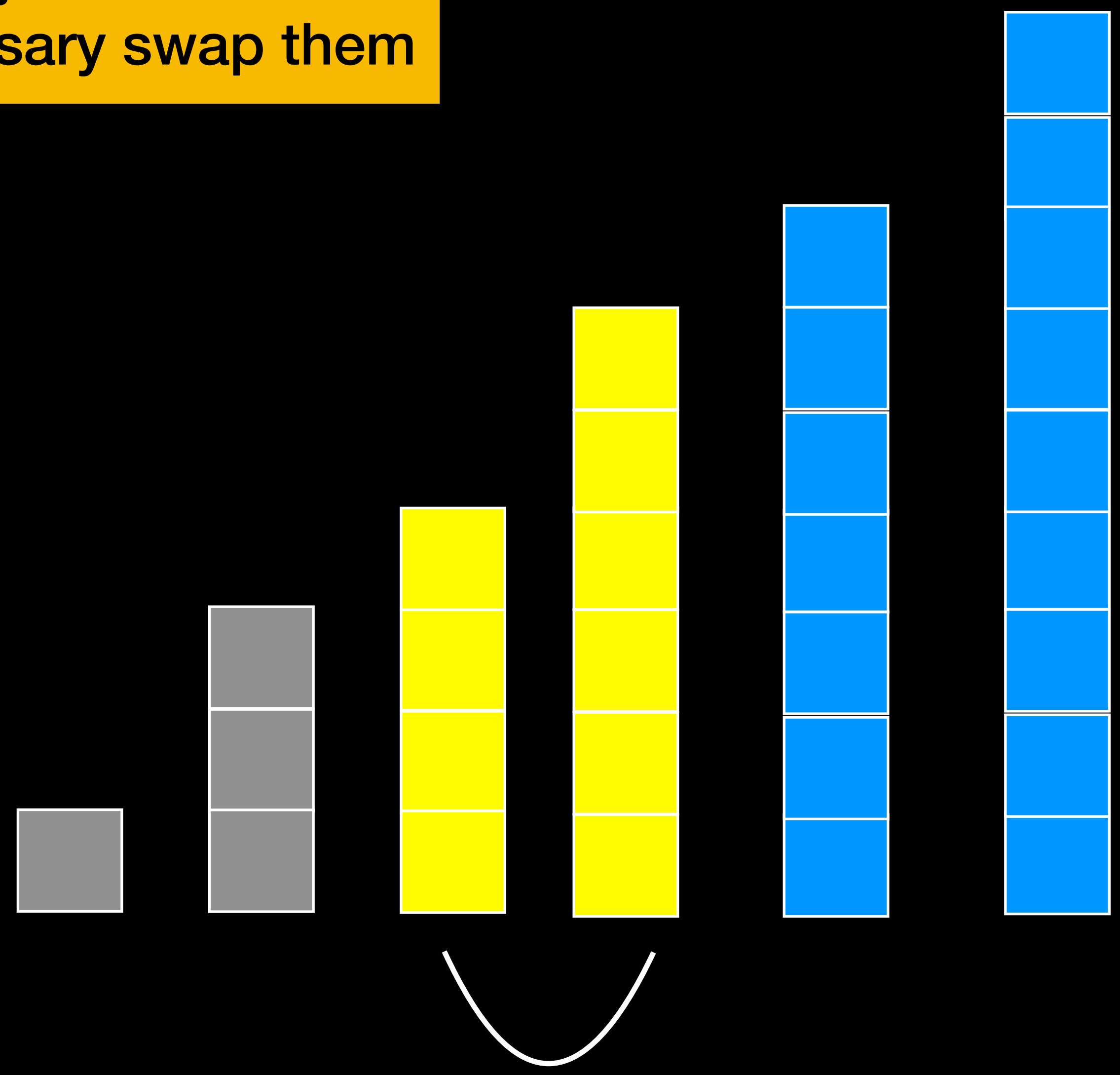
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

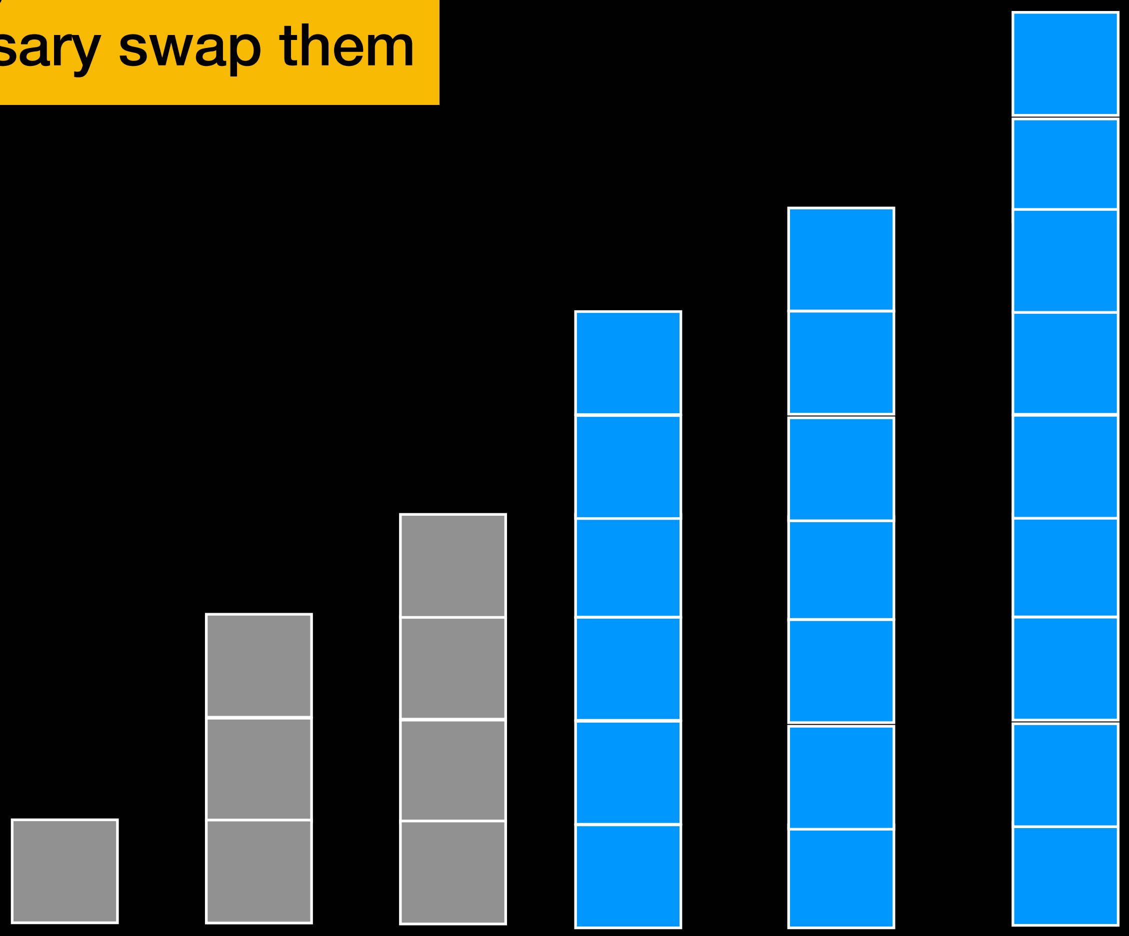
3rd Pass



Bubble Sort

Compare adjacent elements and if necessary swap them

4th Pass:
Sort **n-3**



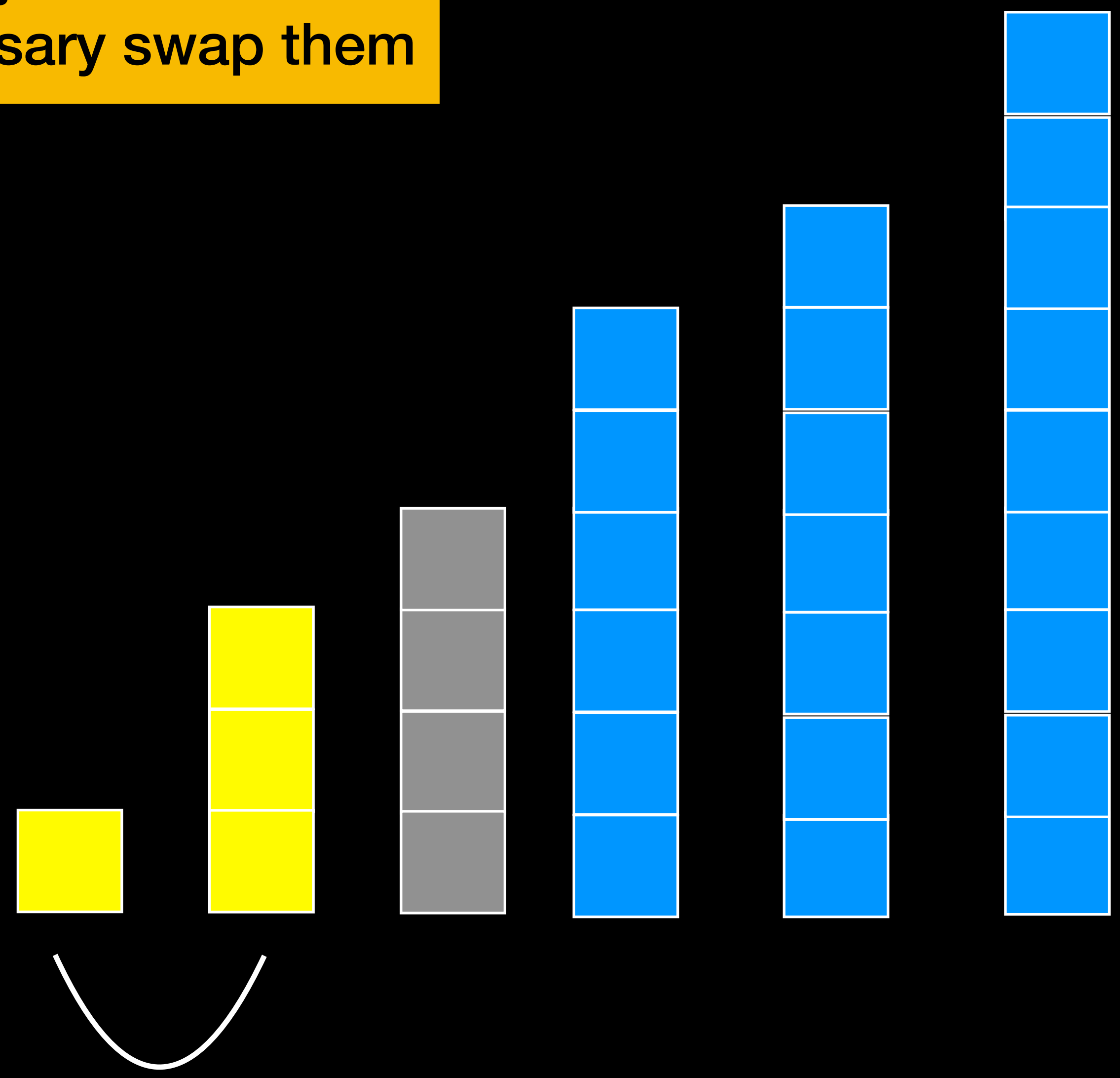
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

4th Pass



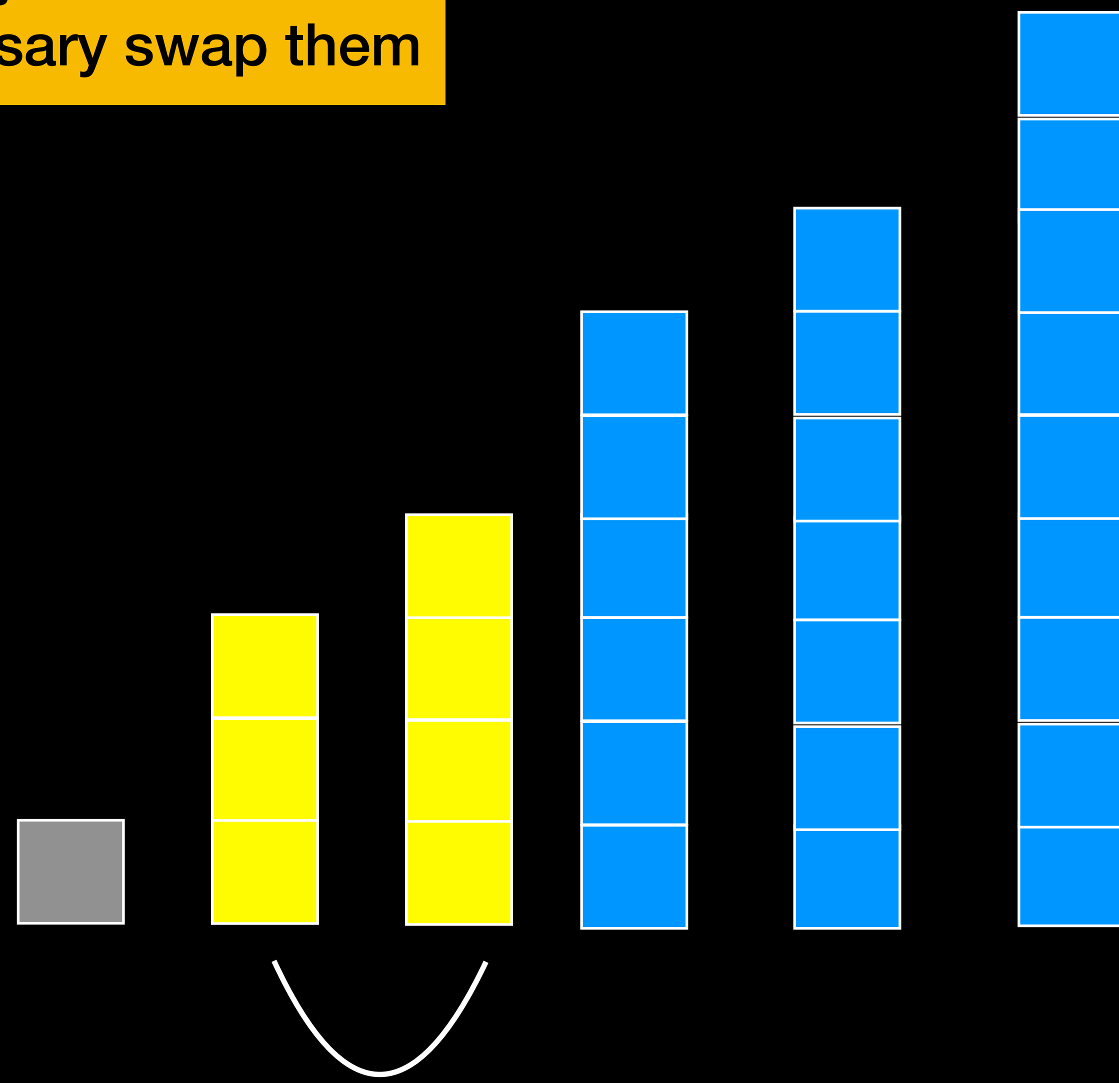
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

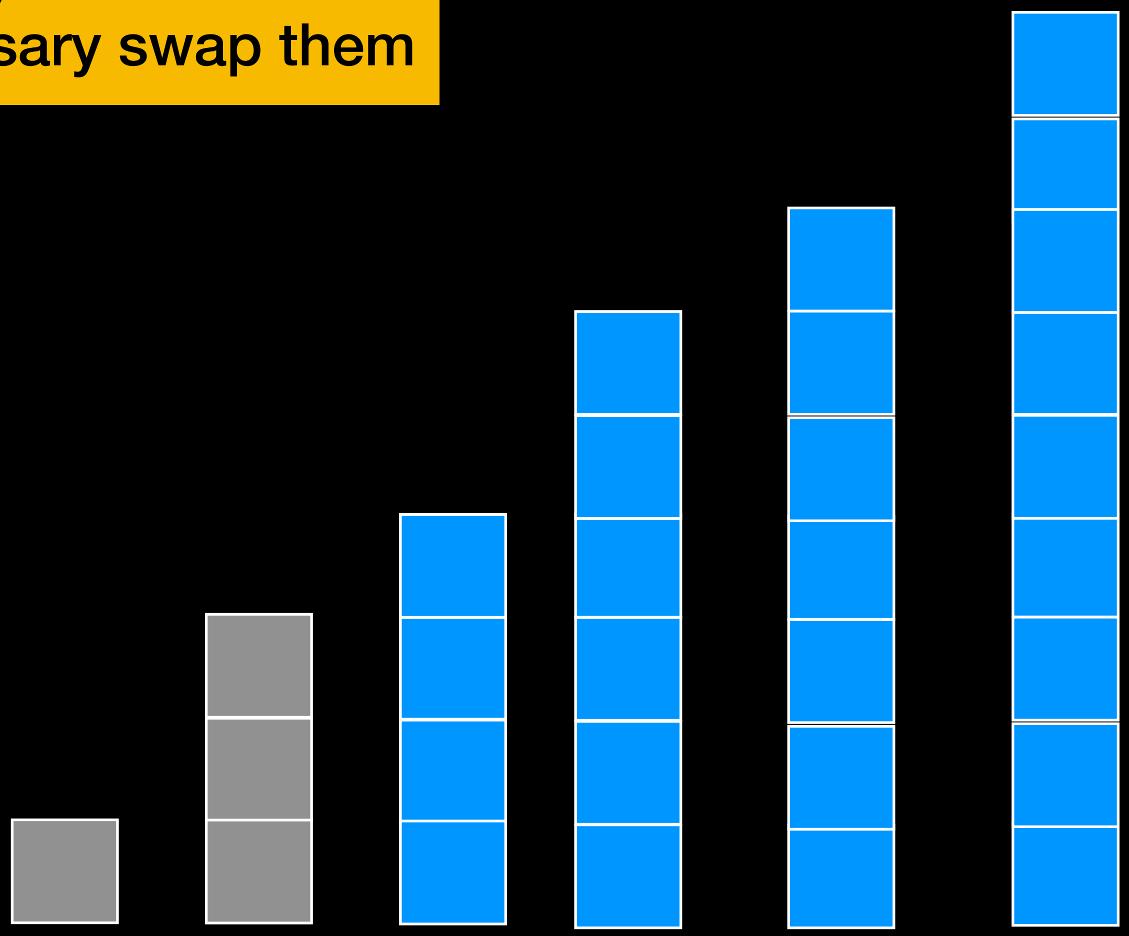
4th Pass



Bubble Sort

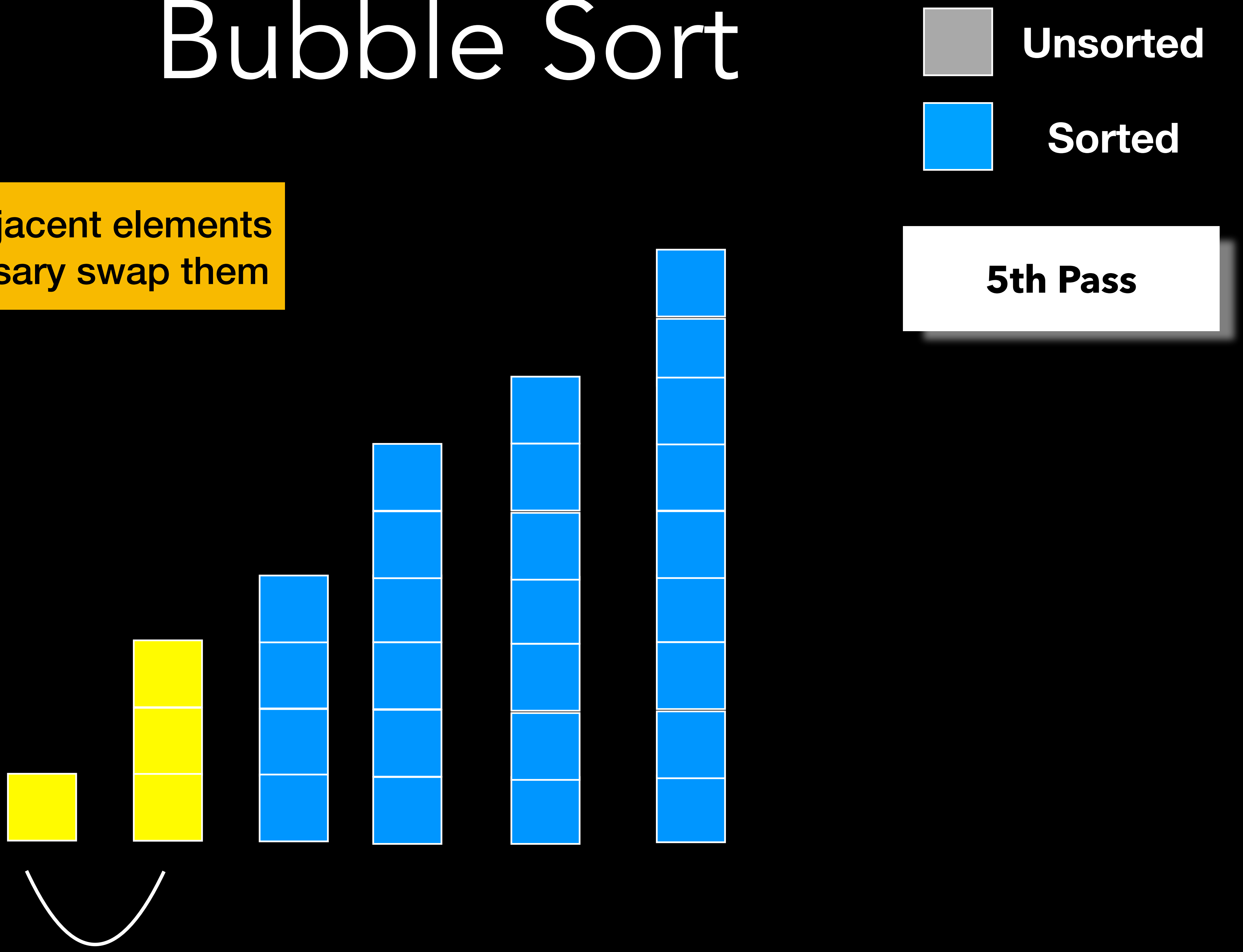
Compare adjacent elements and if necessary swap them

5th Pass:
Sort **n-4**



Bubble Sort

Compare adjacent elements and if necessary swap them



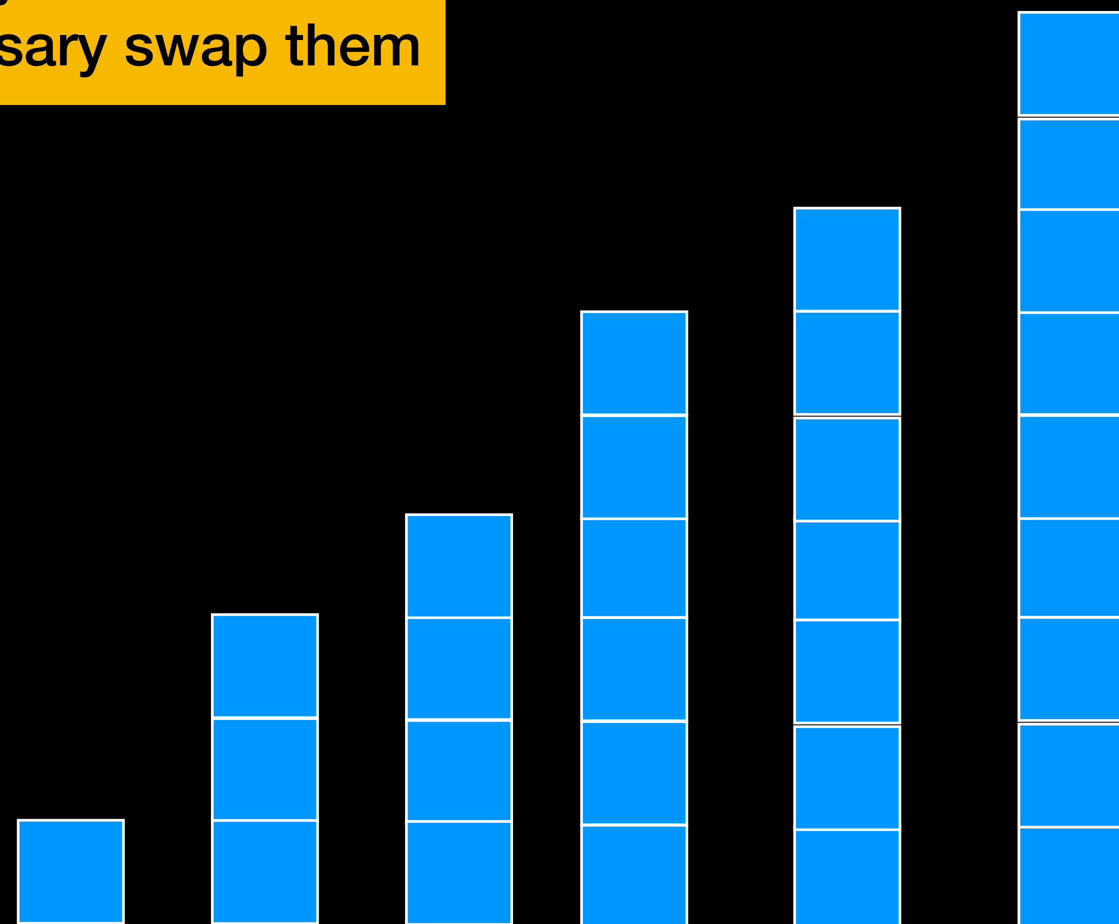
Bubble Sort

Unsorted

Sorted

Compare adjacent elements and if necessary swap them

Done!



Bubble Sort Analysis

How much work?

First pass: **n-1** comparisons and at most **n-1** swaps

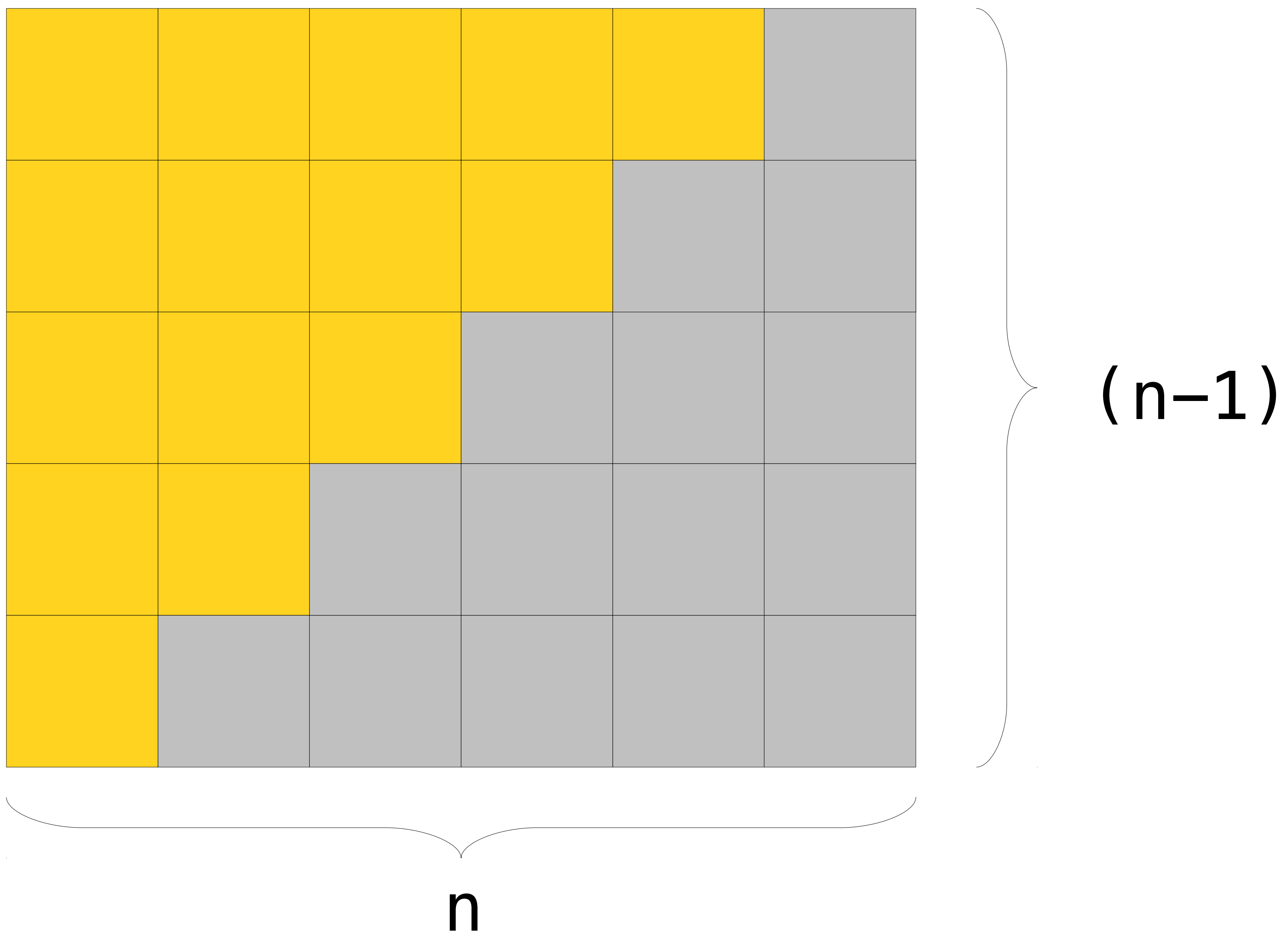
Second pass: **n-2** comparisons and at most **n-2** swaps

Third pass: **n-3** comparisons and at most **n-3** swaps

. . .

Total work: **(n-1) + (n-2) + . . . +1**

$(n-1) + (n-2) + . . . + 2 + 1 = n(n-1)/2$



Bubble Sort Analysis

$T(n) = n(n-1) / 2$ comparisons $+ \; n(n-1) / 2$ swaps $= O(\;)$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(\, n(n-1) / 2 \,) = O(\;)$?

Bubble Sort Analysis

$T(n) = n(n-1) / 2$ comparisons + $n(n-1) / 2$ swaps = $O(\ )$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(\ n(n-1) / 2\ ) = O(\ )$?

$T(n) = 2(\ (n^2-n) / 2\ ) = O(\ )$?

Bubble Sort Analysis

$T(n) = n(n-1) / 2$ comparisons $+ n(n-1) / 2$ swaps $= O(\ )$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(\ n(n-1) / 2\ ) = O(\ )$?

$T(n) = 2(\ (n^2-n) / 2\ ) = O(\ )$?

$T(n) = n^2-n = O(\ )$?

Ignore non-dominant terms

Bubble Sort Analysis

$T(n) = n(n-1) / 2$ comparisons + $n(n-1) / 2$ swaps = $O(\ )$?

A swap is usually more than one operation but this simplification does not change the analysis

$T(n) = 2(\ n(n-1) / 2\ ) = O(\ )$?

$T(n) = 2(\ (n^2-n) / 2\ ) = O(\ )$?

$T(n) = n^2-n = O(\ \mathbf{n^2})$

Bubble Sort run time is $\mathbf{O(n^2)}$

Optimize!

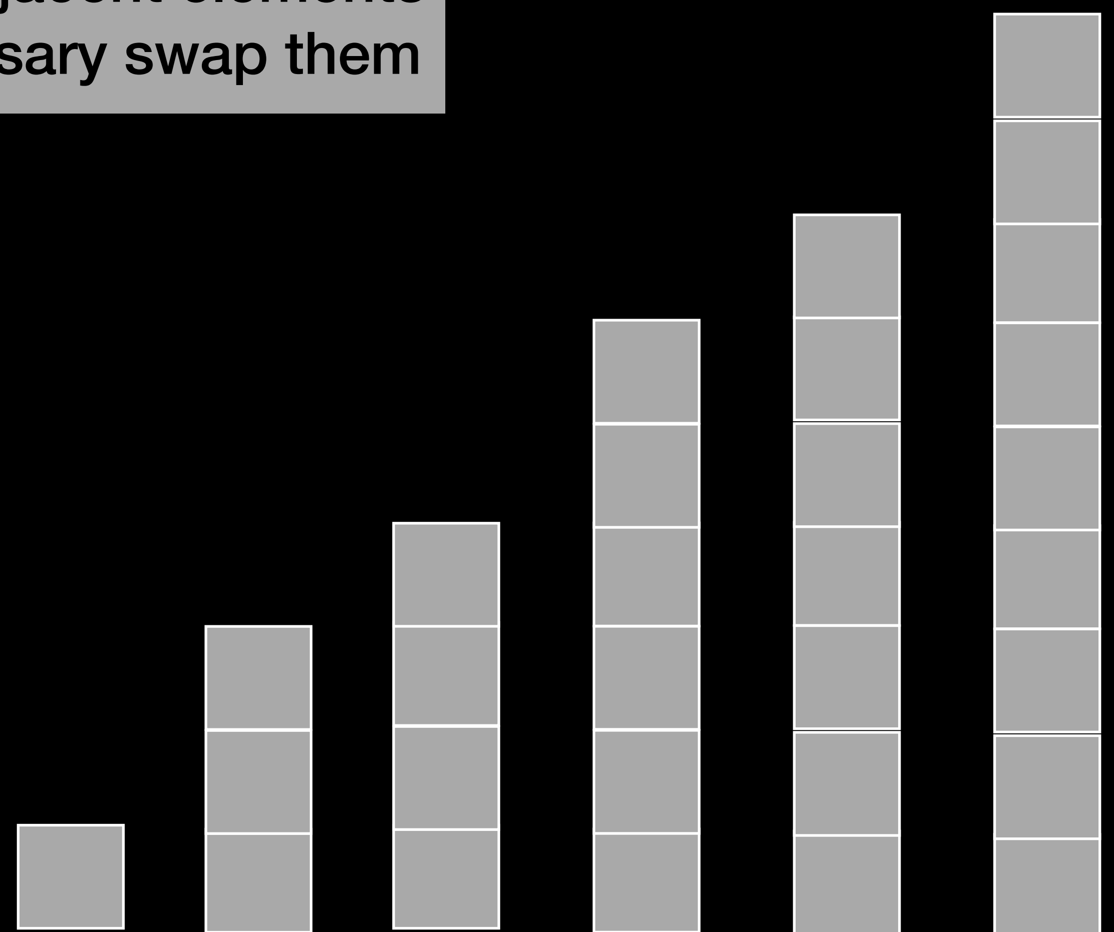
Easy to check:

if there are no swaps in any given phase

stop because it is sorted

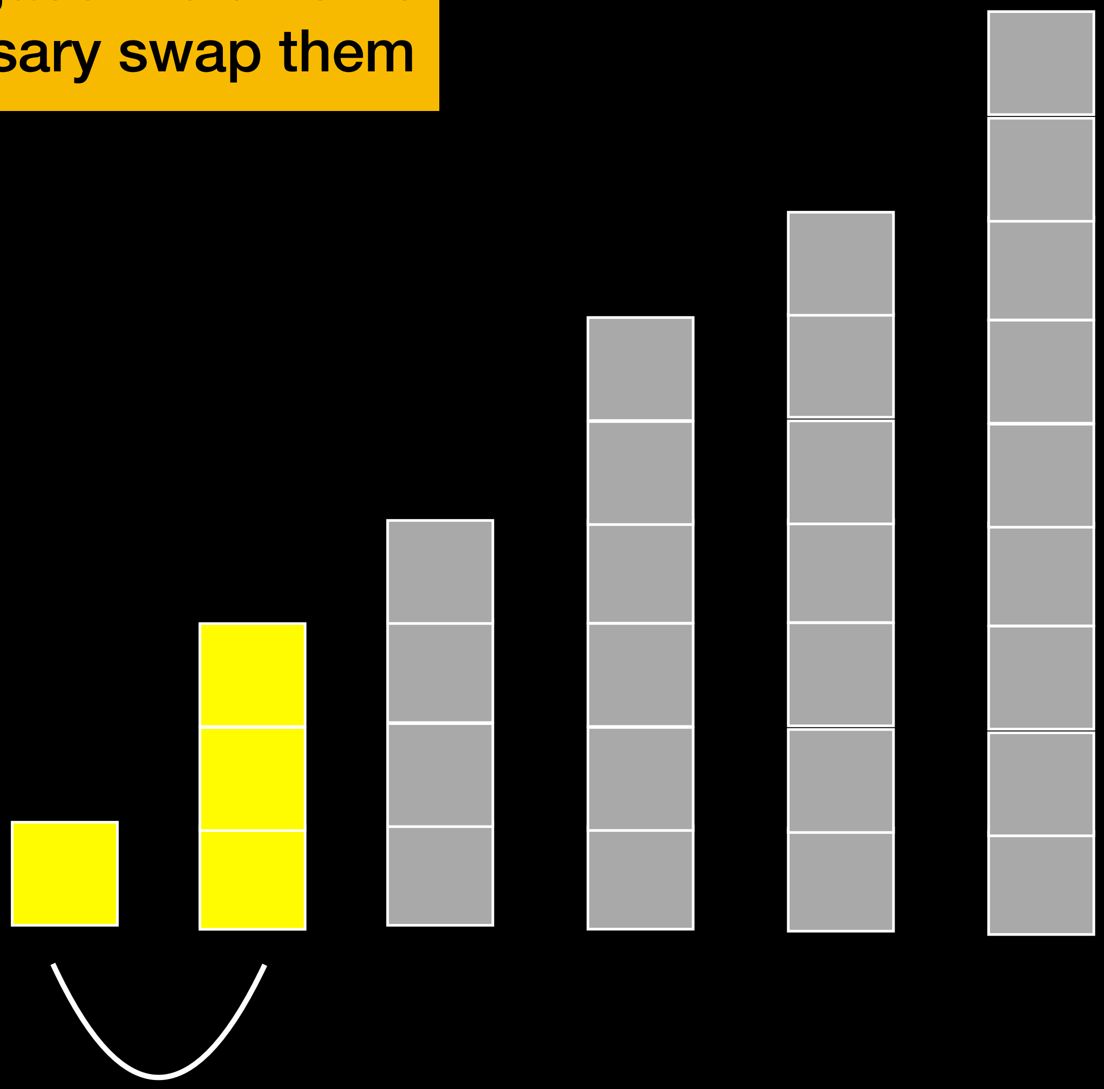
Bubble Sort

Compare adjacent elements and if necessary swap them

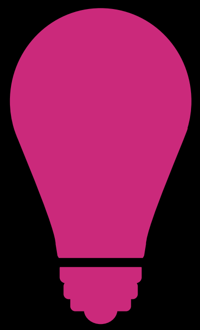


Bubble Sort

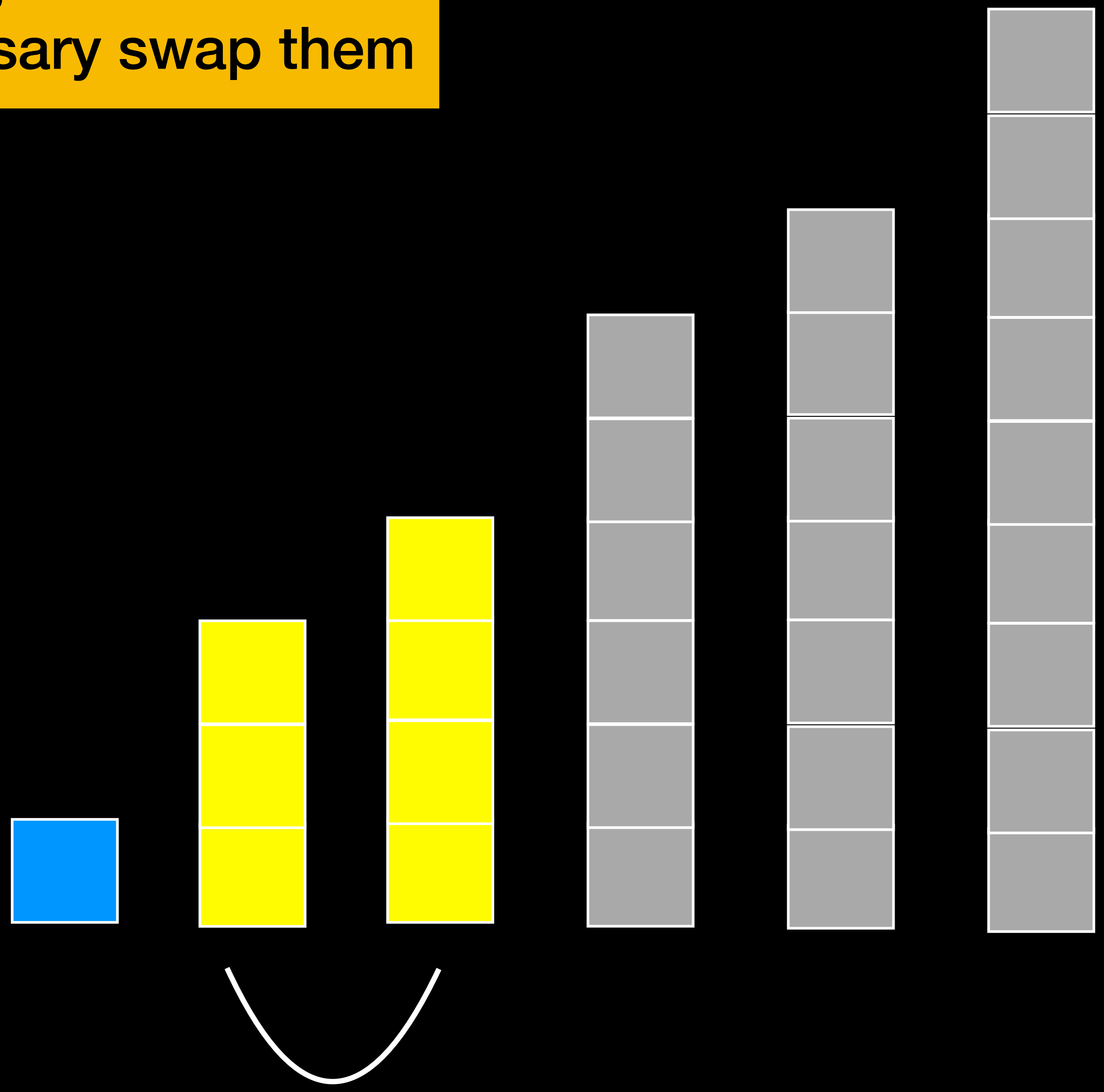
Compare adjacent elements and if necessary swap them



Bubble Sort

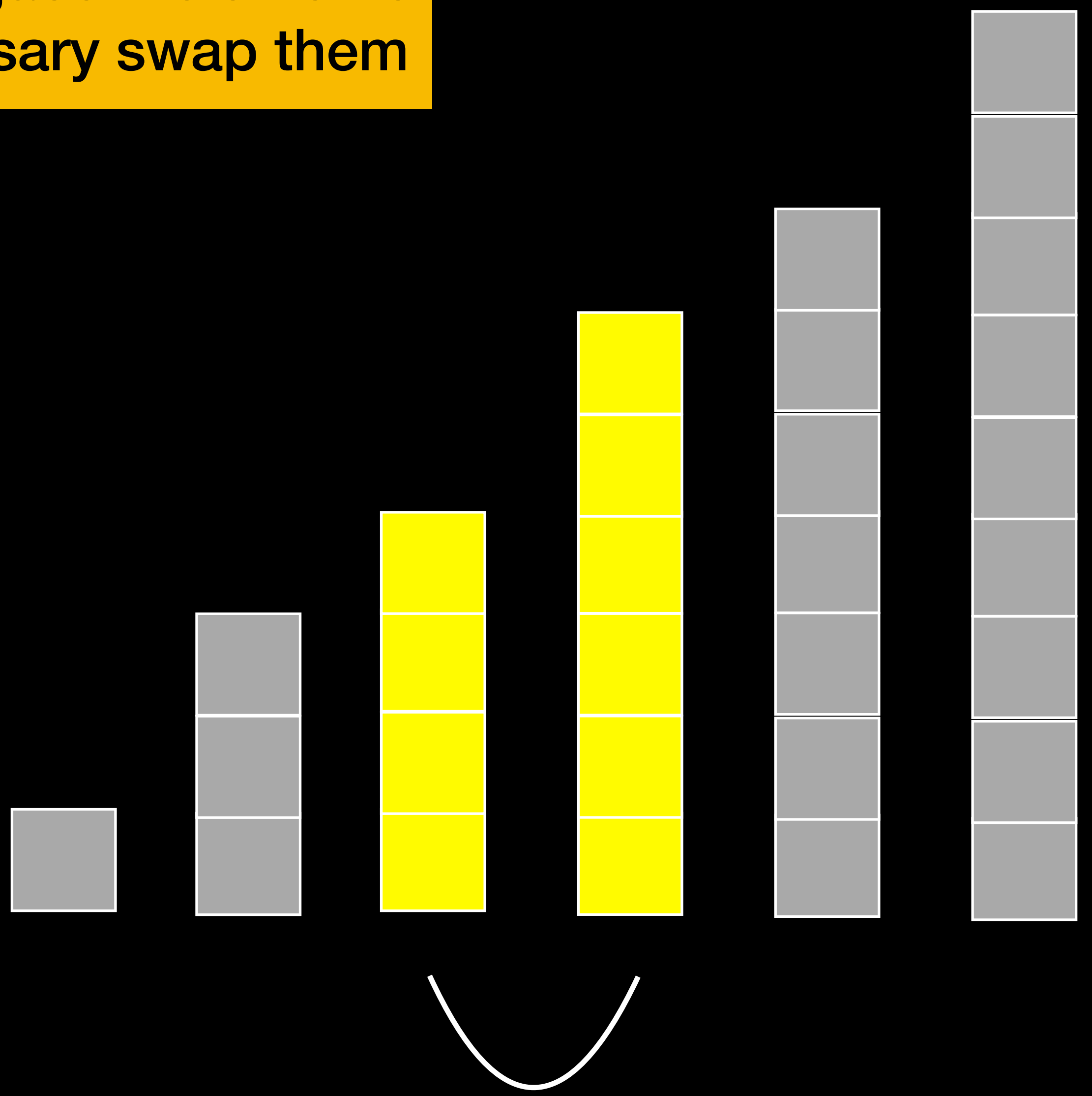


Compare adjacent elements and if necessary swap them



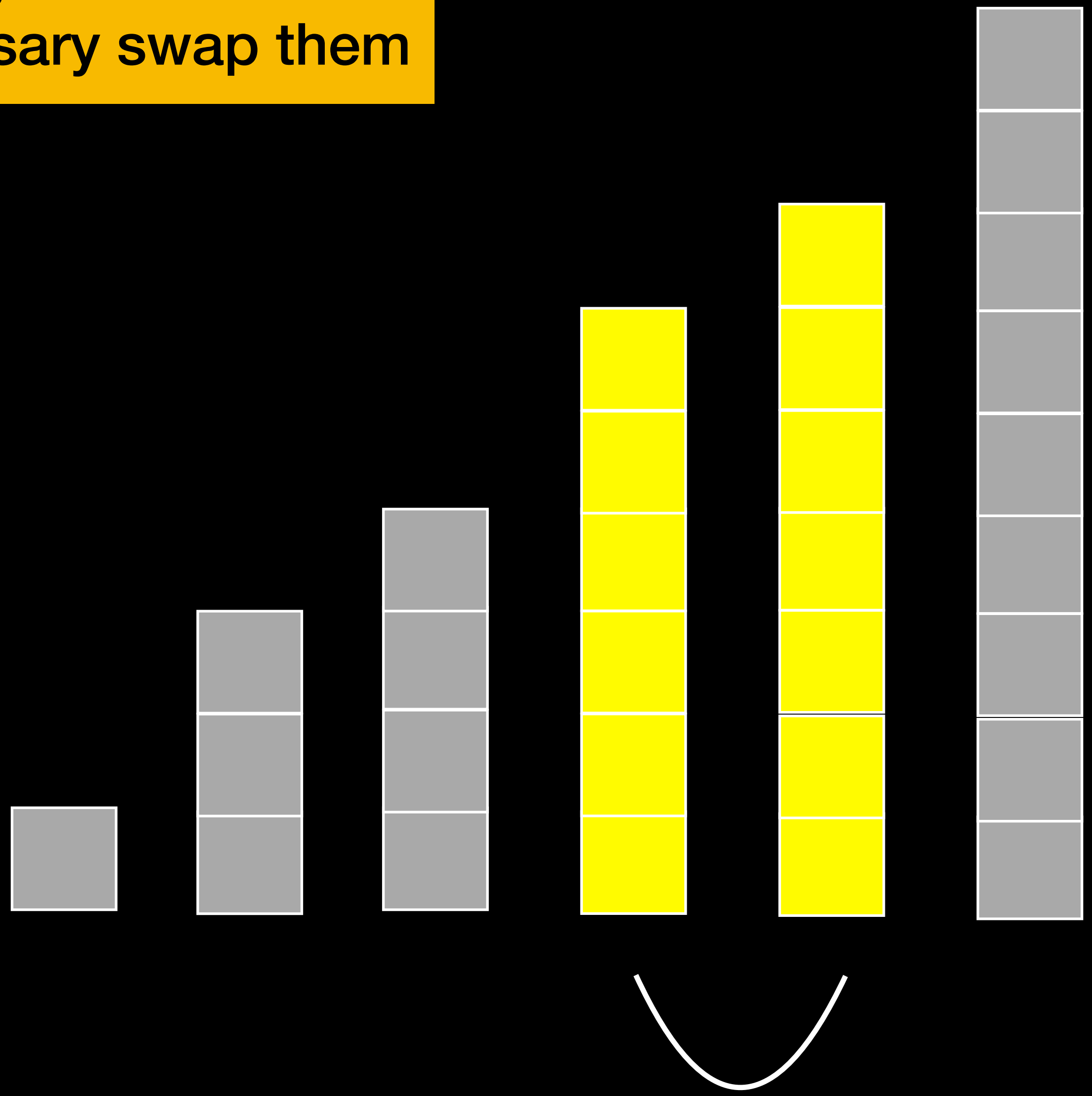
Bubble Sort

Compare adjacent elements and if necessary swap them



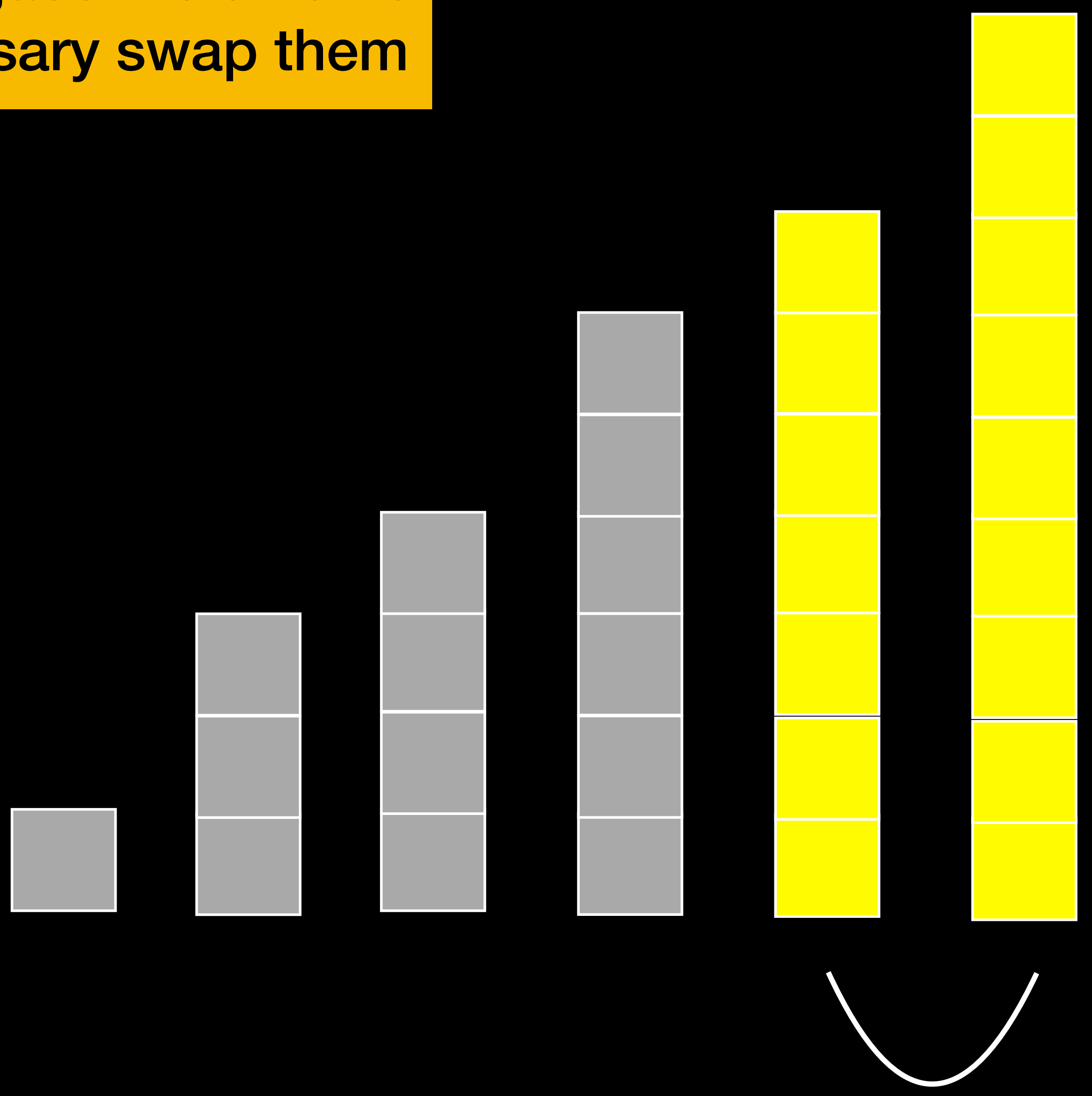
Bubble Sort

Compare adjacent elements and if necessary swap them



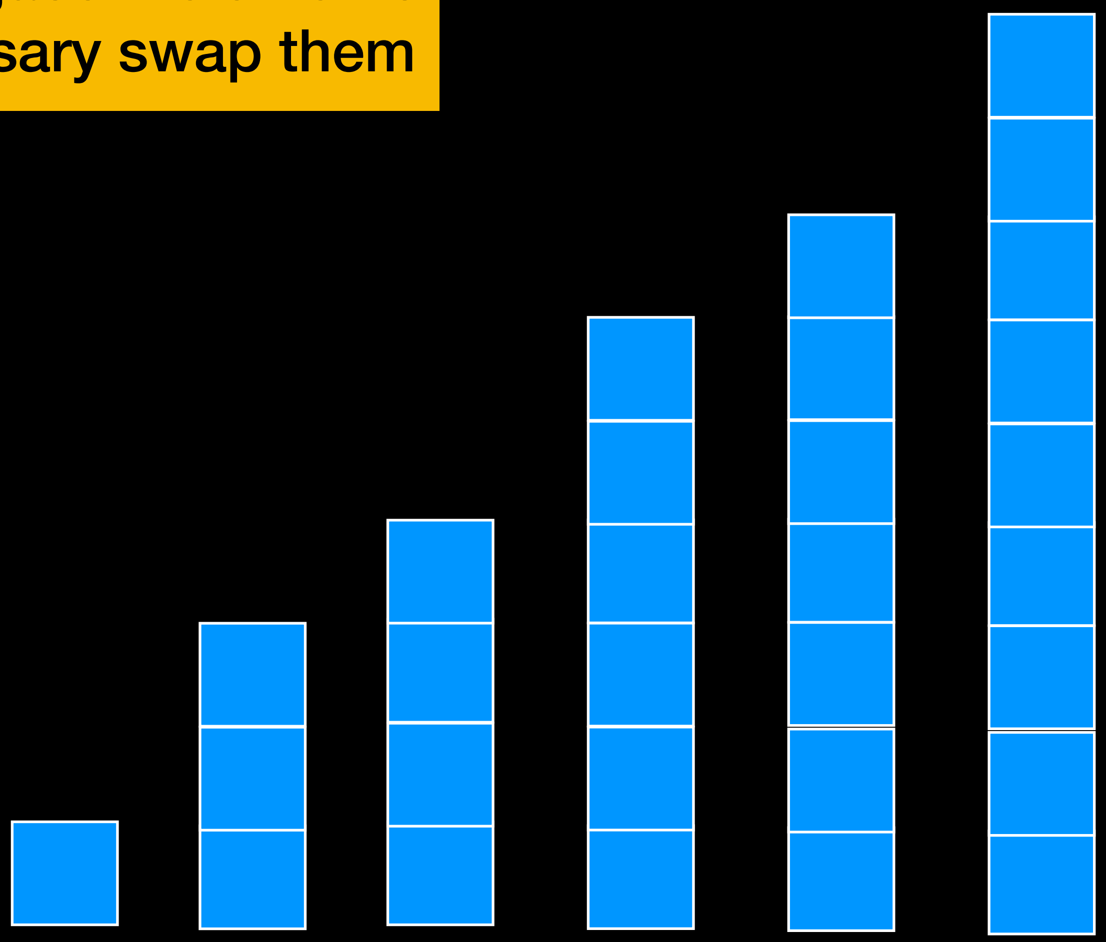
Bubble Sort

Compare adjacent elements and if necessary swap them



Bubble Sort

Compare adjacent elements and if necessary swap them



Bubble Sort Analysis

Execution time DOES depend on initial arrangement of data

Worst case: $O(n^2)$ comparisons and data moves

Best case: $O(n)$ comparisons and data moves

Stable

If array is already sorted bubble sort will stop after first pass and no swaps => good choice for small n and data likely somewhat sorted

Raise your hand if you had Bubble Sort

https://www.youtube.com/watch?v=lyZQPjUT5B4

Insertion Sort

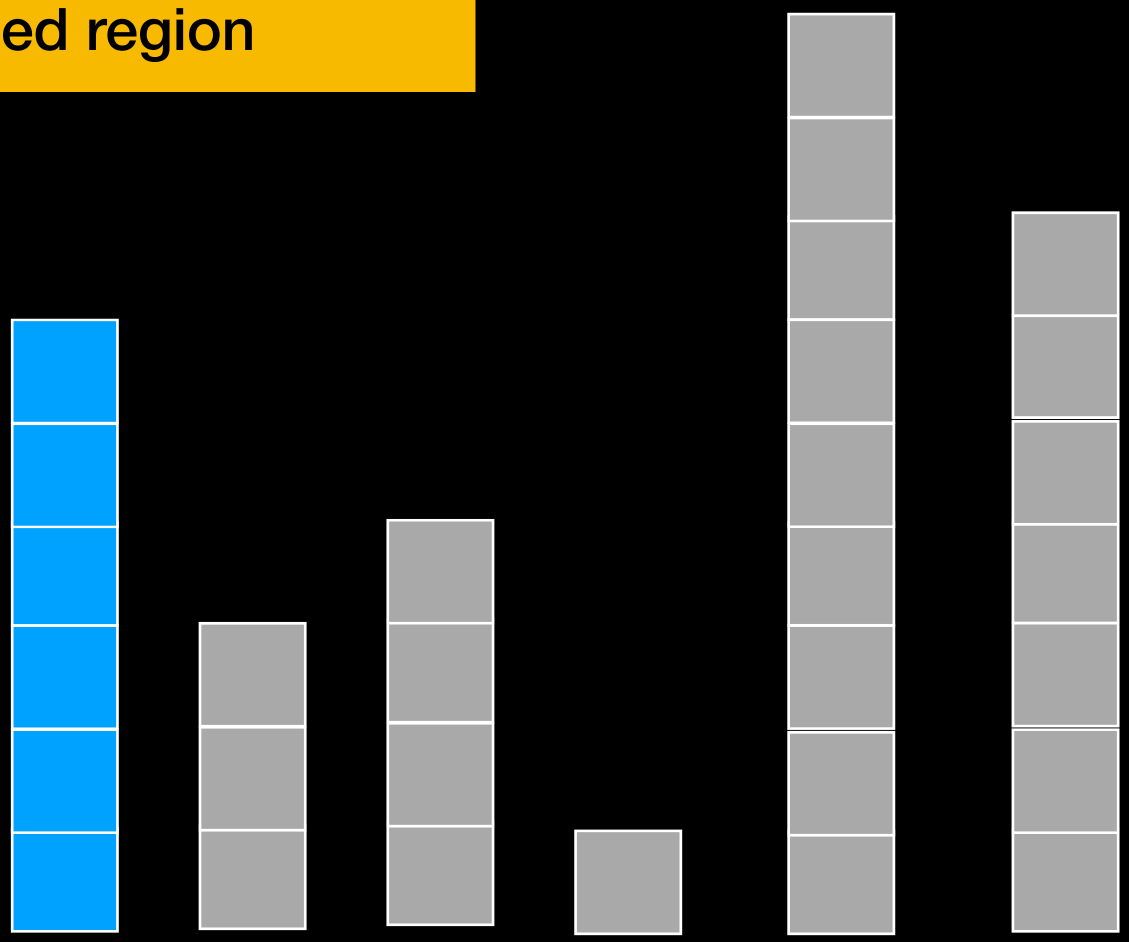
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

Unsorted

Sorted

1st Pass



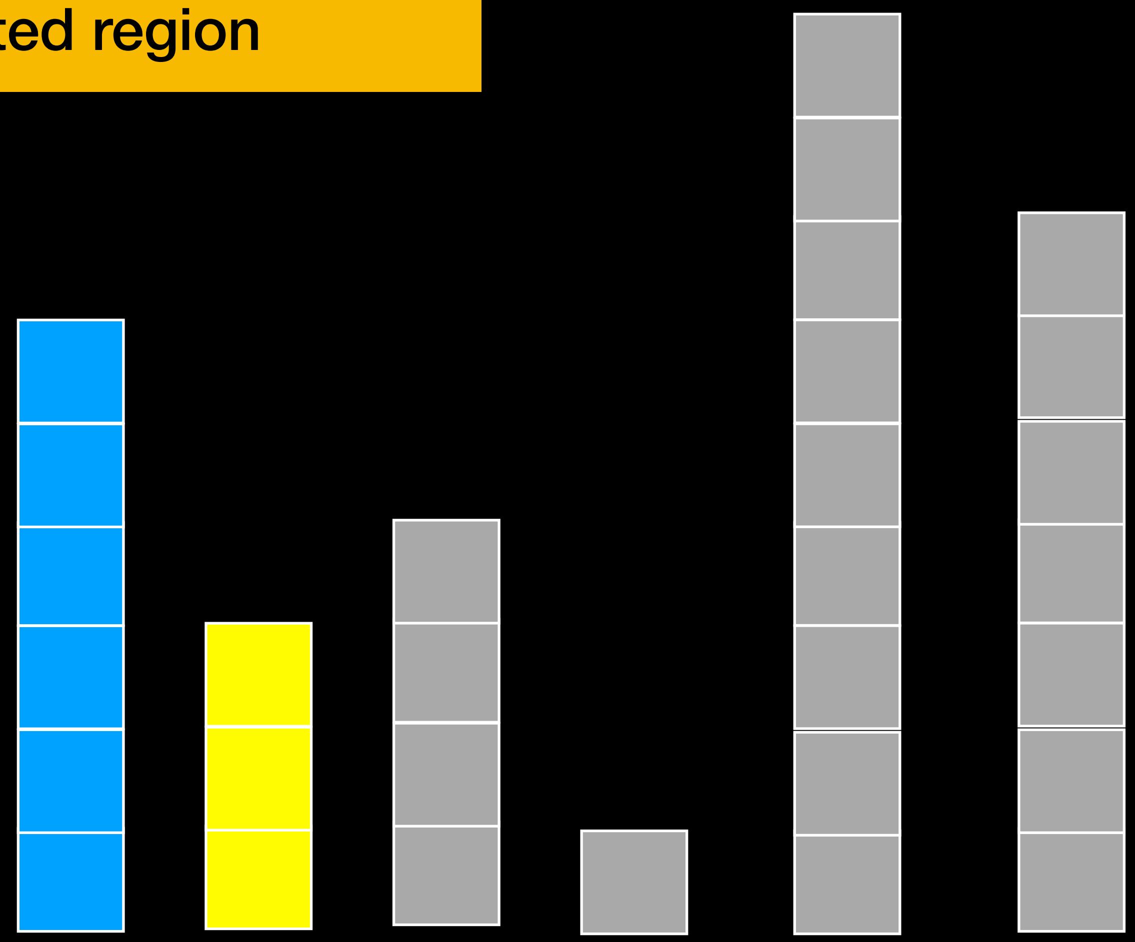
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

1st Pass



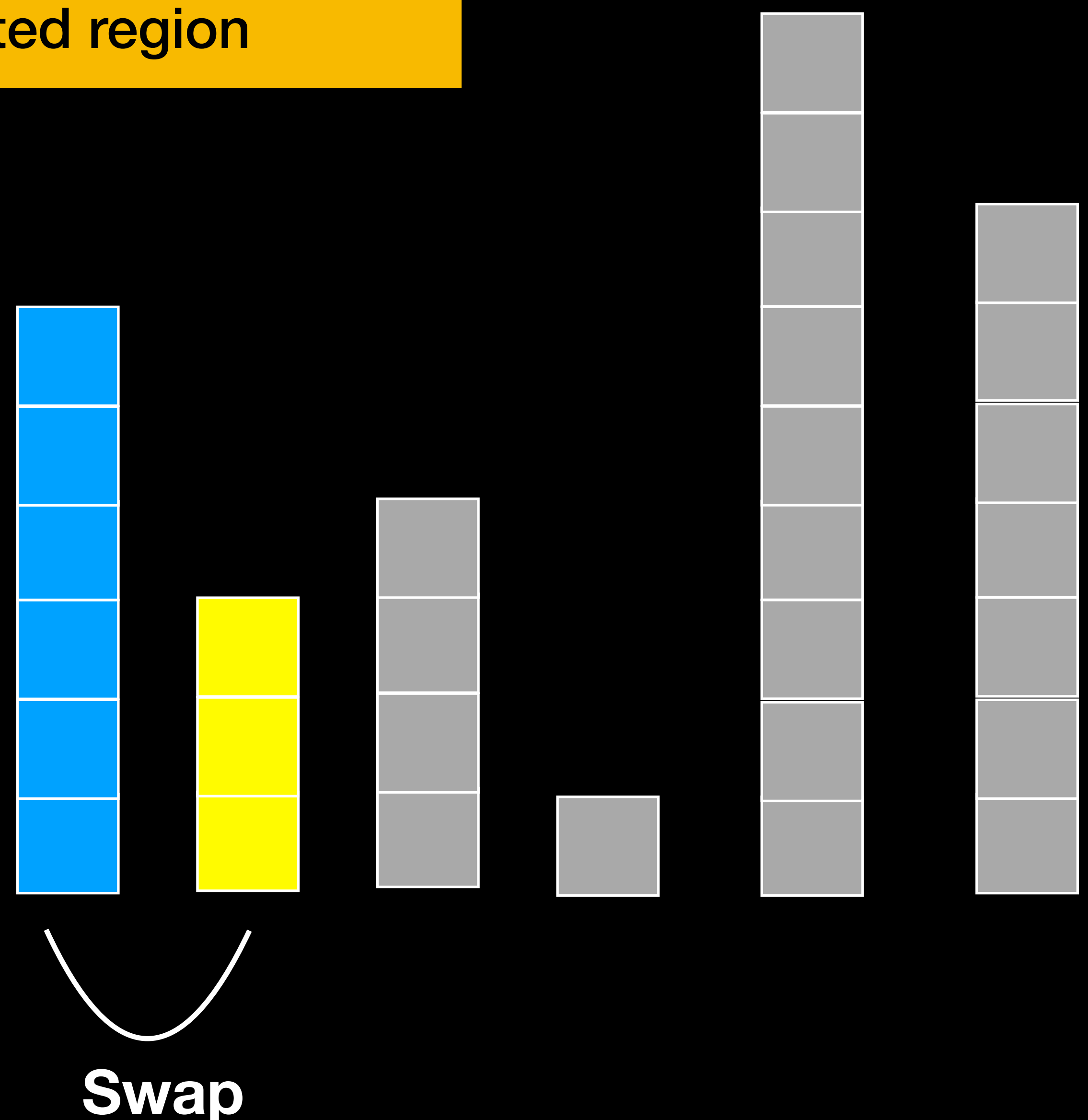
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

1st Pass



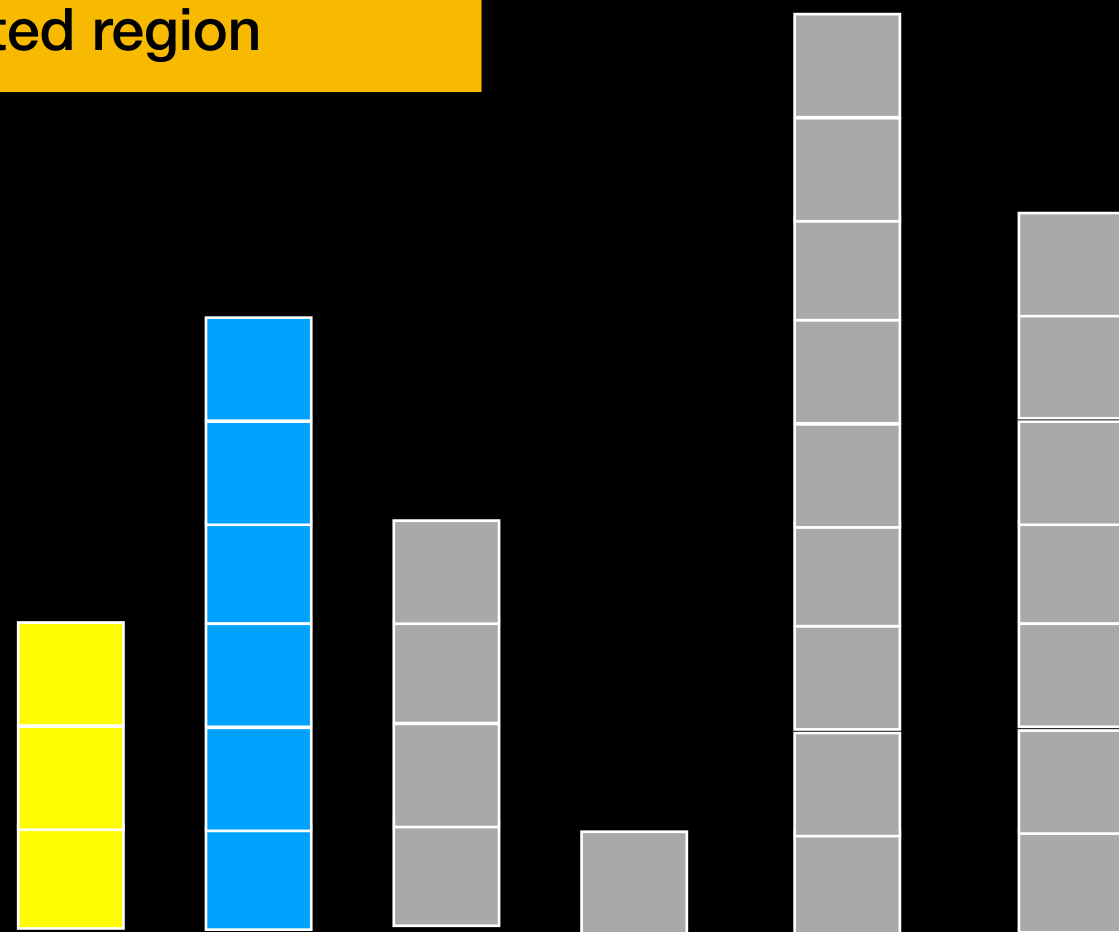
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

Unsorted

Sorted

1st Pass

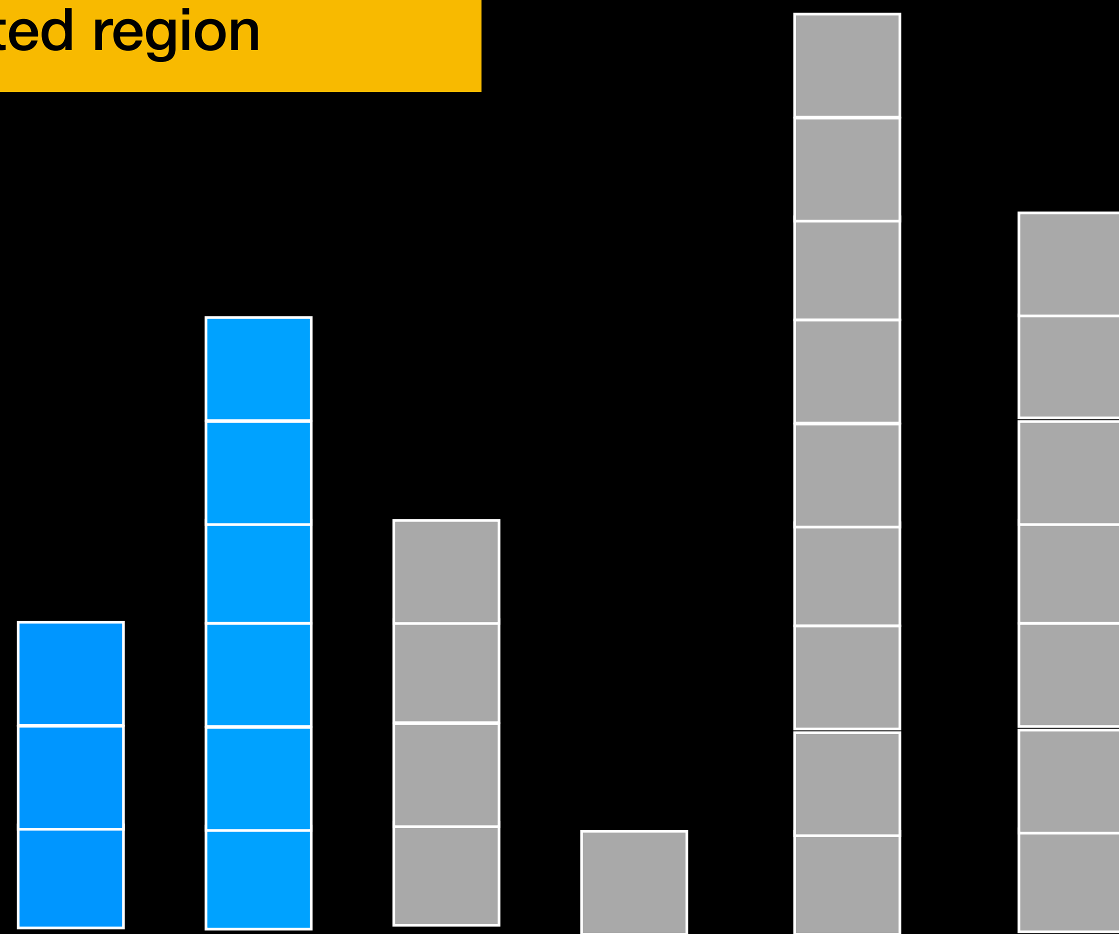


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region



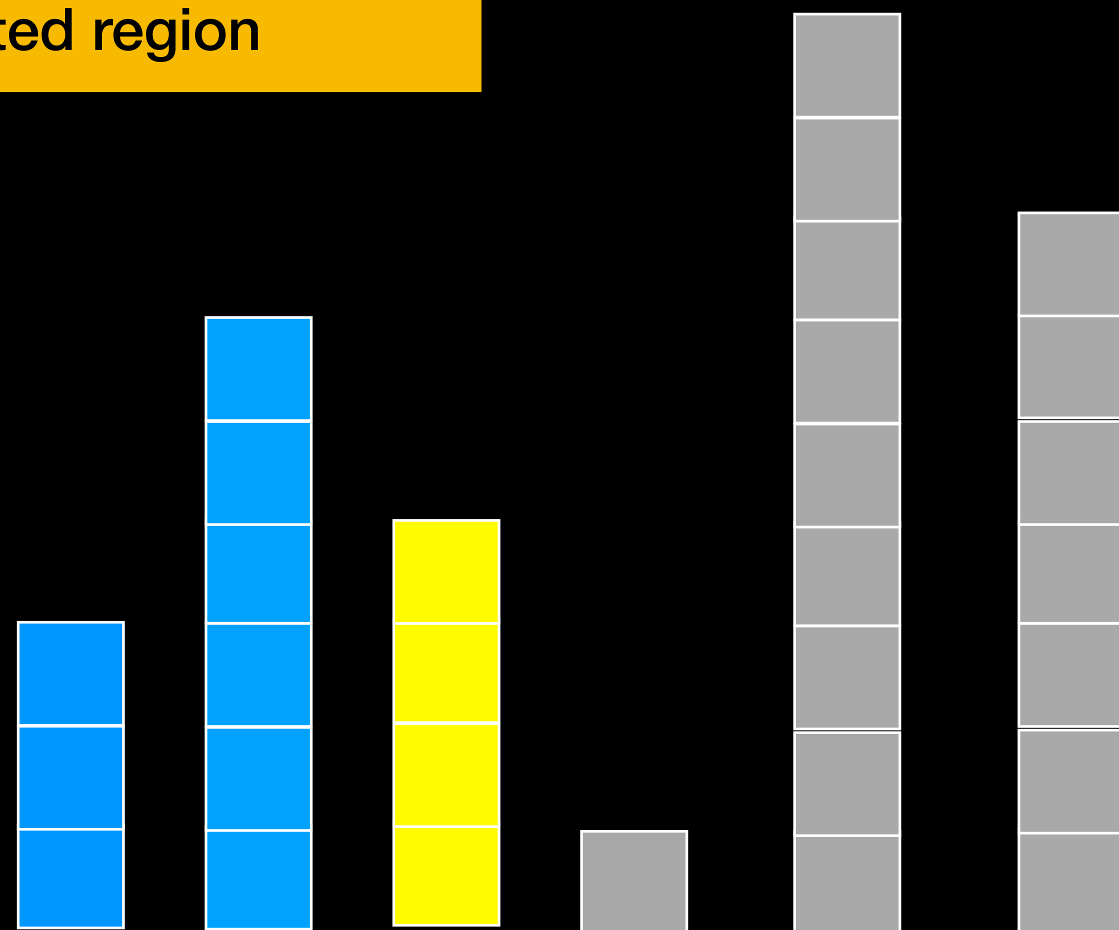
Insertion Sort

Unsorted

Sorted

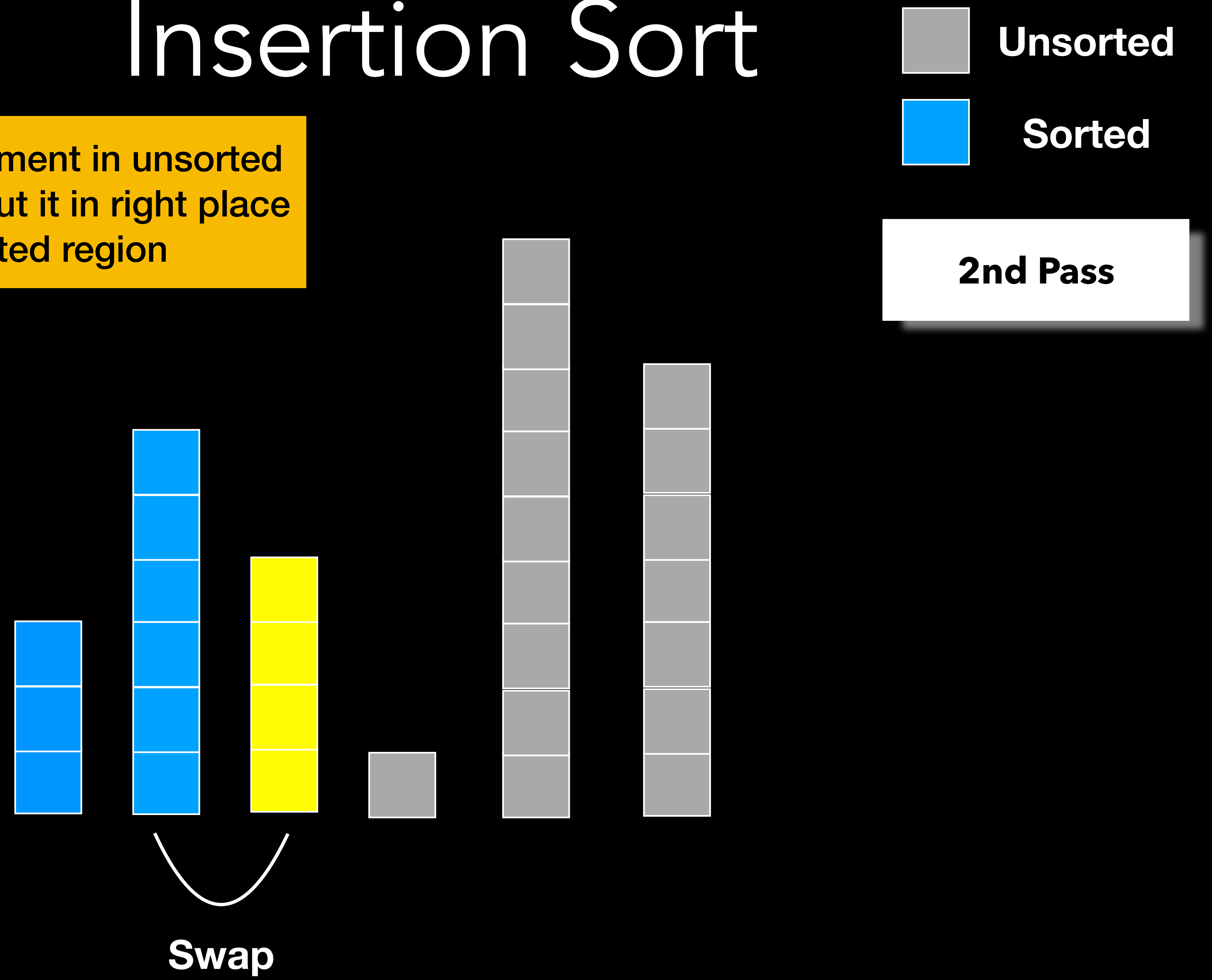
Pick first element in unsorted region and put it in right place in sorted region

2nd Pass



Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region



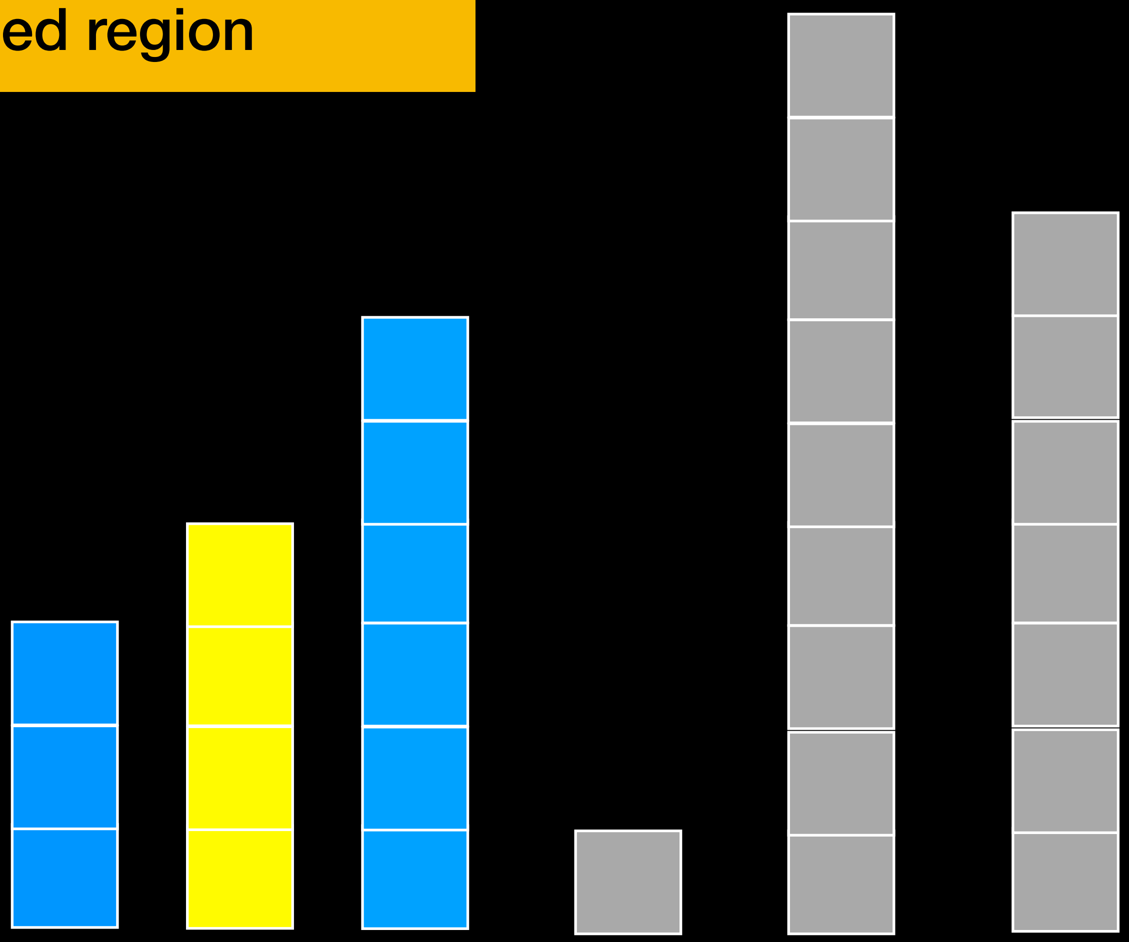
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

2nd Pass



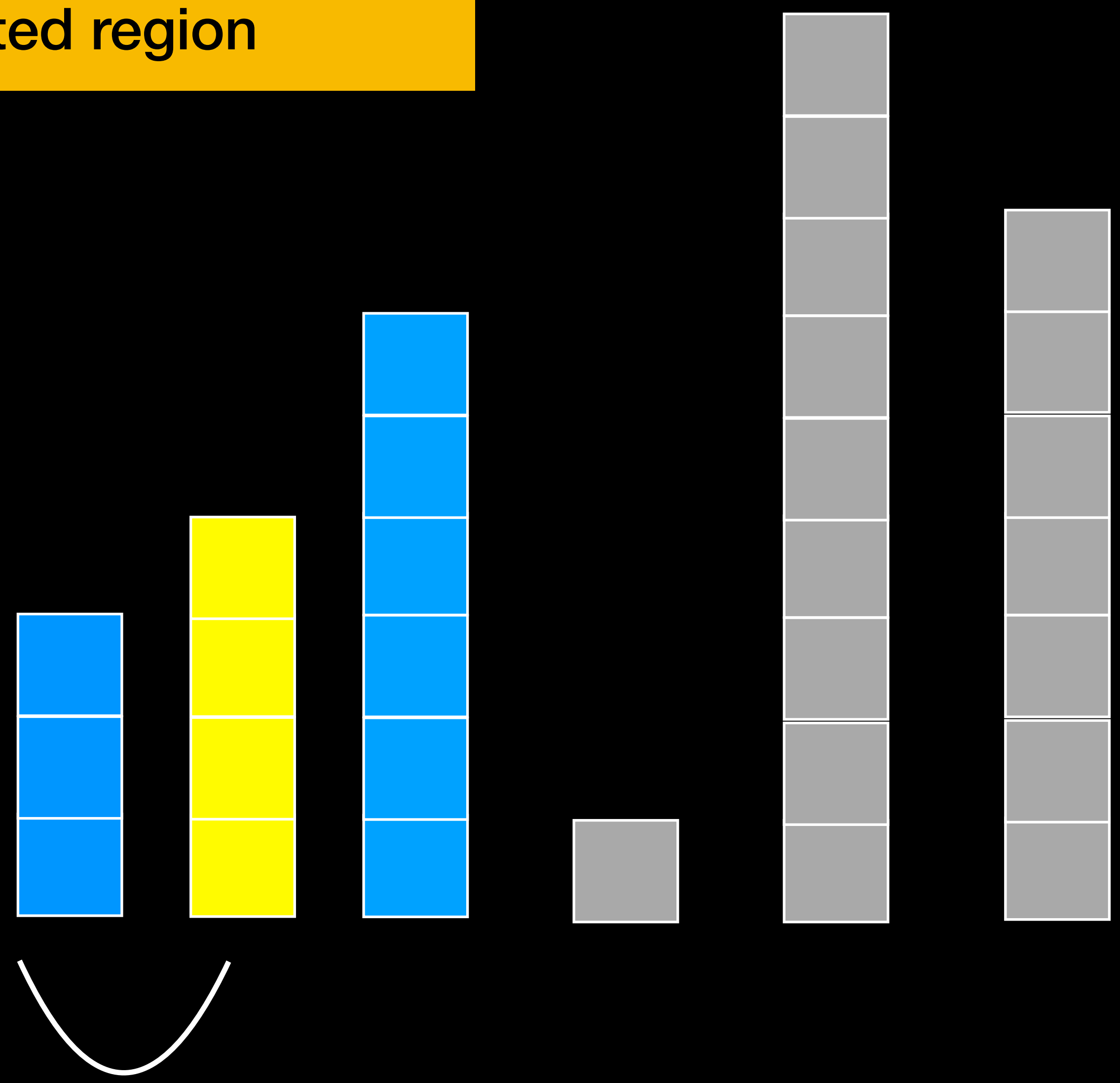
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

Unsorted

Sorted

2nd Pass



Insertion Sort

Unsorted

Sorted

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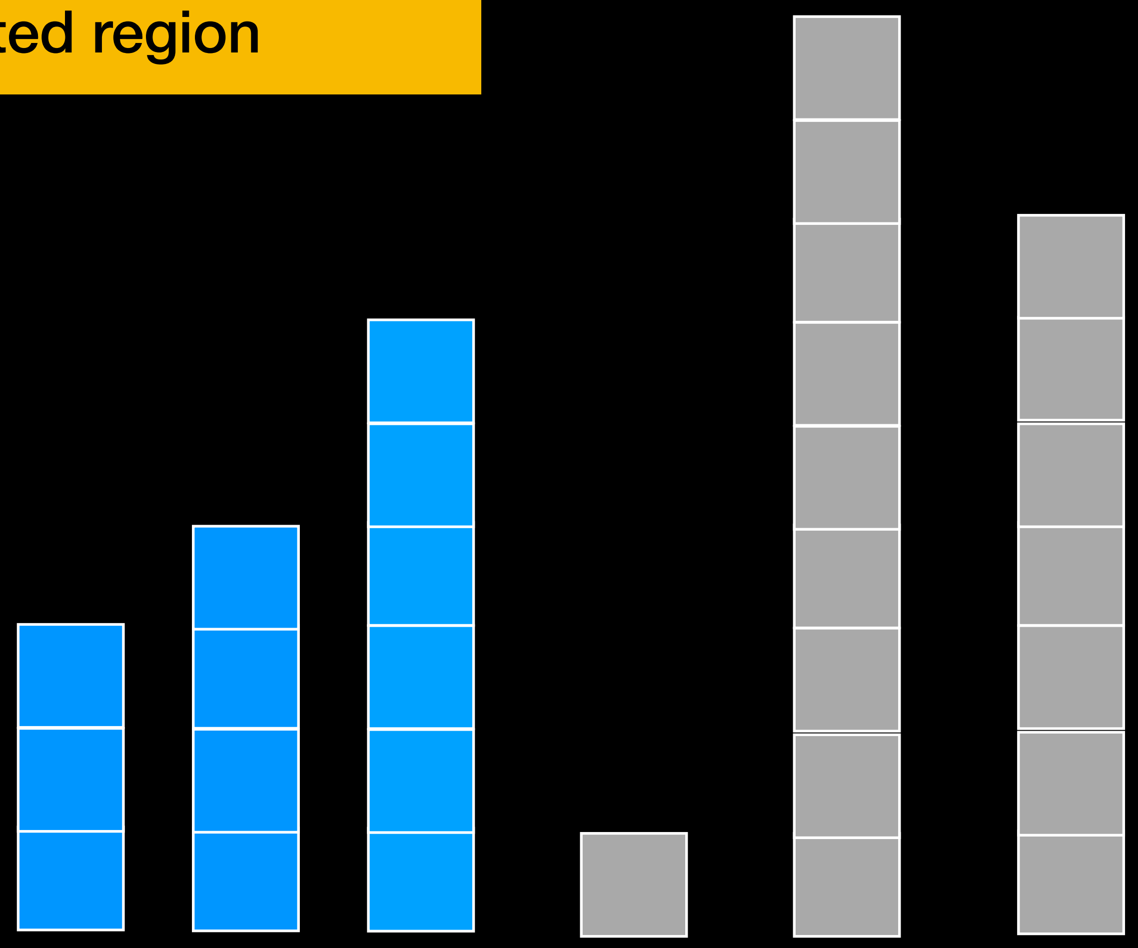


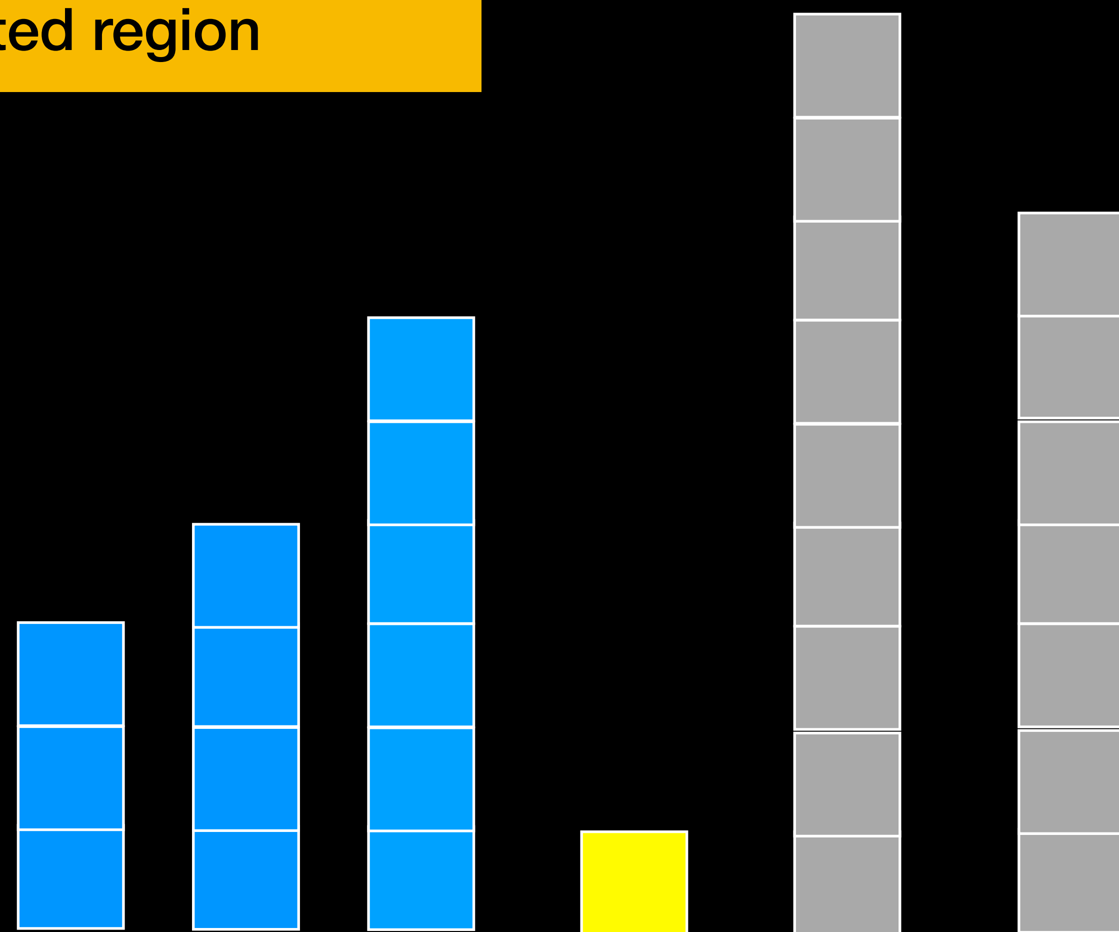
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

3rd Pass



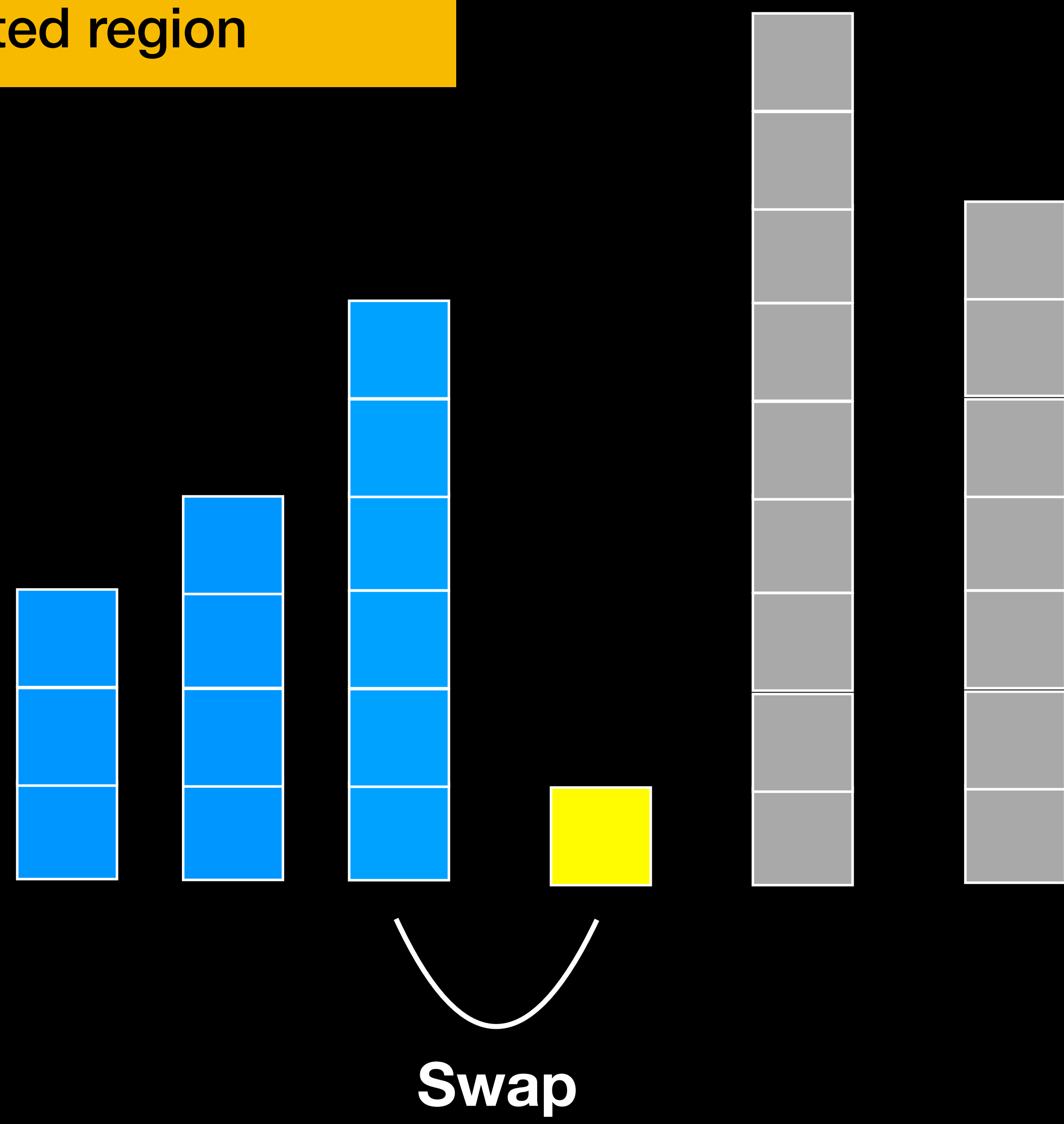
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

3rd Pass



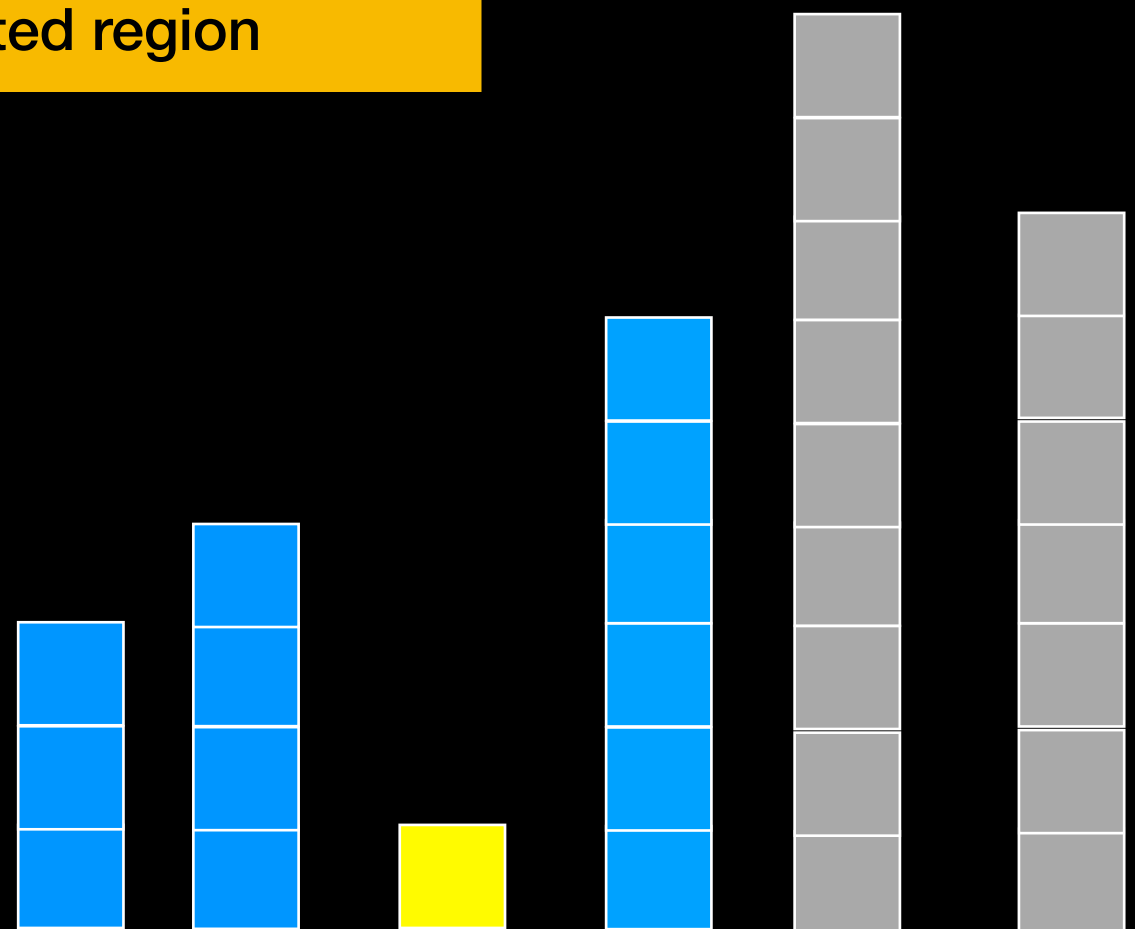
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

3rd Pass



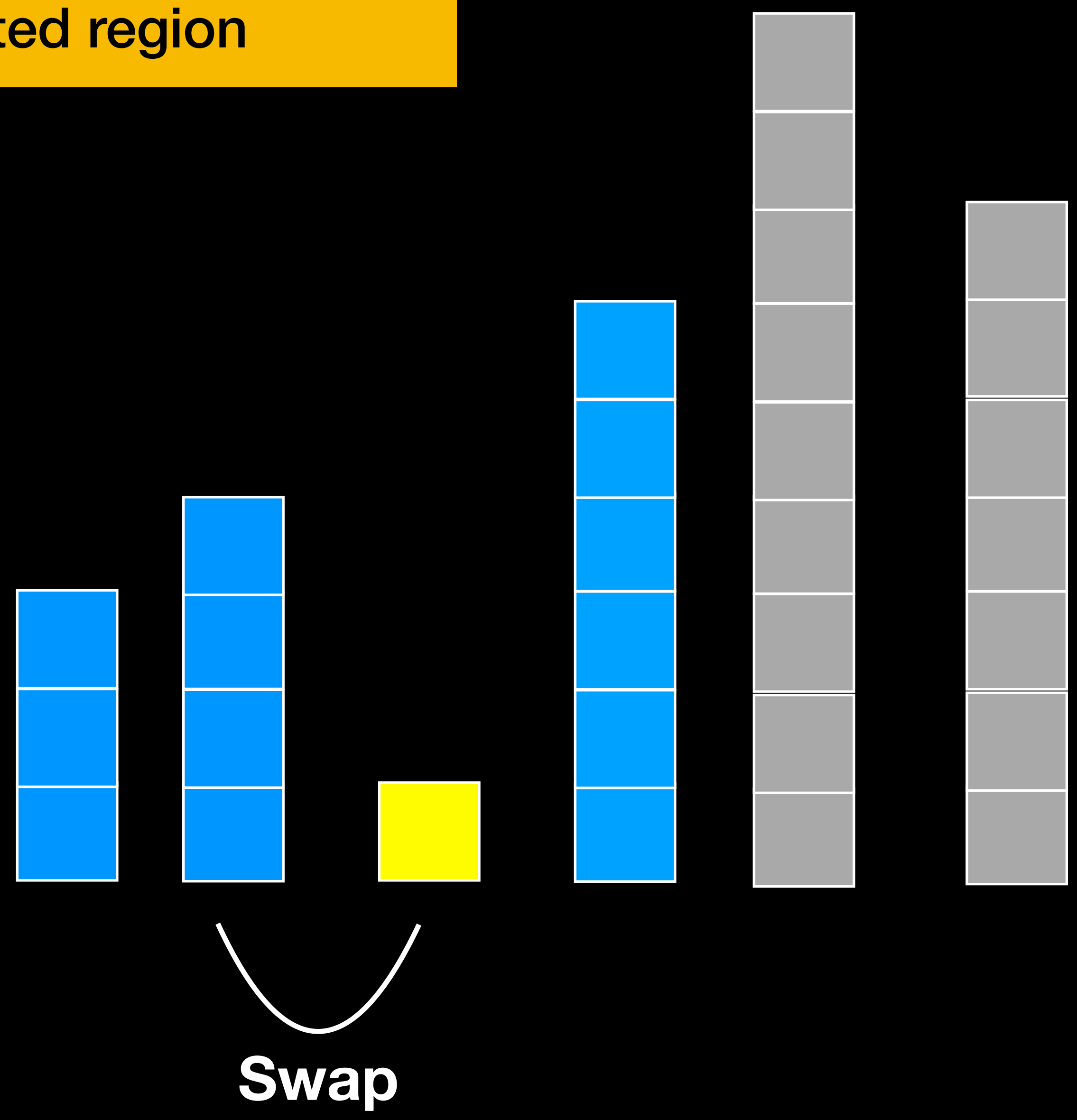
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

3rd Pass



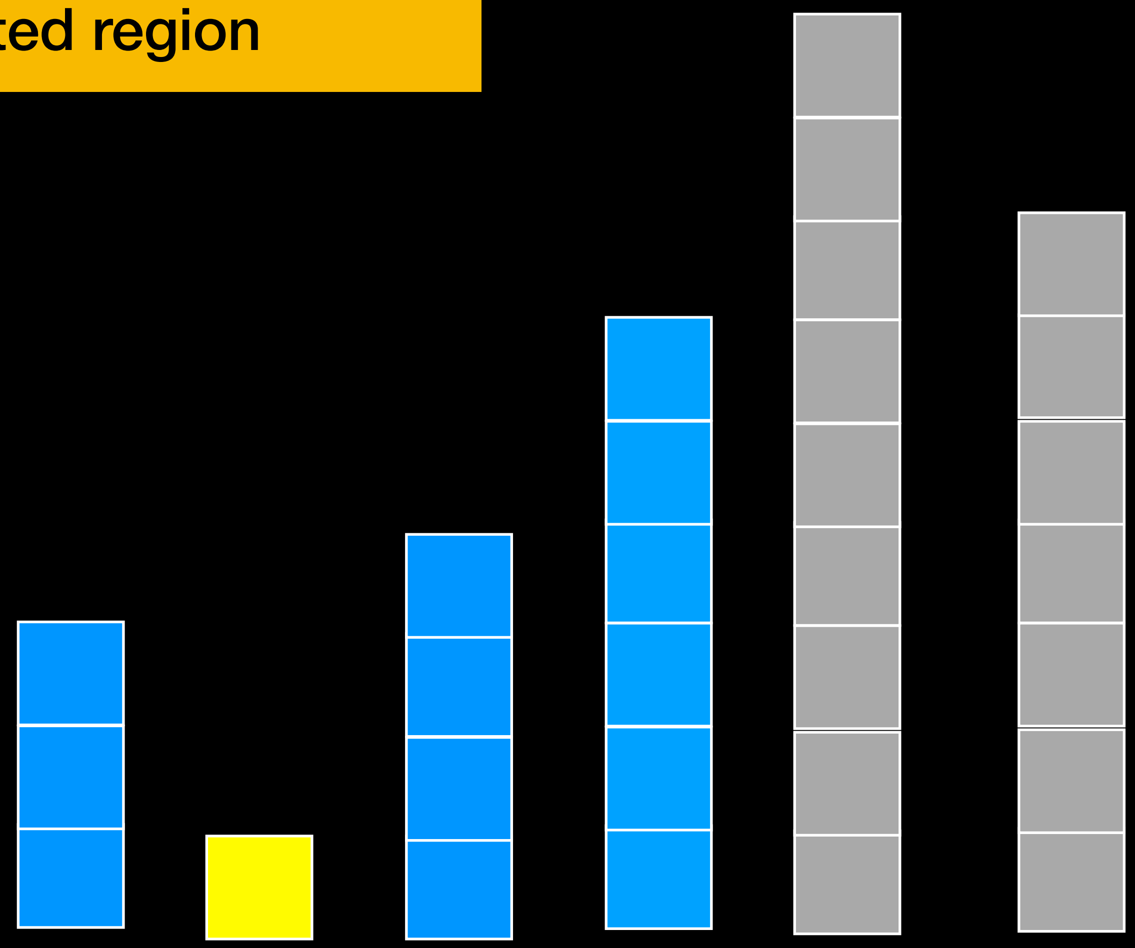
Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

3rd Pass



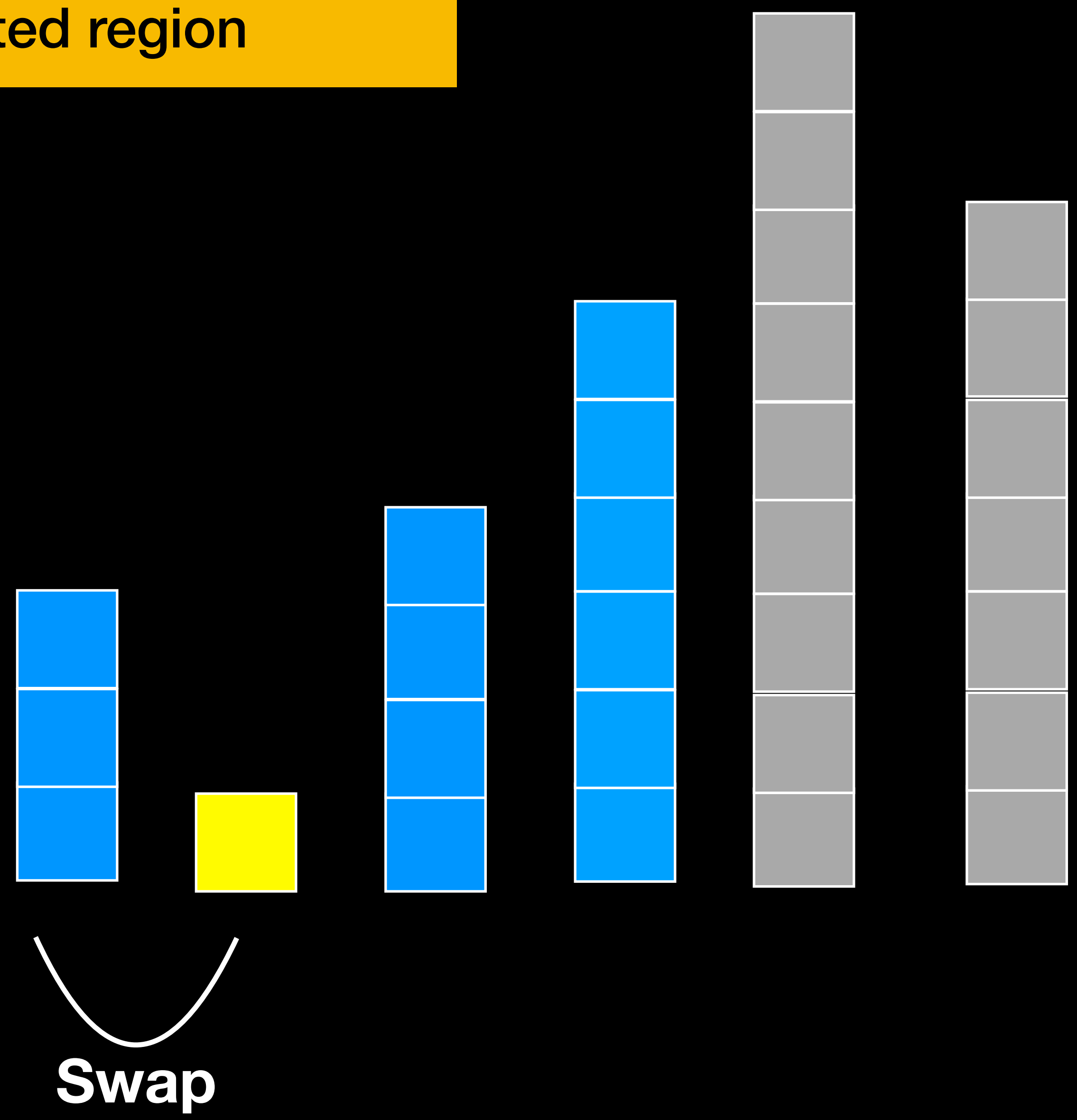
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

Unsorted

Sorted

3rd Pass



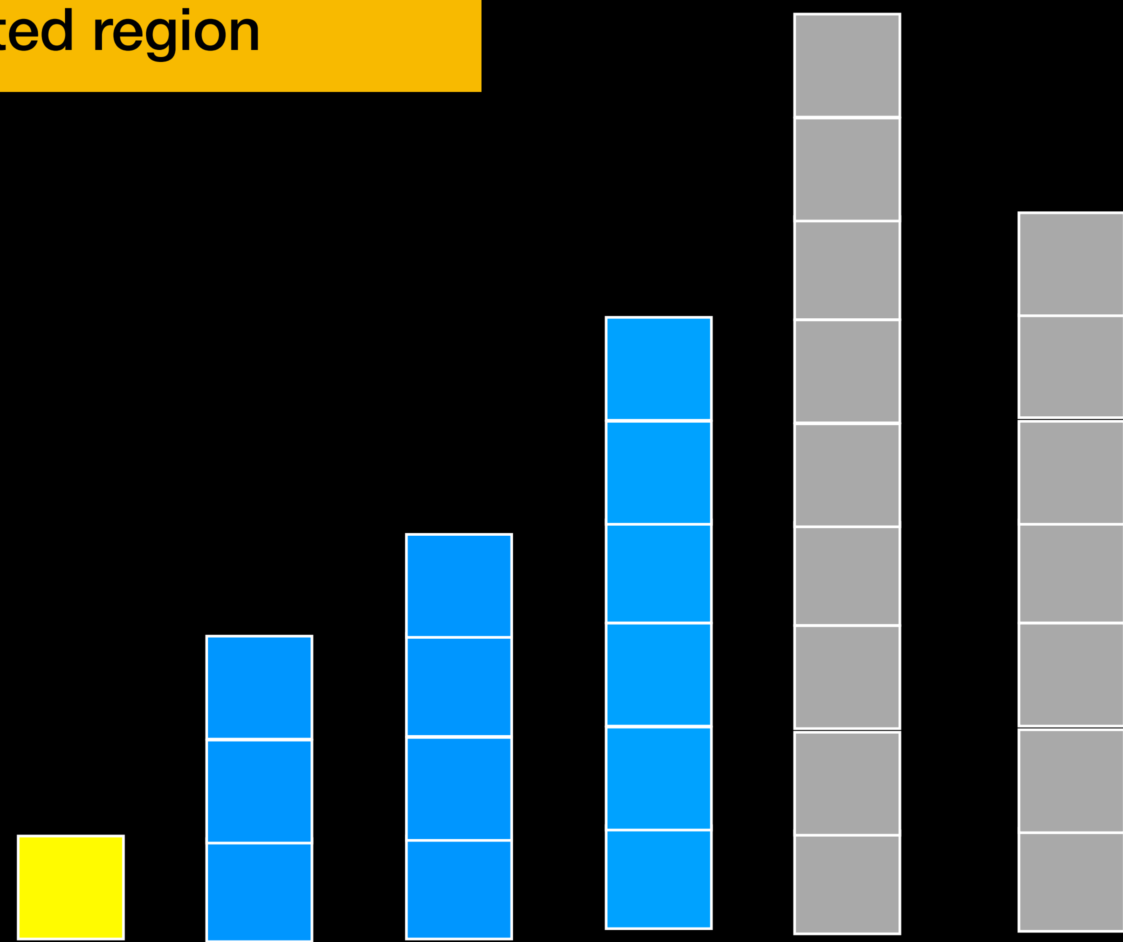
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

Unsorted

Sorted

3rd Pass

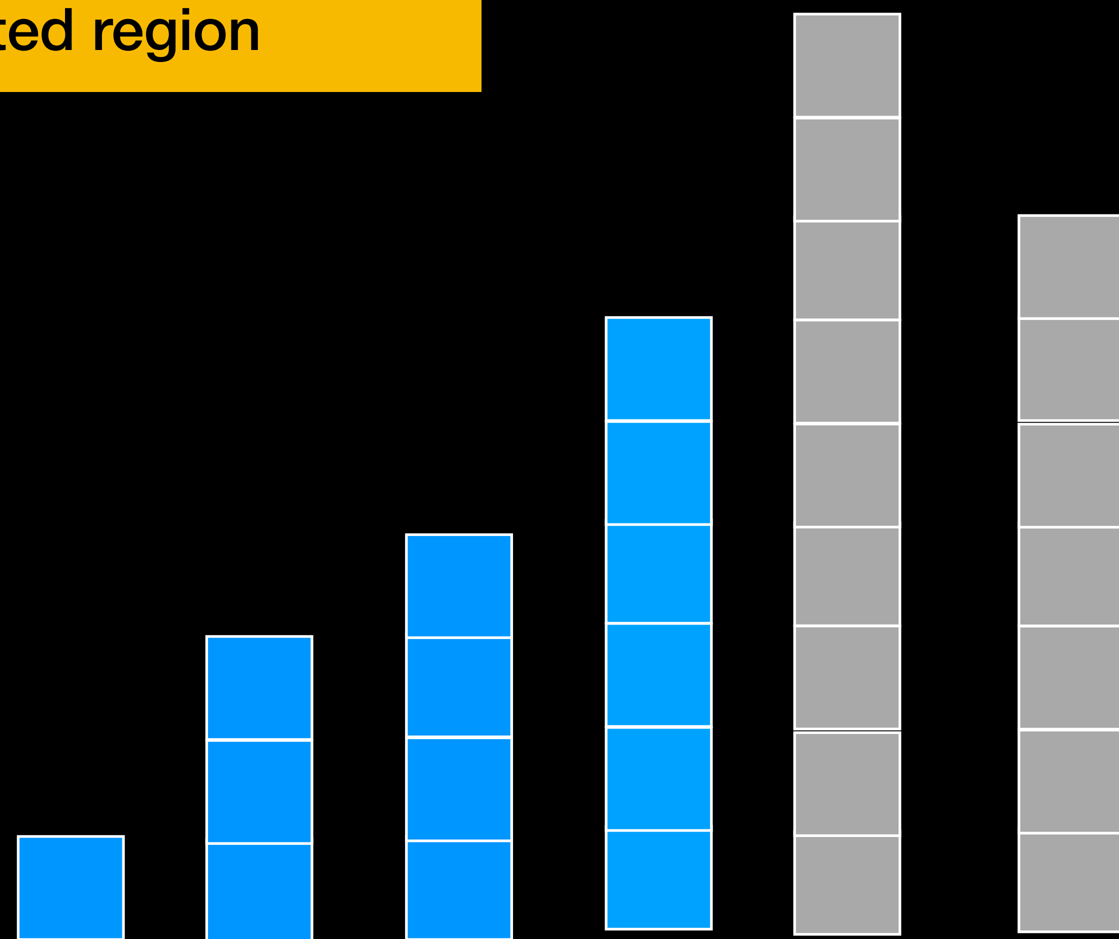


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region



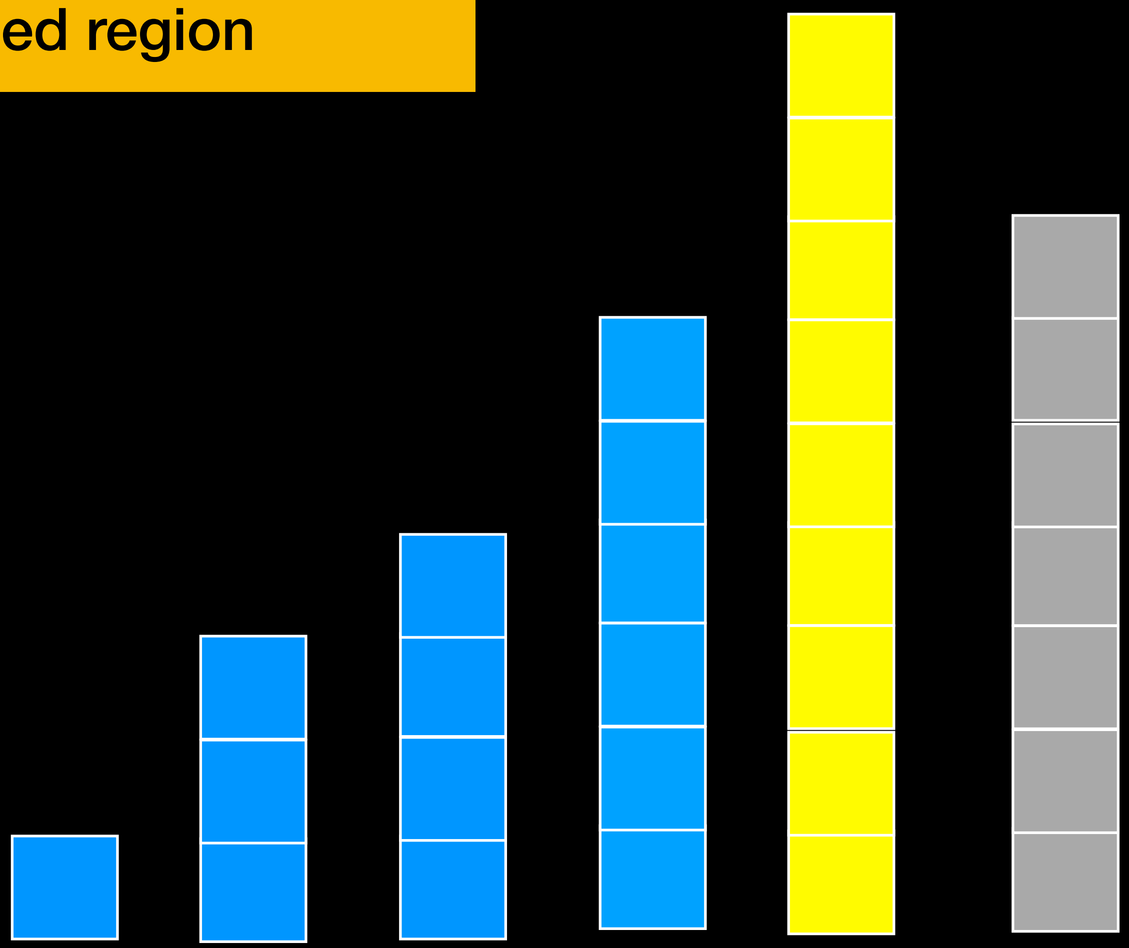
Insertion Sort

Unsorted

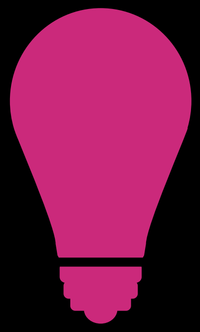
Sorted

Pick first element in unsorted region and put it in right place in sorted region

4th Pass



Insertion Sort

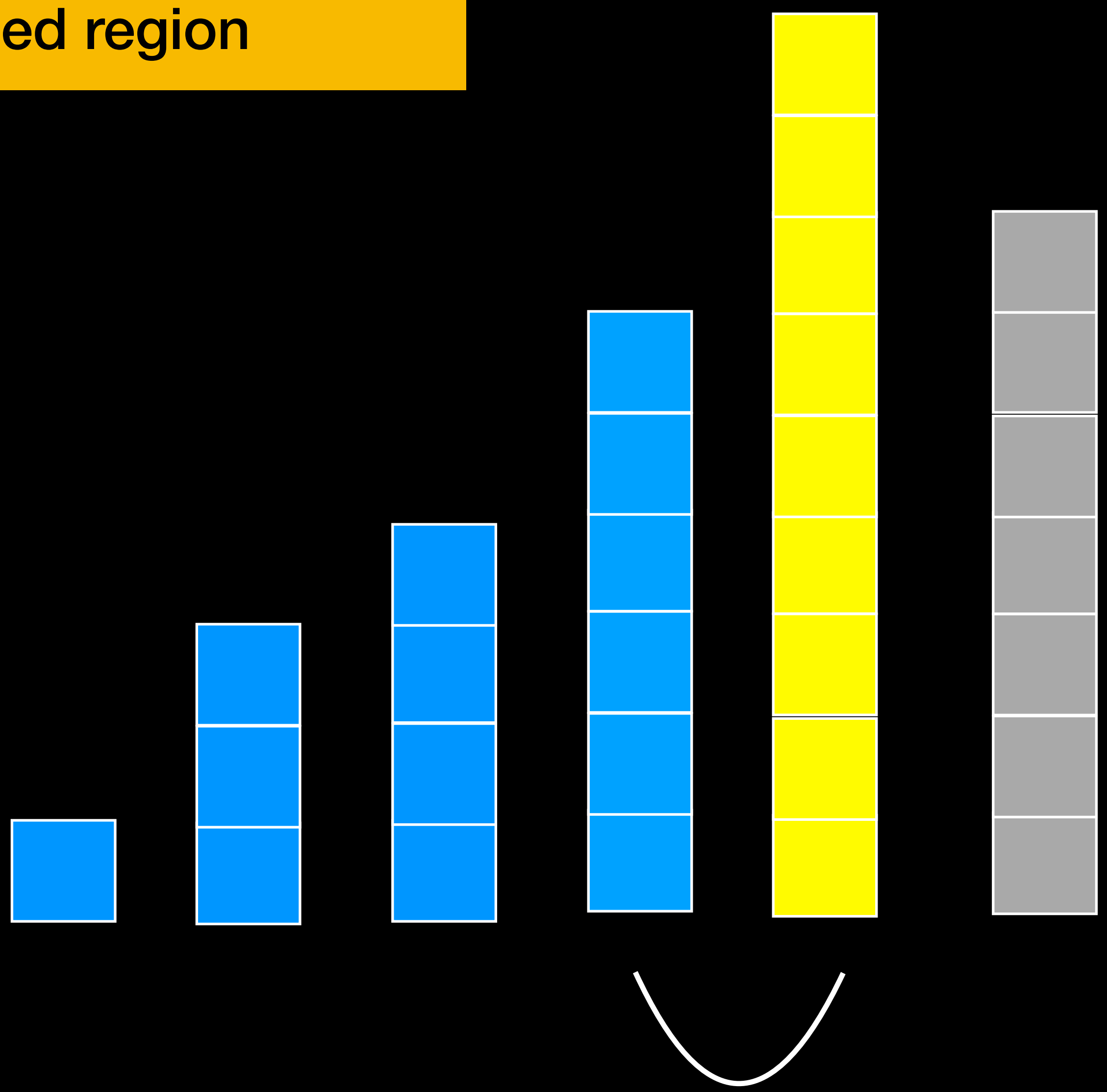


Pick first element in unsorted region and put it in right place in sorted region

Unsorted

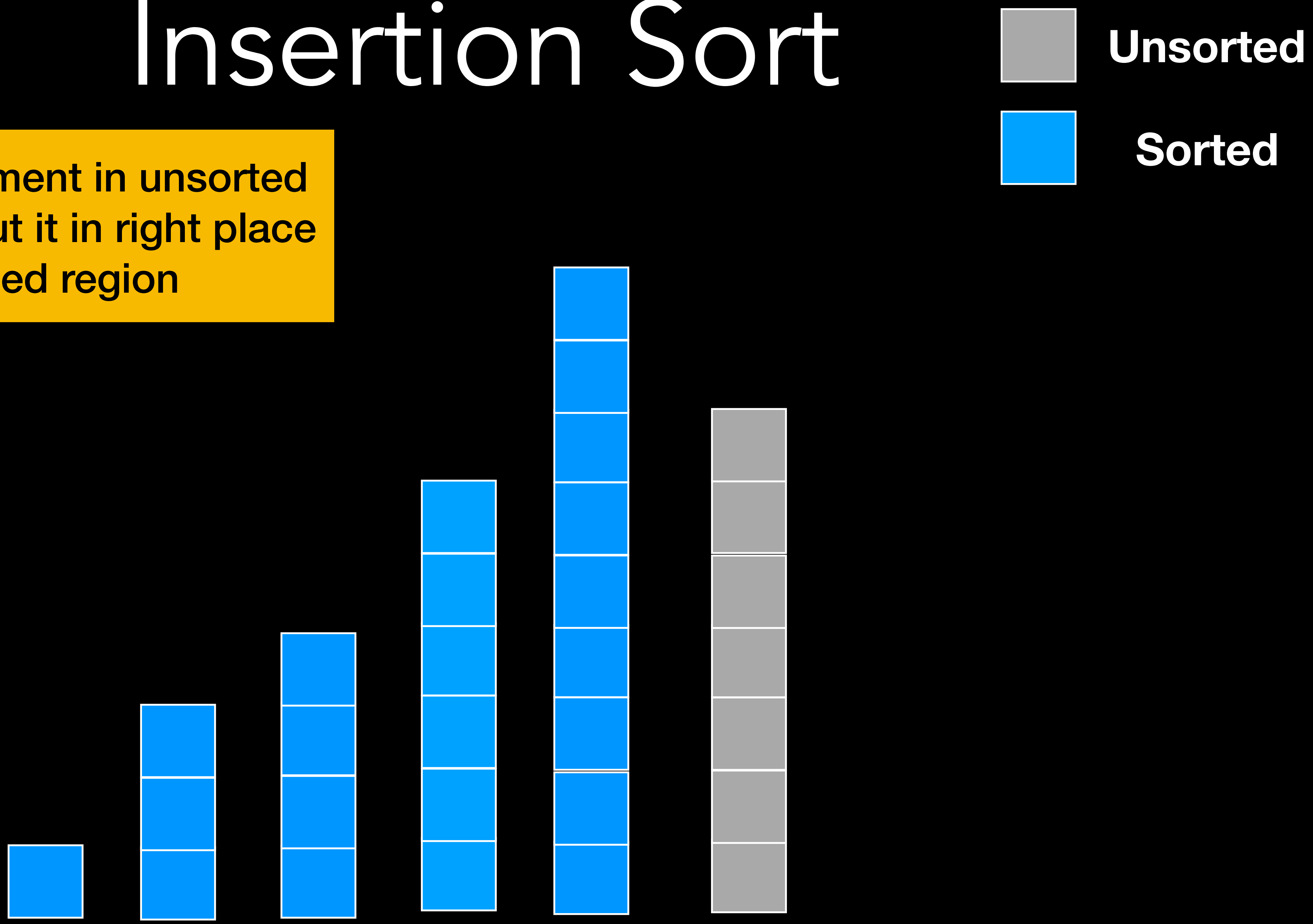
Sorted

4th Pass



Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region



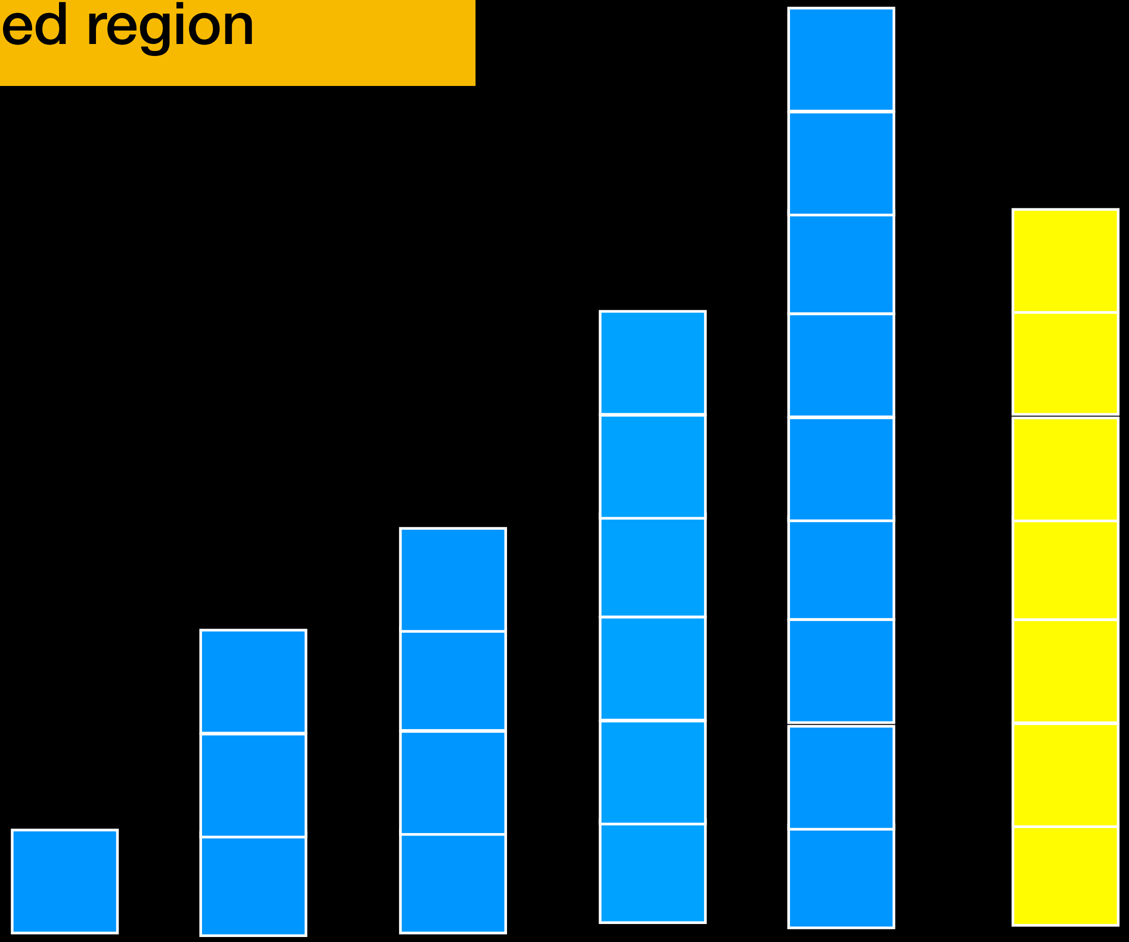
Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

Unsorted

Sorted

5th Pass



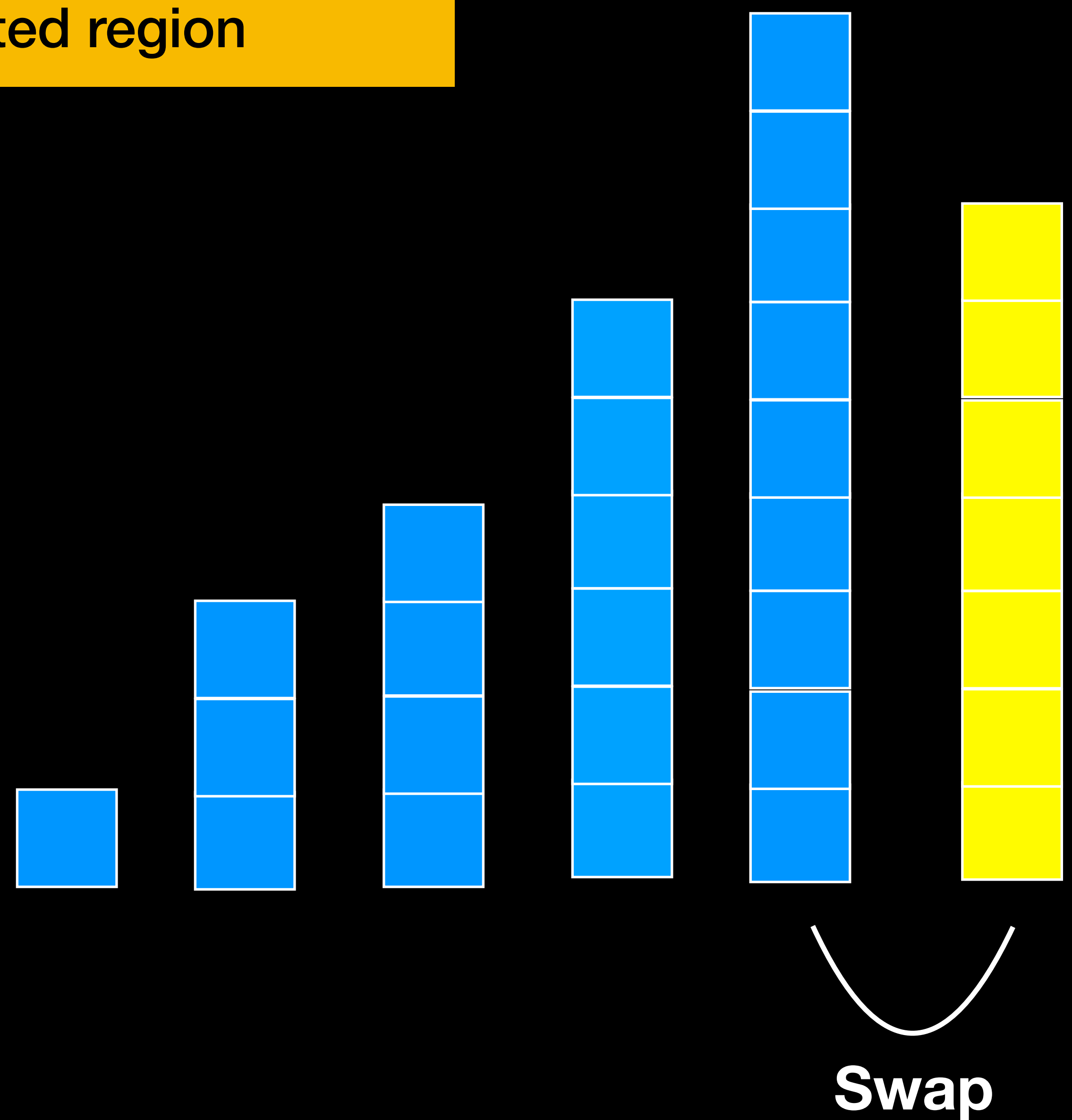
Insertion Sort

Unsorted

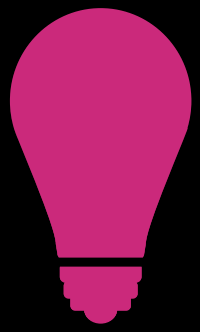
Sorted

Pick first element in unsorted region and put it in right place in sorted region

5th Pass



Insertion Sort

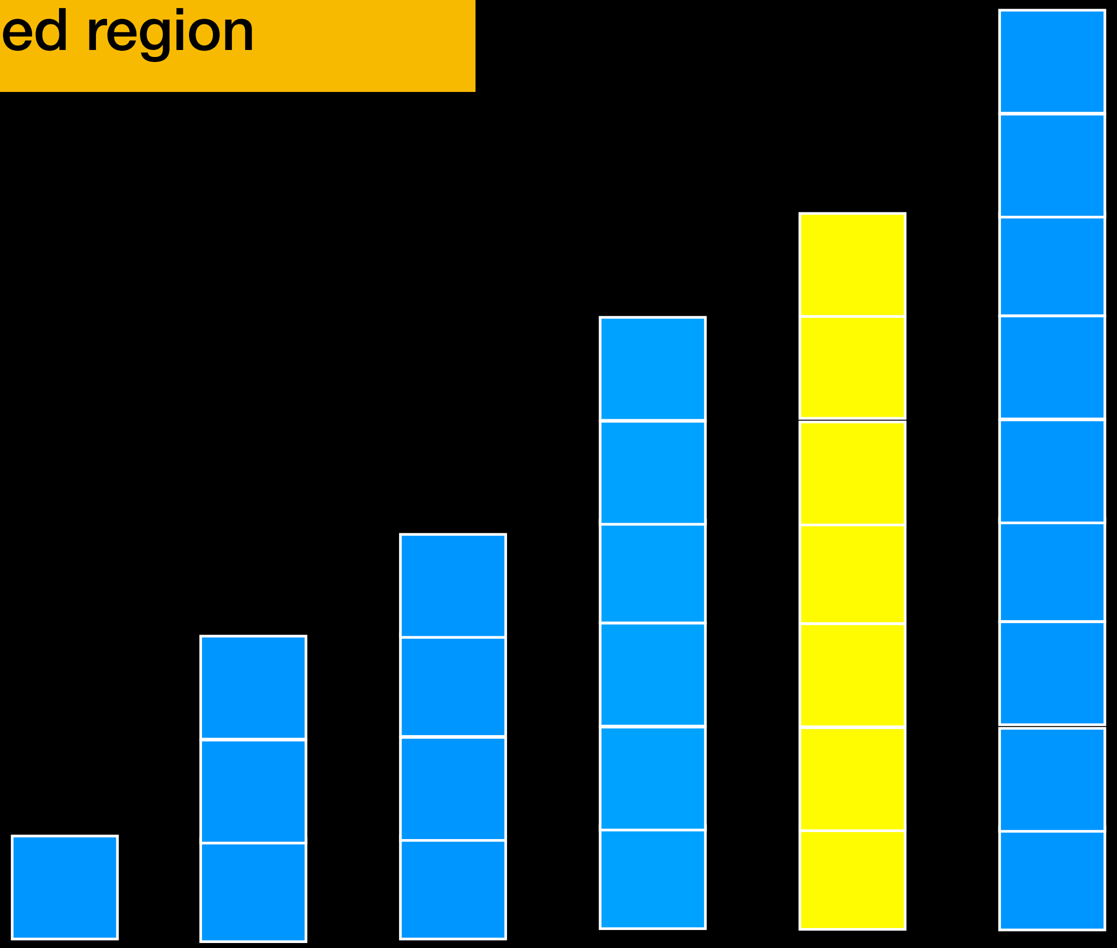


Pick first element in unsorted region and put it in right place in sorted region

Unsorted

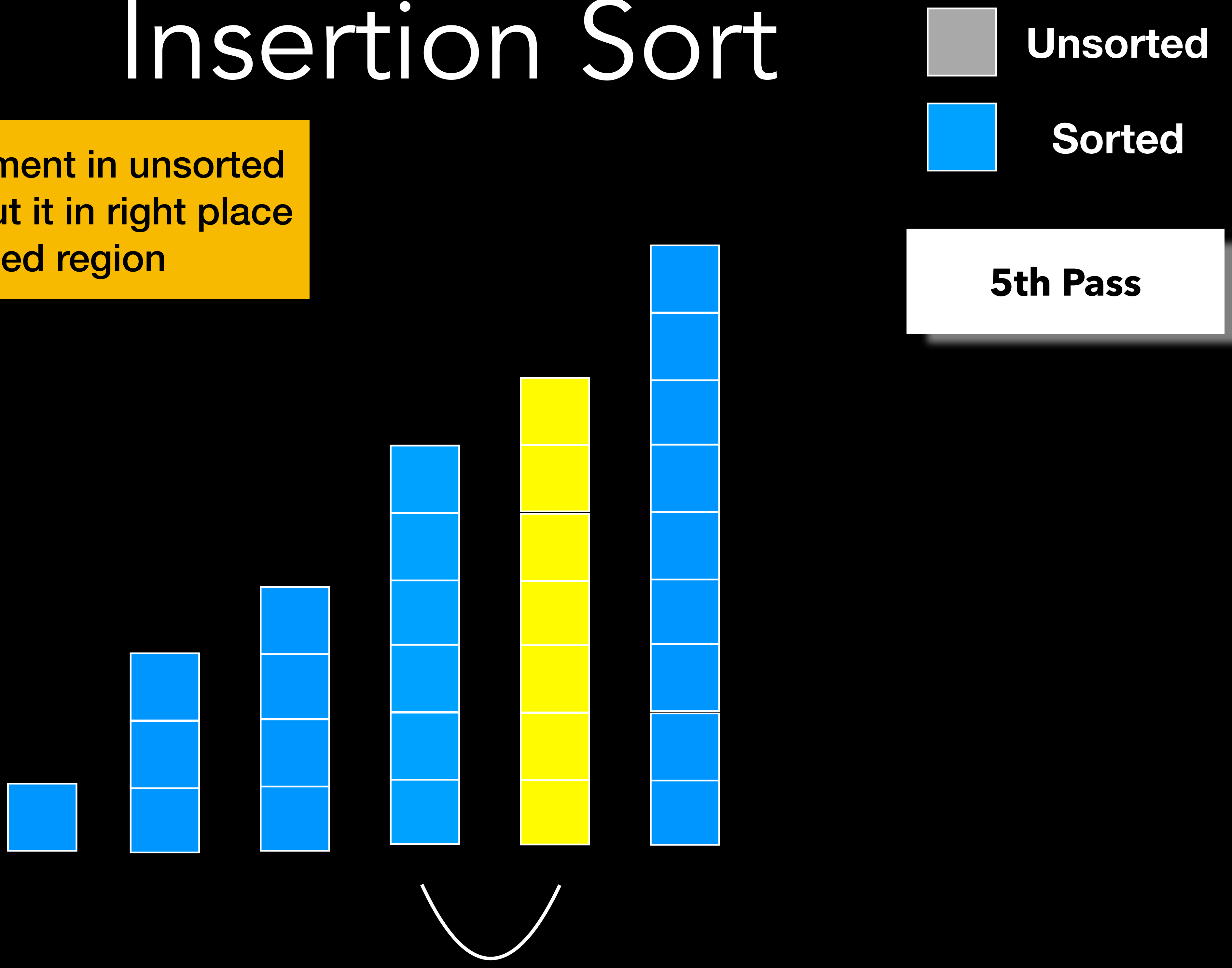
Sorted

5th Pass



Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

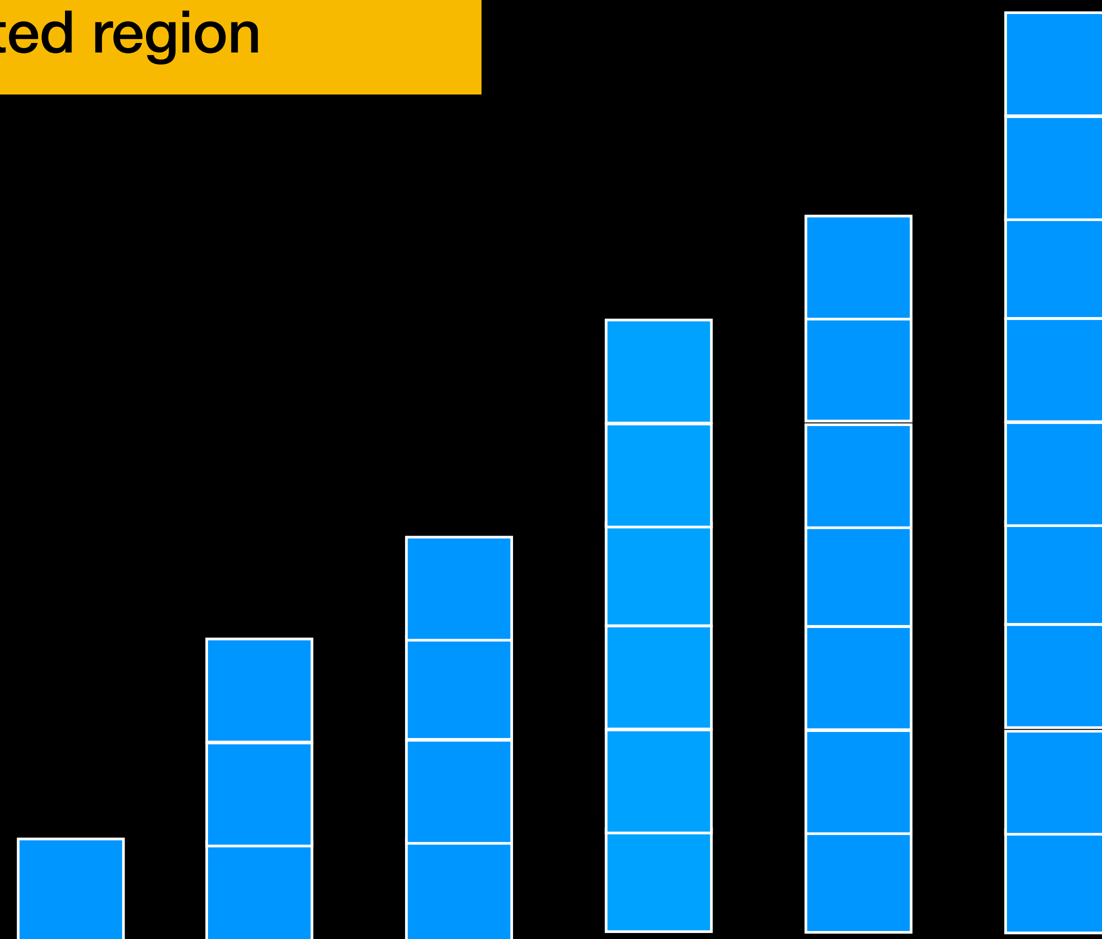


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort Analysis

How much work?

First pass: **1** comparison and at most **1** swap

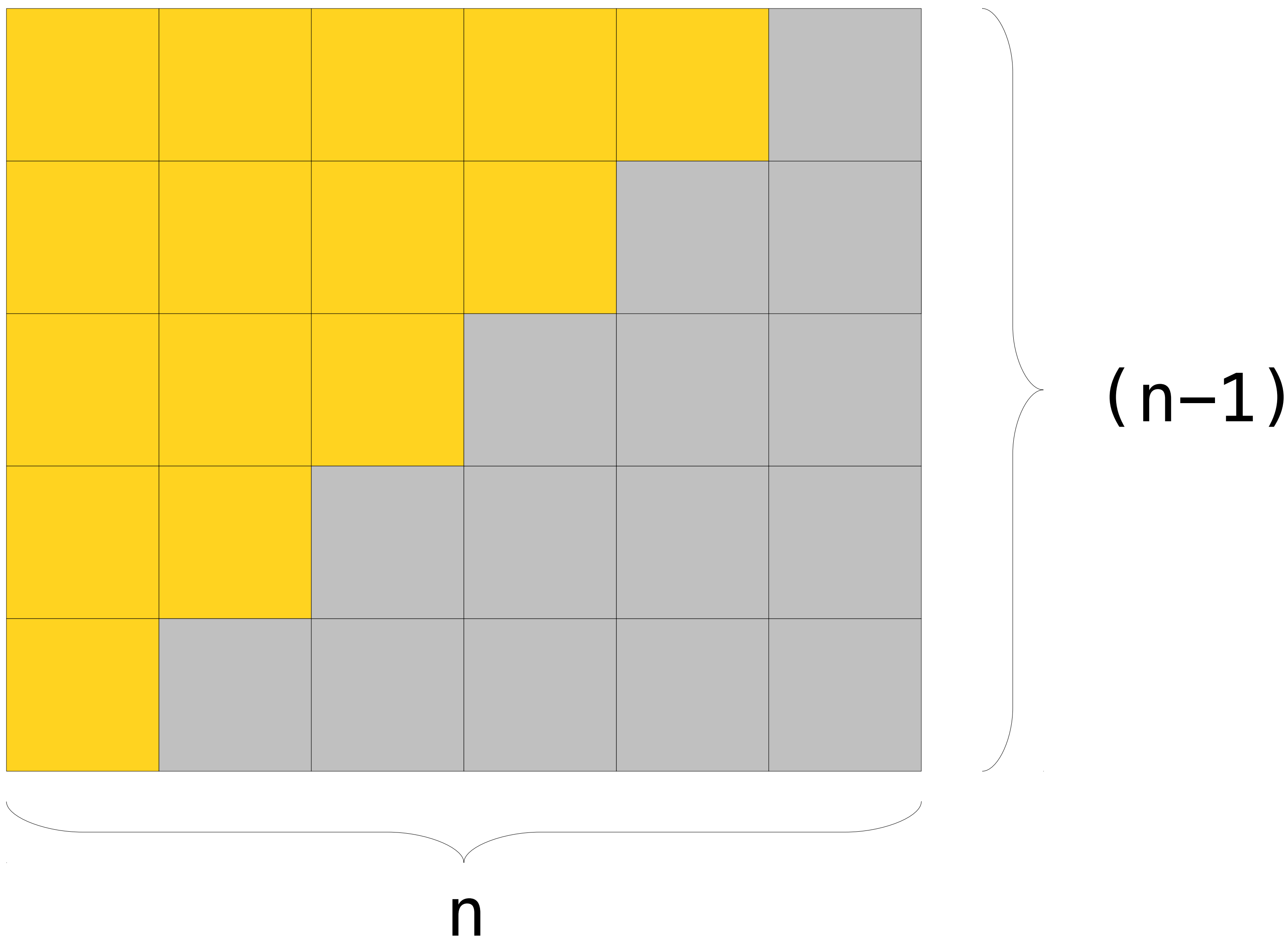
Second pass: at most **2** comparisons and at most **2** swaps

Third pass: at most **3** comparisons and at most **3** swaps

. . .

Total work: **1 + 2 + 3 + . . . + (n-1)**

$1 + 2 + \ldots (n-2) + (n-1) = n(n-1)/2$



Insertion Sort Analysis

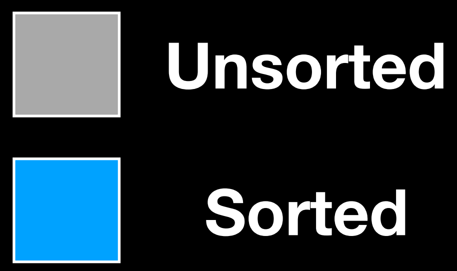
$T(n) = n(n-1) / 2$ comparisons $+ n(n-1) / 2$ swaps $= O(\ )$?

$T(n) = 2(\ (n^2-n) / 2\ ) = O(\ )$?

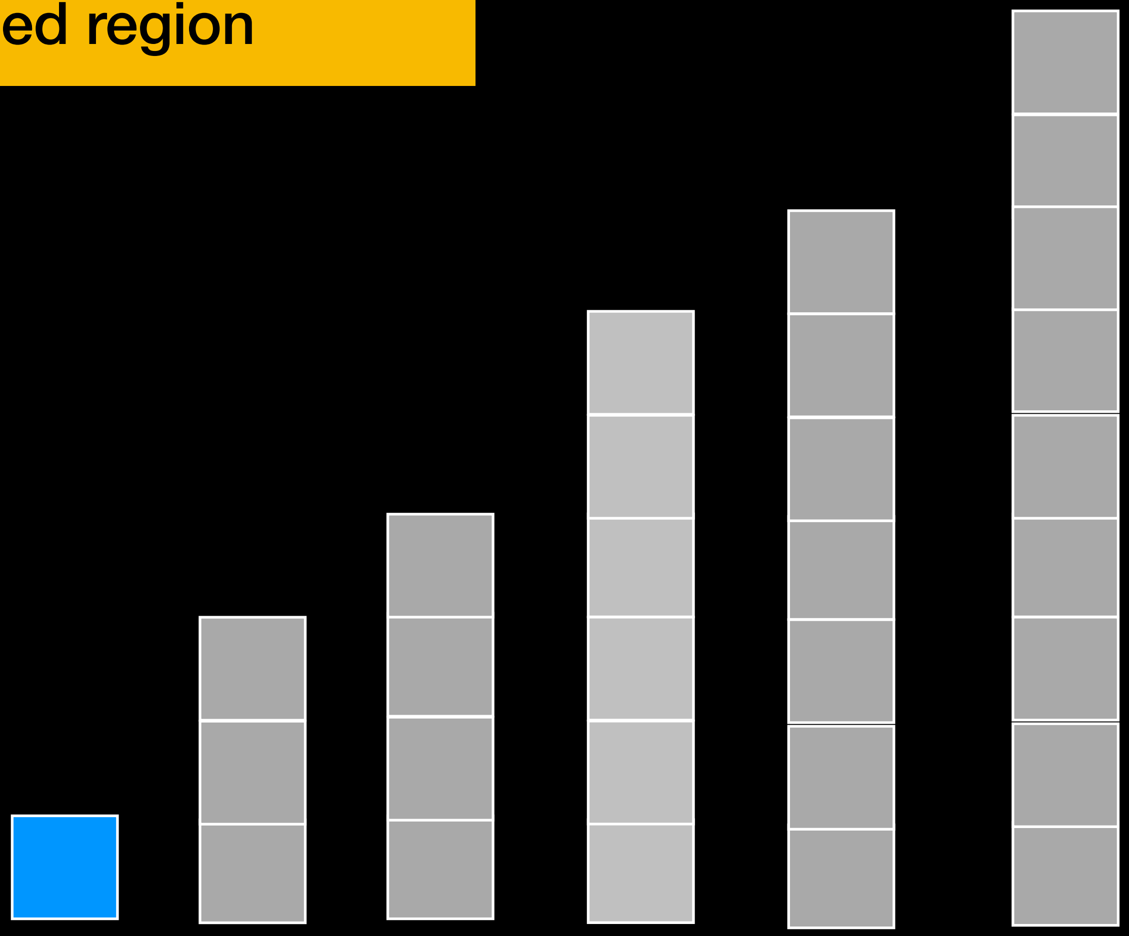
$T(n) = n^2-n = O(\ \mathbf{n^2})$

Insertion Sort run time is $\mathbf{O(n^2)}$

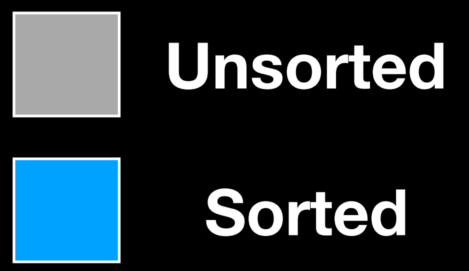
Insertion Sort

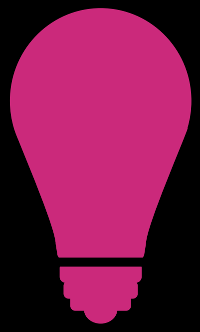


Pick first element in unsorted region and put it in right place in sorted region

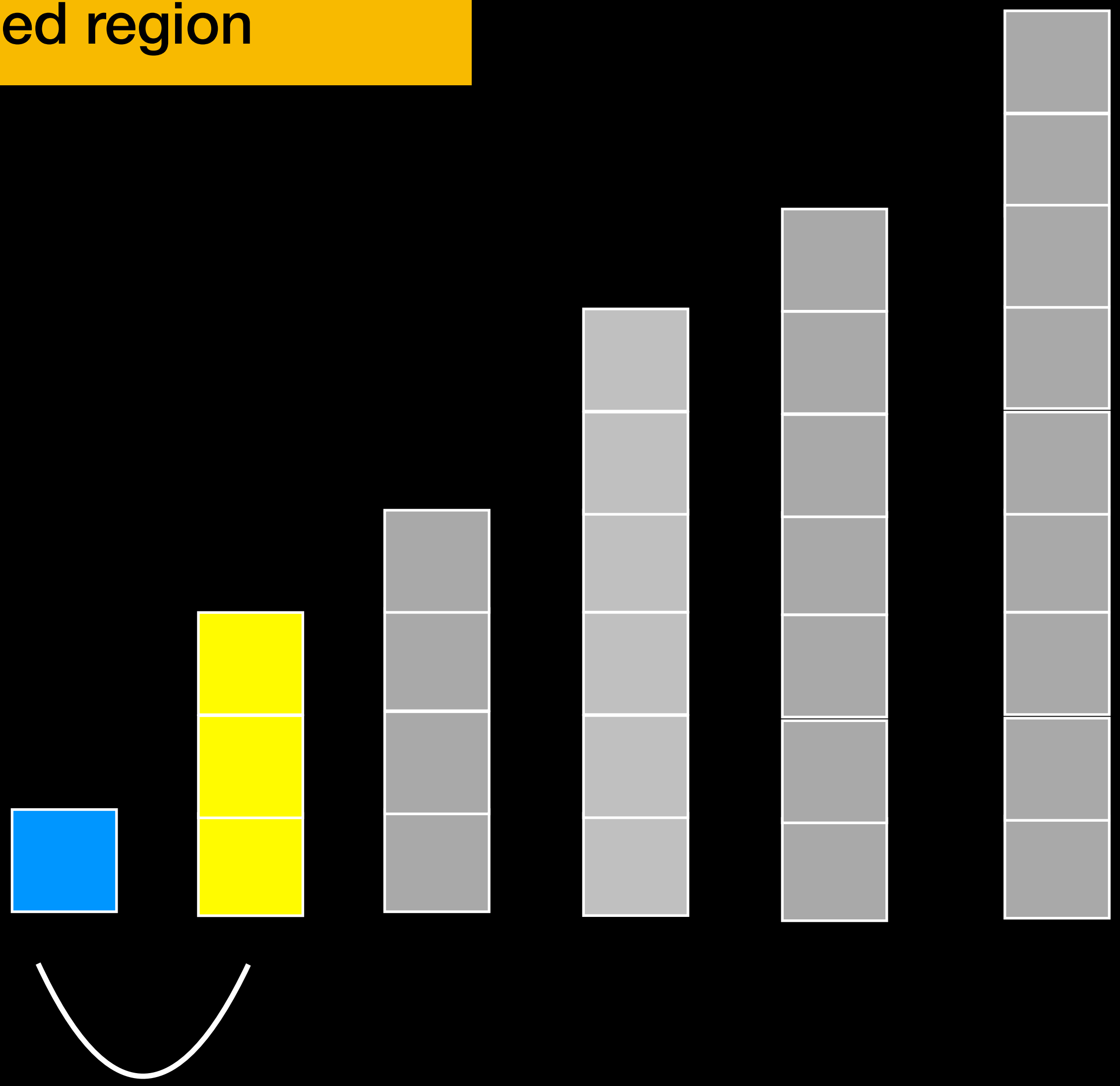


Insertion Sort





Pick first element in unsorted region and put it in right place in sorted region

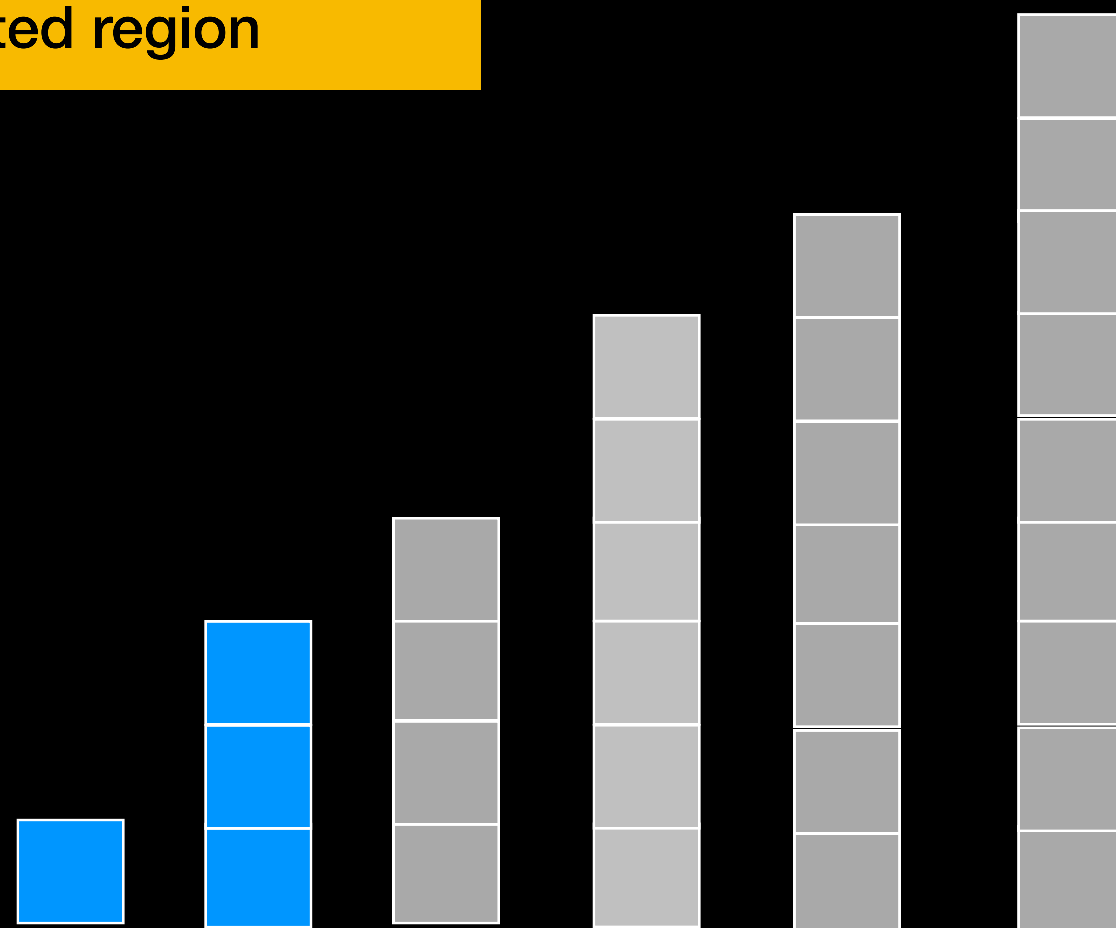


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

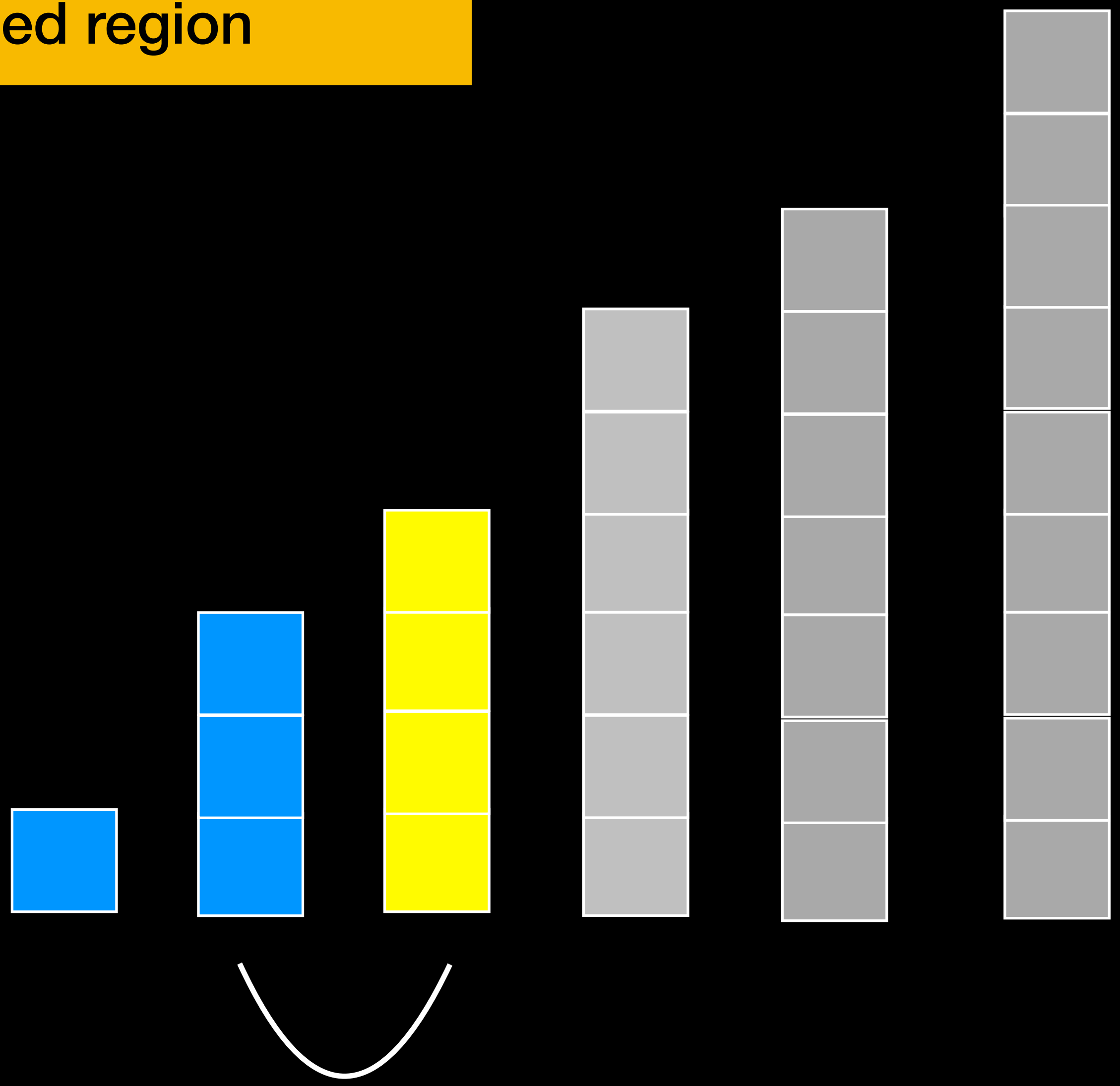


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

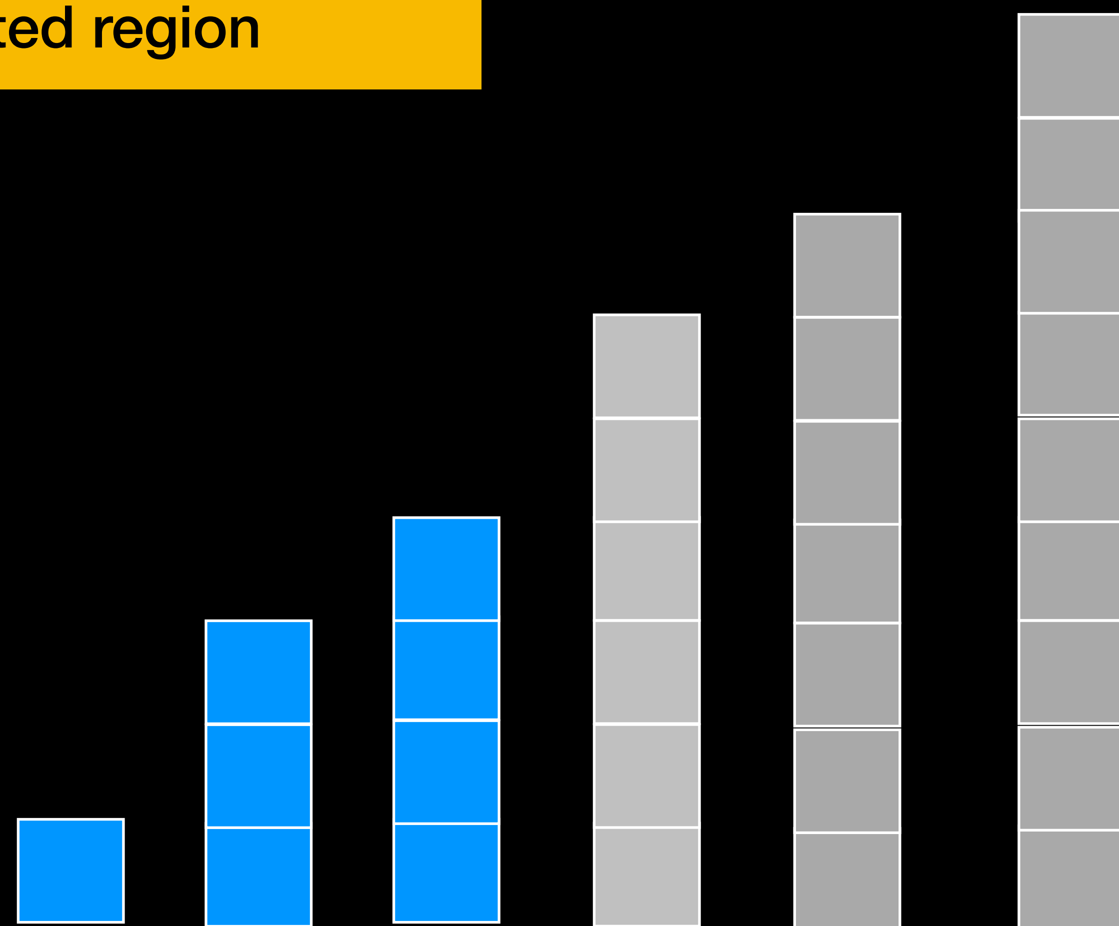


Insertion Sort

Unsorted

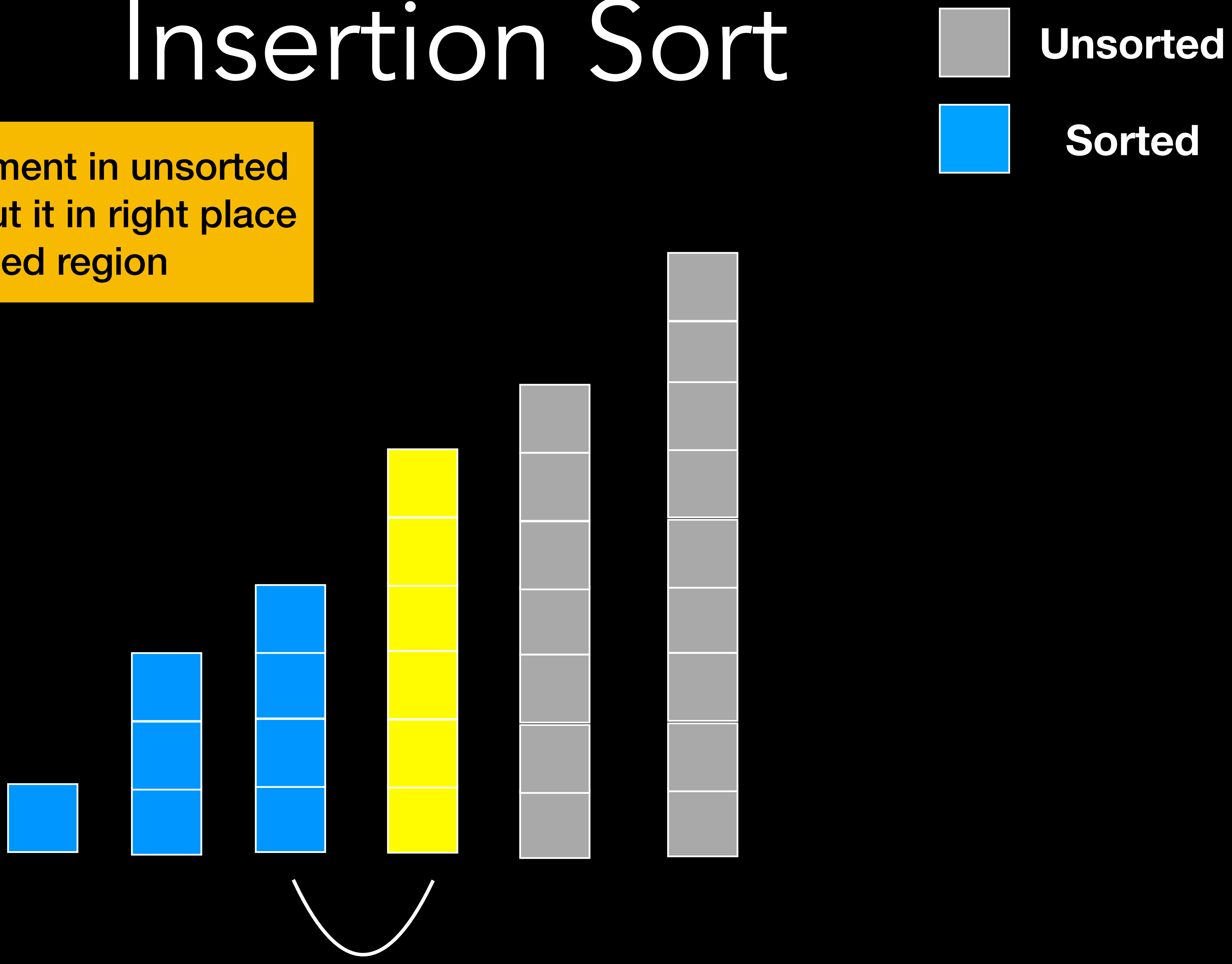
Sorted

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

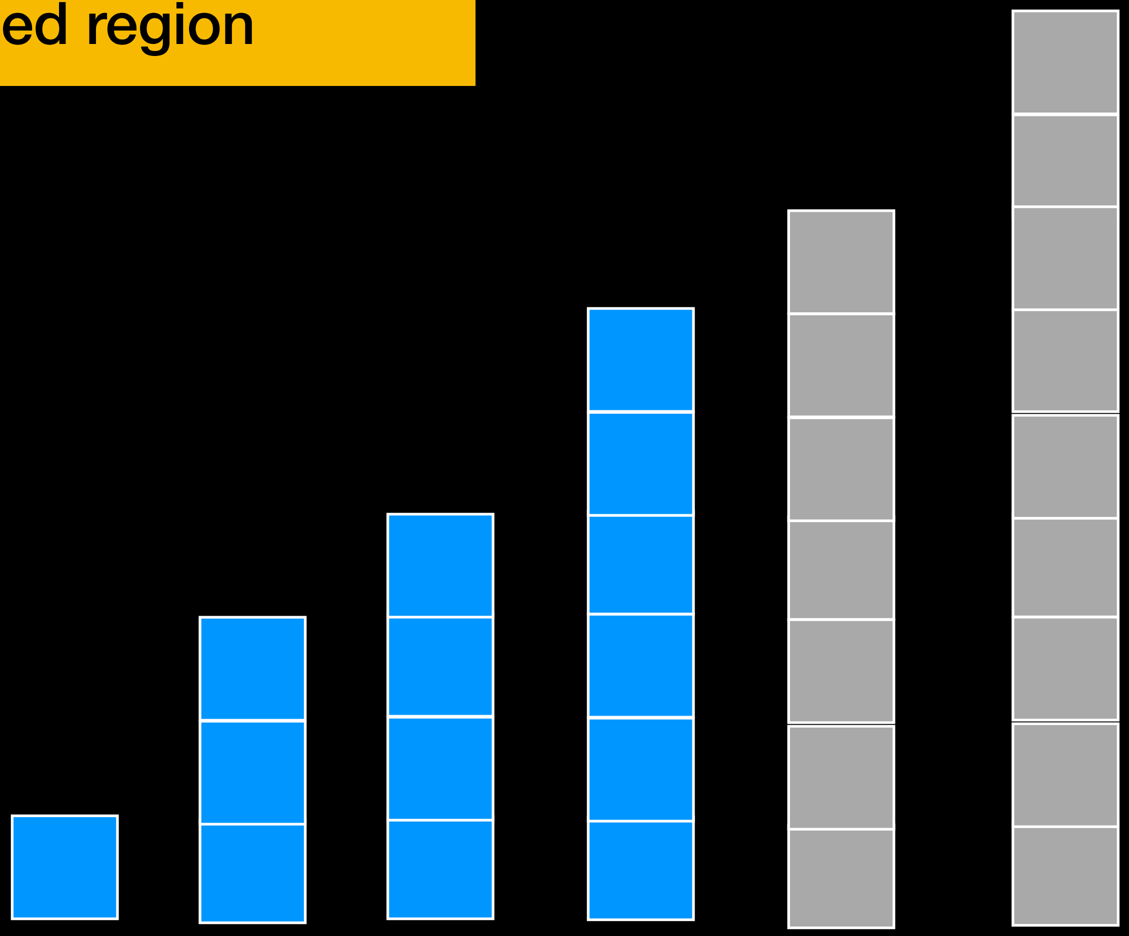


Insertion Sort

Unsorted

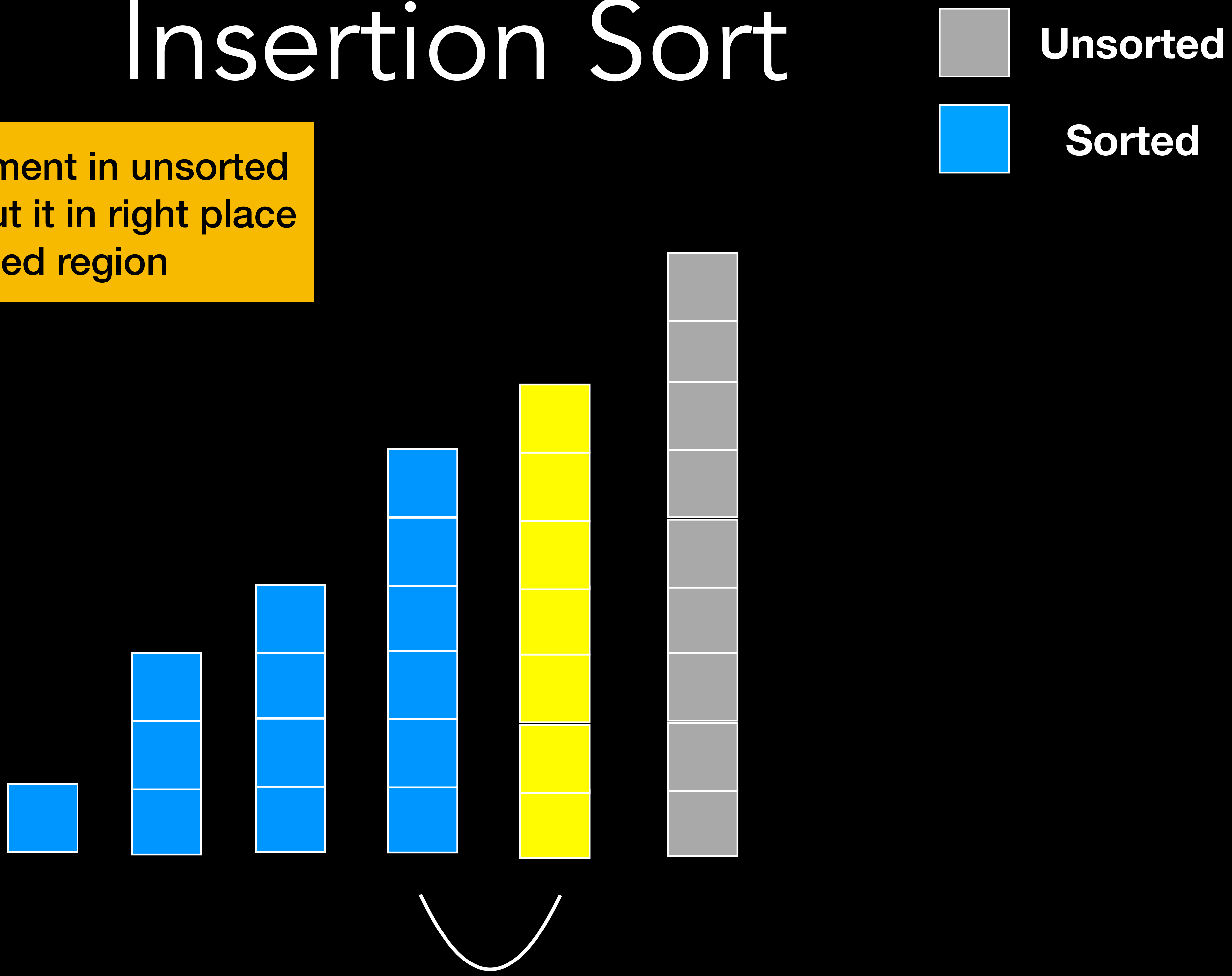
Sorted

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort

Pick first element in unsorted region and put it in right place in sorted region

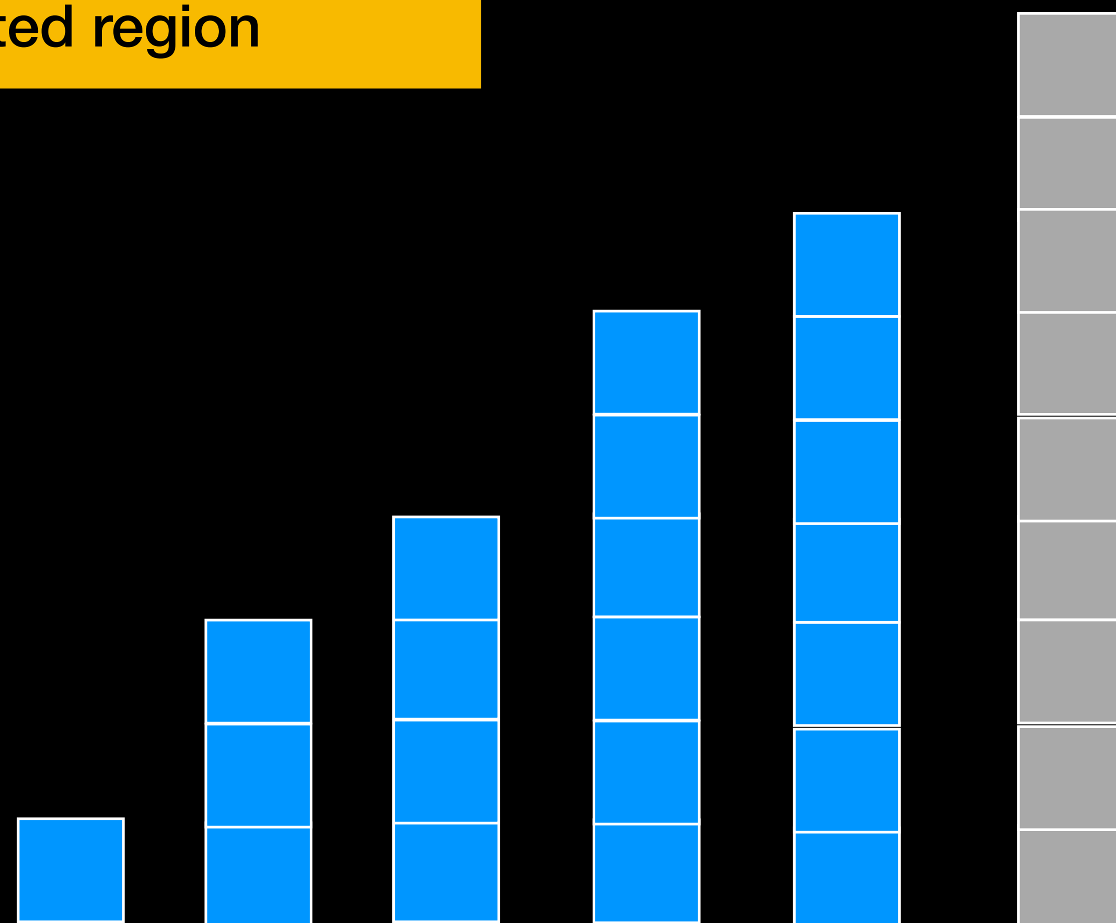


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

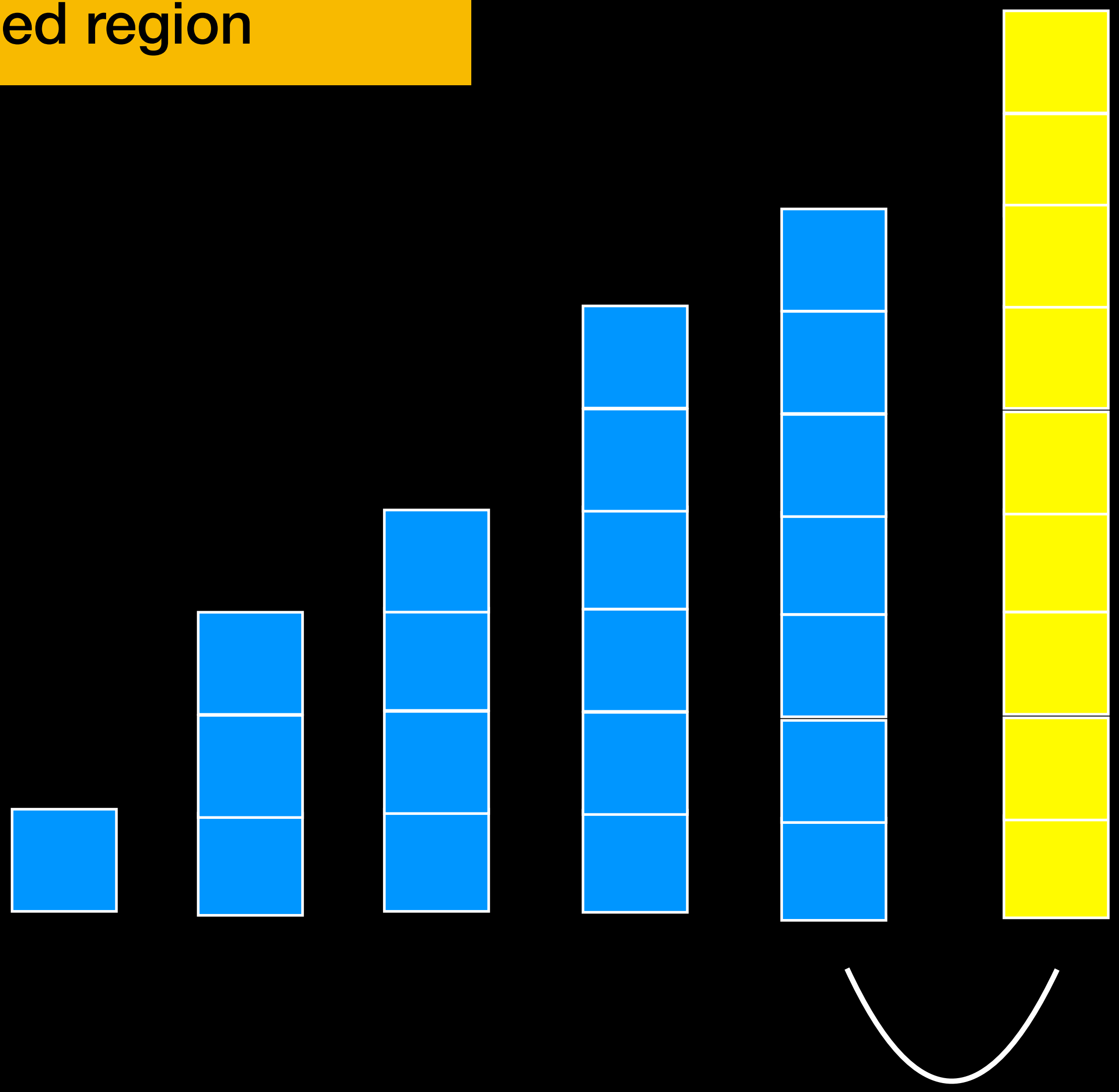


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region

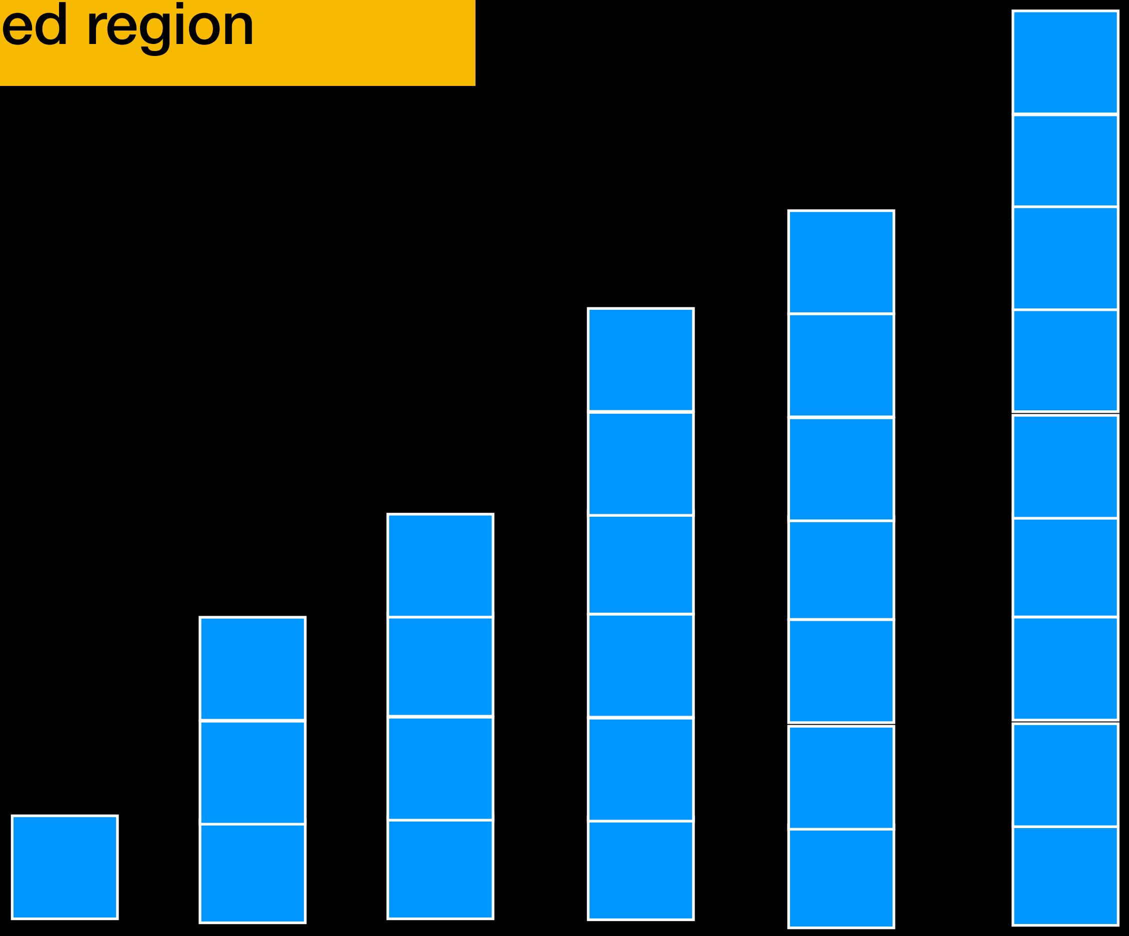


Insertion Sort

Unsorted

Sorted

Pick first element in unsorted region and put it in right place in sorted region



Insertion Sort Analysis

Execution time DOES depend on initial arrangement of data

Worst case: $O(n^2)$ comparisons and data moves

Best case: $O(n)$ comparisons and data moves

Stable

If array is already sorted Insertion sort will do only n comparisons and no swaps => good choice for small n and data likely somewhat sorted

Raise your hand if you had Insertion Sort

What we have so far

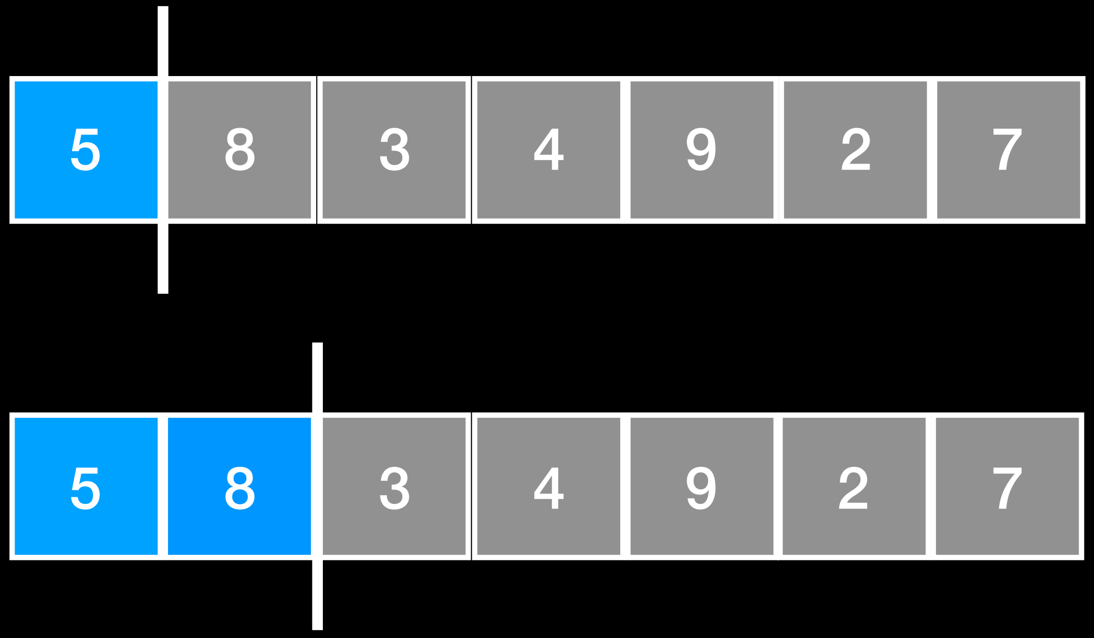
	Worst Case	Best Case
Selection Sort	$O(n^2)$	$O(n^2)$
Bubble Sort	$O(n^2)$	$O(n)$
Insertion Sort	$O(n^2)$	$O(n)$

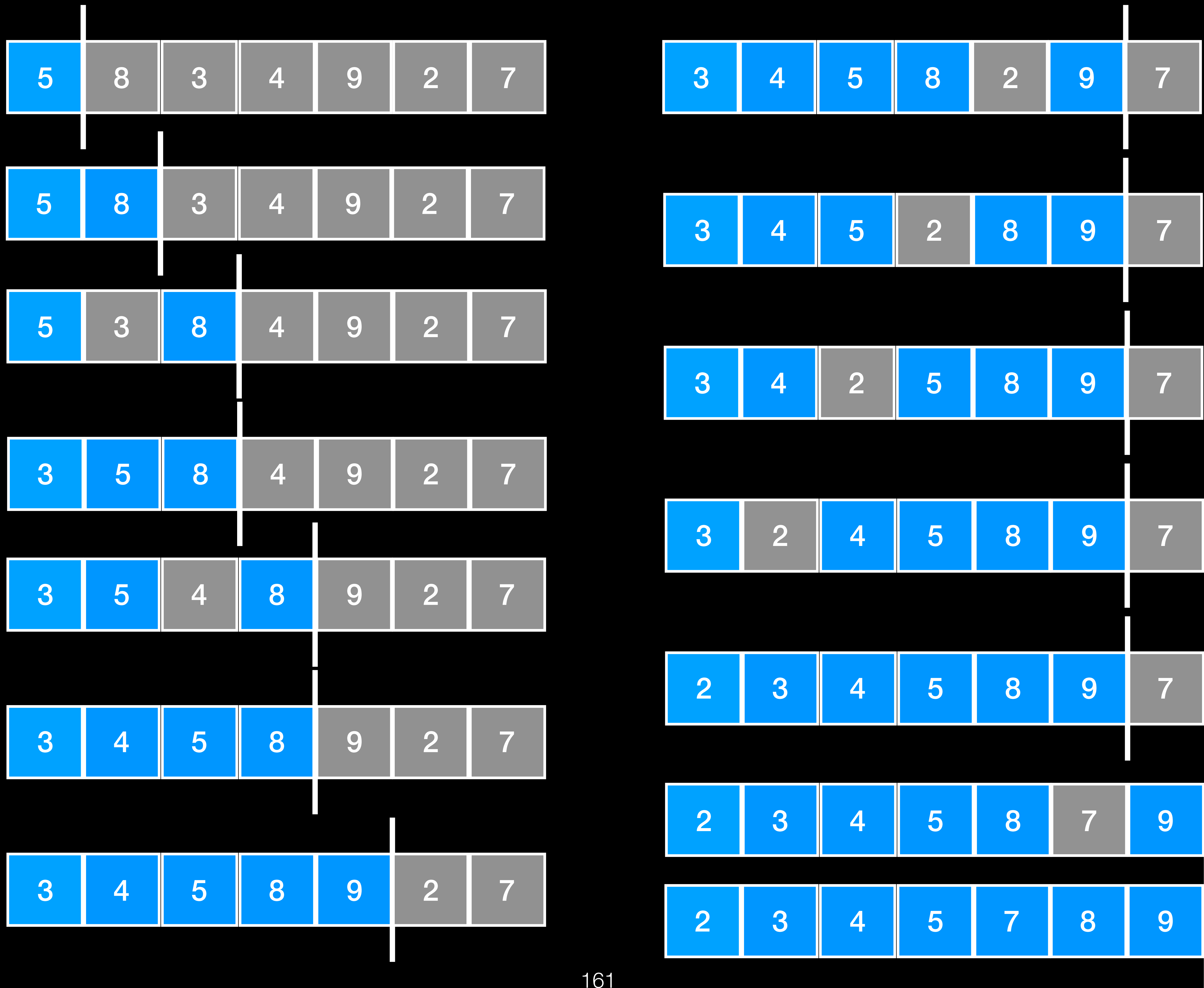
Pick first element in unsorted region and put it in right place in sorted region

Lecture Activity

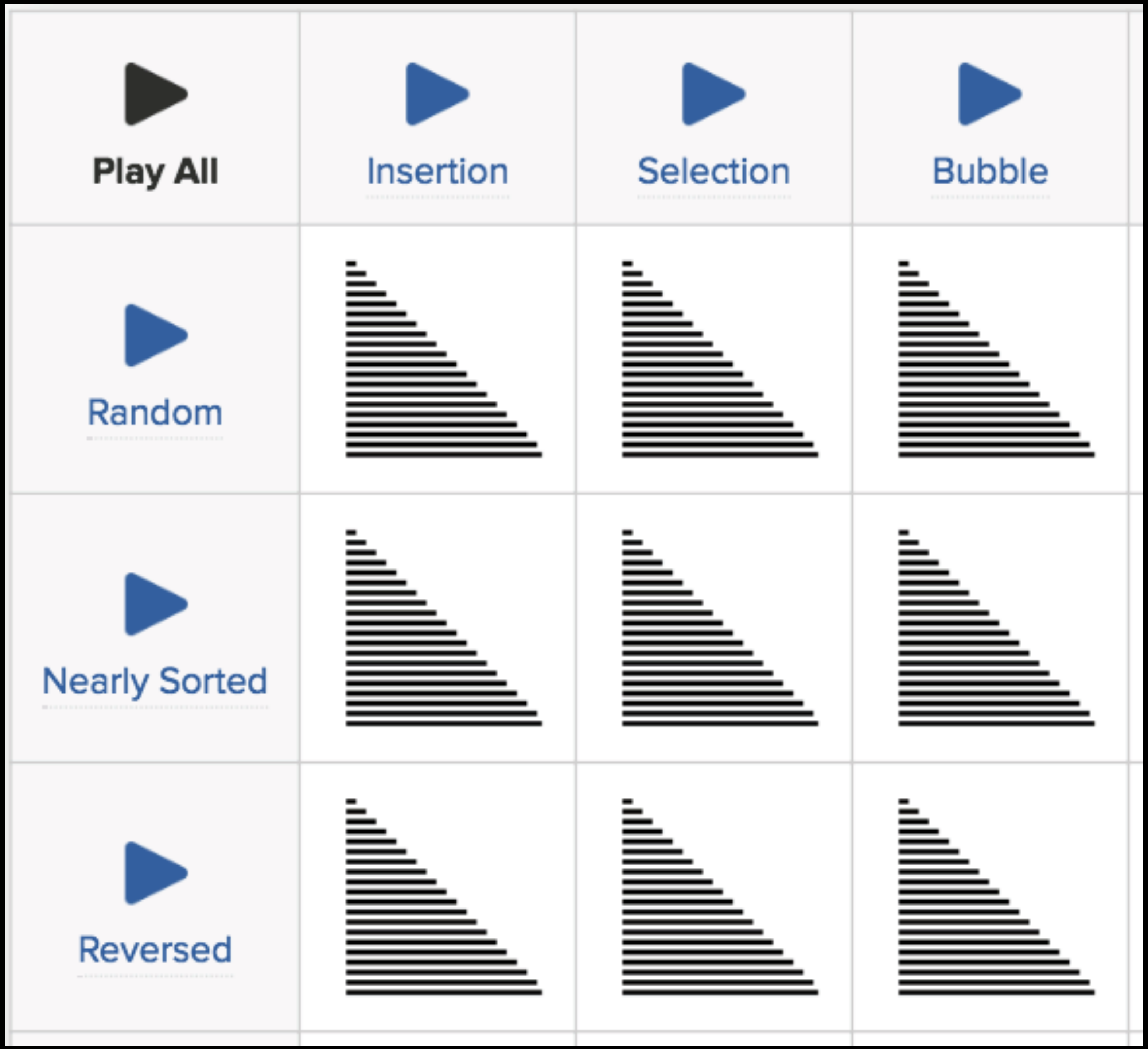
Sort the array using **Insertion Sort**

Show the entire array after each comparison/swap operation and at each step mark clearly the division between the sorted and unsorted portions of the array




5 8 3 4 9 2 7
5 8 3 4 9 2 7
5 3 8 4 9 2 7
3 5 8 4 9 2 7
3 5 4 8 9 2 7
3 4 5 8 9 2 7
3 4 5 8 9 2 7
3 4 5 8 2 9 7
3 4 5 2 8 9 7
3 4 2 5 8 9 7
3 2 4 5 8 9 7
2 3 4 5 8 9 7
2 3 4 5 8 7 9
2 3 4 5 7 8 9

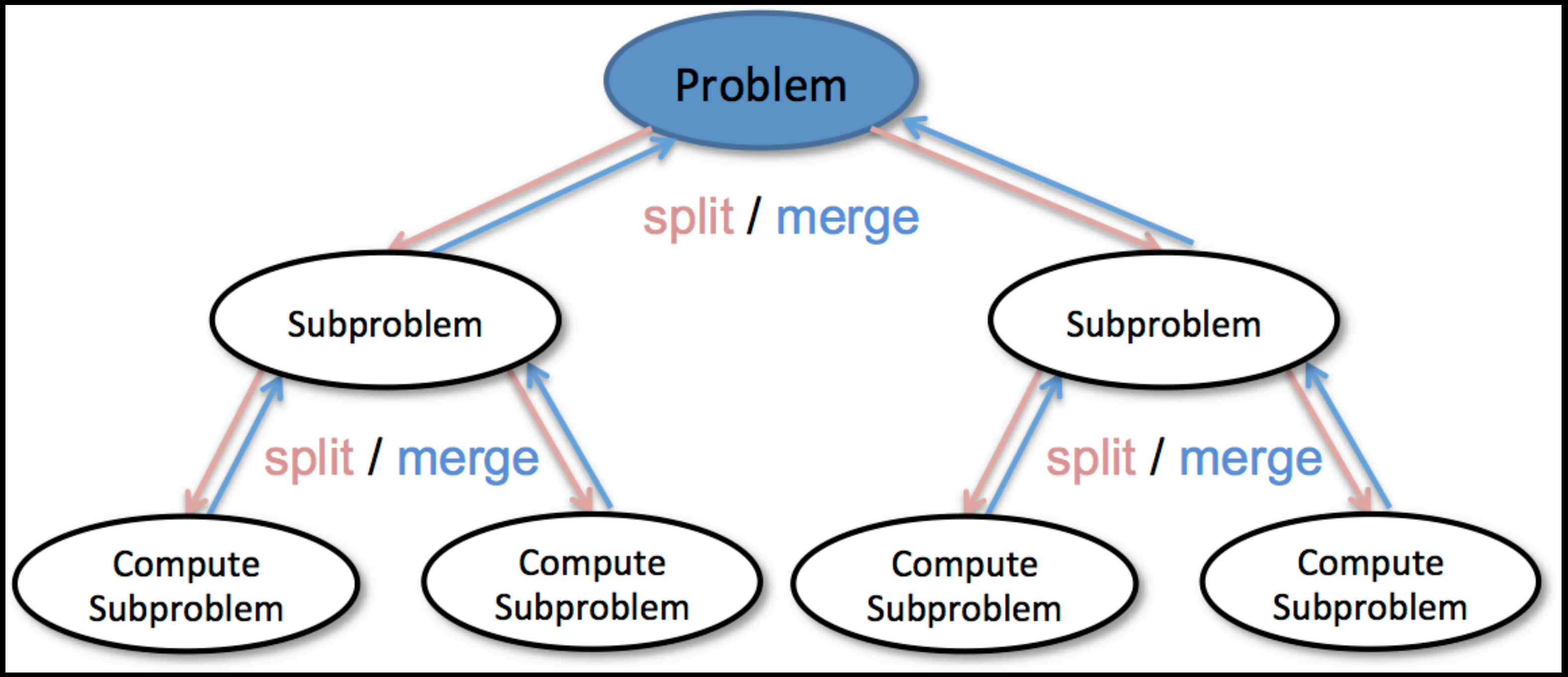
https://www.toptal.com/developers/sorting-algorithms



Can we do better?

Can we do better?

Divide and Conquer!!!



Merge Sort

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

195	599	158	2	260	11	64	932	5

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

$T(^1/_2 n)$

195	599	158	2	260	11	64	932	5

$T(^1/_2 n)$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

195	599	158	2	260	11	64	932	5

$T(^1/_2 n)$ **$T(^1/_2 n)$**

$$(n/2)^2 = n^2/4$$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

100	14	3	43	200	274	523	108	76

195	599	158	2	260	11	64	932	5

$T(\frac{1}{2}n) \approx \frac{1}{4}\, T(n)$ $T(\frac{1}{2}n) \approx \frac{1}{4}\, T(n)$

$(n/2)^2 = n^2/4$

Understanding $O(n^2)$

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

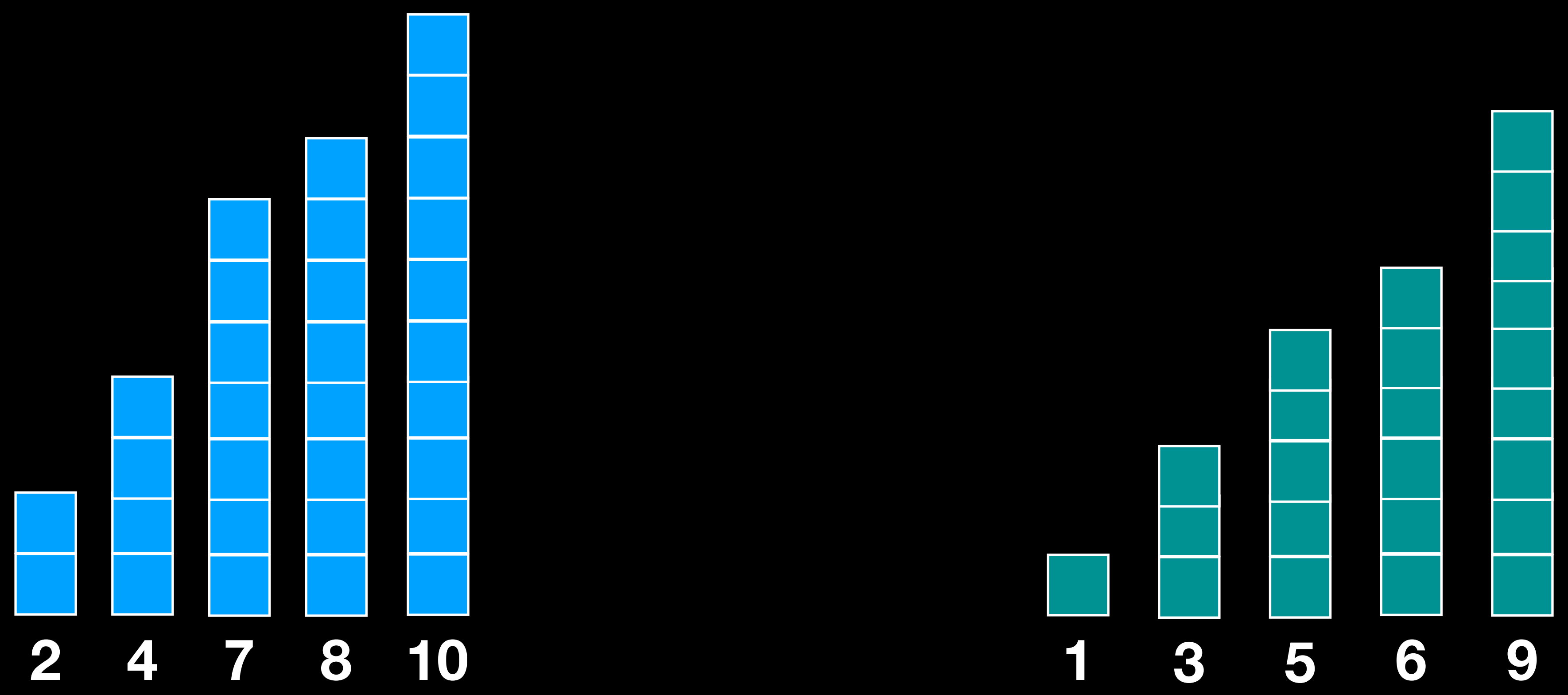
3	14	43	76	100	108	200	274	523

2	5	11	64	158	195	260	599	932

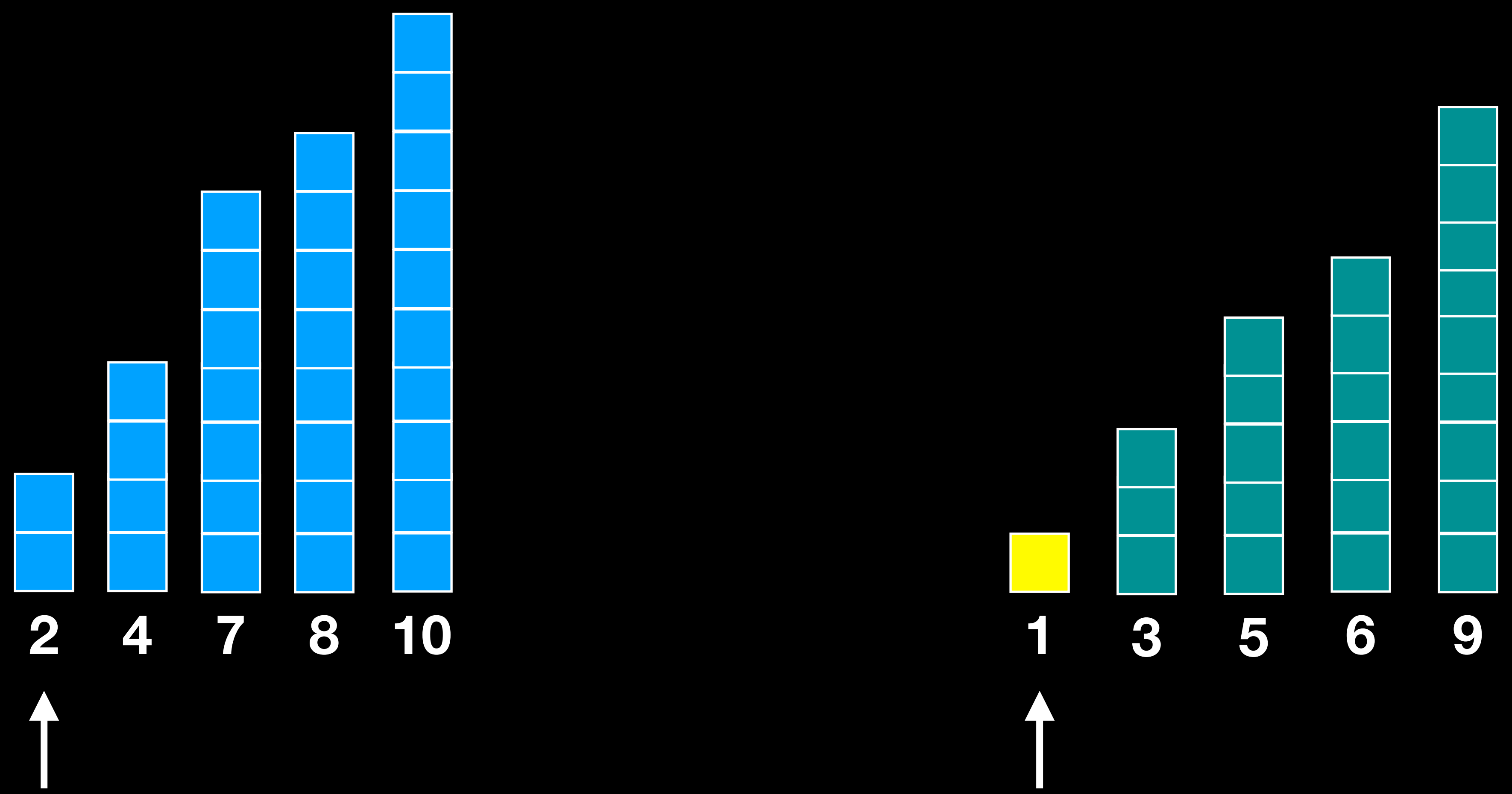
$T(^1/_2 n) \approx {}^1/_4\, T(n)$ $T(^1/_2 n) \approx {}^1/_4\, T(n)$

$(n/2)^2 = n^2/4$

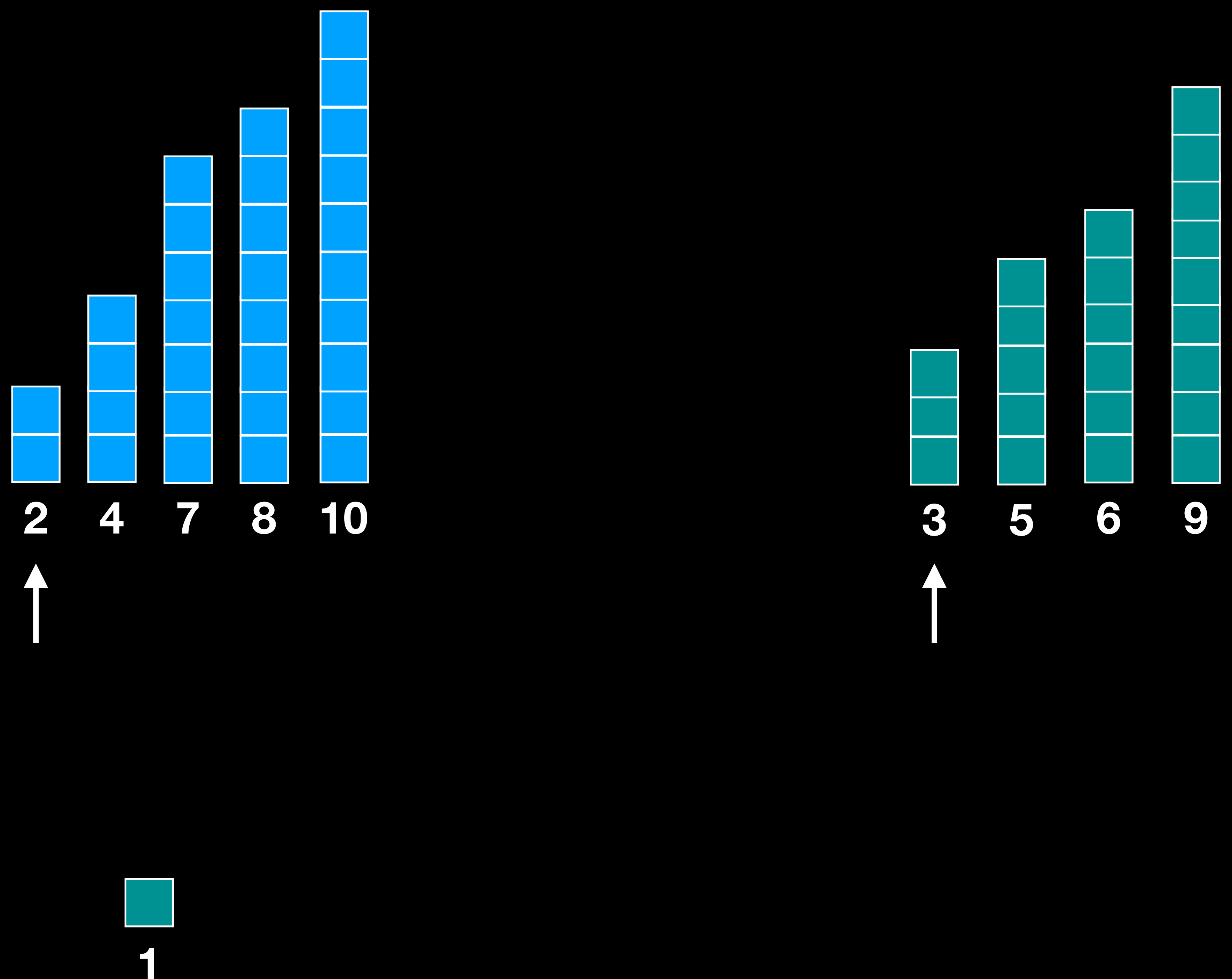
Key Insight: Merge is linear



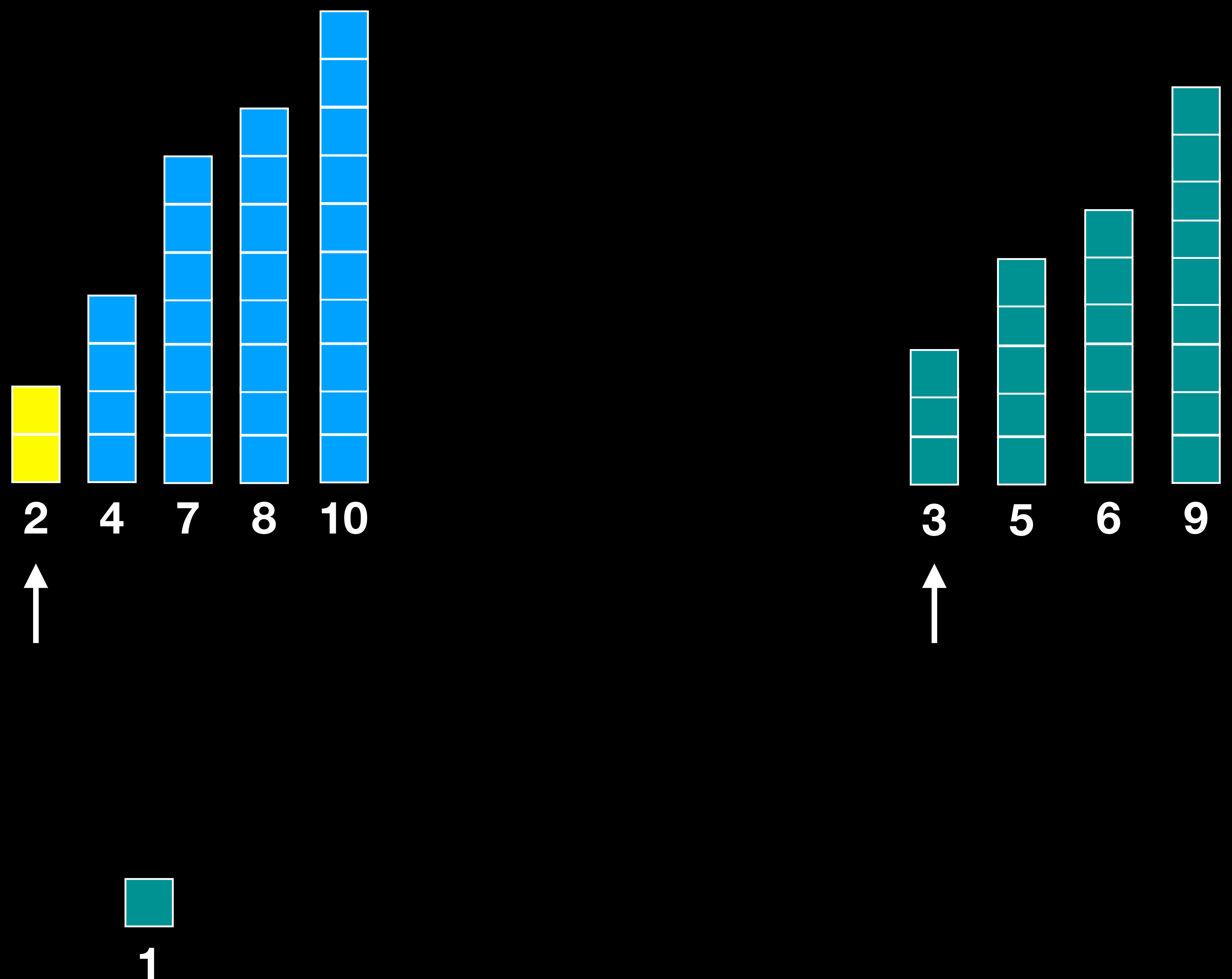
Key Insight: Merge is linear



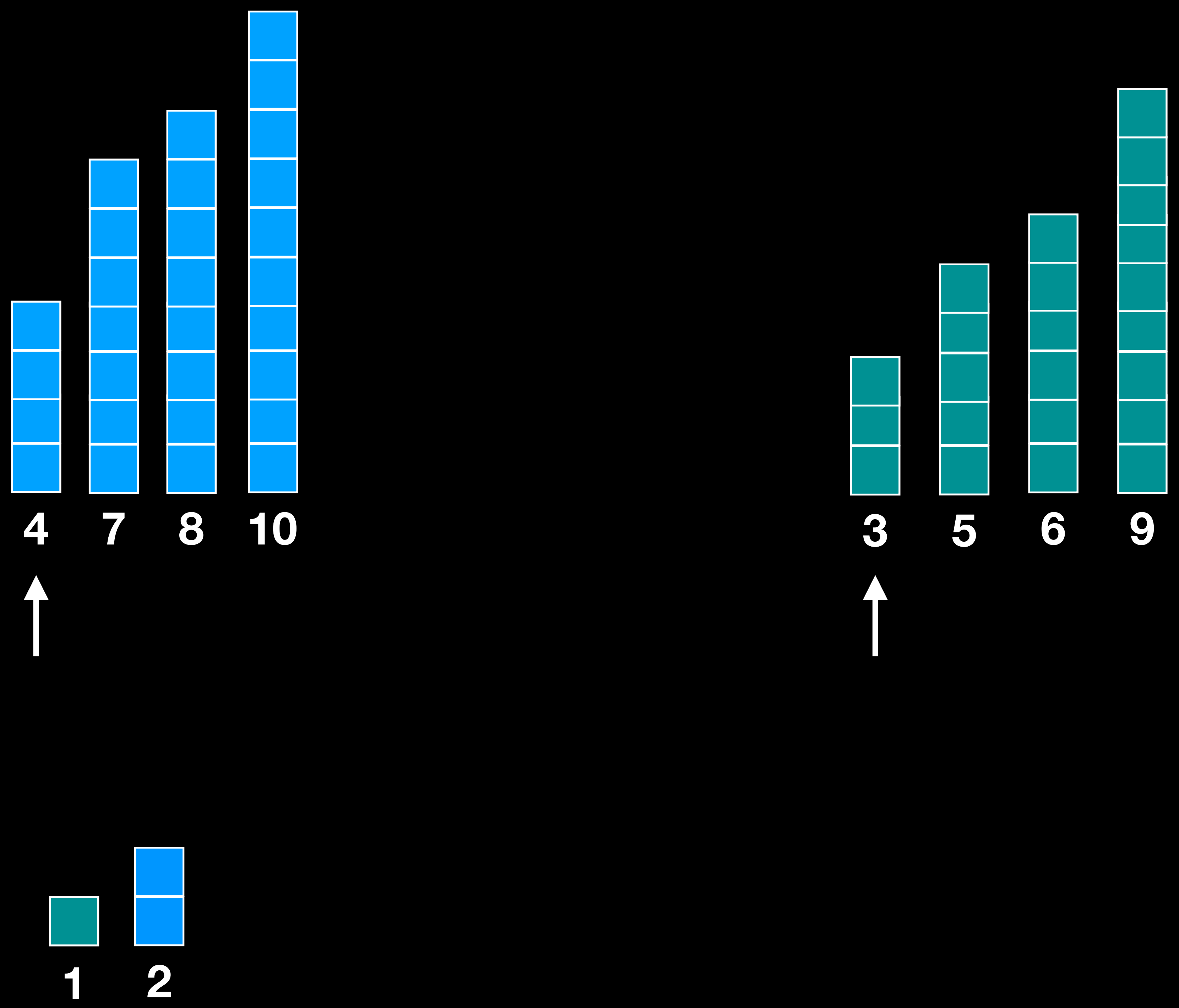
Key Insight: Merge is linear



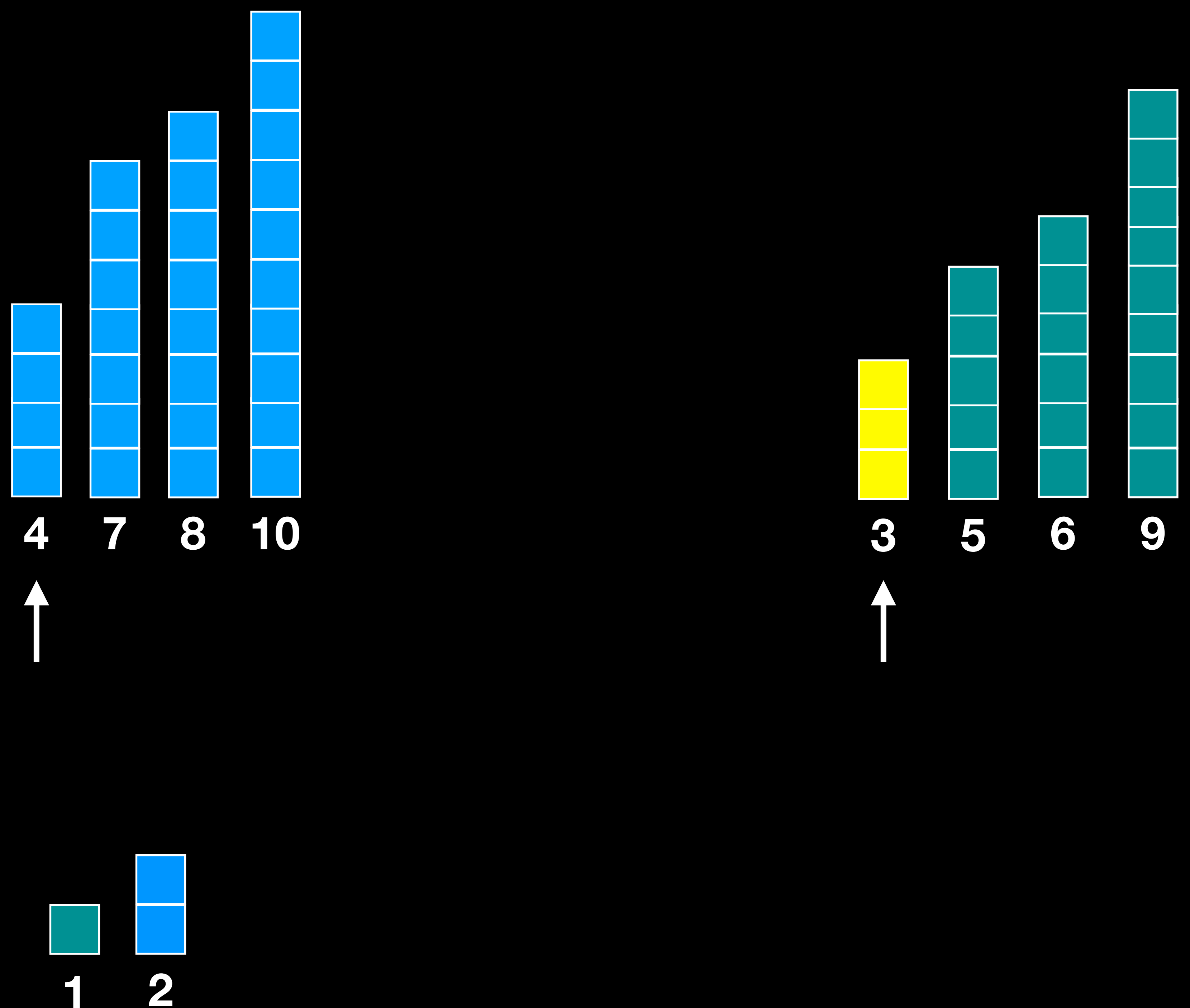
Key Insight: Merge is linear



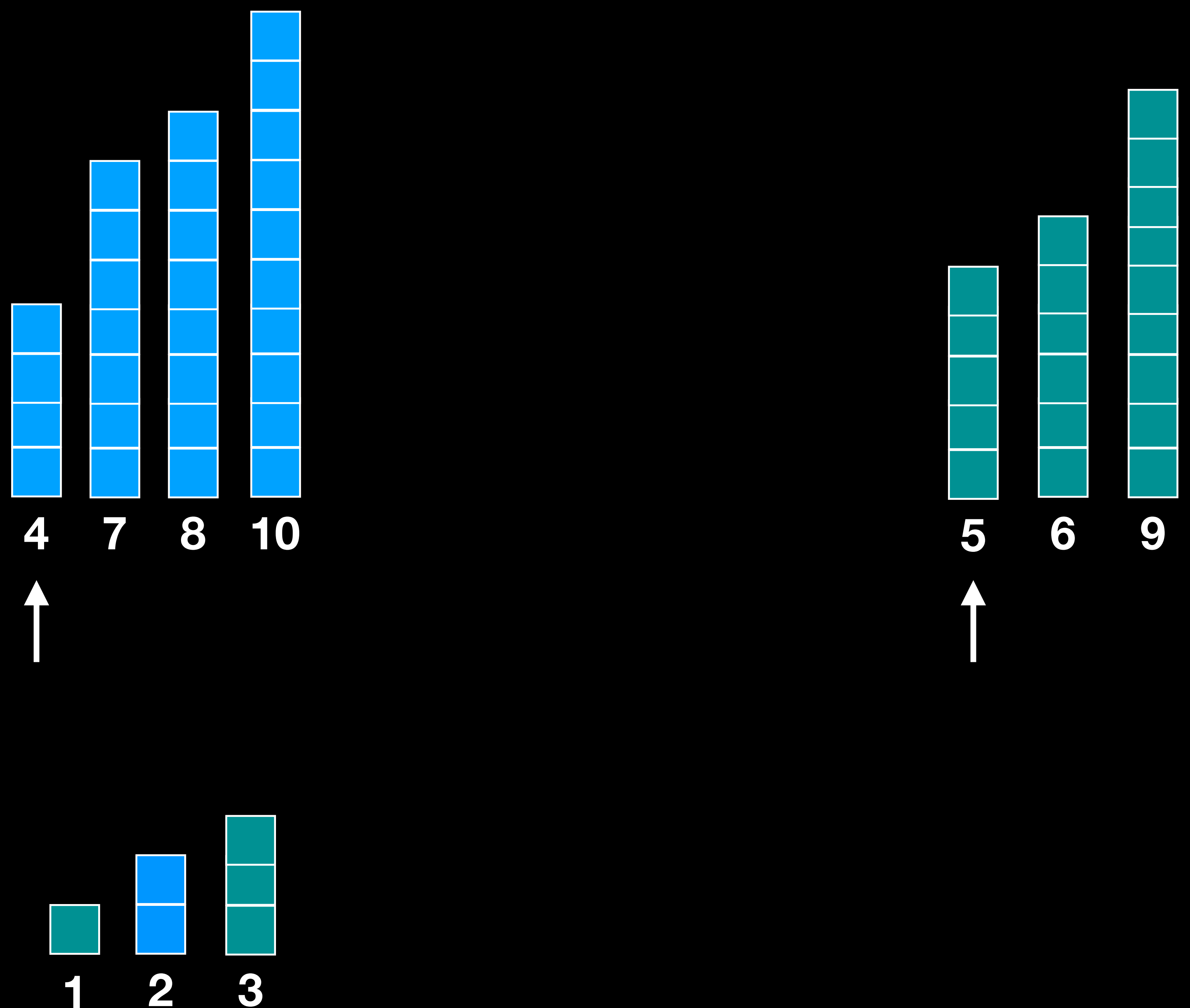
Key Insight: Merge is linear



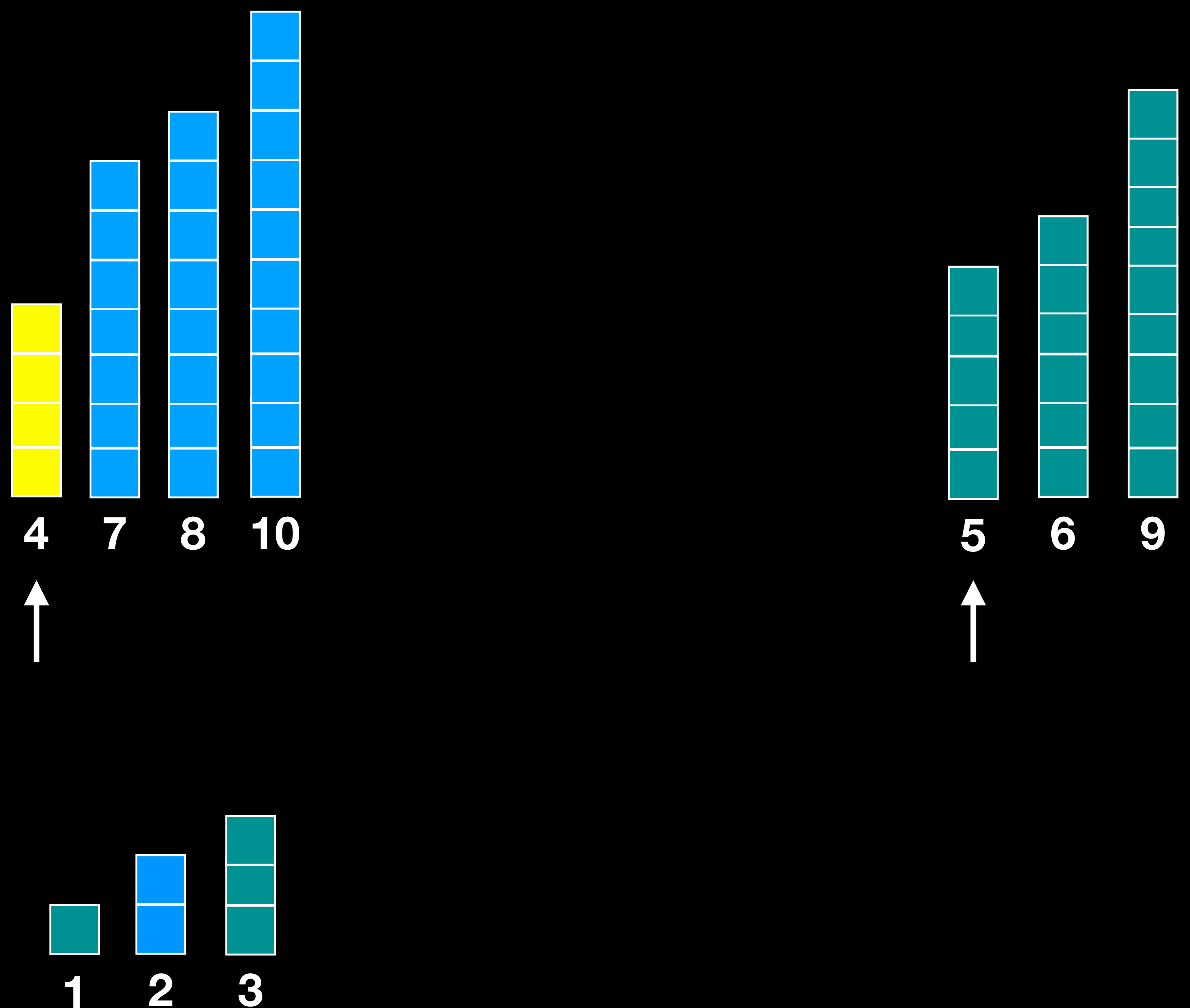
Key Insight: Merge is linear



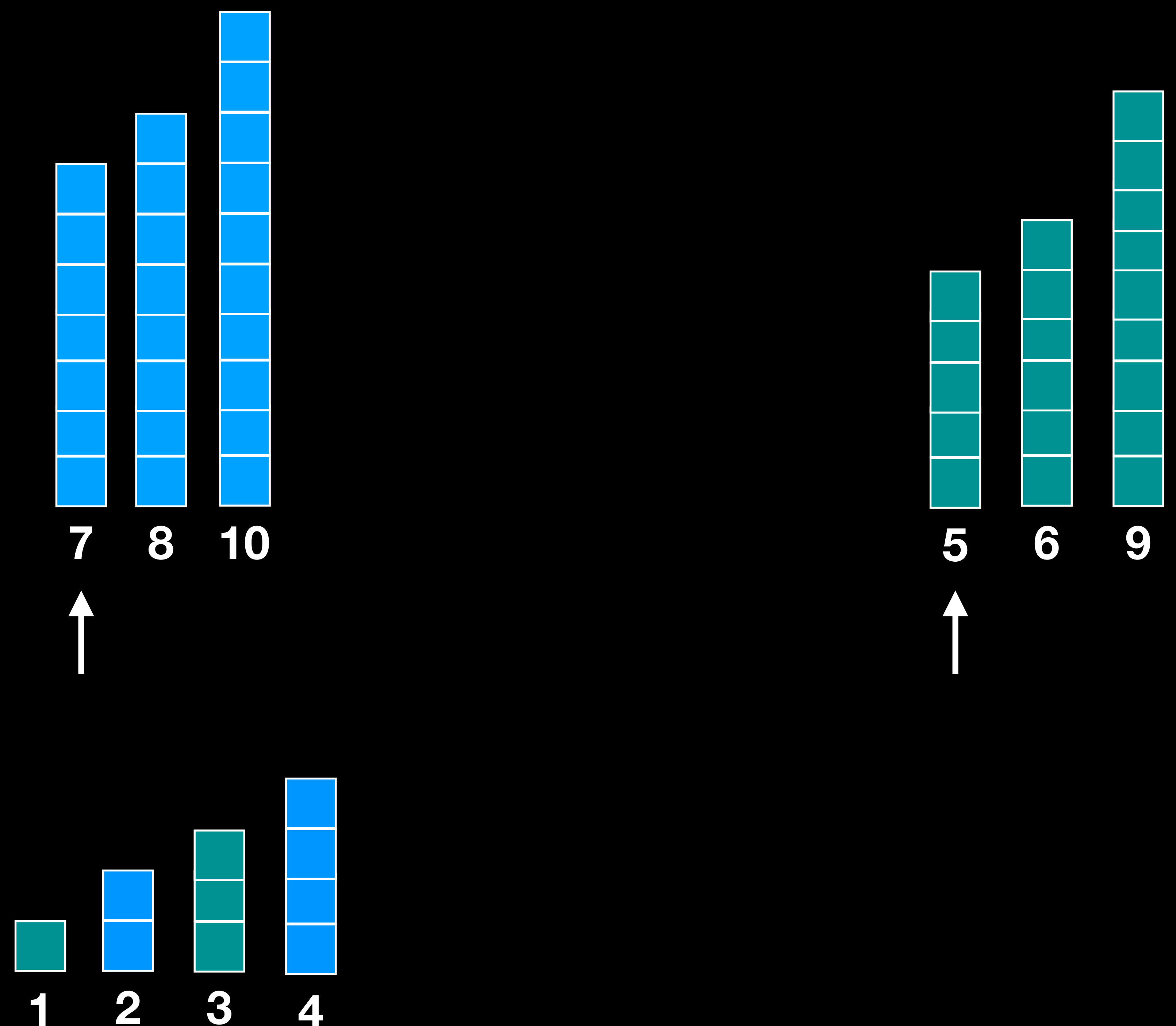
Key Insight: Merge is linear



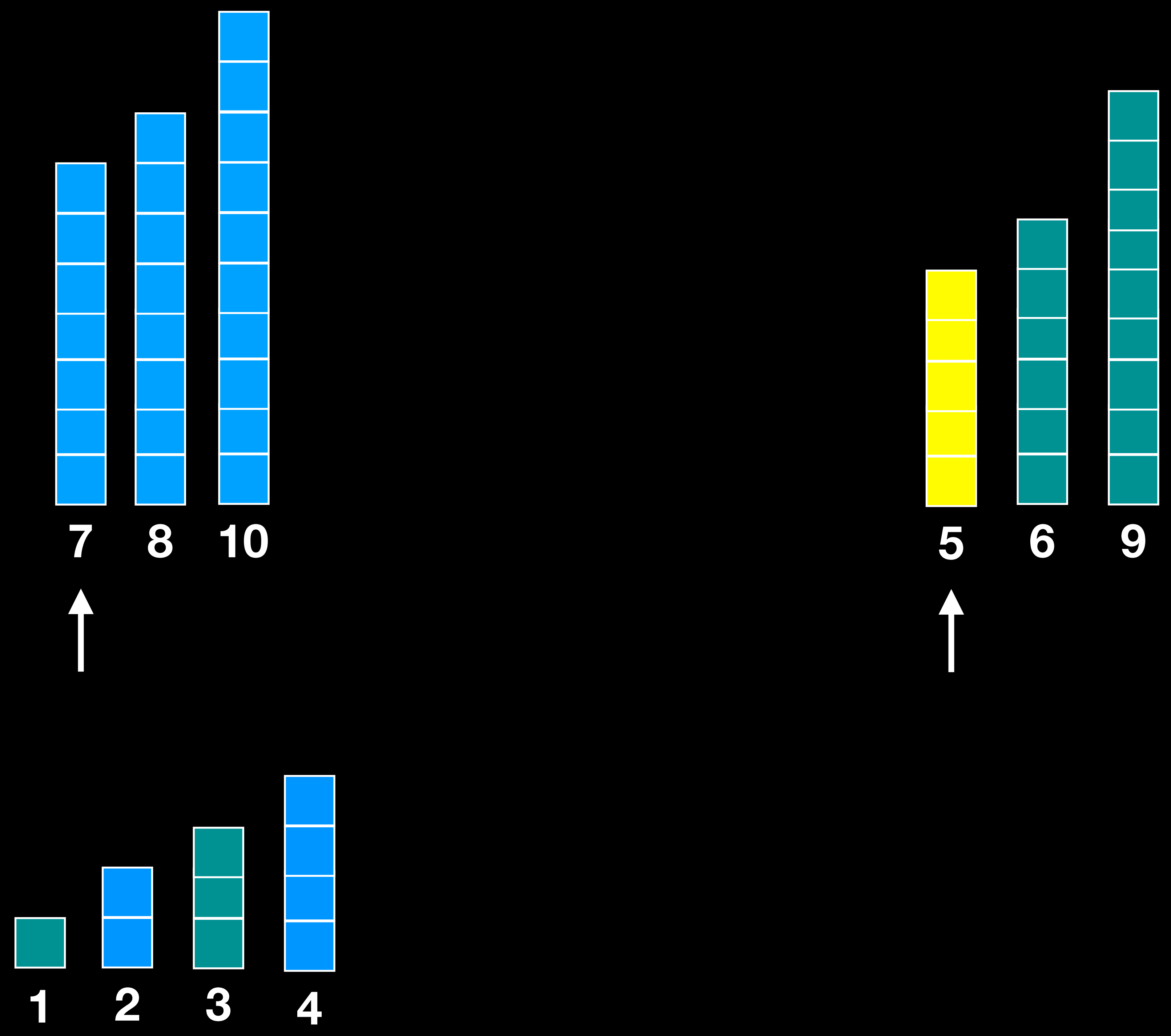
Key Insight: Merge is linear



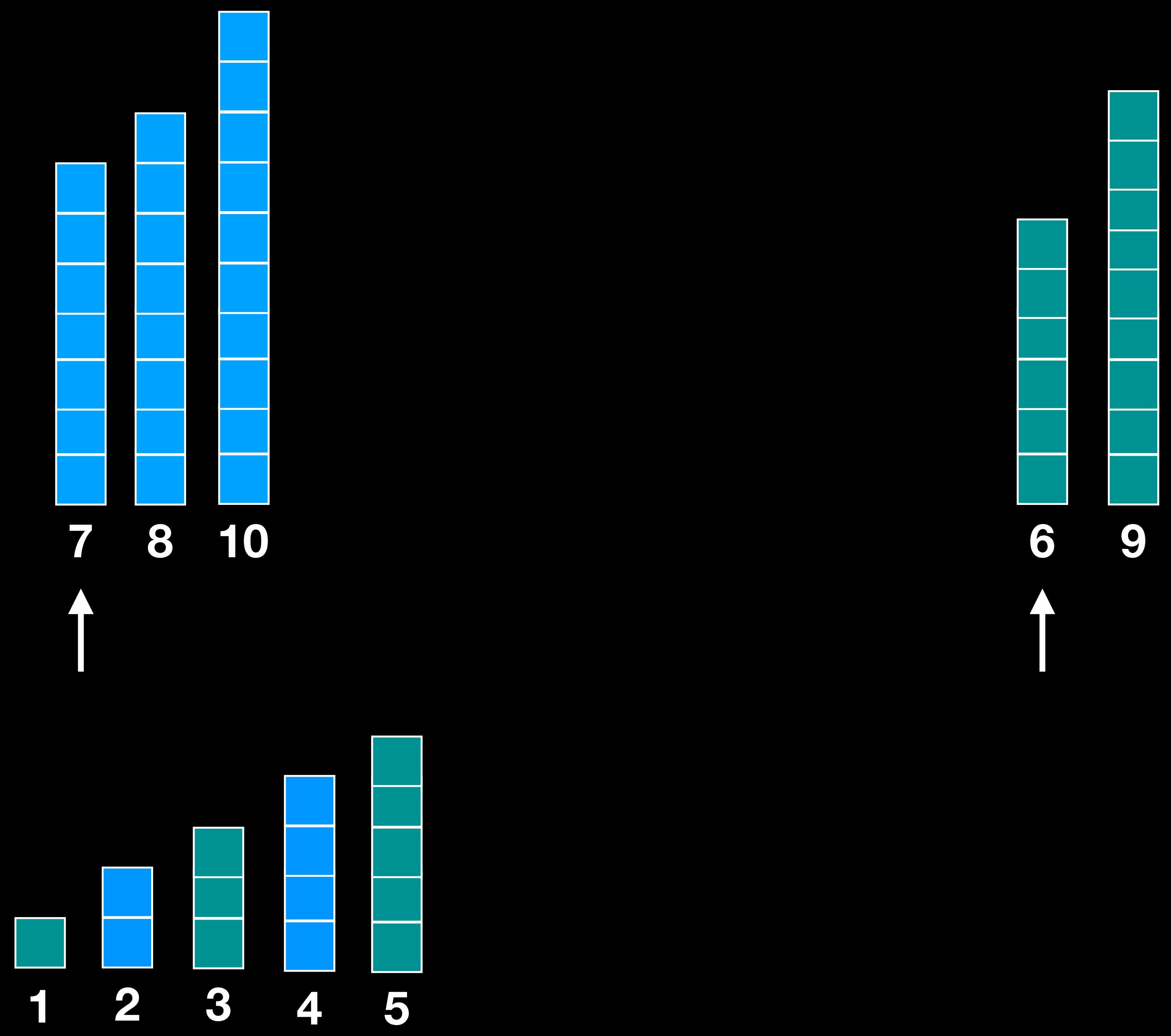
Key Insight: Merge is linear



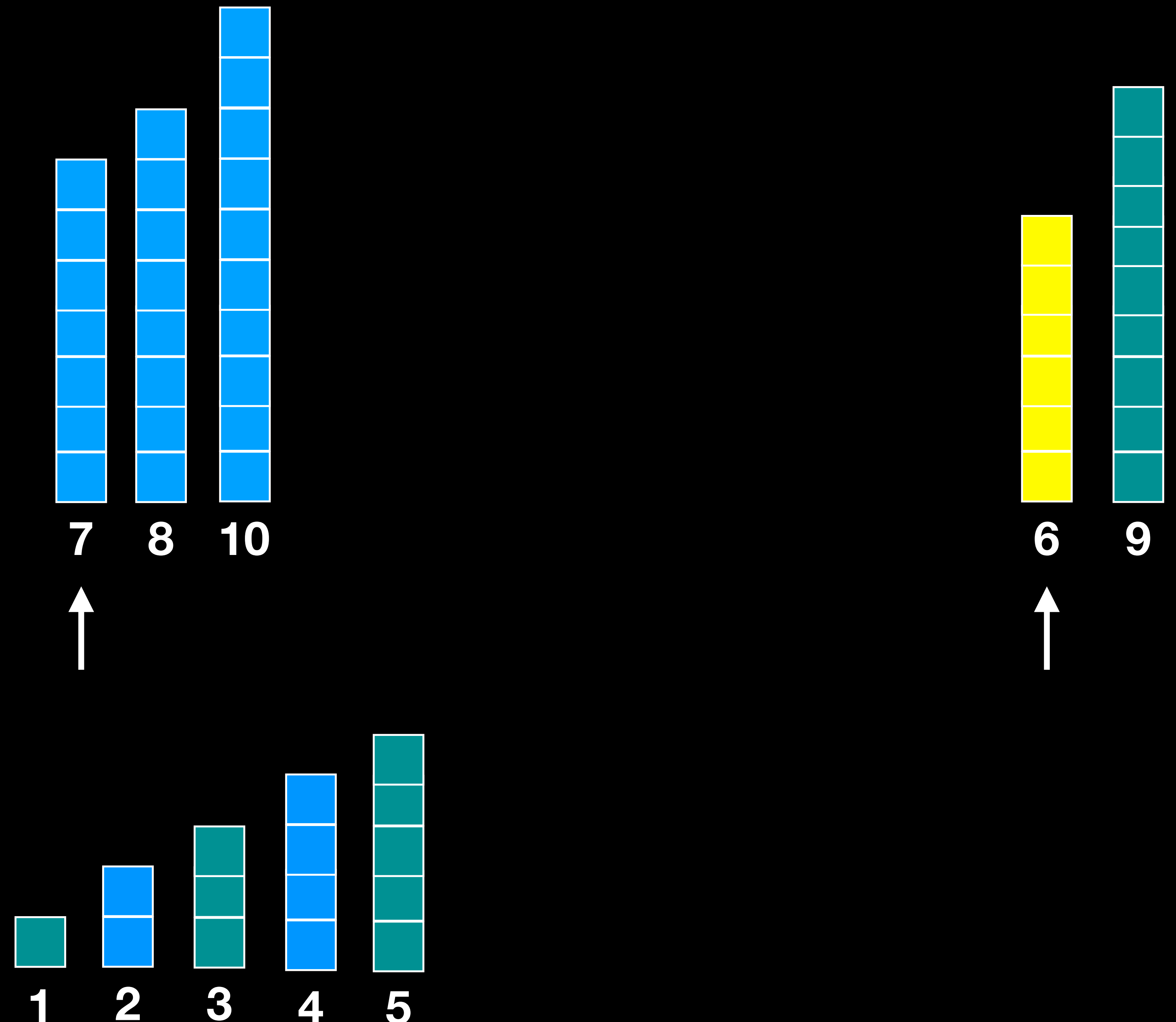
Key Insight: Merge is linear



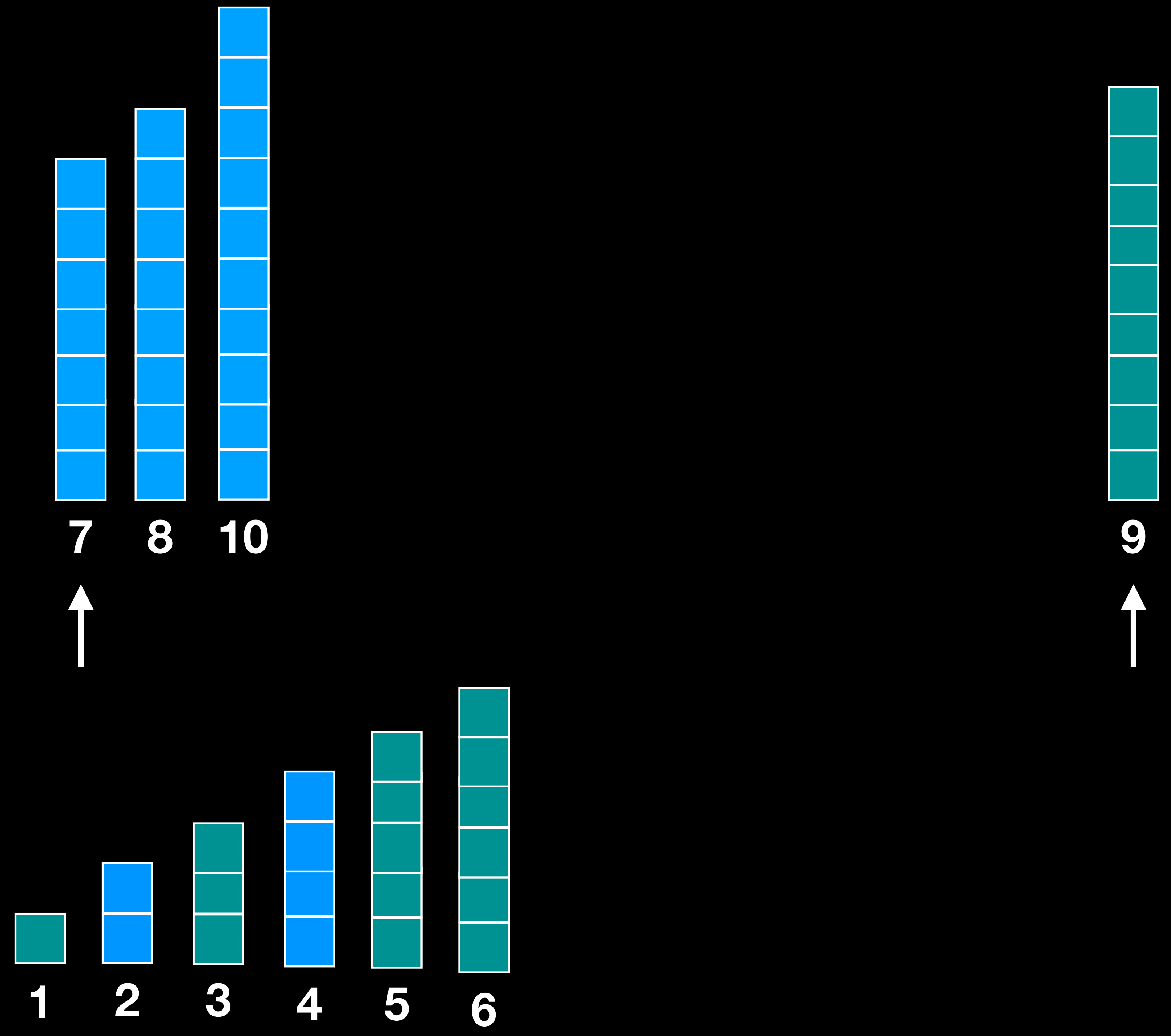
Key Insight: Merge is linear



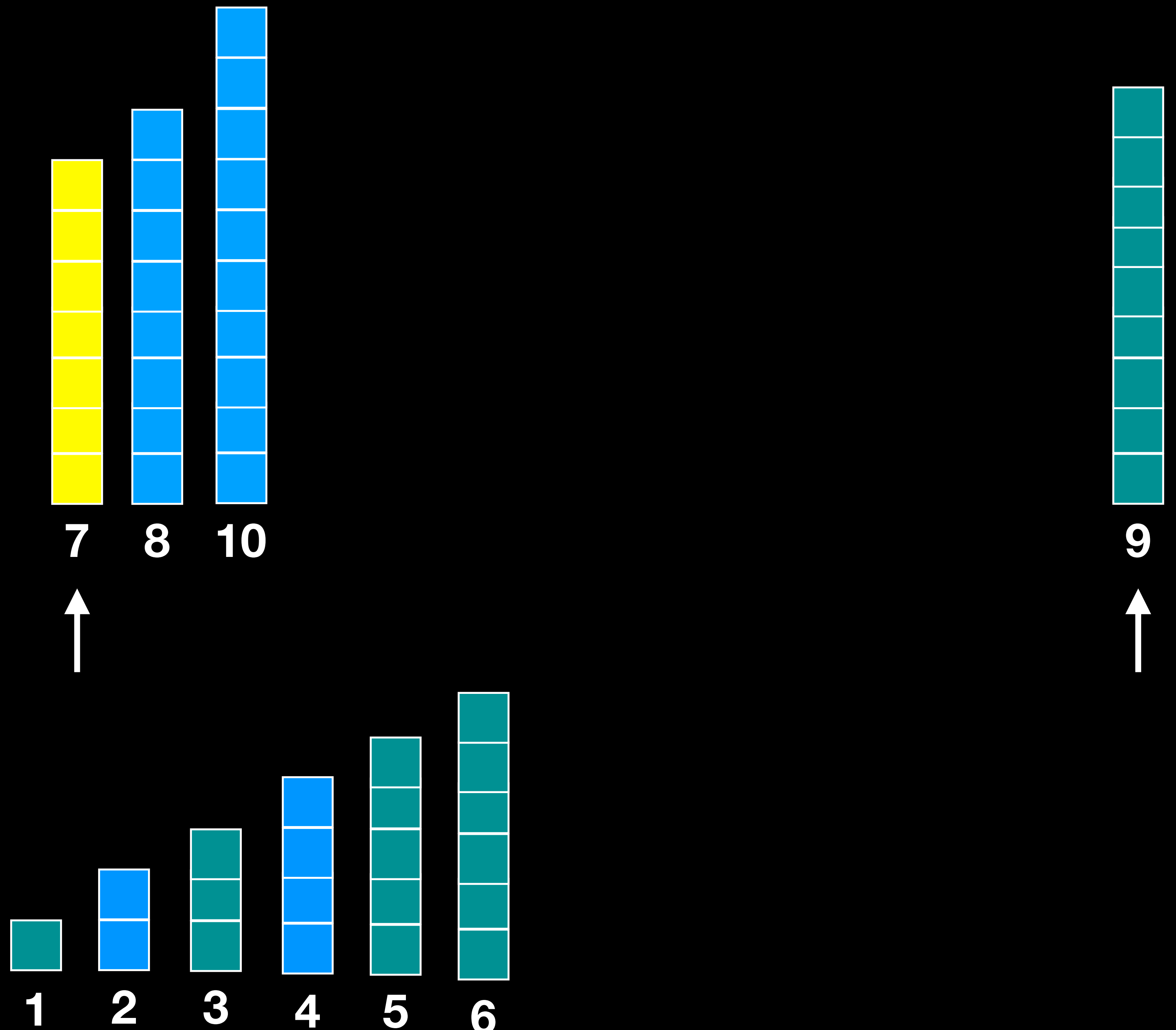
Key Insight: Merge is linear



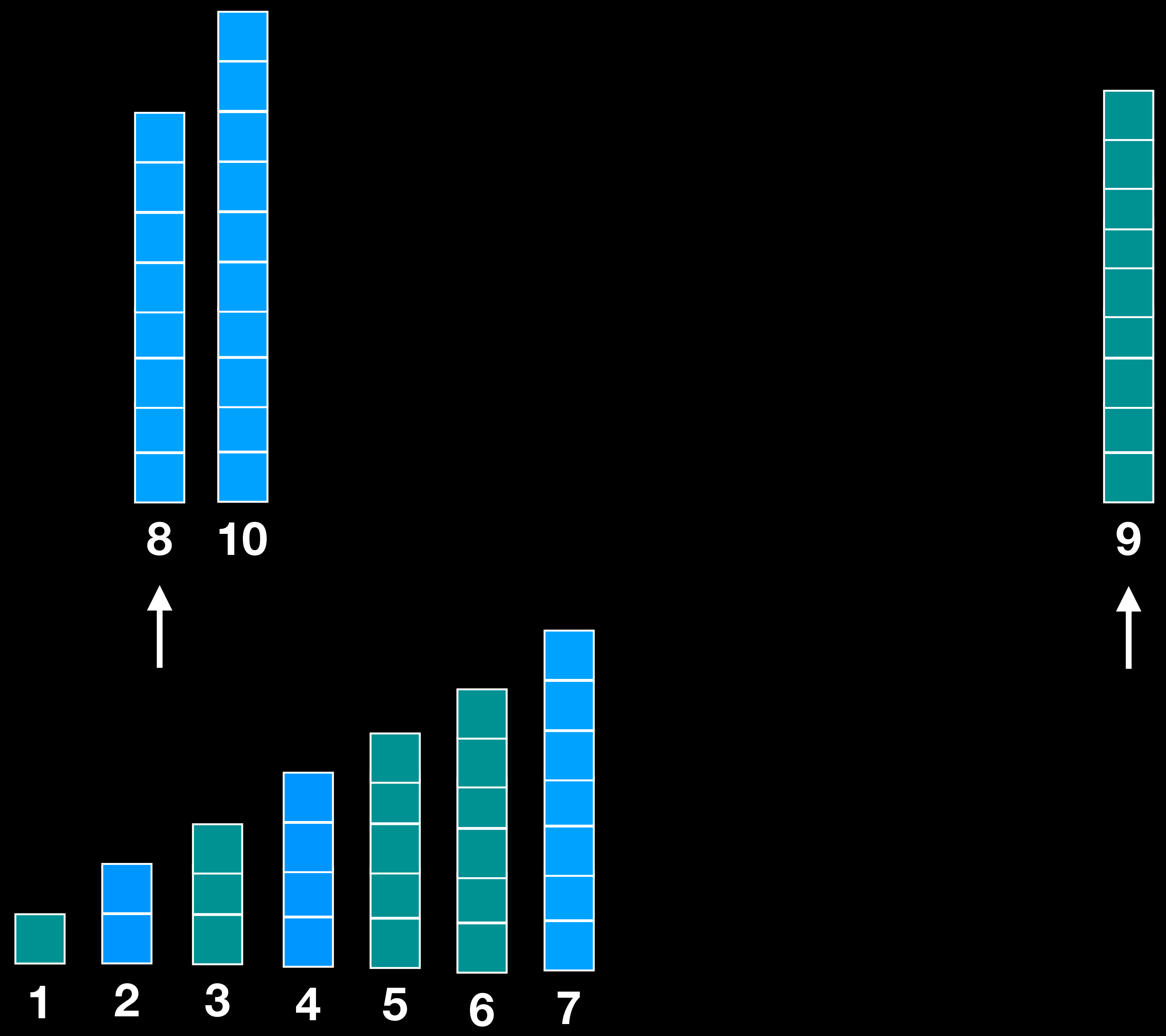
Key Insight: Merge is linear



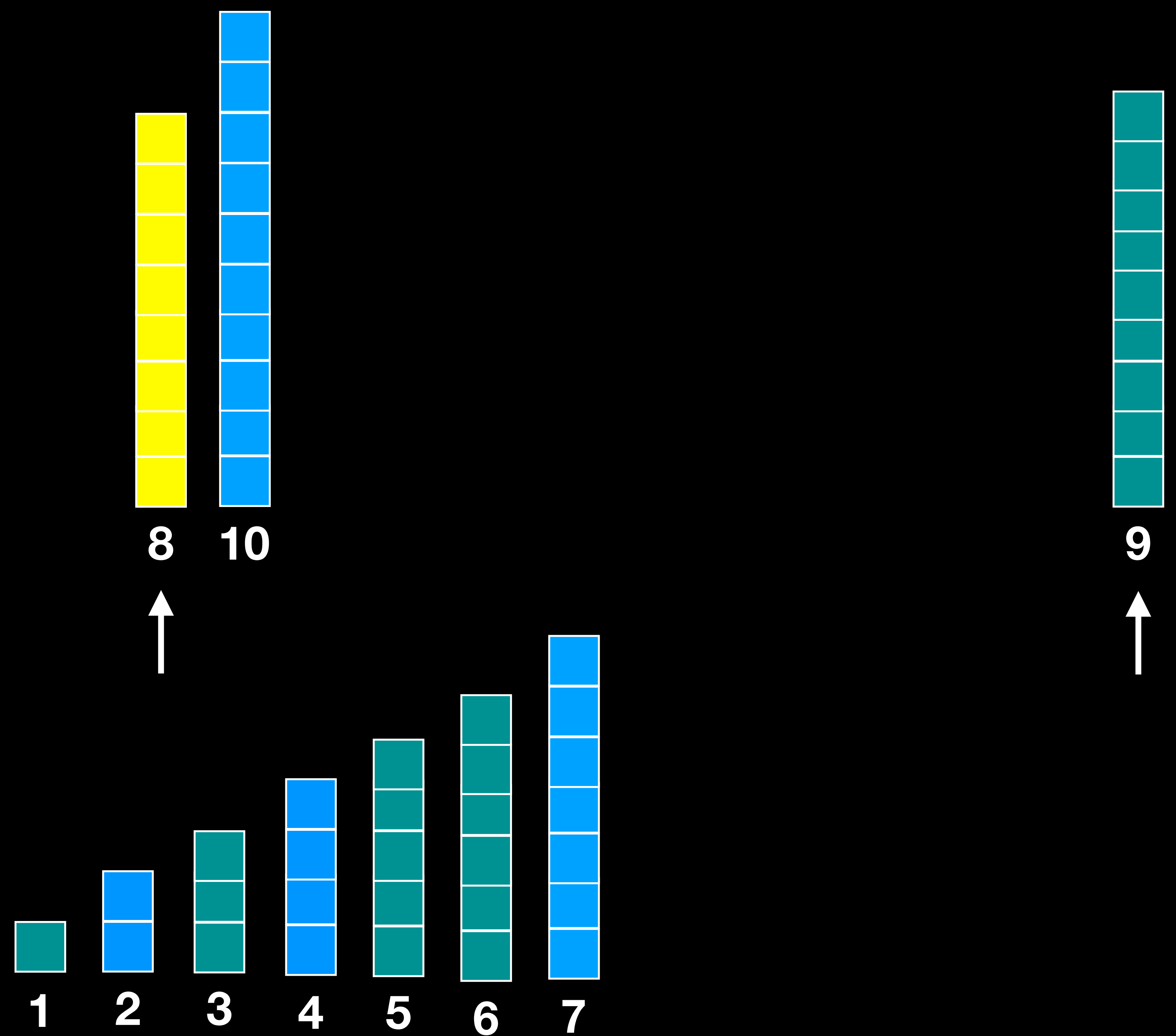
Key Insight: Merge is linear



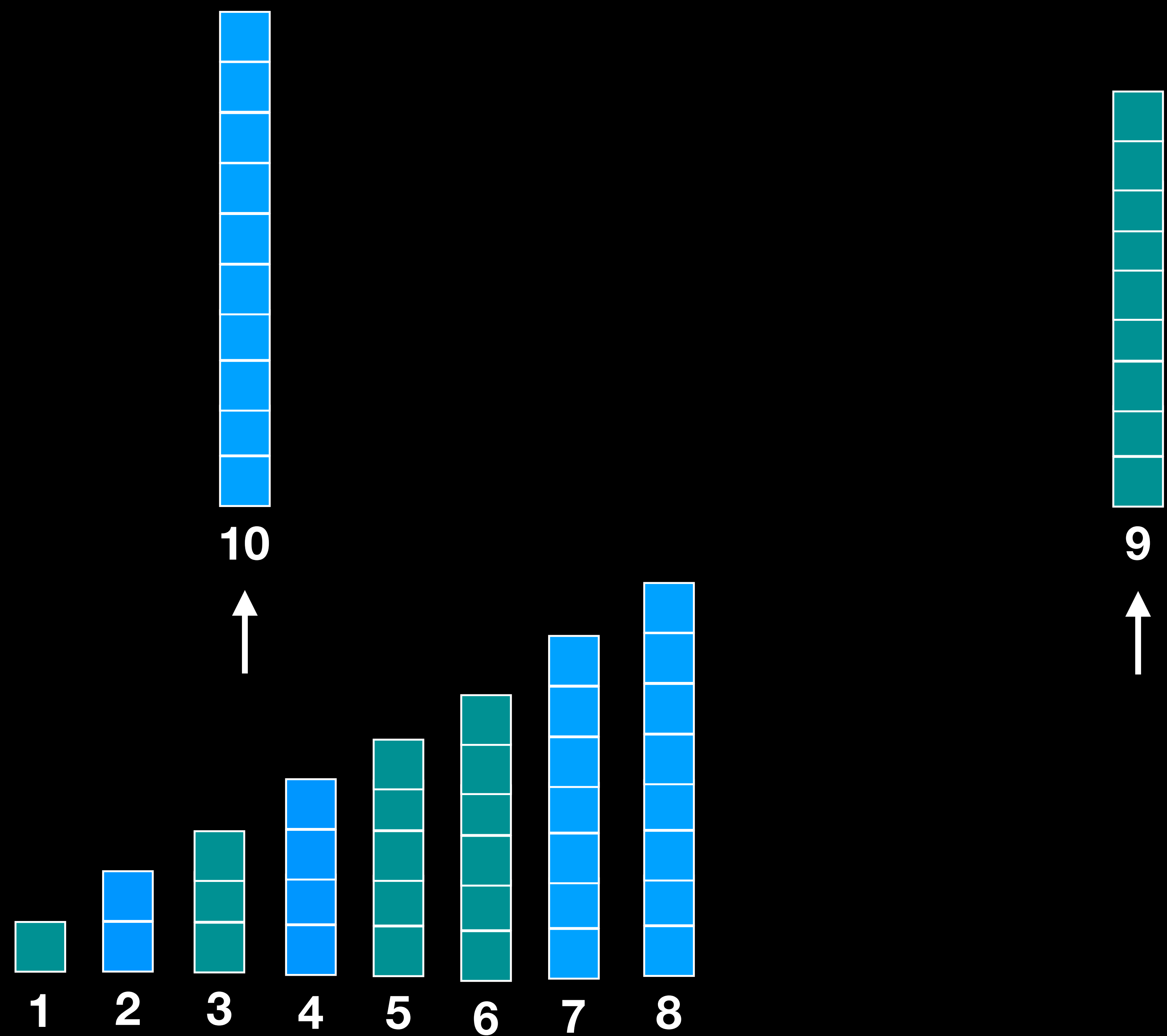
Key Insight: Merge is linear



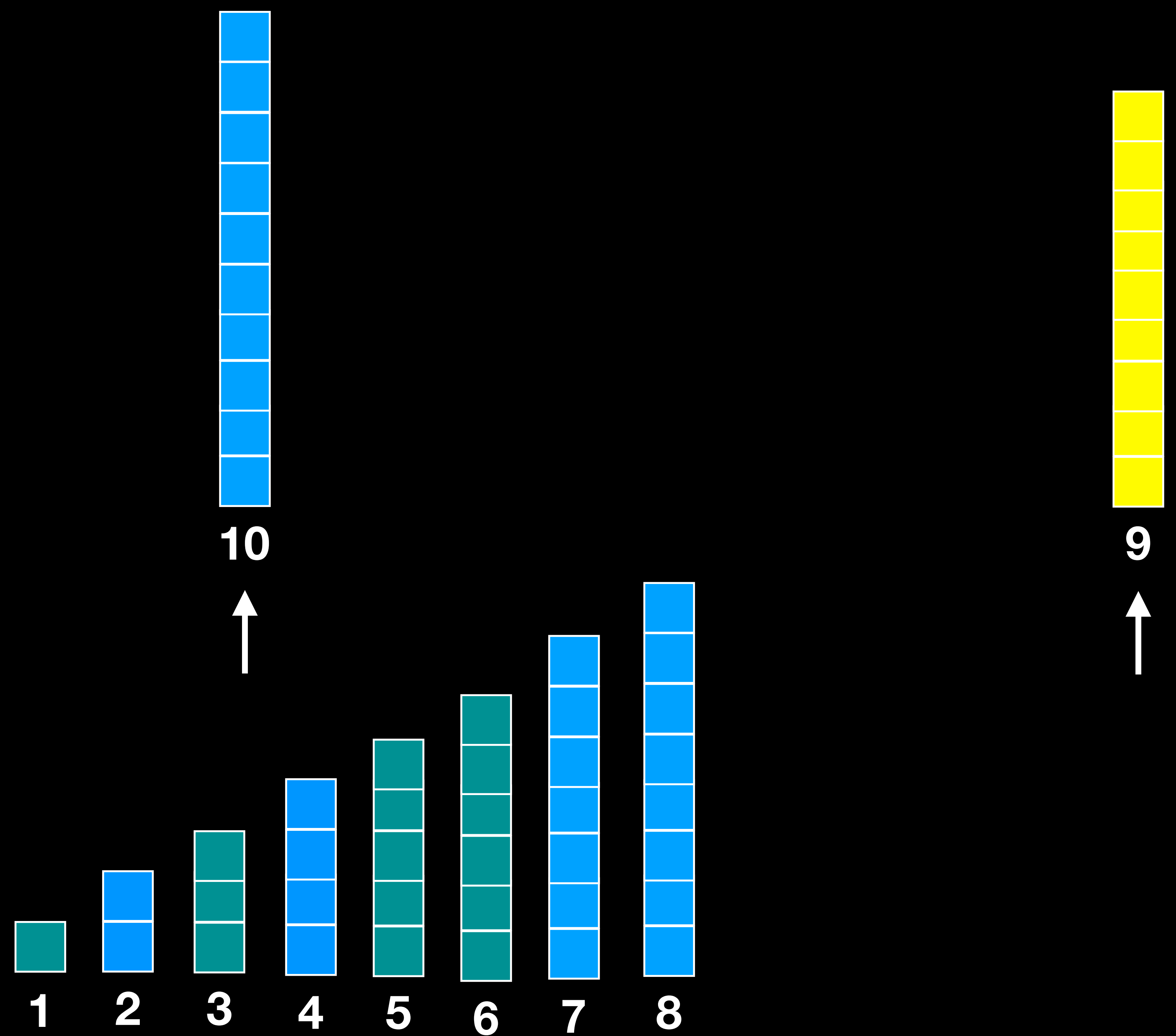
Key Insight: Merge is linear



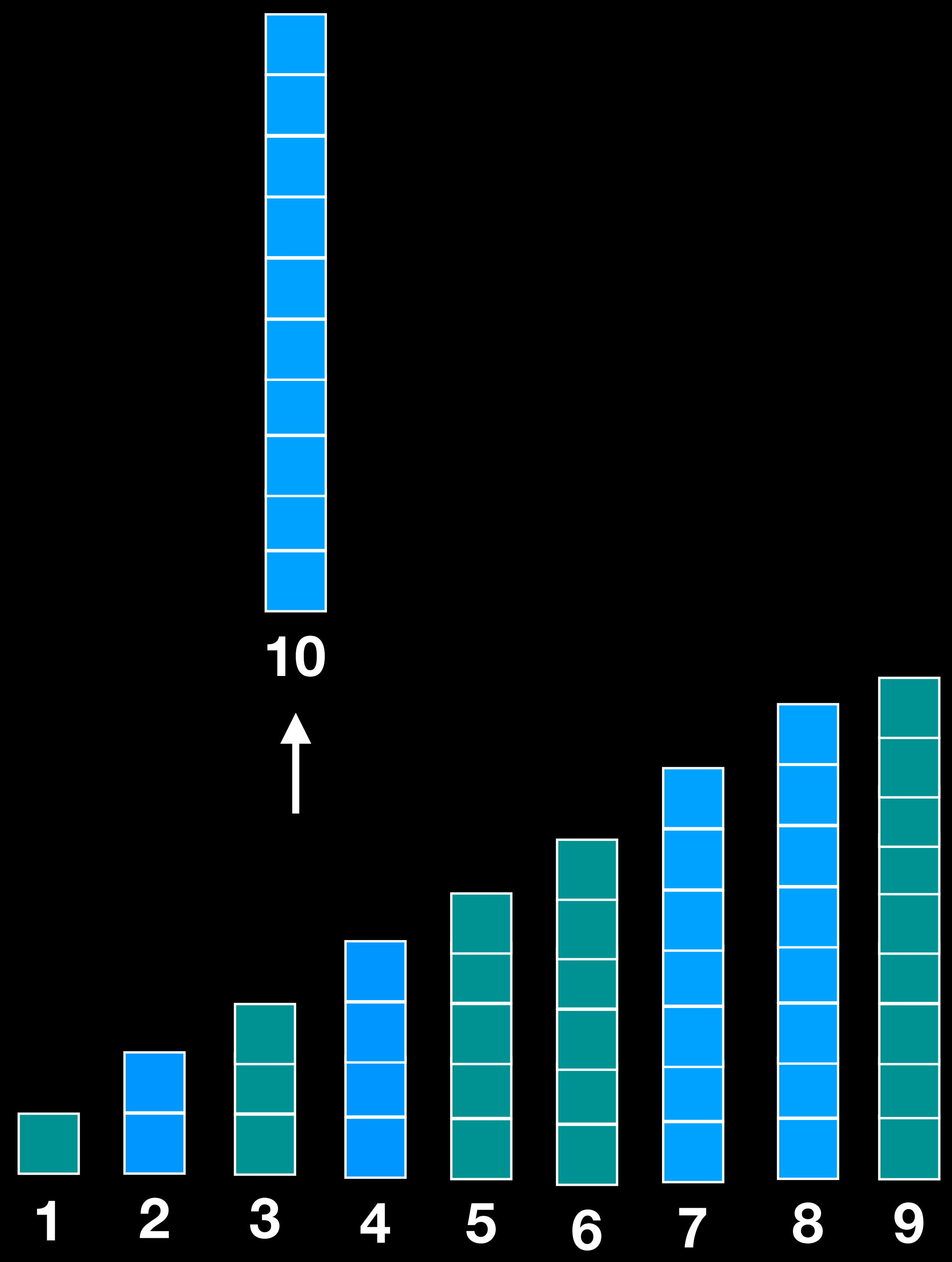
Key Insight: Merge is linear



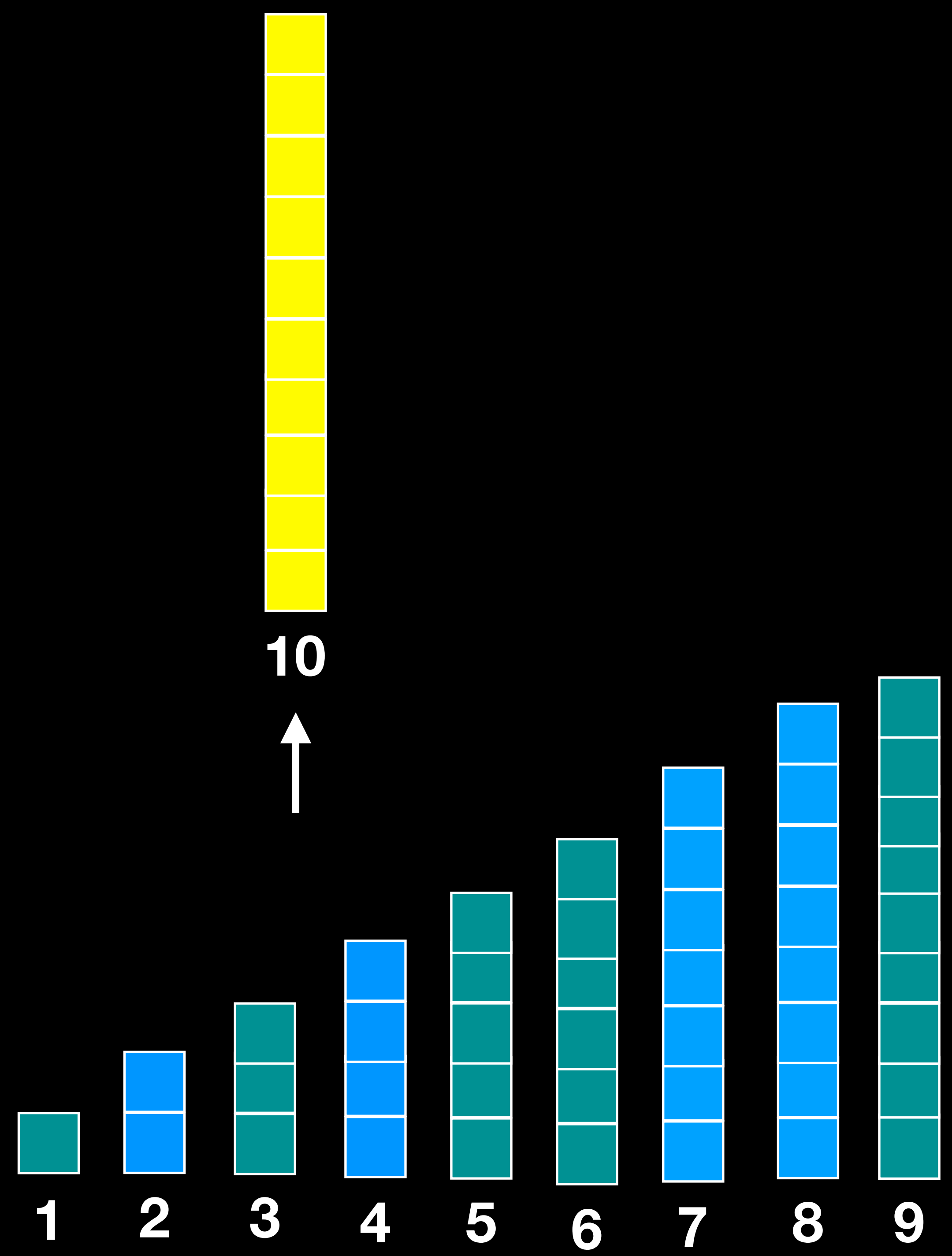
Key Insight: Merge is linear



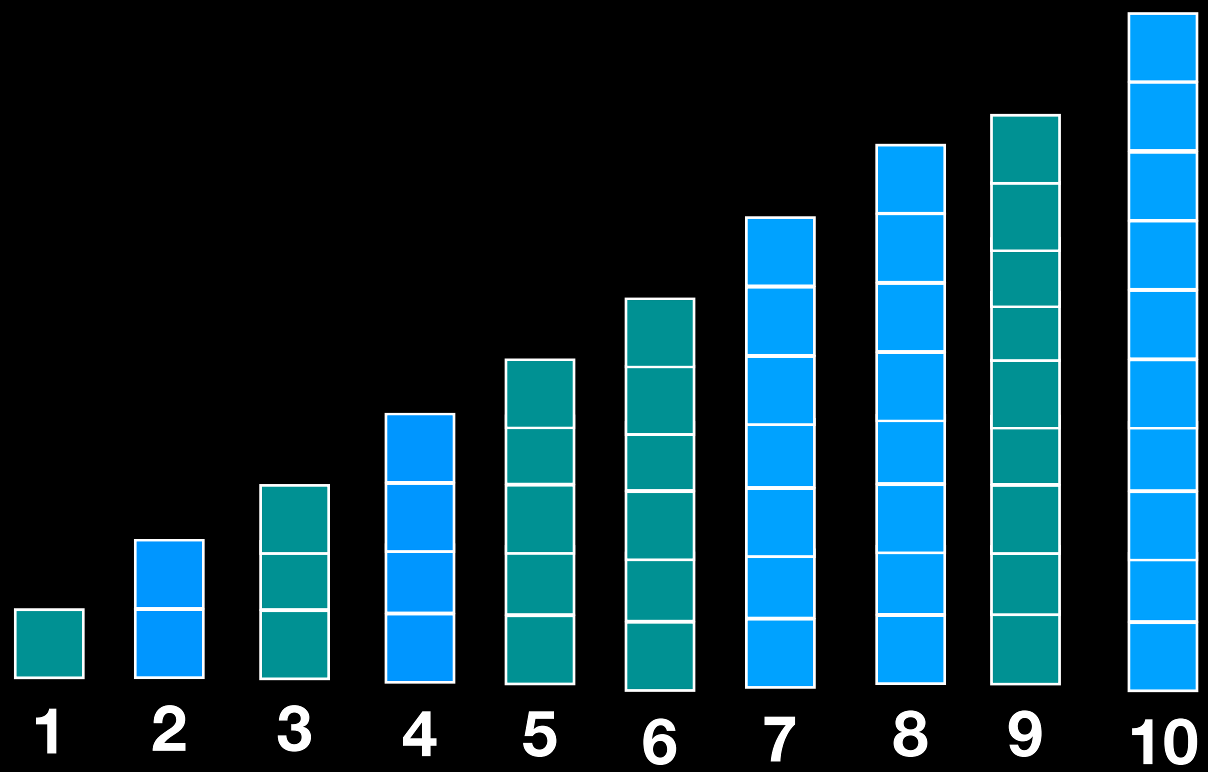
Key Insight: Merge is linear



Key Insight: Merge is linear

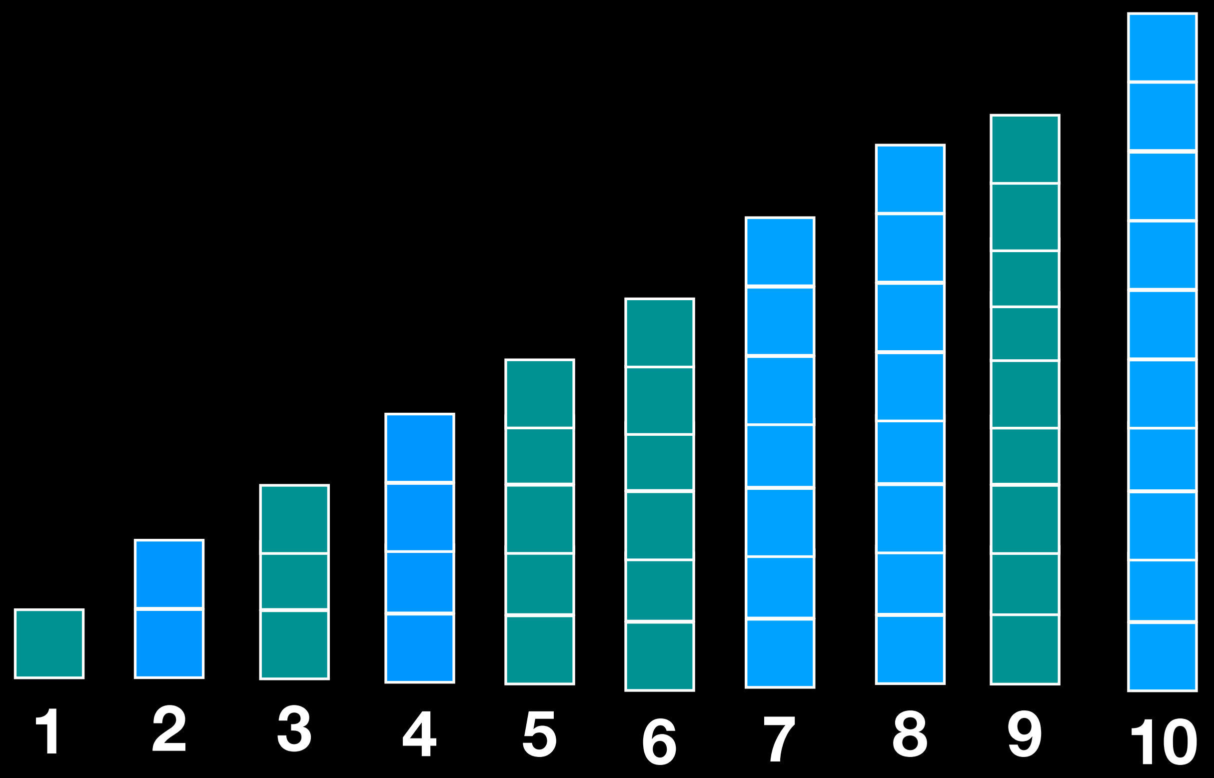


Key Insight: Merge is linear



Key Insight: Merge is linear

Each step makes one comparison and reduces the number of elements to be merged by 1.
If there are *n* total elements to be merged, merging is **O(n)**



Divide and Conquer

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

14	43	76	100	108	200	274	523

11	64	158	195	260	599	932

$T(^1/_2 n) \approx {^1/_4}\, T(n)$

$T(^1/_2 n) \approx {^1/_4}\, T(n)$

Divide and Conquer

100	14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932	5

$T(n)$

14	43	76	100	108	200	274	523

11	64	158	195	260	599	932

$T(^1/_2 n) \approx {}^1/_4\, T(n)$

$T(^1/_2 n) \approx {}^1/_4\, T(n)$

2	3	5	11	14	43	64	76	100	108	158	195	200	260	274	523	599	932

$T(n) \approx {}^1/_2\, T(n) + n$

Speed up insertion sort by a factor of two by splitting in half, sorting separately and merging results!

Divide and Conquer

Splitting in two gives 2x improvement.

Divide and Conquer

Splitting in two gives 2x improvement.

Splitting in four gives 4x improvement.

Divide and Conquer

Splitting in two gives 2x improvement.

Splitting in four gives 4x improvement.

Splitting in eight gives 8x improvement.

Divide and Conquer

Splitting in two gives 2x improvement.

Splitting in four gives 4x improvement.

Splitting in eight gives 8x improvement.

What if we never stop splitting?

Merge Sort

| 14 | 3 | 43 | 200 | 274 | 523 | 108 | 76 | 195 | 599 | 158 | 2 | 260 | 11 | 64 | 932 |

| 14 | 3 | 43 | 200 | 274 | 523 | 108 | 76 | | 195 | 599 | 158 | 2 | 260 | 11 | 64 | 932 |

| 14 | 3 | 43 | 200 | | 274 | 523 | 108 | 76 | | 195 | 599 | 158 | 2 | | 260 | 11 | 64 | 932 |

| 14 | 3 | | 43 | 200 | | 274 | 523 | | 108 | 76 | | 195 | 599 | | 158 | 2 | | 260 | 11 | | 64 | 932 |

| 14 | | 3 | | 43 | | 200 | | 274 | | 523 | | 108 | | 76 | | 195 | | 599 | | 158 | | 2 | | 260 | | 11 | | 64 | | 932 |

Merge Sort

14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932

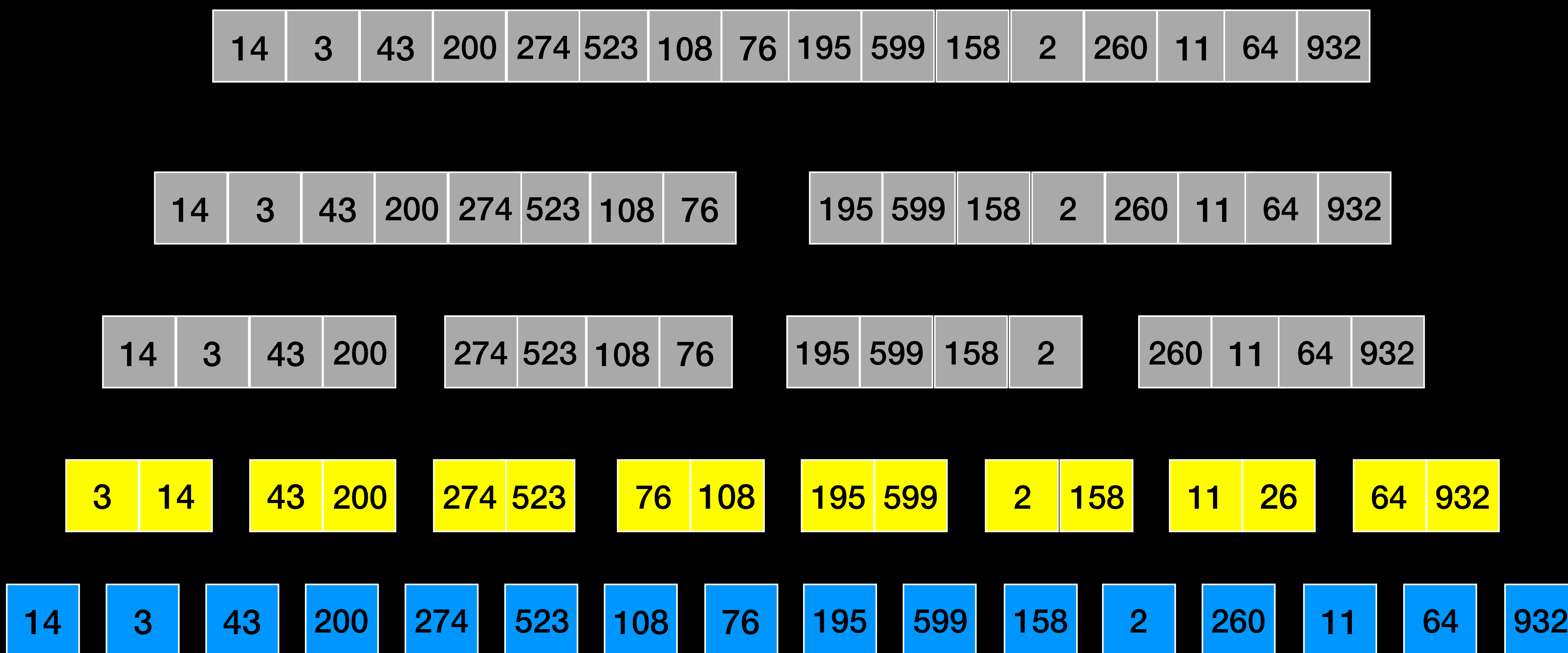
[14, 3, 43, 200, 274, 523, 108, 76] [195, 599, 158, 2, 260, 11, 64, 932]

[14, 3, 43, 200] [274, 523, 108, 76] [195, 599, 158, 2] [260, 11, 64, 932]

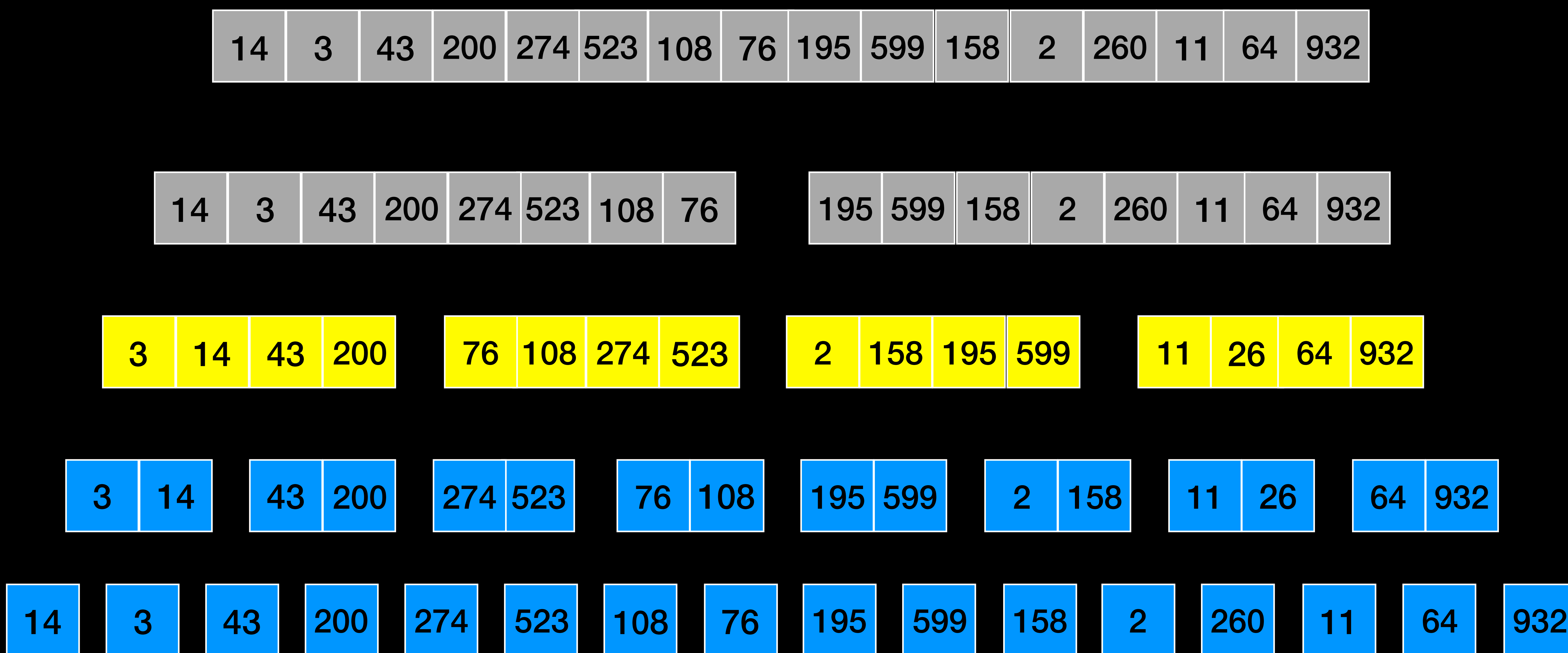
[14, 3] [43, 200] [274, 523] [108, 76] [195, 599] [158, 2] [260, 11] [64, 932]

[14] [3] [43] [200] [274] [523] [108] [76] [195] [599] [158] [2] [260] [11] [64] [932]

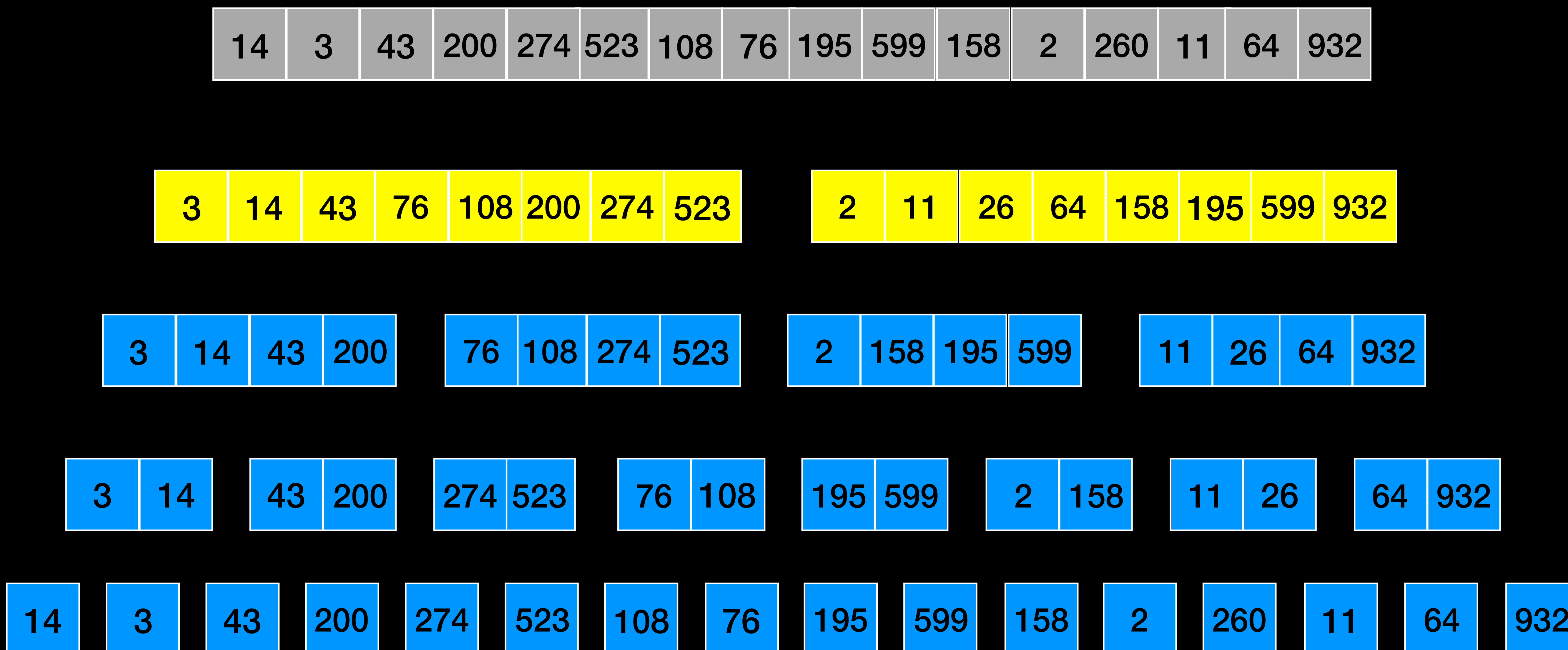
Merge Sort


14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932
14 3 43 200 274 523 108 76
195 599 158 2 260 11 64 932
14 3 43 200
274 523 108 76
195 599 158 2
260 11 64 932
3 14
43 200
274 523
76 108
195 599
2 158
11 26
64 932
14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

Merge Sort

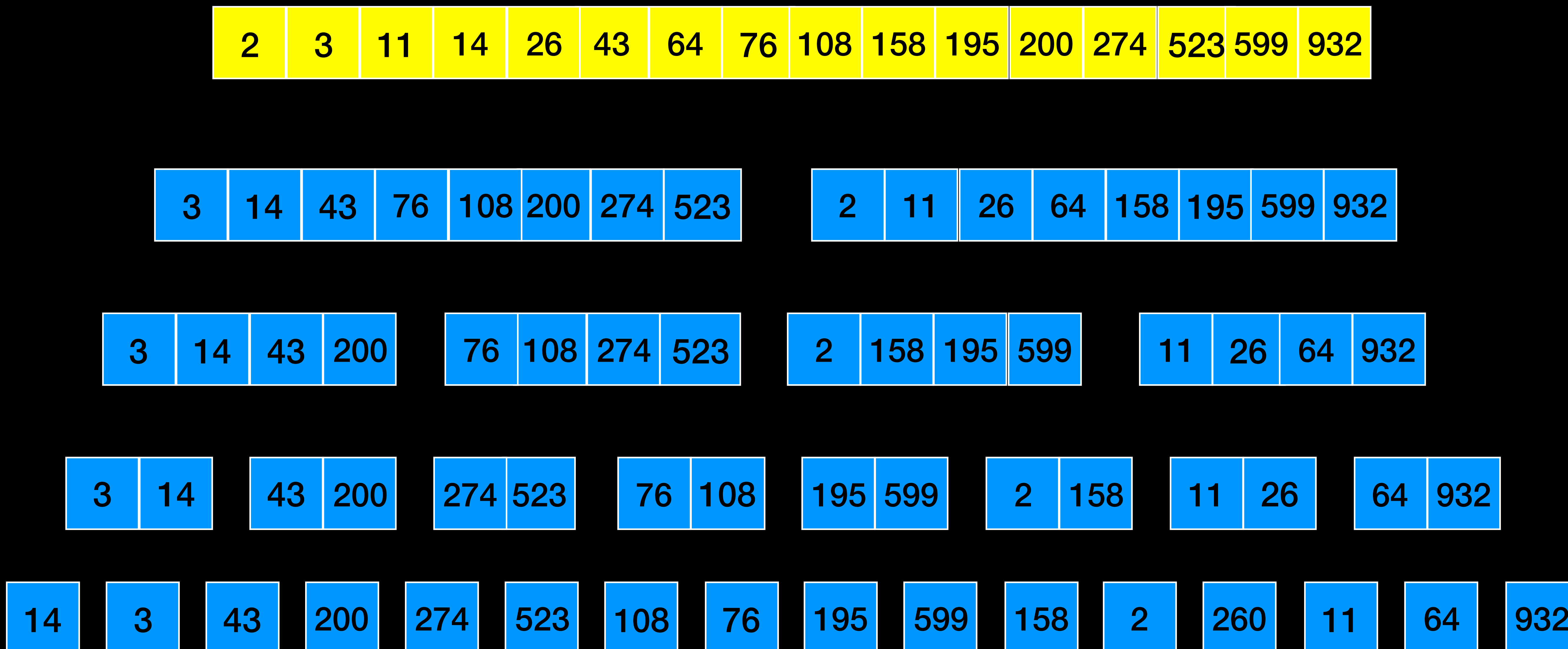

14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932
14 3 43 200 274 523 108 76
195 599 158 2 260 11 64 932
3 14 43 200
76 108 274 523
2 158 195 599
11 26 64 932
3 14
43 200
274 523
76 108
195 599
2 158
11 26
64 932
14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

Merge Sort



14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932
3 14 43 76 108 200 274 523
2 11 26 64 158 195 599 932
3 14 43 200
76 108 274 523
2 158 195 599
11 26 64 932
3 14
43 200
274 523
76 108
195 599
2 158
11 26
64 932
14
3
43
200
274
523
108
76
195
599
158
2
260
11
64
932

Merge Sort


2 3 11 14 26 43 64 76 108 158 195 200 274 523 599 932
3 14 43 76 108 200 274 523
2 11 26 64 158 195 599 932
3 14 43 200
76 108 274 523
2 158 195 599
11 26 64 932
3 14
43 200
274 523
76 108
195 599
2 158
11 26
64 932
14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

Merge Sort

2	3	11	14	26	43	64	76	108	158	195	200	274	523	599	932

3	14	43	76	108	200	274	523

2	11	26	64	158	195	599	932

3	14	43	200

76	108	274	523

2	158	195	599

11	26	64	932

3	14

43	200

274	523

76	108

195	599

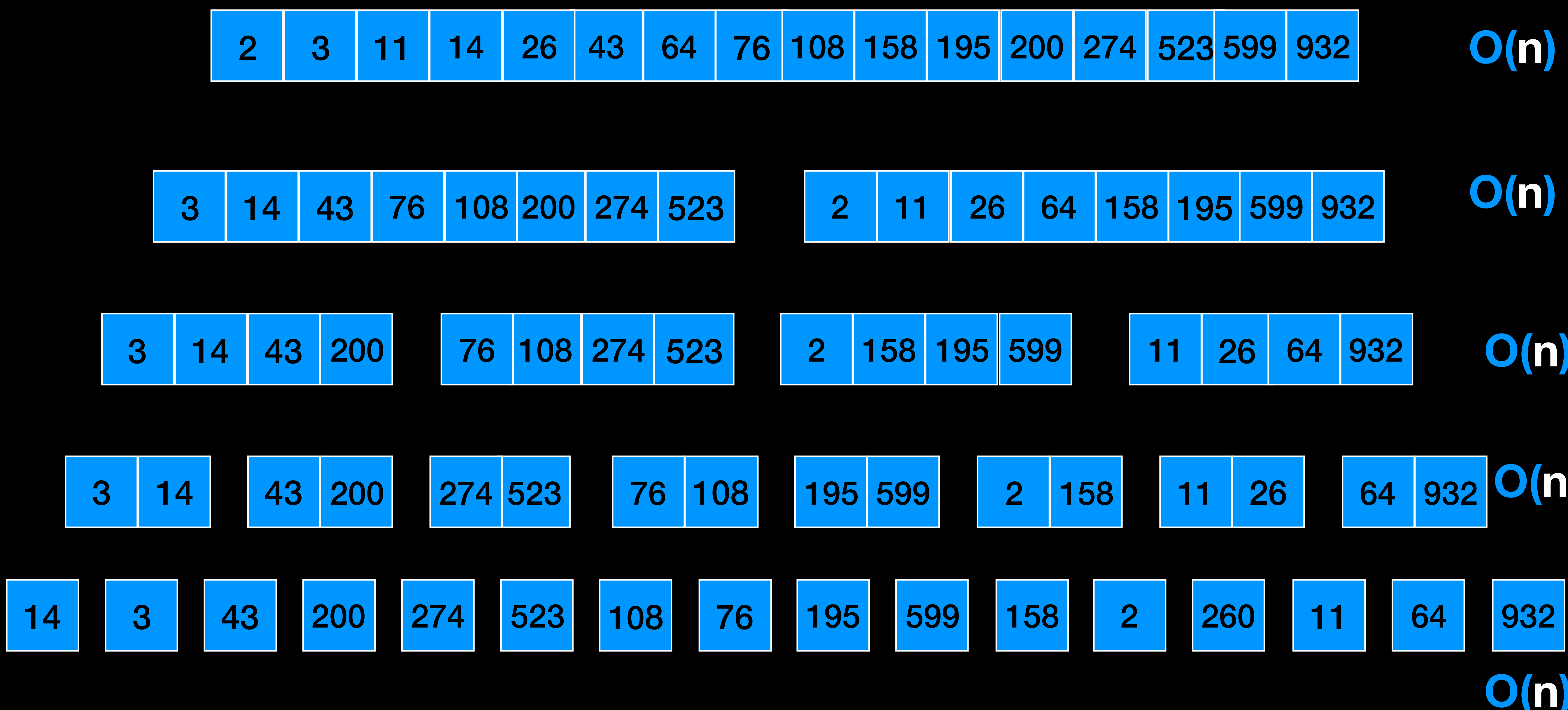
2	158

11	26

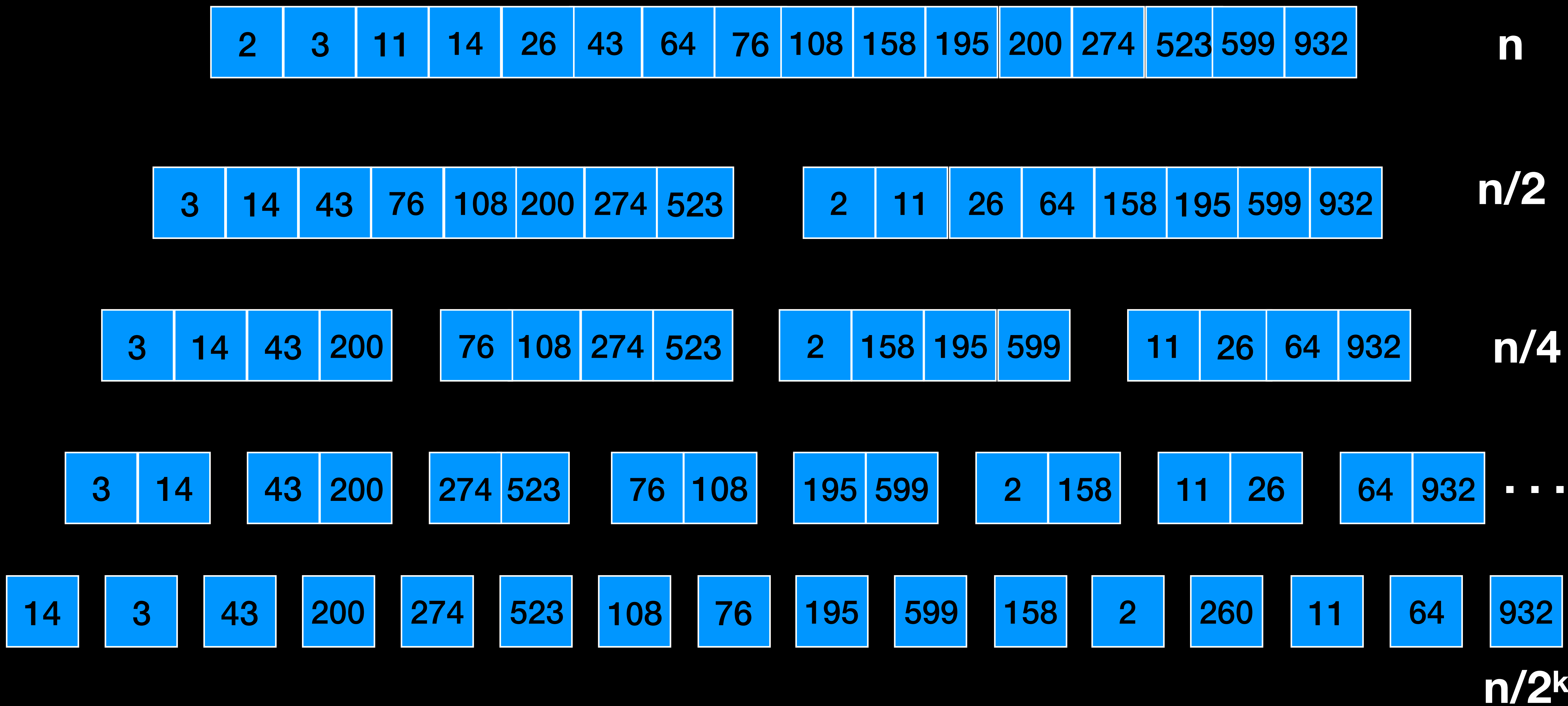
64	932

14 3 43 200 274 523 108 76 195 599 158 2 260 11 64 932

Merge Sort Analysis

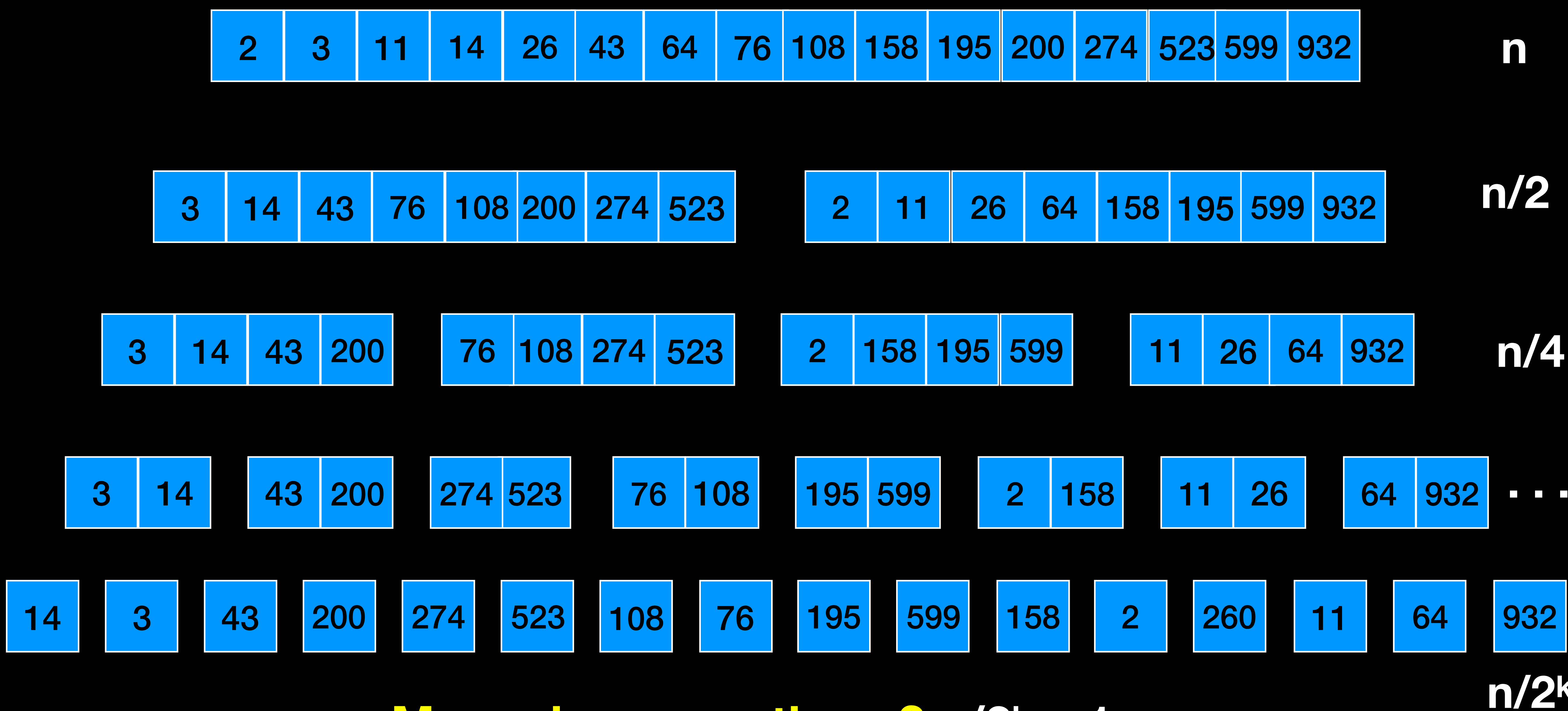


Merge Sort Analysis



Merge how many times?

Merge Sort Analysis

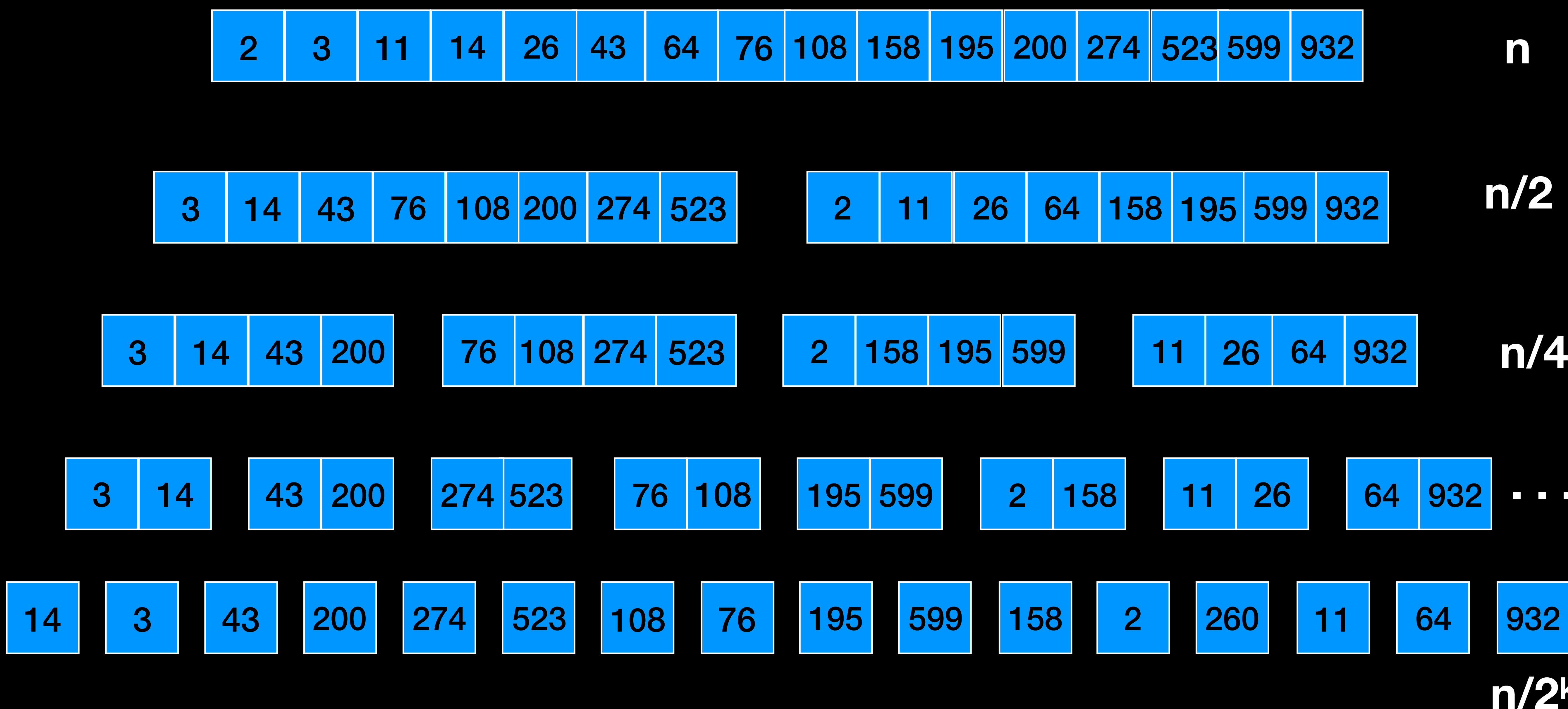


Merge how may times? $n/2^k = 1$

$n = 2^k$

$\log_2 n = k$

Merge Sort Analysis



Merge $\log_2 n$ times

Merge Sort Analysis

2	3	11	14	26	43	64	76	108	158	195	200	274	523	599	932	O(n)

3	14	43	76	108	200	274	523		2	11	26	64	158	195	599	932	O(n)

3	14	43	200		76	108	274	523		2	158	195	599		11	26	64	932	O(n)

3	14		43	200		274	523		76	108		195	599		2	158		11	26		64	932	O(n)

14	3	43	200	274	523	108	76	195	599	158	2	260	11	64	932

O(n)

$\boxed{O(n \log n)}$

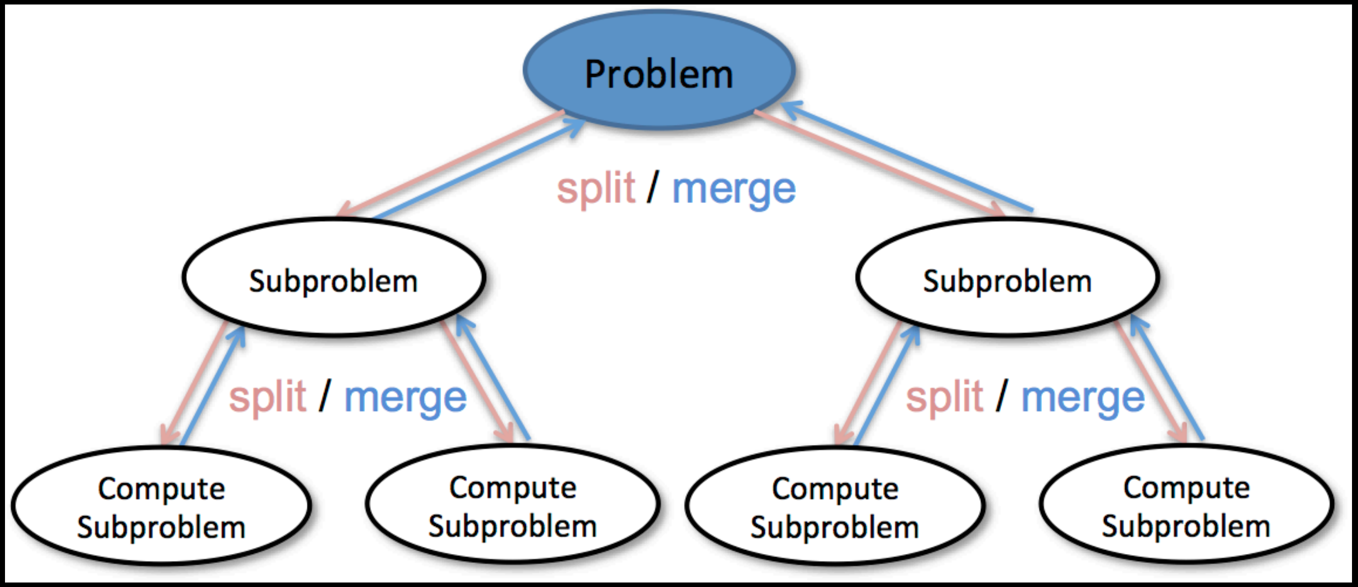
Merge Sort

How would you code this?

Merge Sort

How would you code this?

Hint: Divide and Conquer!!!



Merge Sort

```
void mergeSort(array)
{
    if array size <= 1
        return          //base case
    split array into left_array and right_array
    mergeSort(left_array)
    mergeSort(right_array)

    merge(left_array, right_array, sorted_array)
}
```

Merge Sort Analysis

Execution time does NOT depend on initial arrangement of data

Worst Case: O(n log n) comparisons and data moves

Best Case: O(n log n) comparisons and data moves

Stable

Best we can do with *comparison-based* sorting that does not rely on a data structure in the worst case => can't beat O(n log n)

Space overhead: auxiliary array at each merge step

What we have so far

	Worst Case	Best Case
Selection Sort	$O(n^2)$	$O(n^2)$
Insertion Sort	$O(n^2)$	$O(n)$
Bubble Sort	$O(n^2)$	$O(n)$
Merge Sort	$O(n \log n)$	$O(n \log n)$

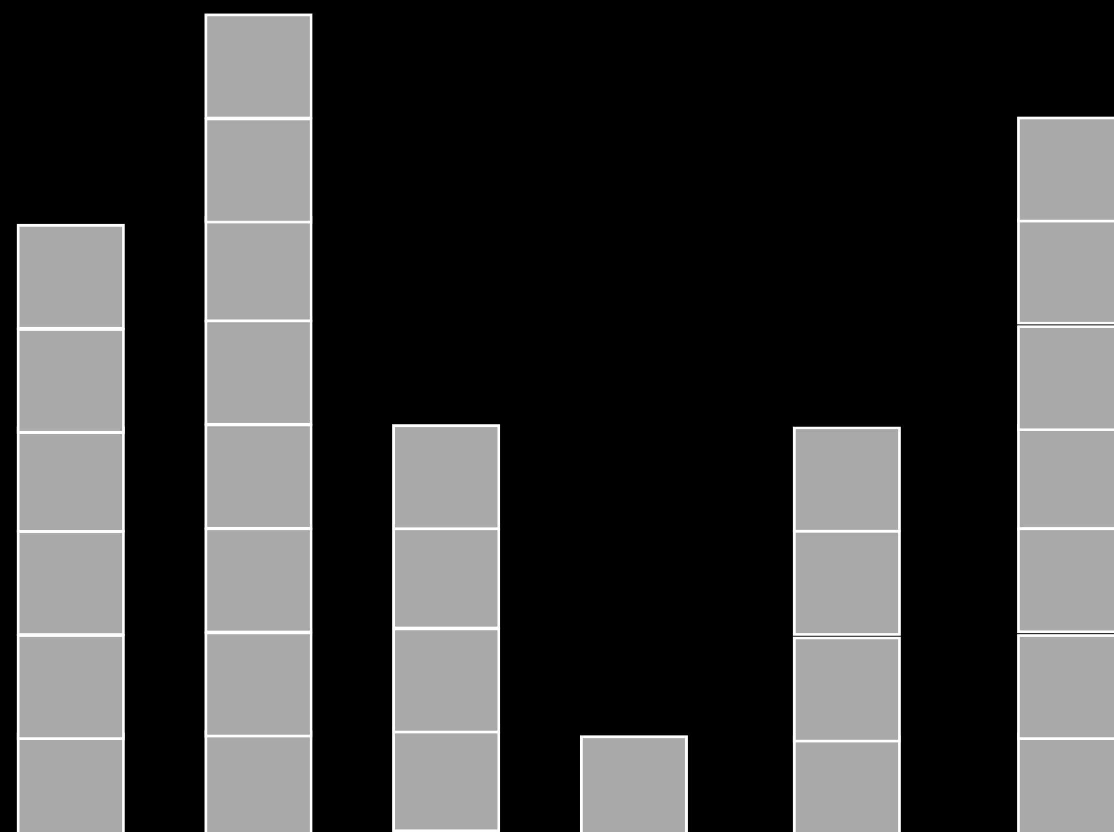
Quick Sort

Quick Sort

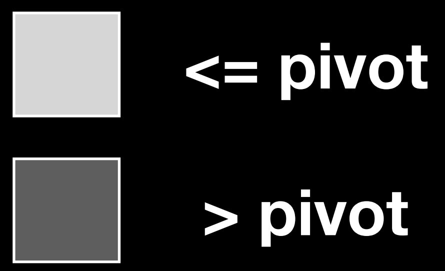
<= pivot

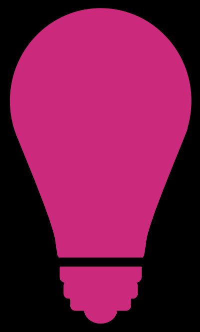
> pivot

Select a **pivot.** Arrange other entries s.t. entries in left partition are ≤ pivot and entries in right partition are > pivot



Quick Sort





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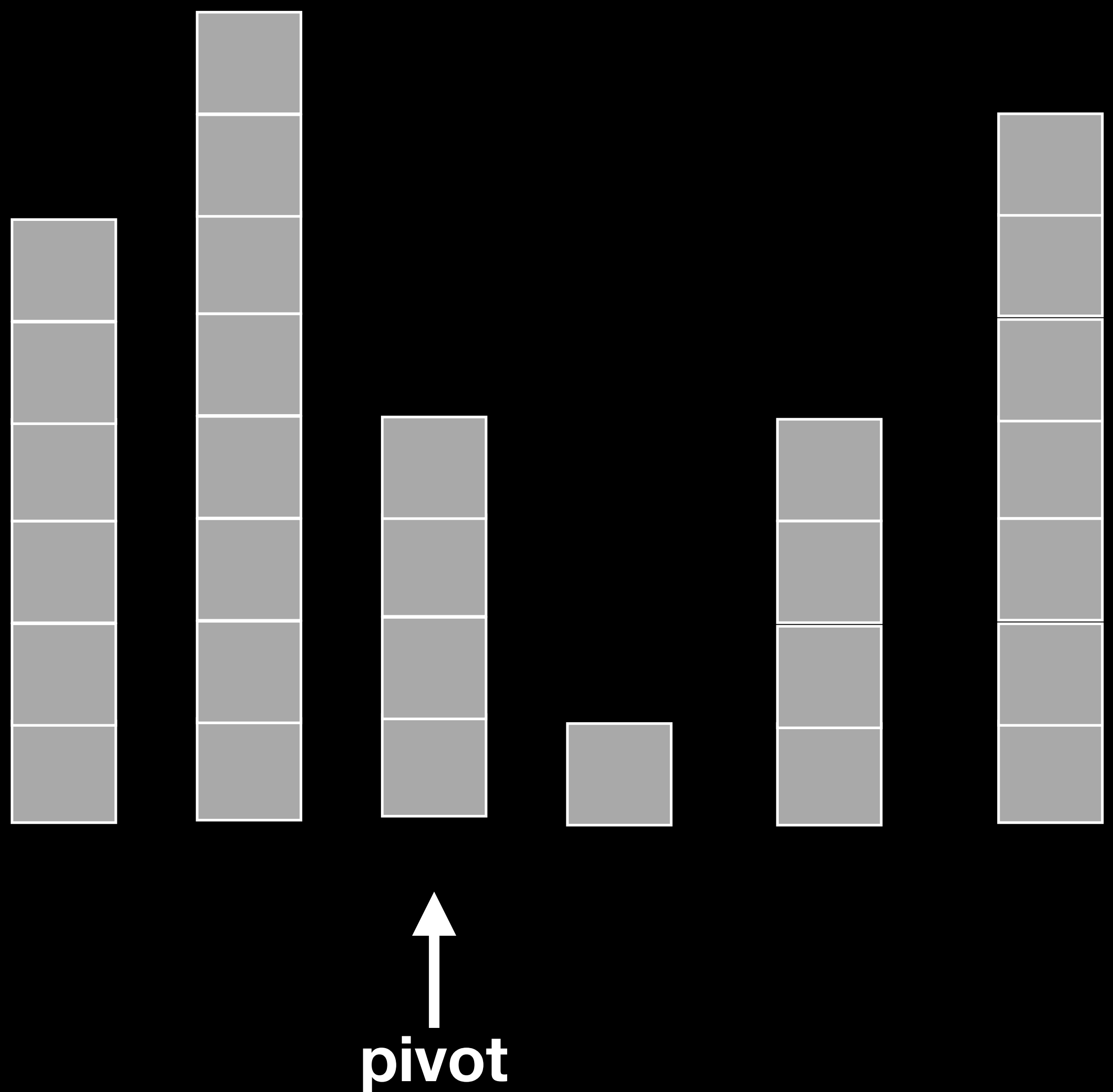

Quick Sort

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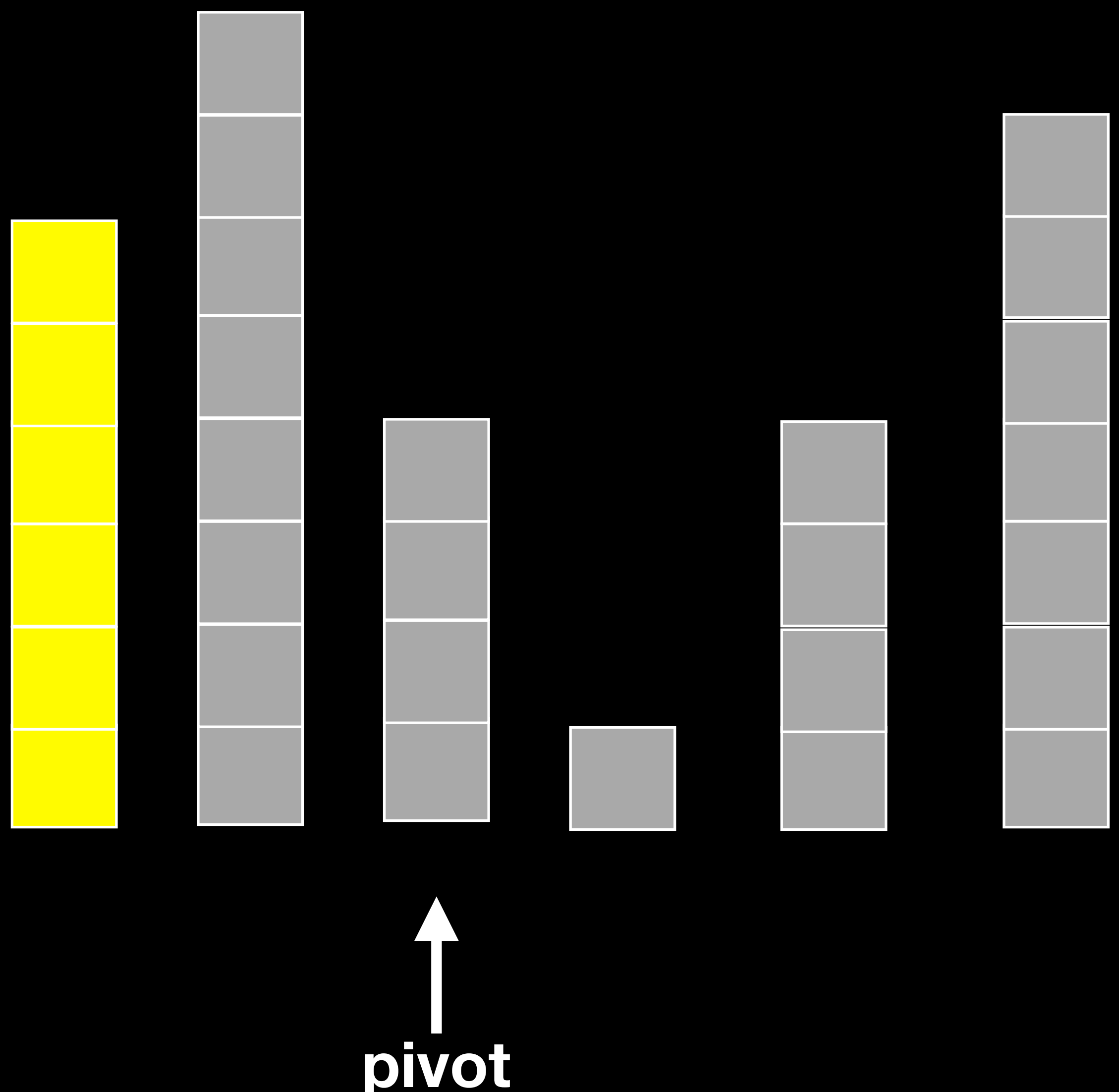
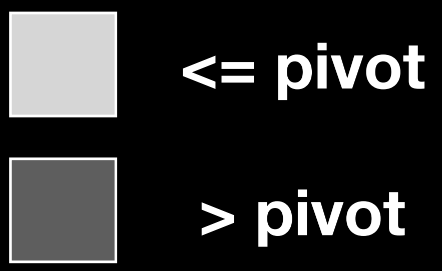


Quick Sort



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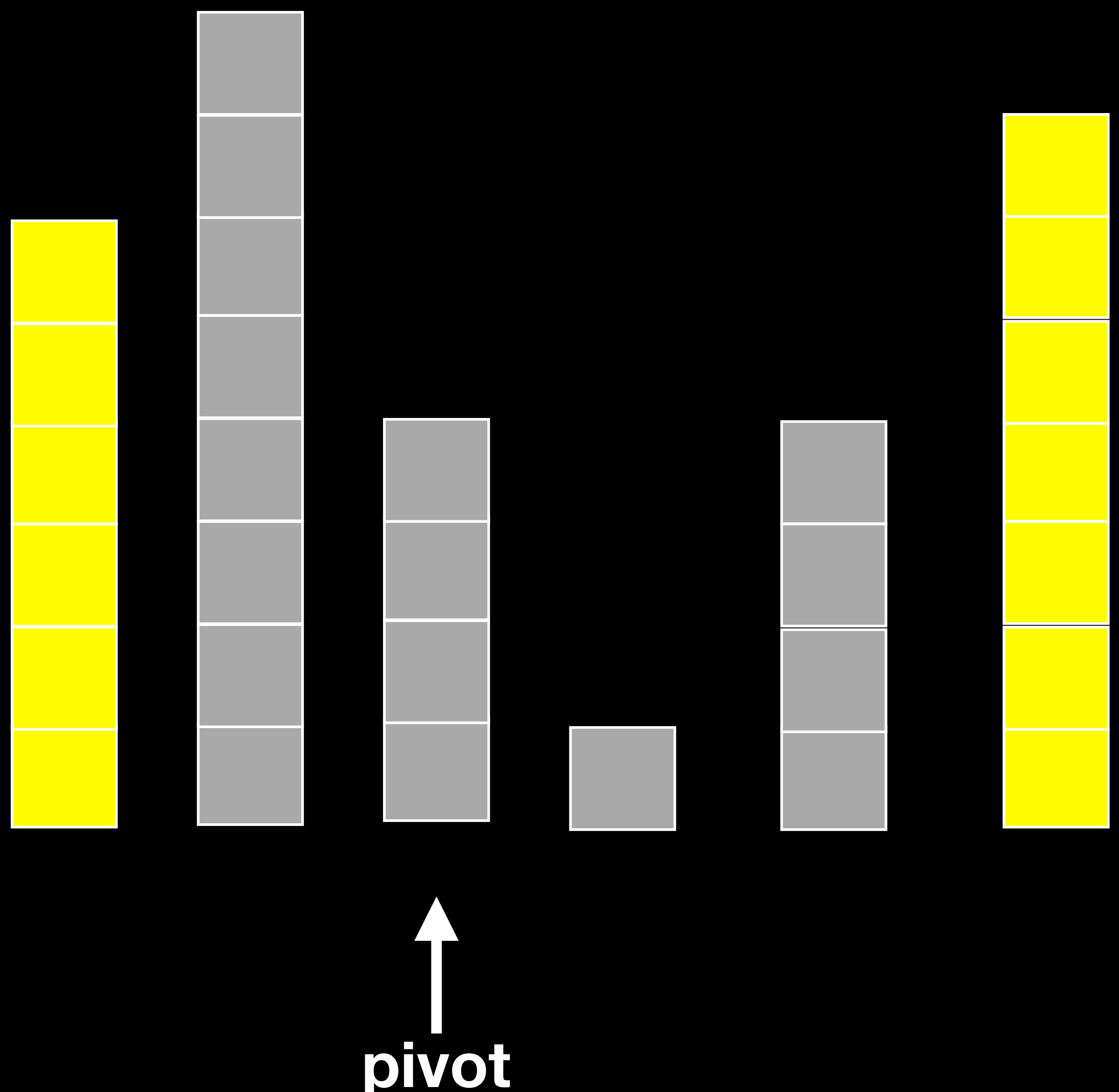
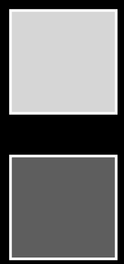


Quick Sort



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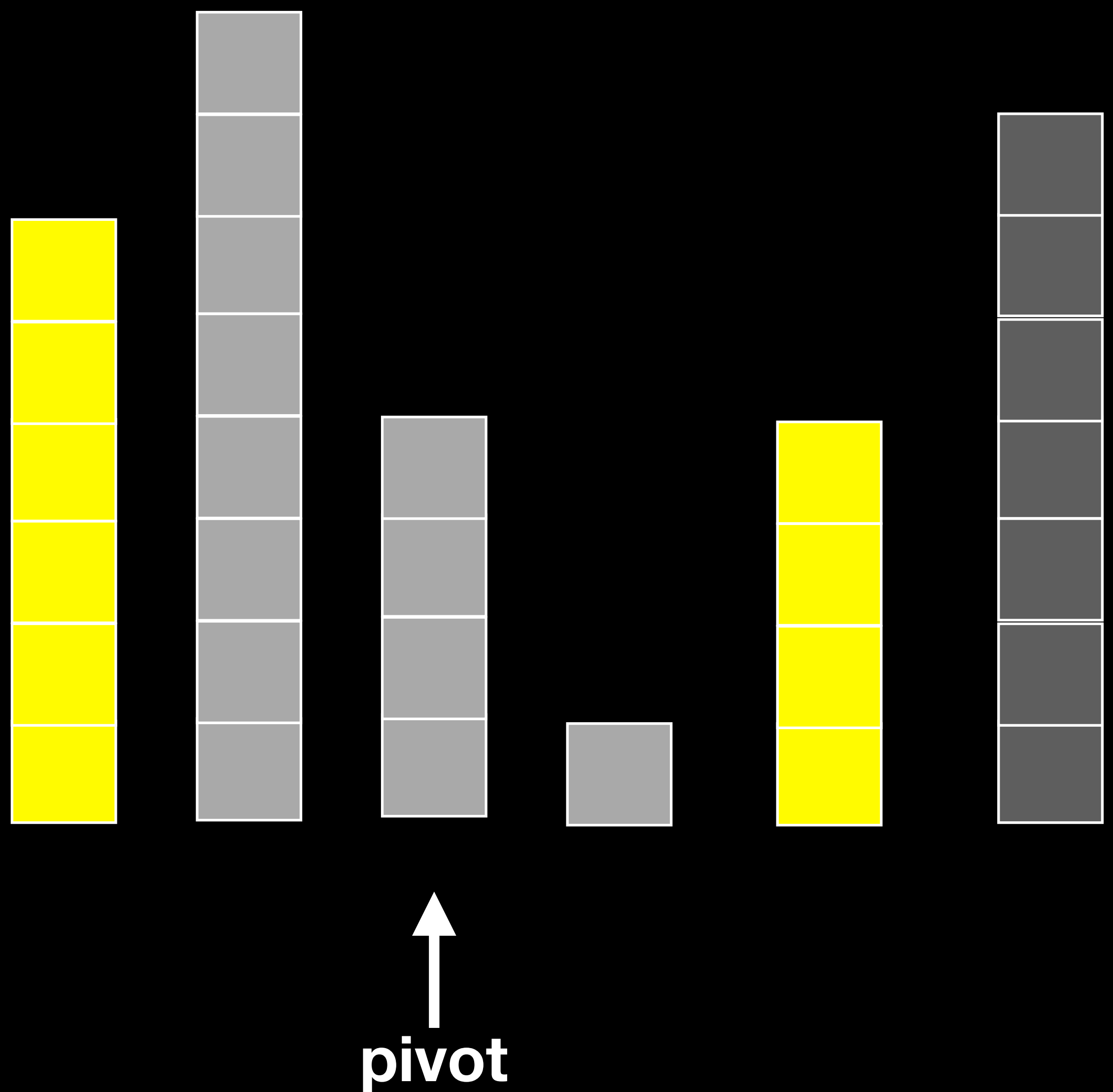

Quick Sort

<= pivot

> pivot

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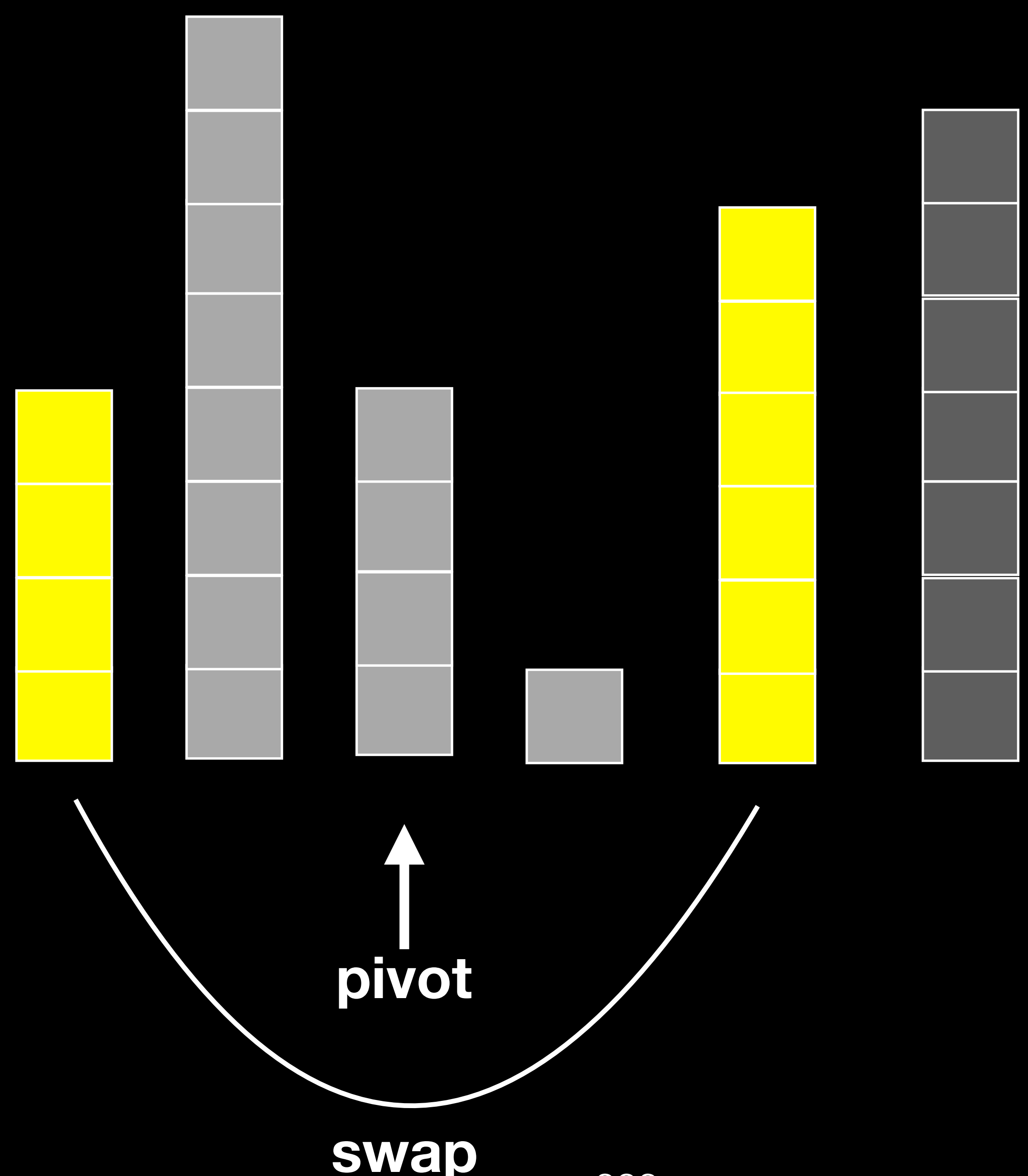

Quick Sort

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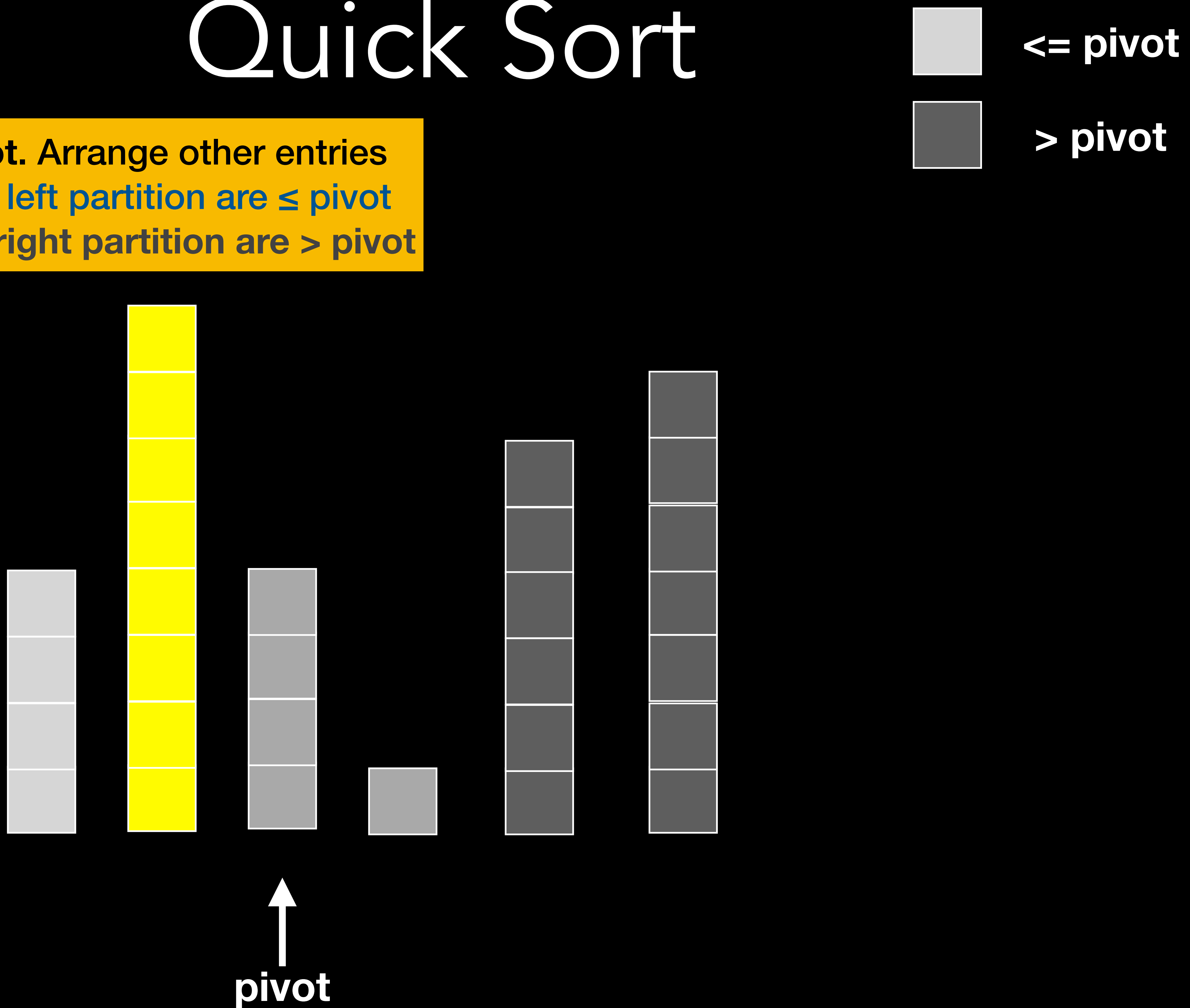
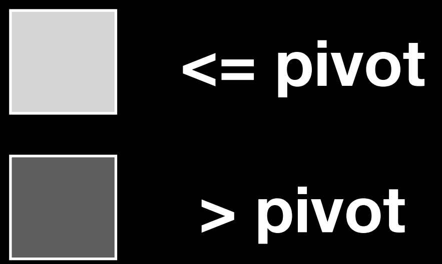


Quick Sort



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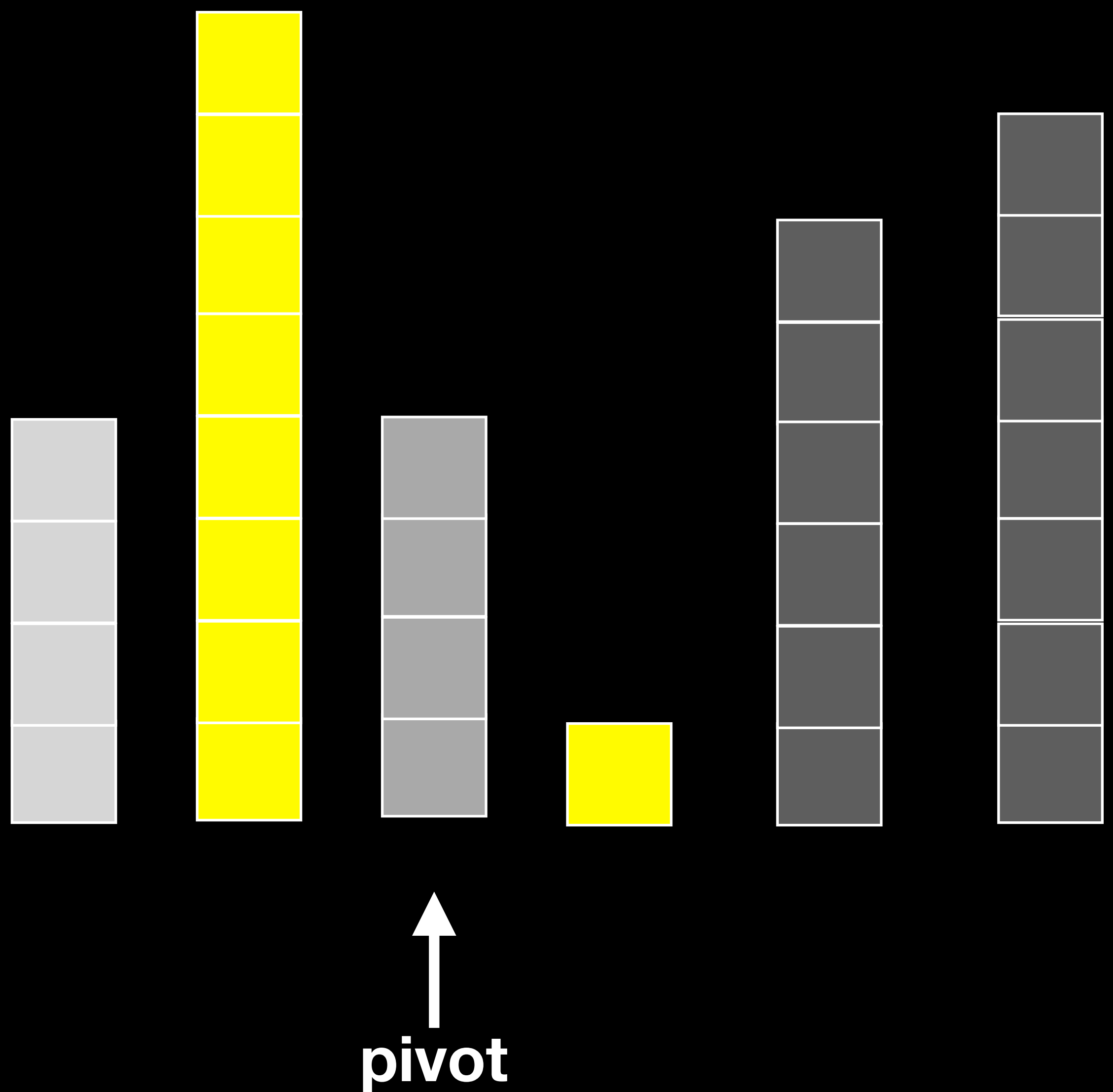
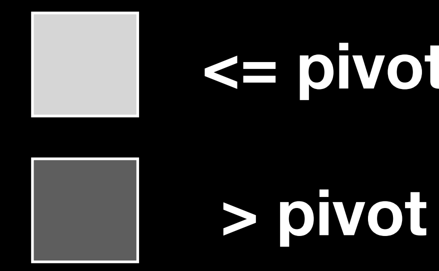


Quick Sort



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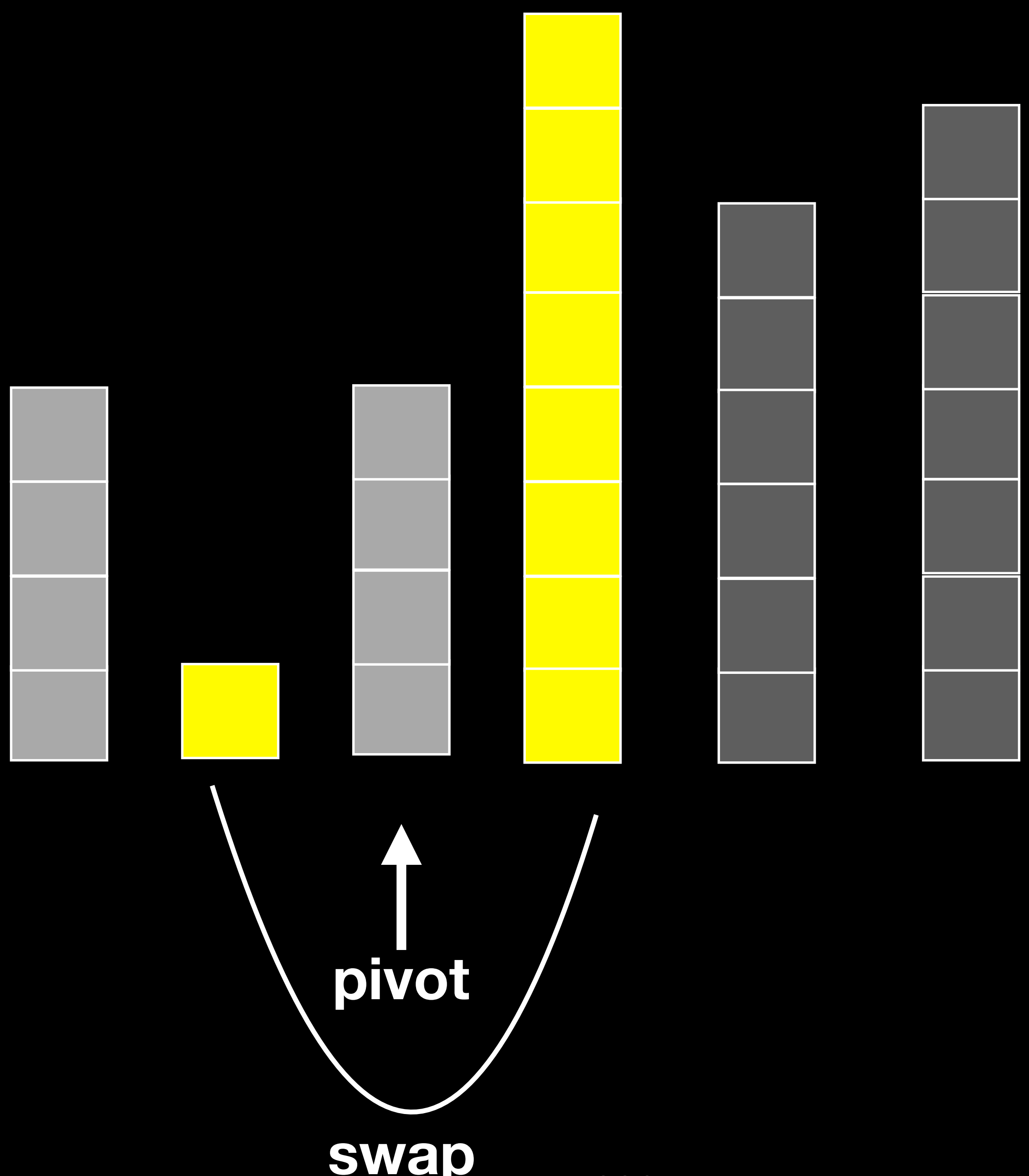

Quick Sort

<= pivot

> pivot

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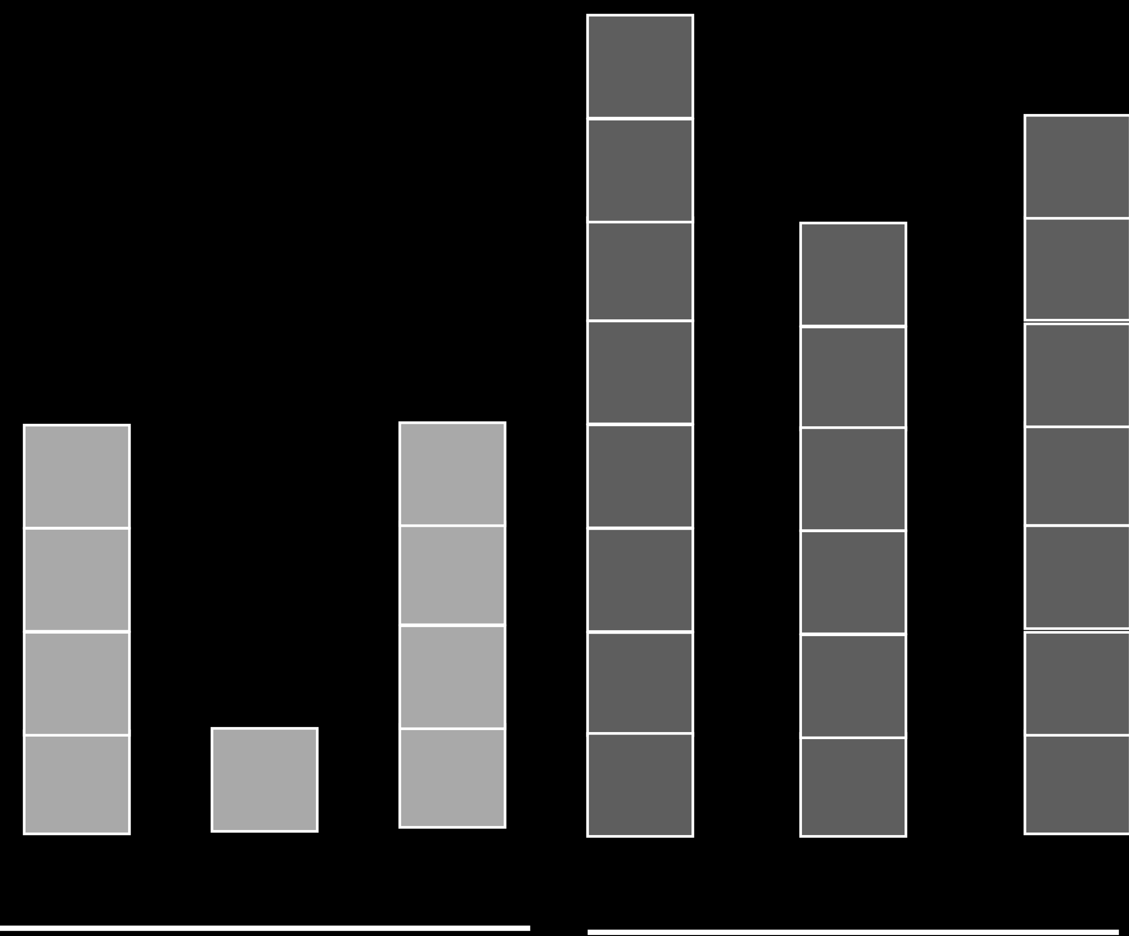


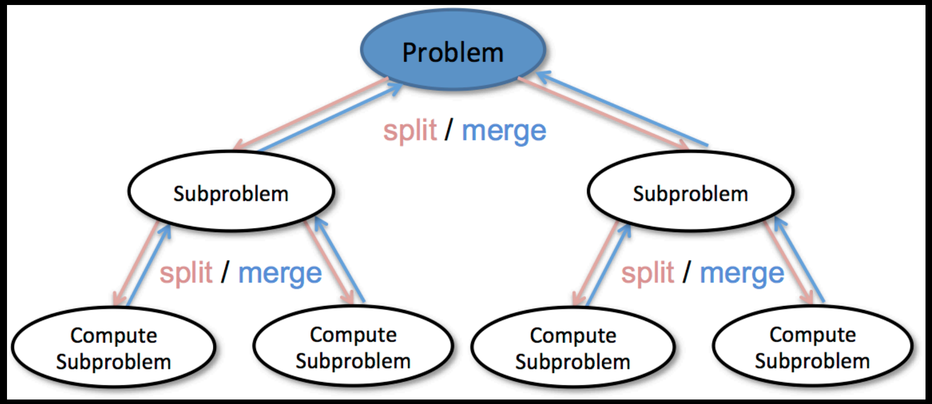
≤ pivot

`quickSort( )`

> pivot

`quickSort( )`

Quick Sort Analysis



Divide and Conquer

n comparisons for each partition

How many subproblems? => Depends on pivot selection

Ideally partition divides problem into two n/2 subproblems for logn recursive calls (Best case)

Possibly (though unlikely) each partition has 1 empty subarray for n recursive calls (Worst case)

How to select pivot?

How to select pivot?

Ideally median

    Need to sort array to find median

Other ideas?

How to select pivot?

Ideally median

Need to sort array to find median

Other ideas?

Pick first, middle, last position and order them making middle the pivot

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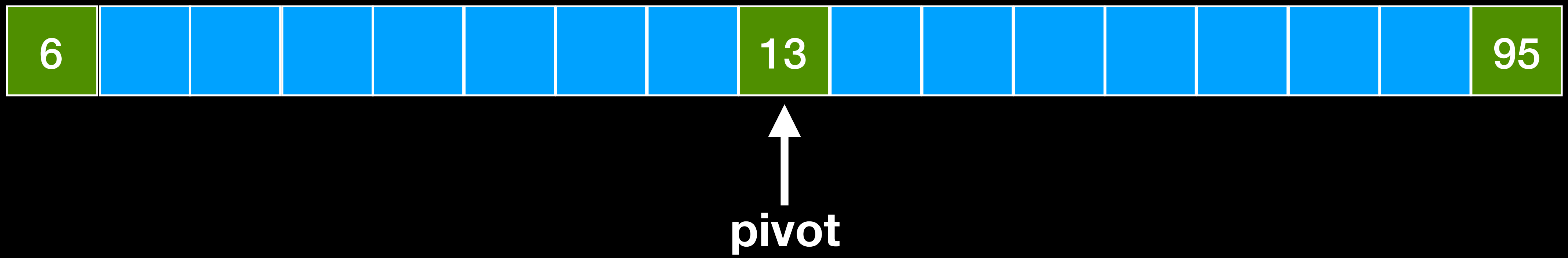


Quick Sort Analysis

Execution time DOES depend on initial arrangement of data AND on PIVOT SELECTION (luck?) => on random data can be faster than Merge Sort

Optimization (e.g. smart pivot selection, speed up base case, iterative instead of recursive implementation) can improve actual runtime -> fastest comparison-based sorting algorithm **on average**

Worst Case: $O(n^2)$ comparisons and data moves

Best Case: $O(n \log n)$ comparisons and data moves

Unstable

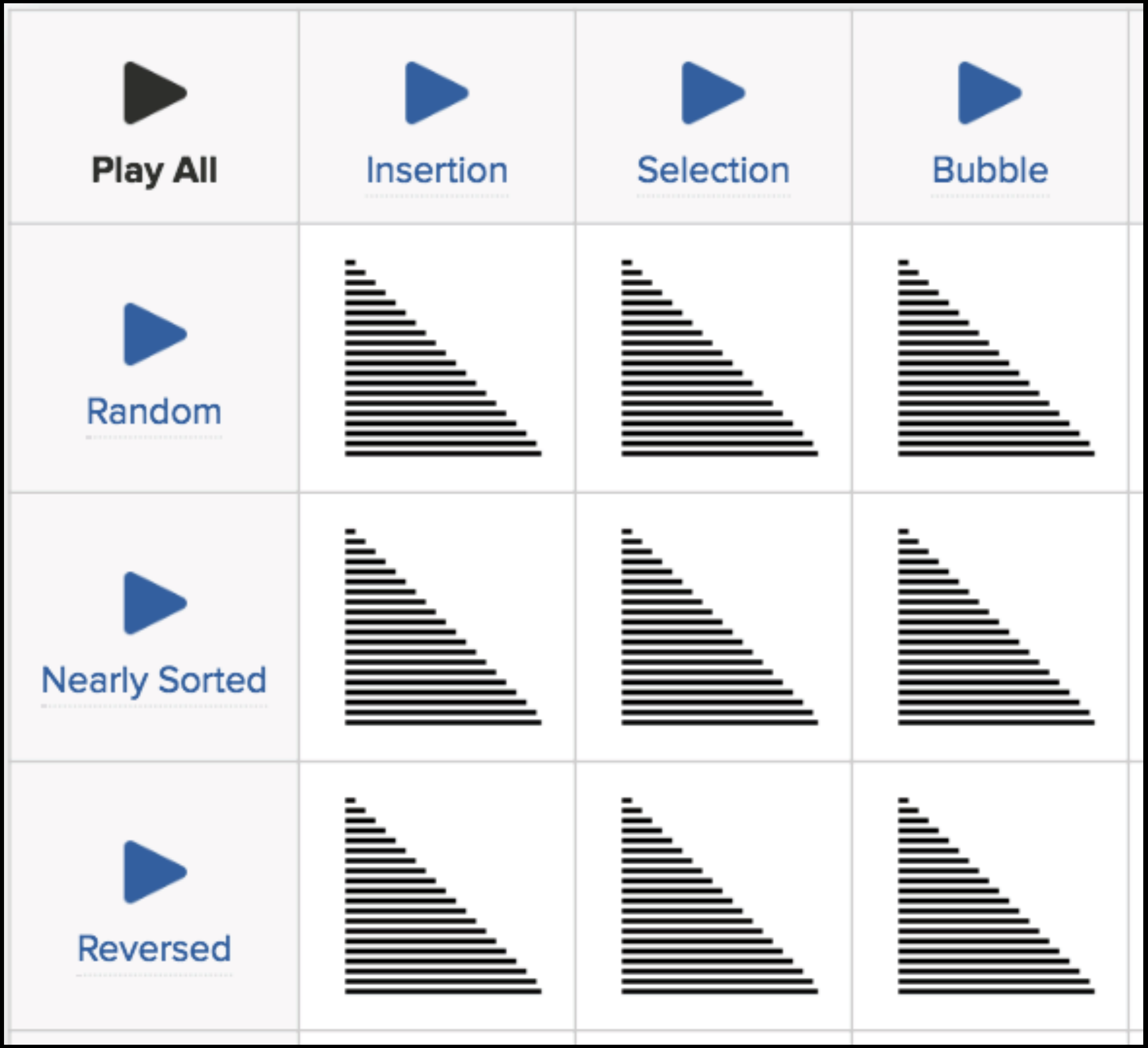
Quick Sort

```
void quickSort(array, first, last)
{
    if last - first + 1 = MIN_SIZE
         //base case with improvement
        insertionSort(array, first, last)
    else
        pivot_index = partition(array, first, last)
        quickSort(array, first, pivot_index - 1)
        quickSort(array, pivot_index + 1, last)

}
```

	Worst Case	Best Case
Selection Sort	$O(n^2)$	$O(n^2)$
Insertion Sort	$O(n^2)$	$O(n)$
Bubble Sort	$O(n^2)$	$O(n)$
Merge Sort	$O(n \log n)$	$O(n \log n)$
Quick Sort	$O(n^2)$	$O(n \log n)$

https://www.toptal.com/developers/sorting-algorithms



https://www.youtube.com/watch?v=kPRA0W1kECg