HODGE THEORETIC REWARD ALLOCATION FOR GENERALIZED COOPERATIVE GAMES ON GRAPHS

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Abstract. We define cooperative games on general graphs and generalize Lloyd S. Shapley's celebrated allocation formula for those games in terms of stochastic path integral driven by the associated Markov chain on each graph. We then show that the value allocation operator, one for each player defined by the stochastic path integral, coincides with the player's component game which is the solution to the least squares (or Poisson's) equation, in light of the combinatorial Hodge decomposition on general weighted graphs. Several motivational examples and applications are presented.

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. For $N \in \mathbb{N}$, we let $[N] := \{1, 2, ..., N\}$ denote the set of players. Let Ξ be an arbitrary finite set representing all possible cooperation states. The typical example is the choice $\Xi := 2^{[N]}$ in the classical work of Shapley [12, 13], where each $S \subseteq [N]$ represents the players involved in the coalition S.

In this paper, each $S \in \Xi$, for instance, might contain more (or less) information than merely the list of players involved in the cooperation S, and this is the reason we want to consider an abstract state space

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Ξ. We assume the null cooperation, denoted by $\emptyset$, is in Ξ; see examples in section (2.4). Now the set of *cooperative games* is defined by

$$\mathcal{G} = \mathcal{G}(\Xi) := \{v : \Xi \to \mathbb{R} \mid v(\emptyset) = 0\}.$$

Thus a cooperative game v assigns a value $v(S)$ for each cooperation S, where the null coalition $\emptyset$ receives zero value. For instance, $S, T \in \Xi$ could both represent the cooperations among the same group of players but working under different conditions, possibly yielding $v(S) \neq v(T)$.

When $\Xi = 2^{[N]}$, L. Shapley considered the question of how to split the grand coalition value $v([N])$ among the players for each game $v \in \mathcal{G}(2^{[N]})$. It is uniquely determined according to the following theorem.

Theorem 1.1 (Shapley [13]). *There exists a unique allocation $v \in \mathcal{G}(2^{[N]}) \mapsto \big(\phi_i(v)\big)_{i\in[N]}$ satisfying the following conditions:*
(i) $\sum_{i\in[N]} \phi_i(v) = v([N])$.
(ii) *If $v\big(S\cup\{i\}\big) = v\big(S\cup\{j\}\big)$ for all $S \subseteq [N]\backslash\{i,j\}$, then $\phi_i(v) = \phi_j(v)$.*
(iii) *If $v\big(S \cup \{i\}\big) - v(S) = 0$ for all $S \subseteq [N] \setminus \{i\}$, then $\phi_i(v) = 0$.*
(iv) $\phi_i(\alpha v + \alpha' v') = \alpha\phi_i(v) + \alpha'\phi_i(v')$ *for all $\alpha, \alpha' \in \mathbb{R}$, $v, v' \in \mathcal{G}(2^{[N]})$.*

Moreover, this allocation is given by the following explicit formula:

$$\phi_i(v) = \sum_{S\subseteq[N]\backslash\{i\}} \frac{|S|!\big(N-1-|S|\big)!}{N!}\Big(v\big(S\cup\{i\}\big) - v(S)\Big). \tag{1.1}$$

The four conditions listed above are often called the *Shapley axioms*. Quoted from [16], they say that [(i) efficiency] the value obtained by the grand coalition is fully distributed among the players, [(ii) symmetry] equivalent players receive equal amounts, [(iii) null-player] a player who contributes no marginal value to any coalition receives nothing, and [(iv) linearity] the allocation is linear in the game values.

(1.1) can be rewritten also quoted from [16]: Suppose the players form the grand coalition by joining, one-at-a-time, in the order defined by a permutation σ of $[N]$. That is, player i joins immediately after the coalition $S_{\sigma,i} = \big\{j \in [N] : \sigma(j) < \sigma(i)\big\}$ has formed, contributing marginal value $v\big(S_{\sigma,i}\cup\{i\}\big) - v(S_{\sigma,i})$. Then $\phi_i(v)$ is the average marginal value contributed by player i over all $N!$ permutations σ, i.e.,

$$\phi_i(v) = \frac{1}{N!} \sum_{\sigma} \Big(v\big(S_{\sigma,i} \cup \{i\}\big) - v(S_{\sigma,i}) \Big). \tag{1.2}$$

Here we notice an important principle, which we may call *Shapley's principle* as in [7], which says the value allocated to player i is based entirely on the marginal values $v\big(S \cup \{i\}\big) - v(S)$ the player i contribute.

The pioneering study of Shapley [12–15] have been followed by many researchers with extensive and diverse literature. For instance, Young [17] and Chun [2] studied Shapley's axioms and suggested its variants. Roth [10] studied the requirement of the utility function for games under which it is unique and equal to the Shapley value. Gul [3] studied the relationship between the cooperative and noncooperative approaches by establishing a framework in which the results of the two theories can be compared. We refer to Roth [11] and Peleg and Sudhölter [8] for more detailed exposition of cooperative game theory.

More recently, the combinatorial Hodge decomposition has been applied to game theory and various economic contexts, for instance Candogan et al. [1], Jiang et al. [5], Stern and Tettenhorst [16]. We refer to Lim [6] for an elementary introduction to the Hodge theory on graphs.

In particular, Stern and Tettenhorst [16] showed that, given a game $v \in \mathcal{G}(2^{[N]})$, there exist *component games* $v_i \in \mathcal{G}(2^{[N]})$ for each player $i \in [N]$ which are naturally defined via the *combinatorial Hodge decomposition*, satisfying $v = \sum_{i \in [N]} v_i$. Moreover, it holds $v_i([N]) = \phi_i(v)$, hence they obtained a new characterization of the Shapley value as the value of the grand coalition in each player's component game.

In this context, the *combinatorial Hodge decomposition* corresponds to the elementary *Fundamental Theorem of Linear Algebra*. For finite-dimensional inner product spaces X, Y and a linear map $\mathrm{d} : X \to Y$ and its adjoint $\mathrm{d}^* : Y \to X$ given by $\langle \mathrm{d}x, y \rangle_Y = \langle x, \mathrm{d}^* y \rangle_X$, FTLA asserts that the orthogonal decompositions hold:

$$X = \mathcal{R}(\mathrm{d}^*) \oplus \mathcal{N}(\mathrm{d}), \qquad Y = \mathcal{R}(\mathrm{d}) \oplus \mathcal{N}(\mathrm{d}^*), \tag{1.3}$$

where $\mathcal{R}(\cdot)$, $\mathcal{N}(\cdot)$ stand for the range and nullspace respectively.

In order to introduce the work of [16] and [7], let us review the setup. Let $G = (V, E)$ be an oriented graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. "Oriented" means at most one of (a, b) and (b, a) is in E for $a, b \in V$. Let $\ell^2(V)$ be the space of functions $V \to \mathbb{R}$ with the (unweighted) inner product

$$\langle u, v\rangle \coloneqq \sum_{a\in V} u(a)v(a). \tag{1.4}$$

Denote by $\ell^2(E)$ the space of functions $E \to \mathbb{R}$ with inner product

$$\langle f, g\rangle \coloneqq \sum_{(a,b)\in E} f(a, b)g(a, b) \tag{1.5}$$

with the convention that, if $f \in \ell^2(E)$ and $(a, b) \in E$, define $f(b, a) \coloneqq -f(a, b)$ for the reverse-oriented edge.

Next, define a linear operator $\mathrm{d}\colon \ell^2(V) \to \ell^2(E)$, the *gradient*, by

$$\mathrm{d}u(a, b) \coloneqq u(b) - u(a). \tag{1.6}$$

Its adjoint $\mathrm{d}^*\colon \ell^2(E) \to \ell^2(V)$, the (negative) *divergence*, is then

$$(\mathrm{d}^* f)(a) = \sum_{b\sim a} f(b, a), \tag{1.7}$$

where $b \sim a$ denotes $(a, b) \in E$ or $(b, a) \in E$, i.e., a, b are adjacent.

Now to study the cooperative games, Stern and Tettenhorst [16] applied the above setup to the *hypercube graph* $G = (V, E)$, where

$$V = 2^{[N]}, \quad E = \big\{\big(S, S\cup\{i\}\big) \in V\times V \mid S \subseteq [N]\setminus\{i\},\ i \in [N]\big\}. \tag{1.8}$$

Note that each vertex $S \subseteq [N]$ may correspond to a vertex of the unit hypercube in $\mathbb{R}^N$, and each edge is oriented in the direction of the inclusion $S \hookrightarrow S\cup\{i\}$. Then for each $i \in [N]$, [16] set $\mathrm{d}_i\colon \ell^2(V) \to \ell^2(E)$ as the following *partial differential operator*

$$\mathrm{d}_i u\big(S, S\cup\{j\}\big) = \begin{cases} \mathrm{d}u\big(S, S\cup\{i\}\big) & \text{if } \ j = i, \\ 0 & \text{if } \ j \neq i. \end{cases} \tag{1.9}$$

Thus $\mathrm{d}_i v \in \ell^2(E)$ encodes the marginal value contributed by player i to the game v, which is a natural object to consider in view of the

Shapley's principle. Indeed, for $v \in \mathcal{G}(2^{[N]})$, Stern and Tettenhorst [16] defined the component game v_i for each $i \in [N]$ as the unique solution in $\mathcal{G}(2^{[N]})$ to the following *least squares* (or *Poisson's*) *equation*[1]

$$\mathrm{d}^*\mathrm{d}v_i = \mathrm{d}^*\mathrm{d}_i v \tag{1.10}$$

and showed that the component games satisfy some natural properties analogous to the Shapley axioms (see [16, Theorem 3.4]). Moreover, by applying the inverse of the *Laplacian* $\mathrm{d}^*\mathrm{d}$ to (1.10), they provided an explicit formula for v_i (see [16, Theorem 3.11]). In addition, [16] discussed the case of weighted hypercube graph, viewing this as modeling variable willingness or unwillingness of players to join certain coalitions.

Most recently Lim [7], inspired by Stern and Tettenhorst [16], proposed a generalization of the Shapley axioms and showed that they completely characterize the component games $(v_i)_{i\in[N]}$ defined by (1.10) for the unweighted hypercube graph. Moreover, in this case, [7] showed $(v_i)_{i\in[N]}$ can be realized by a natural integral representation formula which can be seen as a generalization of the Shapley formula (1.2).

Now the first goal of this paper is to generalize Shapley's coalition space $2^{[N]}$ into the general cooperative state space Ξ. To do this we consider directed graphs $G = (V, E)$ with $V = \Xi$, which can be weighted. For each weighted graph G, we then associate a canonical Markov chain whose transition rates model the probability of which direction the cooperation would progress toward. Then powered by this Markov chain we introduce our main objective of study, the value function $V_i \in \mathcal{G}(\Xi)$ for each player $i \in [N]$, as a stochastic path integral, such that $V_i(S)$ represents the expected total contribution the player i provides toward each cooperation S. This may be seen as a generalization of the Shapley formula for the cooperative games defined on the abstract cooperation

[1]The equation $\mathrm{d}u = f$ is solvable if only if $f \in \mathcal{R}(\mathrm{d})$. When $f \notin \mathcal{R}(\mathrm{d})$, a least squares solution to $\mathrm{d}u = f$ instead solves $\mathrm{d}u = f_1$ where $f = f_1 + f_2$ with $f_1 \in \mathcal{R}(\mathrm{d})$, $f_2 \in \mathcal{N}(\mathrm{d}^*)$ given by FTLA. By applying d^*, we get $\mathrm{d}^*\mathrm{d}u = \mathrm{d}^*f_1 = \mathrm{d}^*f$. Here the substitution $u \to v_i$ and $f \to \mathrm{d}_i v$ yields (1.10).

network $G = (\Xi, E)$. Finally, our main result reveals that the stochastic integral V_i in fact solves the least squares (or Poisson's) equation, yielding that the value functions $(V_i)_{i\in[N]}$ coincide with the component games $(v_i)_{i\in[N]}$ which are defined via the equation (2.5). This justifies the interpretation of the component game value $v_i(S)$ to be a reasonable reward allocation for the player i at the cooperation state S.

For motivation, we illustrate the concepts and generalizations suggested in this paper through examples and applications in section 2.4.

2. Component game, path integral representation of reward allocation, and their coincidence

We begin by defining the inner product space of functions $\ell^2(\Xi)$, $\ell^2(E)$, now possibly weighted. That is, let μ, λ be strictly positive weight functions on Ξ and E respectively, and set $\lambda(T, S) = \lambda(S, T)$ for any $(S, T) \in E$ by convention. Denote by $\ell^2_\mu(\Xi)$ the space of functions $V \to \mathbb{R}$ equipped with the (μ-weighted) inner product

$$\langle u, v\rangle_\mu \coloneqq \sum_{S\in\Xi} \mu(S)u(S)v(S). \tag{2.1}$$

Denote by $\ell^2_\lambda(E)$ the space of functions $E \to \mathbb{R}$ with inner product

$$\langle f, g\rangle_\lambda \coloneqq \sum_{(S,T)\in E} \lambda(S, T)f(S, T)g(S, T) \tag{2.2}$$

with the convention $f(T, S) \coloneqq -f(S, T)$ for the reverse-oriented edge. We would say for $S, T \in \Xi$, there exists a (forward- or reverse-oriented) edge (S, T) if and only if $\lambda(S, T) > 0$. Then we say the weighted graph $(G, \lambda) = ((\Xi, E), \lambda)$ is *connected* if for any distinct $S, T \in \Xi$ there exists a chain of edges $\big((S_k, S_{k+1})\big)_{k=0}^{n-1}$ in E with $S_0 = S$, $S_n = T$. We assume $\emptyset \in \Xi$, so in particular every $S \in \Xi$ is connected with $\emptyset$, for convenience.

2.1. Component games for cooperative game on general graph.

Recall the linear map, gradient, $\mathrm{d} : \ell^2_\mu(\Xi) \to \ell^2_\lambda(E)$ (1.6) between the inner product spaces. We have an adjoint (divergence) d^*, given by

$$\langle \mathrm{d}v, f\rangle_\lambda = \langle v, \mathrm{d}^* f\rangle_\mu. \tag{2.3}$$

It is not hard to find the explicit form of d^*. Let $(\mathbb{1}_S)_{S\in\Xi}$ be the standard basis of $\ell^2(\Xi)$, where $\mathbb{1}_S(T)=1$ if $T=S$ and otherwise 0. Then

$$\mathrm{d}^*f(S)=\frac{\langle \mathbb{1}_S,\mathrm{d}^*f\rangle_\mu}{\mu(S)}=\frac{\langle \mathrm{d}\mathbb{1}_S,f\rangle_\lambda}{\mu(S)}=\sum_{T\sim S}\frac{\lambda(T,S)}{\mu(S)}f(T,S). \tag{2.4}$$

Next we recall the partial differential operator d_i in (1.9). While this is a natural definition for a measure of the contribution of player i for the hypercube graph setup (1.8), it does not seem to readily apply for our general graph G. But the observation here is that d_i may not have to be a linear operator acting on the game space $\mathcal{G}$. Instead, we can be utterly general and define each player's contribution to be an arbitrary element in $\ell^2(E)$. That is, let $\vec{f}=(f_1,...,f_N)\in\otimes_{i=1}^N\ell^2(E)$ denote the N-tuple of player contribution measures, where $f_i(S,T)$ indicates player i's contribution when the cooperation proceeds from S to T.

Given $\vec{f}$, we define the *component game* $v_i\in\mathcal{G}(\Xi)$, for each player i, by the solution to the following least squares equation (cf. (1.10))

$$\mathrm{d}^*\mathrm{d}v_i=\mathrm{d}^*f_i. \tag{2.5}$$

Given an initial condition, (2.5) admits a unique solution so v_i is well defined. This is because G is connected and thus $\mathcal{N}(\mathrm{d})$ is one-dimensional space spanned by the constant game $\mathbb{1}$, defined by $\mathbb{1}(S):=1$ for all $S\in\Xi$. Hence if $\mathrm{d}^*\mathrm{d}v_i=\mathrm{d}^*\mathrm{d}w_i$, then $v_i-w_i\in\mathcal{N}(\mathrm{d})$ but due to the initial condition $v_i(\emptyset)=w_i(\emptyset)=0$ from the assumption $v_i,w_i\in\mathcal{G}(\Xi)$, we have $v_i\equiv w_i$. This is the reason we assume the connectedness of G.

But note that what (2.5) actually determines is the increment dv_i in each connected component of G. Thus by assigning an initial value $v_i(S)$ for some S in each connected component, v_i will be determined in that component via (2.5). Here we shall assume, without loss of generality, G is connected with initial condition $v_i(\emptyset)=0$ for all i. But this is not necessary, and e.g. one may assign any value for $v_i(\emptyset)$ for each i, thereby modeling some sort of inequality at the initial stage.

Let us gather some results regarding the component games, whose proof is analogous to Stern and Tettenhorst [16] and Lim [7].

Proposition 2.1. *Given $(v, (f_i)_i, \mu, \lambda)$ consisting of the cooperative game, contribution measures and weights, the component games $(v_i)_{i\in[N]}$ defined by* (2.5) *satisfy the following:*
- efficiency: *If $\sum_i f_i = \mathrm{d}v$, then $\sum_{i\in[N]} v_i = v$.*
- null-player: *If $f_i \equiv 0$, then $v_i \equiv 0$.*
- linearity: *If we assume $f_i := \mathrm{d}_i v$ for a fixed linear map $\mathrm{d}_i : \ell^2_\mu(\Xi) \to \ell^2_\lambda(E)$, then $(\alpha v + \alpha' v')_i = \alpha v_i + \alpha' v'_i$ for all $\alpha, \alpha' \in \mathbb{R}$ and $v, v' \in \mathcal{G}$.*

Proof. The null-player property is immediate from the defining equation (2.5) and the initial condition $v_i(\emptyset) = 0$. For efficiency, we compute

$$\mathrm{d}^*\mathrm{d} \sum_{i\in[N]} v_i = \sum_{i\in[N]} \mathrm{d}^*\mathrm{d}v_i = \sum_{i\in[N]} \mathrm{d}^* f_i = \mathrm{d}^* \sum_{i\in[N]} f_i = \mathrm{d}^*\mathrm{d}v$$

thus efficiency follows by the unique solvability of (2.5). Finally, linearity follows by the assumed linearity of the map d_i:

$$\begin{aligned}\mathrm{d}^*\mathrm{d}(\alpha v + \alpha' v')_i &= \mathrm{d}^*\mathrm{d}_i(\alpha v + \alpha' v') = \alpha \mathrm{d}^*\mathrm{d}_i v + \alpha' \mathrm{d}^*\mathrm{d}_i v' \\ &= \alpha \mathrm{d}^*\mathrm{d}v_i + \alpha' \mathrm{d}^*\mathrm{d}v'_i = \mathrm{d}^*\mathrm{d}(\alpha v_i + \alpha' v'_i)\end{aligned}$$

yielding $(\alpha v + \alpha' v')_i = \alpha v_i + \alpha' v'_i$ as desired. □

Note that the d_i given in (1.9) is an example of a linear map. Also notice that we do not present a symmetry property analogous to the Shapley theorem 1.1(ii), due to the fact that, unlike the hypercube graph (1.8), a general graph G may not exhibit any obvious symmetry.

Next let us observe that, although the weight μ also affects the divergence d^* as in (2.4), in fact it does not affect the component games.

Lemma 2.2. *Let $f \in \ell^2_\lambda(E)$. Then the solution $v \in \ell^2_\mu(\Xi)$ to the equation $\mathrm{d}^*\mathrm{d}v = \mathrm{d}^* f$ does not depend on the choice of μ.*

Proof. $(\mathrm{d}^*\mathrm{d}v - \mathrm{d}^*f)(S) = \frac{1}{\mu(S)} \sum_{T\sim S} \lambda(T,S)[v(S) - v(T) - f(T,S)]$ shows that $(\mathrm{d}^*\mathrm{d}v - \mathrm{d}^*f)(S) = 0$ if and only if $\sum_{T\sim S} \lambda(T,S)[v(S) - v(T) - f(T,S)] = 0$, showing there is no dependence on μ. □

On the other hand, the solution to $\mathrm{d}^*\mathrm{d}v = \mathrm{d}^* f$ does depend on λ. [16] showcases this with several explicit computations of component games for weighted and unweighted hypercube graph (1.8).

2.2. **Value allocation operator via a stochastic path integral.** In this section we define a reward allocation function V_i for each player i as a stochastic path integral driven by a Markov chain, which is naturally associated to the game graph. We note this construction was given in the case of the uniform ($\lambda \equiv 1$) hypercube graph (1.8) in [7].

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(G, \mu, \lambda, (f_i)_i)$ be the given (connected) weighted graph and contribution measures by the players. In view of (2.4), we define a Markov chain $(X_n^U)_{n\in\mathbb{N}_0}$ on the state space Ξ with $X_0 = U$ (with the convention $X_n = X_n^\emptyset$), equipped with the transition probability $p_{S,T}$ from state S to T as follows:

$$
\begin{aligned}
(2.6) \qquad & p_{S,T} = \frac{\frac{\lambda(S,T)}{\mu(S)}}{\sum_{U\sim S} \frac{\lambda(S,U)}{\mu(S)}} = \frac{\lambda(S,T)}{\sum_{U\sim S} \lambda(S,U)} \quad \text{if} \quad T \sim S, \\
& p_{S,T} = 0 \quad \text{if} \quad T \not\sim S.
\end{aligned}
$$

Thus the weight λ (but not μ) determines to which direction the cooperation is likely to progress. This allows us further flexibility for modeling stochastic cooperation network.

It turns out that the Markov chain (2.6) is *time-reversible*, meaning that there exists the stationary distribution $\pi = (\pi_S)_{S\in\Xi}$ such that

$$
(2.7) \qquad \pi_S p_{S,T} = \pi_T p_{T,S} \quad \text{for all} \quad S, T \in \Xi.
$$

A consequence, which is important to us, is that every loop and its reverse have the same probability, that is (see, e.g., Ross [9])

$$
(2.8) \qquad p_{S,S_1} p_{S_1,S_2} \cdots p_{S_{n-1},S_n} p_{S_n,S} = p_{S,S_n} p_{S_n,S_{n-1}} \cdots p_{S_2,S_1} p_{S_1,S}.
$$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space for the Markov chain. For each $S, T \in \Xi$ and $\omega \in \Omega$, let $\tau_{S,T} = \tau_{S,T}(\omega) \in \mathbb{N}_0$ denote the first (random) time the Markov chain $\big(X_n^S(\omega)\big)_n$ visits T. Given a player's contribution measure $f \in \ell^2(E)$, we define the total contribution of the player along the sample path $\omega \in \Omega$ traveling from S to T by

$$
(2.9) \qquad F_f^S(T) = F_f^S(T)(\omega) := \sum_{n=1}^{\tau_{S,T}(\omega)} f\big(X_{n-1}^S(\omega), X_n^S(\omega)\big).
$$

Now we can define the value function for given $f \in \ell^2(E)$ via the following stochastic path integral driven by the Markov chain (2.6)

$$V_f^S(T) := \int_\Omega F_f^S(T)(\omega) d\mathcal{P}(\omega) = \mathbb{E}[F_f^S(T)]. \tag{2.10}$$

Finally, let us denote $V_i^S := V_{f_i}^S$ for each player $i \in [N]$ given the players' contribution measures $(f_i)_{i\in[N]}$. One may notice that this path integral representation can be seen as a generalization of the Shapley formula (1.2). In particular, $V_i(T) := V_i^{\emptyset}(T)$ represents the expected total contribution the player i provides toward each cooperation T, provided the game starts at the null cooperation state $\emptyset$.

2.3. **The coincidence between the value allocation operator and the component game.** The question is how we can compute the value allocators $(V_i)_{i\in[N]}$ which are described by the stochastic path integral. One could employ some computational methods to simulate the Markov chain and approximate the path integral, for instance.

Or, better yet, our main result of this paper shows that V_i is a valid representation of the component game v_i, that is, $V_i = v_i$ for every player i. This result displays a remarkable connection between stochastic path integrals and combinatorial Hodge theory on general graphs.

Recall that the weight μ on $\ell^2(\Xi)$ is not relevant in either Lemma 2.2 or (2.6), so we will simply set $\mu \equiv 1$ from now on. First, the following lemma establishes a transition formula for the value function V_f.

Lemma 2.3. *Let (G, λ) be any connected weighted graph. For any $S, T, U \in \Xi$ and $f \in \ell^2(E)$, we have $V_f^U(T) - V_f^U(S) = V_f^S(T)$.*

Now we present our main result.

Theorem 2.4. *Let the Markov chain* (2.6) *be defined on each connected component of a weighted graph (G, λ). Then V_f solves the equation*

$$\mathrm{d}^*\mathrm{d}V_f = \mathrm{d}^* f \quad \text{for any} \quad f \in \ell^2(E). \tag{2.11}$$

The theorem tells us when one wants to calculate the value allocation function V_i for the player i given her contribution measure f_i, one can instead compute the least squares solution v_i, which can be easily done via least squares solvers for instance. Conversely, the least squares solution v_i may be approximated by simulating the canonical Markov chain (2.6) on the graph (G, λ) and calculating the contribution aggregator (2.10). Both directions look interesting and potentially useful.

2.4. **Examples and applications.** Let us begin by revisiting the famous *glove game* and the classical Shapley value, quoted from [16].

Example 2.5 (Glove game)**.** *Let $N = 3$, and suppose that player 1 has a left-hand glove, while players 2 and 3 each have a right-hand glove. The players wish to put together a pair of gloves, which can be sold for value 1, while unpaired gloves have no value. That is, $v(S) = 1$ if $S \subseteq N$ contains both a left and a right glove (i.e., player 1 and at least one of players 2 or 3) and $v(S) = 0$ otherwise. The Shapley values are*

$$\phi_1(v) = \frac{2}{3}, \qquad \phi_2(v) = \phi_3(v) = \frac{1}{6}.$$

This is easily seen from (1.2): player 1 contributes marginal value 0 when joining the coalition first (2 of 6 permutations) and marginal value 1 otherwise (4 of 6 permutations) , so $\phi_1(v) = \frac{2}{3}$. Efficiency and symmetry then yield $\phi_2(v) = \phi_3(v) = \frac{1}{6}$.

We present some new examples henceforth.

Example 2.6 (Glove game on an extended graph)**.** *In Shapley's classical coalition game setup, at each stage only one player can join the current coalition, and moreover no one can leave, as can be seen in the Shapley formula (1.2) and the corresponding hypercube graph (1.8).*

Now our general setup can free up these constraints. For example again let $N = 3$, consider the same value function v for the glove game, but now let the game graph $G = (V, E)$ be e.g. such that $V = 2^{[3]}$, and $(S, T) \in E$ iff $S \subsetneq T$. Thus in this setup multiple players can join or leave the coalition simultaneously, e.g., from $\{1\}$ to $\{1, 2, 3\}$ and conversely. But then what is the "$\mathrm{d}_i v$", the individual contribution for such a state transition? [16] sets this as in (1.9), which looks natural for the hypercube graph. But our framework allows for a complete freedom *in the choice of* $f_i = \mathrm{d}_i v$*. Here, for instance, for* $S \subsetneq T$*, we may set*

$$\mathrm{d}_i v\big(S,T\big) := \begin{cases} \frac{1}{|T|-|S|}\big(v(T)-v(S)\big) & \text{if } \ i \in T \setminus S, \\ 0 & \text{if } \ i \in S \end{cases} \tag{2.12}$$

with the usual convention $\mathrm{d}_i v\big(T,S\big) = -\mathrm{d}_i v\big(S,T\big)$*. Thus, in each transition, the surplus* $\mathrm{d}v(S,T) = v(T) - v(S)$ *is equally distributed to the newly incorporated players under this choice of* $\mathrm{d}_i v$*.*

Example 2.7 (Research paper writing game)**.** *We want to free up still another restriction in the classical cooperative game setup, namely, the state space for the game needs to be the* coalition space $2^{[N]}$*. Instead, in our setup, we can consider a general* cooperation state space Ξ*, which does not have to be related with the set of players* $[N]$*.*

To give an example, let Ξ *describe the research progress state space on which the game (reward)* $v : \Xi \to \mathbb{R}$ *is assigned, with the initial state* $\emptyset \in \Xi$ *and the research completion state* $F \in \Xi$*. Let* (G, λ) *be a given game graph with vertices in* Ξ*. Now we define the players contribution measure* $f_i = \mathrm{d}_i v$*, for each edge* $(S,T) \in E$*, by*

$$\mathrm{d}_i v(S,T) = \frac{1}{N}\big(v(T) - v(S)\big). \tag{2.13}$$

Thus, unconditionally, the surplus $\mathrm{d}v(S,T)$ *is equally distributed to all players involved in this game. Since the path integral of a gradient* ($\mathrm{d}v$) *depends only on the initial and terminal points, this clearly implies*

$$V_{\mathrm{d}v}(S) = v(S), \text{ and hence } \ V_i(S) = \frac{v(S)}{N} \ \text{ for all } \ i \in [N] \text{ and } S \in \Xi.$$

In particular $V_i(F) = v(F)/N$*, but not only that, for any research progression path* $\omega \in \Omega$ *towards* F*,* (2.13) *clearly yields*

$$\sum_{n=1}^{\tau_{\emptyset,F}(\omega)} \mathrm{d}_i v\big(X_{n-1}(\omega), X_n(\omega)\big) = \frac{v(F)}{N}$$

meaning that the reward is deterministic and not stochastic.[2]

[2]In managerial perspective, this can cause problem since players may heavily procrastinate. But there is still the driving force from the fact that the research paper writing game v is typically a *pure bargaining game*: $v(S) = 1$ if $S = F$, otherwise 0.

Lastly, we present an application in financial decision problem.

Example 2.8 (Entrepreneur's revenue problem)**.** *In this example let Ξ be the project state space, in which the manager wants to reach the project completion state $F \in \Xi$. The game value $v(U)$ is the manager's revenue if the project ends up in the state U. However at each transition from S to T, the manager has to pay $f_i(S,T)$ to the employee i, since it is her contribution and share. Thus, in this single transition, the manager's surplus is $v(T) - v(S) - \sum_i f_i(S,T)$, which can be positive or negative. Thus we do not impose the efficiency condition* $\mathrm{d}v = \sum_i f_i$ *here, thereby freeing up still another restriction. Moreover, notice that f_i needs not take the form* $\mathrm{d}_i v$*, i.e., f_i needs not depend on the game v.*

Now the manager's revenue problem is, when they start at the initial project state (say $\emptyset$) and if the manager's goal is reaching the project completion state F, what is the expected revenue for the manager?

Observe the answer is $v(F) - \sum_i V_i(F)$, where V_i is defined by the stochastic integral given contribution measures $(f_i)_i$ as in (2.10)*. (So if this is negative, the manager may not want to start the project at all.)*

Moreover the manager may want to recalculate her expected gain or loss in the middle of the project progress. That is, say the current project status is T, and they have come to T from $\emptyset$ through a certain path ω, and thus the manager has paid the payoffs – the path integrals – (2.9) *to the employees. Now the manager may want to calculate the expected gain if she decides to further go on from T to F. This is now given by*

$$v(F) - v(T) - \sum_i V_i^T(F),$$

and the manager can make decisions based on these expected revenue information. And for this, the message of our theorem is that the stochastic integrals V_i^T can be alternatively calculated by solving the least squares equations (2.11)*, and vice versa.*

3. Conclusion

In this paper we reviewed the cooperative game framework of Shapley [12, 13] and its Hodge-theoretic extension by Stern and Tettenhorst

[16] and Lim [7]. These papers regard the cooperative games as value functions on $2^{[N]}$, and [7, 16] consider the differential operators d, d_i defined on the hypercube graph (1.8). Then we proposed that the cooperative games may be defined in a much more general framework of arbitrary weighted game graphs $G = (\Xi, E)$, in which the partial differential d_i is replaced by a general contribution measure $f_i \in \ell^2(E)$. Given f_i for each player i, we proposed a natural value allocation operator V_i, a stochastic path integral driven by the canonical, time-reversible Markov chain on the weighted graph. Then in Theorem 2.4, we verified an intriguing connection of this stochastic integral with the "component game" v_i, which is the solution to the least squares equation (2.5) inspired by the Hodge decomposition (1.3). Now if the efficiency condition $\sum_i f_i = \mathrm{d}v$ holds for a given cooperative game v, then in view of Proposition 2.1, $V_i = v_i$ may be interpreted as a fair and efficient allocation of the cooperation value $v(S)$ to the player i at the cooperation state S, which may be seen as a generalization of the Shapley's value allocation formula (1.2). However, as illustrated in Example 2.8, freeing up the efficiency condition allows us to cover even broader range of problems in economics, finance and other social and physical sciences.

4. Proofs

In the following proof of Lemma 2.3 and Theorem 2.4, one may focus on the fact that the key is the reversibility (2.8) of the Markov chain.

Proof of Lemma 2.3. We first prove a special case $V_f^S(T) = -V_f^T(S)$. For this, consider a general sample path ω of the Markov chain (2.6) starting at S, visiting T, then returning to S. We could divide this journey into four stages:

ω_1: the path returns to S $m \in \mathbb{N}_0$ times while not visiting T yet,
ω_2: the path starts at S and ends at T while not returning to S,
ω_3: the path returns to T $n \in \mathbb{N}_0$ times while not visiting S yet,
ω_4: the path starts at T and ends at S while not returning to T.

Thus $\omega = \omega_1 \circ \omega_2 \circ \omega_3 \circ \omega_4$ is the concatenation of the ω_i's, and the probability of this finite sample path satisfies $\mathcal{P}(\omega) = \mathcal{P}(\omega_1)\mathcal{P}(\omega_2)\mathcal{P}(\omega_3)\mathcal{P}(\omega_4)$.

Now consider a pairing ω' of ω as follows: let ω_1^{-1} be the reversed path of ω_1, that is, if ω_1 visits $T_0 \to T_1 \to \cdots \to T_k$ (where $T_0 = T_k = S$ for ω_1), then ω_1^{-1} visits $T_k \to \cdots \to T_0$. Recall $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1^{-1})$ due to the time-reversibility (2.8). Now define $\omega' := \omega_1^{-1} \circ \omega_2 \circ \omega_3^{-1} \circ \omega_4$. This is another general sample path starting at S, visiting T, then returning to S. We have $\mathcal{P}(\omega) = \mathcal{P}(\omega')$, and moreover,

$$F_f^S(T)(\omega) + F_f^S(T)(\omega') = 2 \sum_{n=1}^{\tau_{S,T}(\omega_2)} f\big(X_{n-1}^S(\omega_2), X_n^S(\omega_2)\big),$$

since the loop ω_1 and its reverse ω_1^{-1} aggregate f with opposite sign, so they cancel out in the above sum. Now consider $\tilde{\omega} := \omega_3 \circ \omega_2^{-1} \circ \omega_1 \circ \omega_4^{-1}$ and $\tilde{\omega}' := \omega_3^{-1} \circ \omega_2^{-1} \circ \omega_1^{-1} \circ \omega_4^{-1}$. $(\tilde{\omega}, \tilde{\omega}')$ then represents a pair of general sample paths starting at T, visiting S, then returning to T. Moreover

$$\begin{aligned} F_f^T(S)(\tilde{\omega}) + F_f^T(S)(\tilde{\omega}') &= 2 \sum_{n=1}^{\tau_{T,S}(\omega_2^{-1})} f\big(X_{n-1}^T(\omega_2^{-1}), X_n^T(\omega_2^{-1})\big) \\ &= -2 \sum_{n=1}^{\tau_{S,T}(\omega_2)} f\big(X_{n-1}^S(\omega_2), X_n^S(\omega_2)\big) \\ &= -(F_f^S(T)(\omega) + F_f^S(T)(\omega')) \end{aligned}$$

since $f \in \ell^2(E)$. Due to the generality of the pair (ω, ω') and its counterpart $(\tilde{\omega}, \tilde{\omega}')$, and $\mathcal{P}(\omega) = \mathcal{P}(\omega') = \mathcal{P}(\tilde{\omega}) = \mathcal{P}(\tilde{\omega}')$ from the reversibility (2.8), the desired identity $V_f^S(T) = -V_f^T(S)$ follows by integration.

Now to show $V_f^U(T) - V_f^U(S) = V_f^S(T)$, we proceed as in [7]:

$$\begin{aligned} F_f^U(T) - F_f^U(S) &= \sum_{n=1}^{\tau_{U,T}} f\big(X_{n-1}^U, X_n^U\big) - \sum_{n=1}^{\tau_{U,S}} f\big(X_{n-1}^U, X_n^U\big) \\ &= \mathbf{1}_{\tau_{U,S}<\tau_{U,T}} \sum_{n=\tau_{U,S}+1}^{\tau_{U,T}} f\big(X_{n-1}^U, X_n^U\big) - \mathbf{1}_{\tau_{U,T}<\tau_{U,S}} \sum_{n=\tau_{U,T}+1}^{\tau_{U,S}} f\big(X_{n-1}^U, X_n^U\big). \end{aligned}$$

By taking expectation, we obtain via the Markov property

$$\begin{aligned}
&\mathbb{E}[F_f^U(T)] - \mathbb{E}[F_f^U(S)] \\
&= \mathcal{P}(\{\tau_{U,S} < \tau_{U,T}\})V_f^S(T) - \mathcal{P}(\{\tau_{U,T} < \tau_{U,S}\})V_f^T(S) \\
&= V_f^S(T)
\end{aligned}$$

which proves the transition formula $V_f^U(T) - V_f^U(S) = V_f^S(T)$. $\square$

Proof of Theorem 2.4. We set $\mu \equiv 1$ without loss of generality. Given $f \in \ell^2(E)$, our aim is to show that V_f solves (2.11). Let $S \in \Xi$, and let $\{T_1, ..., T_n\}$ be the set of all vertices adjacent to S (i.e., either (S, T_k) or (T_k, S) is in E), and set $\Lambda_S = \sum_{k=1}^n \lambda(S, T_k)$. Then by (2.4), (2.6),

$$\mathrm{d}^* f(S)/\Lambda_S = \sum_{k=1}^n p_{S,T_k} f(T_k, S), \text{ and} \tag{4.1}$$

$$\mathrm{d}^*\mathrm{d}V_f(S)/\Lambda_S = \sum_{k=1}^n p_{S,T_k}\big(V_f(S) - V_f(T_k)\big) = \sum_{k=1}^n p_{S,T_k} V_f^{T_k}(S) \tag{4.2}$$

where the last equality is from Lemma 2.3. Now observe that we can interpret (4.2) as the aggregation (2.10) of path integrals of f (2.9) for all loops starting and ending at S, but in this aggregation of f we do not take into account the first move from S to T_k, since this first move is made by the transition rate p_{S,T_k} and not driven by $V_f^{T_k}$. On the other hand, if we aggregate path integrals of f for all loops emanating from S, we get zero due to the reversibility (2.8). Hence we conclude:

$$\begin{aligned}
0 &= \text{aggregation of path integrals of } f \text{ for all loops emanating from } S \\
&= \text{aggregation of path integrals of } f \text{ for all loops, omitting the first moves} \\
&\quad + \text{aggregation of path integrals of } f \text{ for all first moves from } S \\
&= \sum_{k=1}^n p_{S,T_k} V_f^{T_k}(S) + \sum_{k=1}^n p_{S,T_k} f(S, T_k) \\
&= \mathrm{d}^*\mathrm{d}V_f(S)/\Lambda_S - \mathrm{d}^* f(S)/\Lambda_S,
\end{aligned}$$

yielding $\mathrm{d}^*\mathrm{d}V_f(S) = \mathrm{d}^* f(S)$ for all $S \in \Xi$. This completes the proof. $\square$

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