More Recursion

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Today's Plan

Recursion Review

8 Qeens Problem

Permutations

Combinations

Announcements and Syllabus Check

Types of Recursion

Reverse String:

- single recursive call
- Base case: stop => no return value

Dictionary:

- split problem into halves but solve only 1
- Base case: stop => no return value

Fractal Tree:

- split problem into halves and solve both
- Base case: stop => no return value

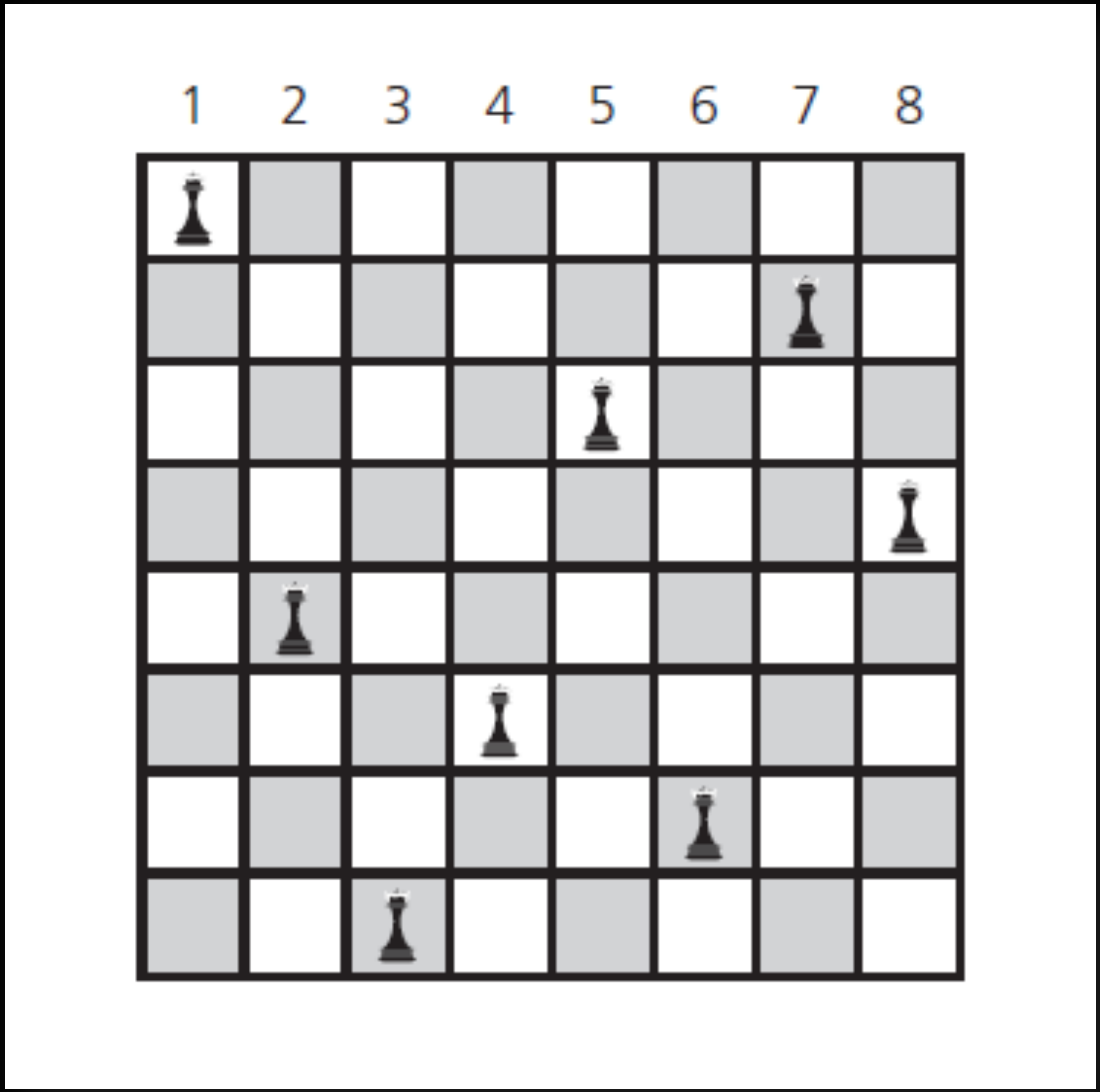
Factorial:

- single recursive call
- Base case: return a value for computation in each recursive call

Recursive Searching and Sorting

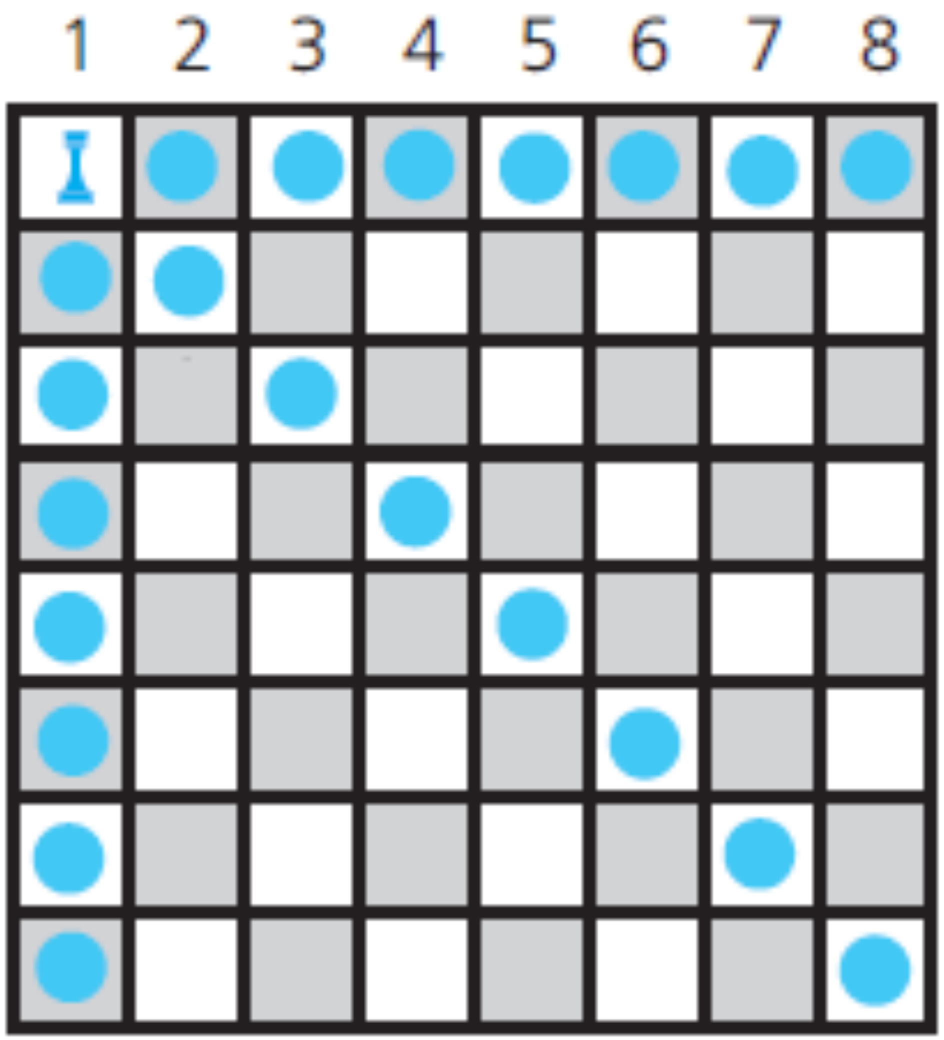
Binary Search	$O(\log n)$	$\Omega(1)$
MergeSort	$O(n \log n)$	$\Omega(n \log n)$
QuickSort	$O(n^2)$	$\Omega(n \log n)$

The Eight Queens Problem



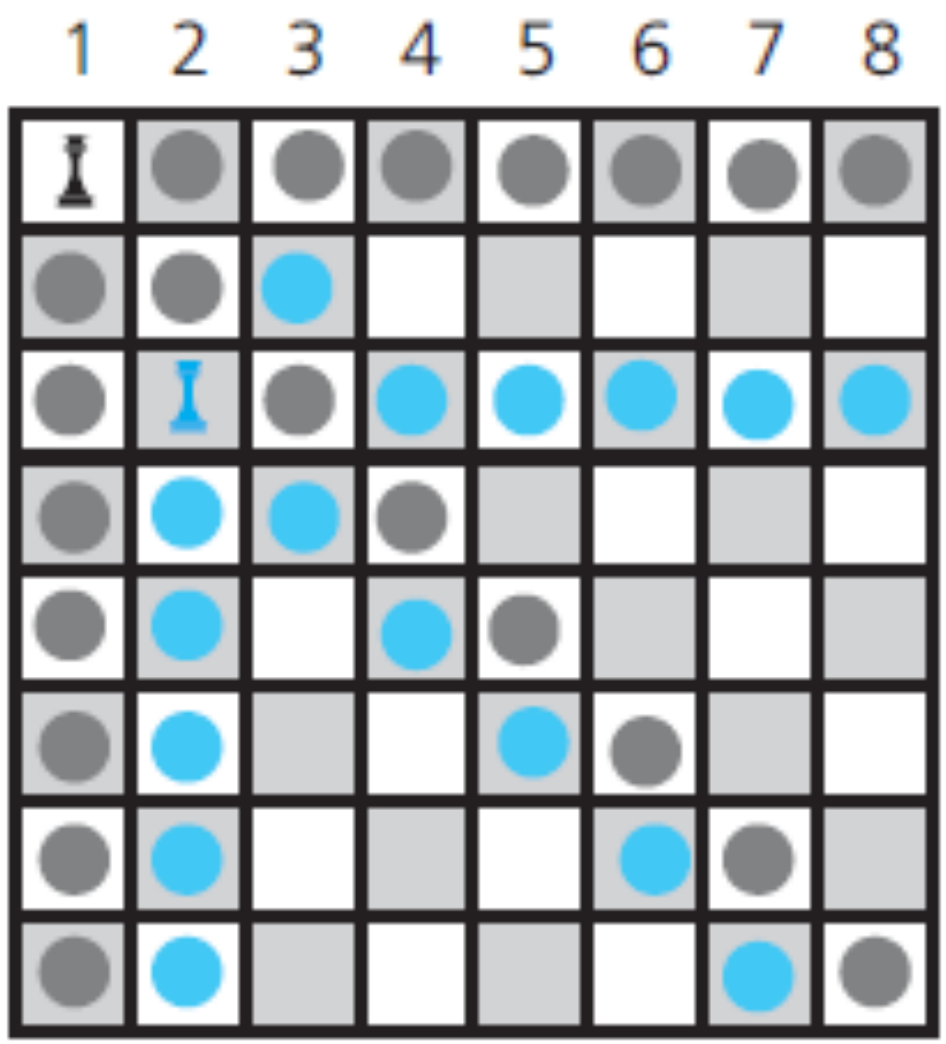
Place 8 Queens on the board s.t. no queen is on the same row, column or diagonal

The Eight Queens Problem



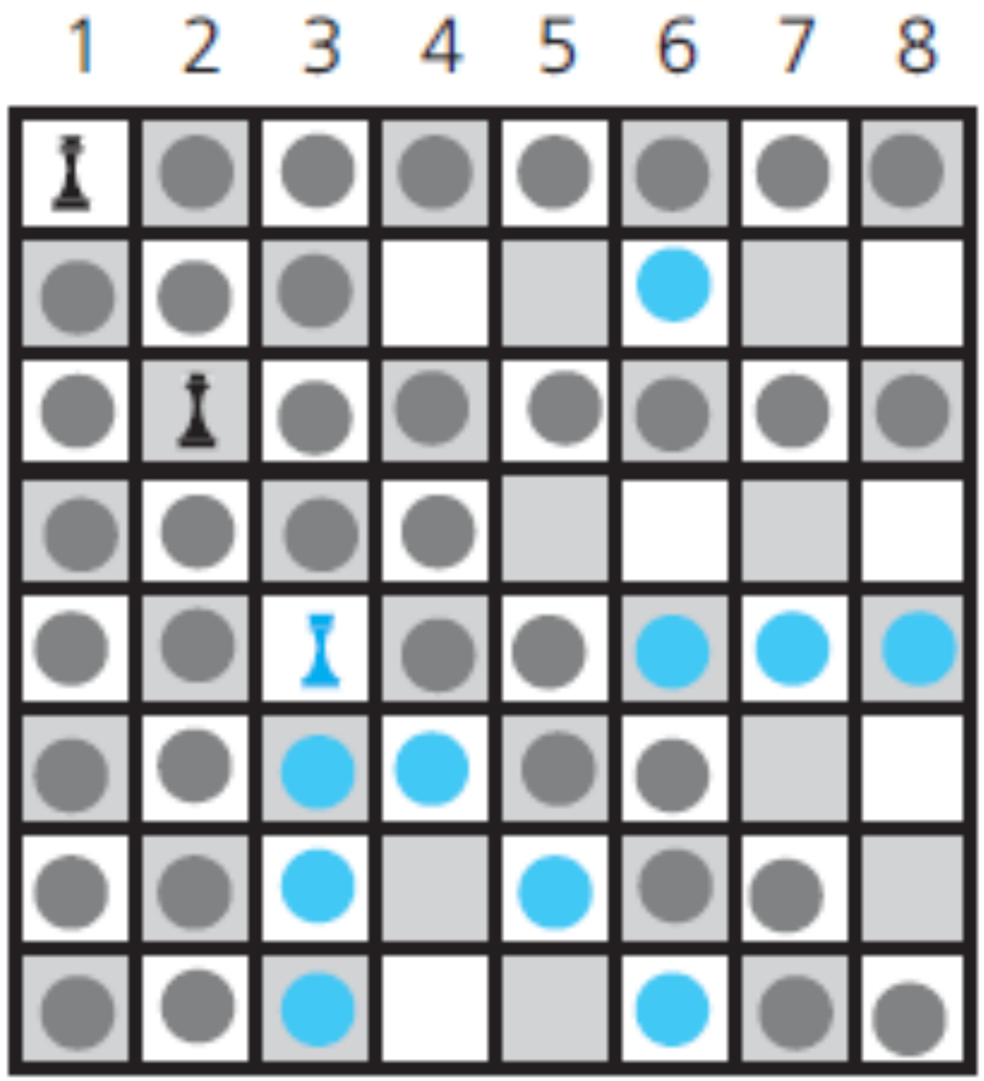
(a) The first queen in column 1

The Eight Queens Problem



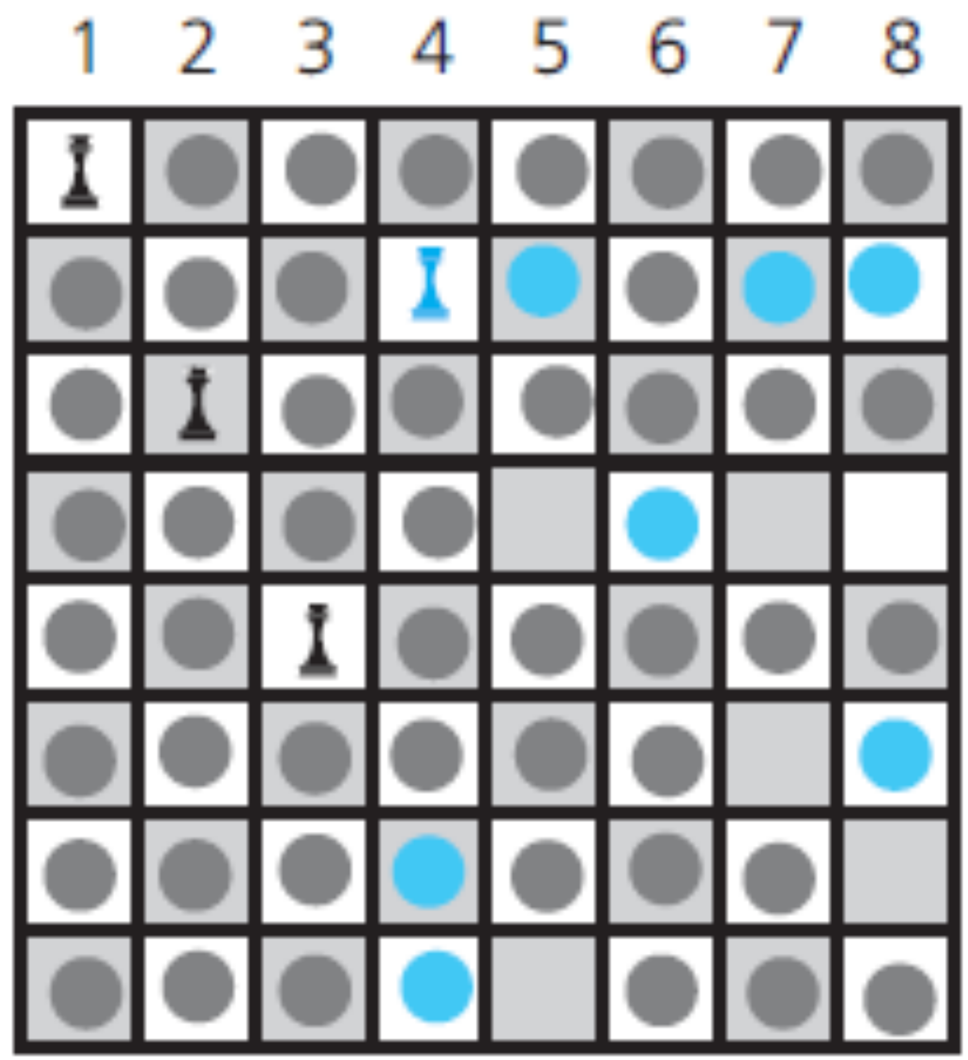
(b) The second queen in column 2

The Eight Queens Problem



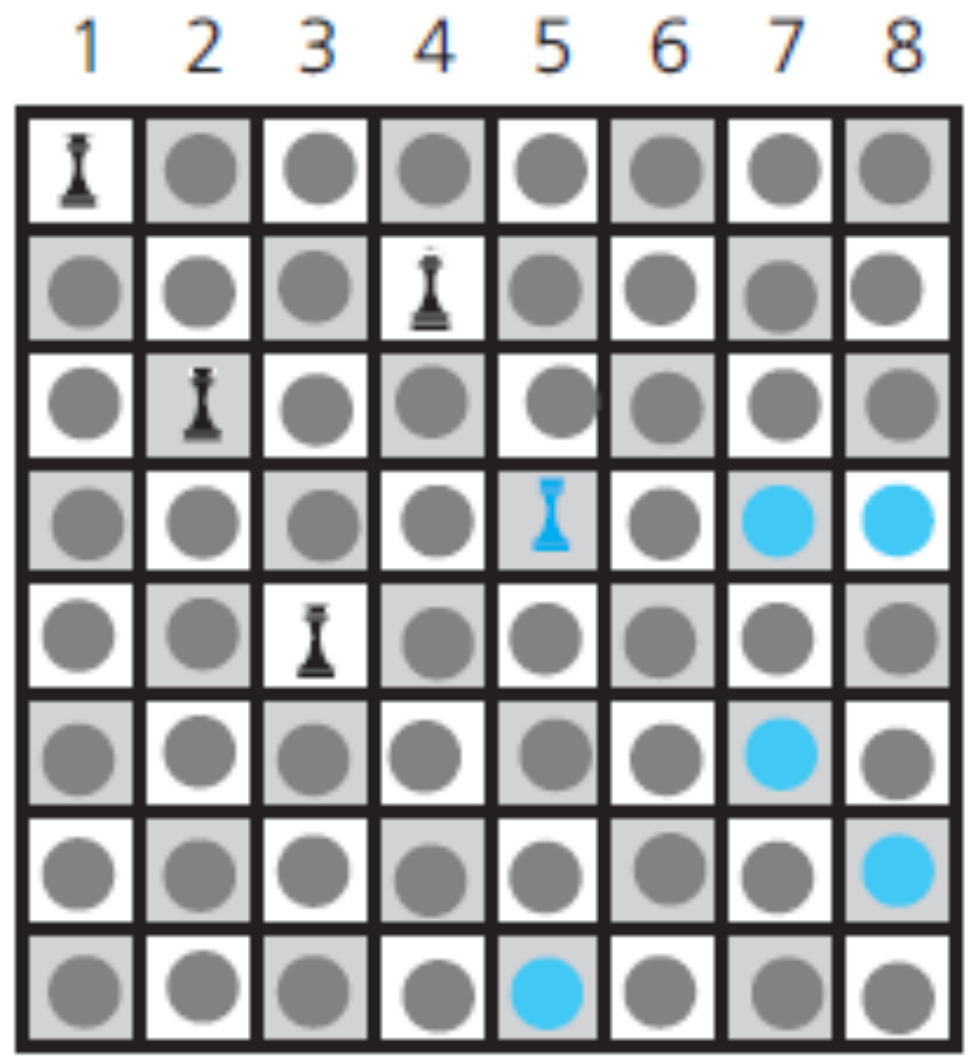
(c) The third queen in column 3

The Eight Queens Problem



(d) The fourth queen in column 4

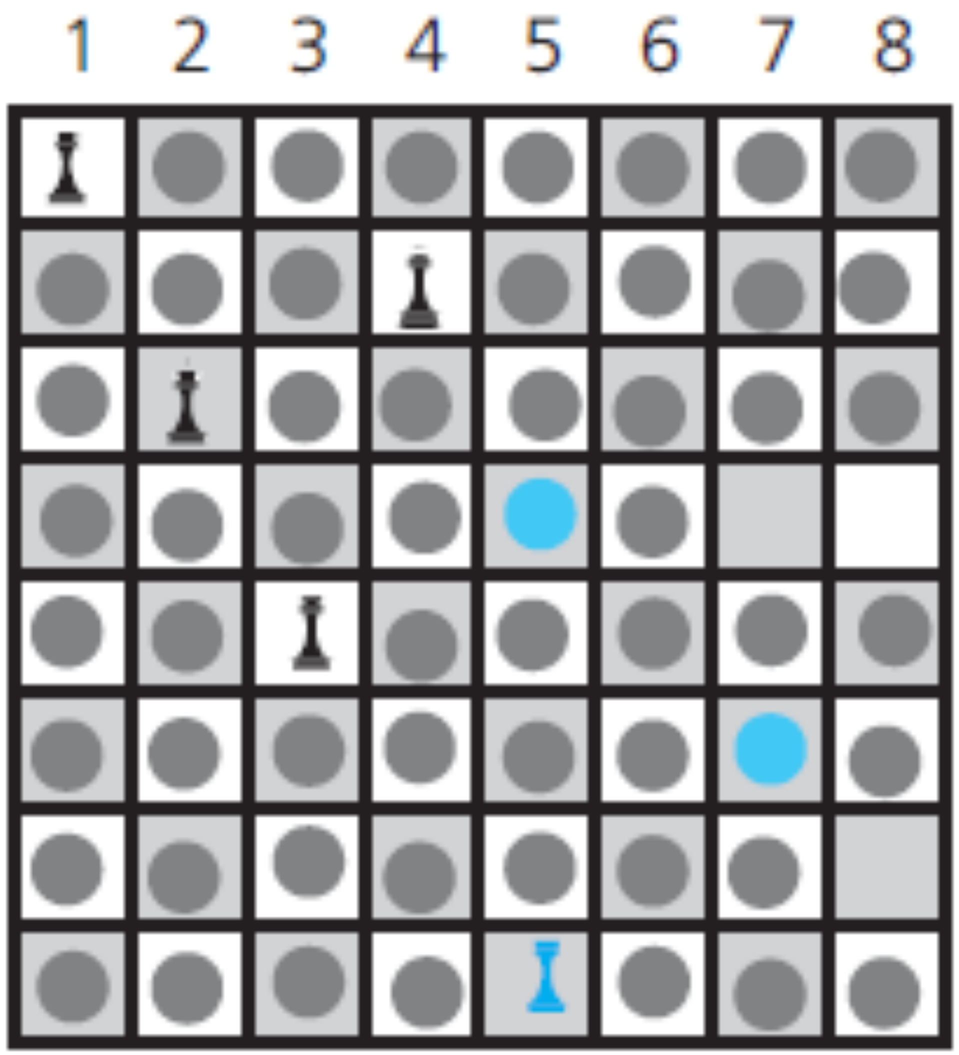
The Eight Queens Problem



(e) The five queens can attack all of column 6

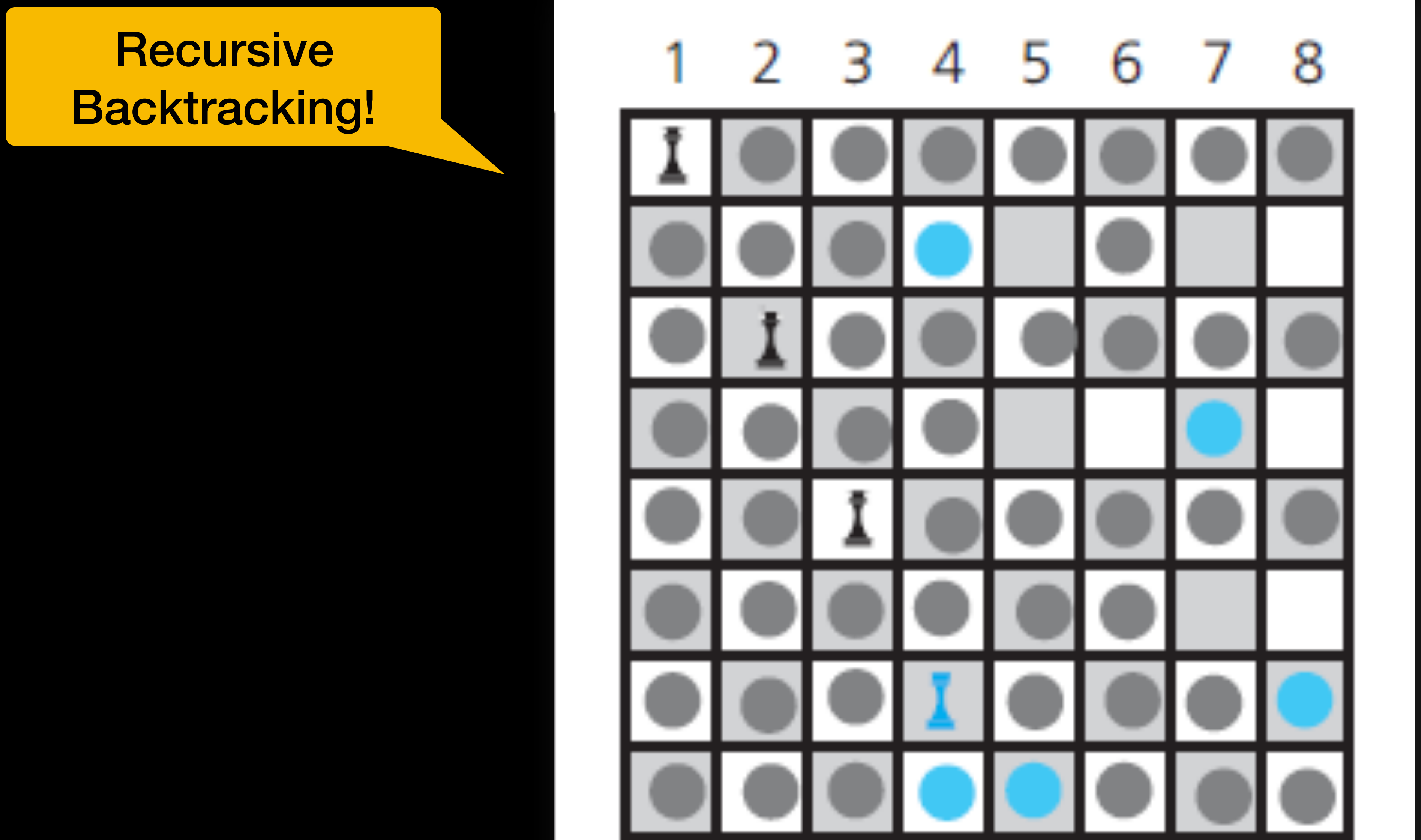
The Eight Queens Problem

Recursive Backtracking!



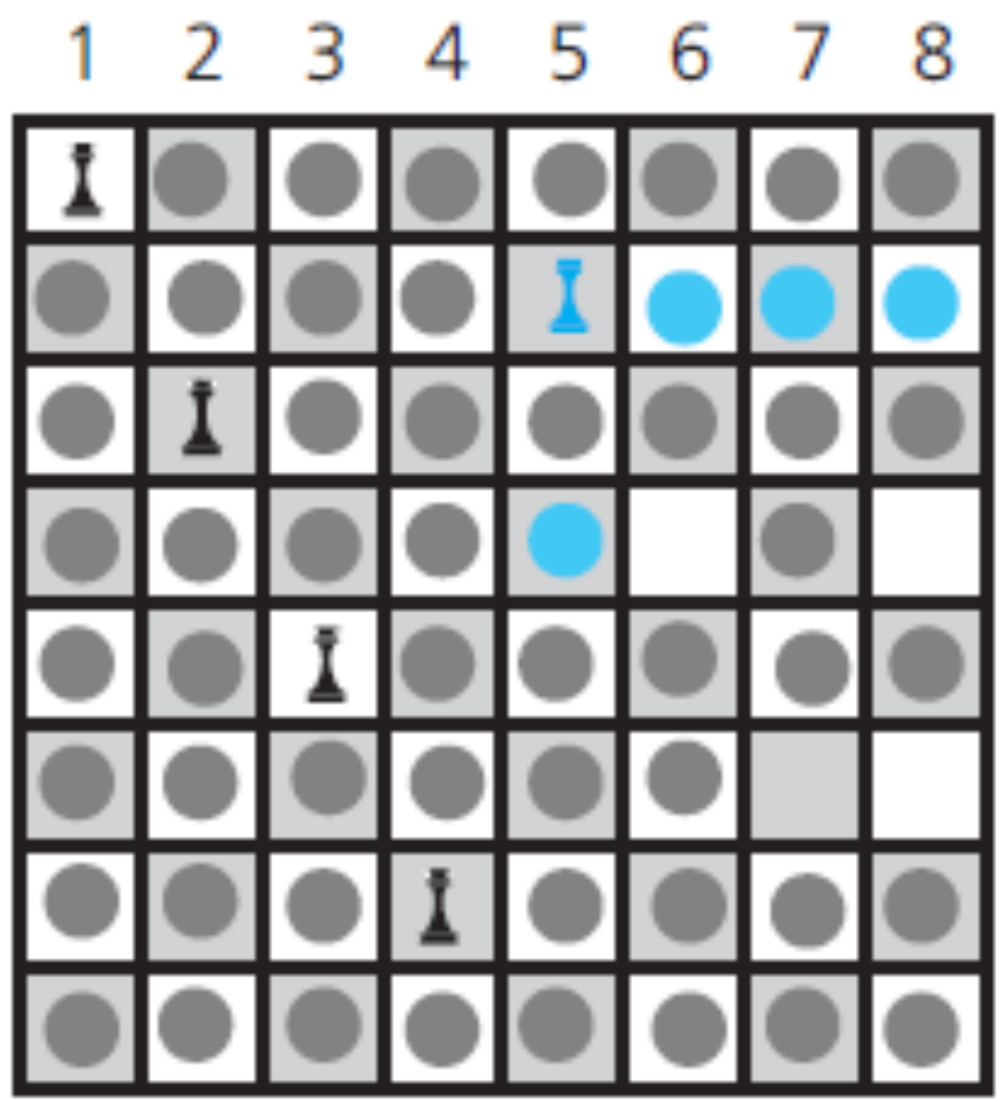
(f) Backtracking to column 5 to try another square for the queen

The Eight Queens Problem



(g) Backtracking to column 4 to try another square for the queen

The Eight Queens Problem



(h) Considering column 5 again

The Eight Queens Problem

```
bool placeQueens(board, column)
{
    if(column > BOARD_SIZE)
        return true; //Problem is solved!
    else{
        while(there are safe squares in this column)
        {
            place queen in next safe  square;
            if(placeQueen(board, column+1)) //recursively look forward
                return true; //queen safely placed
        }
        return false; //recursive backtracking
}
```

Find Permutations

A	B	C	D

A	B	C	D
A	B	D	C
A	C	B	D
A	C	D	B
A	D	B	C
A	D	C	B

B	A	C	D
B	A	D	C
B	C	A	D
B	C	D	A
B	D	A	C
B	D	C	A

C	A	B	D
C	A	D	B
C	B	A	D
C	B	D	A
C	D	A	B
C	D	B	A

D	A	B	C
D	A	C	B
D	B	A	C
D	B	C	A
D	C	A	B
D	C	B	A

Find Permutations

A	B	C	D

A	B	C	D	B	A	C	D	C	A	B	D	D	A	B	C
A	B	D	C	B	A	D	C	C	A	D	B	D	A	C	B
A	C	B	D	B	C	A	D	C	B	A	D	D	B	A	C
A	C	D	B	B	C	D	A	C	B	D	A	D	B	C	A
A	D	B	C	B	D	A	C	C	D	A	B	D	C	A	B
A	D	C	B	B	D	C	A	C	D	B	A	D	C	B	A

Find Permutations

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A	D	B	C
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B	A	D	C
B	C	A	D
B	C	D	A
B	D	A	C
B	D	C	A

C	A	B	D
C	A	D	B
C	B	A	D
C	B	D	A
C	D	A	B
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D	A	B	C
D	A	C	B
D	B	A	C
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Find Permutations

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B	A	D	C
B	C	A	D
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Find Permutations

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Find Permutations

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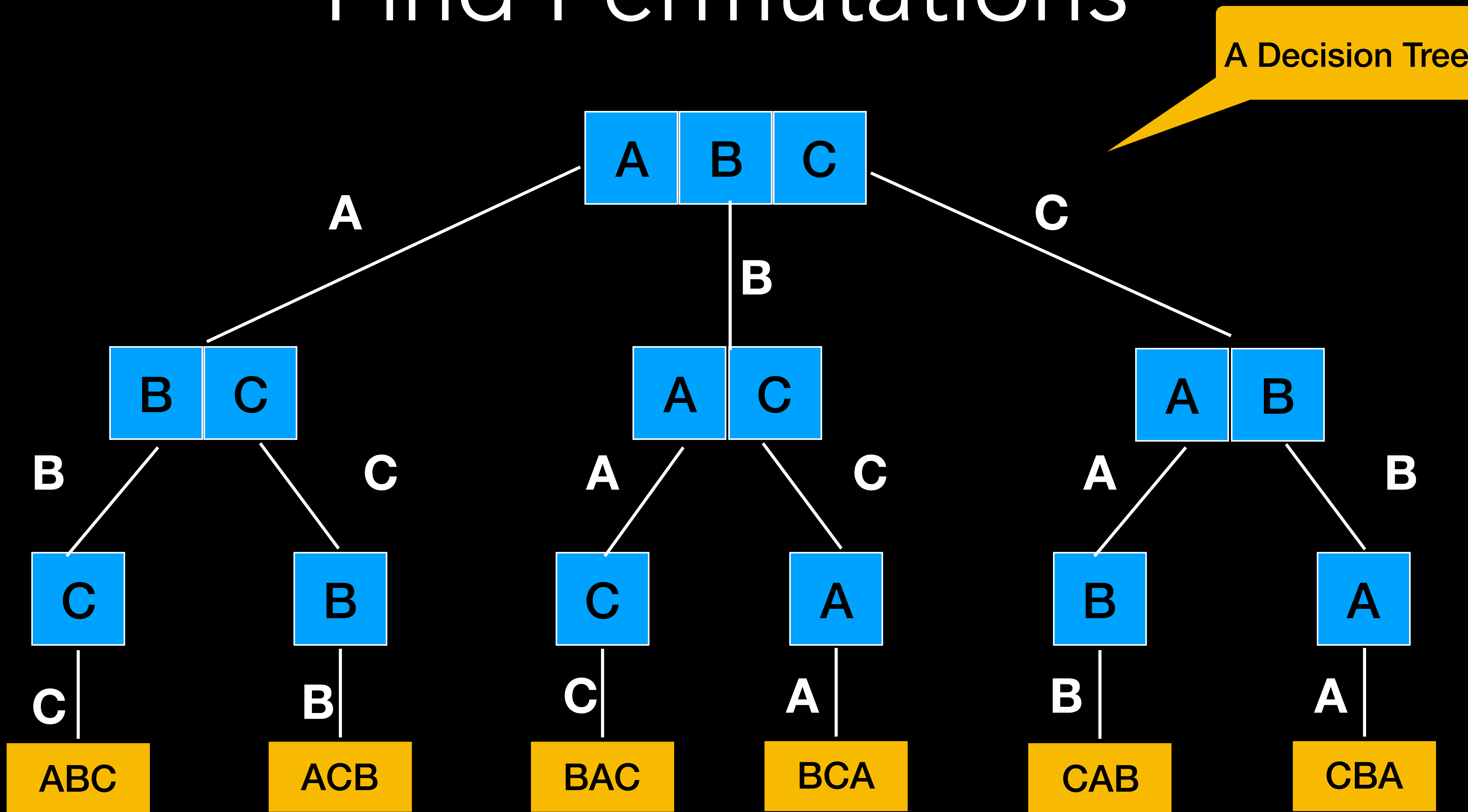
Find Permutations



```
vector<string> generatePermutations(string word)
{
    vector<string> result;
    if (word.length() == 0)
    {
        // The empty string has only itself as a permutation
        result.push_back(word);
        return result;
    }
    for (int i = 0; i < word.length(); i++)
    {
        // The word without the ith letter
        string shorter_word = word.substr(0, i) + word.substr(i + 1);
        vector<string> shorter_permutations =
                                       generatePermutations(shorter_word);

        // Add the ith letter to the front of all permutations
        // of the shorter word
        for (int j = 0; j < shorter_permutations.size(); j++)
        {
            string longer_word = word[i] + shorter_permutations[j];
            result.push_back(longer_word);
        }
    }
    return result;
}
```

Exam Drill:
Analyze the worst-case time complexity of this algorithm
O(?)

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    }
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}
```

```
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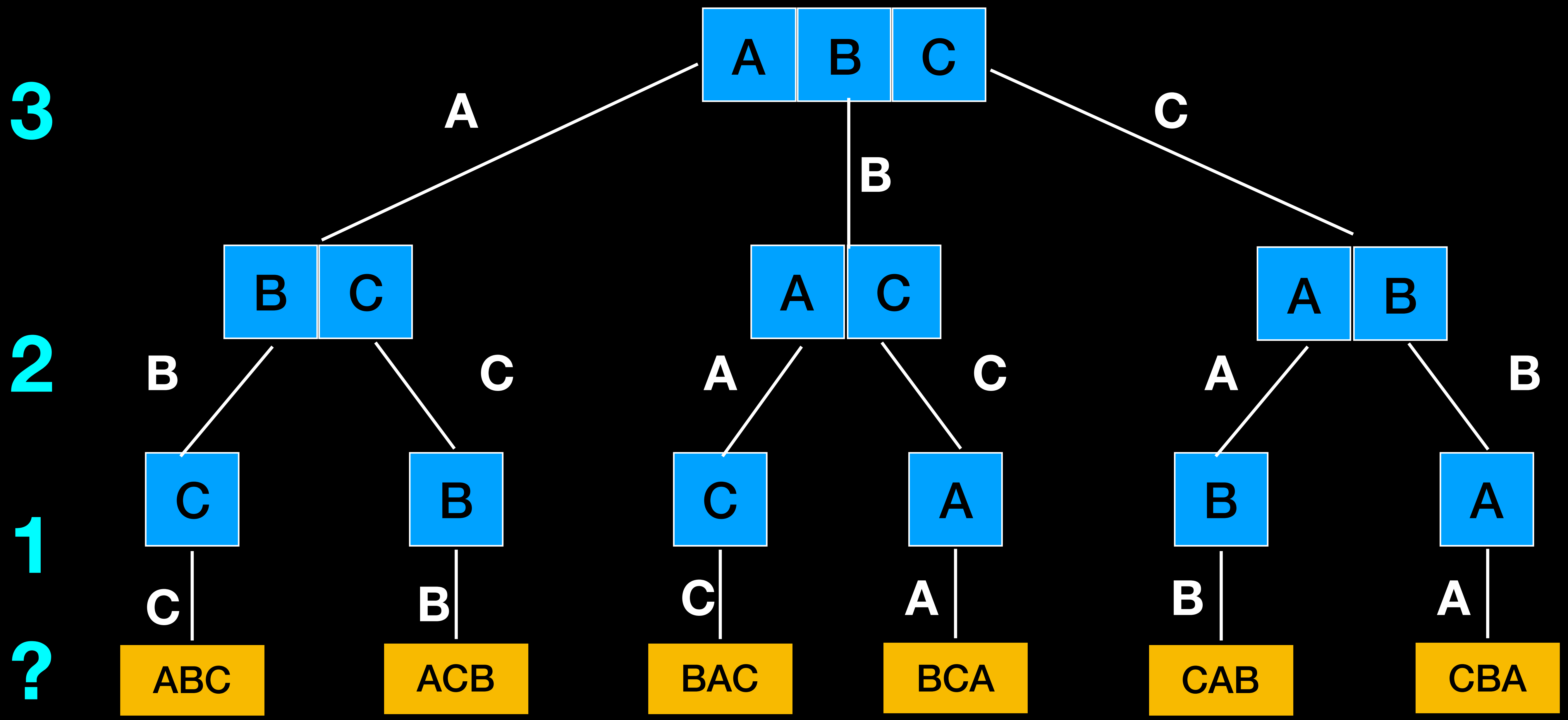
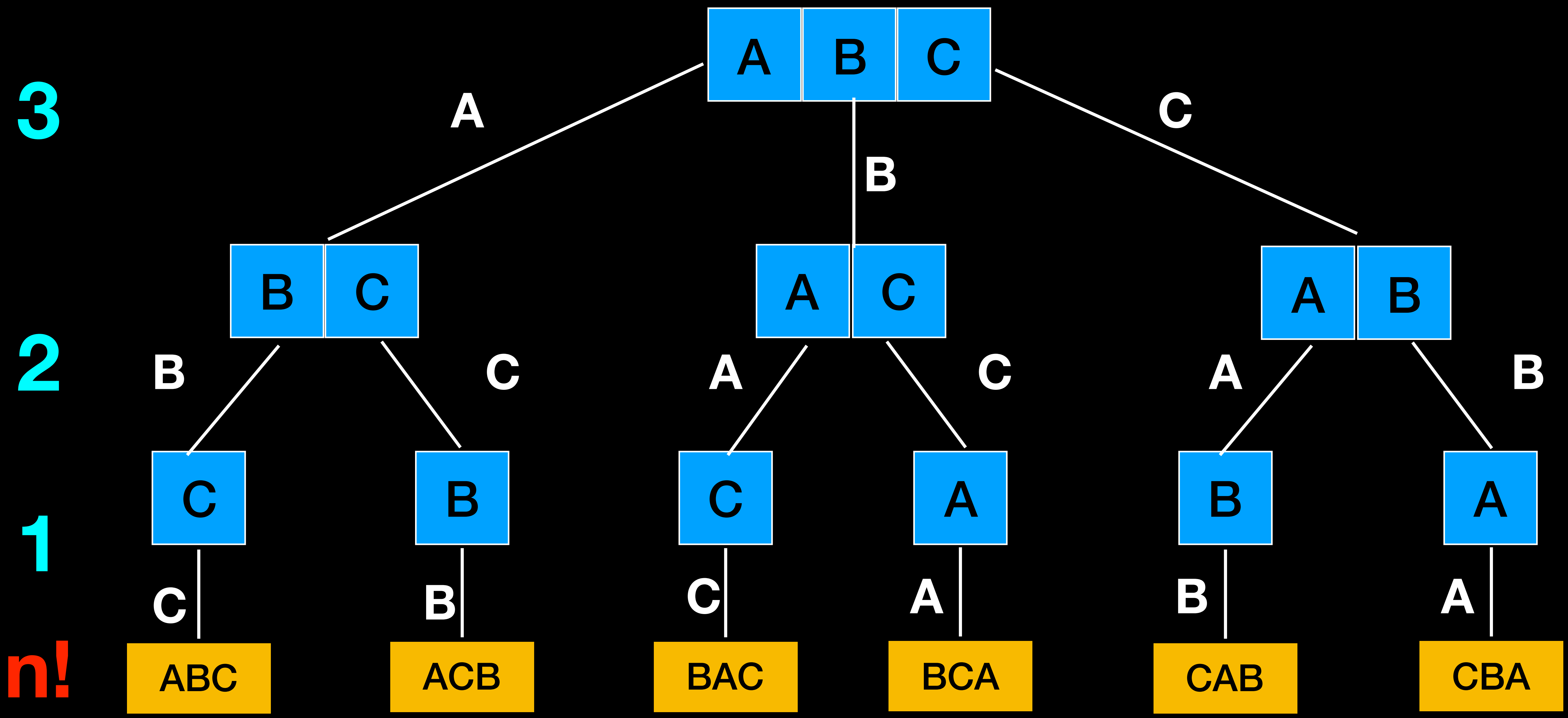
            string longer_word = word[i] + shorter_permutations[j];
            result.push_back(longer_word);
        }
    }
    return result;
}
```

k n k ? n k

Find Permutations



Find Permutations



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              result.push_back(longer_word);
          }
      }
      return result;
}
```

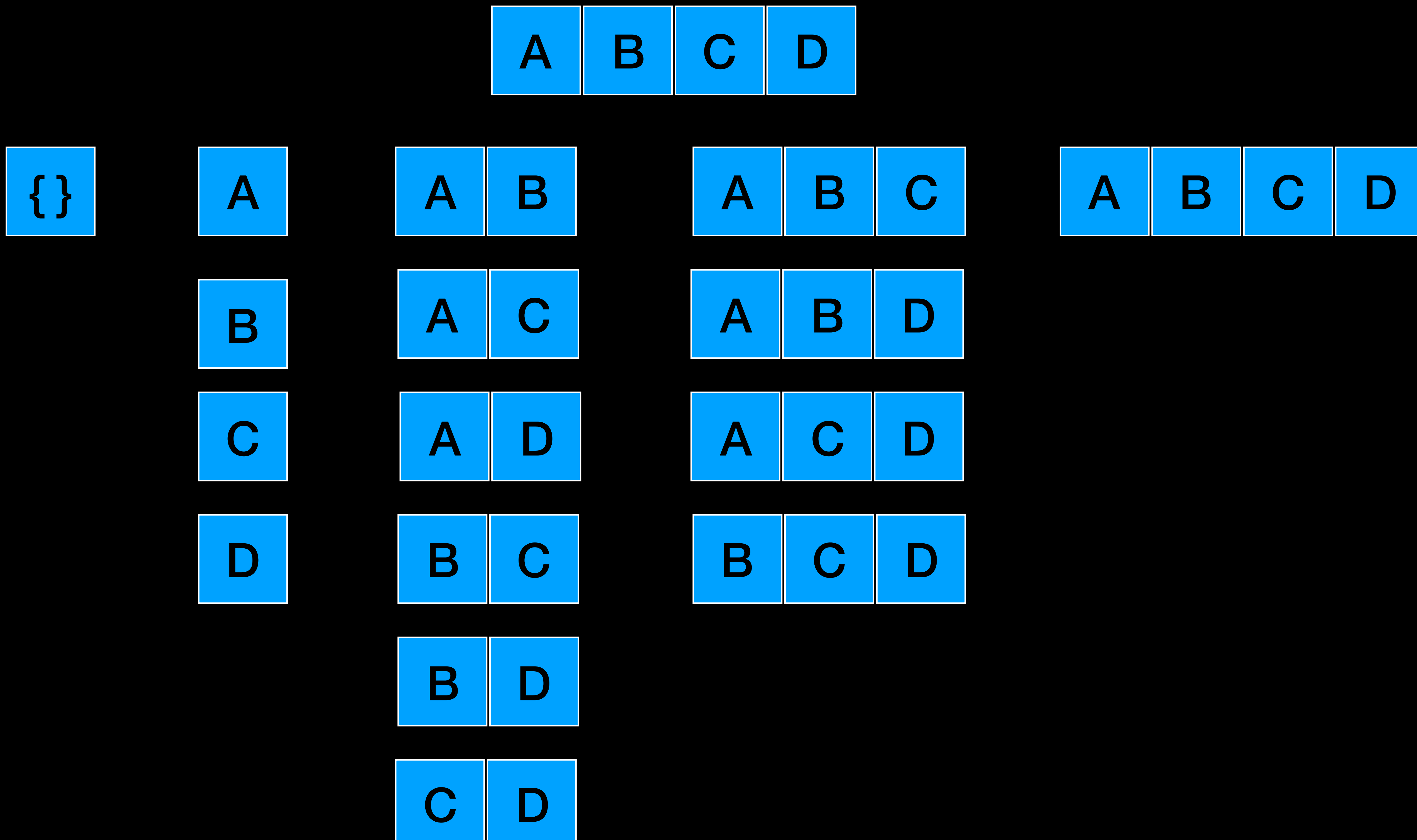
$$T(n) = O(\, n!\, n^2 \,)$$

Recursive Decision Tree

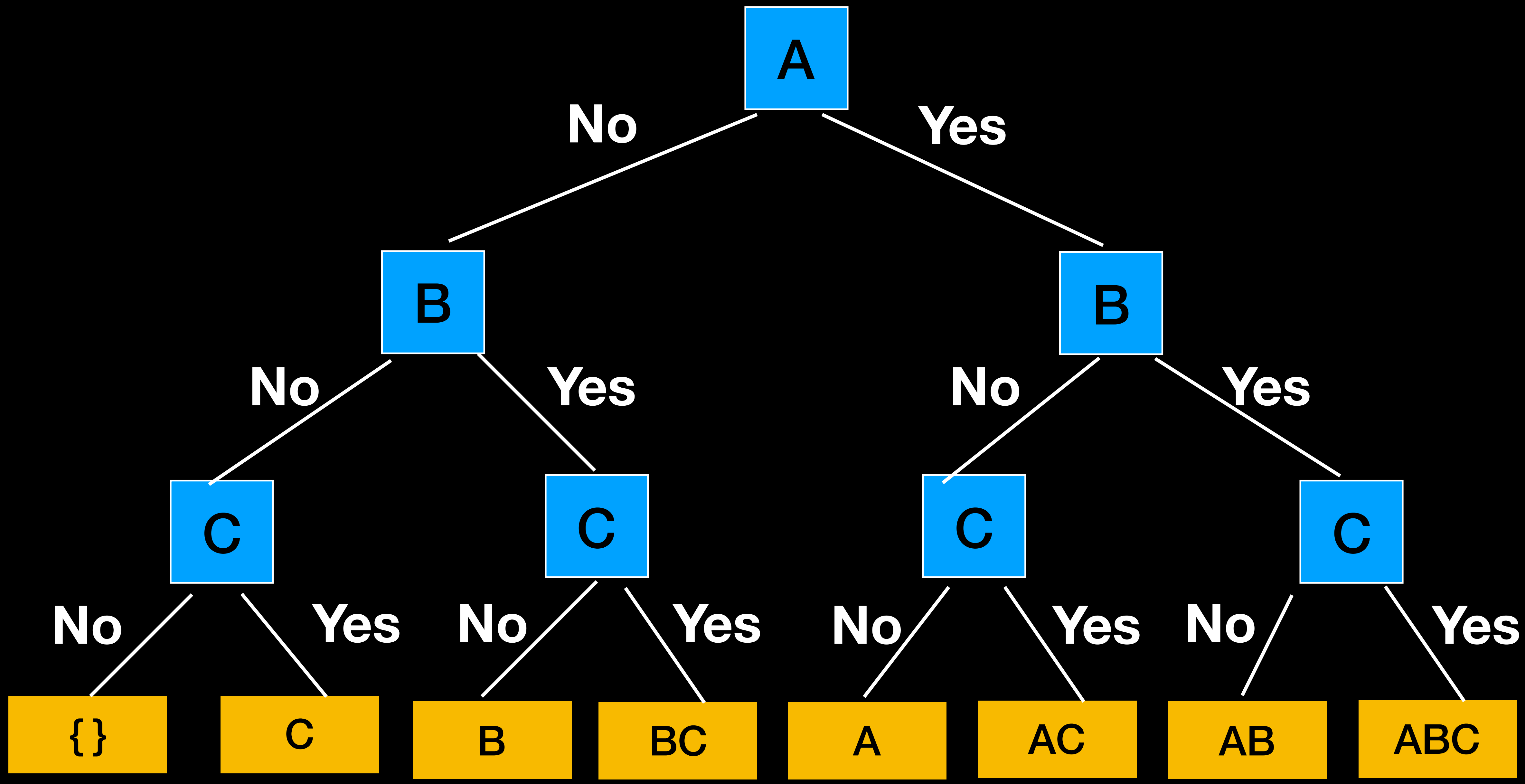
```
void exploreFrom(current_state, decisions_made) {
    if (all decisions have been made) { //base case
        output the result of the decisions we've made;
    } else {
        for (each decision we can make) {
            exploreFrom(result of making that decision,
                        decisions_made + this_decision);
        }
    }
}
```

Generally, if you can express a problem solution with a decision tree you can translate it into a recursive algorithm

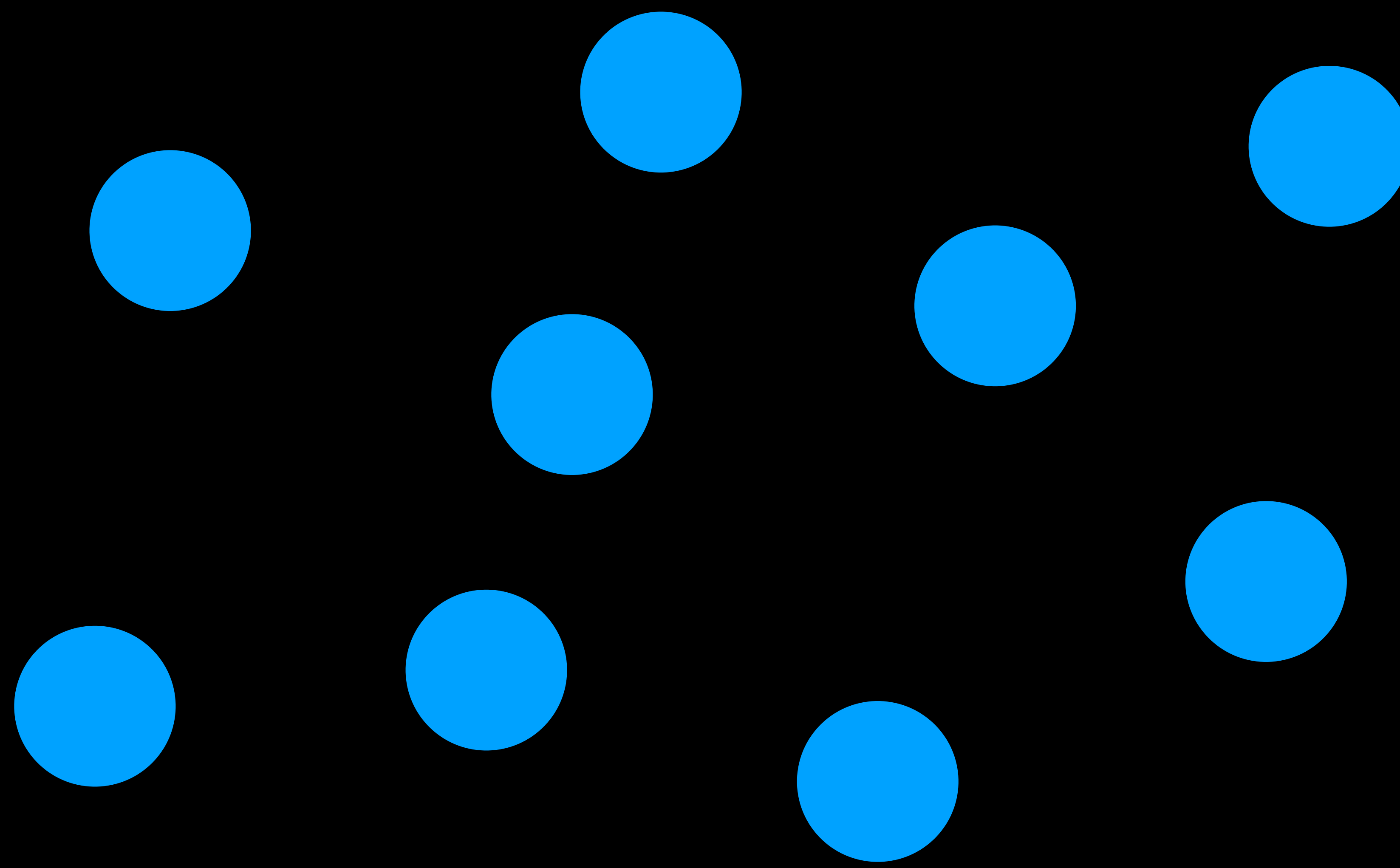
Find Combinations



Find All Combinations



Find Combinations (n choose k)



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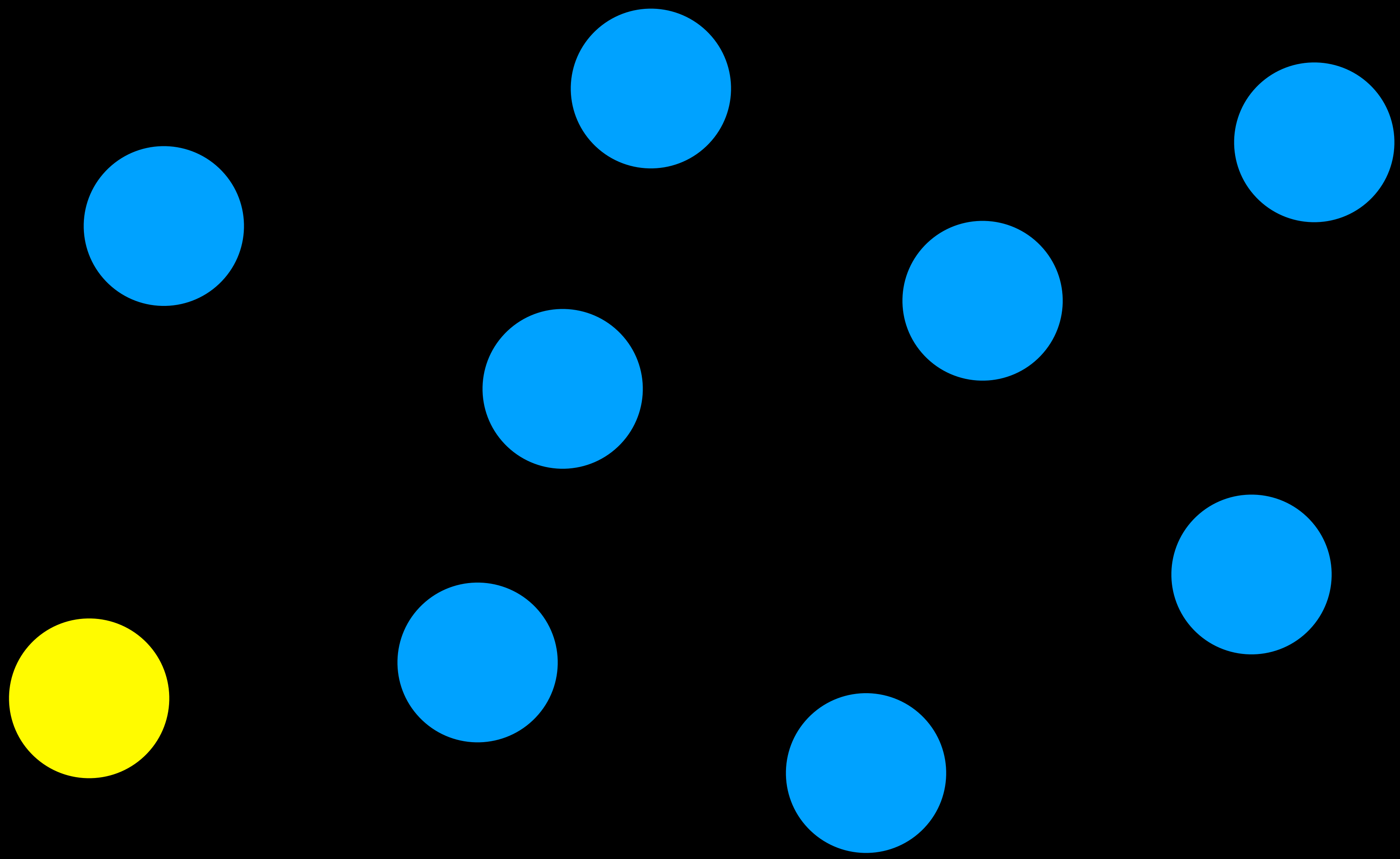

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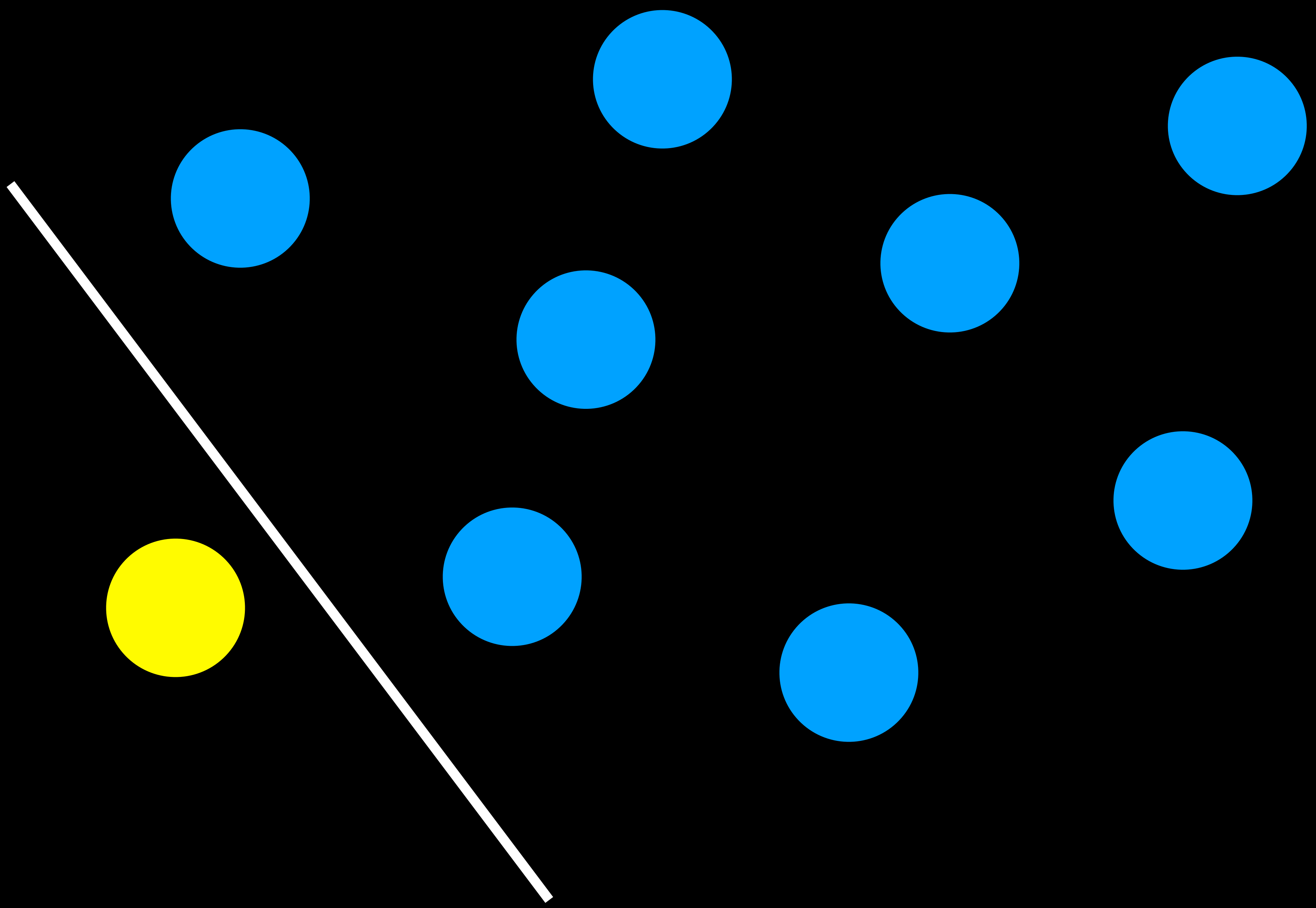

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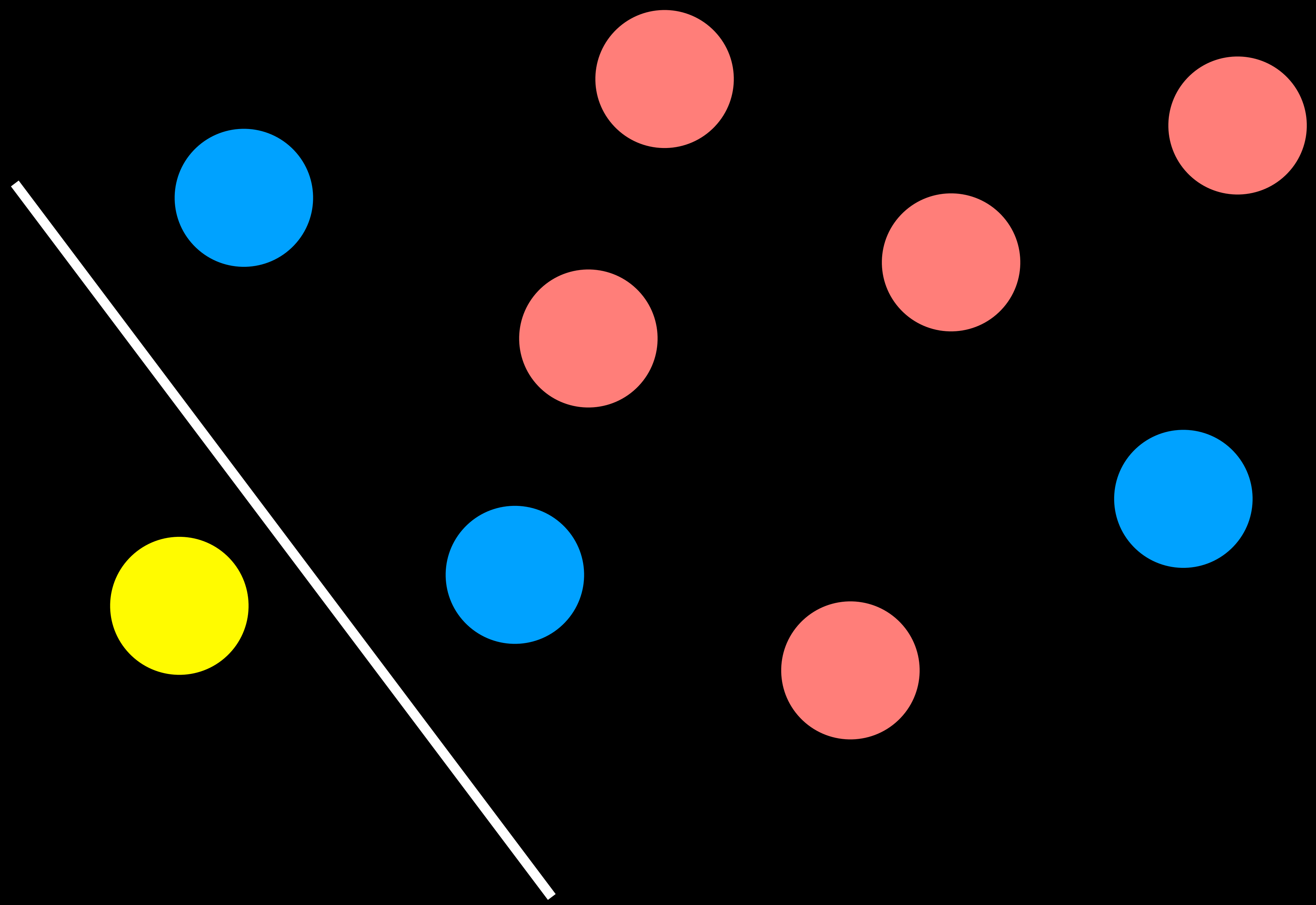

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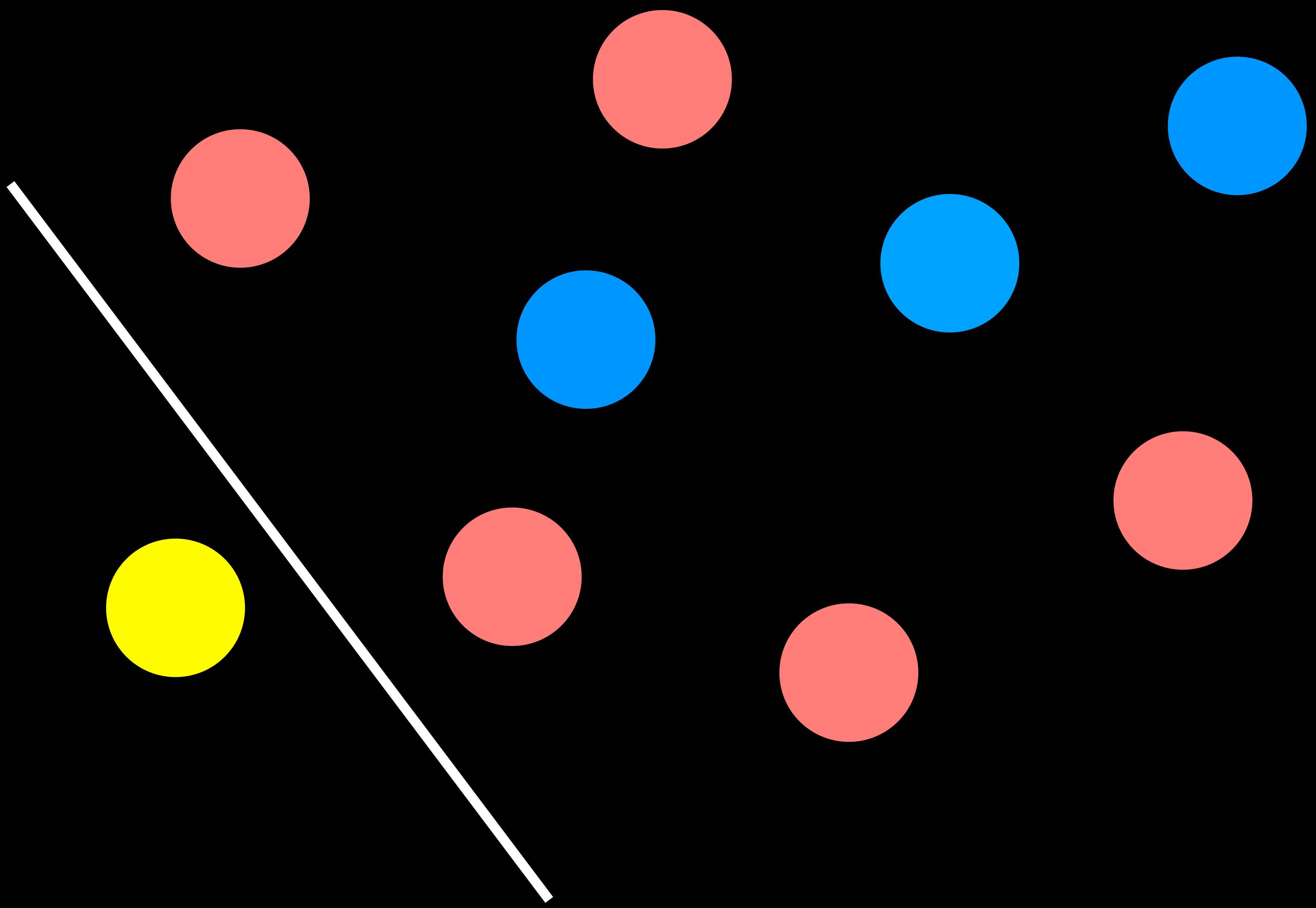

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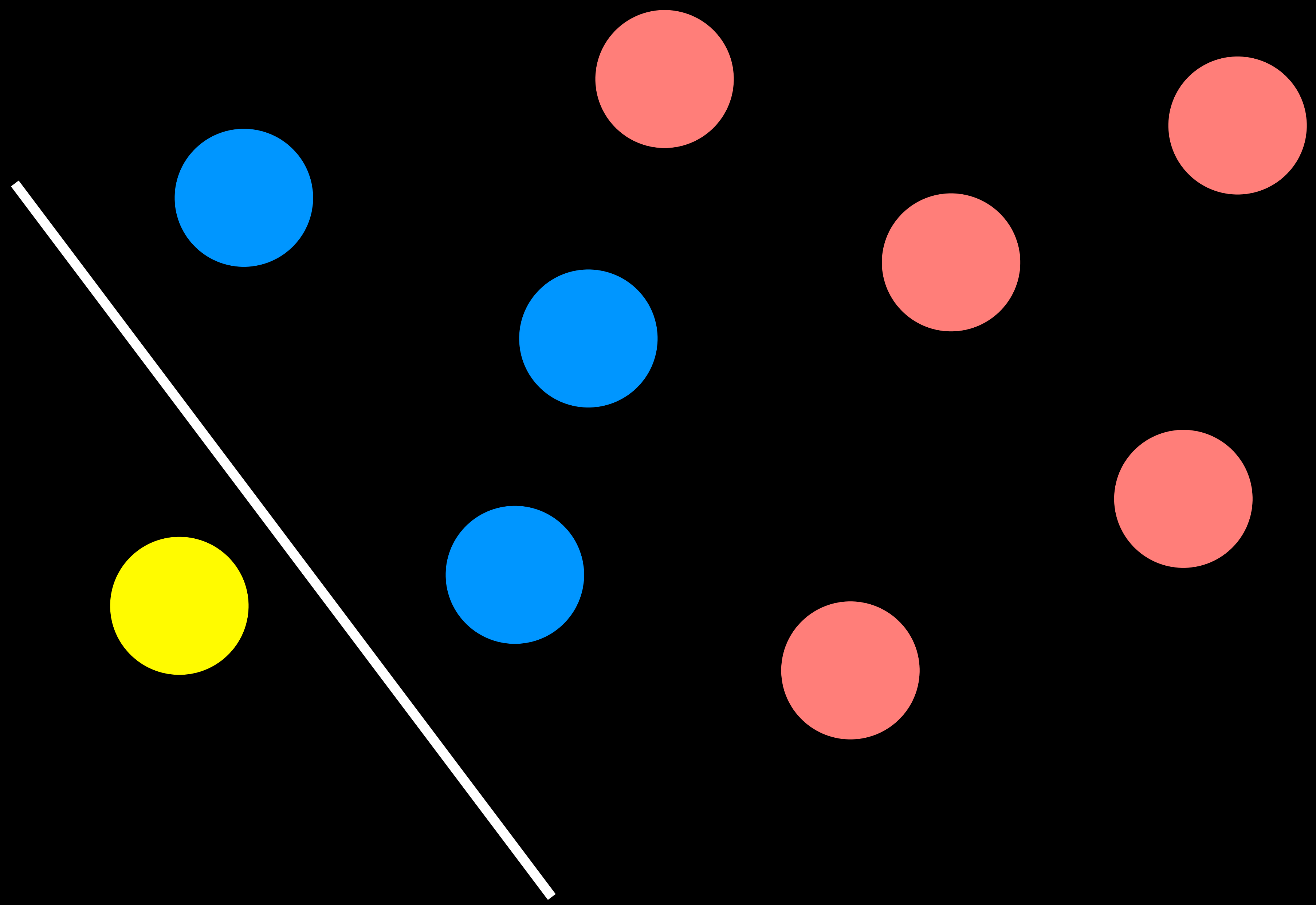

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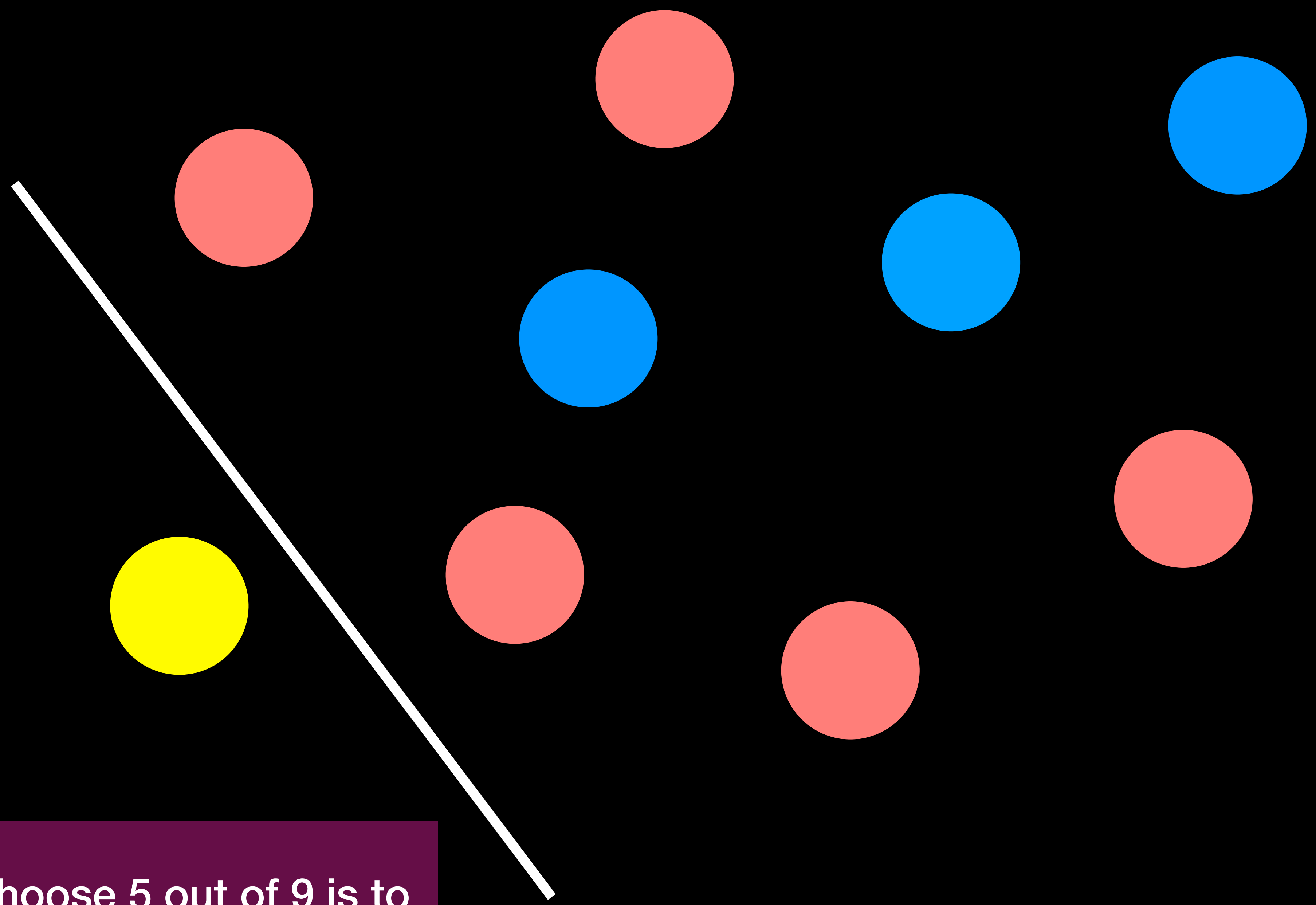
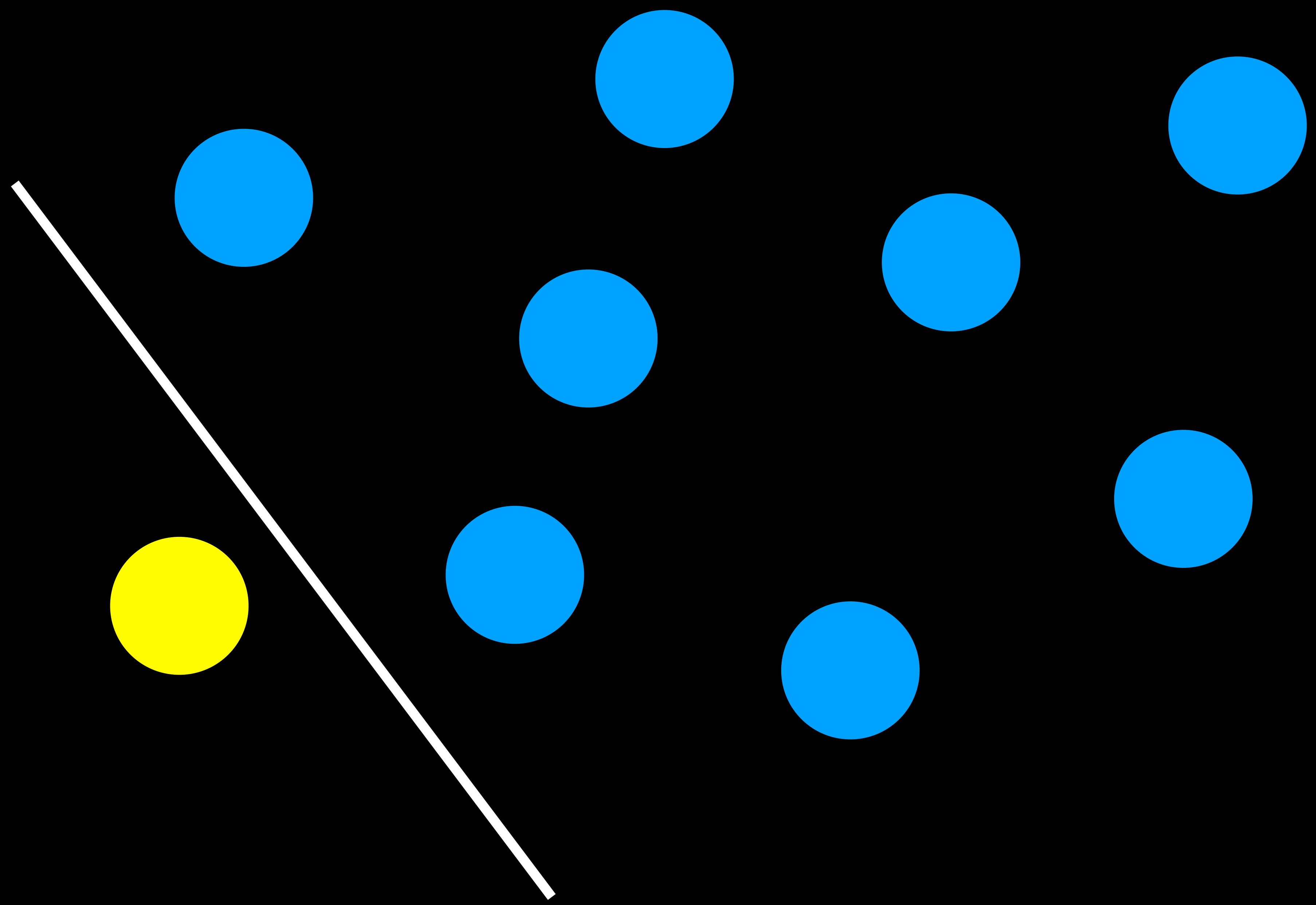


One way to choose 5 out of 9 is to
Exclude 1 and choose 5 out of 8

Find Combinations (n choose k)



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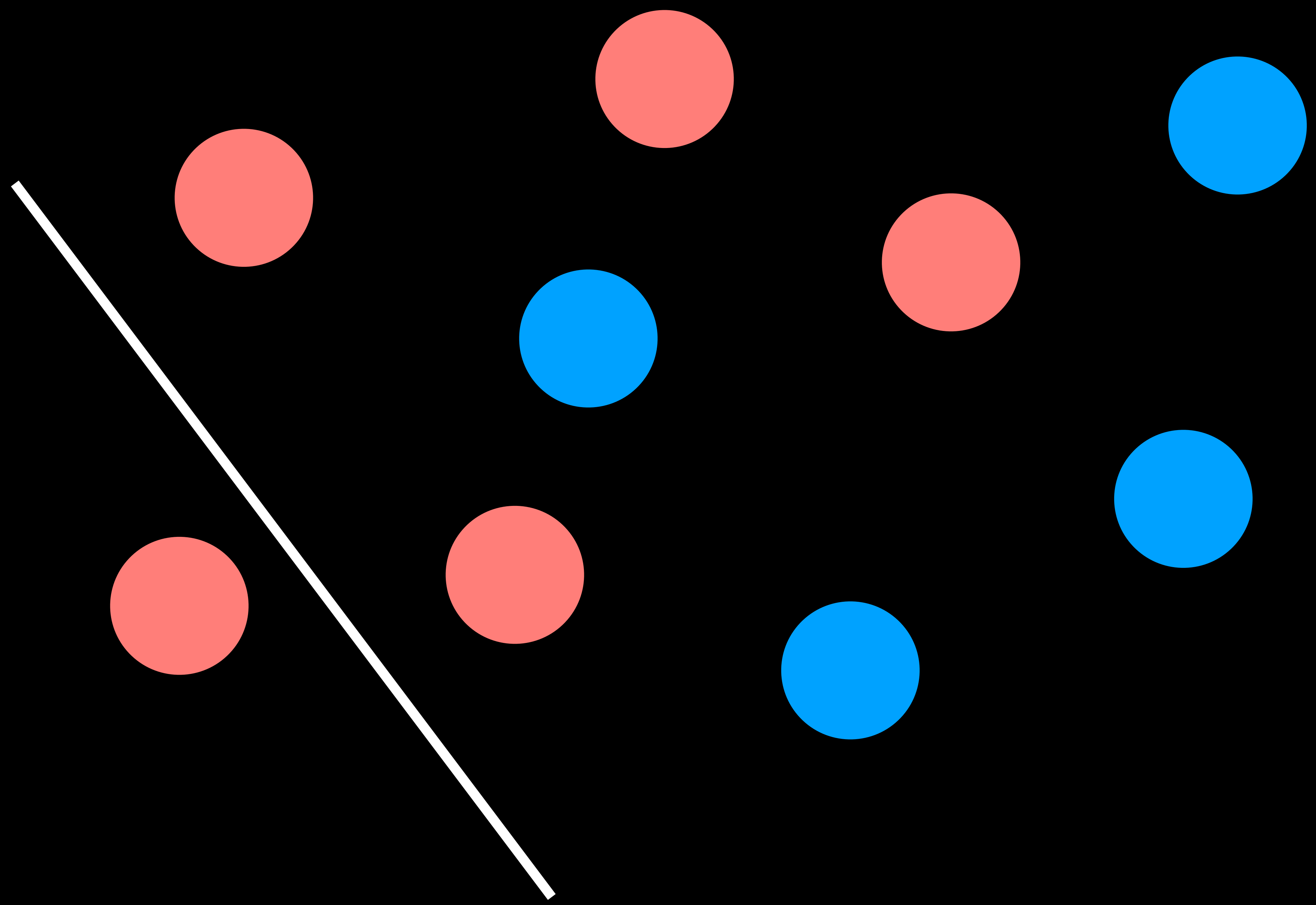
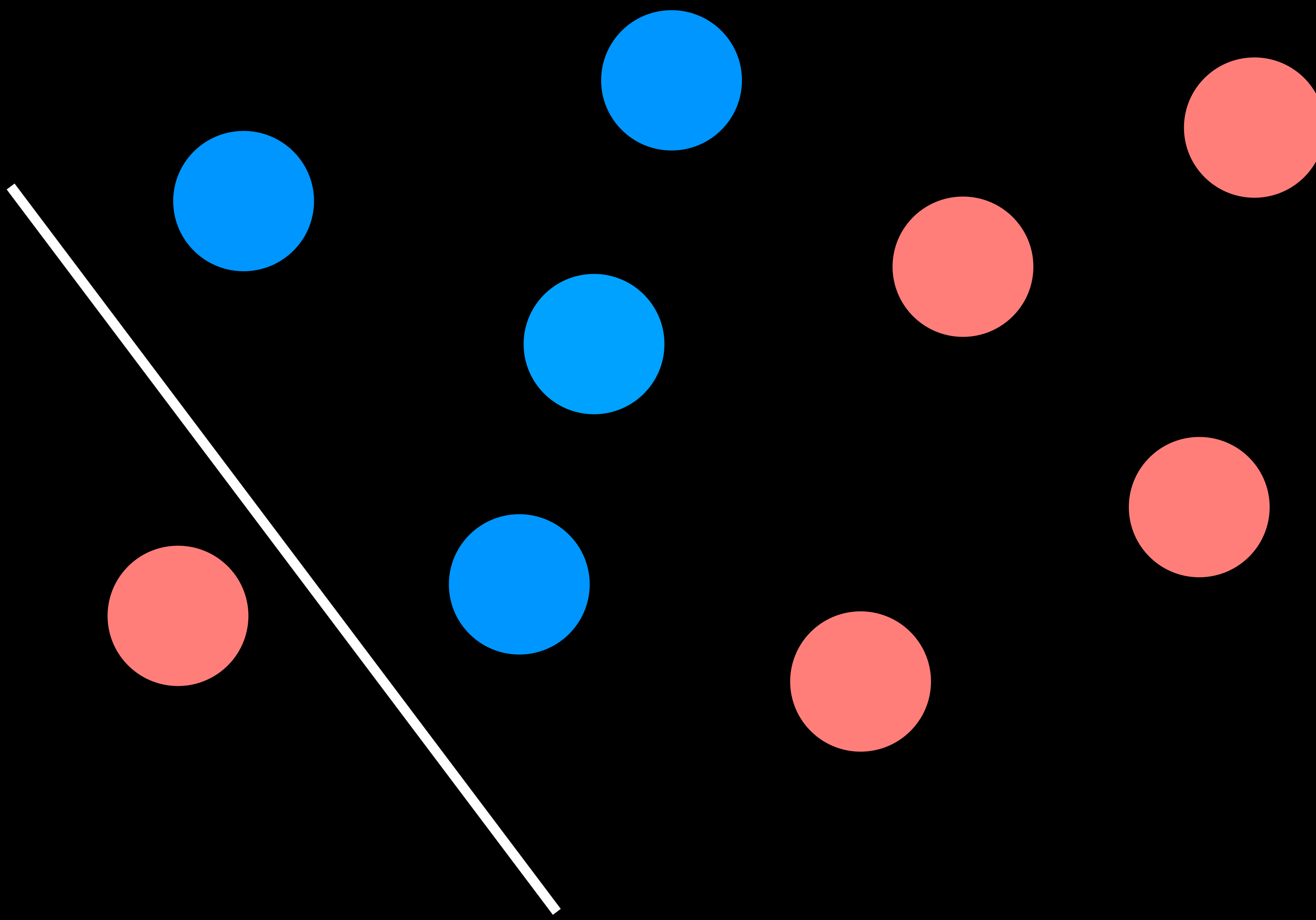
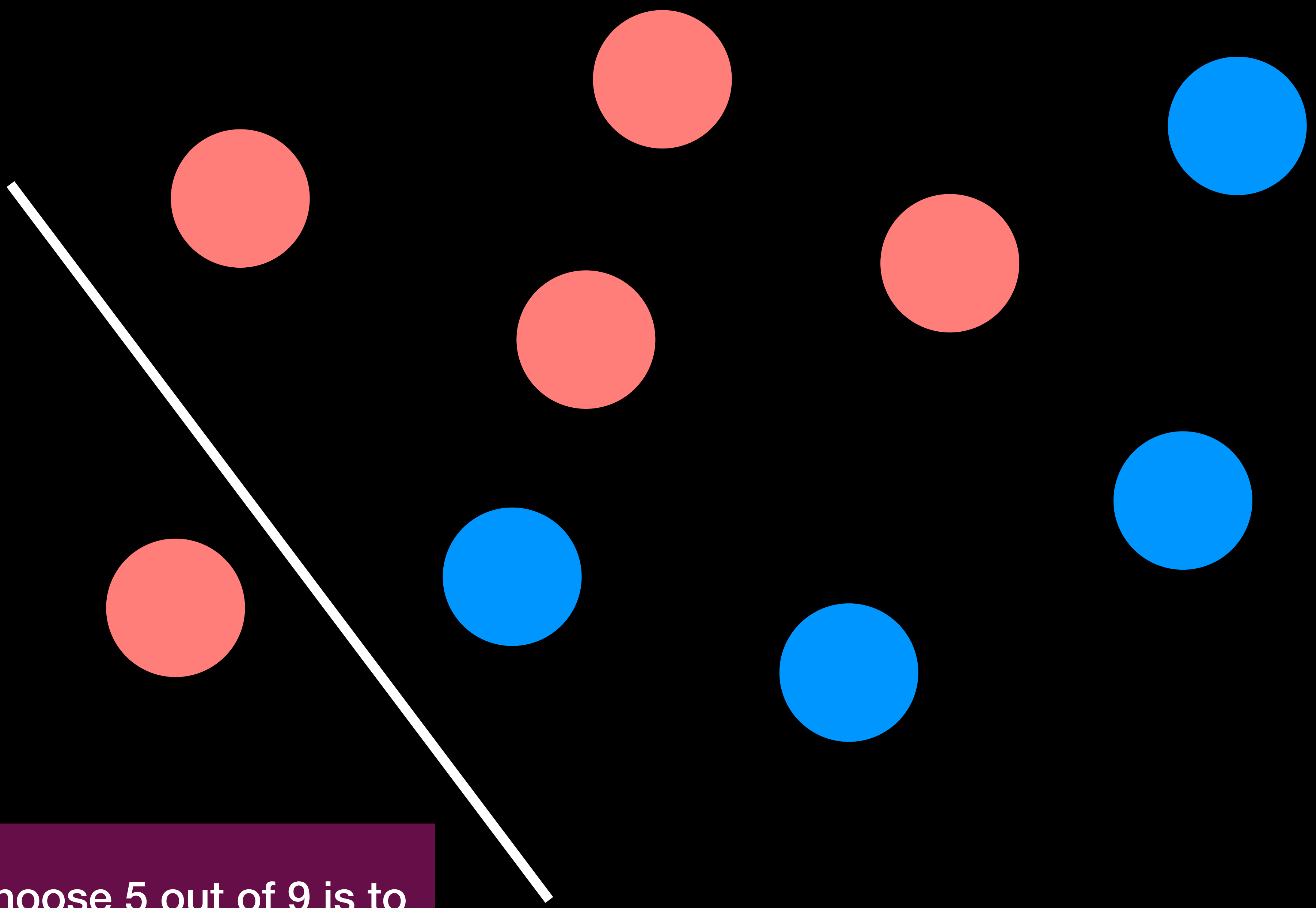


Find Combinations (n choose k)



Find Combinations (n choose k)



One way to choose 5 out of 9 is to
Include 1 and choose 4 out of 8

Find Combinations

```
chooseK(sequence,  k)
{
    if (k == 0) or (sequence.length() == k)
        return sequence; //basecase

    combinations_k = chooseK(sequence_without_first, k);
    combinations_k-1 = chooseK(sequence_without_first, k-1);
    append first element to combinations_k-1;
    return combinations_k + combinations_k-1;

}
```

Find Combinations

Analysis

```
chooseK(sequence,  k)
{
    if (k == 0) or (sequence.length() == k)
        return sequence; //basecase

    combinations_k = chooseK(sequence_without_first, k);
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}
```

Find Combinations

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    return combinations_k + combinations_k-1;

}
```

T(n-1, k)

T(n-1, k-1)

$$T(n, k) = T(n-1, k) + T(n-1, k-1) + c$$

$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

$$T(n, k) = \boxed{T(n-2, k) + T(n-2, k - 1)}$$

$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

$$T(n, k) = \boxed{T(n-2, k) + T(n-2, k - 1)} + \boxed{T(n-2, k-1) + T(n-2, k-2)} + 2c$$

$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

$$T(n, k) = \boxed{T(n-2, k) + T(n-2, k - 1)} + \boxed{T(n-2, k-1) + T(n-2, k-2)} + 2c$$

$$T(n, k) = T(n-2, k) + 2T(n-2, k - 1) + T(n-2, k-2) + 2c$$

$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

$$T(n, k) = \boxed{T(n-2, k) + T(n-2, k - 1)} + \boxed{T(n-2, k-1) + T(n-2, k-2)} + 2c$$

$$T(n, k) = \boxed{T(n-2, k)} + \boxed{2T(n-2, k - 1)} + \boxed{T(n-2, k-2)} + 2c$$

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$$T(n, k) = \boxed{T(n-3, k) + T(n-3, k-1)}$$

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$$T(n, k) = \boxed{T(n-3, k) + T(n-3, k-1)} + \boxed{2T(n-3, k-1) + 2T(n-3, k-2)} +$$

$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

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$$T(n, k) = \boxed{T(n-3, k) + T(n-3, k-1)} + \boxed{2T(n-3, k - 1) + 2T(n-3, k - 2)} + \boxed{T(n-3, k-2) + T(n-3, k-3)} + 3c$$

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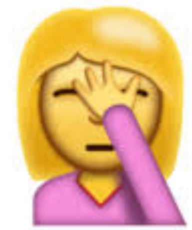
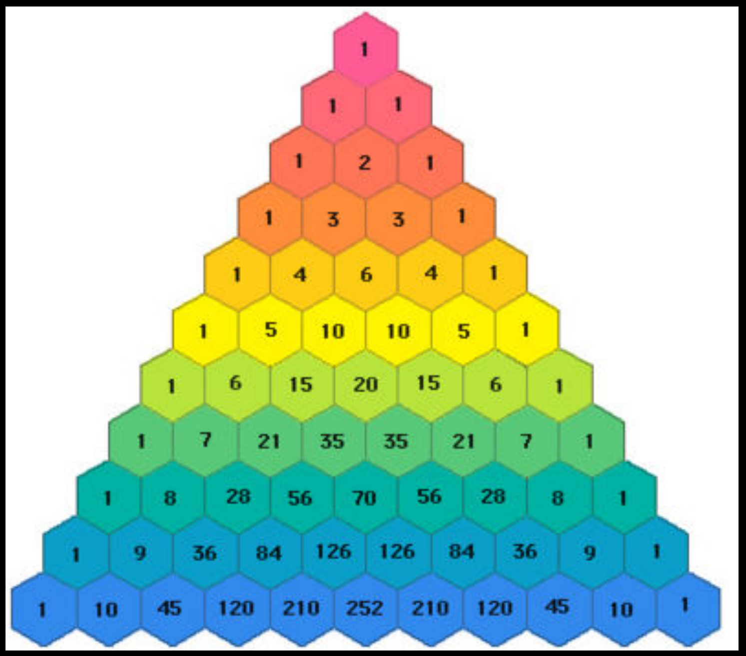
$$T(n, k) = \boxed{T(n-2, k)} + \boxed{2T(n-2, k - 1)} + \boxed{T(n-2, k-2)} + 2c$$

$$\begin{aligned} T(n, k) = {} & \boxed{T(n-3, k) + T(n-3, k-1)} + \boxed{2T(n-3, k - 1) + 2T(n-3, k - 2)} + \\ & \boxed{T(n-3, k-2) + T(n-3, k-3)} + 3c \end{aligned}$$

$$T(n, k) = T(n-3, k) + 3T(n-3, k - 1) + 3T(n-3, k - 2) + T(n-3, k-3) + 3c$$

Of Course!!!
It's nChoosek !!!





$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

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$$T(n, k) = T(n-3, k) + 3T(n-3, k - 1) + 3T(n-3, k - 2) + T(n-3, k-3) + 3c$$

Binomial Coefficients

But we are looking for Big-O!!!

$$T(n, k) = \boxed{T(n-1, k)} + \boxed{T(n-1, k-1)} + c$$

$$T(n, k) = \boxed{T(n-2, k) + T(n-2, k-1)} + \boxed{T(n-2, k-1) + T(n-2, k-2)} + 2c$$

$$T(n, k) = \boxed{T(n-2, k)} + \boxed{2T(n-2, k-1)} + \boxed{T(n-2, k-2)} + 2c$$

$$T(n, k) = \boxed{T(n-3, k) + T(n-3, k-1)} + \boxed{2T(n-3, k-1) + 2T(n-3, k-2)} + \boxed{T(n-3, k-2) + T(n-3, k-3)} + 3c$$

$$T(n, k) = T(n-3, k) + 3T(n-3, k-1) + 3T(n-3, k-2) + T(n-3, k-3) + 3c$$

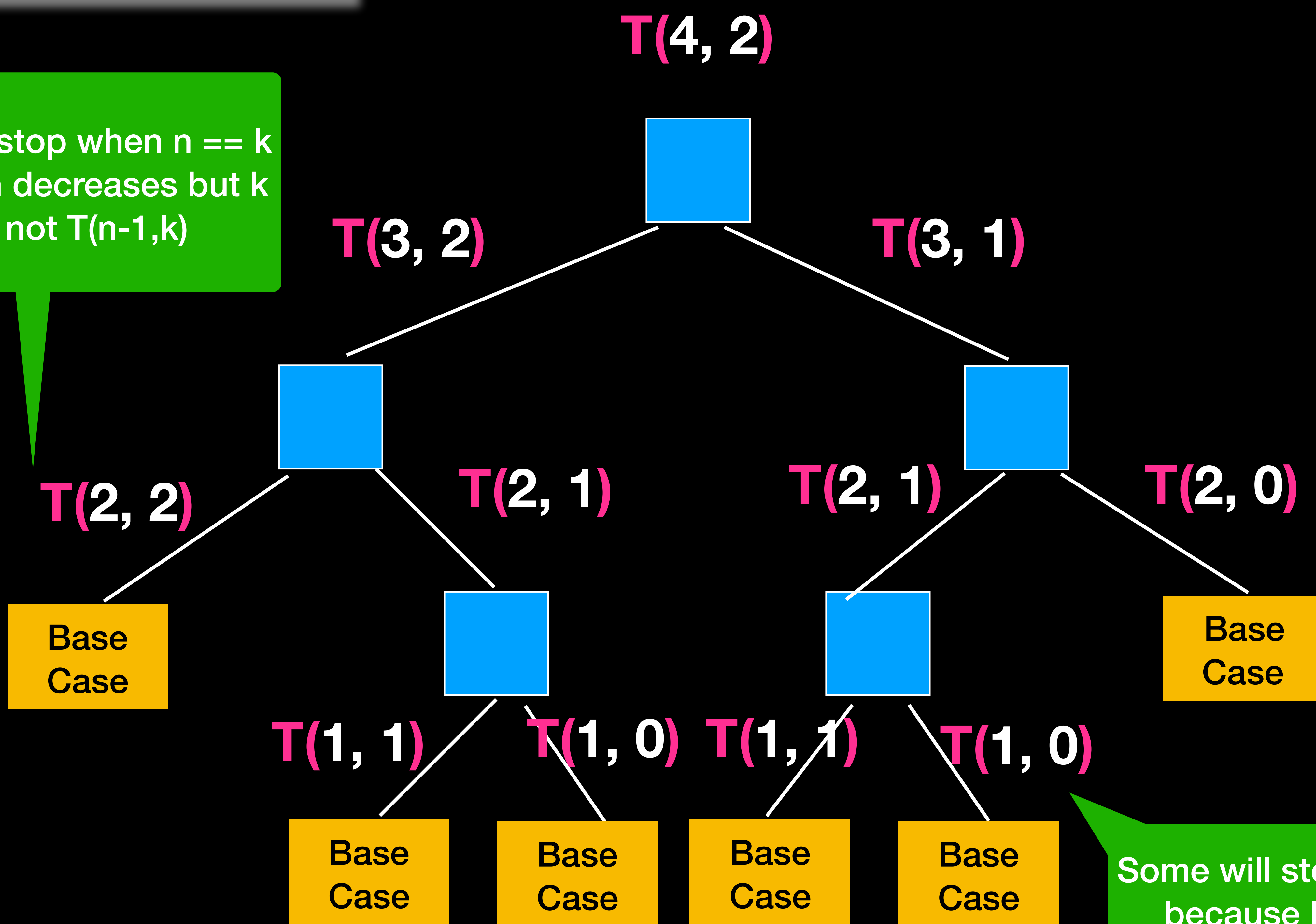
Some will stop when n == 0

All +
We are looking for
dominant term

Some will stop when k == n

We know n > k

Some will stop when n == k because n decreases but k does not T(n-1,k)

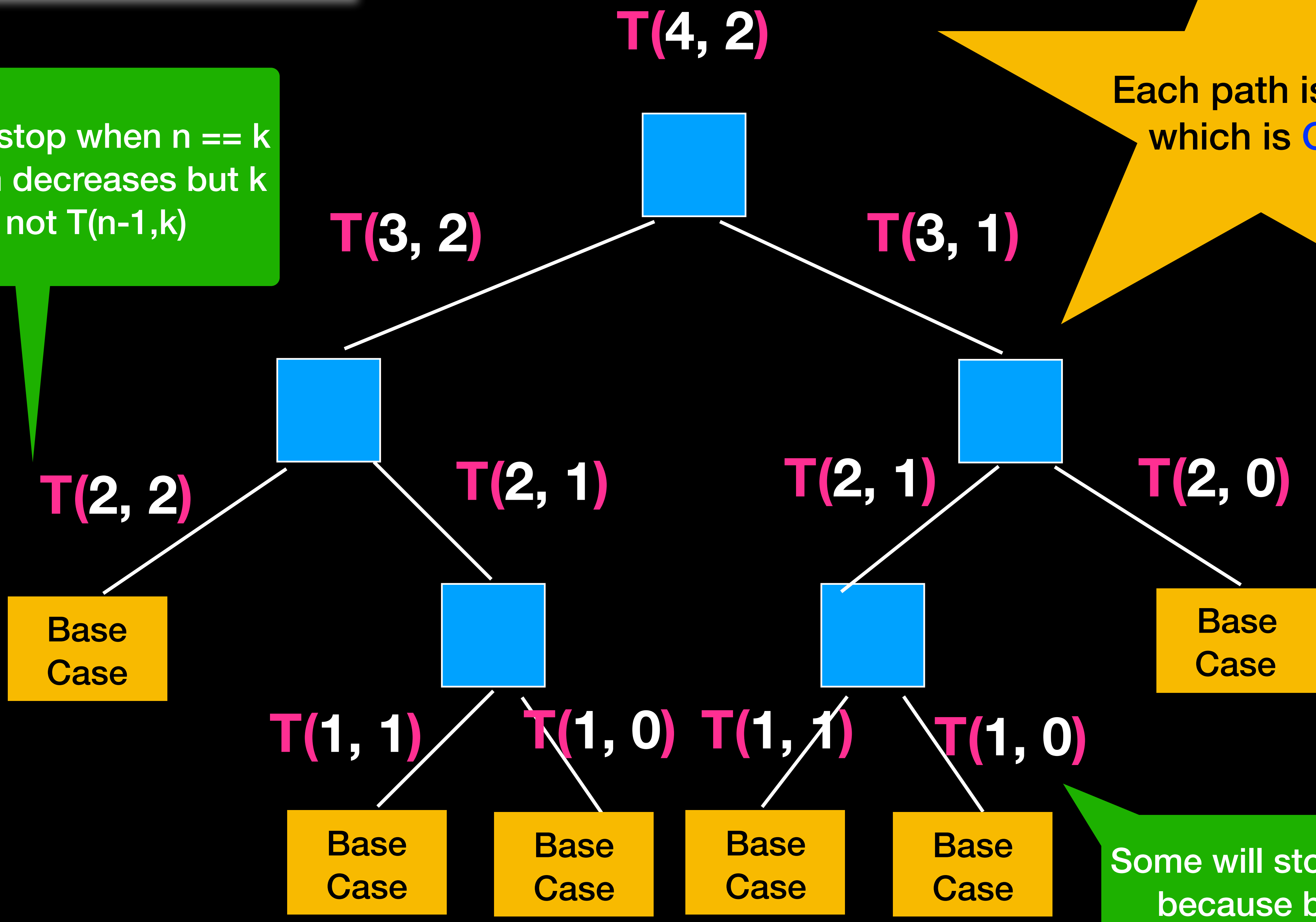


Some will stop when k == 0 because both n and k decrease but k < n T(n-1,k-1)

We know n > k

Some will stop when n == k because n decreases but k does not T(n-1,k)

Each path is O(k) which is O(n)



Some will stop when k == 0 because both n and k decrease but k < n T(n-1,k-1)

Find Combinations

```
chooseK(sequence,  k)
{
    if (k == 0) or (sequence.length() == k)
        return sequence; //basecase

    combinations_k = chooseK(sequence_without_first, k);
    combinations_k-1 = chooseK(sequence_without_first, k-1);
    append first element to combinations_k-1;
    return combinations_k + combinations_k-1;

}
```

T(n-1, k)

T(n-1, k-1)

$$T(n, k) = c_1 O(n) + c_2 O(n) + O(1)$$

Find Combinations

```
chooseK(sequence,  k)
{
    if (k == 0) or (sequence.length() == k)
        return sequence; //basecase

    combinations_k = chooseK(sequence_without_first, k);
    combinations_k-1 = chooseK(sequence_without_first, k-1);
    append first element to combinations_k-1;
    return combinations_k + combinations_k-1;

}
```

$$T(n) = O(\ n\ )$$

Count Combinations

```
int countCombinations(int n, int k)
{
    if ( (k == 0) || (k == n))
        return 1;
    else
        return countCombinations(n-1, k-1) +
                          countCombinations((n-1), k);
}
```