Trees

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Today’s Plan

Trees

Binary Tree ADT

Binary Search Tree ADT

Announcements

Office hours on Thursday this week, 10am - 12pm

Project 5 out today, due Friday May 10

Workshop: How to write a Personal Essay for Nationally Competitive Fellowship Applications

June 2 - 5

721 Hunter

More info in flyer on Blackboard and course webpage announcements

ADT Operations we have seen so far

Bag, List, Stack, Queue

Add data to collection
Remove data from collection
Retrieve data from collection

Stack and Queue always **position based**

Bag, retrieval always **value based** (there are no positions)

List has **both**.

For all of them, data organization is linear

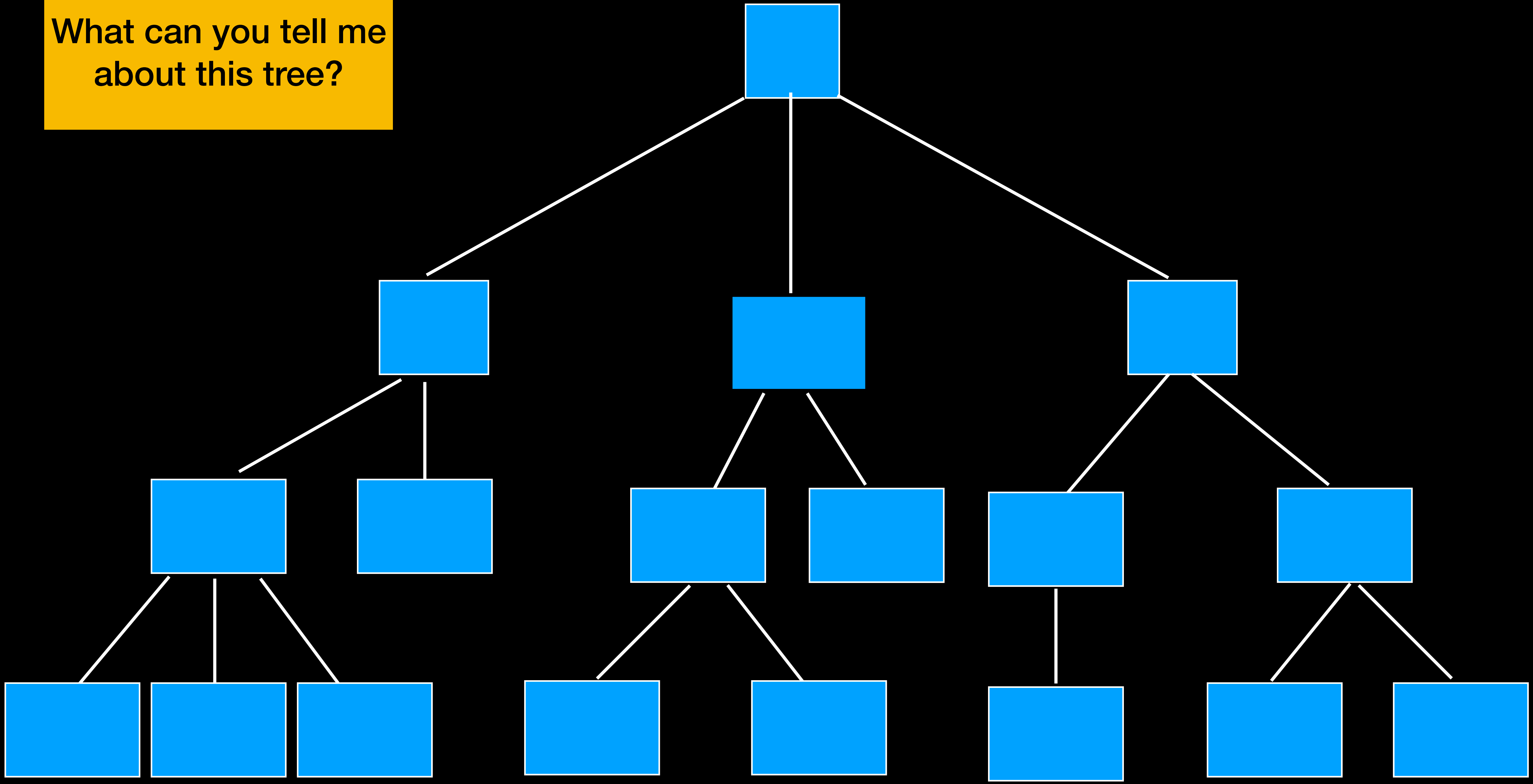
Tree

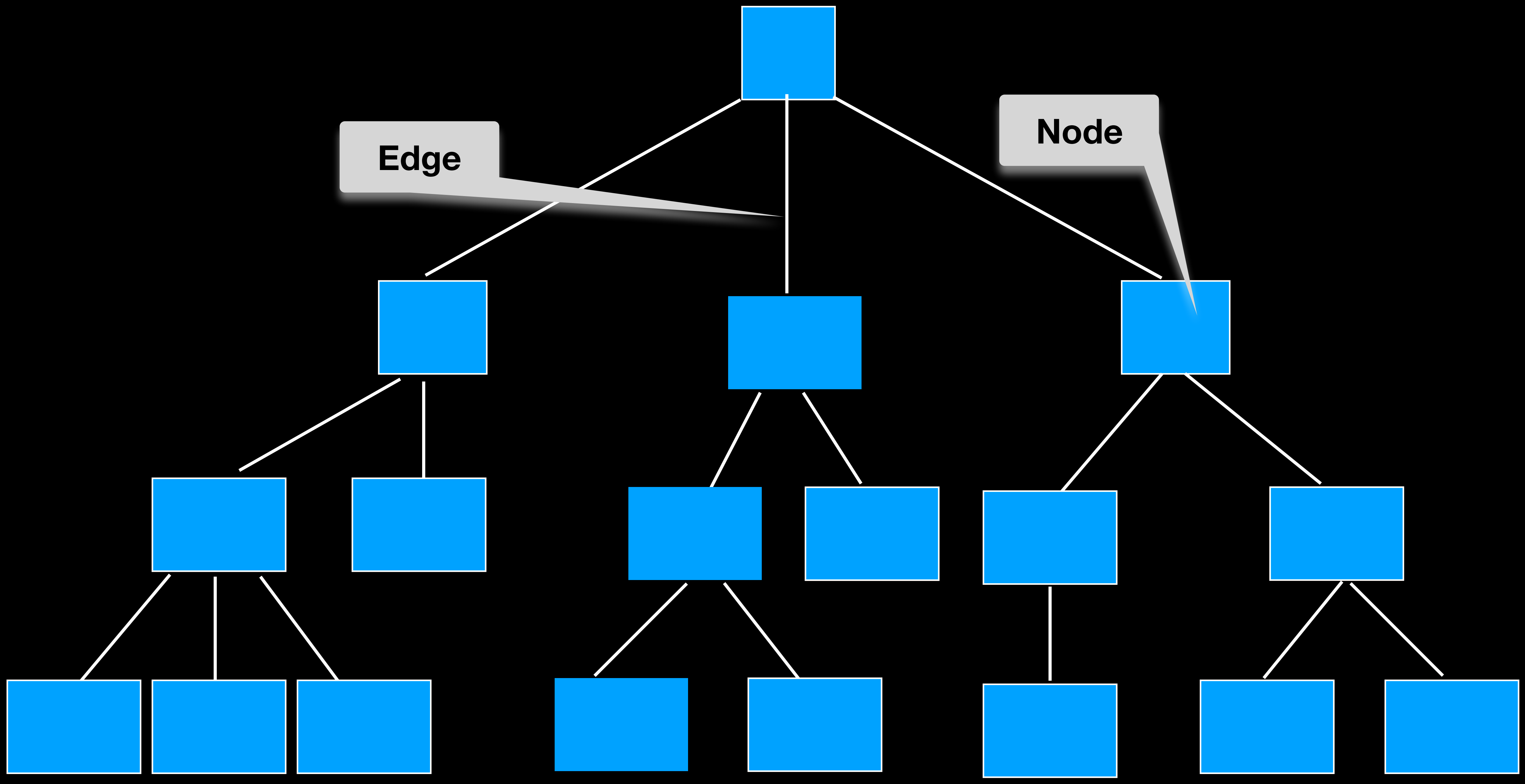
Non-linear

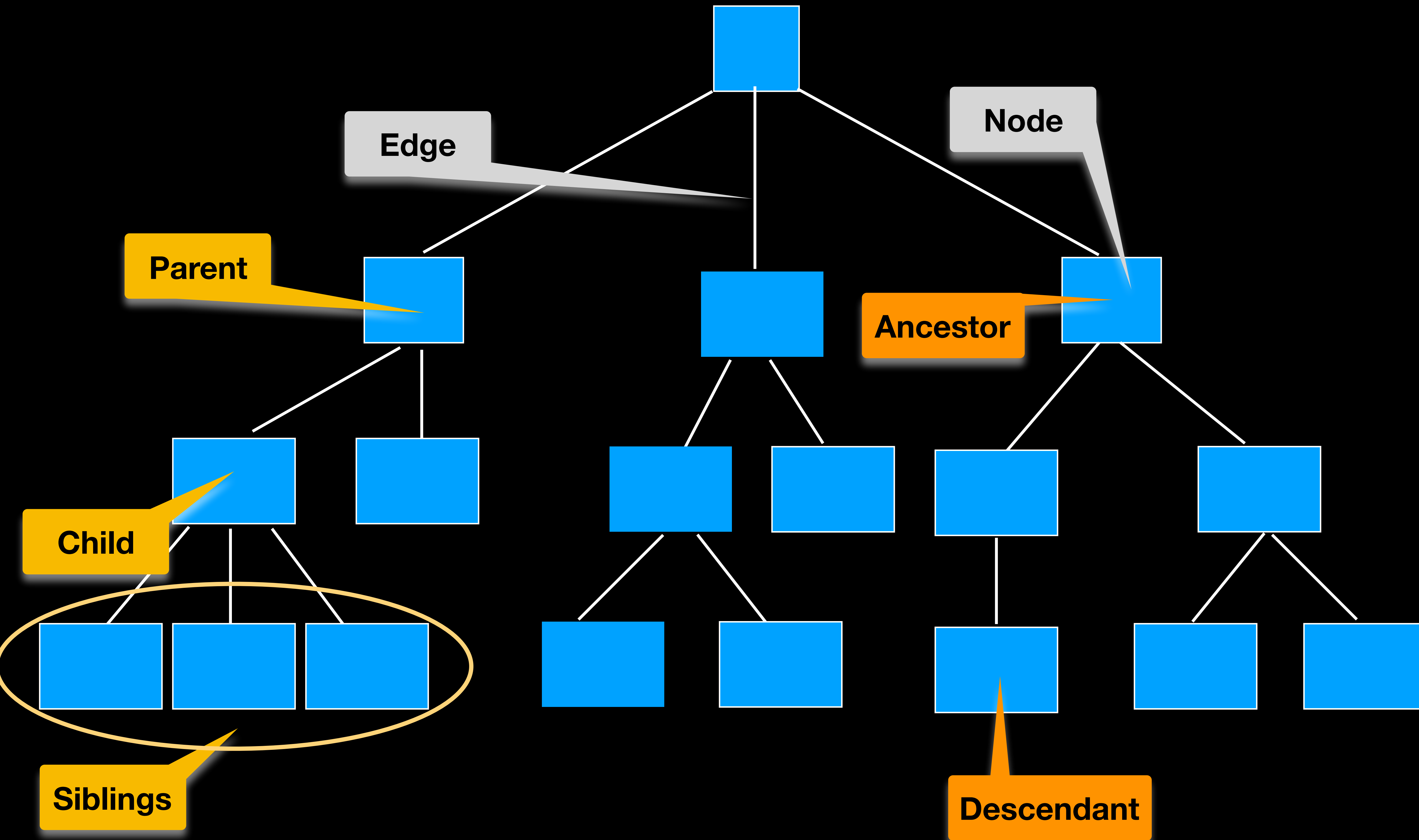
Can represent relationships

Hierarchical (directional) organization

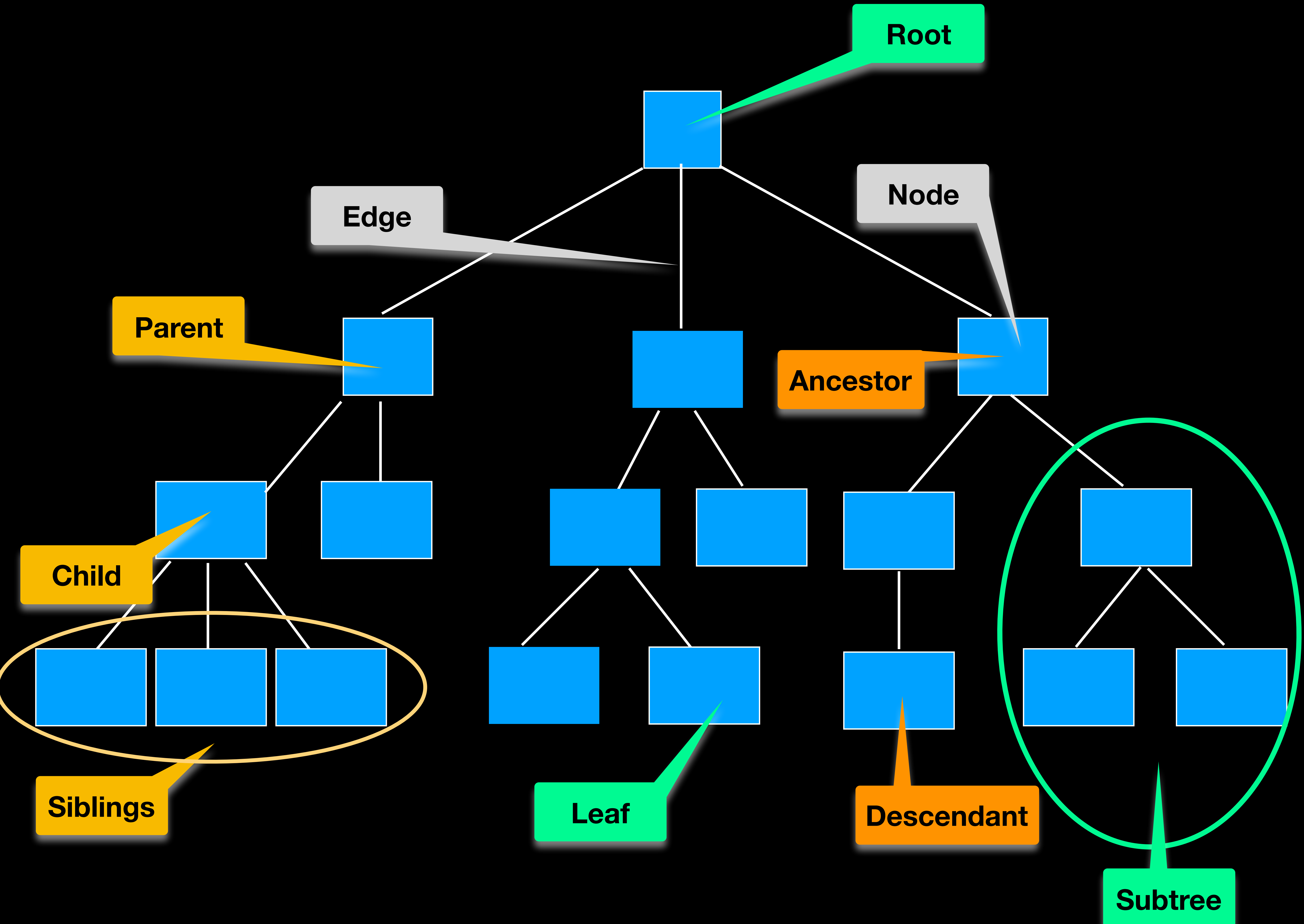
What can you tell me about this tree?



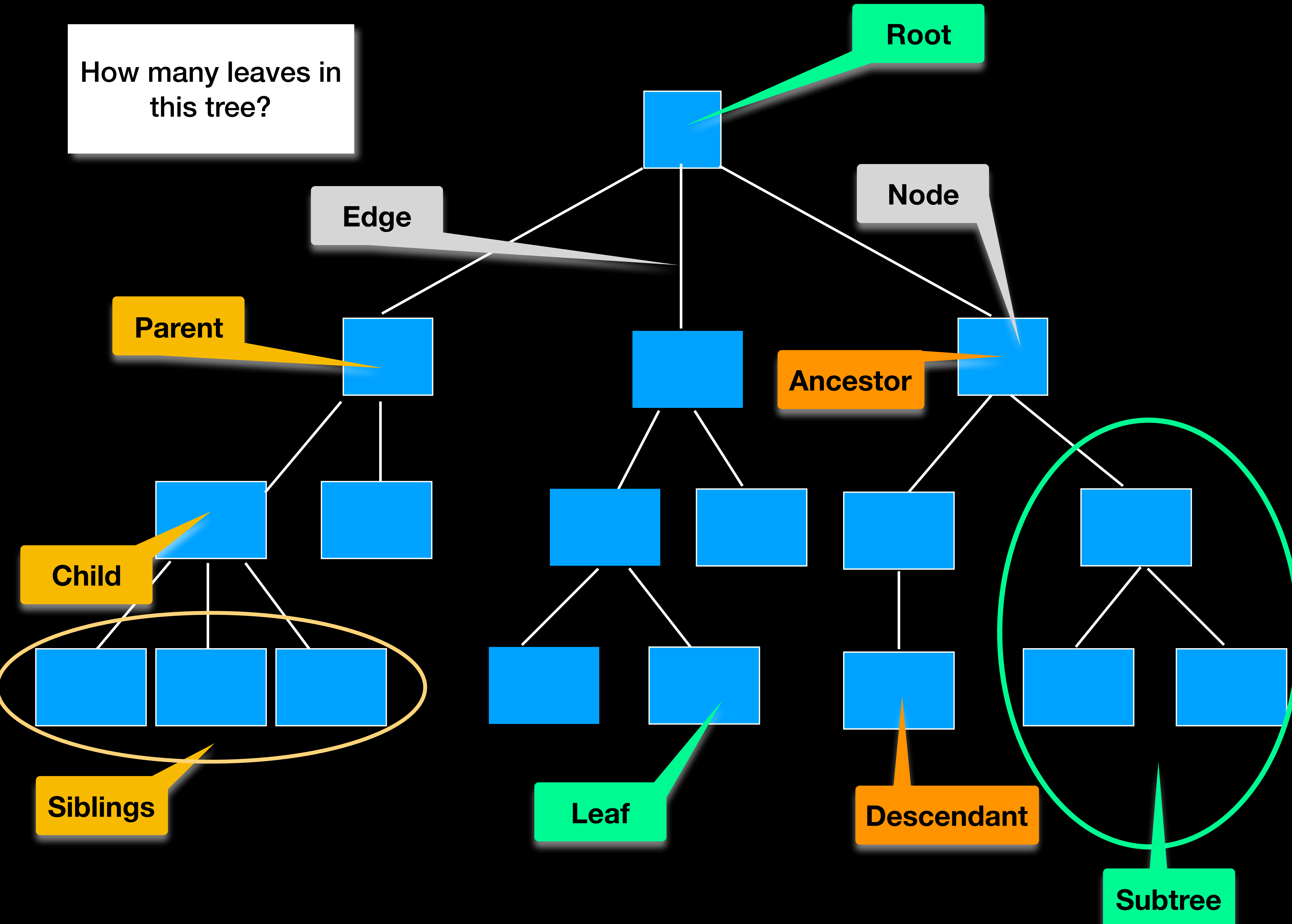

Edge
Node

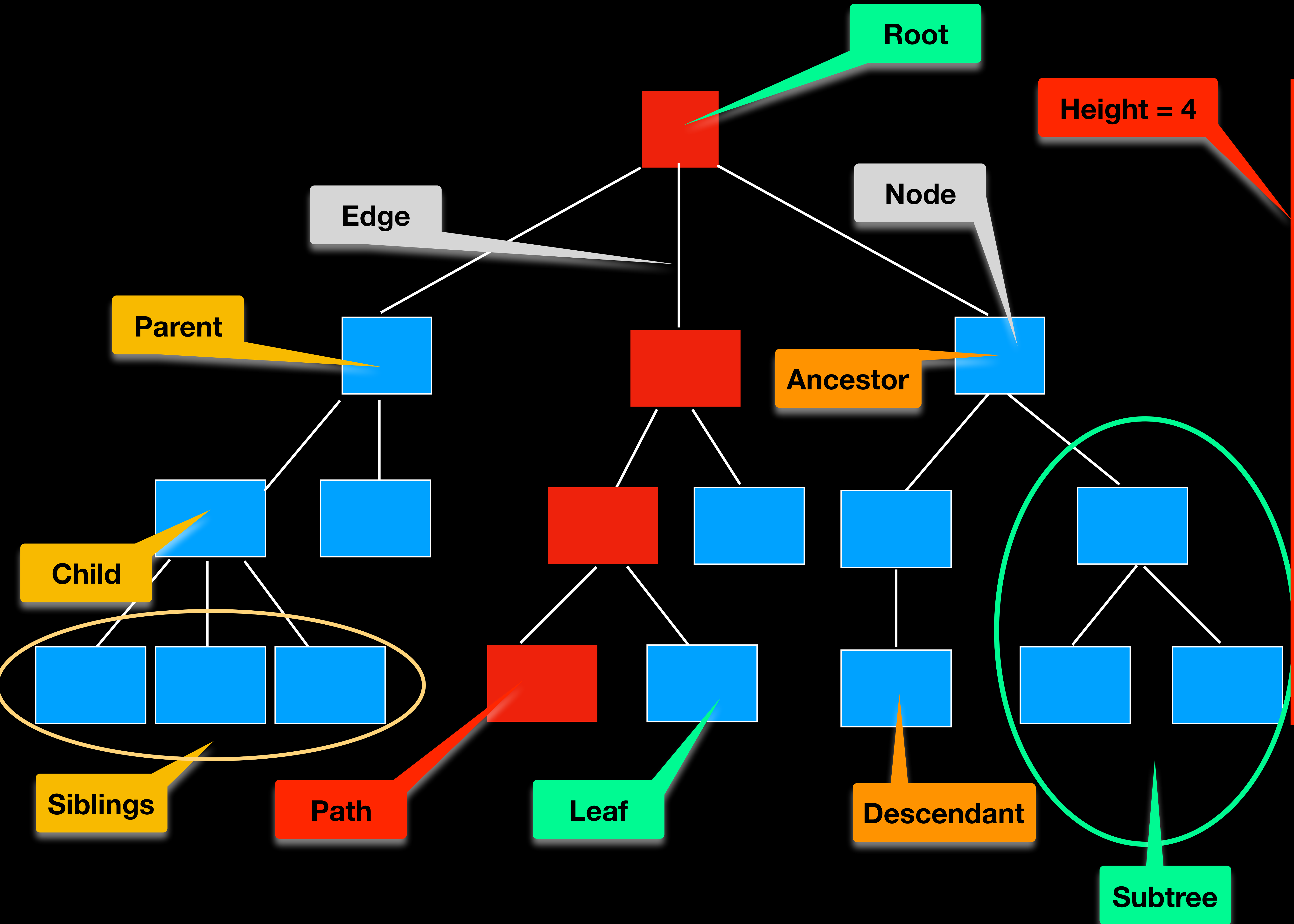


Edge
Node
Parent
Ancestor
Child
Siblings
Descendant


Root
Edge
Node
Parent
Ancestor
Child
Siblings
Leaf
Descendant
Subtree

How many leaves in this tree?



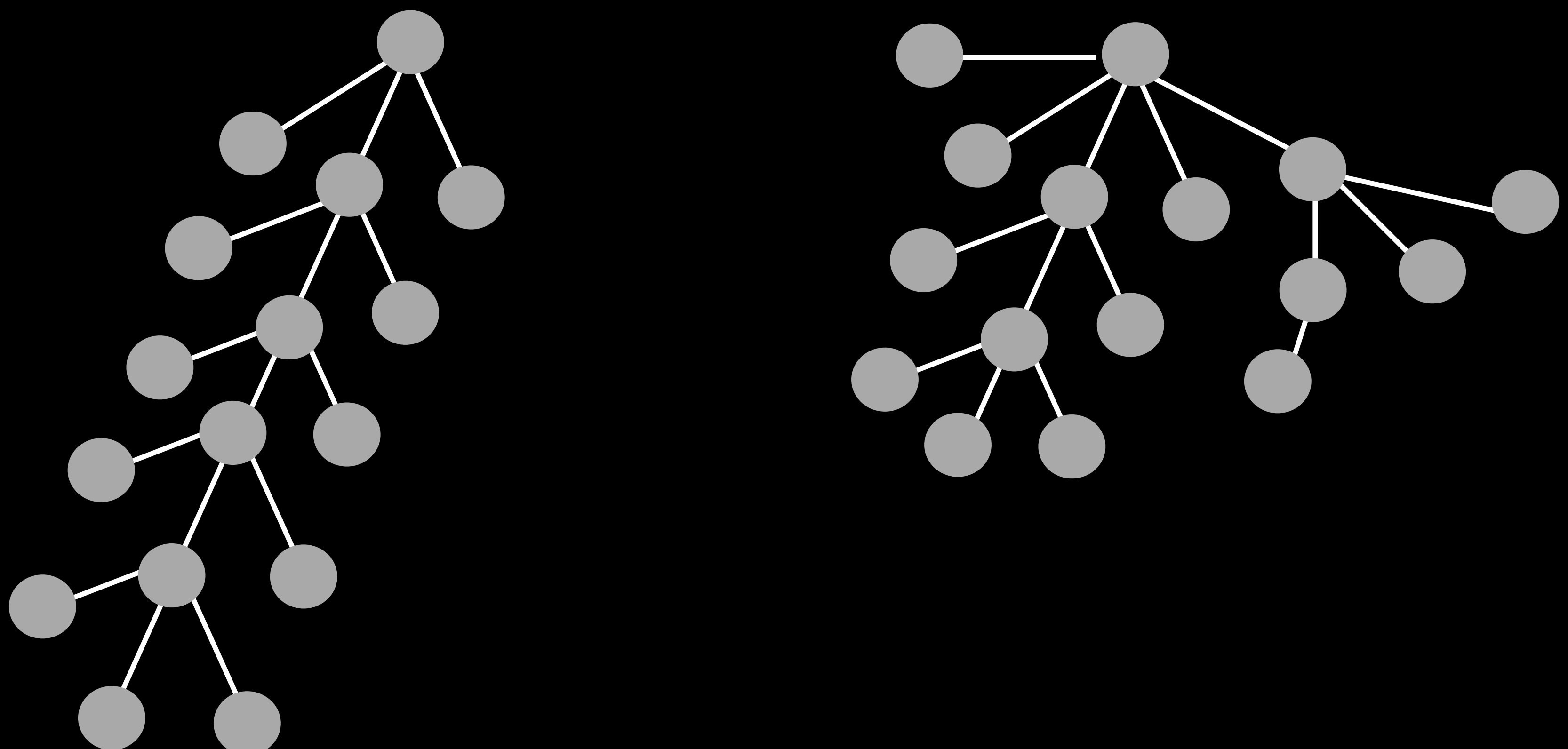

Root
Height = 4
Edge
Node
Parent
Ancestor
Child
Siblings
Path
Leaf
Descendant
Subtree

Path: a sequence of nodes c_1, c_2, ..., c_k where c_{i+1} is a child of c_i.

Height: the **number of nodes** in the *longest* path from the root to a leaf.

Subtree: the subtree rooted at node *n* is the tree formed by taking *n* as the root node and including all its descendants.

Different shapes/structures



Both n = 16
Both 11 leaves
Different height

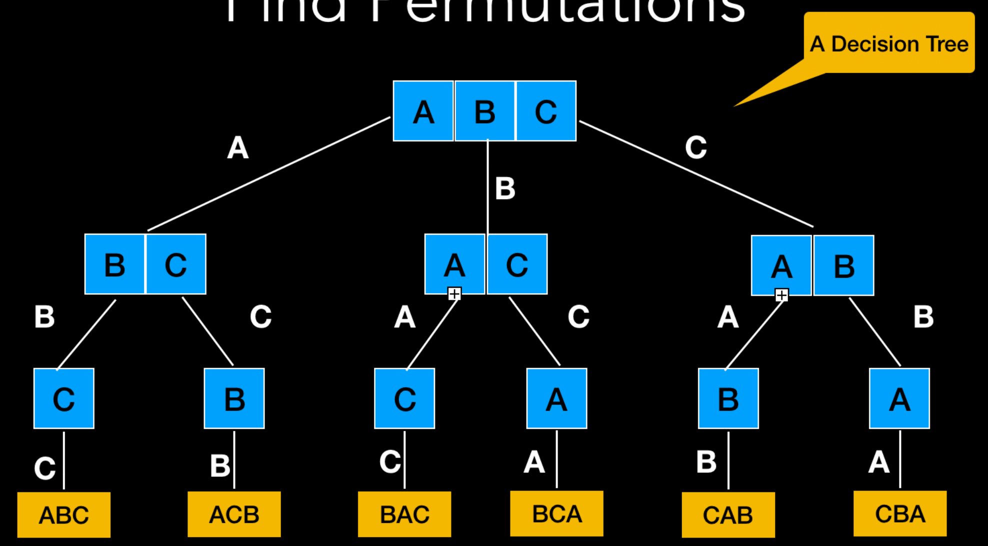
We have already seen Trees!

Mostly as a “thinking tool”

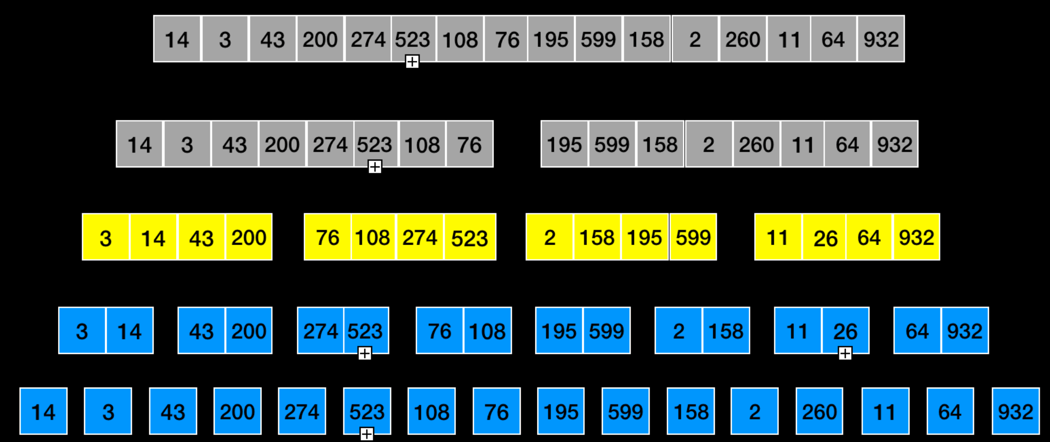
- Decision Trees

- Divide and Conquer

Find Permutations



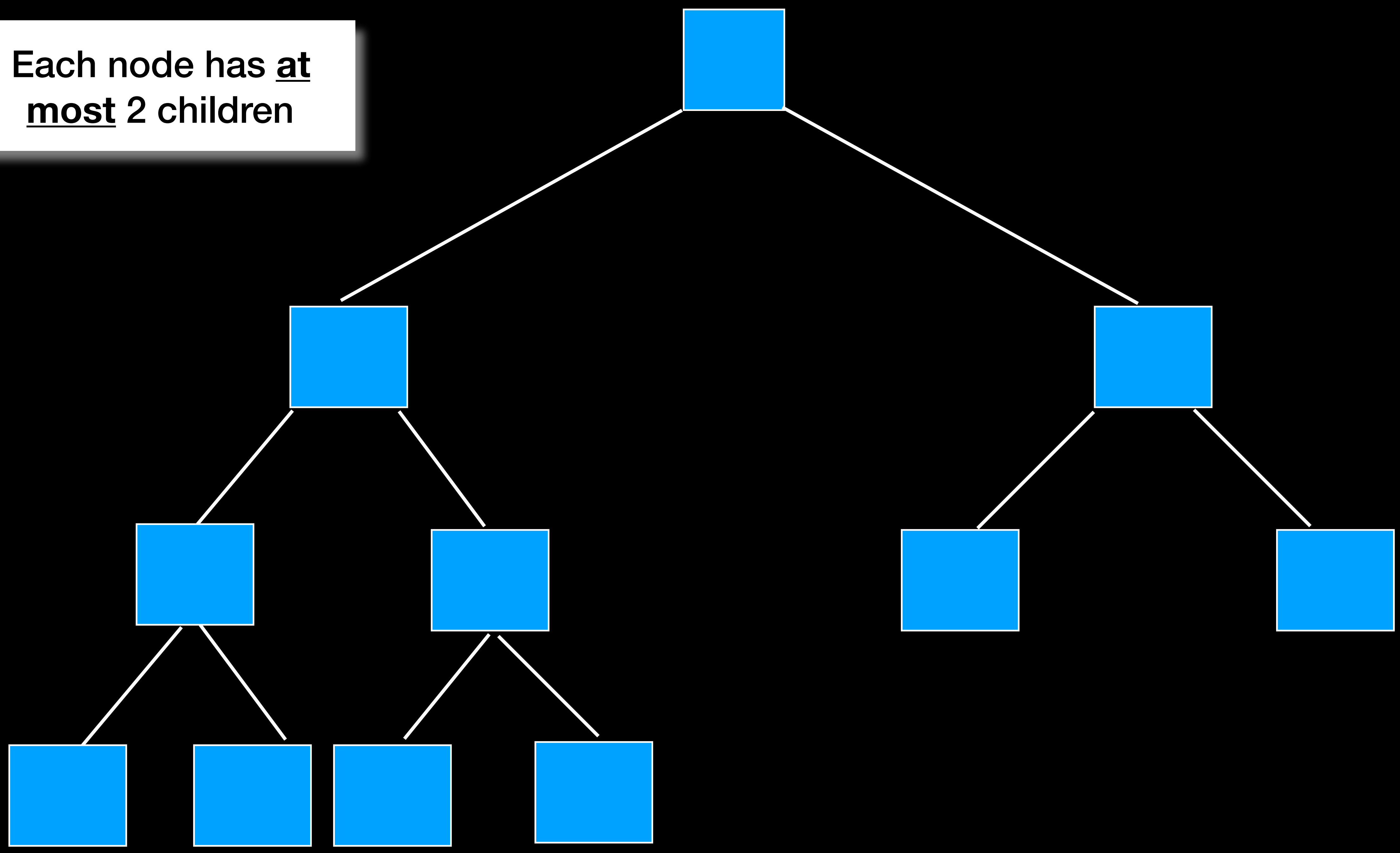
Merge Sort



Binary Tree ADT

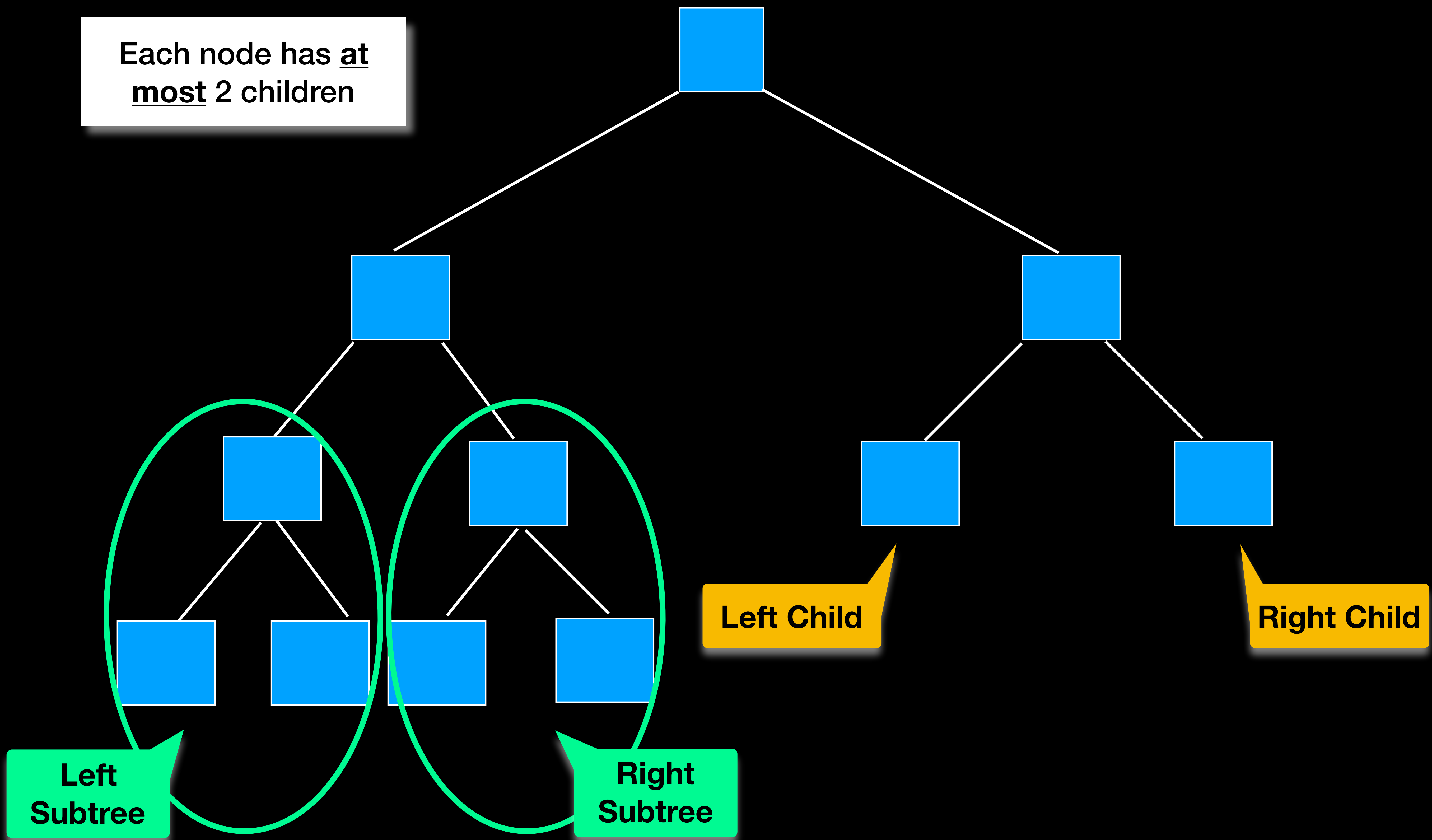
Binary Tree

Each node has **at most** 2 children

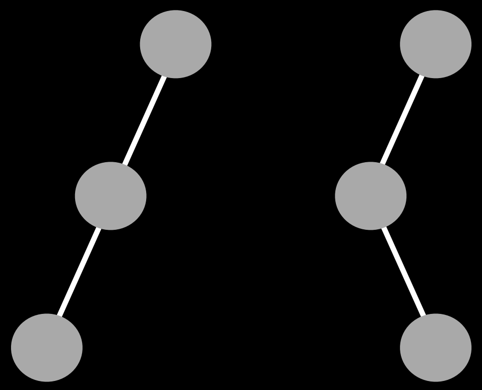


Binary Tree

Each node has **<u>at most</u>** 2 children



Different shapes/structures



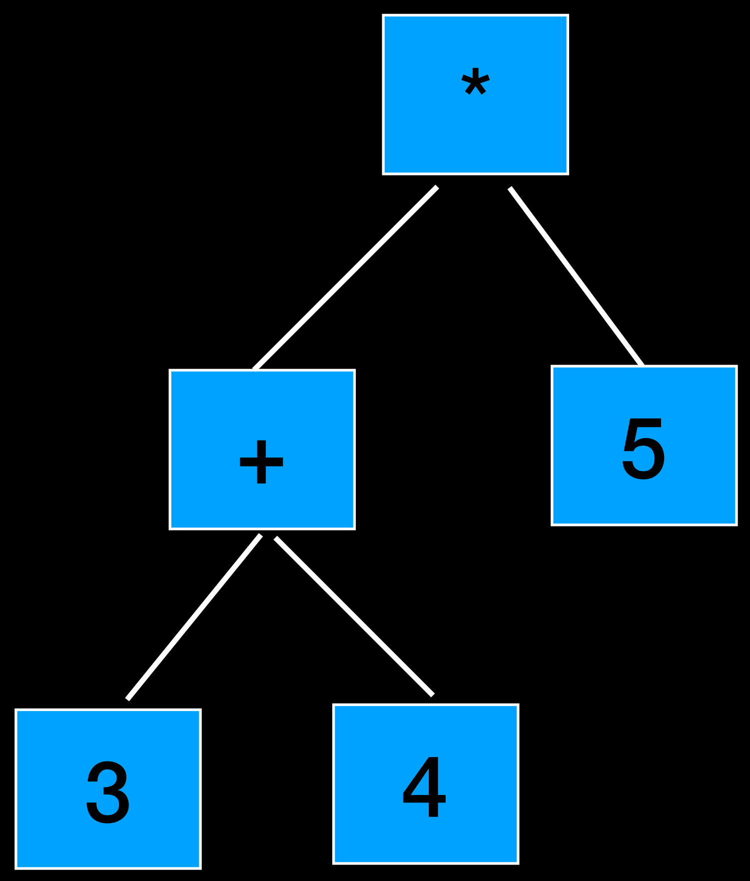
Both h = 3 and one leaf
But different

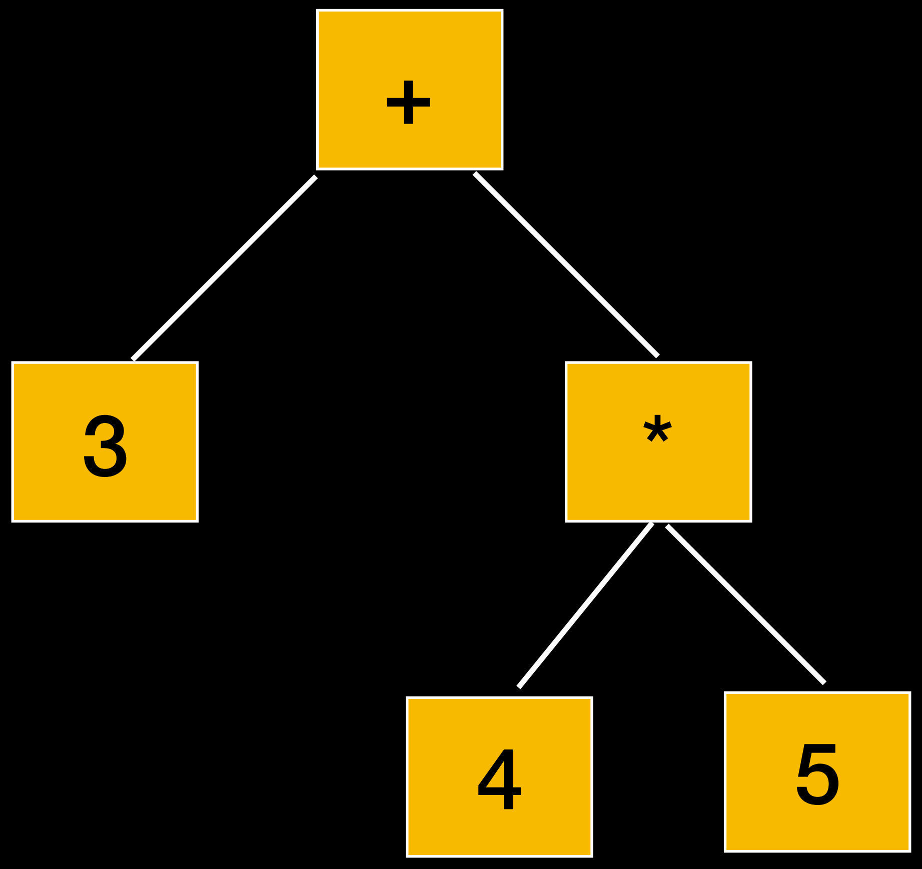
Binary Tree Applications

Algebraic Expressions

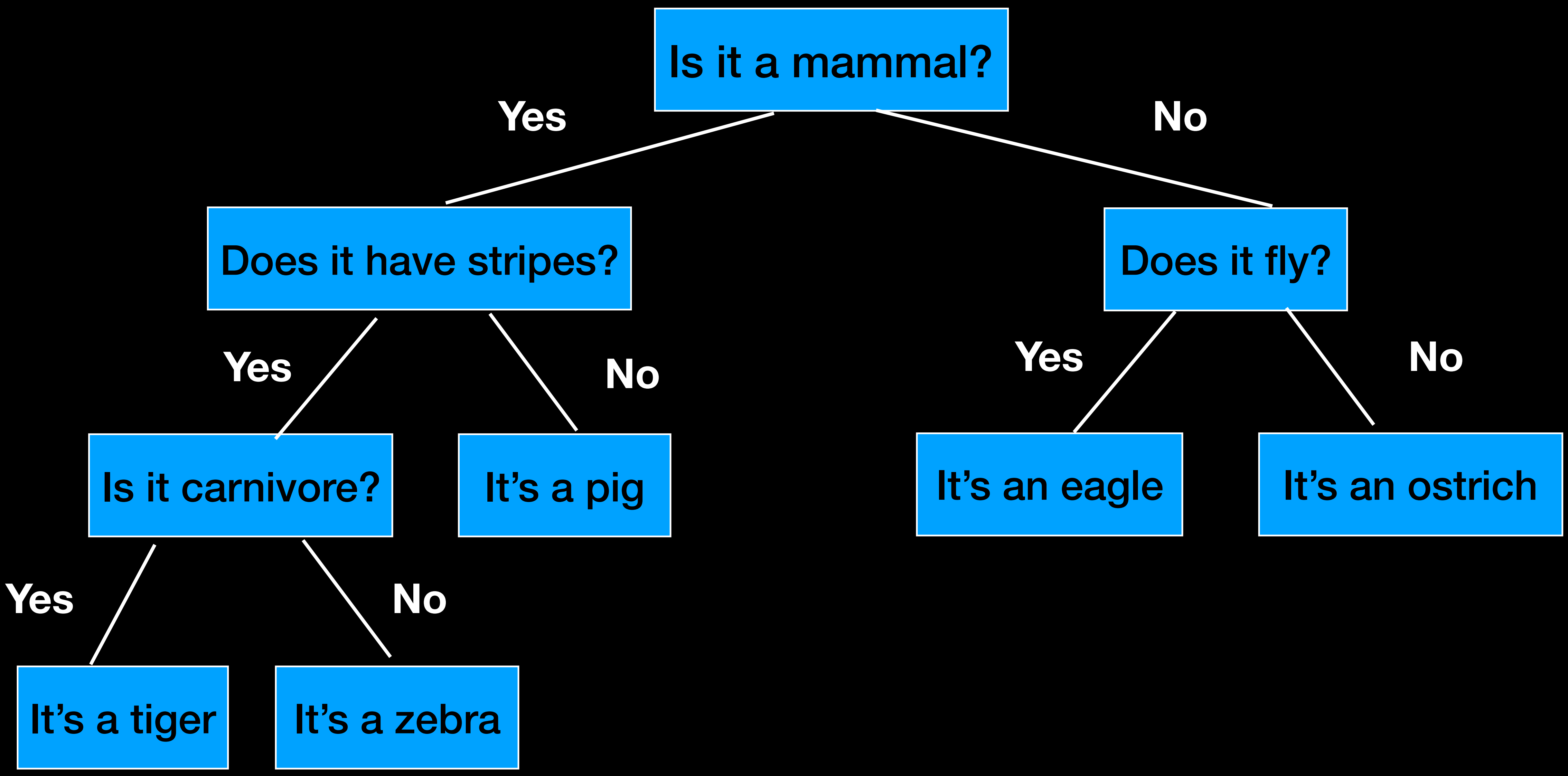
(3 + 4) * 5

3 + 4 * 5





Decision Tree



Huffman Tree

Huffman Encoding Compression Algorithm (**Huffman Encoding**):
"In 1951, David A. Huffman for his MIT Information Theory class term paper hit upon the idea of using a **frequency-sorted binary tree** and quickly proved this method the most efficient."

IDEA: Encode symbols into a sequence of bits s.t. most frequent symbols have shortest encoding

Not encryption but compression => use shortest code for most frequent symbols

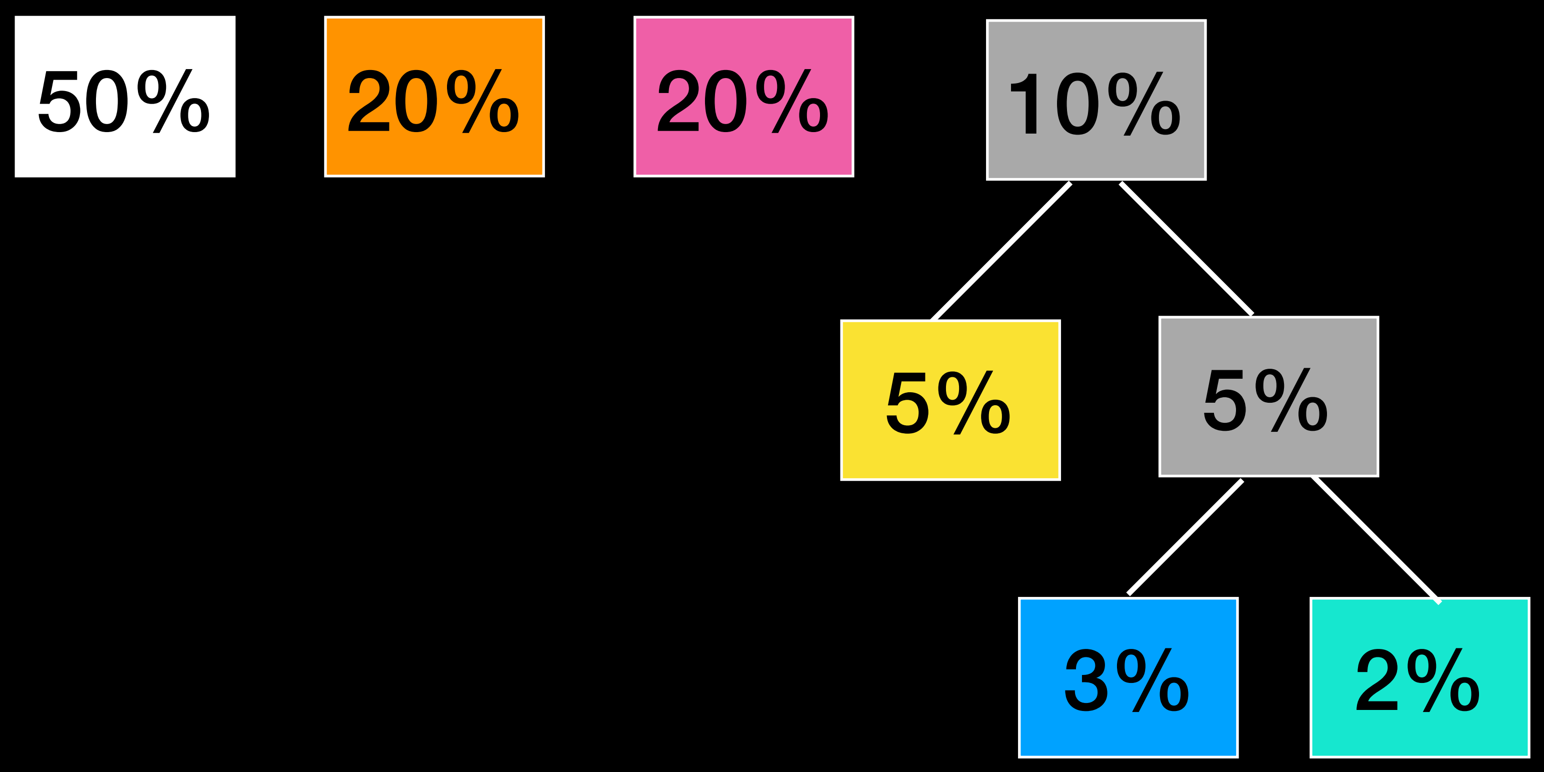
No codeword is prefix to another codeword (i.e. if a symbol is encoded as 00 no other codeword can start with 00)

Huffman Tree

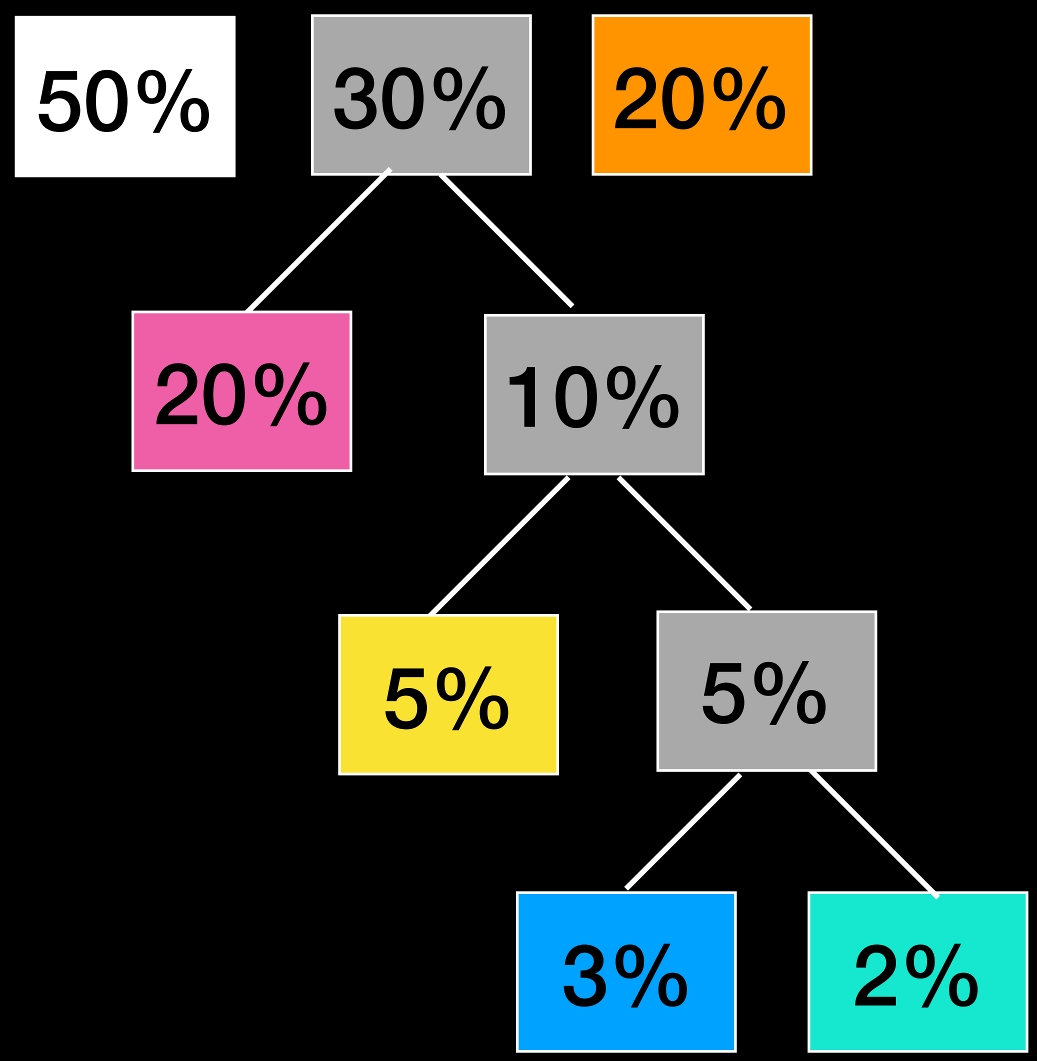
50%	20%	20%	5%	3%	2%

Huffman Tree

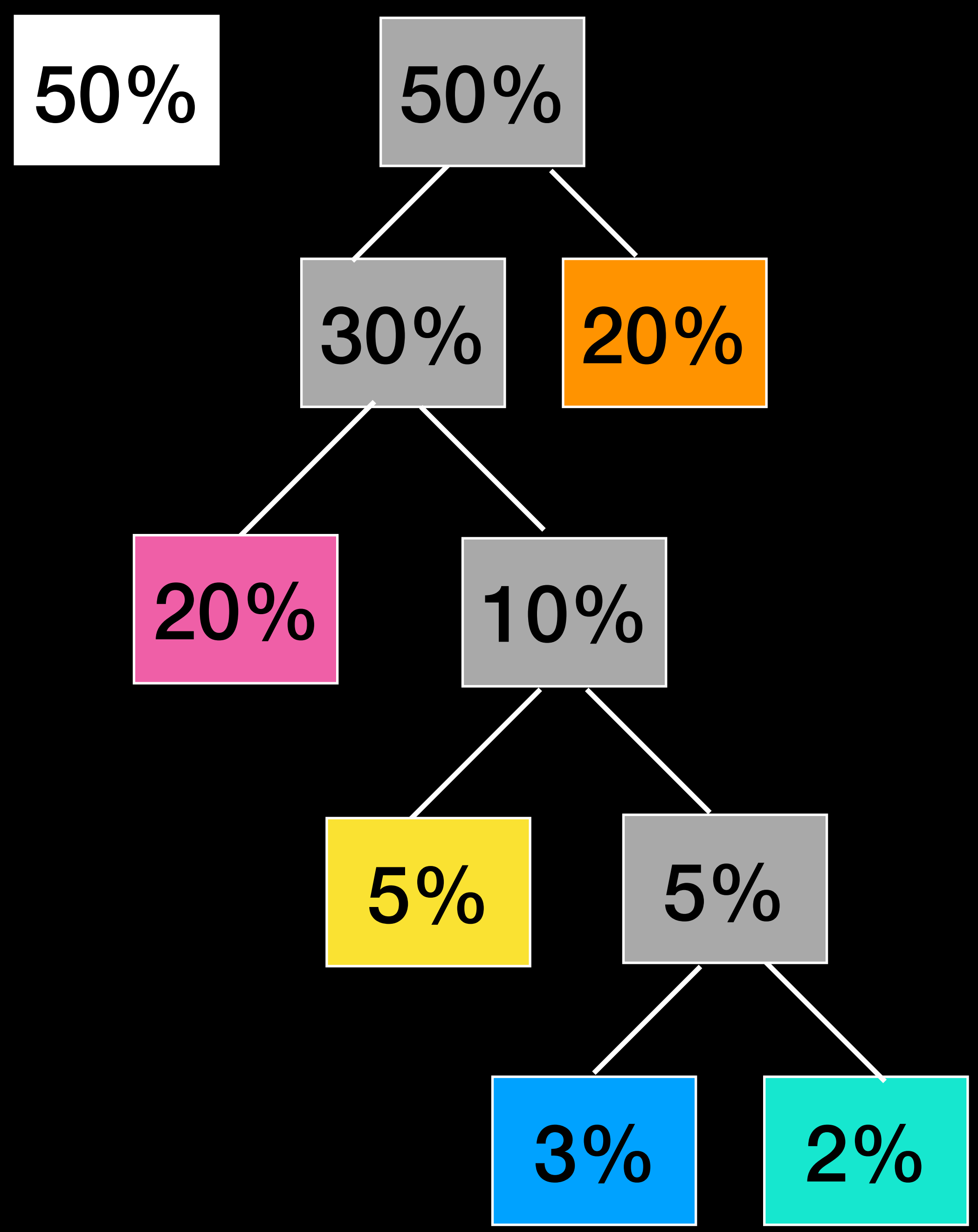
Huffman Tree



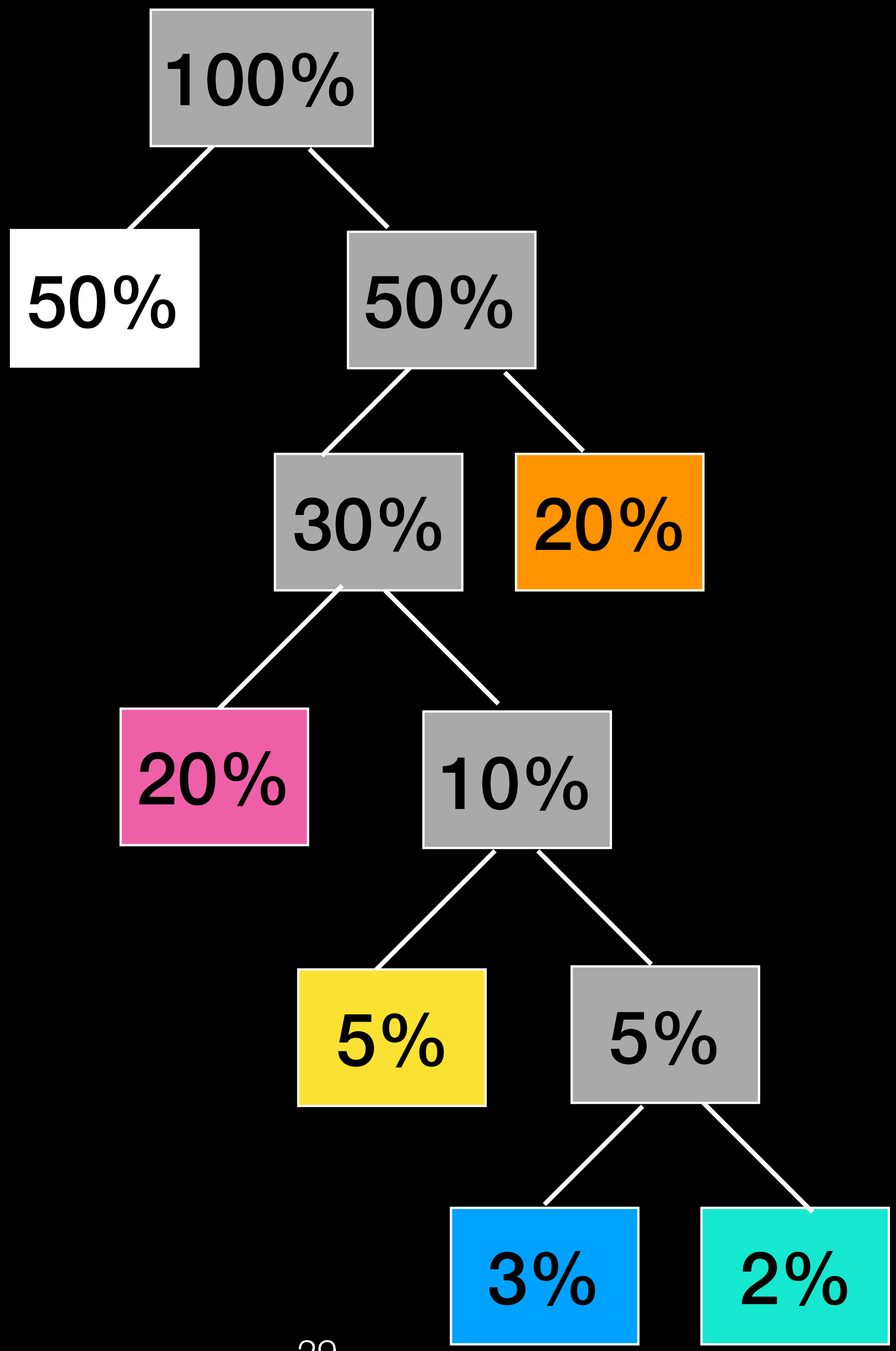
Huffman Tree



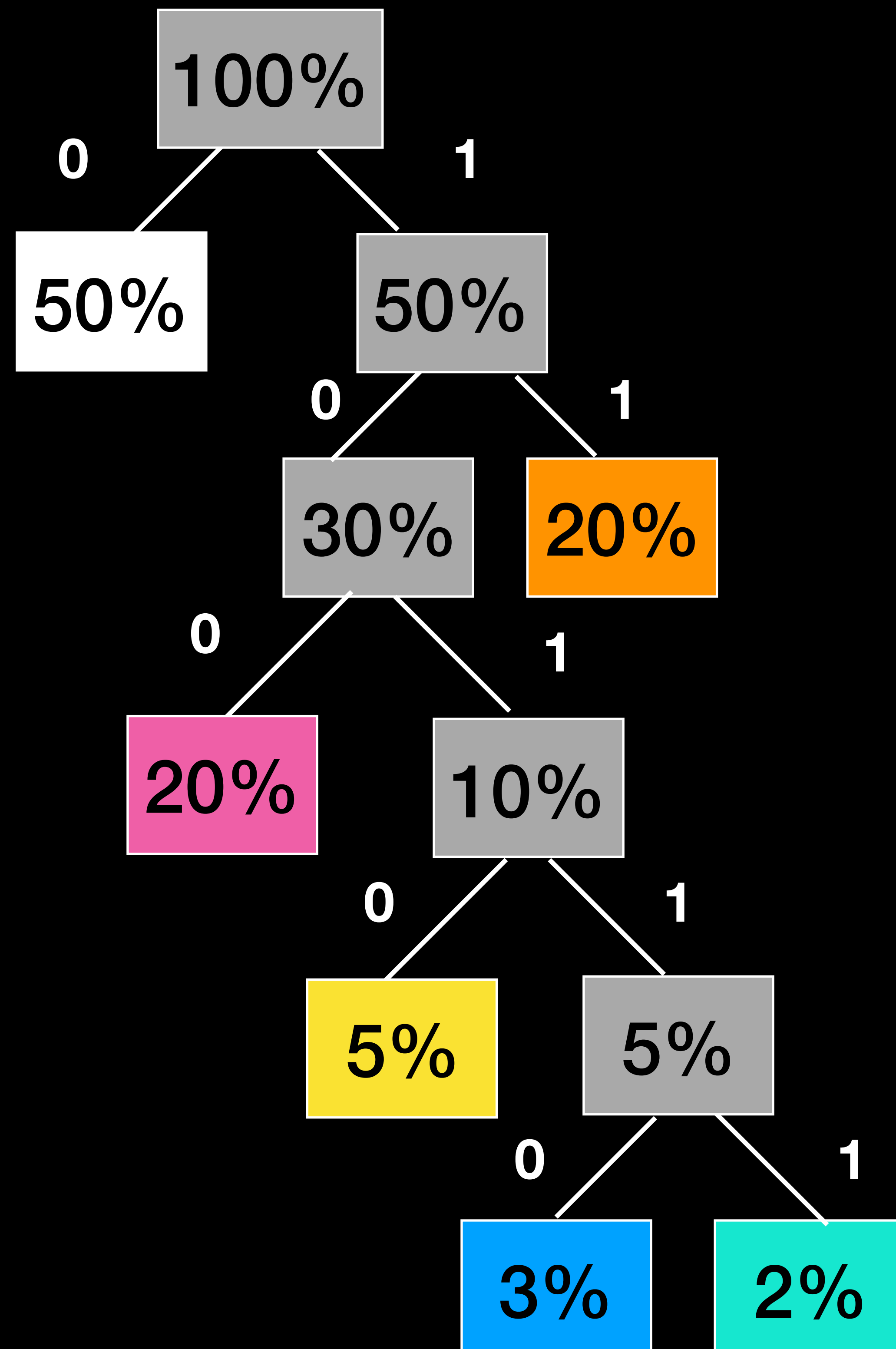
Huffman Tree



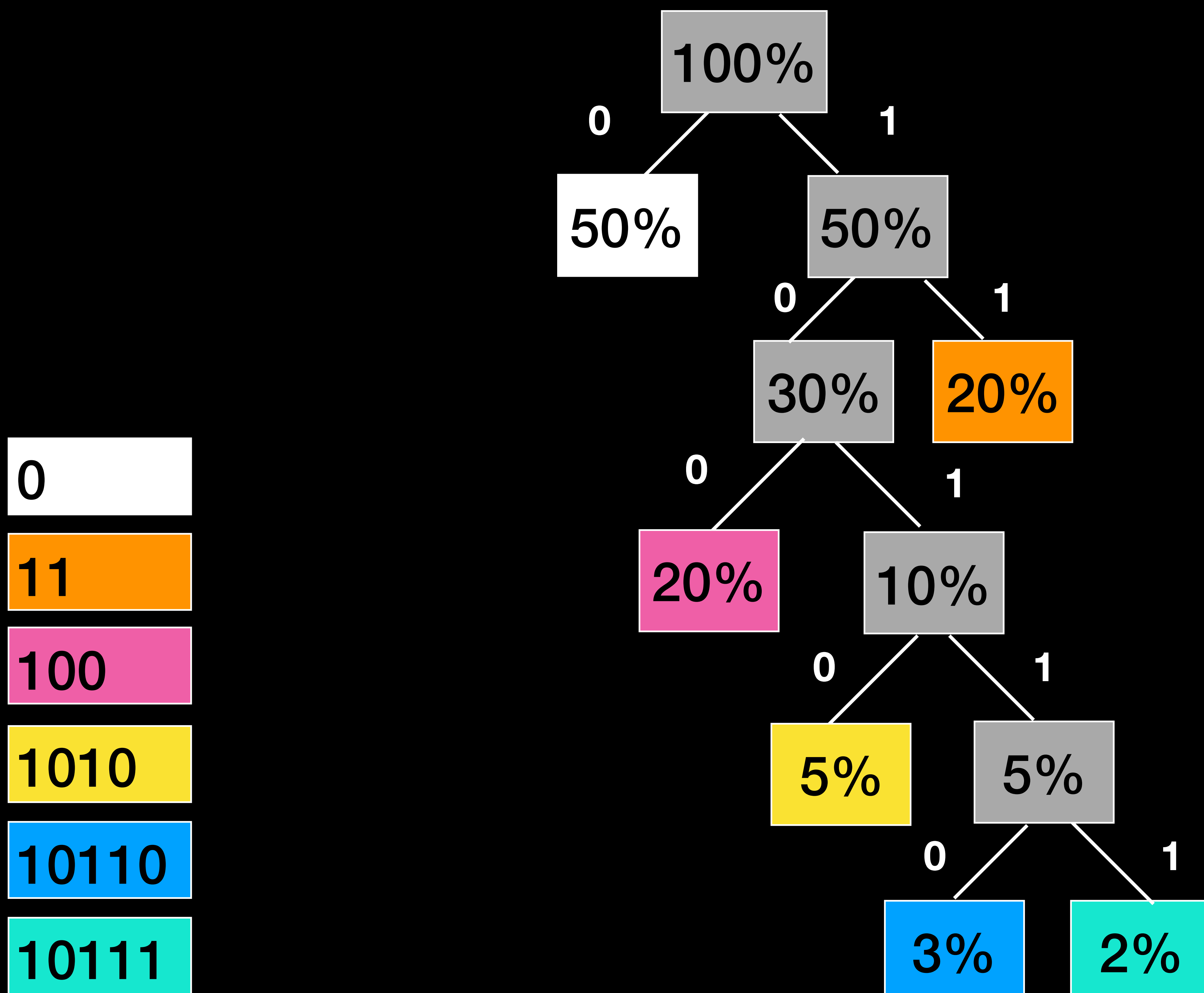
Huffman Tree



Huffman Tree



Huffman Tree

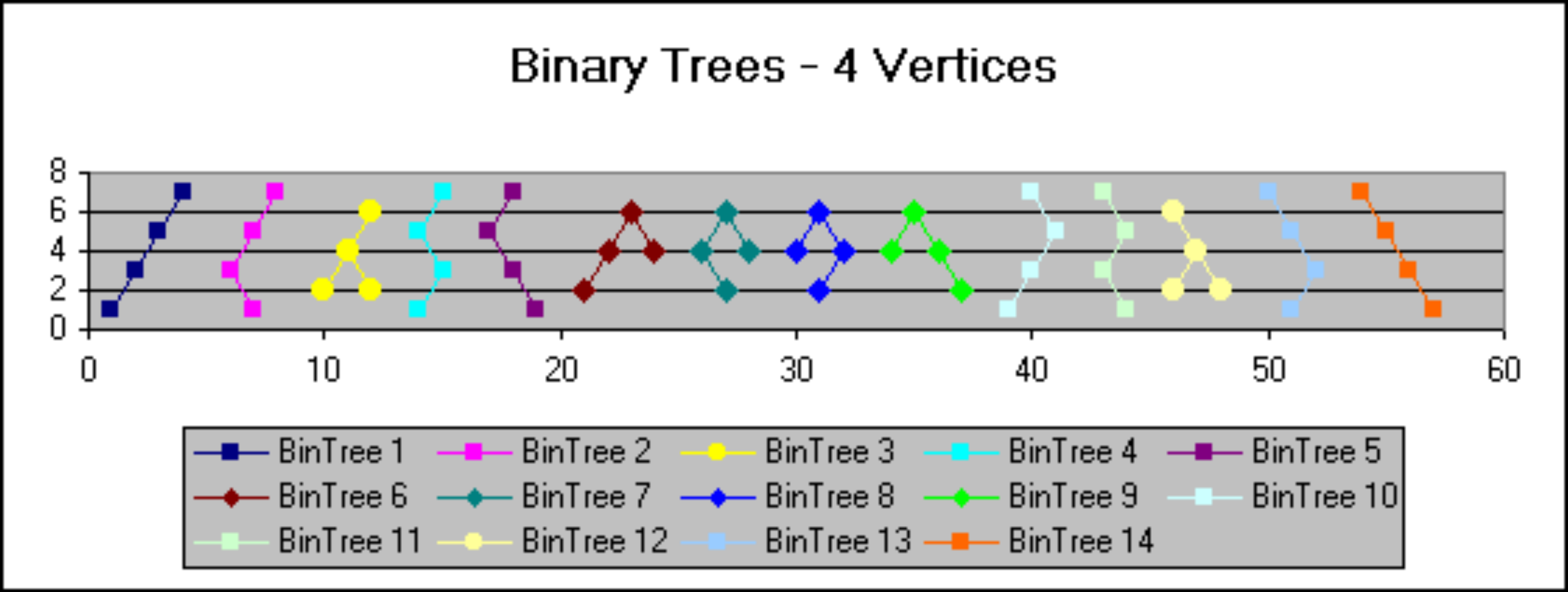


Lecture Activity

Think about structure!

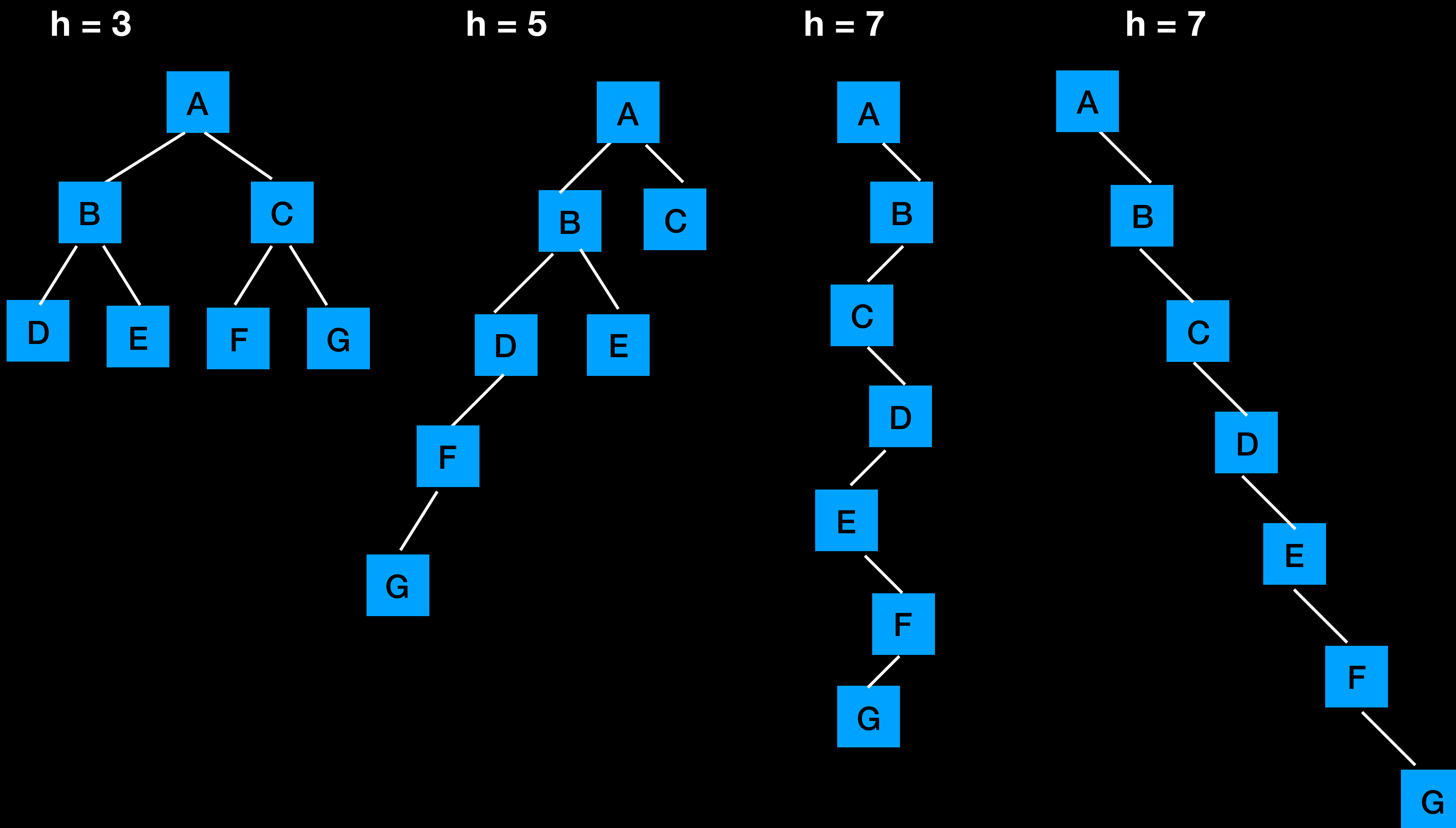
Draw **ALL POSSIBLE** binary trees with 4 nodes

Label each tree with its <u>height</u> and <u>number of leaves</u>.



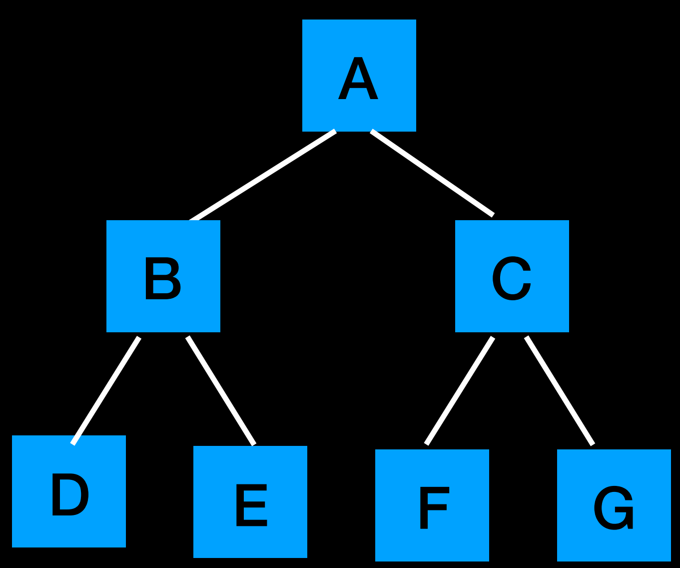
IMGAGE FROM : http://www.durangobill.com/BinTrees.html

Tree Structure

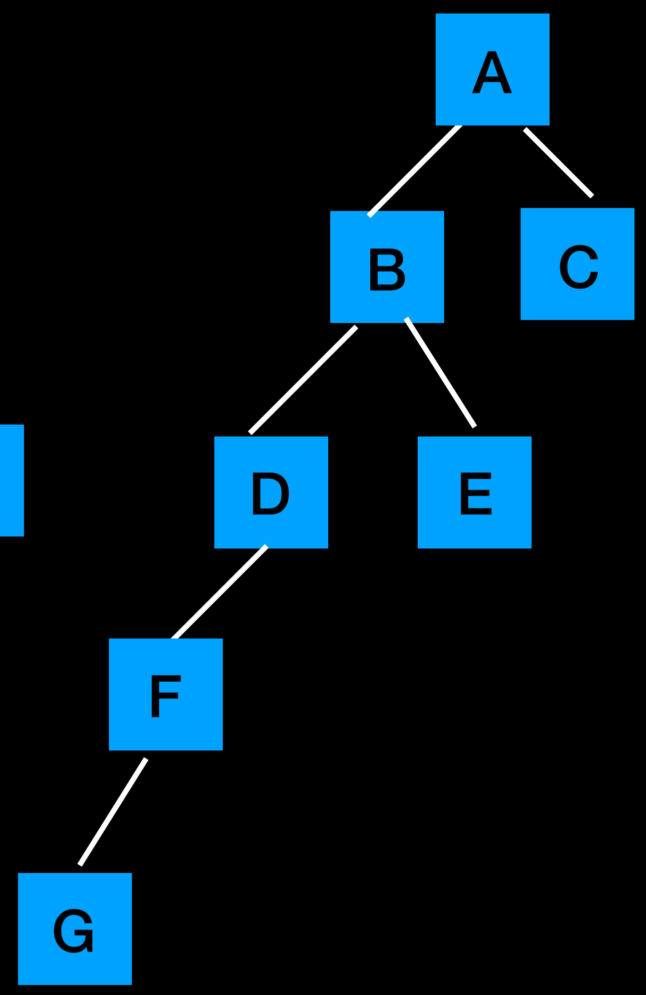


Tree Structure

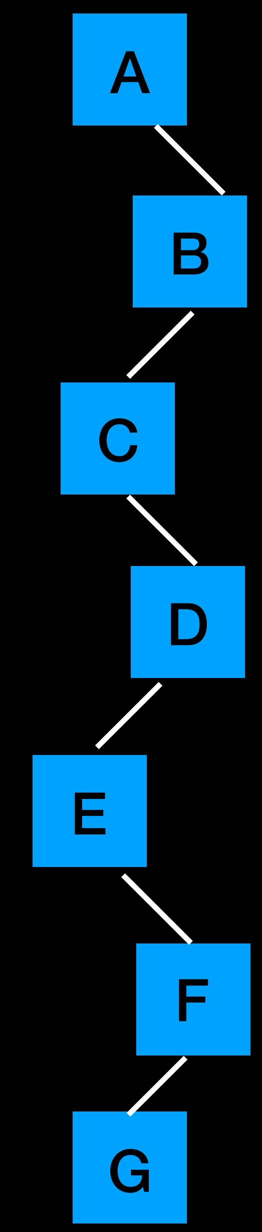
h = 3



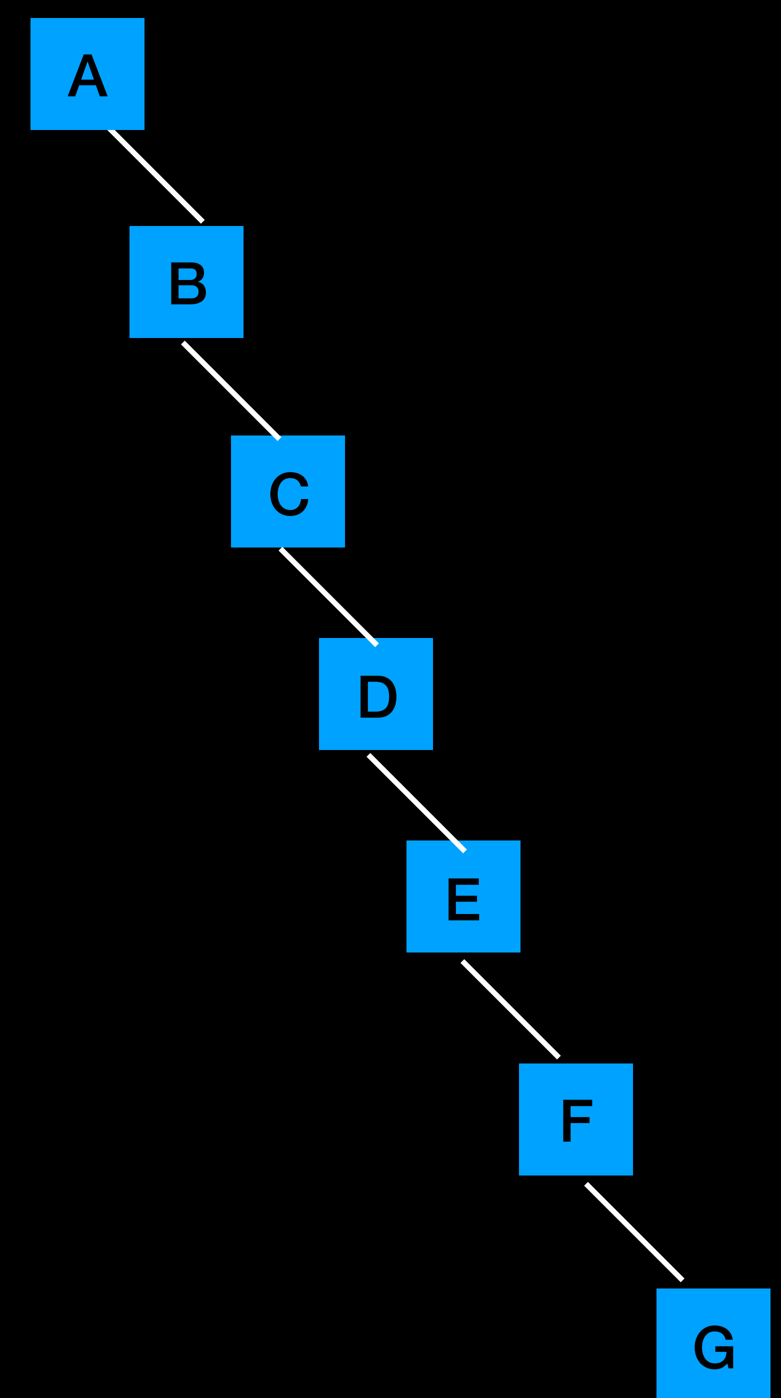
h = 5



h = 7

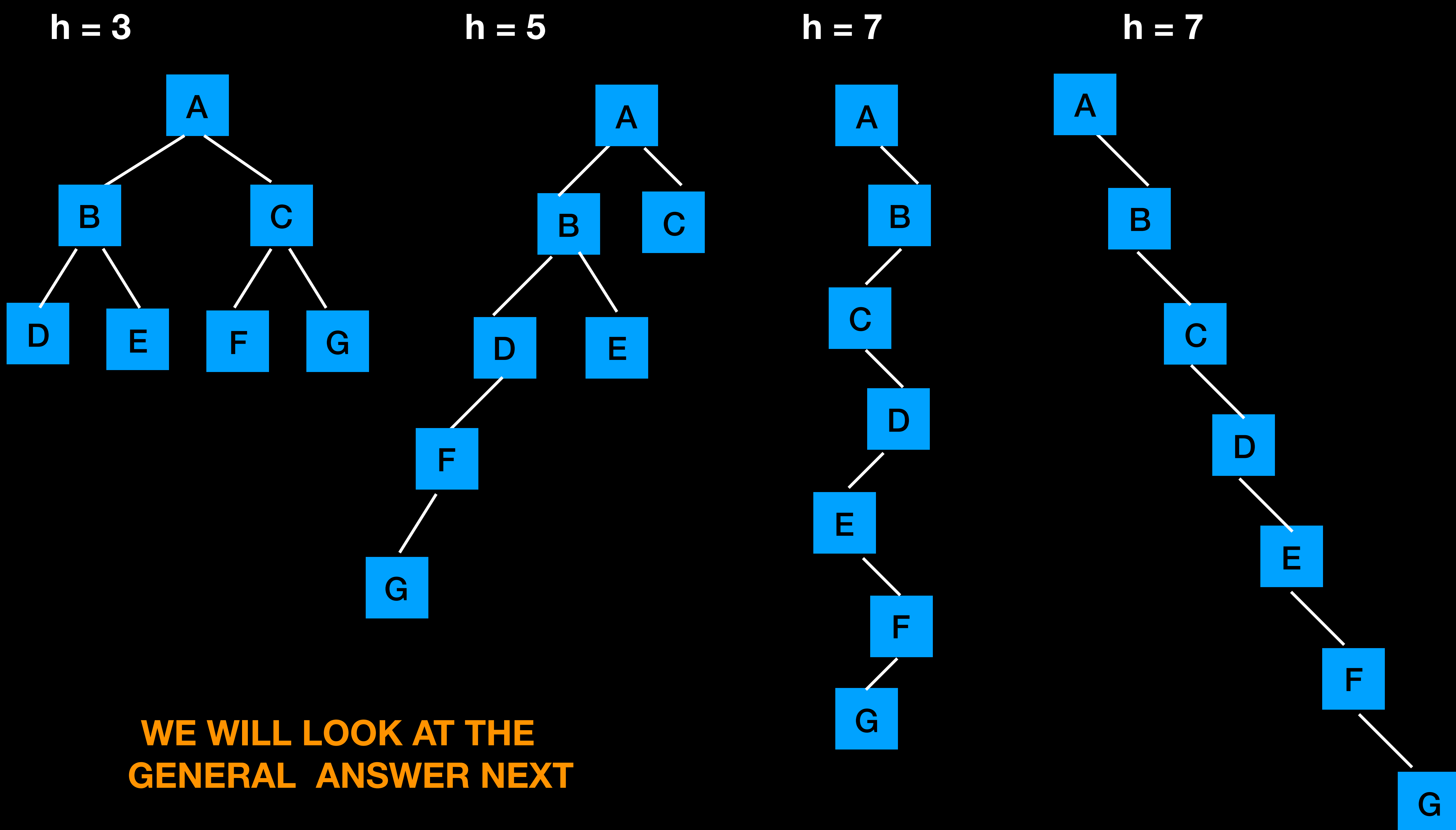


h = 7



What is the maximum (minimum) height of a tree with 7 nodes?

Tree Structure



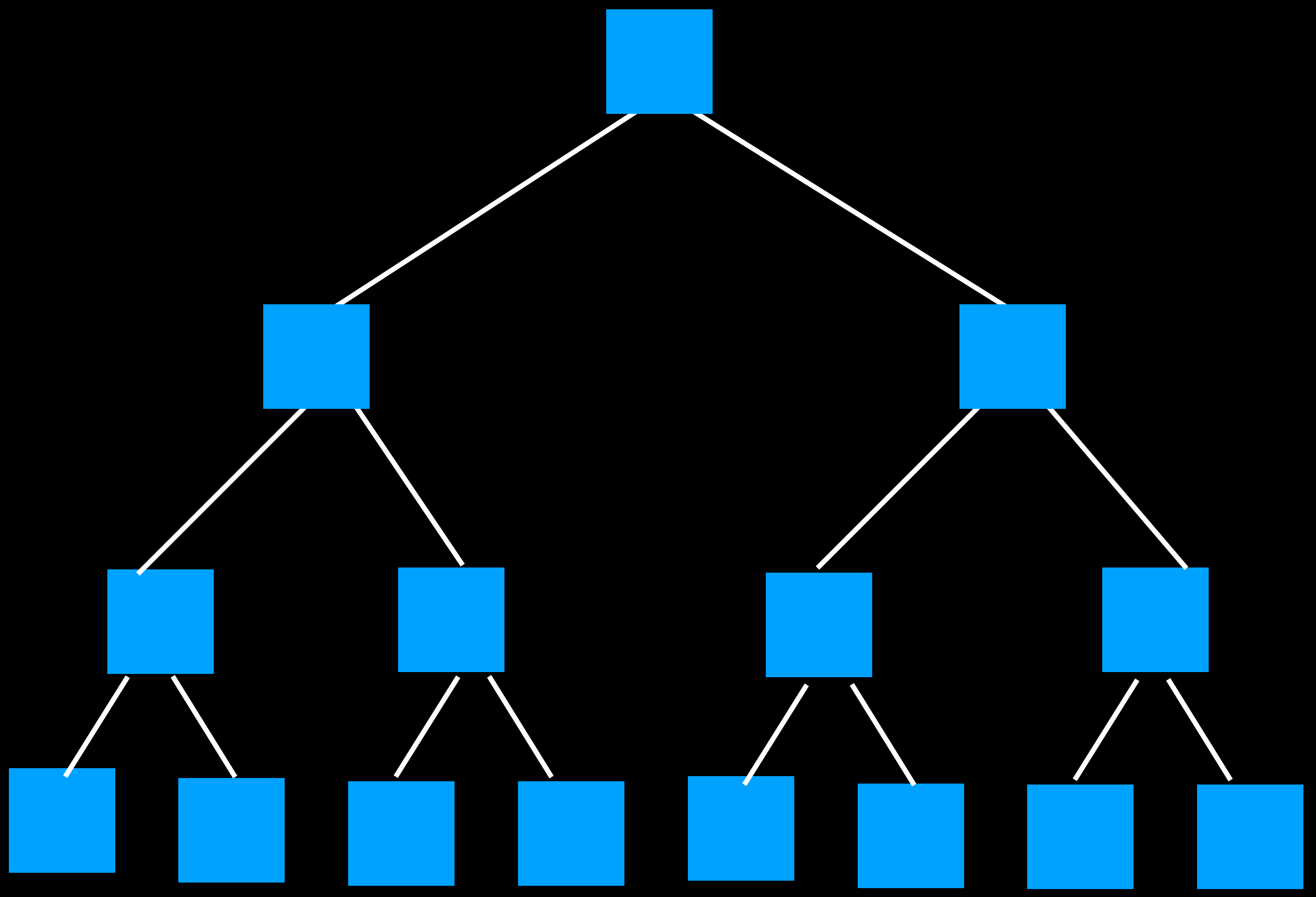
WE WILL LOOK AT THE GENERAL ANSWER NEXT

Full Binary Tree

Every node that is not a leaf has exactly 2 children

Every node has left and right subtrees of same size

All leaves are at same level *h*



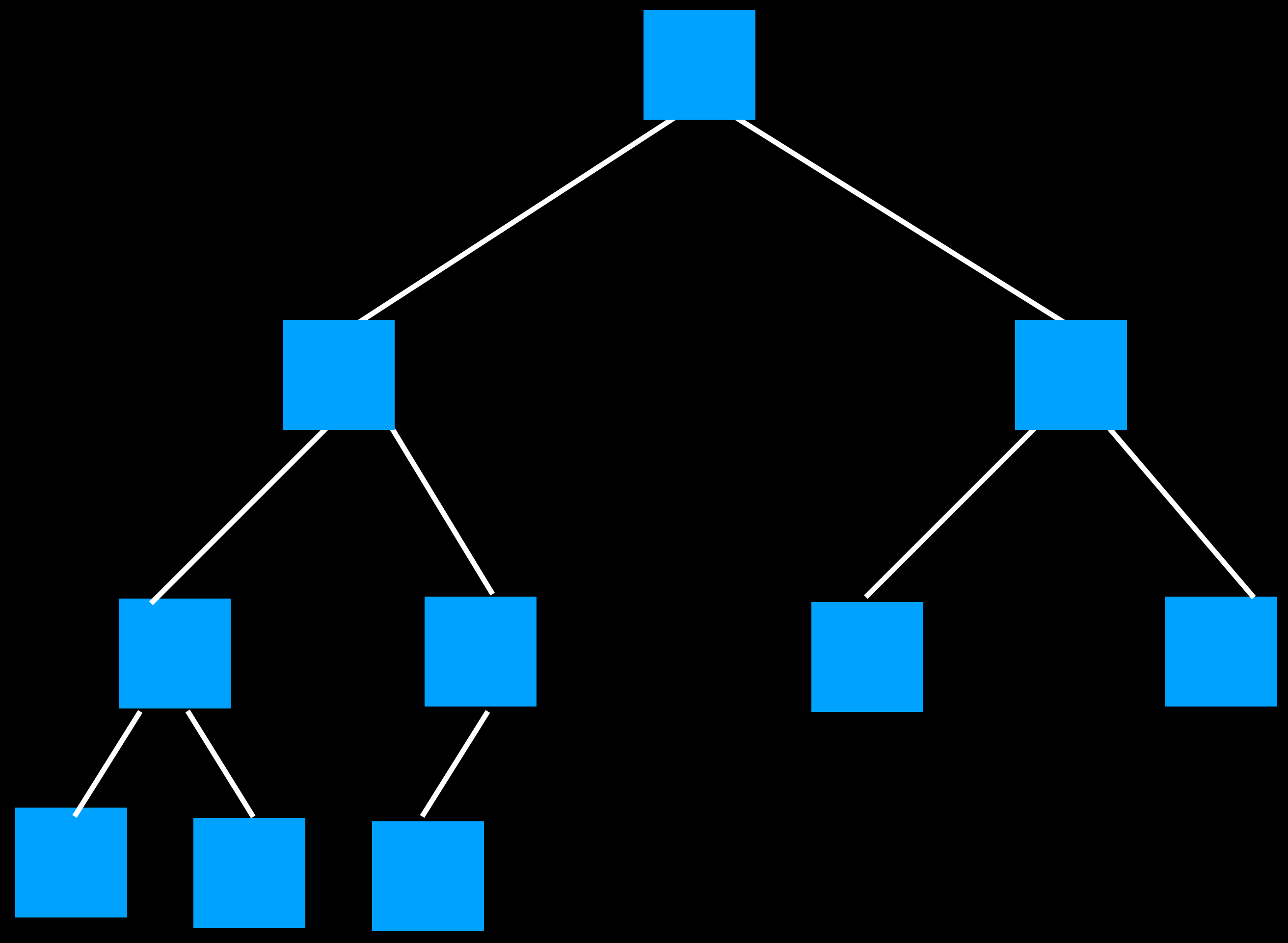
Complete Binary Tree

A three that is full up to level *h-1*, with level *h* filled in from left to right

All nodes at levels *h-2* and above have exactly 2 children

When a node at level *h-1* has children, all nodes to its left have exactly 2 children

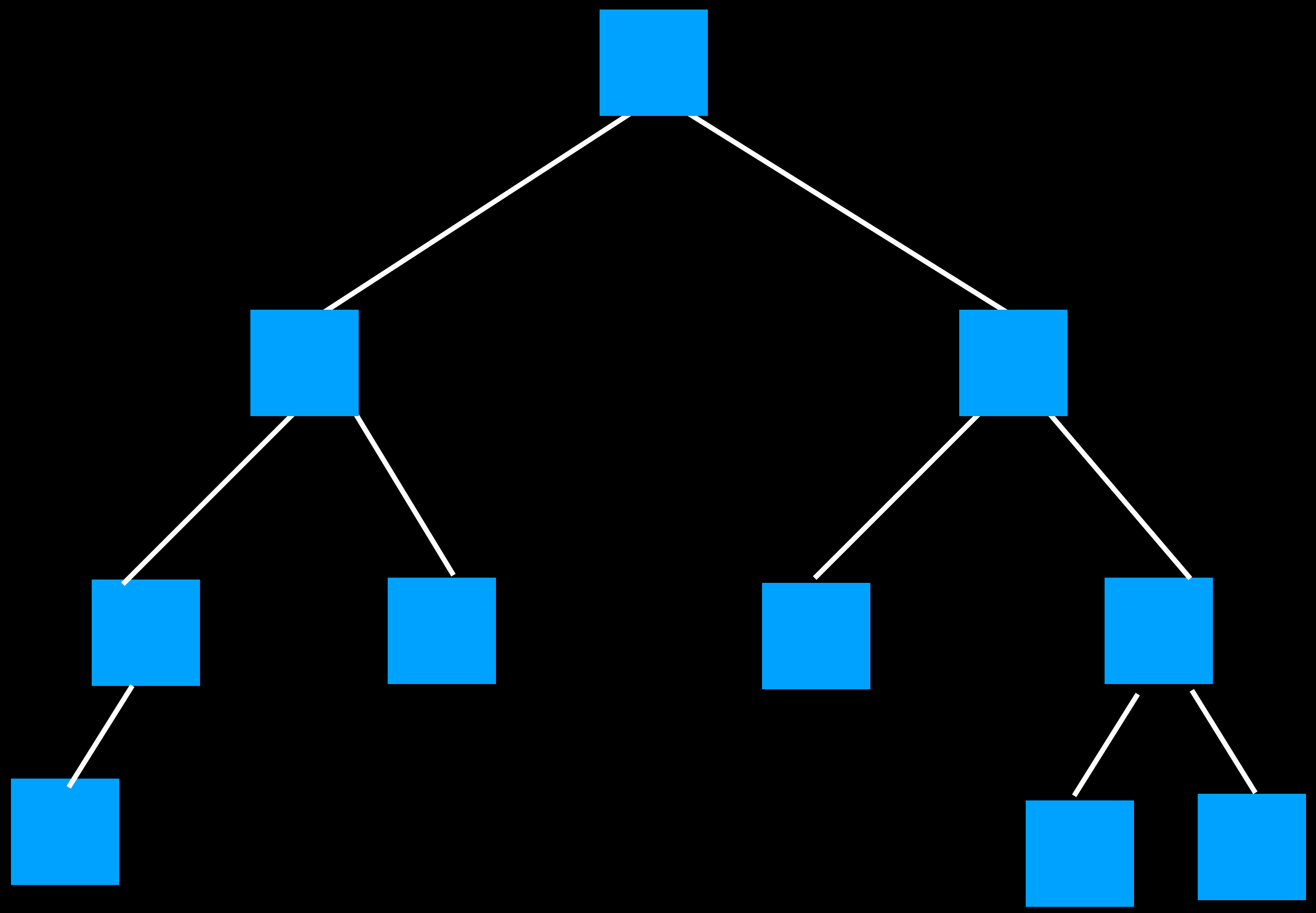
When a node at level *h-1* has one child, it is a left child



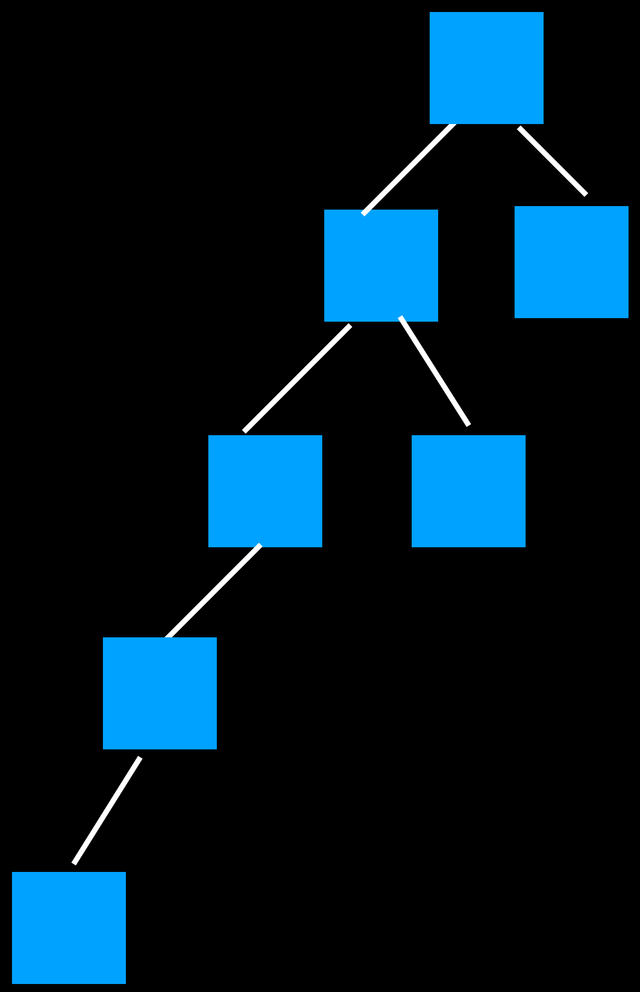
(Height) Balanced Binary Tree

For any node, its left and right subtrees differ in height by no more than 1

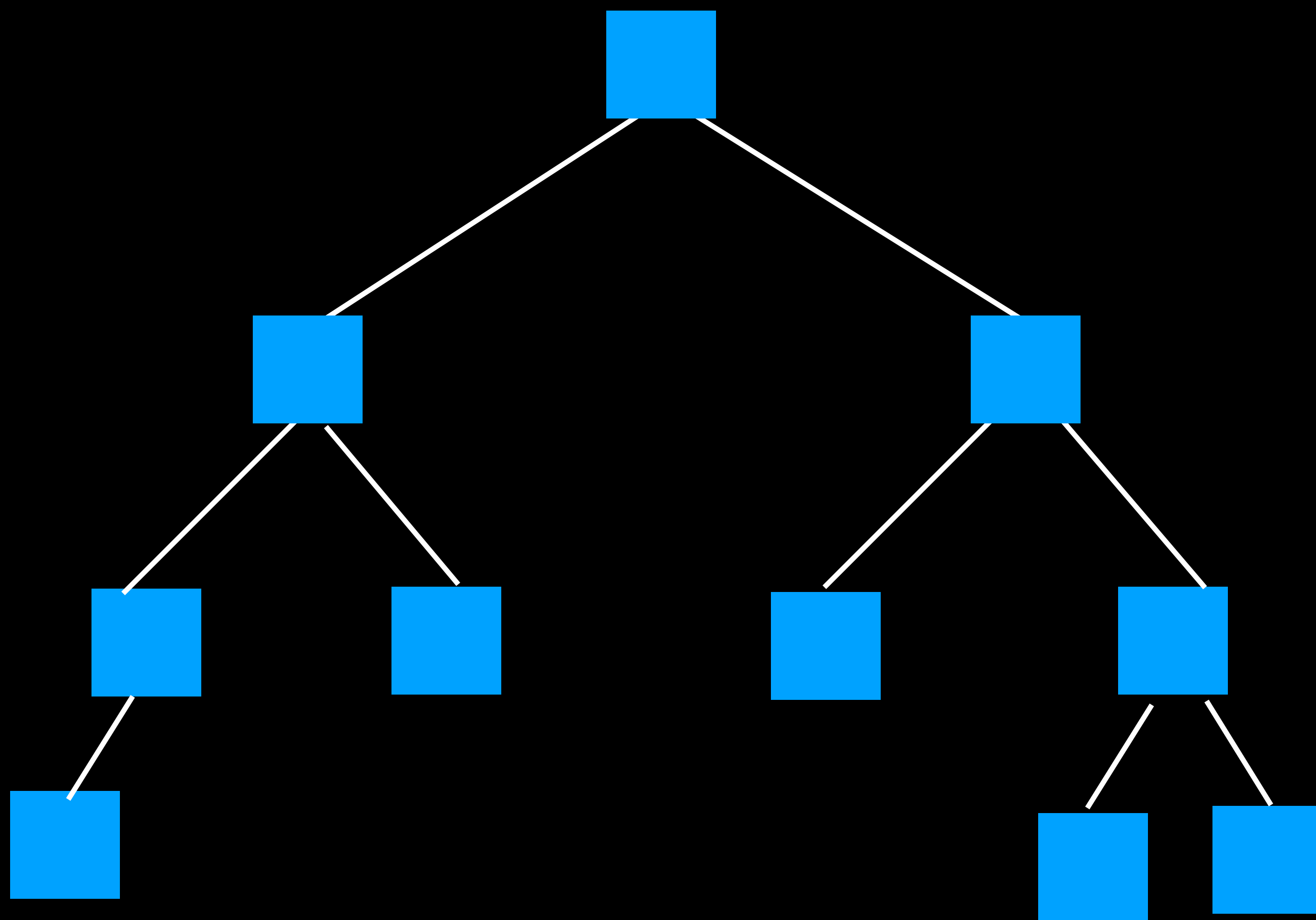
All paths from root to leaf differ in length by at most 1



Unbalanced



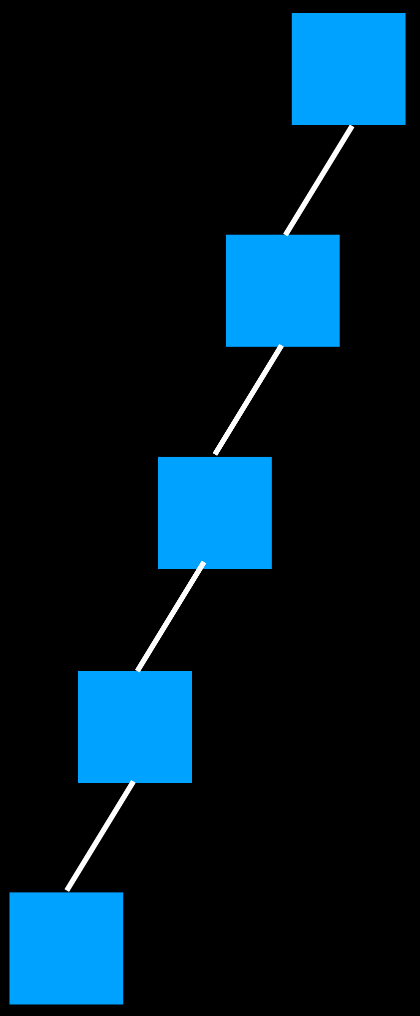
Balanced



Maximum Height

n nodes
every node 1 child
$h = n$

Essentially a chain



Minimum Height

Binary tree of height h can have up to $n = 2^h - 1$

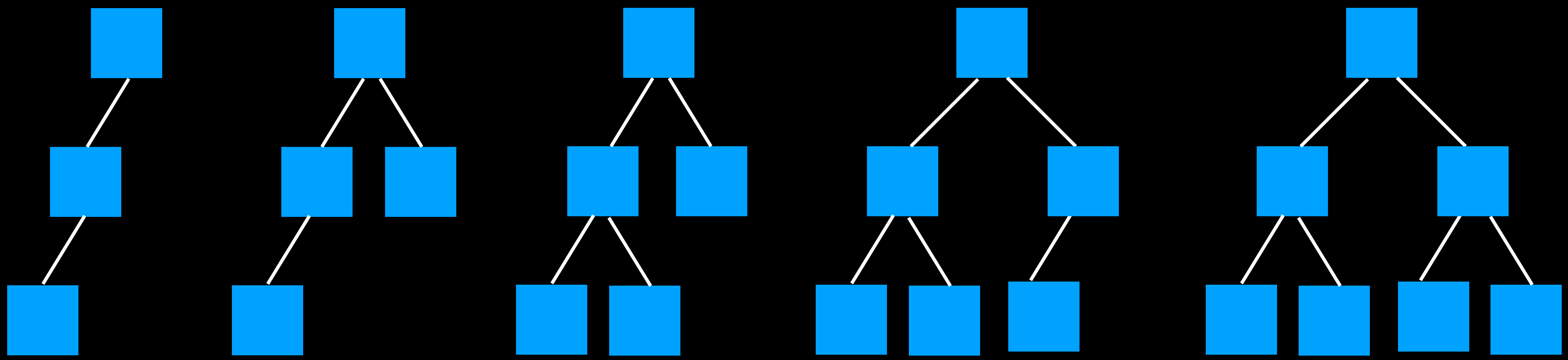
For example for $h = 3$, $1 + 2 + 4 = 7 = 2^3 - 1$

$h = \log_2(n+1)$ for a full binary tree

For example:

1,000 nodes h ≈ 10 (1,000 ≈ 2^{10})

1,000,000 nodes h ≈ 20 (10^6 ≈ 2^{20})



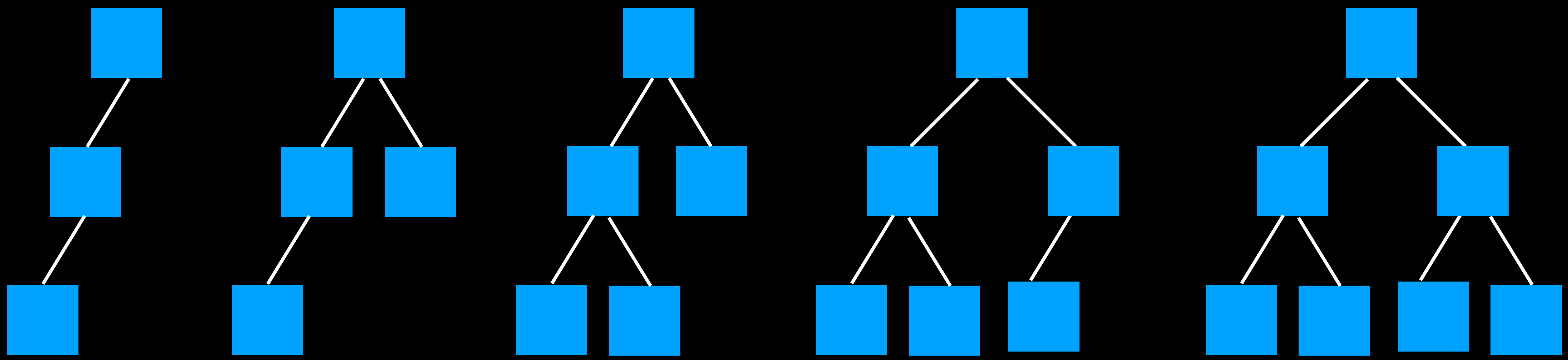
Minimum Height

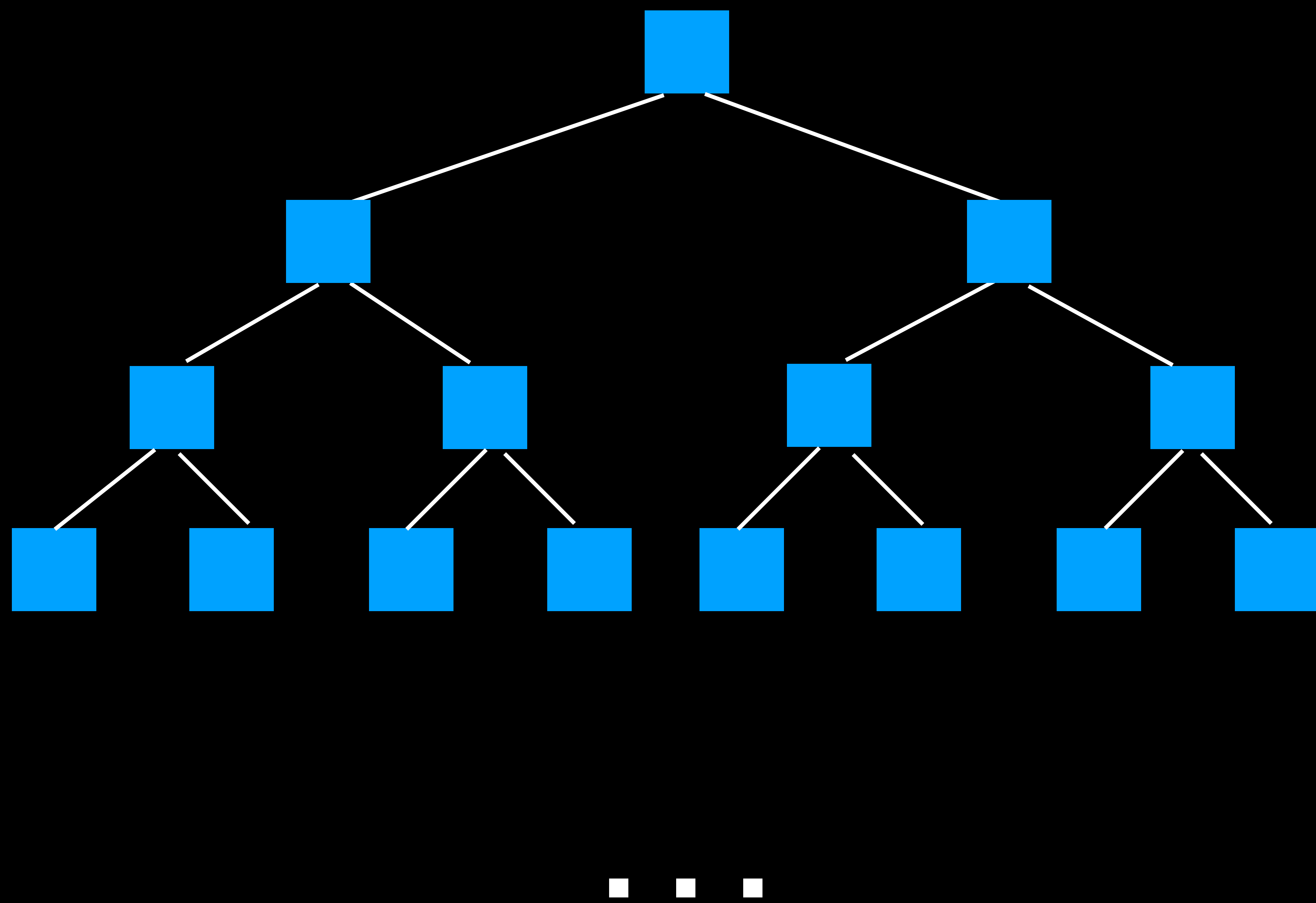
Binary tree of height *h* can have up to *n* = $2^h - 1$
For example for *h* = 3, 1 + 2 + 4 = 7 = $2^3 - 1$
h = $\log_2 (n+1)$ for a full binary tree

For example:
1,000 nodes h ≈ 10 (1,000 ≈ 2^{10})
1,000,000 nodes h ≈ 20 (10^6 ≈ 2^{20})

Recall analysis of Divide and Conquer algorithms

Important when we will be looking for things in trees given some order!!!





In a full tree:

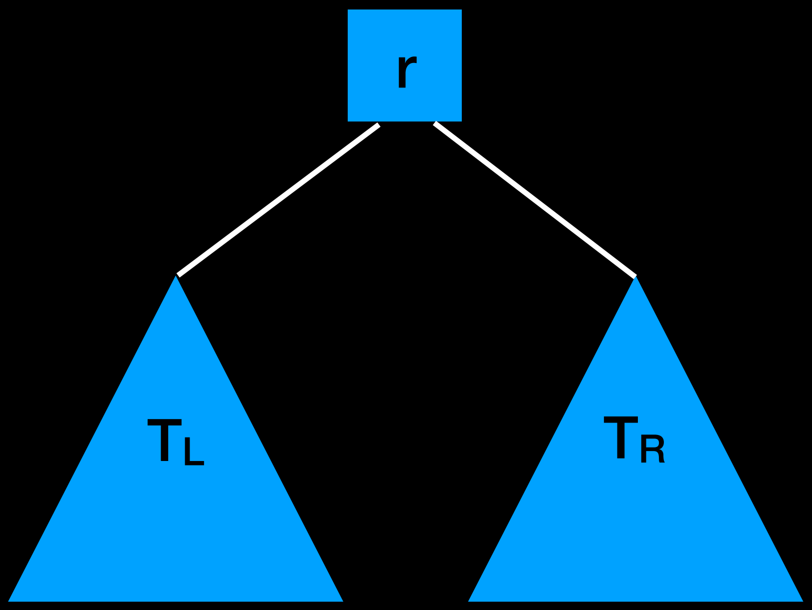
h	n @ level	Total n
1	$1 = 2^0$	$1 = 2^1 - 1$
2	$2 = 2^1$	$3 = 2^2 - 1$
3	$4 = 2^2$	$7 = 2^3 - 1$
4	$8 = 2^3$	$15 = 2^4 - 1$
h	2^{h-1}	$2^h - 1$

Binary Tree Traversals

Visit (retrieve, print, modify …) **every node** in the tree

Essentially visit the root as well as it’s subtrees

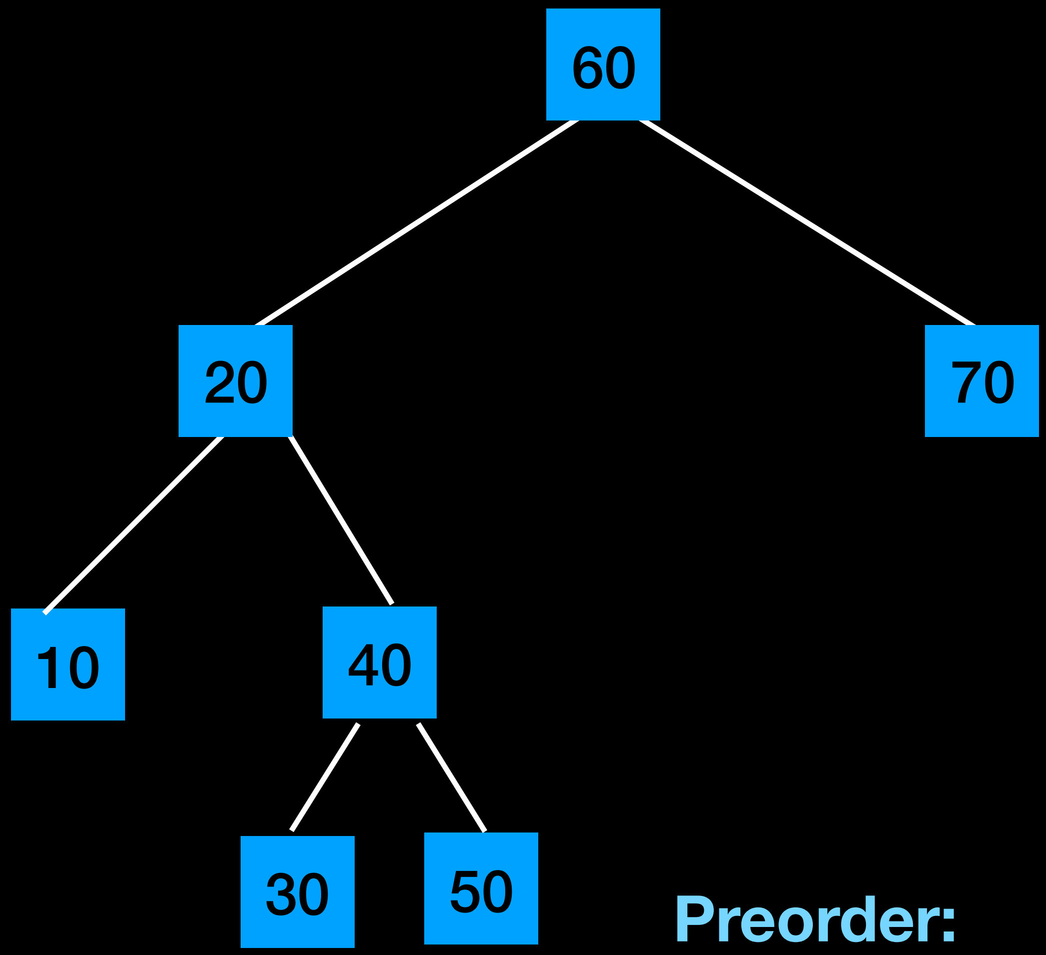
Order matters!!!

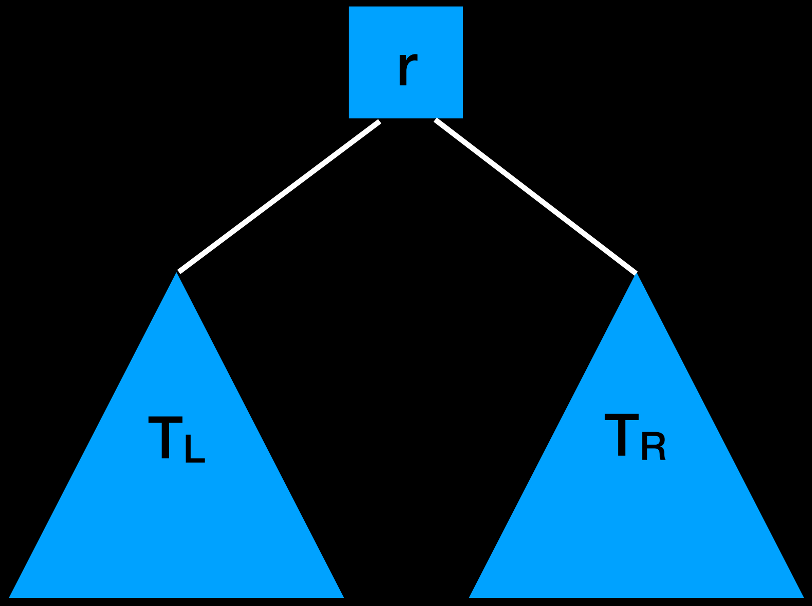


Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```





Preorder:

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

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if (T is not empty) //implicit base case
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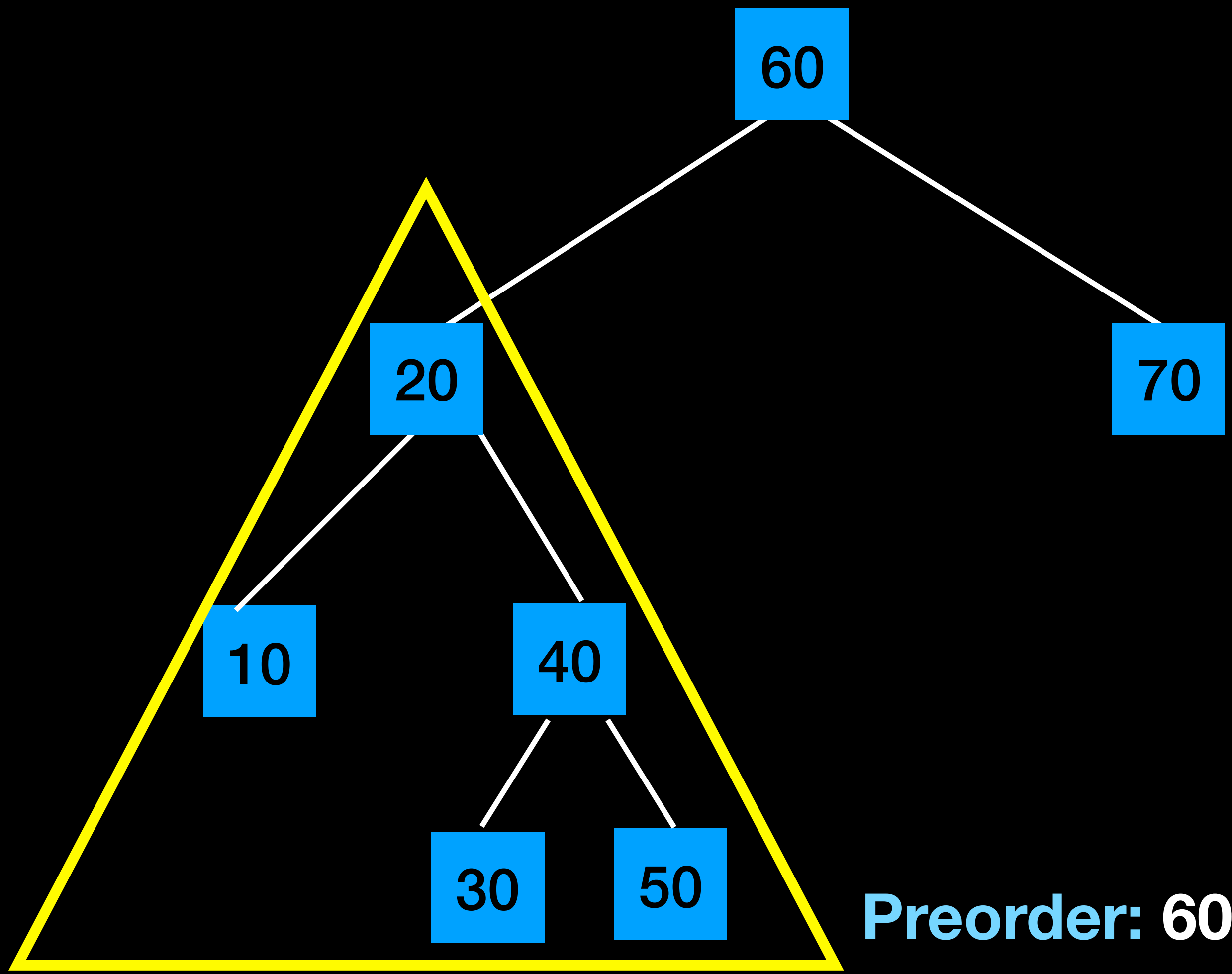
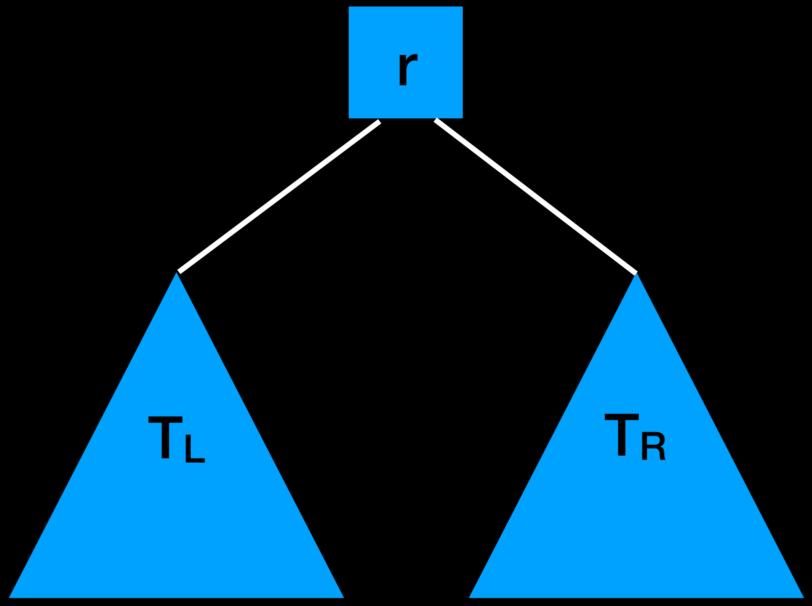
}
```



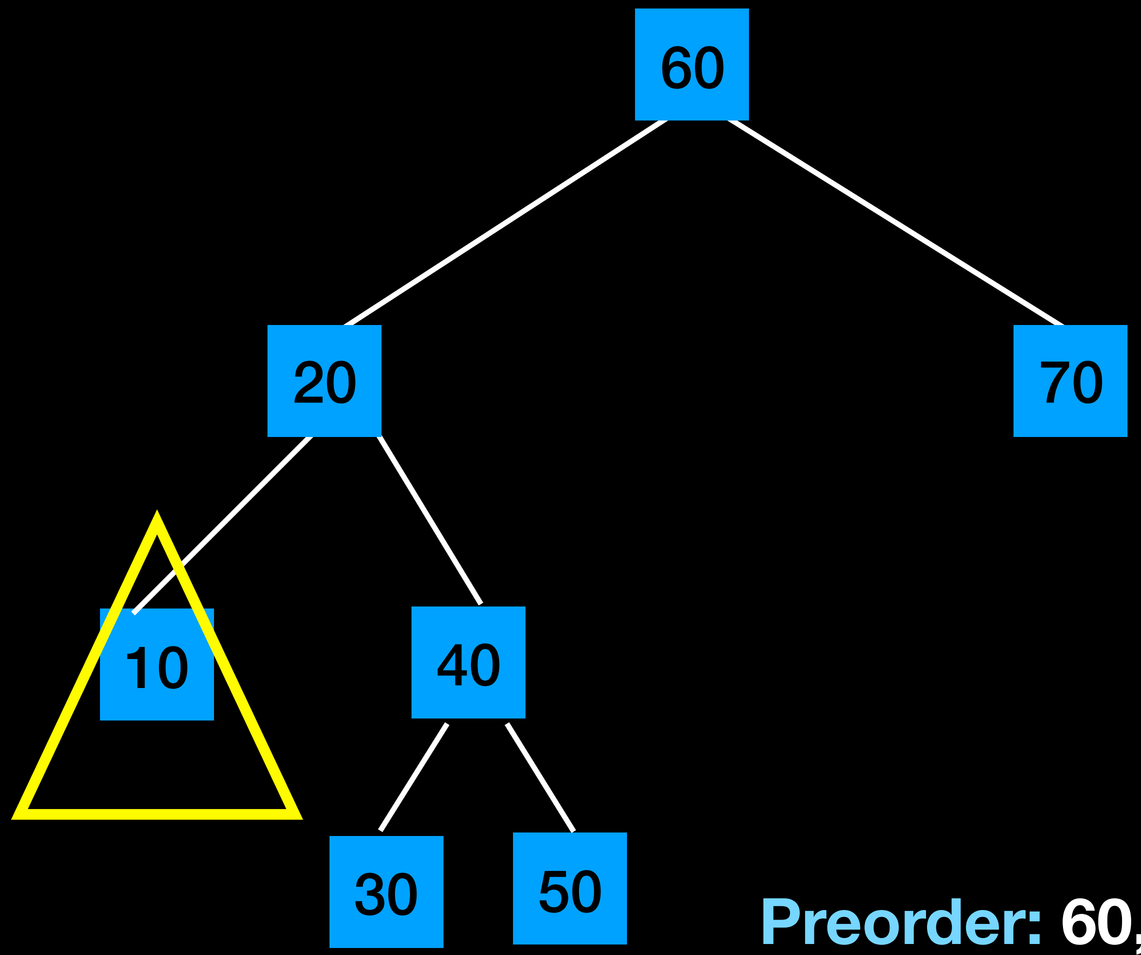
Preorder: 60



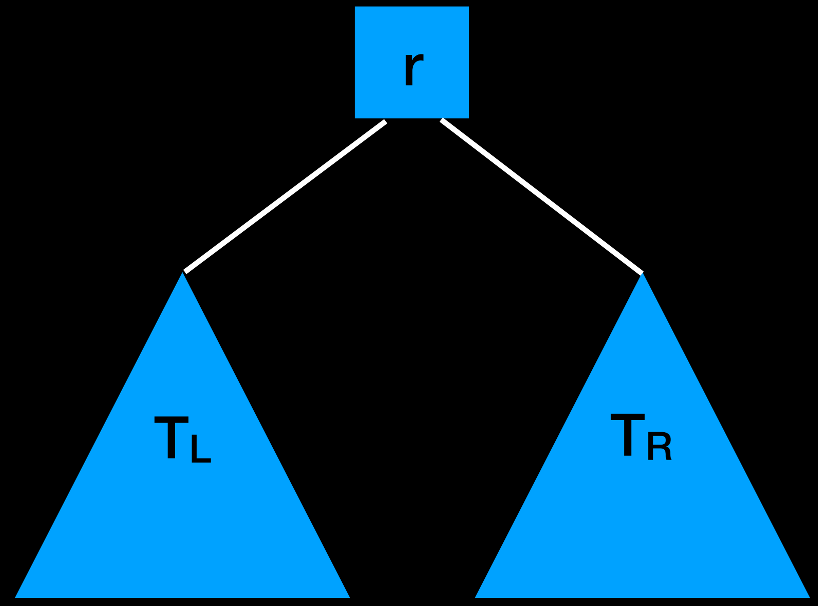
Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



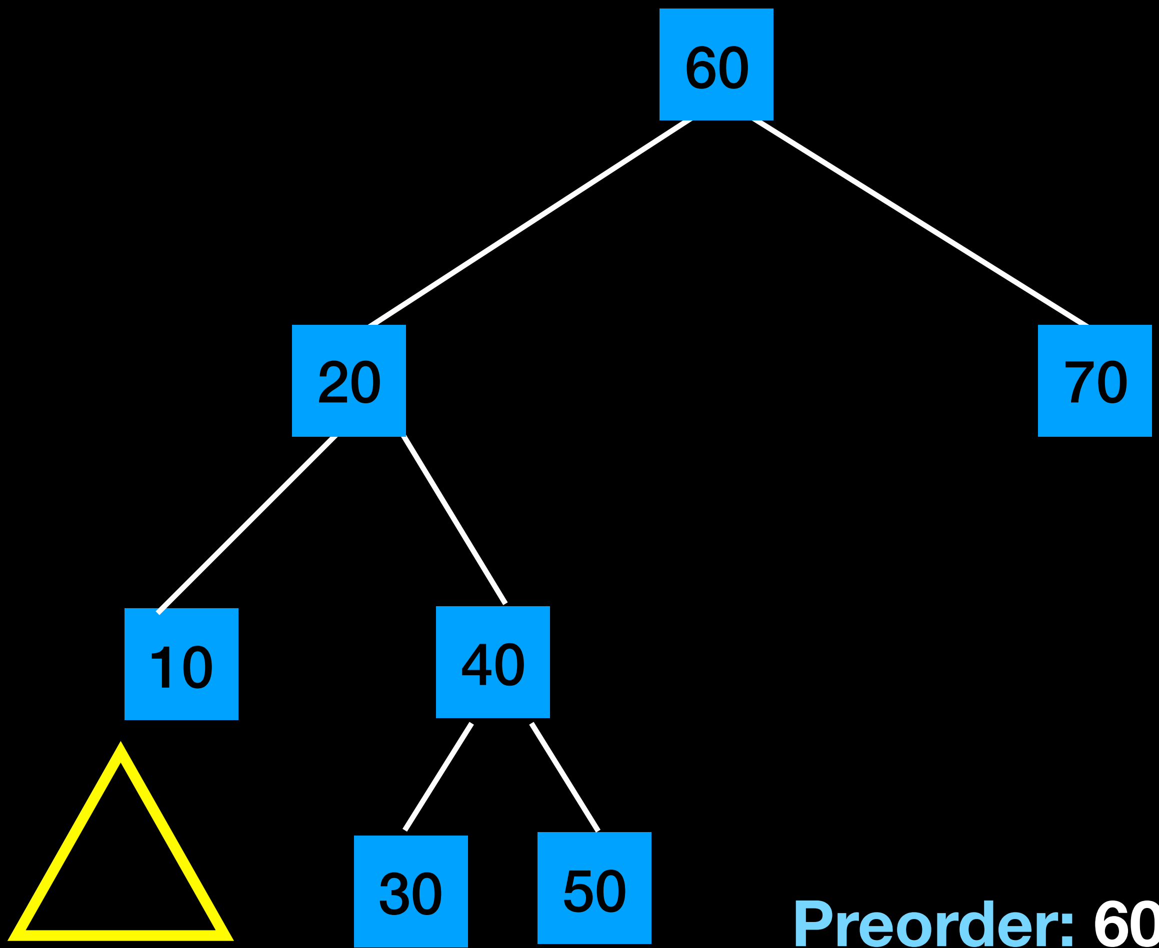
Preorder: **60, 20**

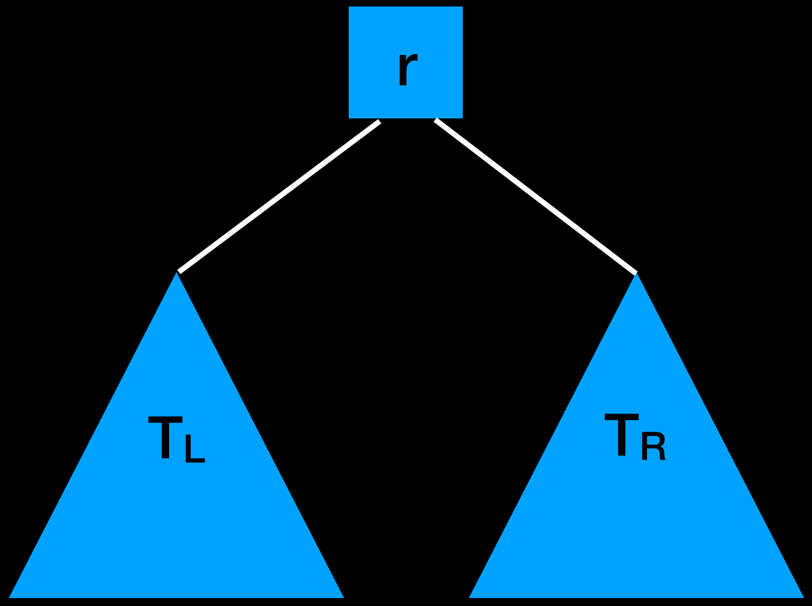


Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse T_L
    traverse T_R
}
```





Preorder: 60, 20, 10

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

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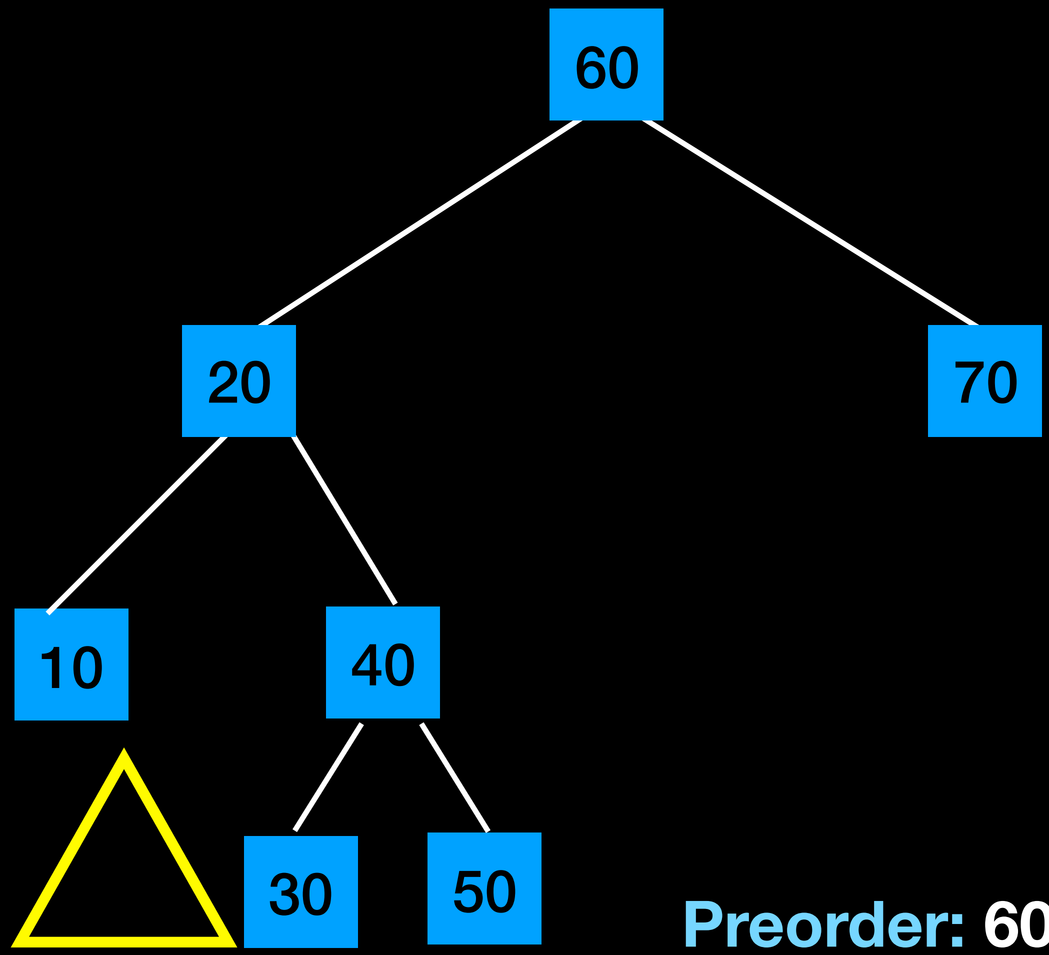
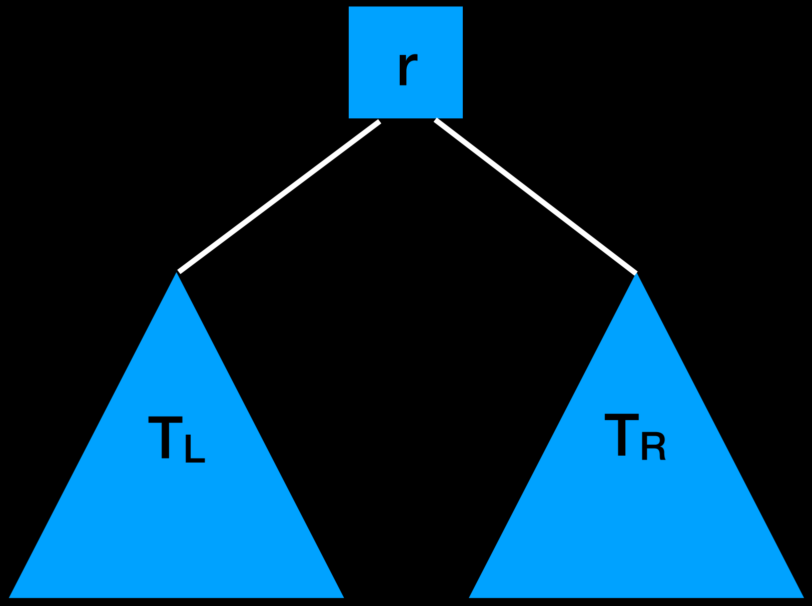
    traverse TL
    traverse TR
}
```





Preorder: 60, 20, 10

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

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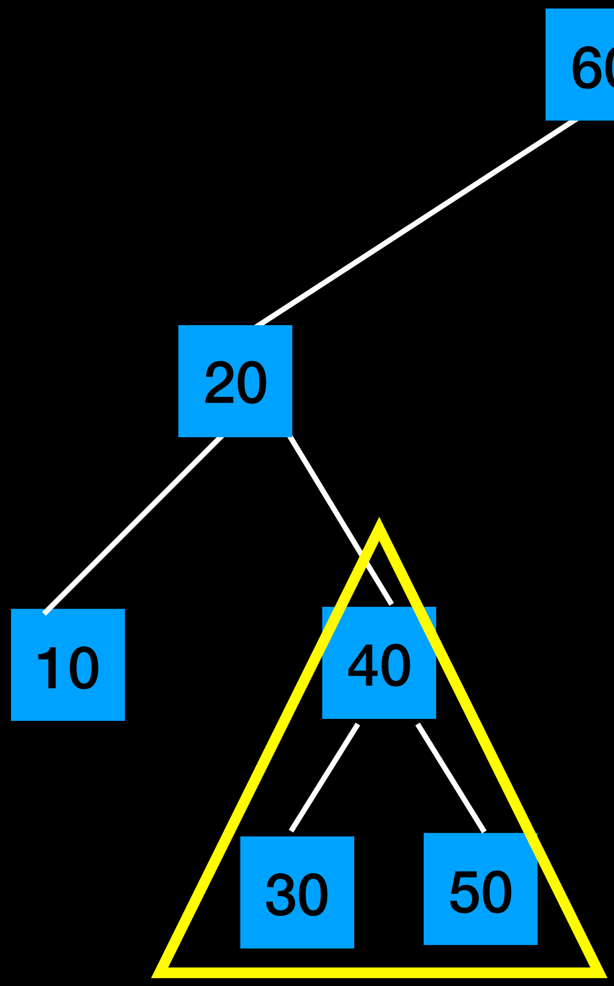
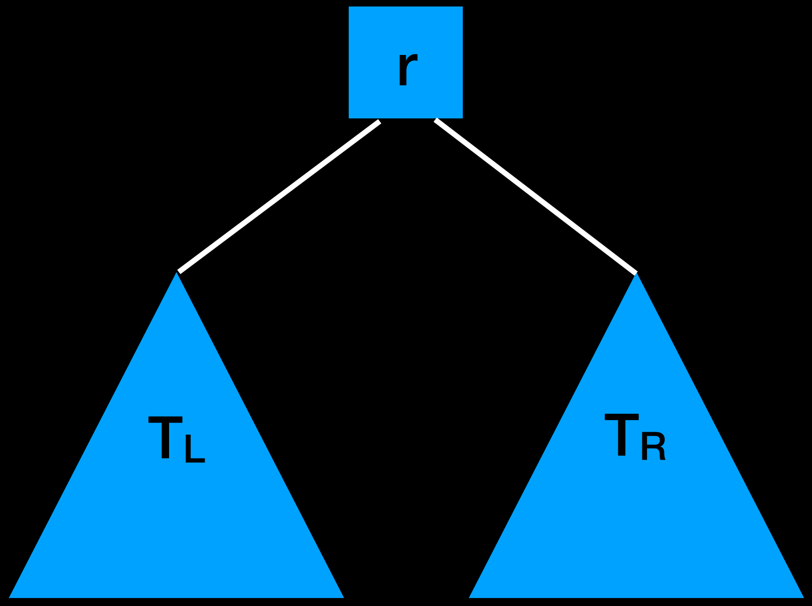
    traverse TR
}
```





Preorder: 60, 20, 10

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

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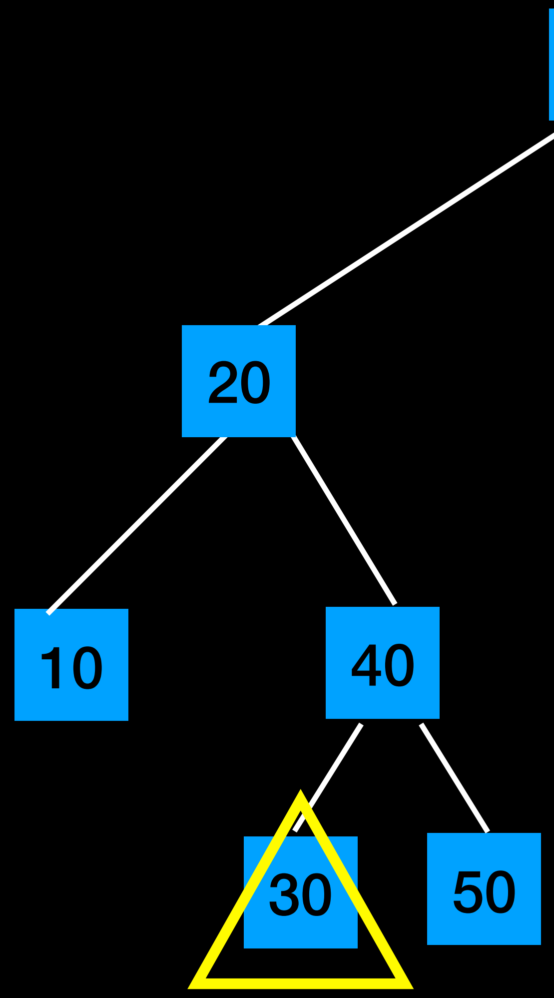
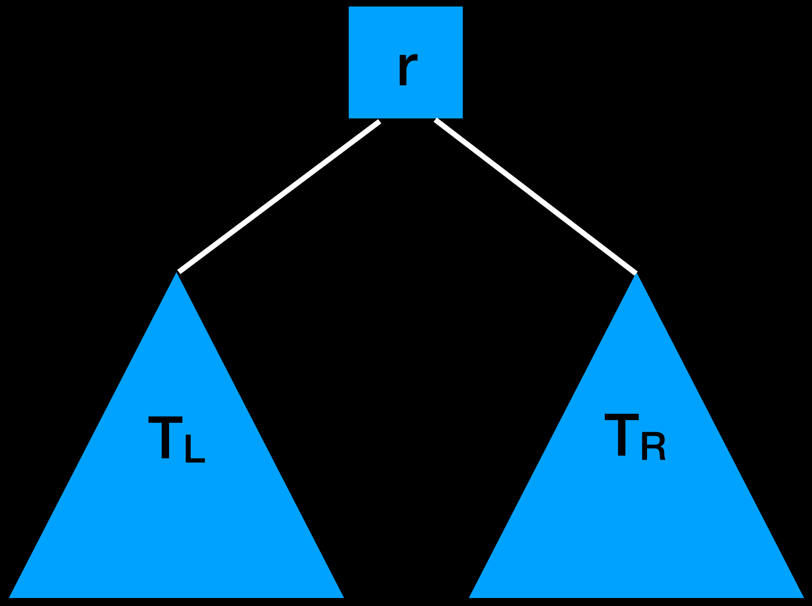
    traverse TR
}
```



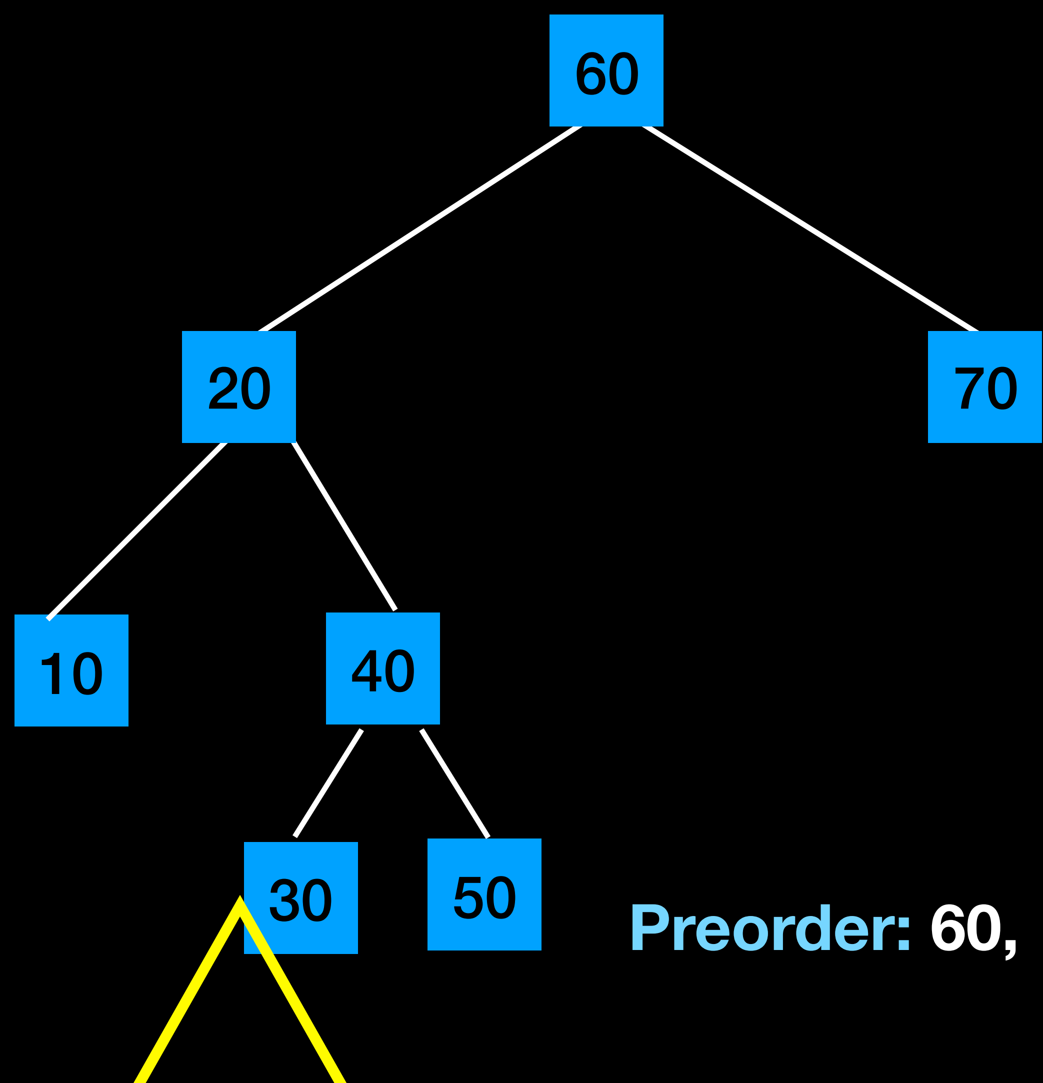


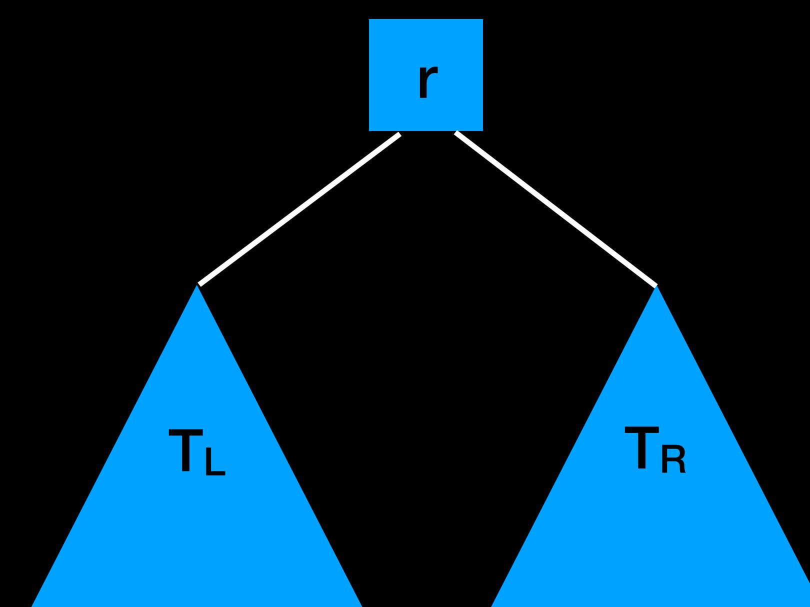
Preorder: 60, 20, 10, 40

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse T_L
    traverse T_R
}
```





Preorder: 60, 20, 10, 40, 30

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

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if (T is not empty) //implicit base case
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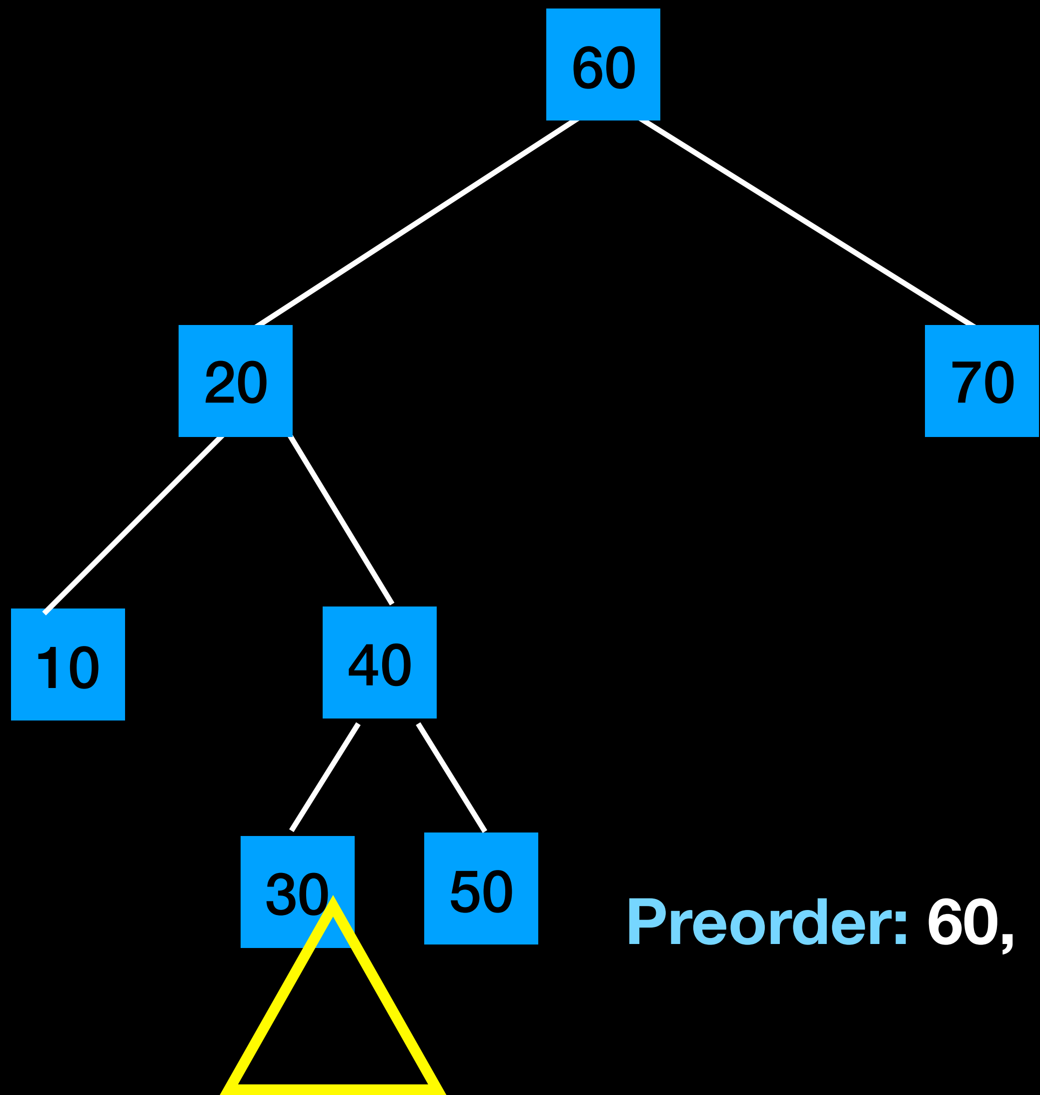
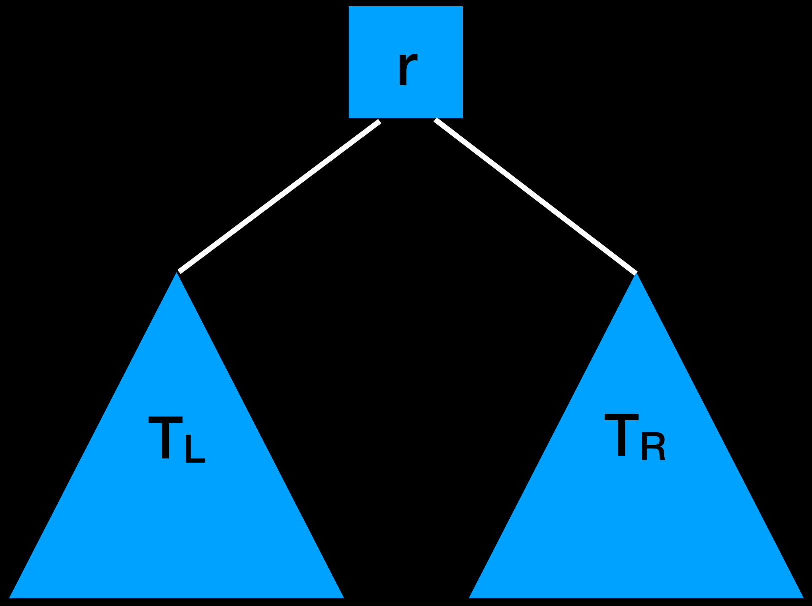
}
```



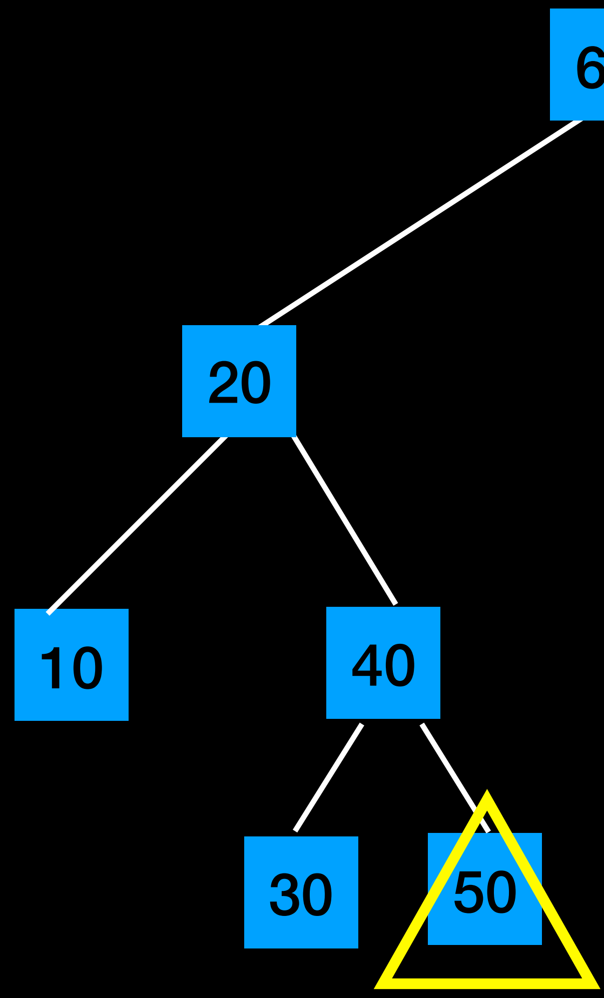


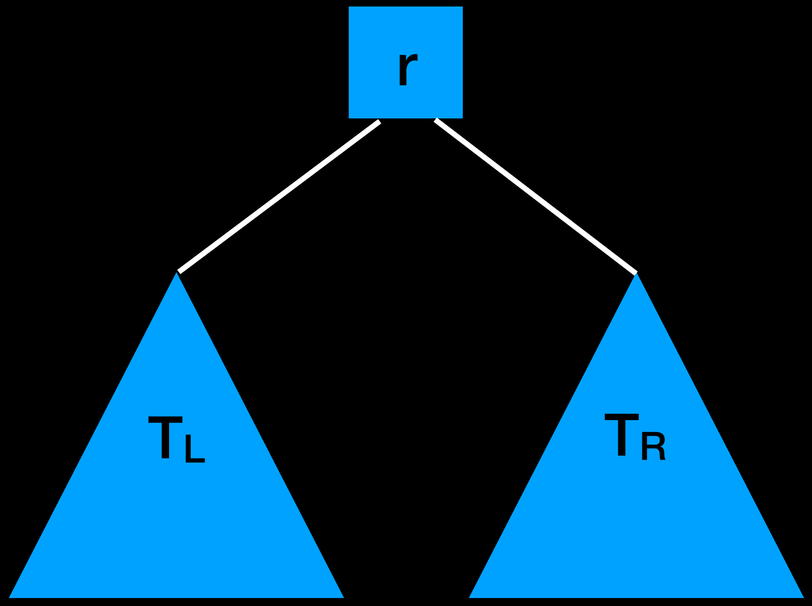
Preorder: 60, 20, 10, 40, 30

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



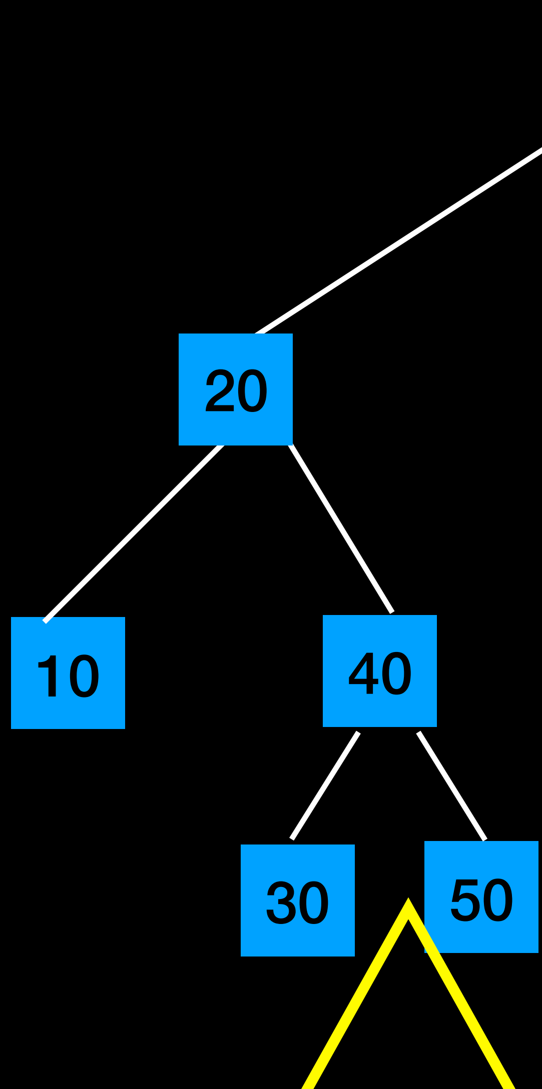


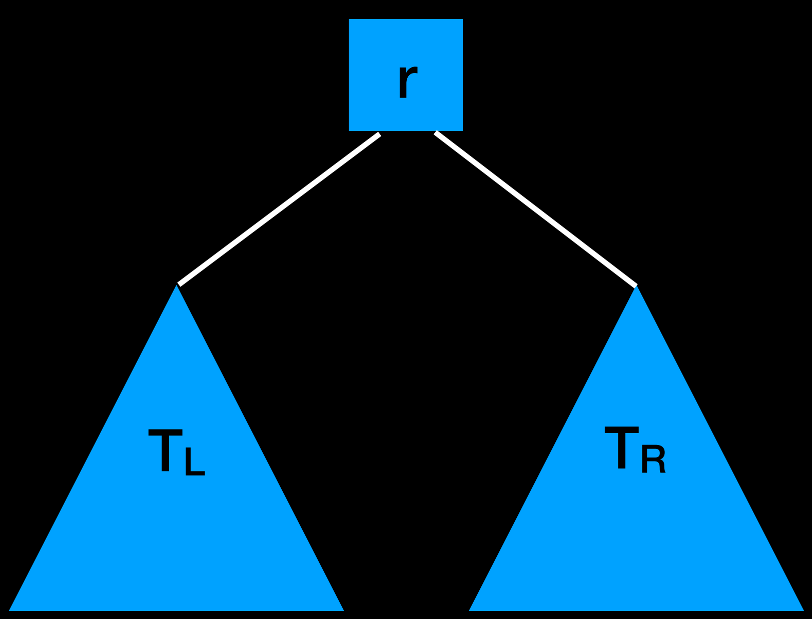
Preorder: 60, 20, 10, 40, 30

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



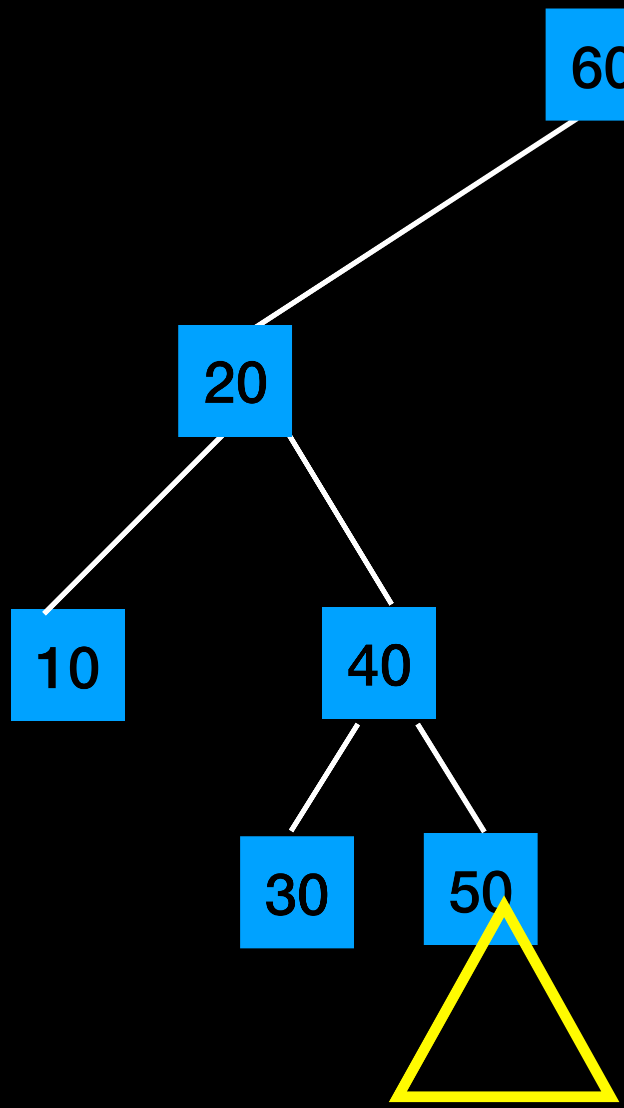


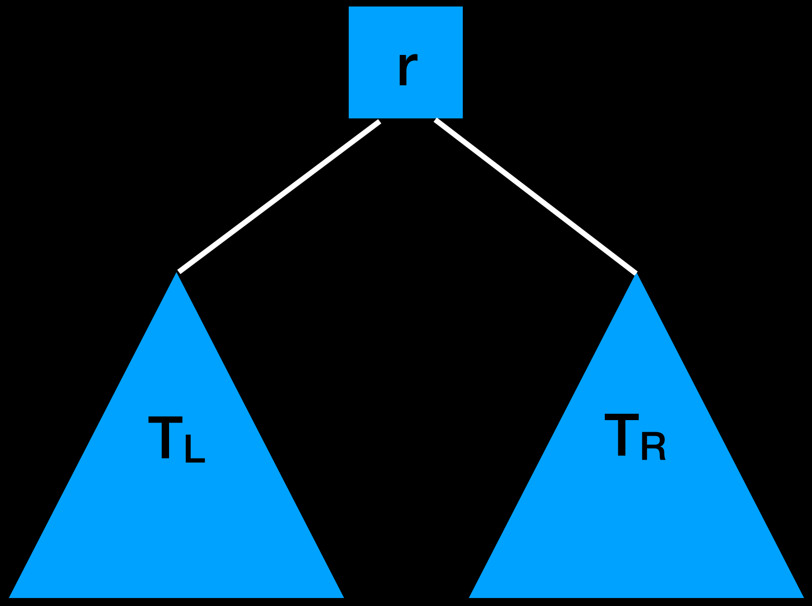
Preorder: 60, 20, 10, 40, 30, 50

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



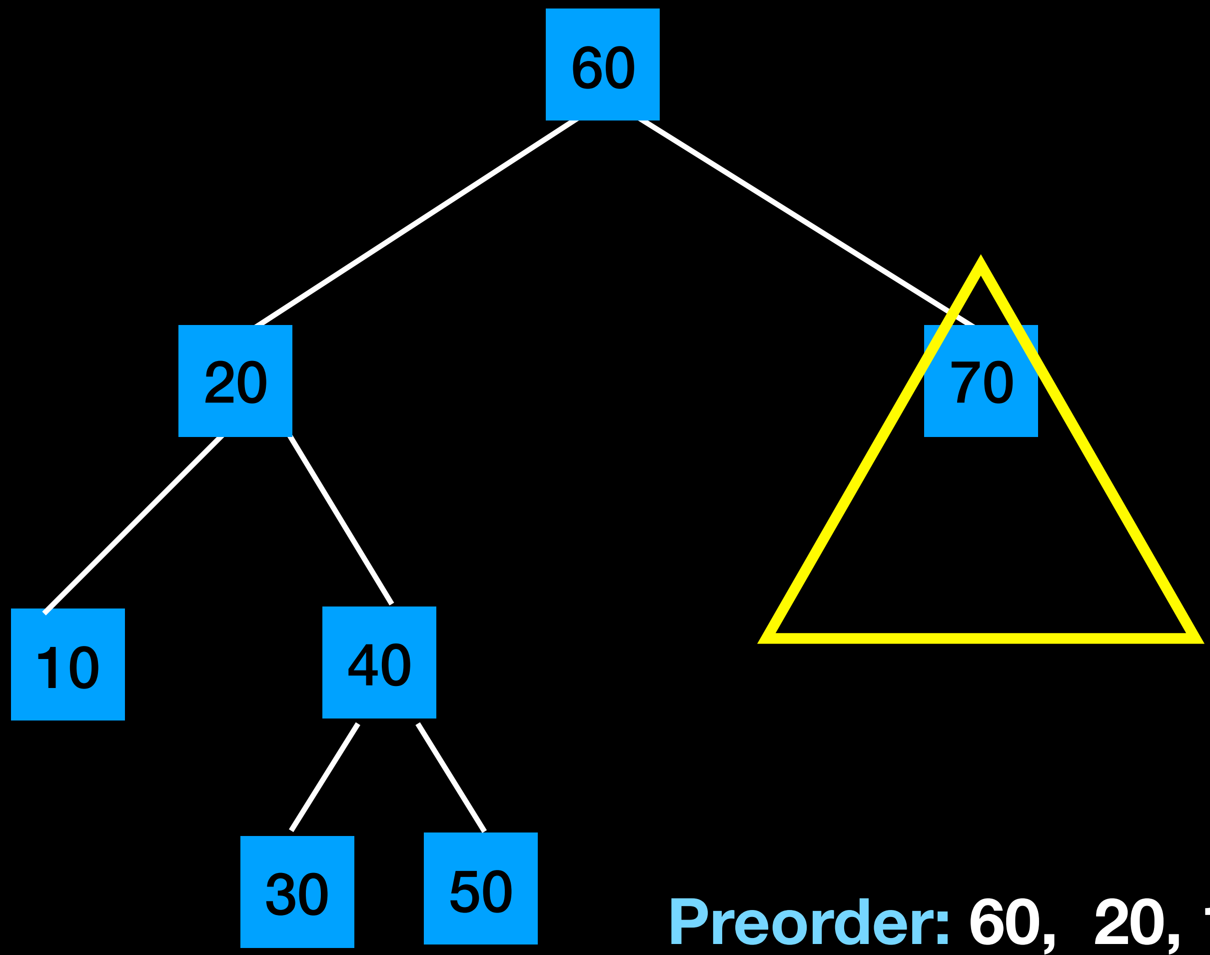


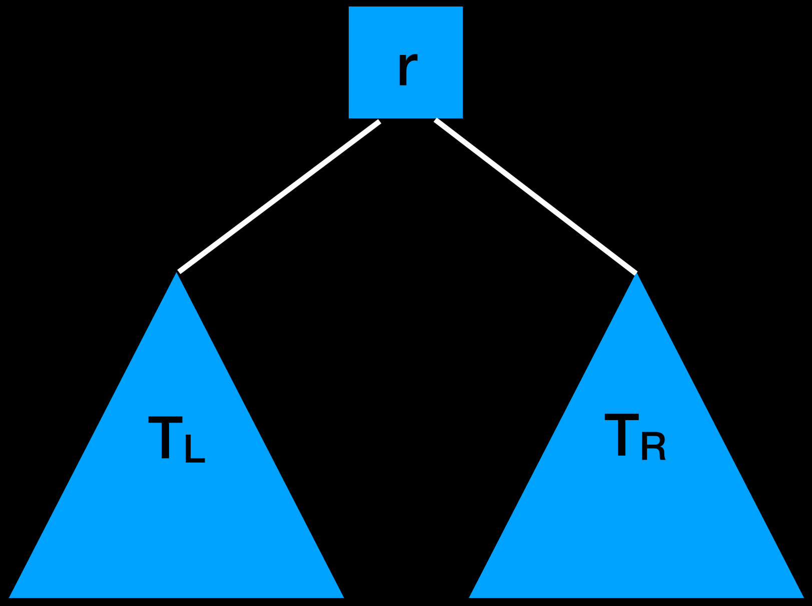
Preorder: 60, 20, 10, 40, 30, 50

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



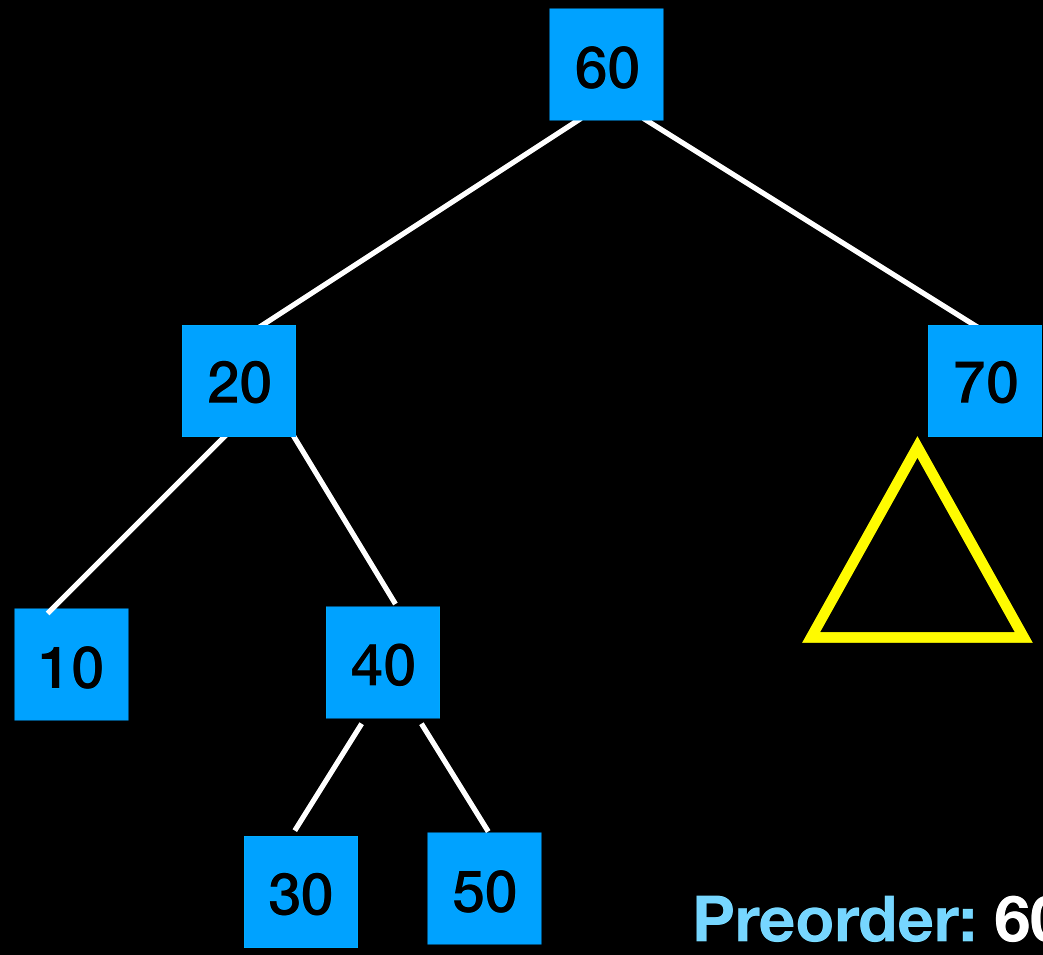


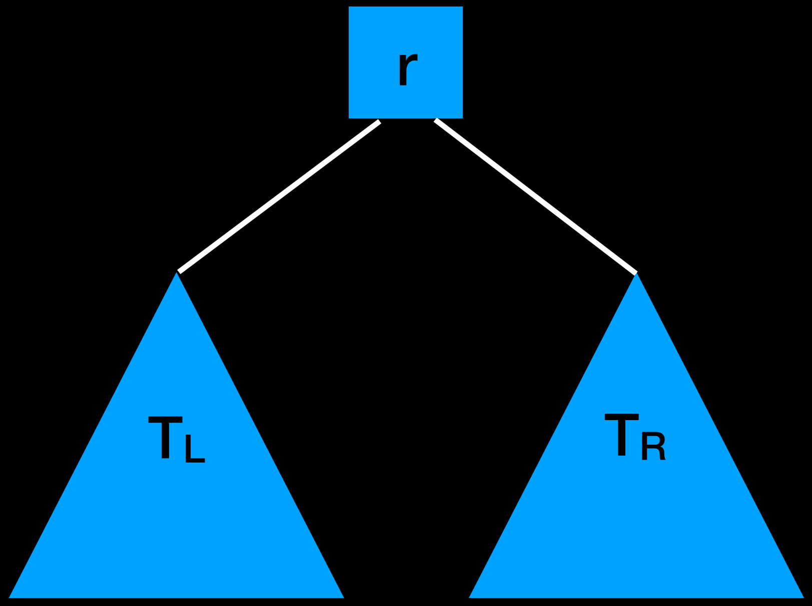
Preorder: 60, 20, 10, 40, 30, 50

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



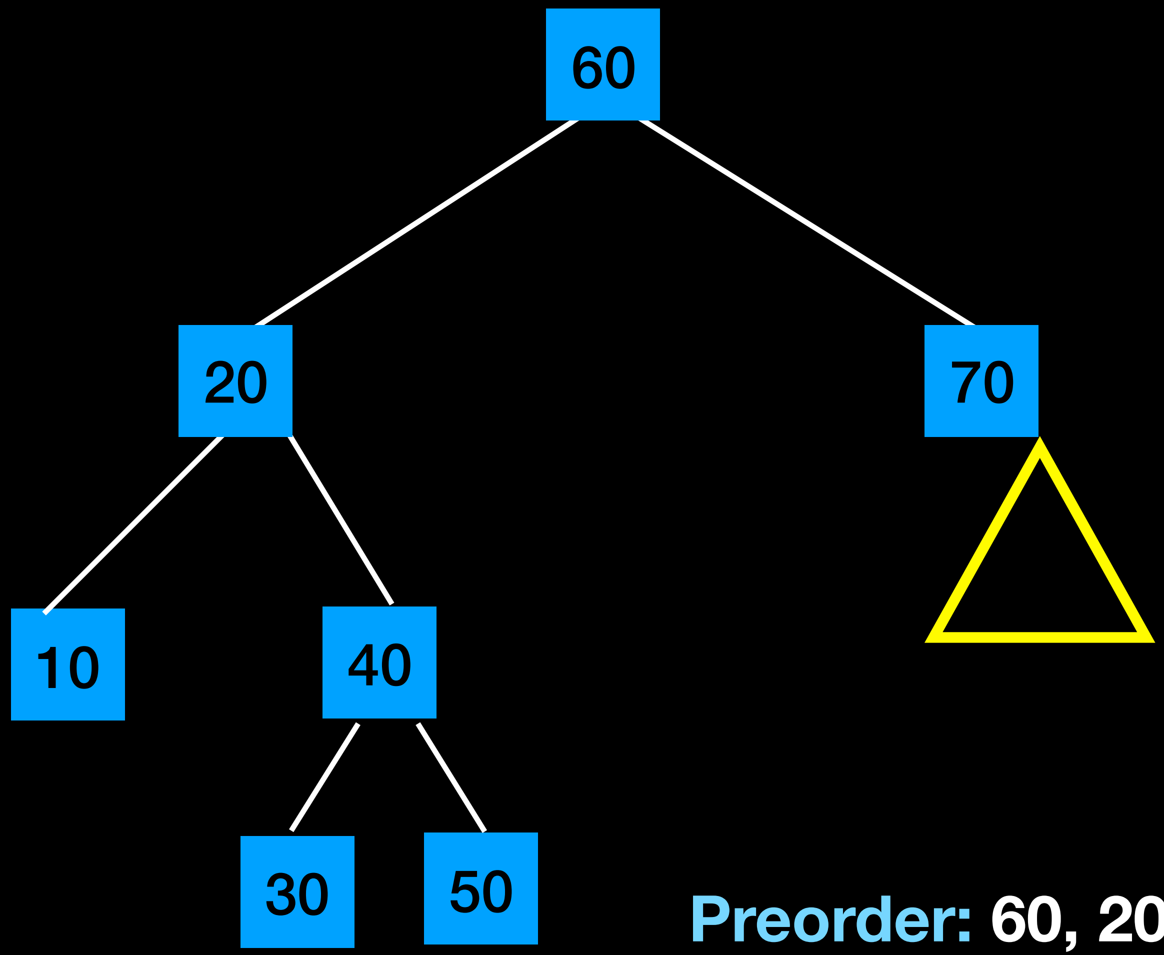


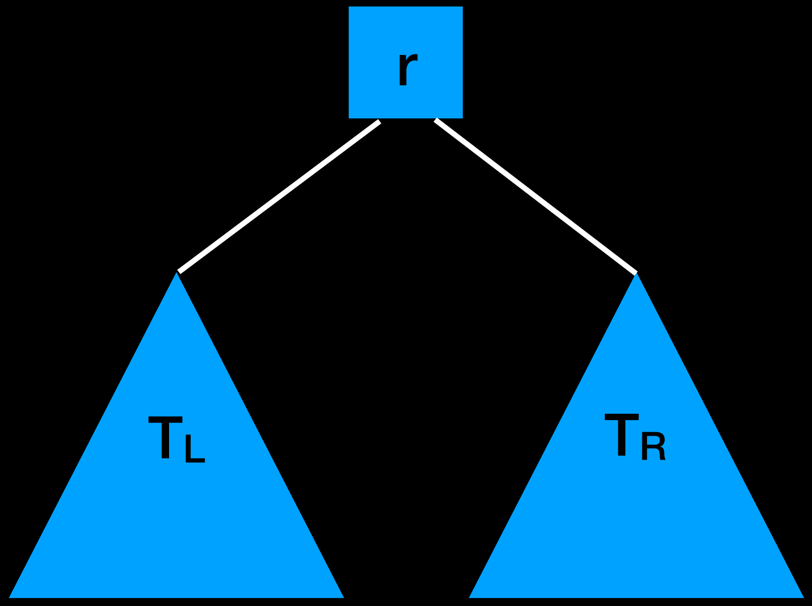
Preorder: 60, 20, 10, 40, 30, 50, 70

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



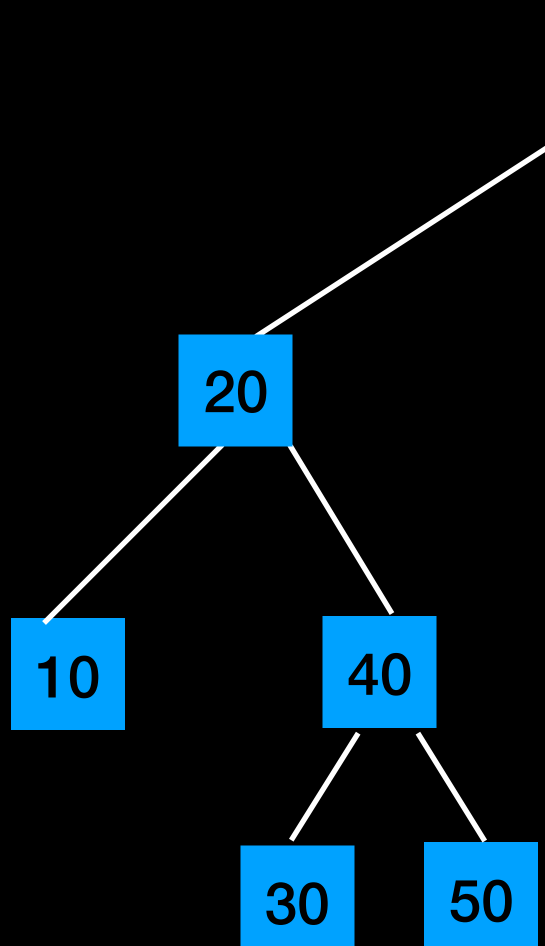


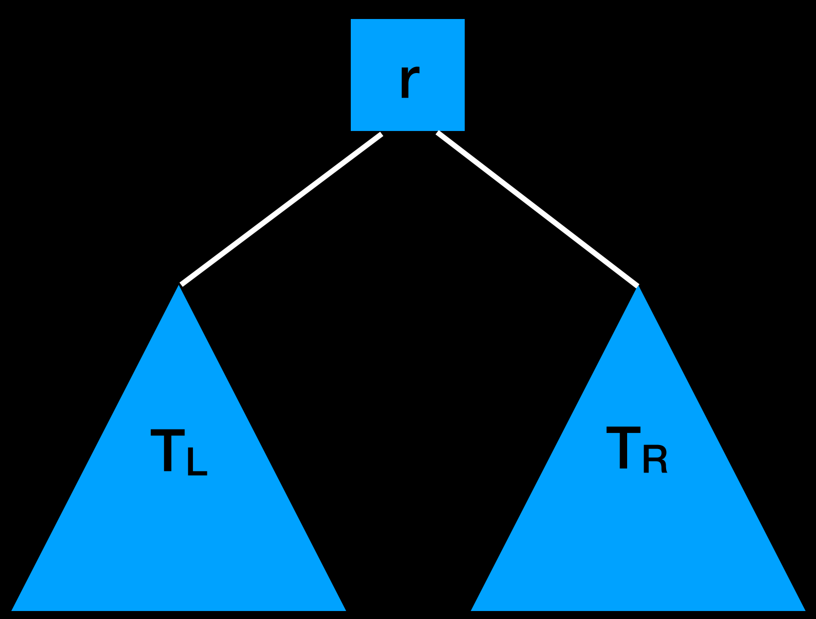
Preorder: 60, 20, 10, 40, 30, 50, 70

Visit (retrieve, print, modify …) every node in the tree

Preorder Traversal:

```
if (T is not empty) //implicit base case
{
    visit the root r
    traverse TL
    traverse TR
}
```



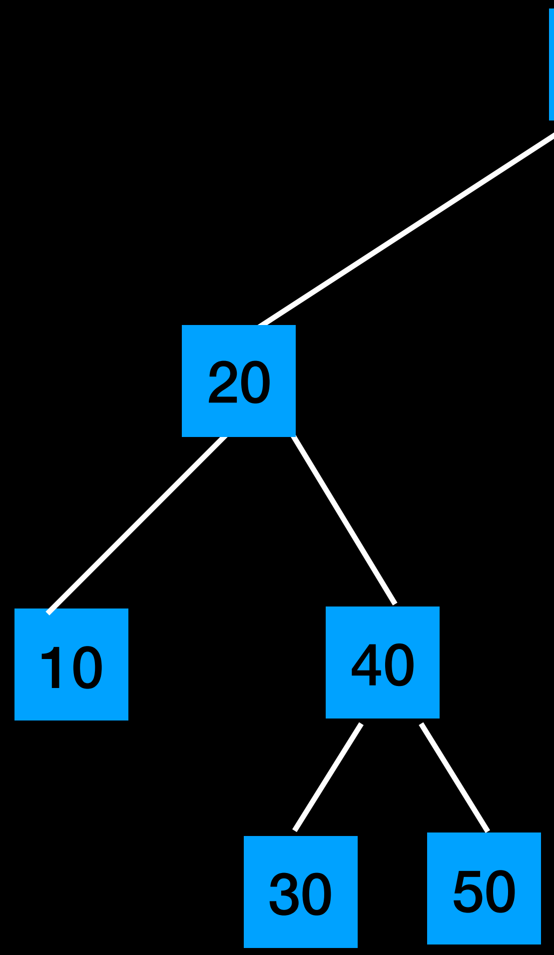


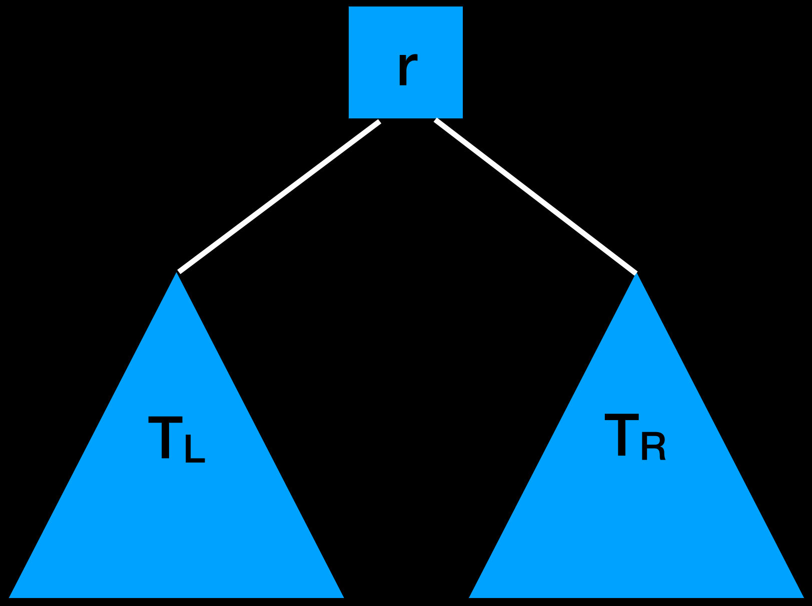
Preorder: 60, 20, 10, 40, 30, 50, 70

Visit (retrieve, print, modify …) every node in the tree

Inorder Traversal:

```
if (T is not empty) //implicit base case
{
    traverse T_L
    visit the root r
    traverse T_R
}
```



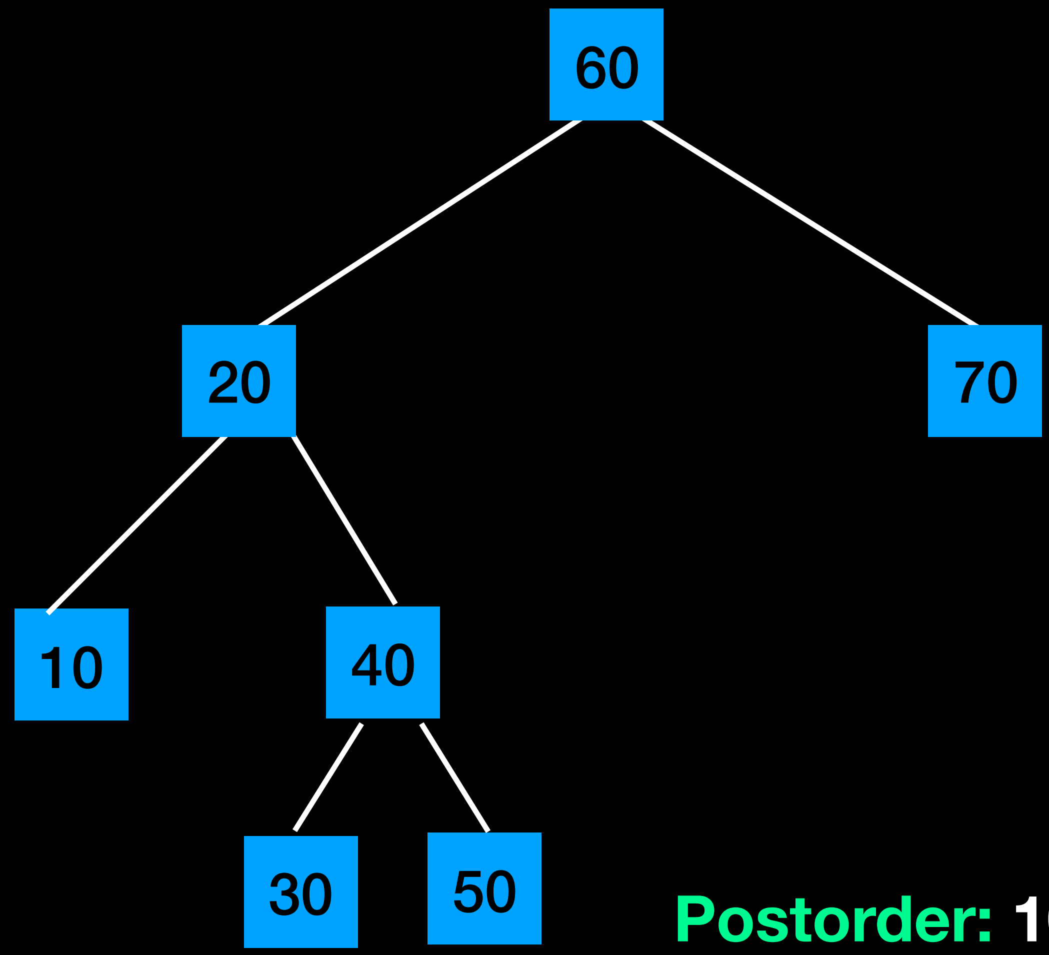


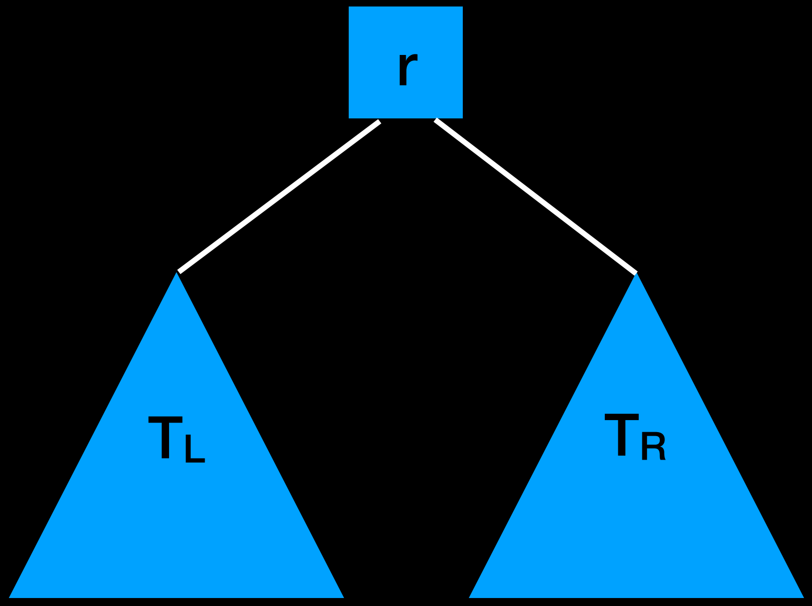
Inorder: **10, 20, 30, 40, 50, 60, 70**

Visit (retrieve, print, modify …) every node in the tree

Postorder Traversal:

```
if (T is not empty) //implicit base case
{
    traverse TL
    traverse TR
    visit the root r
}
```





Postorder: **10, 30, 50, 40, 20, 70, 60**

BinaryTree ADT Operations

```cpp
#ifndef BinaryTree_H_
#define BinaryTree_H_

template<class T>
class BinaryTree
{

public:
    BinaryTree(); // constructor
    BinaryTree(const BinaryTree<T>& tree); // copy constructor
    ~BinaryTree(); // destructor
    bool isEmpty() const;
    size_t getHeight() const;
    size_t getNumberOfNodes() const;
    void add(const T& new_item);
    void remove(const T& new_item);
    T find(const T& item) const;
    void clear();

    void preorderTraverse(Visitor<T>& visit) const;
    void inorderTraverse(Visitor<T>& visit) const;
    void postorderTraverse(Visitor<T>& visit) const;

    BinaryTree& operator= (const BinaryTree<T>& rhs);

private: // implementation details here

}; // end BST

#include "BinaryTree.cpp"
#endif // BinaryTree_H_
```

```cpp
#ifndef BinaryTree_H_
#define BinaryTree_H_

template<class T>
class BinaryTree
{

public:
     BinaryTree(); // constructor
     BinaryTree(const BinaryTree<T>& tree); // copy constructor
     ~BinaryTree(); // destructor
     bool isEmpty() const;
     size_t getHeight() const;
     size_t getNumberOfNodes() const;
     void add(const T& new_item);
     void remove(const T& new_item);
     T find(const T& item) const;
     void clear();

     void preorderTraverse(Visitor<T>& visit) const;
     void inorderTraverse(Visitor<T>& visit) const;
     void postorderTraverse(Visitor<T>& visit) const;

     BinaryTree& operator= (const BinaryTree<T>& rhs);

private: // implementation details here

}; // end BST

#include "BinaryTree.cpp"
#endif // BinaryTree_H_
```

How might you add
Will determine the tree structure

We will talk about this
when we implement it

Considerations

Recall

Remember our Bag ADT?

- Array implementation
- Linked Chain implementation
- Assume no duplicates

Find an element: $O(n)$

Remove: Find element and if there remove it $O(n)$

Add: Check if element is there and if not add it $O(n)$

Recall

Remember our Bag ADT?

- Array implementation
- Linked Chain implementation
- Assume no duplicates

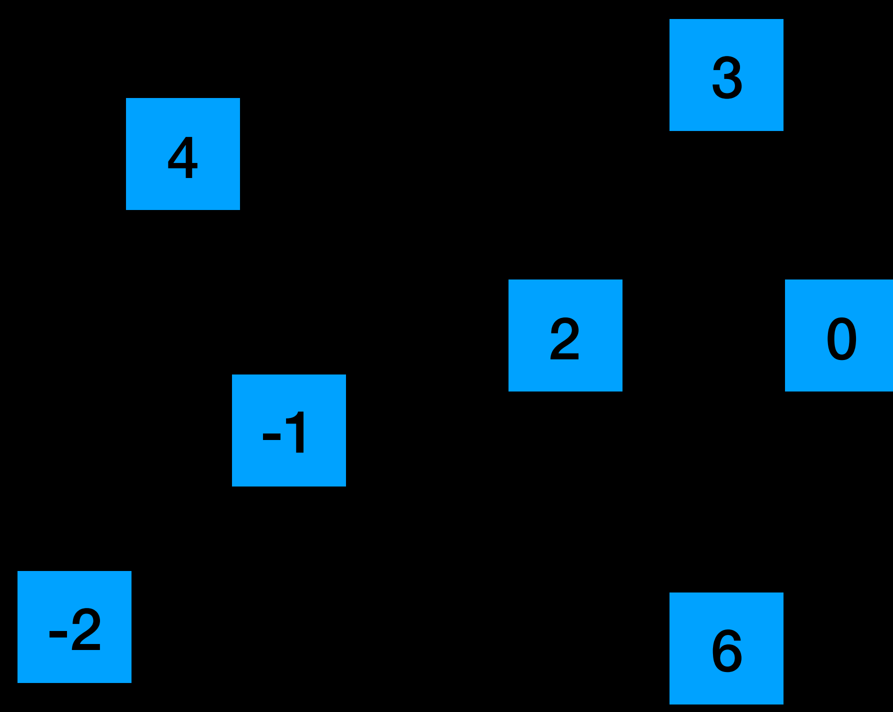
Can we do better?

Find an element: O(n)

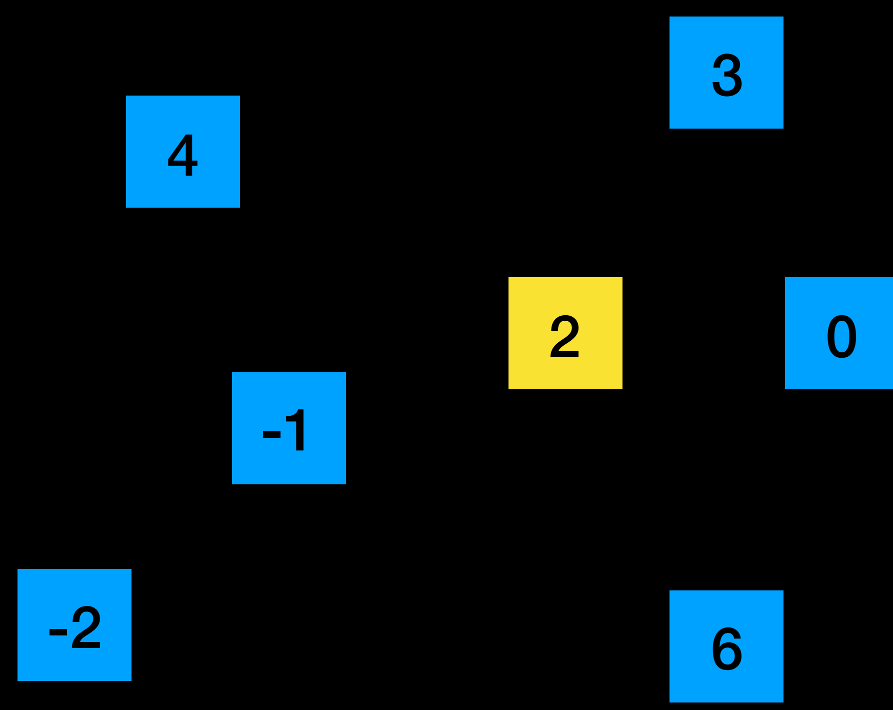
Remove: Find element and if there remove it O(n)

Add: Check if element is there and if not add it O(n)

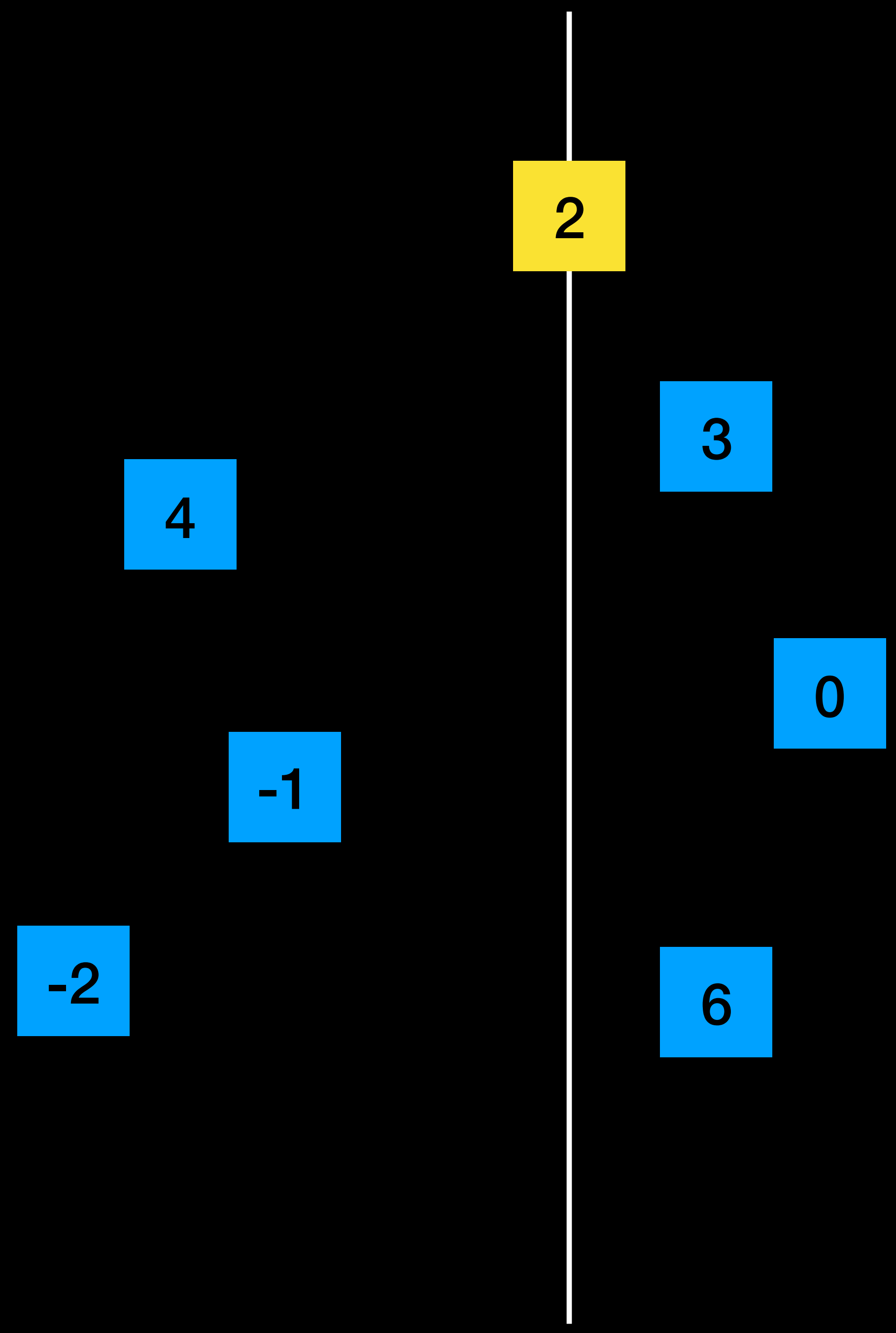
A Different Approach



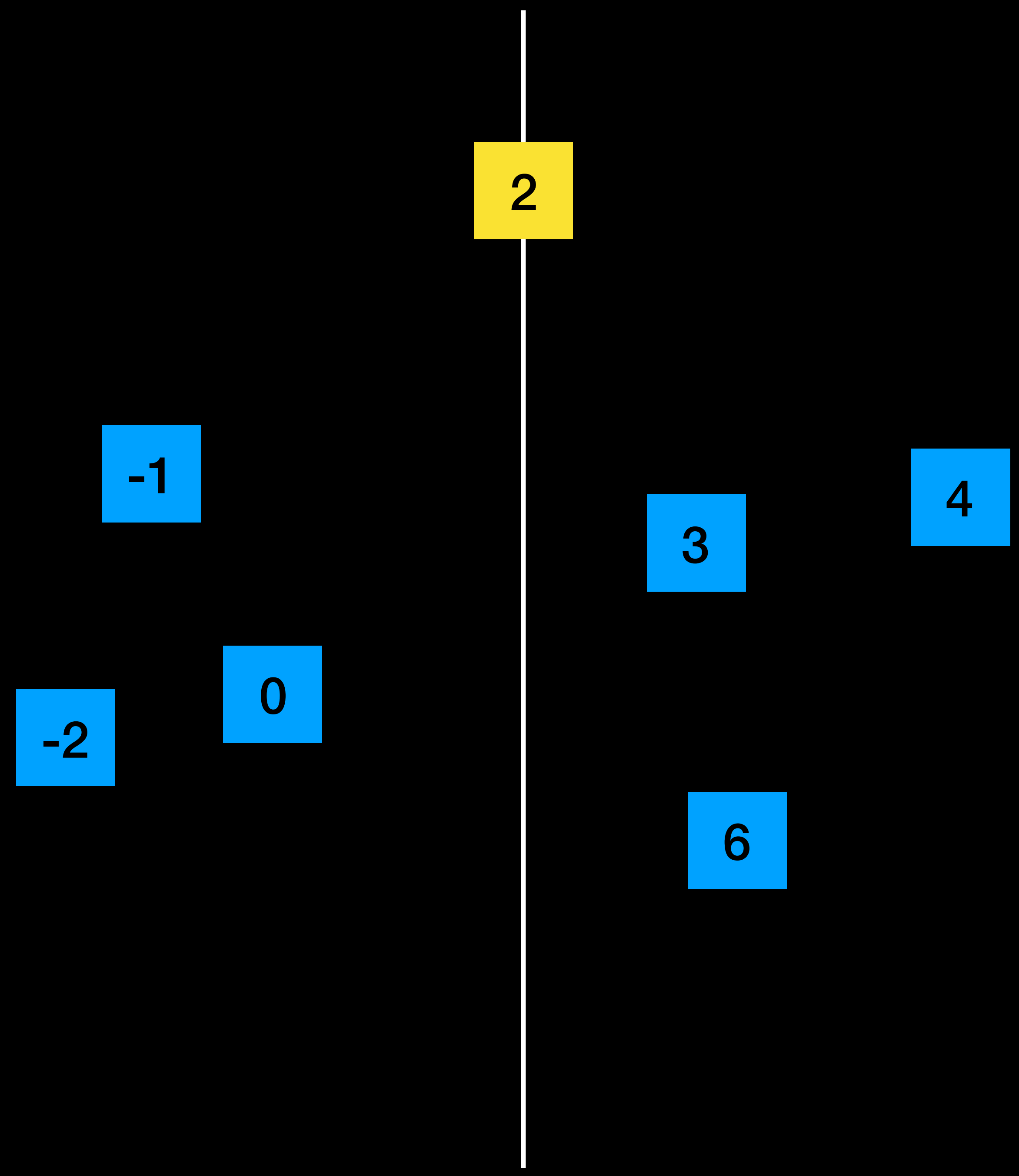
A Different Approach



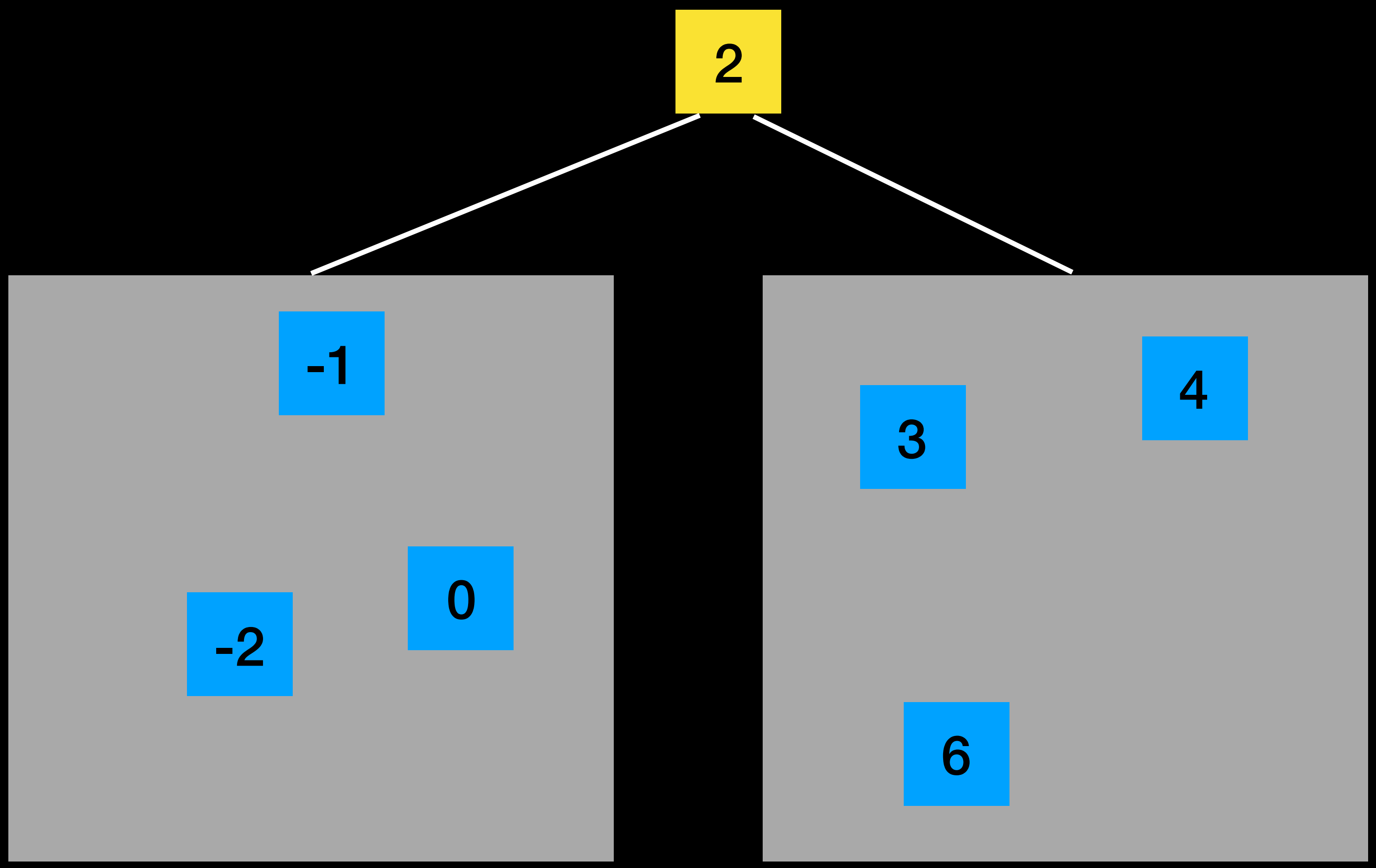
A Different Approach



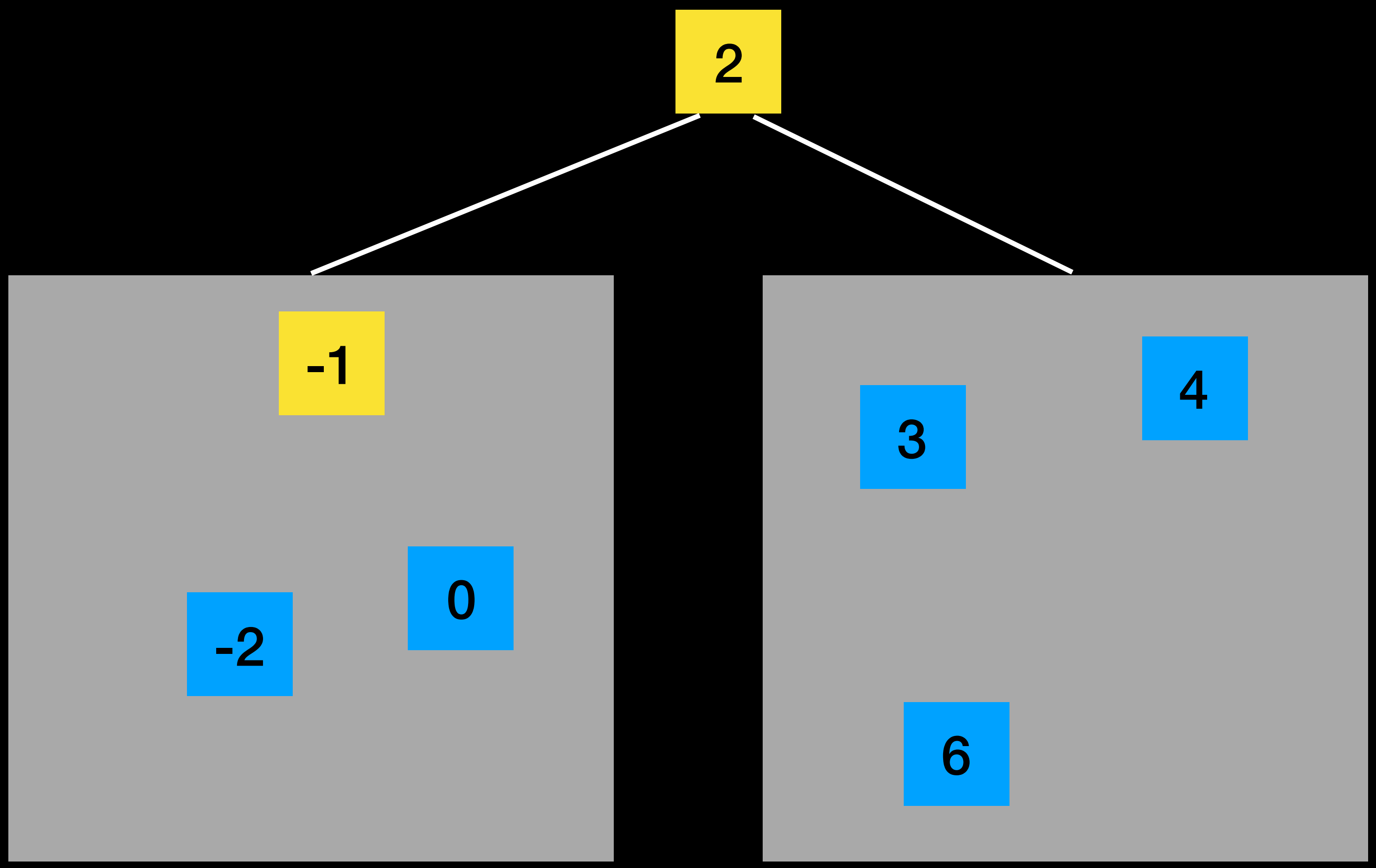
A Different Approach



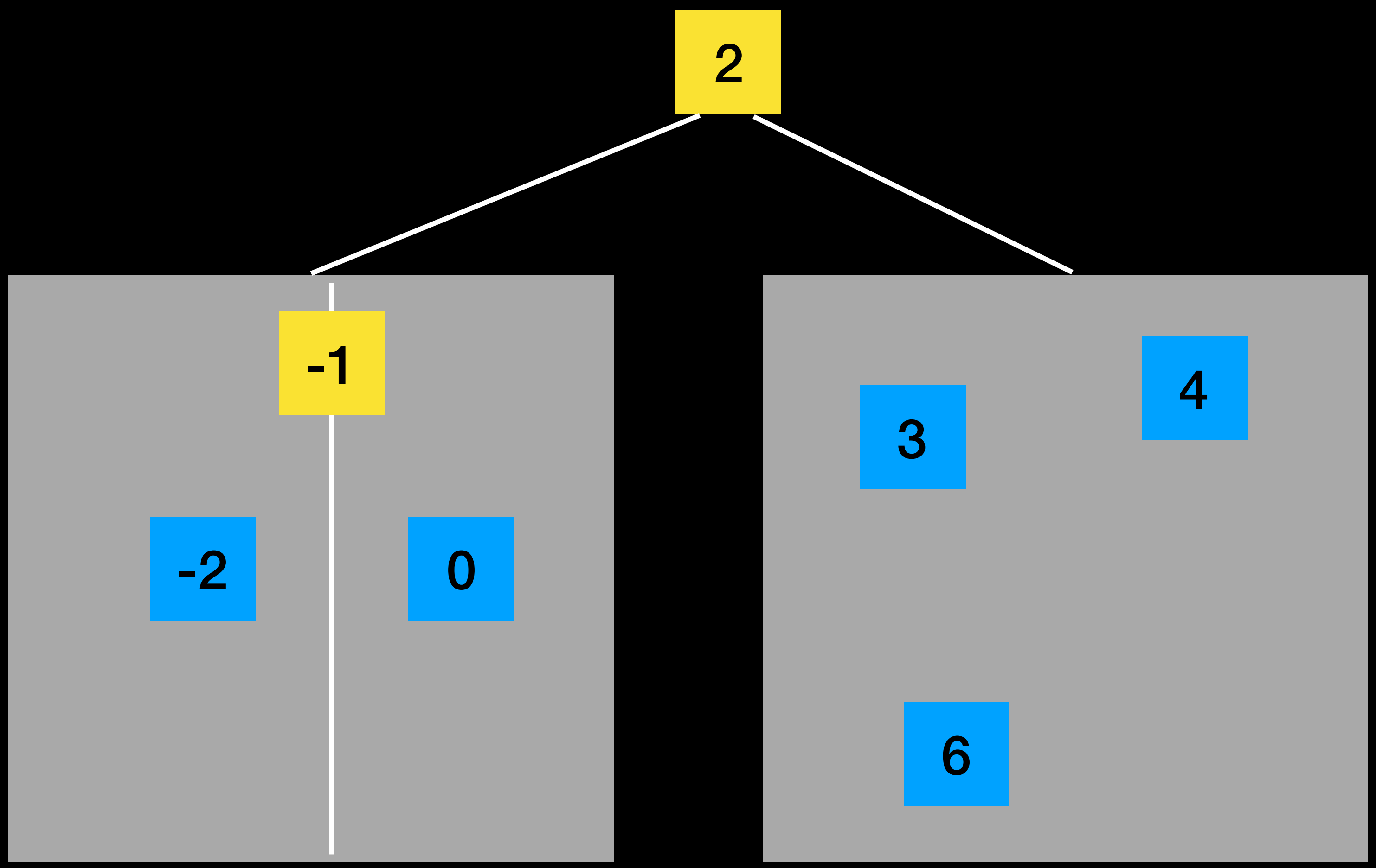
A Different Approach



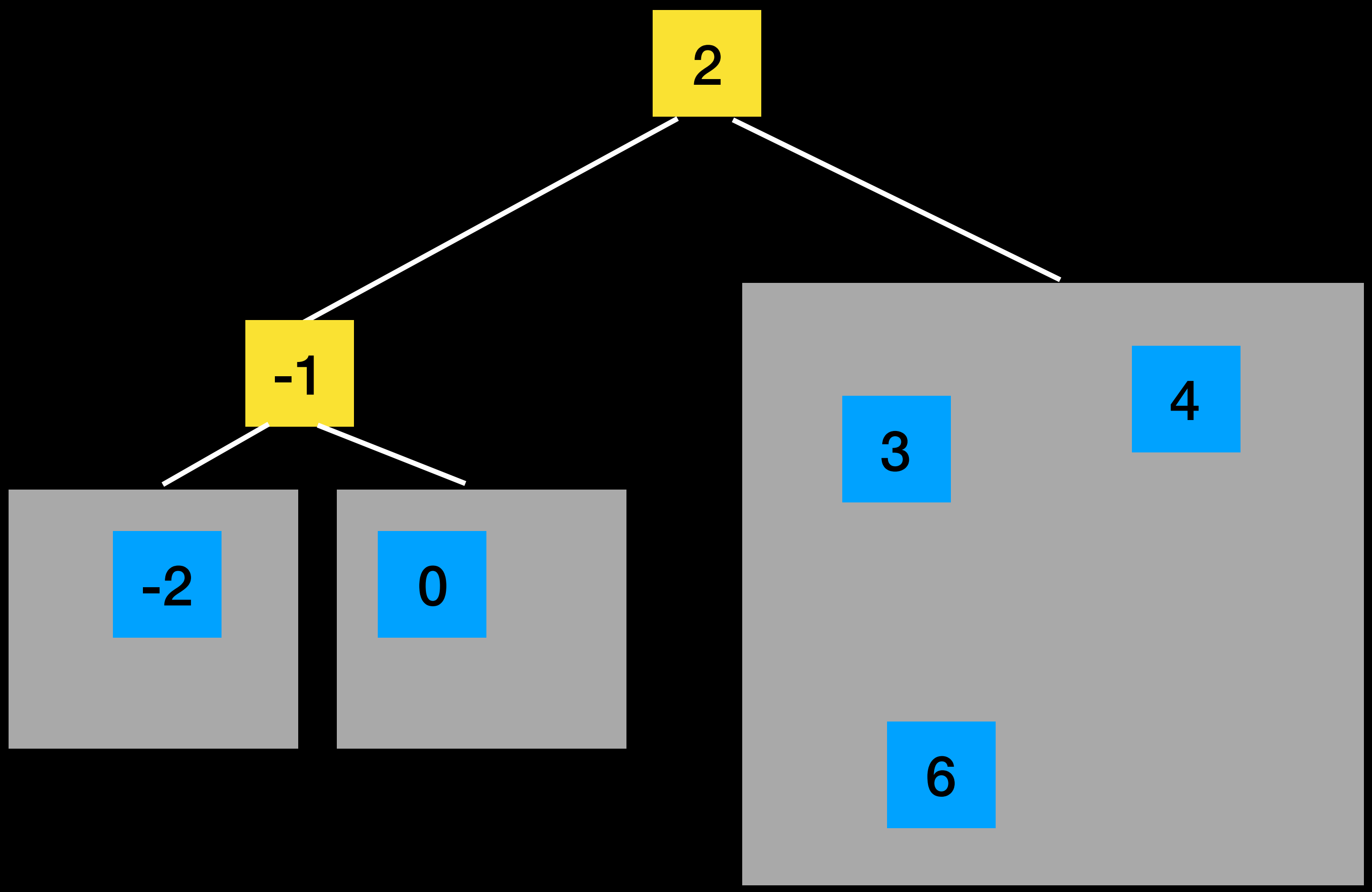
A Different Approach



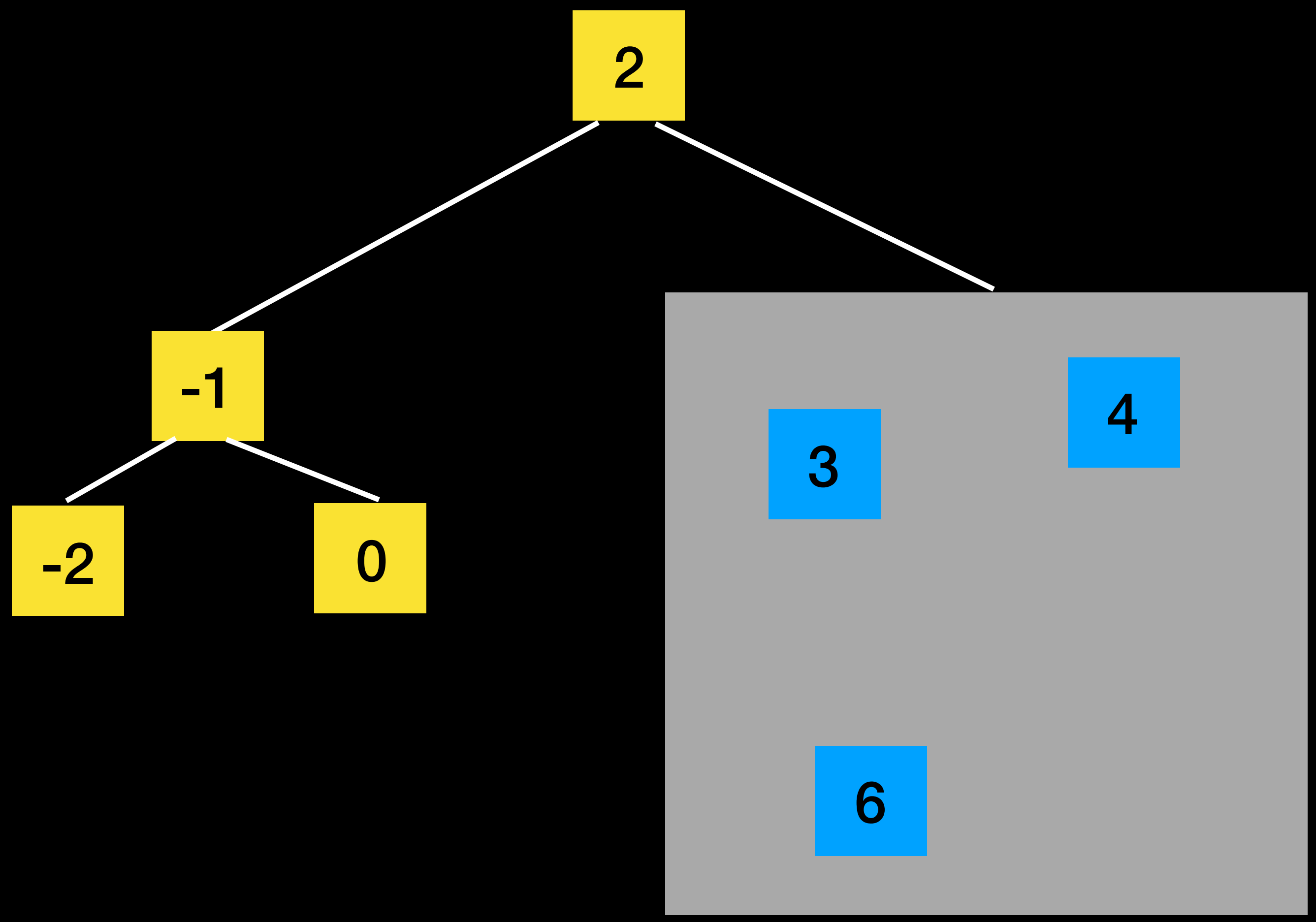
A Different Approach



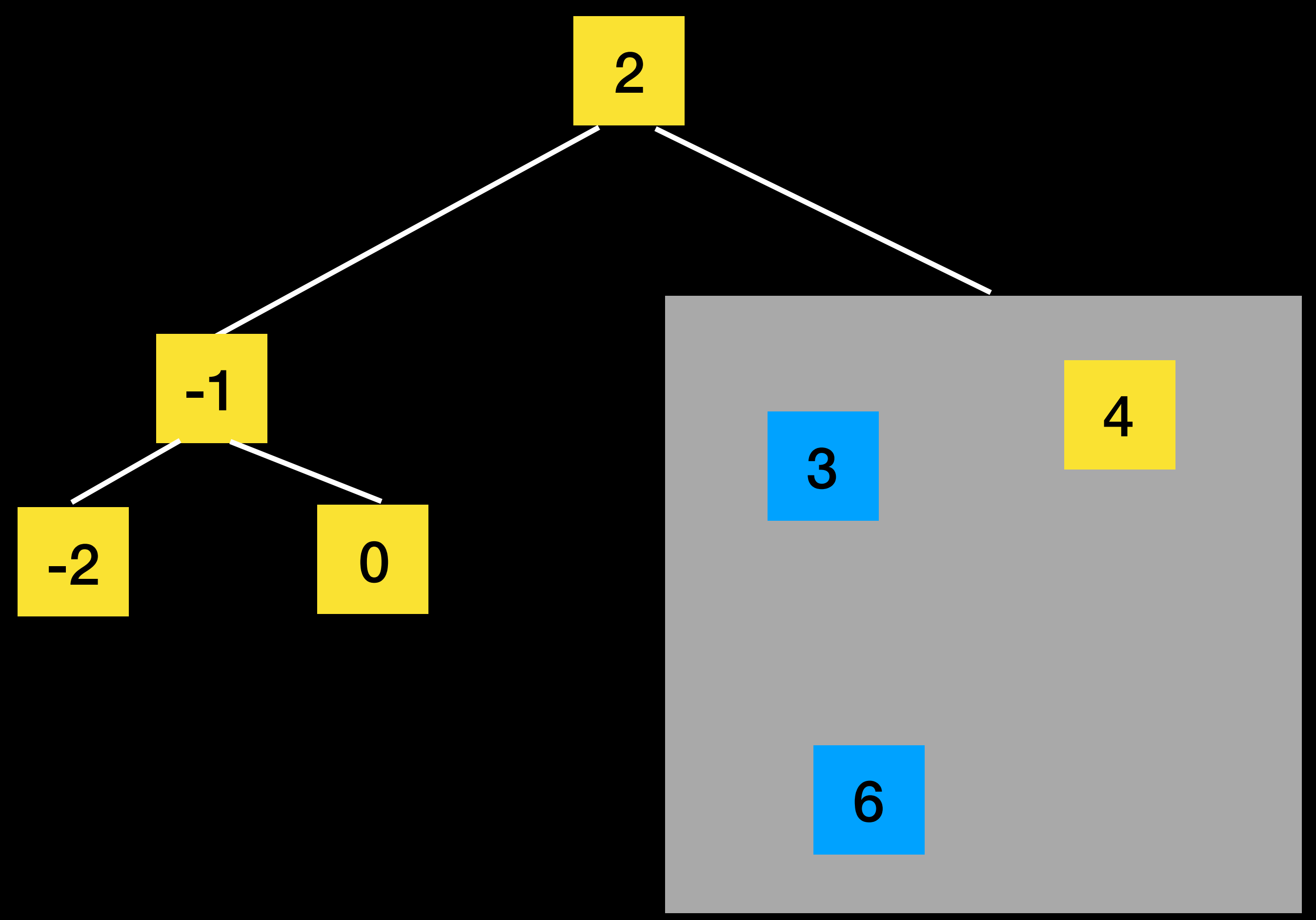
A Different Approach



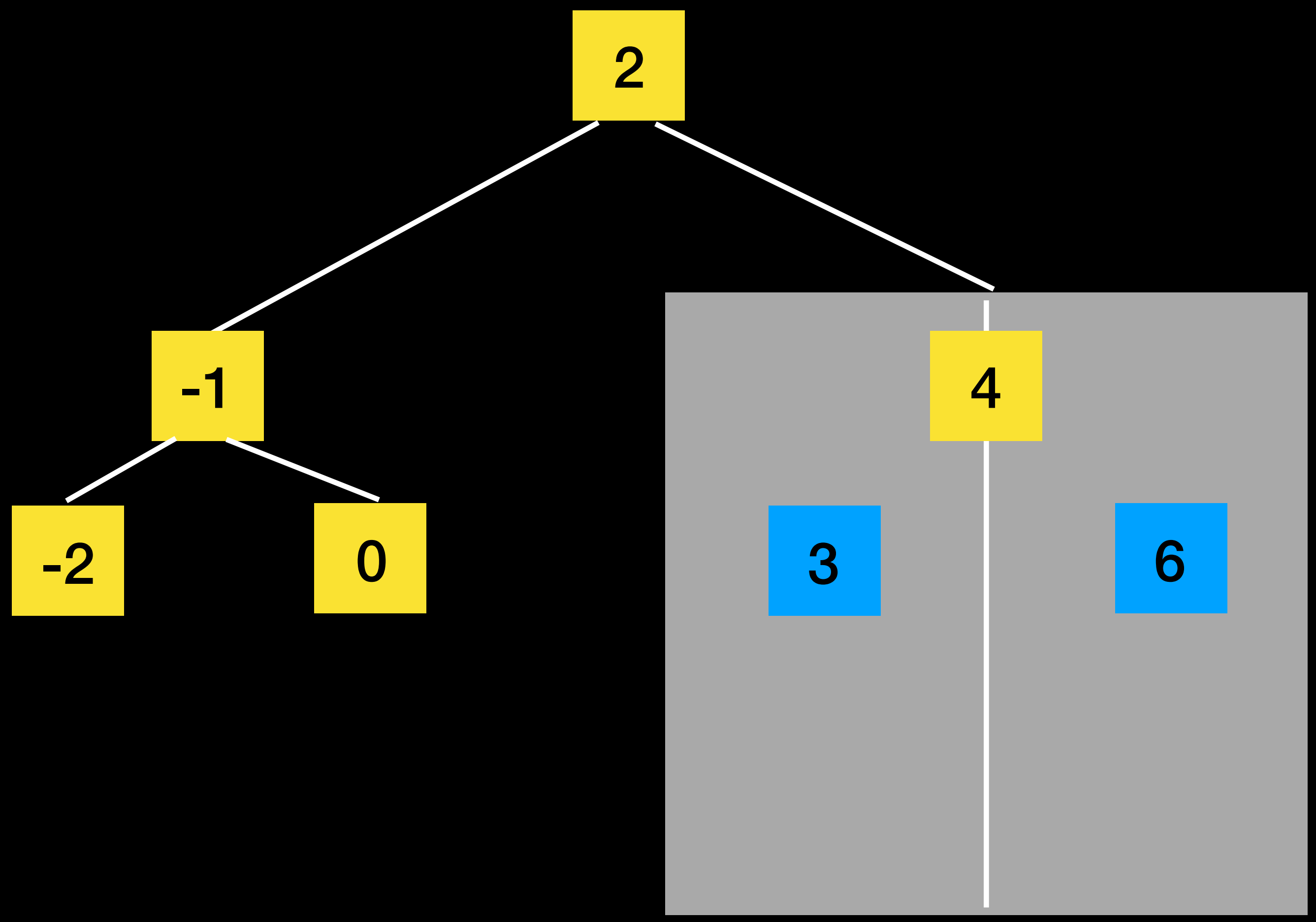
A Different Approach



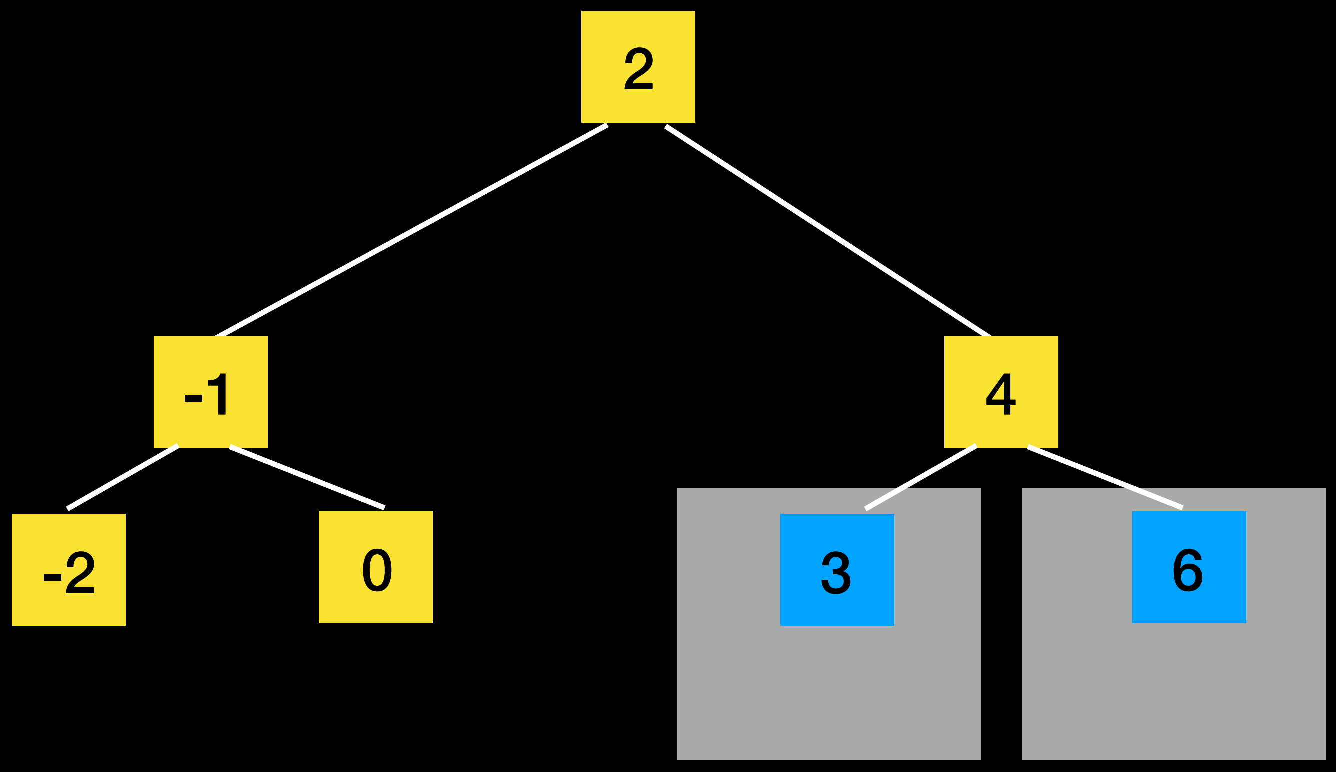
A Different Approach



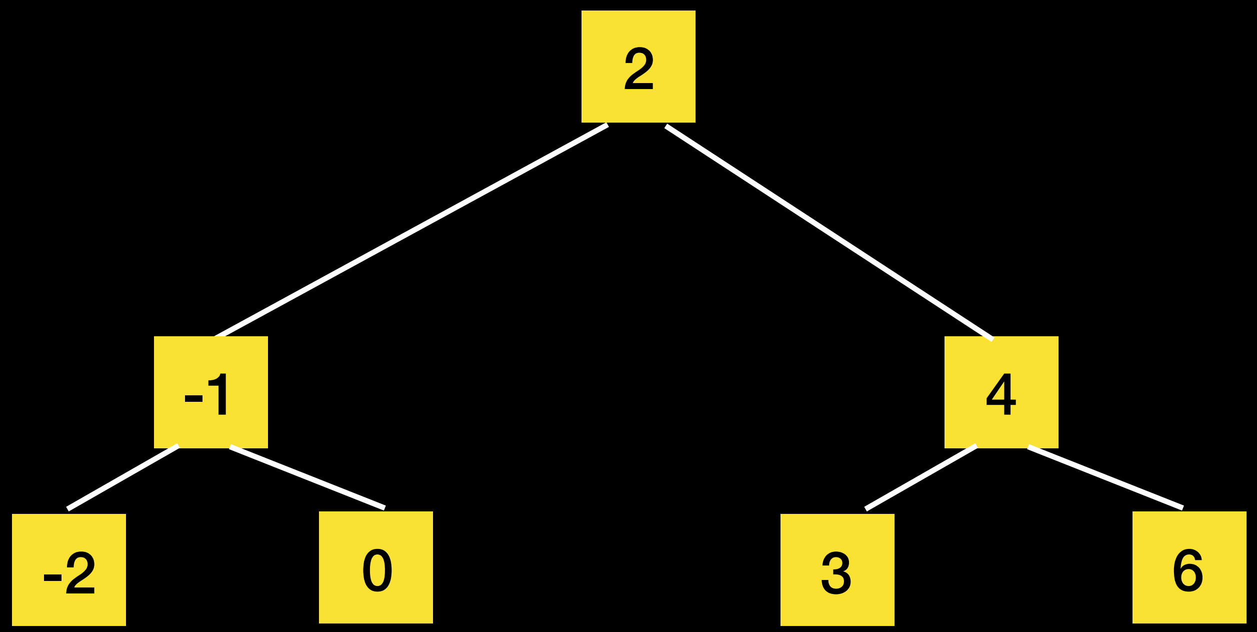
A Different Approach



A Different Approach

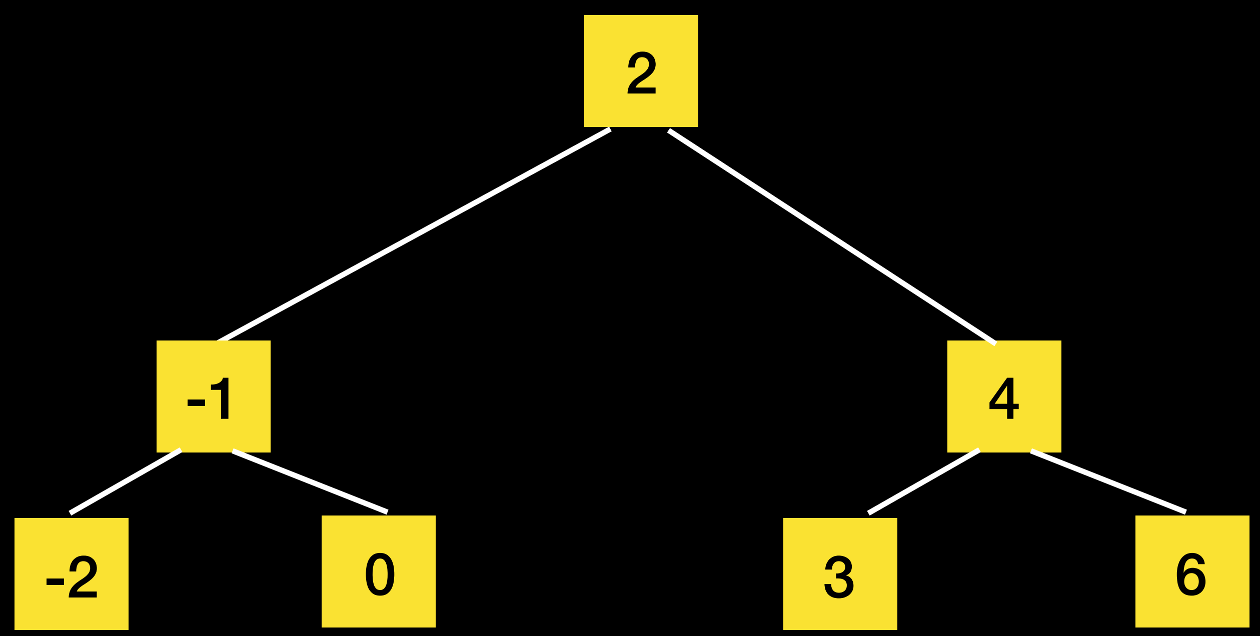


A Different Approach



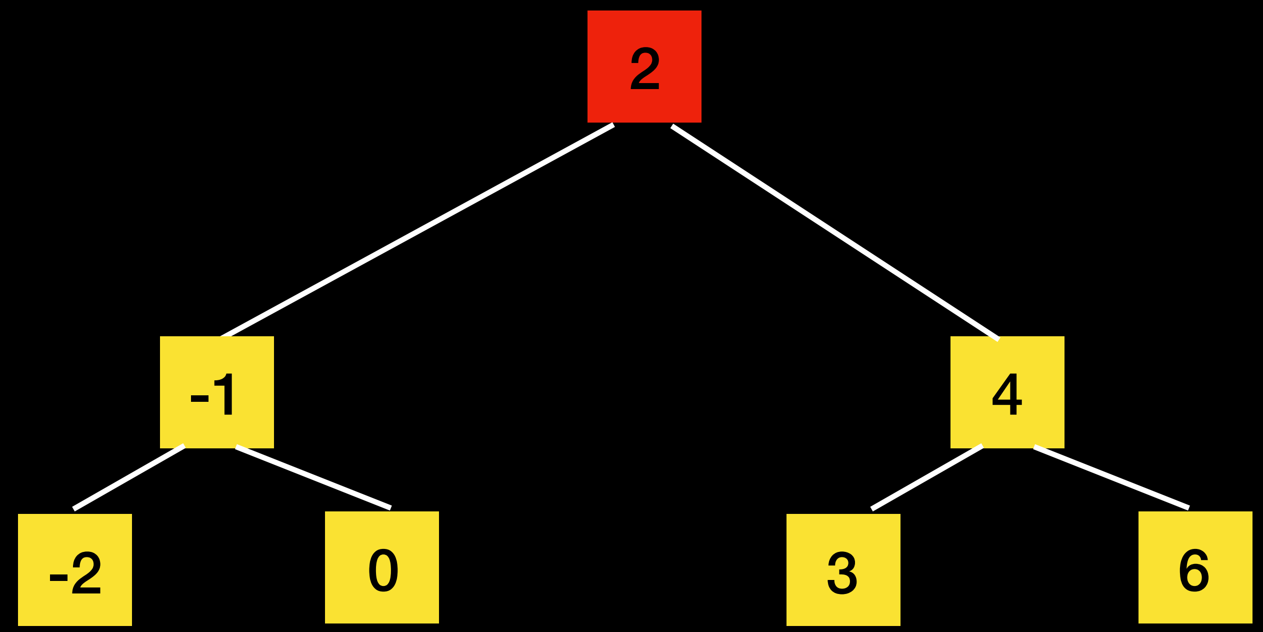
A Different Approach

Find 5



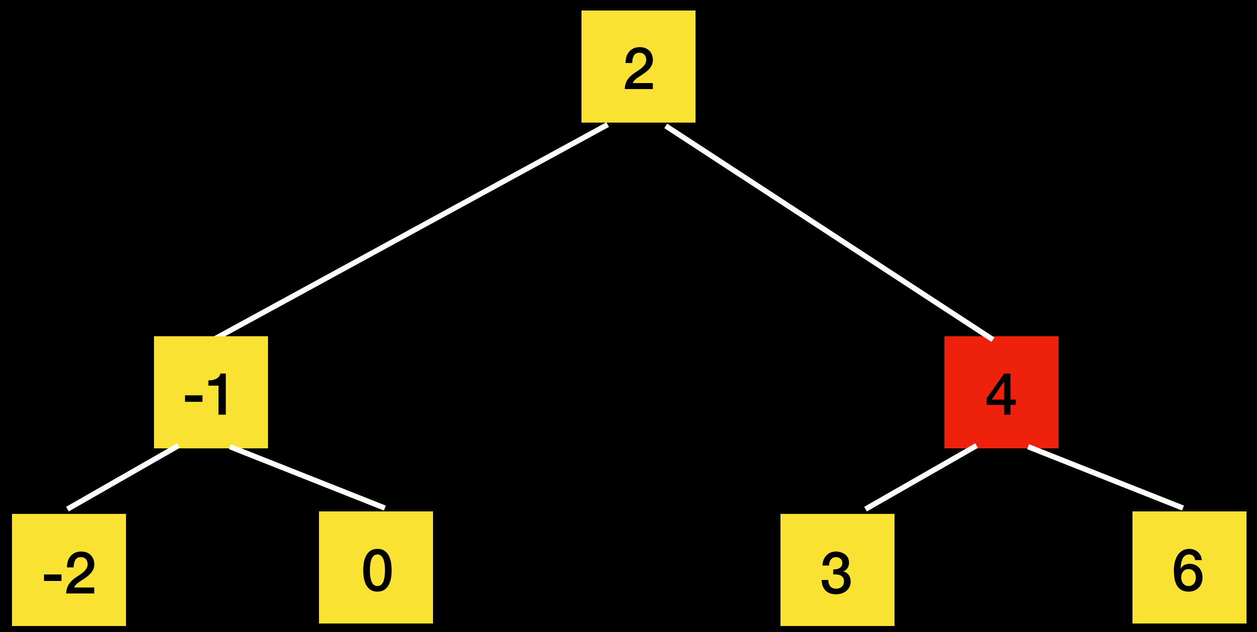
A Different Approach

Find 5



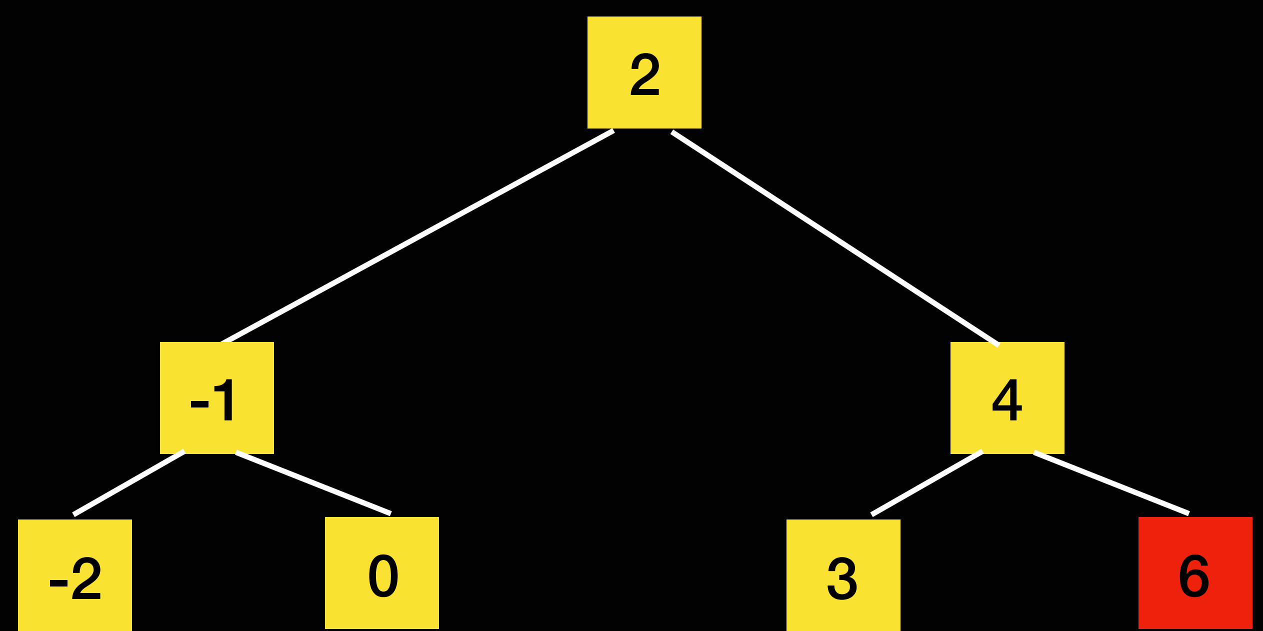
A Different Approach

Find 5



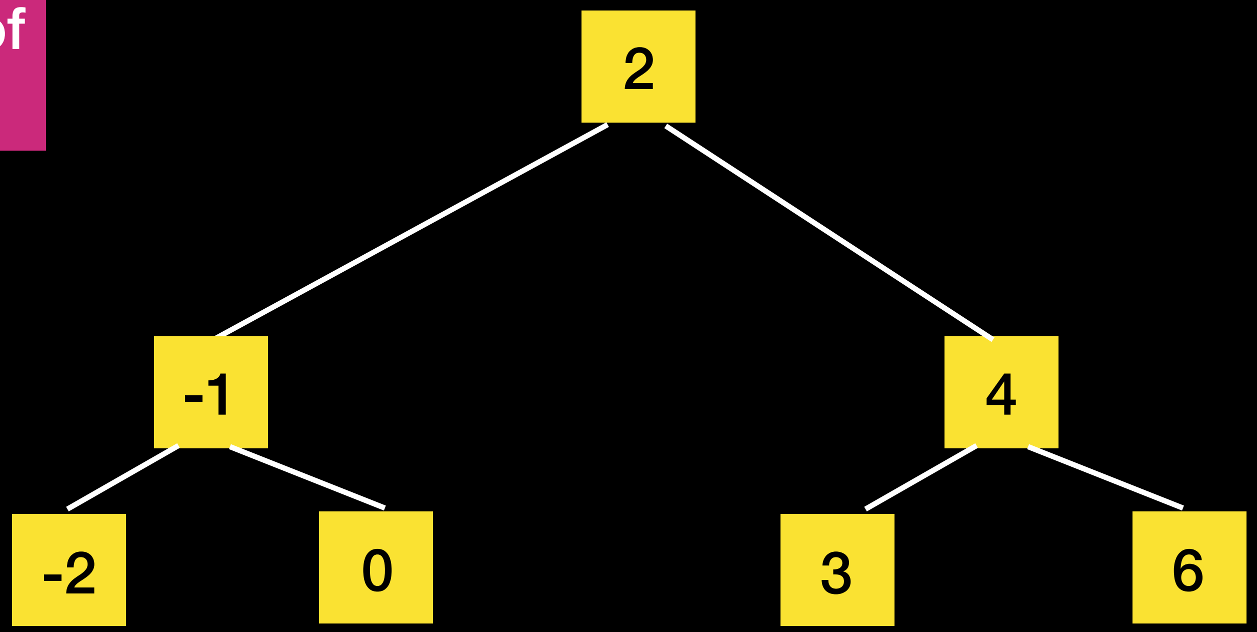
A Different Approach

Find 5



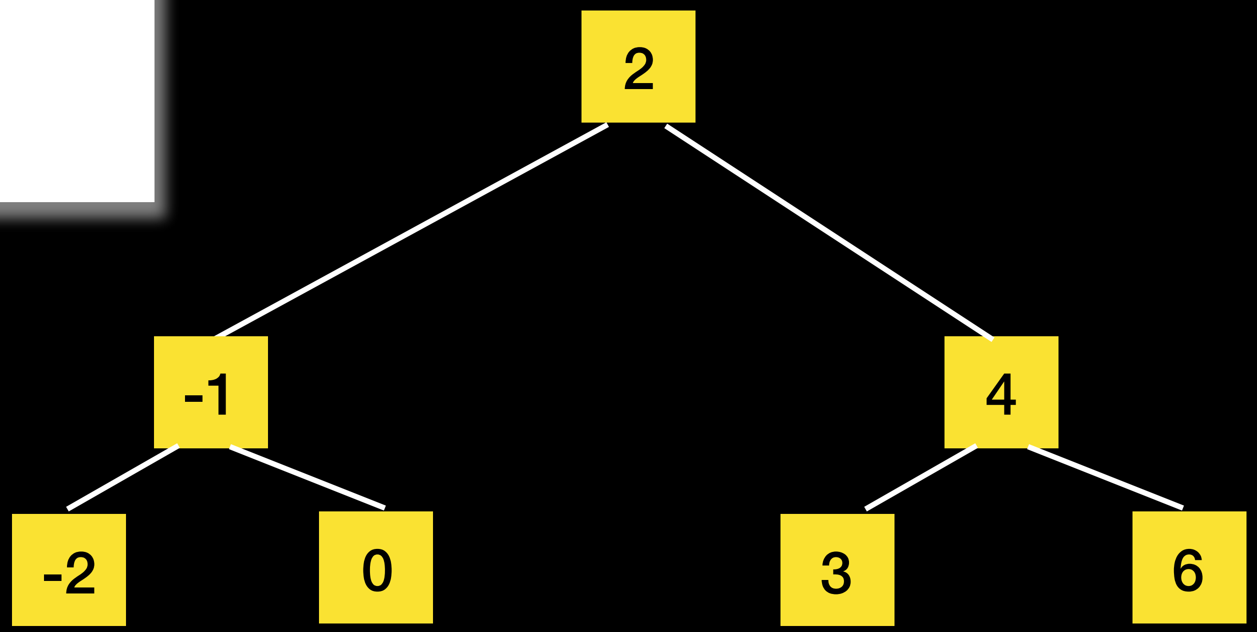
A Different Approach

What's special about the shape of this tree?



Binary Search Tree

Structural Property:
For each node n
n > all values in T_L
n < all values in T_R



BST Formally

Let S be a set of values upon which a total ordering relation $<$, is defined. For example, S can be a set of numbers.
A **binary search tree** (**BST**) **T** for the ordered set (S,$<$) is a binary tree with the following properties:

- Each node of T has a value. If p and q are nodes, then we write $p < q$ to mean that the value of p is less than the value of q.

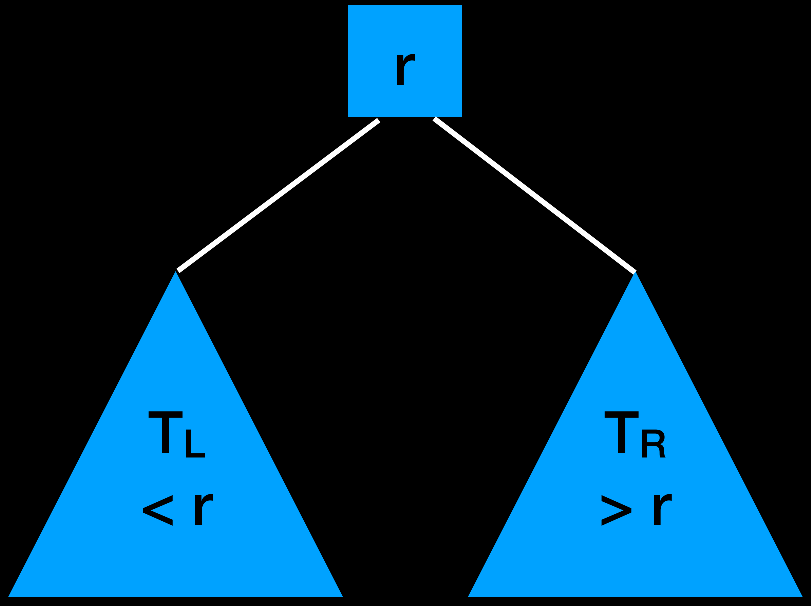
- For each node $n \in T$, if p is a node in the left subtree of n, then $p < n$.

- For each node $n \in T$, if p is a node in the right subtree of n, then $n < p$.

- For each element $s \in S$ there exists a node $n \in T$ such that $s = n$.

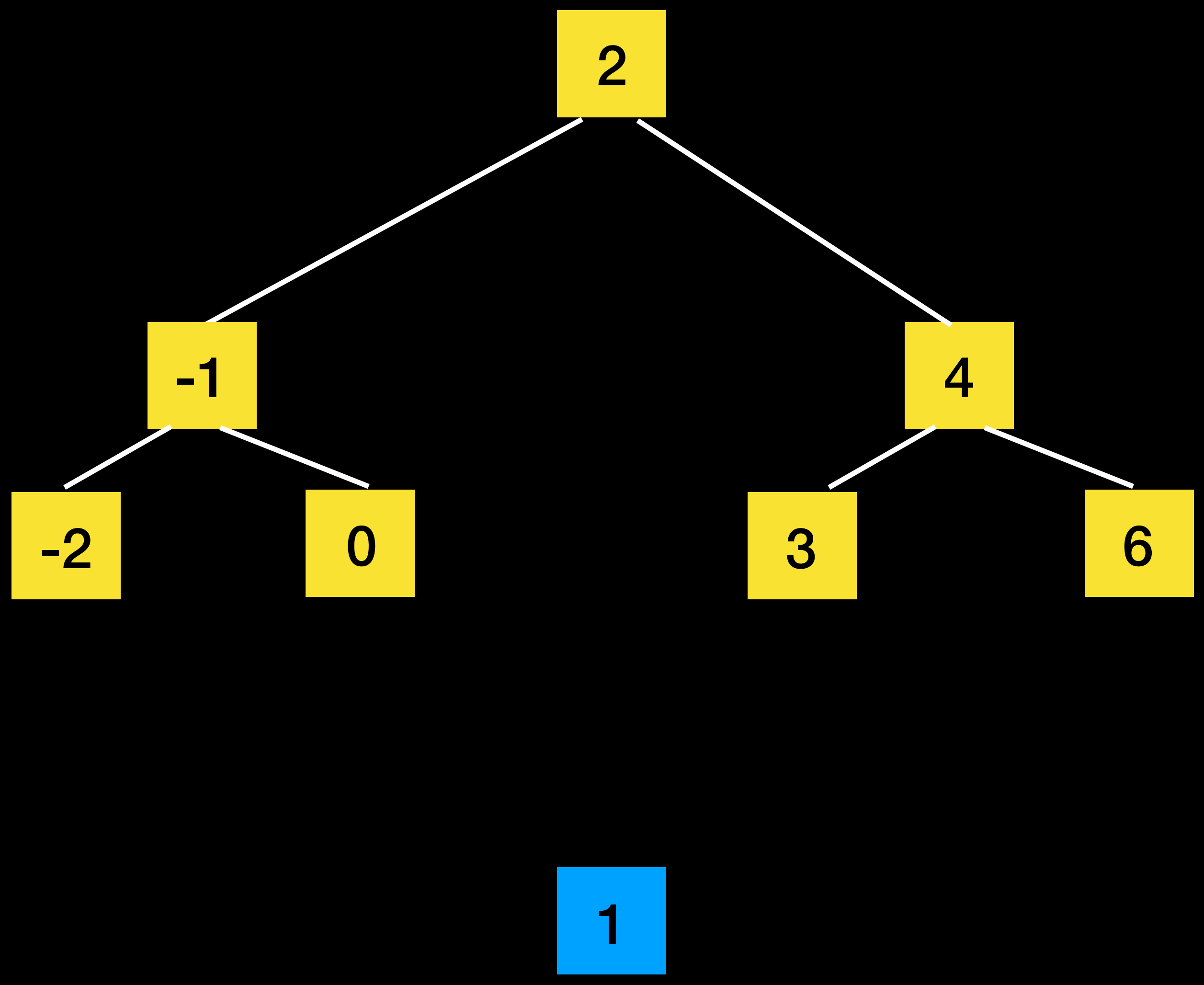
Binary Search Tree

Structural Property:
For each node n
n > all values in T_L
n < all values in T_R

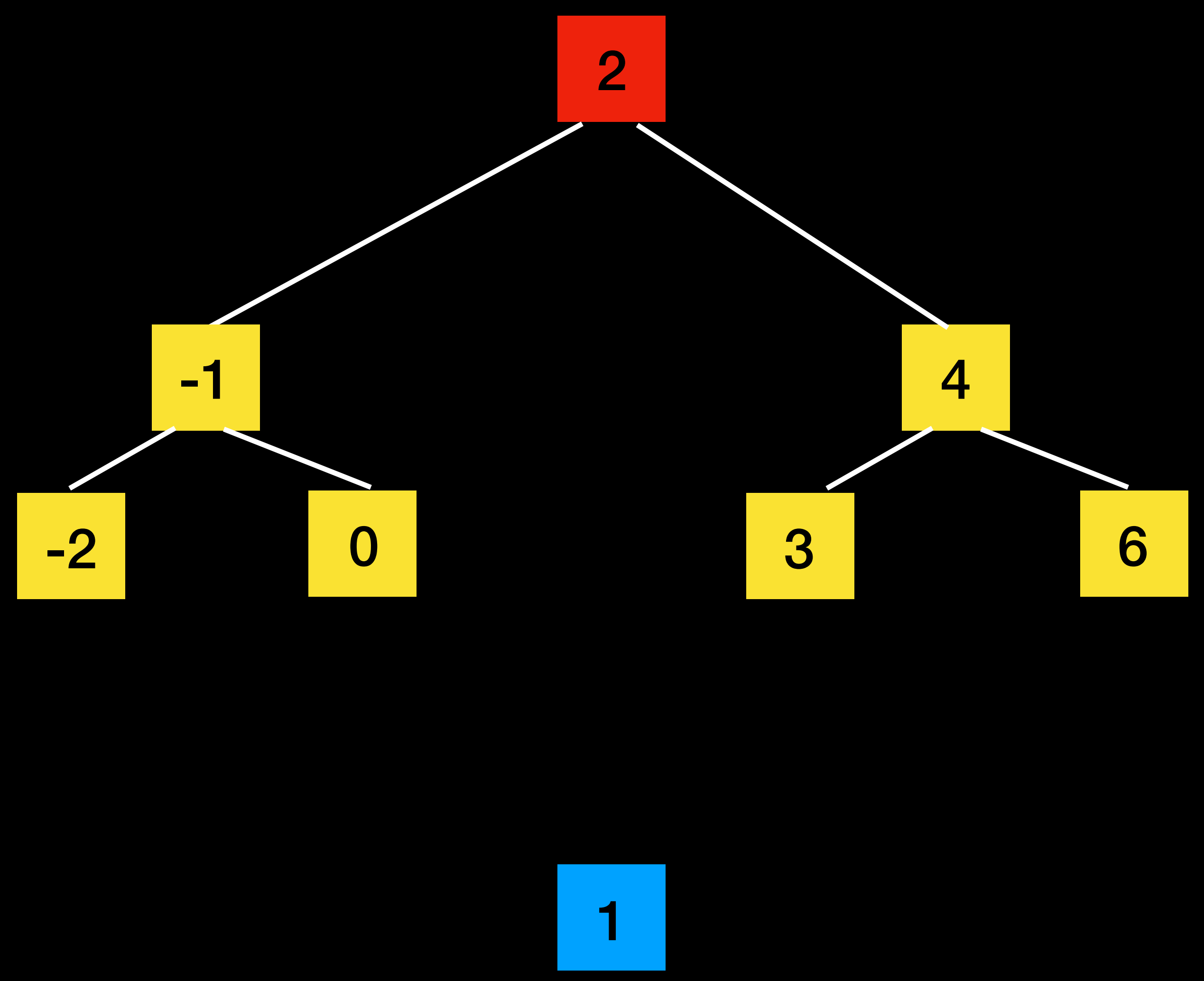



```
search(bs_tree, item)
{
    if (bs_tree is empty) //base case
        item not found
    else if (item == root)
        return root
    else if (item < root)
        search(TL , item)
    else // item > root
        search(TR , item)
}
```

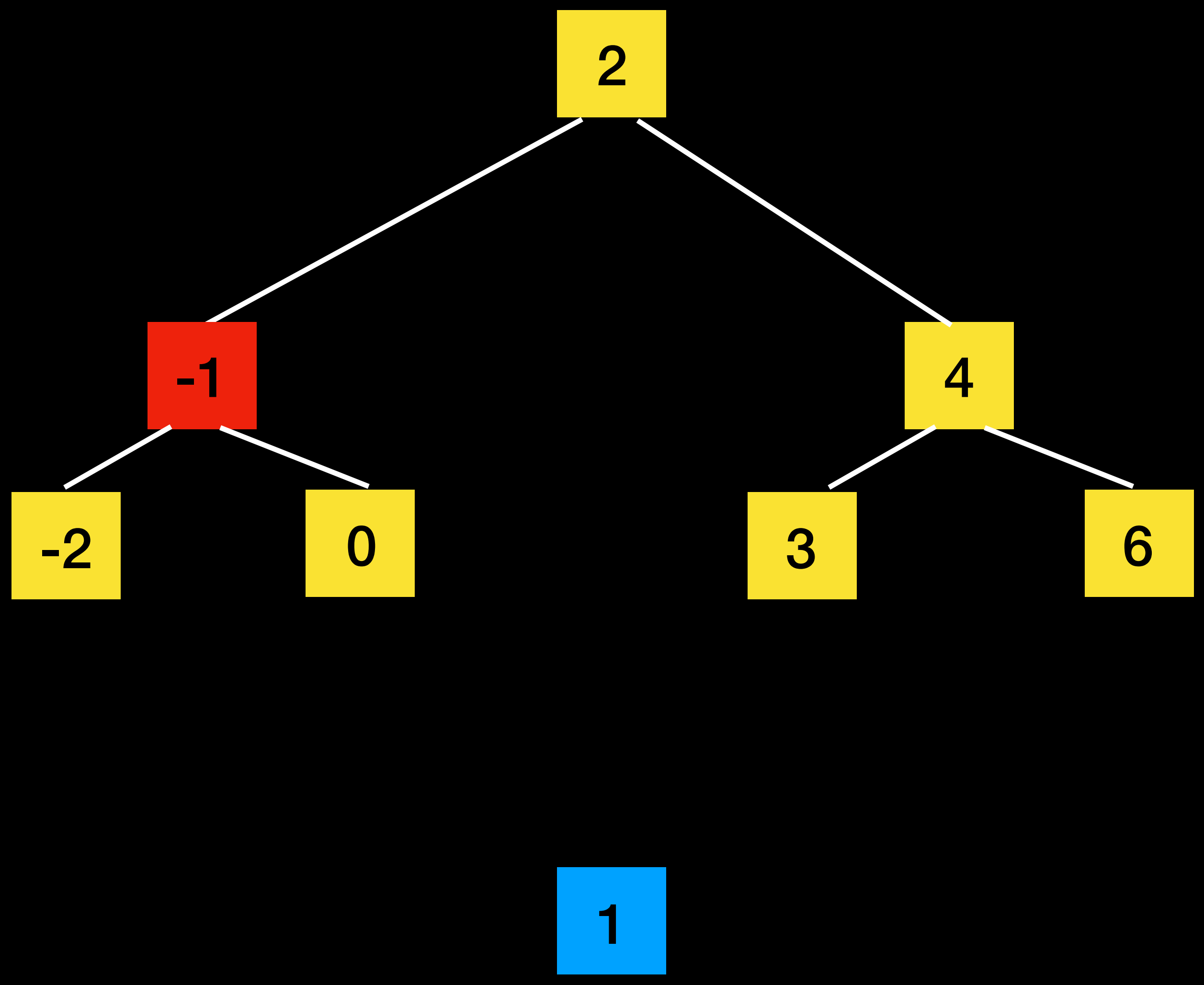
Inserting into a BST



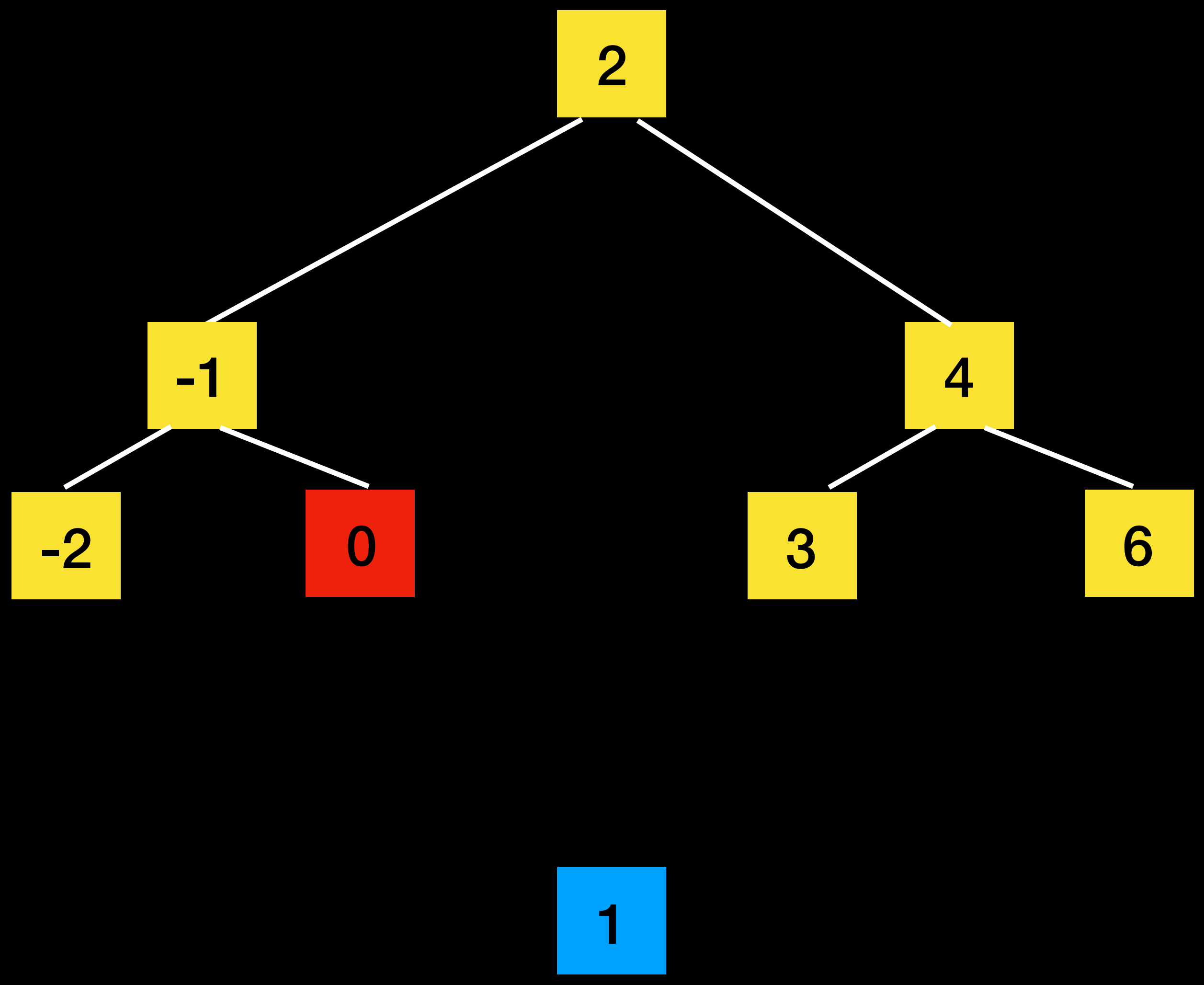
Inserting into a BST



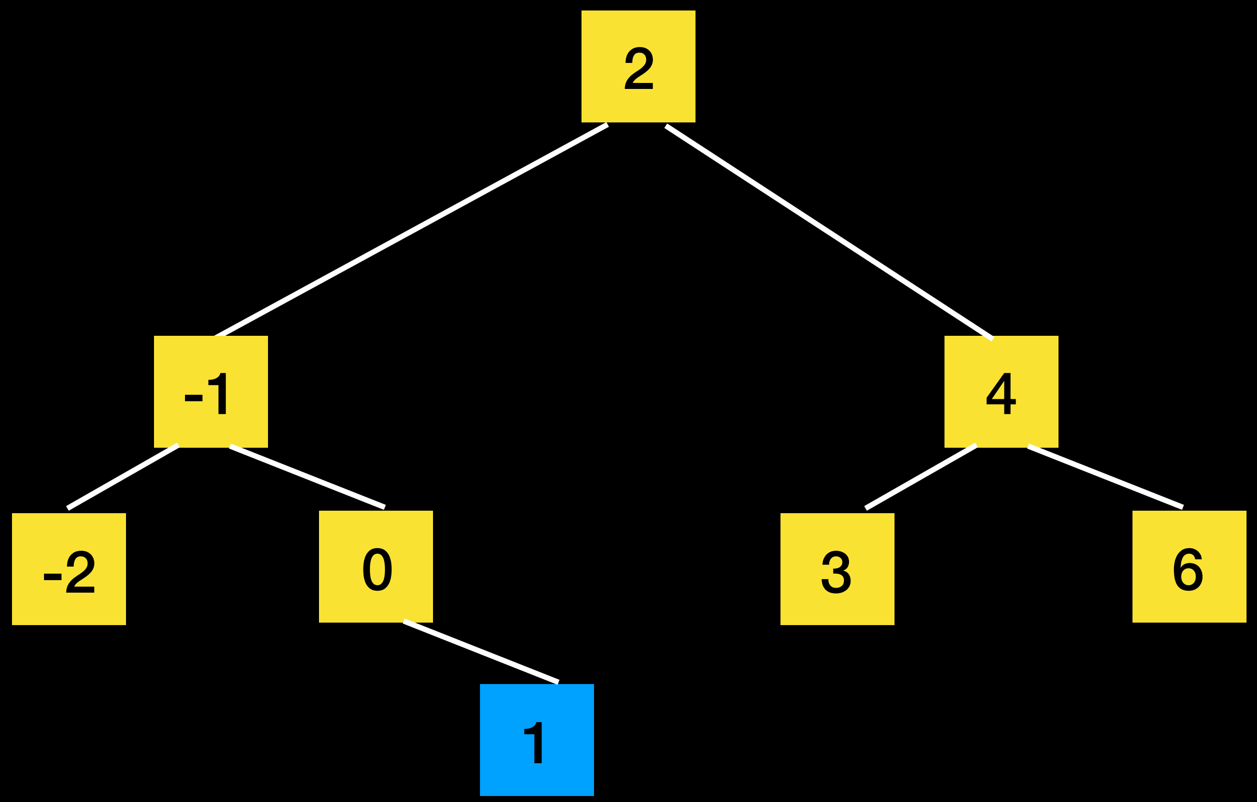
Inserting into a BST



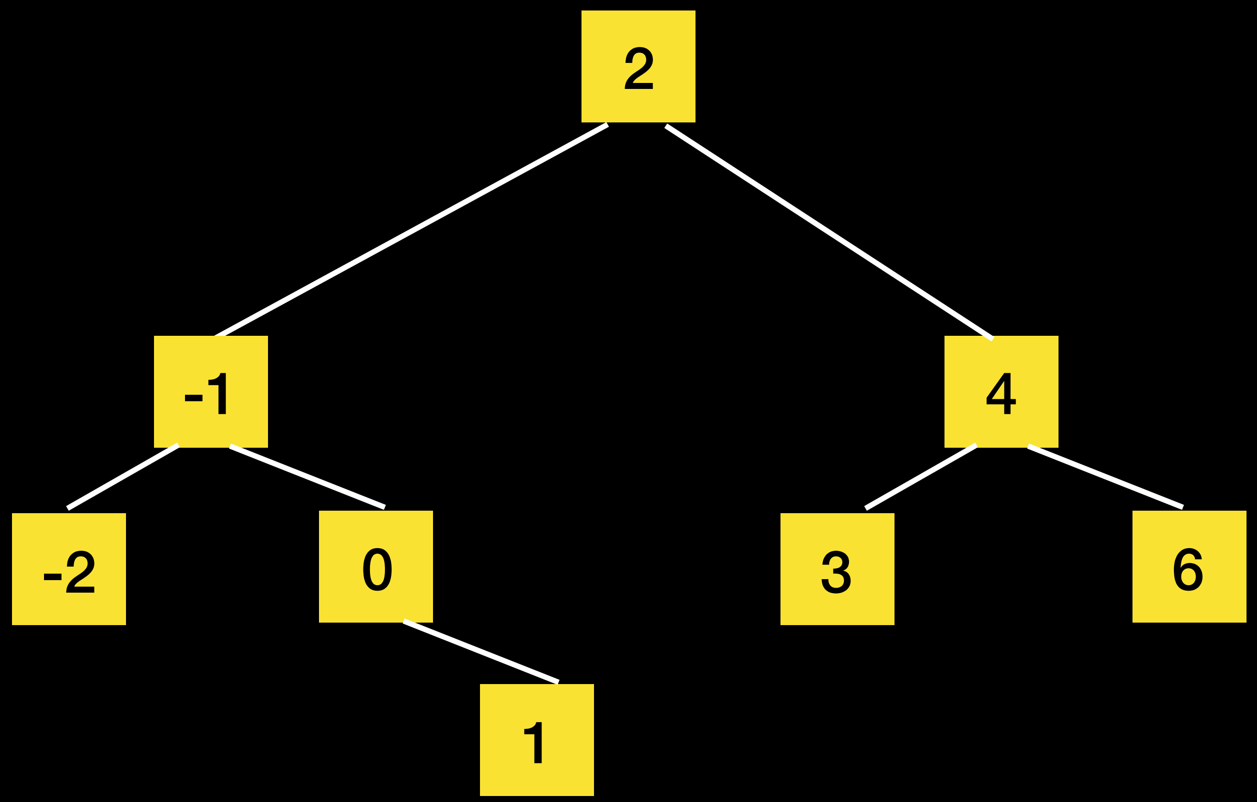
Inserting into a BST



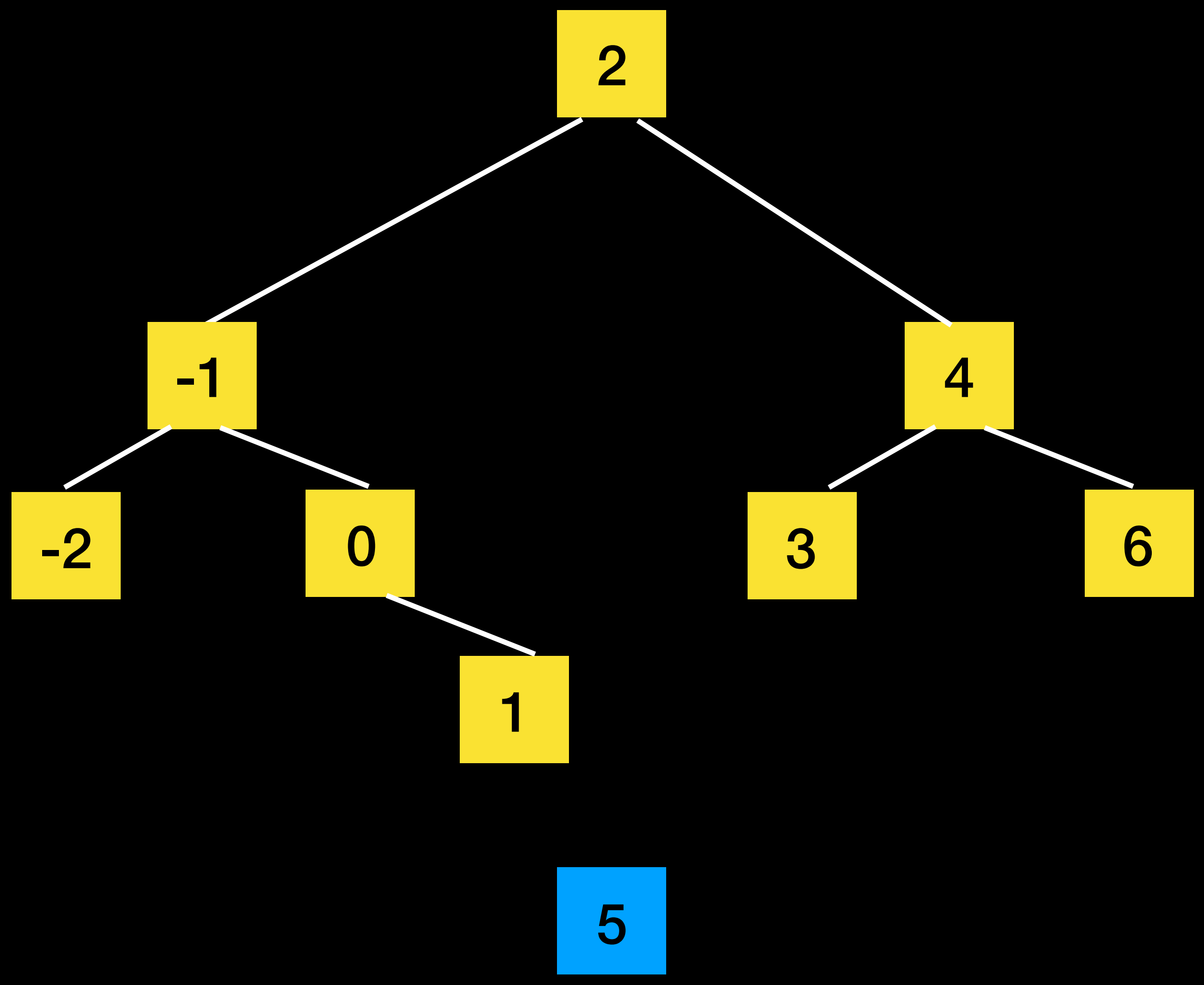
Inserting into a BST



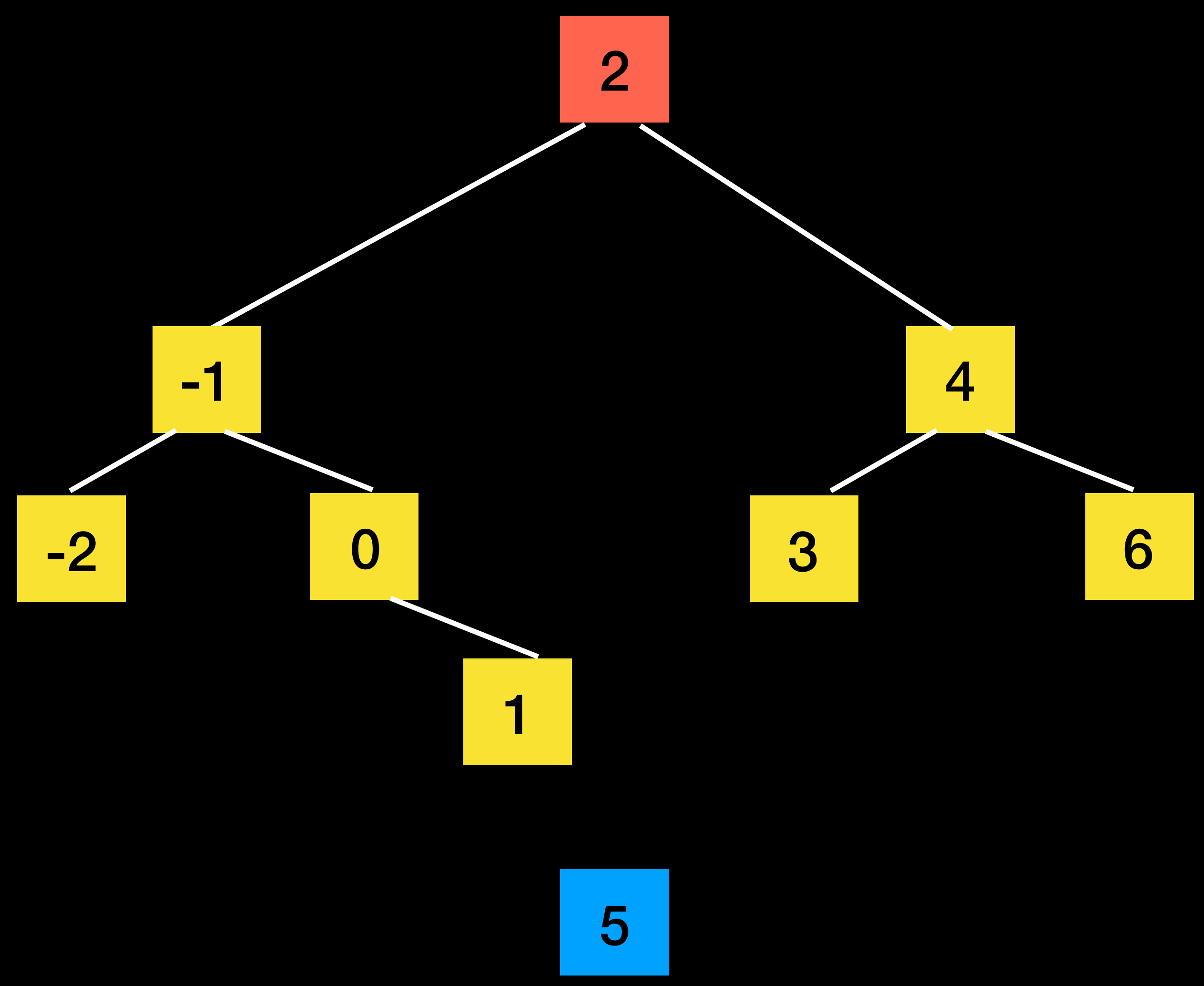
Inserting into a BST



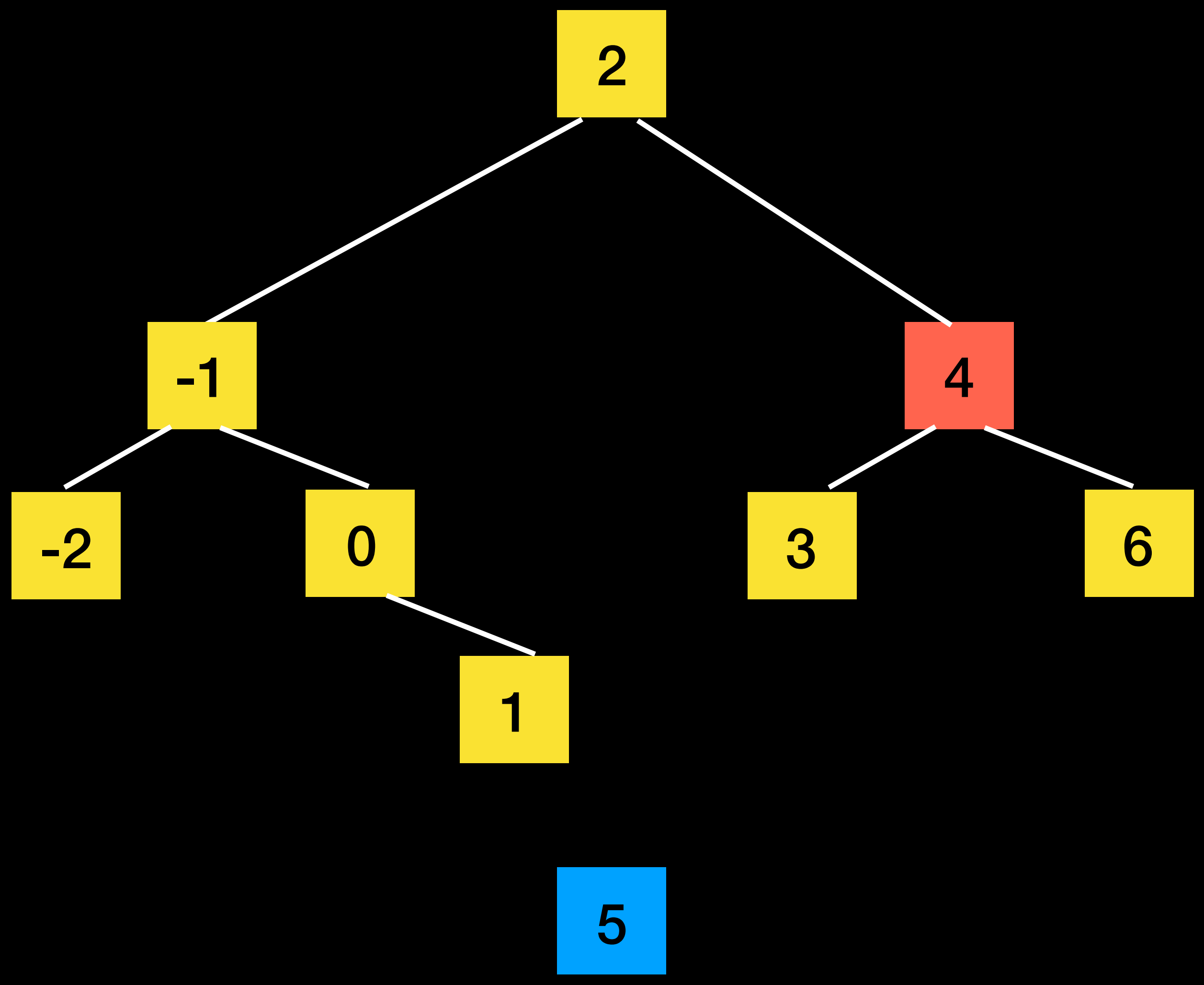
Inserting into a BST



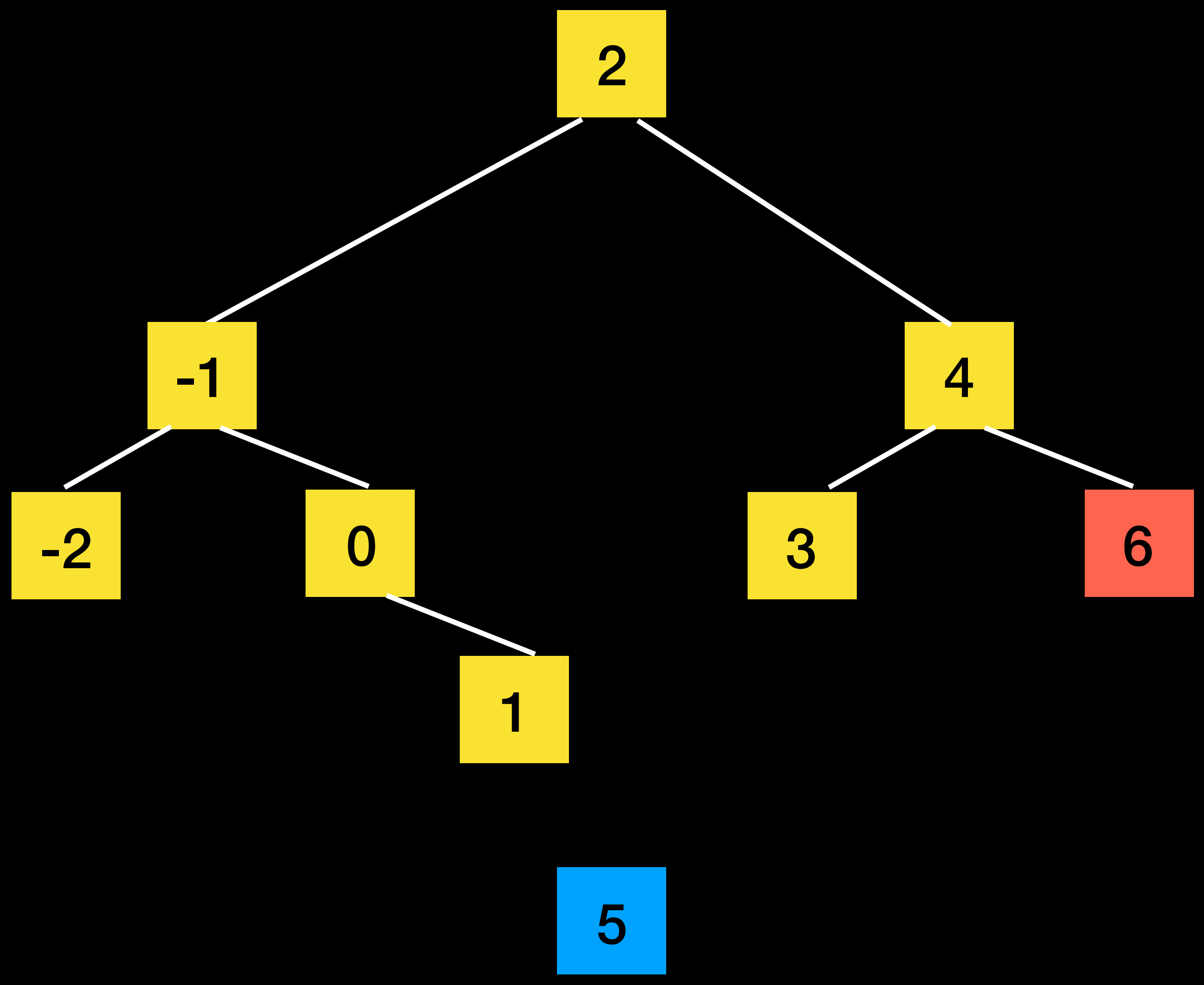
Inserting into a BST



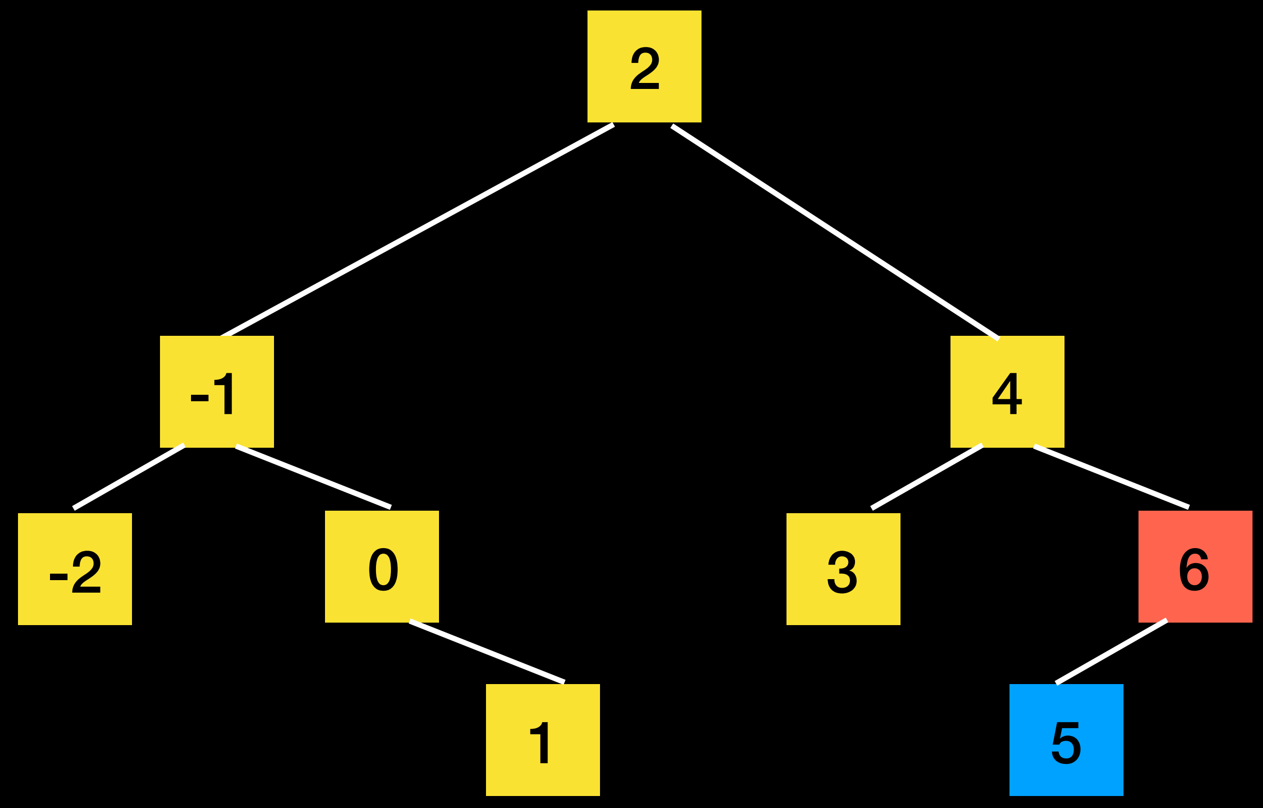
Inserting into a BST



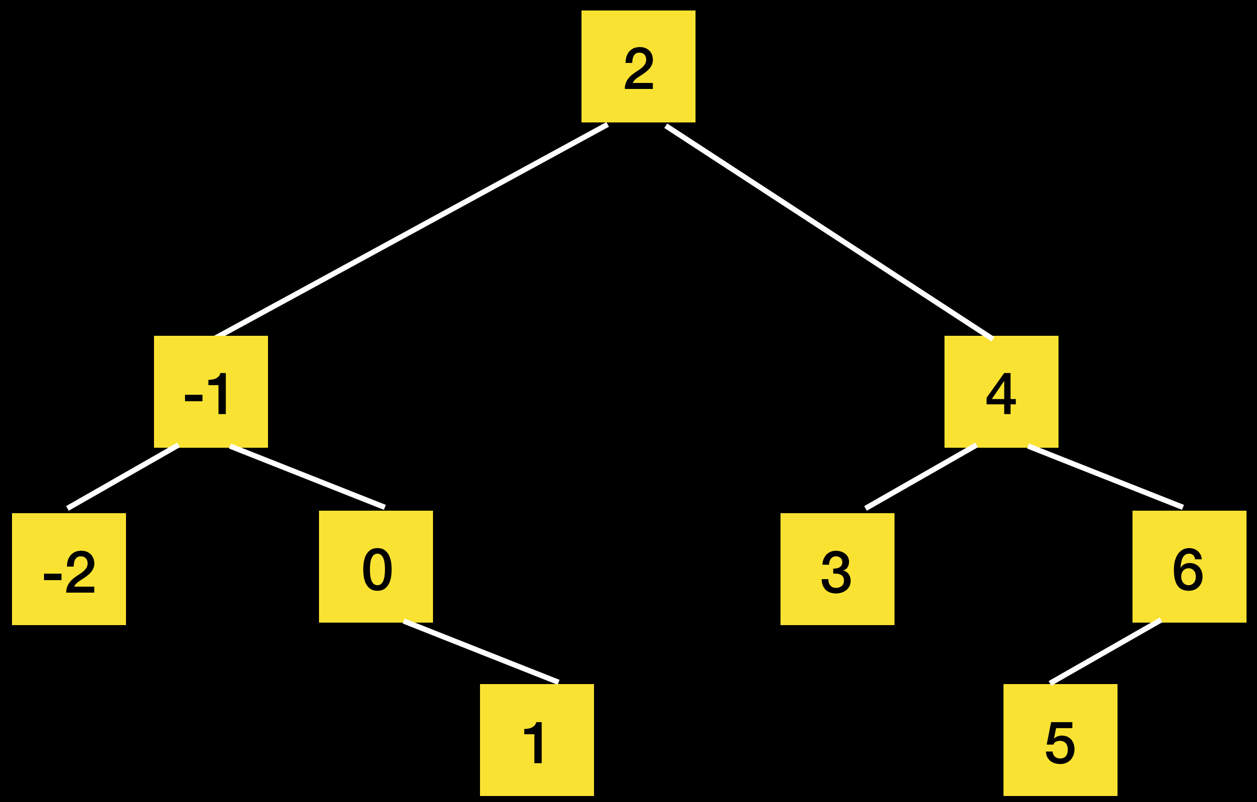
Inserting into a BST



Inserting into a BST

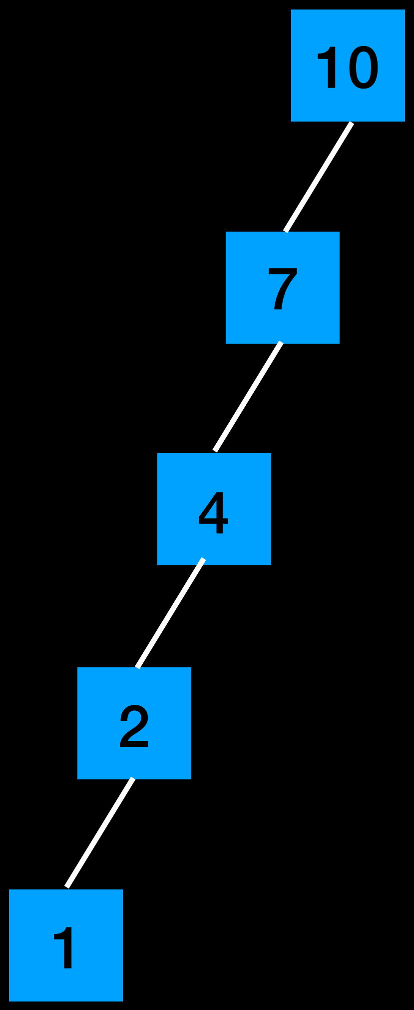


Inserting into a BST

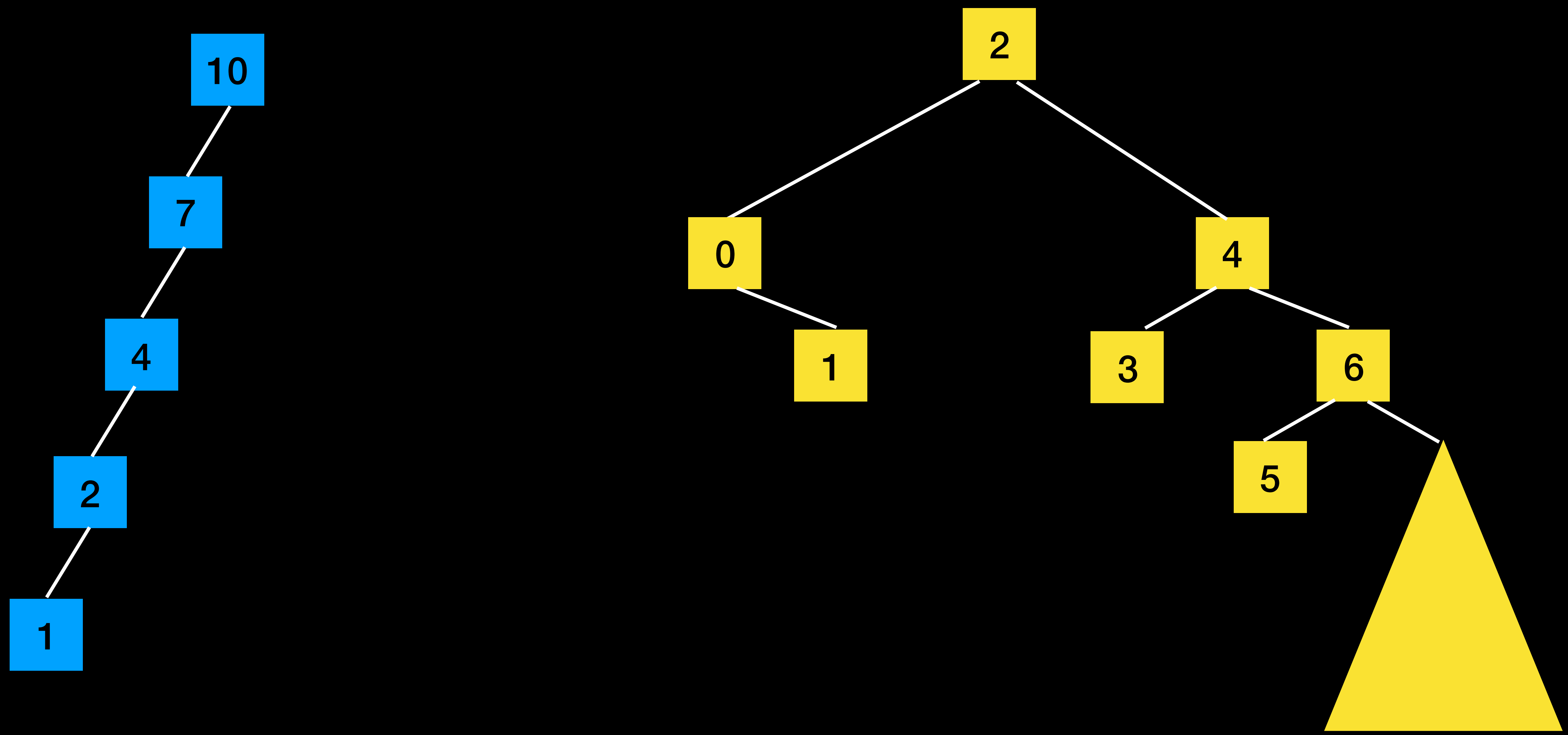


You **Grow** a tree with BST property, you **don't get to restructure it**
(Self-balancing BST and AVL trees will do that, perhaps in CSCI 335)

Growing a BST



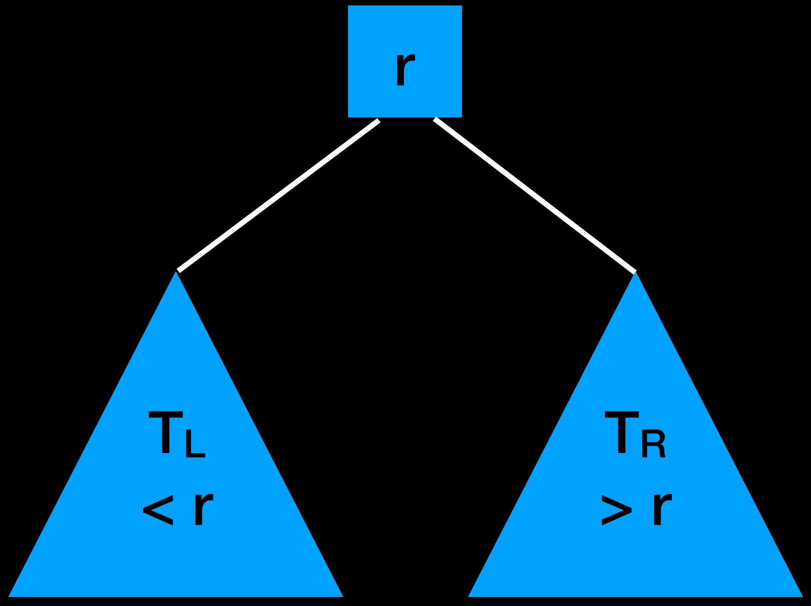
Growing a BST



Lecture Activity

Write **pseudocode** to insert an item into a BST

Inserting into a BST

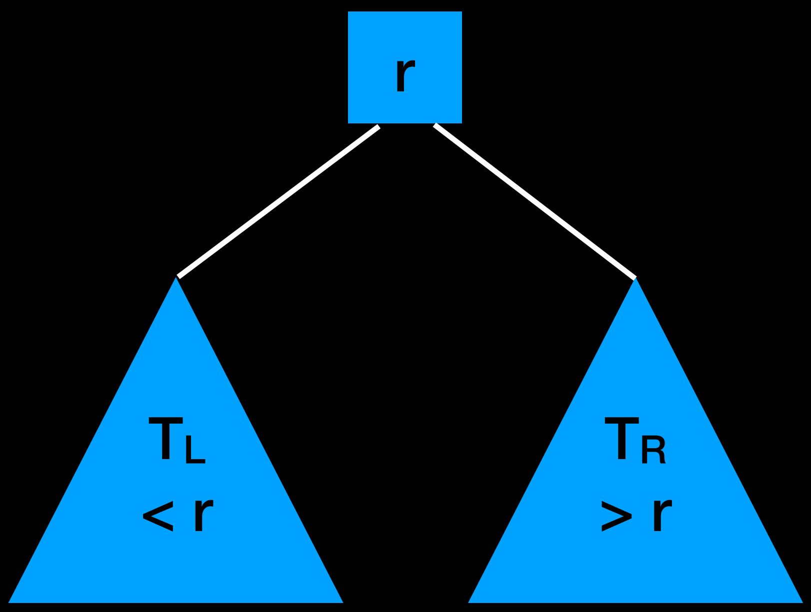



```
add(bs_tree, item)
{
    if (bs_tree is empty) //base case
        make item the root
    else if (item < root)
        add(TL , item)
    else // item > root
        add(TR , item)
}
```

Traversing a BST

Same as traversing any binary tree

Which type of traversal is special for a BST?

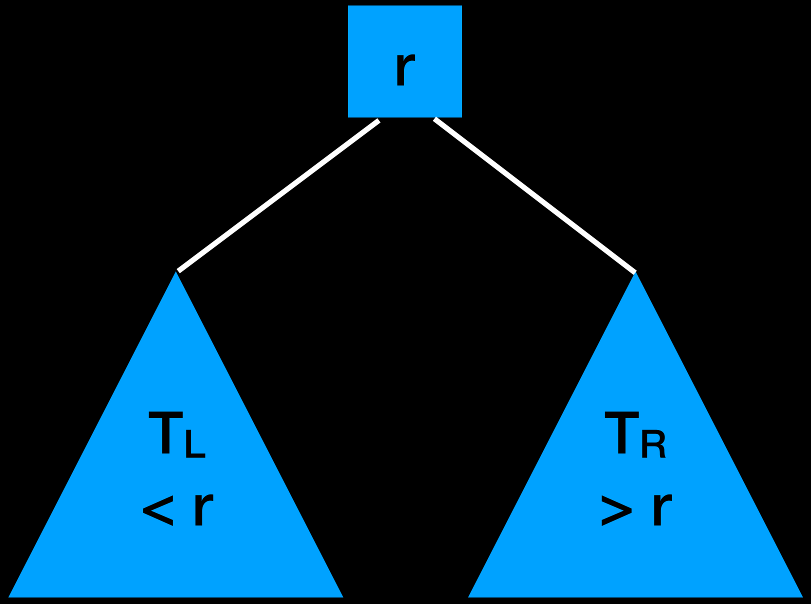


Traversing a BST

Same as traversing any binary tree

```
inorder(bs_tree)
{
    //implicit base case
    if (bs_tree is not empty)
    {
        inorder(T_L)
        visit the root
        inorder(T_R)
    }
}
```

Visits nodes in sorted ascending order



Efficiency of BST

Searching is key to most operations

Think about the structure and height of the tree

Efficiency of BST

Searching is key to most operations

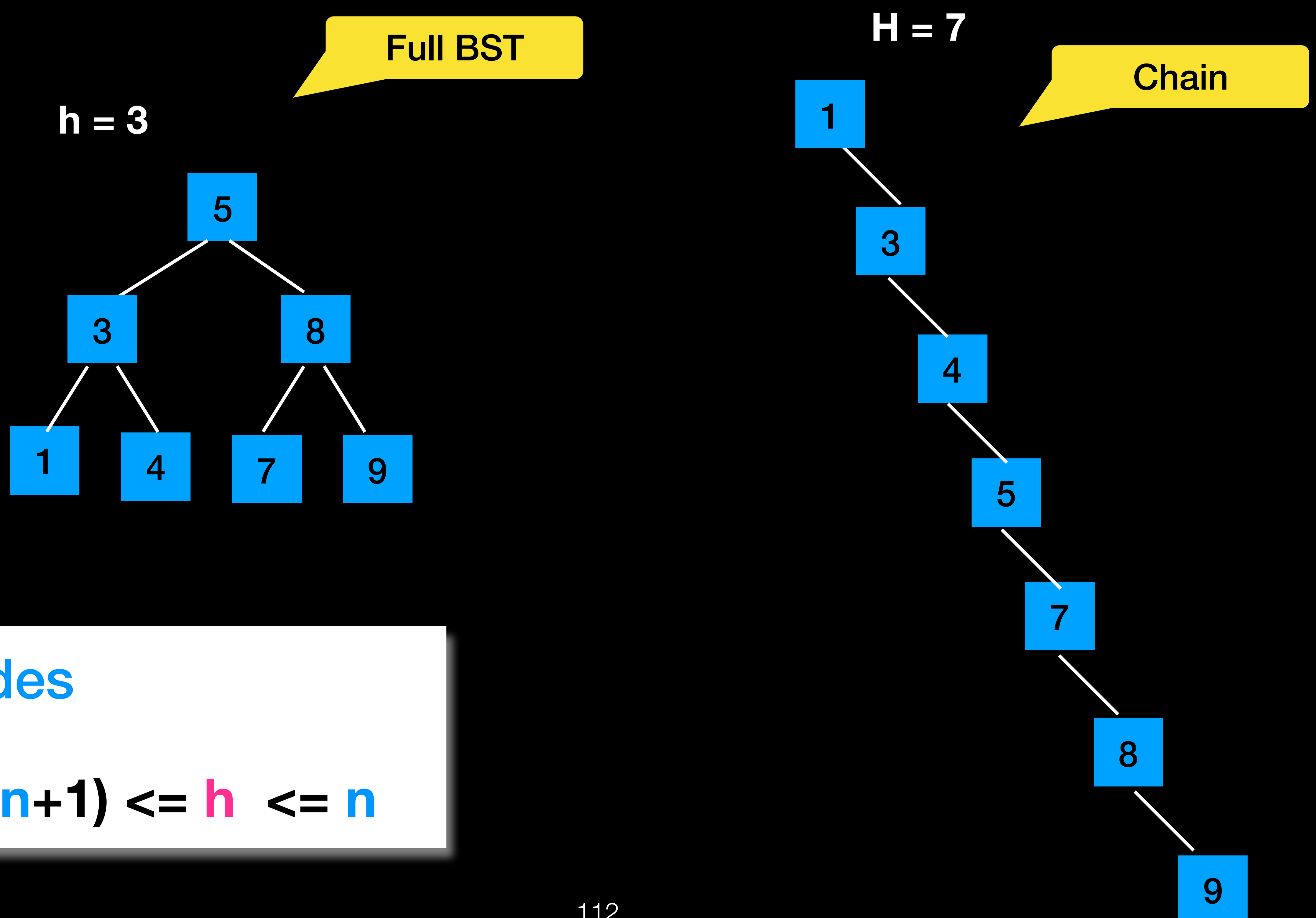
Think about the structure and height of the tree

O(h)

What is the maximum height?

What is the minimum height?

Tree Structure



n nodes

$\log_2(n+1) <= h <= n$

Operation	In Full Tree	Worst-case
Search	$\log_2(n+1)$	$O(h)$
Add	$\log_2(n+1)$	$O(h)$
Remove	$\log_2(n+1)$	$O(h)$
Traverse	n	$O(n)$

BST Operations

```
#ifndef BST_H_
#define BST_H_

template<class T>
class BST
{

public:
     BST(); // constructor
     BST(const BST<T>& tree); // copy constructor
     ~ BST(); // destructor
     bool isEmpty() const;
     size_t getHeight() const;
     size_t getNumberOfNodes() const;
     void add(const T& new_item);
     void remove(const T& new_item);
     T find(const T& item) const;
     void clear();

     void preorderTraverse(Visitor<T>& visit) const;
     void inorderTraverse(Visitor<T>& visit) const;
     void postorderTraverse(Visitor<T>& visit) const;

     BST& operator= (const BST<T>& rhs);

private: // implementation details here
}; // end BST

#include "BST.cpp"
#endif // BST_H_
```

Looks a lot like a BinaryTree

Might you inherit from it?

What would you override?

We will talk about this when we implement it