HODGE THEORETIC REWARD ALLOCATION FOR GENERALIZED COOPERATIVE GAMES ON GRAPHS

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Abstract. We define cooperative games on general graphs and generalize Lloyd S. Shapley's celebrated allocation formula for those games in terms of stochastic path integral driven by the associated Markov chain on each graph. We then show that the value allocation operator, one for each player defined by the stochastic path integral, coincides with the player's component game provided by the combinatorial Hodge decomposition on general weighted graphs.

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. For $N \in \mathbb{N}$, let $[N] := \{1, 2, ..., N\}$ denote the set of players. Let Ξ be an arbitrary finite set representing all possible cooperation states. The typical example is the choice $\Xi := 2^{[N]}$ in the classical work of Shapley [12, 13], where each $S \subseteq [N]$ may represent the players involved in the coalition S.

In this paper, each $S \in \Xi$, for instance, might contain more (or less) information than merely the list of players involved in the cooperation S, and this is the reason we want to consider an abstract state space Ξ. We may assume that the null cooperation, denoted by $\emptyset$, is in Ξ.

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Now the set of *cooperative games* is defined by

$$\mathcal{G} = \mathcal{G}(\Xi) := \{v : \Xi \to \mathbb{R} \mid v(\emptyset) = 0\}.$$

Thus a cooperative game v assigns a value $v(S)$ for each cooperation S, where the null coalition $\emptyset$ receives zero value. For instance, $S, T \in \Xi$ could both represent the cooperations among the same group of players but working under different conditions, possibly yielding $v(S) \neq v(T)$.

When $\Xi = 2^{[N]}$, L. Shapley considered the question of how to split the grand coalition value $v([N])$ among the players for each game $v \in \mathcal{G}(2^{[N]})$. It is uniquely determined according to the following theorem.

Theorem 1.1 (Shapley [13])**.** *There exists a unique allocation* $v \in \mathcal{G}(2^{[N]}) \mapsto \big(\phi_i(v)\big)_{i\in[N]}$ *satisfying the following conditions:*
(i) $\sum_{i\in[N]} \phi_i(v) = v([N])$.
(ii) *If* $v\big(S\cup\{i\}\big) = v\big(S\cup\{j\}\big)$ *for all* $S \subseteq [N]\setminus\{i,j\}$*, then* $\phi_i(v) = \phi_j(v)$.
(iii) *If* $v\big(S \cup \{i\}\big) - v(S) = 0$ *for all* $S \subseteq [N] \setminus \{i\}$*, then* $\phi_i(v) = 0$.
(iv) $\phi_i(\alpha v + \alpha' v') = \alpha\phi_i(v) + \alpha'\phi_i(v')$ *for all* $\alpha, \alpha' \in \mathbb{R}$*,* $v, v' \in \mathcal{G}(2^{[N]})$.

Moreover, this allocation is given by the following explicit formula:

$$\phi_i(v) = \sum_{S\subseteq[N]\setminus\{i\}} \frac{|S|!\big(N-1-|S|\big)!}{N!}\Big(v\big(S\cup\{i\}\big) - v(S)\Big). \tag{1.1}$$

The four conditions listed above are often called the *Shapley axioms*. Quoted from [16], they say that [(i) efficiency] the value obtained by the grand coalition is fully distributed among the players, [(ii) symmetry] equivalent players receive equal amounts, [(iii) null-player] a player who contributes no marginal value to any coalition receives nothing, and [(iv) linearity] the allocation is linear in the game values.

(1.1) can be rewritten also quoted from [16]: Suppose the players form the grand coalition by joining, one-at-a-time, in the order defined by a permutation σ of $[N]$. That is, player i joins immediately after the coalition $S_{\sigma,i} = \big\{j \in [N] : \sigma(j) < \sigma(i)\big\}$ has formed, contributing marginal value $v\big(S_{\sigma,i}\cup\{i\}\big) - v(S_{\sigma,i})$. Then $\phi_i(v)$ is the average marginal value contributed by player i over all $N!$ permutations σ, i.e.,

$$\phi_i(v) = \frac{1}{N!}\sum_{\sigma}\Big(v\big(S_{\sigma,i}\cup\{i\}\big) - v(S_{\sigma,i})\Big). \tag{1.2}$$

Here we notice an important principle, which we may call *Shapley's principle* as in [7], which says the value allocated to player i is based entirely on the marginal values $v\big(S\cup\{i\}\big)-v(S)$ the player i contribute.

The pioneering study of Shapley [12–15] have been followed by many researchers with extensive and diverse literature. For instance, Young [17] and Chun [2] studied Shapley's axioms and suggested its variants. Roth [10] studied the requirement of the utility function for games under which it is unique and equal to the Shapley value. Gul [3] studied the relationship between the cooperative and noncooperative approaches by establishing a framework in which the results of the two theories can be compared. We refer to Roth [11] and Peleg and Sudhölter [8] for more detailed exposition of cooperative game theory.

More recently, the combinatorial Hodge decomposition has been applied to game theory and various economic contexts, for instance Candogan et al. [1], Jiang et al. [5], Stern and Tettenhorst [16]. We refer to Lim [6] for an elementary introduction to the Hodge theory on graphs.

In particular, Stern and Tettenhorst [16] showed that, given a game $v \in \mathcal{G}(2^{[N]})$, there exist *component games* $v_i \in \mathcal{G}(2^{[N]})$ for each player $i \in [N]$ which are naturally defined via the *combinatorial Hodge decomposition*, satisfying $v = \sum_{i\in[N]} v_i$. Moreover, it holds $v_i([N]) = \phi_i(v)$, hence they obtained a new characterization of the Shapley value as the value of the grand coalition in each player's component game.

In this context, the *combinatorial Hodge decomposition* corresponds to the elementary *Fundamental Theorem of Linear Algebra*. For finite-dimensional inner product spaces X, Y and a linear map $\mathrm{d} : X \to Y$ and its adjoint $\mathrm{d}^* : Y \to X$ given by $\langle \mathrm{d}x, y\rangle_Y = \langle x, \mathrm{d}^* y\rangle_X$, FTLA asserts that the orthogonal decompositions hold:

$$X = \mathcal{R}(\mathrm{d}^*) \oplus \mathcal{N}(\mathrm{d}), \qquad Y = \mathcal{R}(\mathrm{d}) \oplus \mathcal{N}(\mathrm{d}^*), \tag{1.3}$$

where $\mathcal{R}(\cdot)$, $\mathcal{N}(\cdot)$ stand for the range and nullspace respectively.

In order to introduce the work of [16] and [7], let us review the setup. Let $G = (V, E)$ be an oriented graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. "Oriented" means at most one of (a, b) and (b, a) is in E for $a, b \in V$. Let $\ell^2(V)$ be the space of functions

$V \to \mathbb{R}$ with the (unweighted) inner product

$$\langle u, v \rangle \coloneqq \sum_{a \in V} u(a)v(a). \tag{1.4}$$

Denote by $\ell^2(E)$ the space of functions $E \to \mathbb{R}$ with inner product

$$\langle f, g \rangle \coloneqq \sum_{(a,b) \in E} f(a,b)g(a,b) \tag{1.5}$$

with the convention that, if $f \in \ell^2(E)$ and $(a,b) \in E$, define $f(b,a) \coloneqq -f(a,b)$ for the reverse-oriented edge.

Next, define a linear operator $\mathrm{d}\colon \ell^2(V) \to \ell^2(E)$, the *gradient*, by

$$\mathrm{d}u(a,b) \coloneqq u(b) - u(a). \tag{1.6}$$

Its adjoint $\mathrm{d}^*\colon \ell^2(E) \to \ell^2(V)$, the (negative) *divergence*, is then

$$(\mathrm{d}^* f)(a) = \sum_{b \sim a} f(b,a), \tag{1.7}$$

where $b \sim a$ denotes $(a,b) \in E$ or $(b,a) \in E$, i.e., a, b are adjacent.

Now to study the cooperative games, Stern and Tettenhorst [16] applied the above setup to the *hypercube graph* $G = (V, E)$, where

$$V = 2^{[N]}, \quad E = \bigl\{ \bigl(S, S \cup \{i\}\bigr) \in V \times V \mid S \subseteq [N] \setminus \{i\},\ i \in [N] \bigr\}. \tag{1.8}$$

Note that each vertex $S \subseteq [N]$ may correspond to a vertex of the unit hypercube in $\mathbb{R}^N$, and each edge is oriented in the direction of the inclusion $S \hookrightarrow S \cup \{i\}$. Then for each $i \in [N]$, [16] set $\mathrm{d}_i\colon \ell^2(V) \to \ell^2(E)$ as the following *partial differential operator*

$$\mathrm{d}_i u\bigl(S, S \cup \{j\}\bigr) = \begin{cases} \mathrm{d}u\bigl(S, S \cup \{i\}\bigr) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases} \tag{1.9}$$

Thus $\mathrm{d}_i v \in \ell^2(E)$ encodes the marginal value contributed by player i to the game v, which is a natural object to consider in view of the Shapley's principle. Indeed, given $v \in \mathcal{G}(2^{[N]})$, Stern and Tettenhorst [16] defined the component game v_i for each $i \in [N]$ as the unique solution in $\mathcal{G}(2^{[N]})$ to the following *least squares* equation[1]

$$\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^*\mathrm{d}_i v \tag{1.10}$$

and showed that the component games satisfy some natural properties analogous to the Shapley axioms (see [16, Theorem 3.4]). Moreover, by applying the inverse of the *Laplacian* $\mathrm{d}^*\mathrm{d}$ to (1.10), they provided an explicit formula for v_i (see [16, Theorem 3.11]). In addition, [16] discusses the case of weighted hypercube graph, viewing this as modeling variable willingness or unwillingness of players to join certain coalitions.

Most recently the author [7], inspired by Stern and Tettenhorst [16], proposed a generalization of the Shapley axioms and showed that they completely characterize the component games $(v_i)_{i\in[N]}$ defined by (1.10) for the unweighted hypercube graph. Moreover, in this case, [7] showed $(v_i)_{i\in[N]}$ can be realized by a natural integral representation formula which may be seen as a generalization of the Shapley formula (1.2).

Now the first goal of this paper is to generalize Shapley's coalition space $2^{[N]}$ into the general cooperative state space Ξ. To do this we consider directed graphs $G = (V, E)$ with $V = \Xi$, which can be weighted. For each weighted graph G, we then associate a canonical Markov chain whose transition rates model the probability of which direction the cooperation would progress toward. Then in terms of this Markov chain we define the value function $V_i \in \mathcal{G}(\Xi)$, for each player $i \in [N]$, as a stochastic path integral such that $V_i(S)$ represents the expected total contribution the player i provides toward each cooperation S. This may be seen as a generalization of the Shapley formula for the cooperative games defined on the abstract cooperation network $G = (\Xi, E)$. Finally, we show that the value functions $(V_i)_{i\in[N]}$ in fact coincide with the component games $(v_i)_{i\in[N]}$ which are defined via the least squares equation (2.5). This may provide a justification for the interpretation

[1]The equation $\mathrm{d}u = f$ is solvable if only if $f \in \mathcal{R}(\mathrm{d})$. When $f \notin \mathcal{R}(\mathrm{d})$, a least squares solution to $\mathrm{d}u = f$ instead solves $\mathrm{d}u = f_1$ where $f = f_1 + f_2$ with $f_1 \in \mathcal{R}(\mathrm{d})$, $f_2 \in \mathcal{N}(\mathrm{d}^*)$ given by FTLA. By applying d^*, we get $\mathrm{d}^*\mathrm{d}u = \mathrm{d}^*f_1 = \mathrm{d}^*f$. The substitution $u \to v_i$ and $f \to \mathrm{d}_i v$ then yields (1.10).

of the component game value $v_i(S)$ as a reasonable reward allocation for the player i at the cooperation state S, as noted by Lim [7].

2. Component game, path integral representation of reward allocation, and their coincidence

We begin by defining the inner product space of functions $\ell^2(\Xi)$, $\ell^2(E)$, now possibly weighted. That is, let μ, λ be strictly positive weight functions on Ξ and E respectively, and set $\lambda(T,S) = \lambda(S,T)$ for any $(S,T) \in E$ by convention. Denote by $\ell^2_\mu(\Xi)$ the space of functions $V \to \mathbb{R}$ equipped with the (μ-weighted) inner product

$$\langle u, v\rangle_\mu \coloneqq \sum_{S\in\Xi} \mu(S)u(S)v(S). \tag{2.1}$$

Denote by $\ell^2_\lambda(E)$ the space of functions $E \to \mathbb{R}$ with inner product

$$\langle f, g\rangle_\lambda \coloneqq \sum_{(S,T)\in E} \lambda(S,T)f(S,T)g(S,T) \tag{2.2}$$

with the convention $f(T,S) \coloneqq -f(S,T)$ for the reverse-oriented edge. We would say for $S,T \in \Xi$, there exists a (forward- or reverse-oriented) edge (S,T) if and only if $\lambda(S,T) > 0$. Then we say the weighted graph $(G,\lambda) = ((\Xi,E),\lambda)$ is *connected* if for any distinct $S,T \in \Xi$ there exists a chain of edges $\big((S_k,S_{k+1})\big)_{k=0}^{n-1}$ in E with $S_0 = S$ and $S_n = T$. We may assume $\emptyset \in \Xi$, so in particular every $S \in \Xi$ is connected with $\emptyset$.

2.1. Component games for cooperative game on general graph.

Recall the linear map, gradient, $\mathrm{d} : \ell^2_\mu(\Xi) \to \ell^2_\lambda(E)$ (1.6) between the inner product spaces. We have an adjoint (divergence) d^*, given by

$$\langle \mathrm{d}v, f\rangle_\lambda = \langle v, \mathrm{d}^* f\rangle_\mu. \tag{2.3}$$

It is not hard to find the explicit form of d^*. Let $(\mathbb{1}_S)_{S\in\Xi}$ be the standard basis of $\ell^2(\Xi)$, where $\mathbb{1}_S(T) = 1$ if $T = S$ and otherwise 0. Then

$$\mathrm{d}^* f(S) = \frac{\langle \mathbb{1}_S, \mathrm{d}^* f\rangle_\mu}{\mu(S)} = \frac{\langle \mathrm{d}\mathbb{1}_S, f\rangle_\lambda}{\mu(S)} = \sum_{T\sim S} \frac{\lambda(T,S)}{\mu(S)} f(T,S). \tag{2.4}$$

Next we recall the partial differential operator d_i in (1.9). While this is a natural definition for a measure of the contribution of player i for the hypercube graph setup (1.8), it does not seem to readily apply for our general graph G. But the observation here is that d_i may not have to be a linear operator acting on the game space $\mathcal{G}$. Instead, we can be utterly general and define each player's contribution to be an arbitrary element in $\ell^2(E)$. That is, let $\vec{f} = (f_1, ..., f_N) \in \otimes_{i=1}^N \ell^2(E)$ denote the N-tuple of player contribution measures, where $f_i(S, T)$ indicates player i's contribution when the cooperation proceeds from S to T.

Given $\vec{f}$, we define the *component game* $v_i \in \mathcal{G}(\Xi)$, for each player i, by the solution to the following least squares equation (cf. (1.10))

$$\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^* f_i. \tag{2.5}$$

Note that (2.5) is uniquely solvable so v_i is well defined. This is because G is connected and thus $\mathcal{N}(\mathrm{d})$ is one-dimensional space spanned by the constant game $\mathbb{1}$, defined by $\mathbb{1}(S) := 1$ for all $S \in \Xi$. Hence if $\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^*\mathrm{d}w_i$, then $v_i - w_i \in \mathcal{N}(\mathrm{d})$ but due to the initial condition $v_i(\emptyset) = w_i(\emptyset) = 0$ from the assumption $v_i, w_i \in \mathcal{G}(\Xi)$, we have $v_i \equiv w_i$. This is why we want to assume the connectedness of G.

But note that what (2.5) actually determines is the increment dv_i in each connected component of G. Thus by assigning an initial value $v_i(S)$ for some S in each connected component, v_i will be determined in that component via (2.5). Here we shall assume, without loss of generality, G is connected with initial condition $v_i(\emptyset) = 0$ for all i.[2]

Let us gather some results regarding the component games, whose proof is analogous to Stern and Tettenhorst [16] and Lim [7].

Proposition 2.1. *Given* $(v, (f_i)_i, \mu, \lambda)$ *consisting of the cooperative game, contribution measures and weights, the component games* $(v_i)_{i\in[N]}$ *defined by* (2.5) *satisfy the following:*

- efficiency: *If* $\sum_i f_i = \mathrm{d}v$, *then* $\sum_{i\in[N]} v_i = v$.
- null-player: *If* $f_i \equiv 0$, *then* $v_i \equiv 0$.
- linearity: *If we assume* $f_i := \mathrm{d}_i v$ *for a fixed linear map* $\mathrm{d}_i : \ell^2_\mu(\Xi) \to \ell^2_\lambda(E)$, *then* $(\alpha v + \alpha' v')_i = \alpha v_i + \alpha' v'_i$ *for all* $\alpha, \alpha' \in \mathbb{R}$ *and* $v, v' \in \mathcal{G}$.

[2]One might instead consider assigning any value for $v_i(\emptyset)$ for each i satisfying $\sum_i v_i(\emptyset) = v(\emptyset)$, thereby modeling some sort of inequality at the initial stage.

Proof. The null-player property is immediate from the defining equation (2.5) and the initial condition $v_i(\emptyset) = 0$. For efficiency, we compute

$$\mathrm{d}^*\mathrm{d}\sum_{i\in[N]} v_i = \sum_{i\in[N]} \mathrm{d}^*\mathrm{d}v_i = \sum_{i\in[N]} \mathrm{d}^* f_i = \mathrm{d}^* \sum_{i\in[N]} f_i = \mathrm{d}^*\mathrm{d}v$$

thus efficiency follows by the unique solvability of (2.5). Finally, linearity follows by the assumed linearity of the map d_i:

$$\begin{aligned}\mathrm{d}^*\mathrm{d}(\alpha v + \alpha' v')_i &= \mathrm{d}^*\mathrm{d}_i(\alpha v + \alpha' v') = \alpha \mathrm{d}^*\mathrm{d}_i v + \alpha' \mathrm{d}^*\mathrm{d}_i v' \\ &= \alpha \mathrm{d}^*\mathrm{d}v_i + \alpha' \mathrm{d}^*\mathrm{d}v_i' = \mathrm{d}^*\mathrm{d}(\alpha v_i + \alpha' v_i')\end{aligned}$$

yielding $(\alpha v + \alpha' v')_i = \alpha v_i + \alpha' v_i'$ as desired. □

Note that the d_i given in (1.9) is an example of a linear map. Also note that we do not present a symmetry property analogous to the Shapley theorem 1.1(ii), due to the fact that, unlike the hypercube graph (1.8), a general graph G may not exhibit any obvious symmetry.

Next let us observe that, although the weight μ also affects the divergence d^* as in (2.4), in fact it does not affect the component games.

Lemma 2.2. *Let $f \in \ell^2_\lambda(E)$. Then the solution $v \in \ell^2_\mu(\Xi)$ to the equation $\mathrm{d}^*\mathrm{d}v = \mathrm{d}^* f$ does not depend on the choice of μ.*

Proof. Observe

$$\begin{aligned}(\mathrm{d}^*\mathrm{d}v - \mathrm{d}^* f)(S) &= \frac{1}{\mu(S)} \sum_{T\sim S} \lambda(T,S)[\mathrm{d}v(T,S) - f(T,S)] \\ &= \frac{1}{\mu(S)} \sum_{T\sim S} \lambda(T,S)[v(S) - v(T) - f(T,S)].\end{aligned}$$

This shows $(\mathrm{d}^*\mathrm{d}v - \mathrm{d}^* f)(S) = 0$ if and only if $\sum_{T\sim S} \lambda(T,S)[v(S) - v(T) - f(T,S)] = 0$, showing there is no dependence on μ. □

On the other hand, the solution to $\mathrm{d}^*\mathrm{d}v = \mathrm{d}^* f$ does depend on λ. [16] showcases this with several explicit computations of component games for weighted and unweighted hypercube graph (1.8).

2.2. **Value allocation operator via a stochastic path integral.** In this section we define a reward allocation function V_i for each player i as a stochastic path integral driven by a Markov chain, which is naturally associated to the game graph. We note this construction was given in the case of the uniform ($\lambda \equiv 1$) hypercube graph (1.8) in [7].

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(G, \mu, \lambda, (f_i)_i)$ be the given (connected) weighted graph and contribution measures by the players. In view of (2.4), we define a Markov chain $(X_n^U)_{n\in\mathbb{N}_0}$ on the state space Ξ with $X_0 = U$ (with the convention $X_n = X_n^{\emptyset}$), equipped with the transition probability $p_{S,T}$ from state S to T as follows:

$$
\begin{aligned}
(2.6) \qquad & p_{S,T} = \frac{\frac{\lambda(S,T)}{\mu(S)}}{\sum_{U\sim S} \frac{\lambda(S,U)}{\mu(S)}} = \frac{\lambda(S,T)}{\sum_{U\sim S} \lambda(S,U)} \quad \text{if} \quad T \sim S, \\
& p_{S,T} = 0 \quad \text{if} \quad T \not\sim S.
\end{aligned}
$$

Thus the weight λ (but not μ) determines to which direction the cooperation is likely to progress. This allows us further flexibility for modeling stochastic cooperation network.

It turns out that the Markov chain (2.6) is *time-reversible*, meaning that there exists the stationary distribution $\pi = (\pi_S)_{S\in\Xi}$ such that

$$
(2.7) \qquad \pi_S p_{S,T} = \pi_T p_{T,S} \quad \text{for all} \quad S, T \in \Xi.
$$

A consequence, which is important to us, is that every loop and its reverse have the same probability, that is (see, e.g., Ross [9])

$$
(2.8) \qquad p_{S,S_1} p_{S_1,S_2} \dots p_{S_{n-1},S_n} p_{S_n,S} = p_{S,S_n} p_{S_n,S_{n-1}} \dots p_{S_2,S_1} p_{S_1,S}.
$$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space for the Markov chain. For each $S, T \in \Xi$ and $\omega \in \Omega$, let $\tau_{S,T} = \tau_{S,T}(\omega) \in \mathbb{N}_0$ denote the first time the Markov chain $\big(X_n^S(\omega)\big)_n$ visits T. Given a player's contribution measure $f \in \ell^2(E)$, we define the total contribution of the player along the sample path $\omega \in \Omega$ moving from the state S to T by

$$
(2.9) \qquad F_f^S(T) = F_f^S(T)(\omega) := \sum_{n=1}^{\tau_{S,T}(\omega)} f\big(X_{n-1}^S(\omega), X_n^S(\omega)\big).
$$

Now we can define the value function for given $f \in \ell^2(E)$ via the following stochastic path integral driven by the Markov chain (2.6)

$$
(2.10) \qquad V_f^S(T) := \int_\Omega F_f^S(T)(\omega) d\mathcal{P}(\omega) = \mathbb{E}[F_f^S(T)].
$$

Finally, let us denote $V_i^S := V_{f_i}^S$ for each player $i \in [N]$ given the players' contribution measures $(f_i)_{i\in[N]}$. One may notice that this path integral representation can be seen as a generalization of the Shapley formula (1.2). In particular, $V_i(T) := V_i^{\emptyset}(T)$ represents the expected total contribution the player i provides toward each cooperation T, provided the game starts at the null cooperation $\emptyset$.

2.3. **The coincidence between the value allocation operator and the component game.** The question is how we can compute the value allocators $(V_i)_{i\in[N]}$ which are described by the stochastic path integral. One could employ some computational methods to simulate the Markov chain and approximate the path integral, for instance.

Or, better yet, our main result of this paper shows that V_i is a valid representation of the component game v_i, that is, $V_i = v_i$ for every player i. This result displays a remarkable connection between stochastic path integrals and combinatorial Hodge theory on general graphs.

Recall that the weight μ on $\ell^2(\Xi)$ is not relevant in either Lemma 2.2 or (2.6), so we will simply set $\mu \equiv 1$ from now on. First, the following lemma establishes a transition formula for the value function V_f.

Lemma 2.3. *Let (G, λ) be any connected weighted graph. For any $S, T, U \in \Xi$ and $f \in \ell^2(E)$, we have $V_f^U(T) - V_f^U(S) = V_f^S(T)$.*

Now we present our main result.

Theorem 2.4. *Let the Markov chain* (2.6) *be defined on each connected component of a weighted graph (G, λ). Then it holds*

$$\mathrm{d}^*\mathrm{d}V_f = \mathrm{d}^* f \quad \textit{for any} \quad f \in \ell^2(E). \tag{2.11}$$

The theorem tells us when one wants to calculate the value allocation function V_i for the player i given her contribution measure f_i, one can instead compute the least squares solution v_i, which can be easily done via least squares solvers for instance. Conversely, the least squares solution v_i may be approximated by simulating the canonical Markov chain (2.6) on the graph (G, λ) and calculating the contribution aggregator (2.10). Both directions look interesting and potentially useful.

3. Conclusion

In this paper we reviewed the cooperative game framework of Shapley [12, 13] and its Hodge-theoretic extension by Stern and Tettenhorst [16] and Lim [7]. These papers regard the cooperative games as value functions on $2^{[N]}$, and [7, 16] consider the differential operators d, d_i defined on the hypercube graph (1.8). Then we proposed that the cooperative games may be defined in a much more general framework of arbitrary weighted game graphs $G = (\Xi, E)$, in which the partial differential d_i is replaced by a general contribution measure $f_i \in \ell^2(E)$. Given f_i for each player i, we proposed a natural value allocation operator V_i, a stochastic path integral driven by the canonical, time-reversible Markov chain on the weighted graph. Then we verified an intriguing connection of this stochastic integral with the "component game" v_i, which is the solution to the least squares equation (2.5) given by the Hodge decomposition (1.3). Finally, if the condition $\sum_i f_i = \mathrm{d}v$ holds for a given cooperative game v, then in view of Proposition 2.1, $V_i = v_i$ may be interpreted as a fair and efficient allocation of the cooperation value $v(S)$ to the player i at the cooperation state S, which may be seen as a generalization of the Shapley's value allocation formula (1.2).

4. Proofs

Proof of Lemma 2.3. We first prove a special case $V_f^S(T) = -V_f^T(S)$. For this, consider a general sample path ω of the Markov chain (2.6) starting at S, visiting T, then returning to S. We could divide this journey into four stages:
ω_1: the path returns to S $m \in \mathbb{N}_0$ times while not visiting T yet,
ω_2: the path starts at S and ends at T while not returning to S,
ω_3: the path returns to T $n \in \mathbb{N}_0$ times while not visiting S yet,
ω_4: the path starts at T and ends at S while not returning to T.
Thus $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3 \otimes \omega_4$ is the concatenation of the ω_i's, and the probability of this finite sample path satisfies $\mathcal{P}(\omega) = \mathcal{P}(\omega_1)\mathcal{P}(\omega_2)\mathcal{P}(\omega_3)\mathcal{P}(\omega_4)$.

Now consider a pairing ω' of ω as follows: let ω_1^{-1} be the reversed path of ω_1, that is, if ω_1 visits $T_0 \to T_1 \to \cdots \to T_k$ (where $T_0 = T_k = S$

for ω_1), then ω_1^{-1} visits $T_k \to \cdots \to T_0$. Recall $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1^{-1})$ due to the time-reversibility (2.8). Now define $\omega' := \omega_1^{-1} \otimes \omega_2 \otimes \omega_3^{-1} \otimes \omega_4$. This is another general sample path starting at S, visiting T, then returning to S. We have $\mathcal{P}(\omega) = \mathcal{P}(\omega')$, and moreover,

$$F_f^S(T)(\omega) + F_f^S(T)(\omega') = 2 \sum_{n=1}^{\tau_{S,T}(\omega_2)} f\big(X_{n-1}^S(\omega_2), X_n^S(\omega_2)\big),$$

since the loop ω_1 and its reverse ω_1^{-1} aggregate f with opposite sign, so they cancel out in the above sum. Now consider $\tilde{\omega} := \omega_3 \otimes \omega_2^{-1} \otimes \omega_1 \otimes \omega_4^{-1}$ and $\tilde{\omega}' := \omega_3^{-1} \otimes \omega_2^{-1} \otimes \omega_1^{-1} \otimes \omega_4^{-1}$. $(\tilde{\omega}, \tilde{\omega}')$ then represents a pair of general sample paths starting at T, visiting S, then returning to T. Moreover

$$\begin{aligned}
F_f^T(S)(\tilde{\omega}) + F_f^T(S)(\tilde{\omega}') &= 2 \sum_{n=1}^{\tau_{T,S}(\omega_2^{-1})} f\big(X_{n-1}^T(\omega_2^{-1}), X_n^T(\omega_2^{-1})\big) \\
&= -2 \sum_{n=1}^{\tau_{S,T}(\omega_2)} f\big(X_{n-1}^S(\omega_2), X_n^S(\omega_2)\big) \\
&= -(F_f^S(T)(\omega) + F_f^S(T)(\omega'))
\end{aligned}$$

since $f \in \ell^2(E)$. Due to the generality of the pair (ω, ω') and its counterpart $(\tilde{\omega}, \tilde{\omega}')$, and $\mathcal{P}(\omega) = \mathcal{P}(\omega') = \mathcal{P}(\tilde{\omega}) = \mathcal{P}(\tilde{\omega}')$ from the reversibility (2.8), the desired identity $V_f^S(T) = -V_f^T(S)$ follows by integration.

Now to show $V_f^U(T) - V_f^U(S) = V_f^S(T)$, we proceed as in [7]:

$$\begin{aligned}
F_f^U(T) - F_f^U(S) &= \sum_{n=1}^{\tau_{U,T}} f\big(X_{n-1}^U, X_n^U\big) - \sum_{n=1}^{\tau_{U,S}} f\big(X_{n-1}^U, X_n^U\big) \\
&= \mathbf{1}_{\tau_{U,S} < \tau_{U,T}} \sum_{n=\tau_{U,S}+1}^{\tau_{U,T}} f\big(X_{n-1}^U, X_n^U\big) - \mathbf{1}_{\tau_{U,T} < \tau_{U,S}} \sum_{n=\tau_{U,T}+1}^{\tau_{U,S}} f\big(X_{n-1}^U, X_n^U\big).
\end{aligned}$$

By taking expectation, we obtain via the Markov property

$$\begin{aligned}
\mathbb{E}[F_f^U(T)] - \mathbb{E}[F_f^U(S)] &= \mathcal{P}(\{\tau_{U,S} < \tau_{U,T}\}) V_f^S(T) - \mathcal{P}(\{\tau_{U,T} < \tau_{U,S}\}) V_f^T(S) \\
&= V_f^S(T)
\end{aligned}$$

which proves the transition formula $V_f^U(T) - V_f^U(S) = V_f^S(T)$. $\square$

Proof of Theorem 2.4. We set $\mu \equiv 1$ without loss of generality. Given $f \in \ell^2(E)$, our aim is to show that V_f solves (2.11). Let $S \in \Xi$, and let $\{T_1, ..., T_n\}$ be the set of all vertices adjacent to S (i.e., either (S, T_k) or (T_k, S) is in E), and set $\Lambda_S = \sum_{k=1}^n \lambda(S, T_k)$. Then by (2.4), (2.6),

$$\mathrm{d}^* f(S)/\Lambda_S = \sum_{k=1}^n p_{S,T_k} f(T_k, S), \text{ and} \tag{4.1}$$

$$\mathrm{d}^*\mathrm{d}V_f(S)/\Lambda_S = \sum_{k=1}^n p_{S,T_k} \big(V_f(S) - V_f(T_k)\big) = \sum_{k=1}^n p_{S,T_k} V_f^{T_k}(S) \tag{4.2}$$

where the last equality is from Lemma 2.3. Now observe that we can interpret (4.2) as the aggregation (2.10) of path integrals of f (2.9) for all loops starting and ending at S, but in this aggregation of f we do not take into account the first move from S to T_k, since this first move is made by the transition rate p_{S,T_k} and not driven by $V_f^{T_k}$. On the other hand, if we aggregate path integrals of f for all loops emanating from S, we get zero due to the reversibility (2.8). Hence we conclude:

$$\begin{aligned}
0 &= \text{aggregation of path integrals of } f \text{ for all loops emanating from } S\\
&= \text{aggregation of path integrals of } f \text{ for all loops, omitting the first moves}\\
&\quad + \text{aggregation of path integrals of } f \text{ for all first moves from } S\\
&= \sum_{k=1}^n p_{S,T_k} V_f^{T_k}(S) + \sum_{k=1}^n p_{S,T_k} f(S, T_k)\\
&= \mathrm{d}^*\mathrm{d}V_f(S)/\Lambda_S - \mathrm{d}^* f(S)/\Lambda_S,
\end{aligned}$$

yielding $\mathrm{d}^*\mathrm{d}V_f(S) = \mathrm{d}^* f(S)$ for all $S \in \Xi$. □

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