OPTIMAL EXERCISE DECISION OF AMERICAN OPTIONS UNDER MODEL UNCERTAINTY

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Abstract. Given the marginal distribution information of the underlying asset price at two future times T_1 and T_2, we consider the problem of determining a model-free upper bound on the price of a class of American options that must be exercised at either T_1 or T_2. The model uncertainty consistent with the given marginal information is described as the martingale optimal transport problem. We show that if the American option payoff satisfies a suitable convexity condition, then any option exercise scheme associated with any market model that jointly maximizes the expected option payoff must be nonrandomized.

Keywords: Robust finance, American option, Hedging, Martingale, Optimal transport, Duality, Dual attainment, Infinite-dimensional linear programming
MSC2010 Classification: 90Bxx, 90Cxx, 49Jxx, 49Kxx, 60Dxx, 60Gxx

1. Introduction

This paper was mainly inspired by Hobson and Norgilas [15], Aksamit, Deng, Obłój and Tan [1], as well as Beiglböck and Juillet [5] and Beiglböck, Nutz and Touzi [6]. A related problem in continuous time setup was studied in Bayraktar, Cox and Stoev [2]. We consider two future times $0 < T_1 < T_2$ and an asset price process (X, Y), where X, Y represents the asset price at time T_1, T_2, respectively. Let $\mathcal{P}(\mathcal{X})$ denote the set of all probability measures/distributions over a set $\mathcal{X}$ with finite first moment. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ be probability measures in convex order:

$$\mu \preceq_c \nu \quad \text{if} \quad \mu(f) \leq \nu(f) \quad \text{for every convex function } f \text{ on } \mathbb{R},$$

where $\mu(f) := \mathbb{E}_\mu[f(X)] = \int f(x)\mu(dx)$. We consider market models that are defined by the following set of martingale transports from μ to ν:

$$\mathcal{M}(\mu,\nu) = \{\pi \in \mathcal{P}(\mathbb{R}^2)\,|\,\pi = \mathrm{Law}(X,Y), \mathbb{E}_\pi[Y|X] = X, \mathrm{Law}(X) = \mu, \mathrm{Law}(Y) = \nu\}.$$

Date: October 19, 2023.

In finance, each $\pi \in \mathcal{M}(\mu,\nu)$ represents a feasible joint law of the price (X,Y) given the marginal information μ,ν in the (two-period) market, under which (X,Y) is a martingale, written as $\mathbb{E}_\pi[Y|X] = X$. It is well known that the condition $\mu \preceq_c \nu$ is equivalent to $\mathcal{M}(\mu,\nu) \neq \emptyset$. We refer to [7, 8, 10, 11] for further background.

[15] considered the cost function which describes an American option payoff

$$c = (c_1, c_2) = (c_1(x), c_2(x,y)), \quad c_1, c_2 \in \mathbb{R}, \tag{1.1}$$

such that if an obligee (option holder) selects c_1, she receives the payout $c_1(X)$, otherwise she receives the payout $c_2(X,Y)$. Thus, in the former case, her payout is determined at time 1, whereas it is determined at time 2 in the latter. We assume she can make this choice conditional on the price $X = x$, and that she can also randomize her choice, which is represented by a Borel function $s : \mathbb{R} \to [0,1]$, which means that given $X = x$, she chooses c_1 with probability $s(x)$, otherwise c_2 with probability $1 - s(x)$. Given a function $s : \mathbb{R} \to \mathbb{R}$ and a measure μ on $\mathbb{R}$, define a measure $s\mu$ by $s\mu(B) = \int_B s(x)\mu(dx)$. Since μ is fixed, the choice of a randomization s is equivalent to the choice of $0 \le \mu_1 \le \mu$,[1] such that $\mu_1 + \mu_2 = \mu$ with $\mu_2 := \mu - \mu_1$, thus $s_1 := s$, $s_2 := 1 - s$ equals the Radon–Nikodym derivative $\frac{d\mu_1}{d\mu}, \frac{d\mu_2}{d\mu}$ μ-a.s., respectively. This leads us to consider the optimization problem

$$P_c := \sup_{\pi \in \mathcal{M}(\mu,\nu)} \sup_{\mu_1 \le \mu} \; \mathbb{E}_{\gamma_1}[c_1] + \mathbb{E}_{\gamma_2}[c_2], \tag{1.2}$$

where for a given $\pi = \pi_x \otimes \mu \in \mathcal{M}(\mu,\nu)$,[2] we define $\gamma_l = \pi_x \otimes \mu_l$, $l = 1, 2$, such that $\gamma_1 + \gamma_2 = \pi$ and that γ_1 and γ_2 share the same kernel $\{\pi_x\}_x$ inherited from π.

In view of the obligor (the person responsible for the payment of the option), a solution (π, μ_1) to (1.2) represents a worst-possible market scenario π combined with the option exercise scheme μ_1, yielding the maximum expected payout P_c.

We will assume the following regularity condition on c throughout the paper.

[A] Throughout the paper, we assume that c_1, c_2 are continuous, $\mu \preceq_c \nu$, and that the marginals μ,ν satisfy the following condition: there exist continuous functions

[1] All measures/distributions in this paper are assumed to be non-negative.

[2] Any $\pi = \mathrm{Law}(X,Y) \in \mathcal{P}(\mathbb{R}^2)$, representing the joint law of the random variables X and Y, can be written as $\pi = \pi_x \otimes \mathrm{Law}(X)$, where $\pi_x \in \mathcal{P}(\mathbb{R})$ is called a kernel of π with respect to $\mathrm{Law}(X)$. π_x represents the conditional distribution of Y given $X = x$, i.e., $\pi_x(B) = \mathcal{P}(Y \in B \,|\, X = x)$ for all Borel set $B \subseteq \mathbb{R}$. Note that $\pi = \pi_x \otimes \mu \in \mathcal{M}(\mu,\nu)$ iff $\int y\, \pi_x(dy) = x$ μ-a.e. x.

$v \in L^1(\mu)$, $w \in L^1(\nu)$ such that $|c_1| + |c_2| \le v(x) + w(y)$. Note that this implies $\big|\sum_l \mathbb{E}_{\gamma_l}[c_l]\big| \le \sum_l \mathbb{E}_{\gamma_l}[|c_l|] \le \sum_l \mathbb{E}_\pi[|c_l|] \le \mu(v) + \nu(w) < \infty$ for any $\pi \in \mathcal{M}(\mu,\nu)$. This in turn implies that the problem (1.2) is attained (i.e., admits an optimizer) by a standard argument in the calculus of variations [19].

[15] considered a specific cost called an American put, whose payoff is given by

$$c_1(x) = (K_1 - x)^+, \quad c_2(x,y) = c_2(y) = (K_2 - y)^+, \quad K_1 > K_2, \tag{1.3}$$

and considered those option exercise schemes which are *pure*, or *non-randomized*; that is, [15] assumed that the obligee can only choose a Borel set $B \subseteq \mathbb{R}$ in which she selects c_1 if $x \in B$ and c_2 otherwise. In terms of μ_1, notice that this is equivalent to the statement that μ_1 and μ_2 are mutually disjoint, written as $\mu_1 \perp \mu_2$ (while $\mu_1 + \mu_2 = \mu$). In other words, [15] assumed that μ_1, μ_2 must saturate μ on their respective supports. In addition, [15] assumed that μ is continuous, i.e., has no atoms. Under these assumptions, [15] showed that an optimal market model π for the problem (1.2) is given by the *left-curtain coupling* (see [5, 12, 15] for more details about this interesting martingale transport) along with an optimal exercise strategy B, and furthermore, the cheapest superhedge can be derived.

Now we'd like to shift our focus and ask, "Under what conditions must the optimal option exercise be pure?" That is, when will an optimal μ_1 saturate μ, or equivalently, achieve $\mu_1 \perp \mu_2$? Note that the problem (1.2) can be rewritten as

$$P_c = \sup_{\mu_1 \le \mu} P_c(\mu_1), \quad \text{where} \quad P_c(\mu_1) := \sup_{\pi \in \mathcal{M}(\mu,\nu)} \mathbb{E}_{\gamma_1}[c_1] + \mathbb{E}_{\gamma_2}[c_2], \tag{1.4}$$

where $\gamma_l = \pi_x \otimes \mu_l$, $l = 1, 2$. As a result, we break down the problem (1.2), which has a non-convex domain,[3] into subproblems $P_c(\mu_1)$ in which (μ_1, μ_2) is fixed, and as a result, has a convex subdomain. We now present our main result.

Theorem 1.1. *Assume* [**A**]. *Suppose that $c_2(x,y)$ is strictly convex in y, $c_1(x) \neq c_2(x,x)$ μ-a.s., and ν is absolutely continuous with respect to the Lebesgue measure. Then every solution (π, μ_1) to the problem* (1.4) *satisfies $\mu_1 \perp \mu - \mu_1$.*

[3] A feasible pair (π, μ_1) in (1.2) can be seen equivalent to the corresponding (π_1, π_2). Now even if (π_1, π_2), (π_1', π_2') are feasible (i.e., sharing the same kernel respectively), the convex combination $(\frac{\pi_1+\pi_1'}{2}, \frac{\pi_2+\pi_2'}{2})$ may not share the same kernel thus infeasible, unless $\mu_1 = \mu_1'$ and $\mu_2 = \mu_2'$.

We note that the condition $c_1(x) > c_2(x,x)$ is natural because, if $c_1(x) \le c_2(x,x)$ and c_2 is convex in y, it is always optimal to choose $c_2(x,y)$ by Jensen's inequality $c_2(x,x) \le \int c_2(x,y)\pi_x(dy)$. Theorem 1.1 says that in this case, every optimal exercise, or stopping, is nonrandomized. Evidently, the problem (1.2) can be viewed as an optimal stopping problem, in which the option holder either stops at time 1 and receives the sure reward $c_1(x)$, or goes and receives the reward $c_2(x,y)$ (which is stochastic at time 1) at time 2. This naturally places the theorem in the context of the vast literature on the Skorokhod embedding problem [4,13,18], with the key difference that we now face uncertainty in the family of models $\mathcal{M}(\mu,\nu)$. We note that such model uncertainty was also considered in [2,9] in continuous time setup.

We also note that in the optimal transport literature, the absolute continuity of μ is typically assumed in order to derive non-randomizing solutions, known as Monge solutions. Continuity of μ was also assumed in [15]. In contrast, Theorem 1.1 assumes the absolute continuity of ν, while making no assumptions about μ.

The remainder of the paper is structured as follows. The theorem will be proved utilizing three propositions, two of which are related to duality (no duality gap), and one of which is a dual attainment result. They will be discussed in Section 2. Section 3 then presents proofs of the results.

2. Duality

In this section, we consider cost functions more general than (1.1), such as

$$\vec{c} = (c_1, c_2, ..., c_L),\ \ c_l = c_l(x,y) \in \mathbb{R},\ \ l = 1,2,...,L. \tag{2.1}$$

We assume c_l are continuous, and $\sum_{l=1}^{L} |c_l(x,y)| \le v(x) + w(y)$ for some continuous functions $v \in L^1(\mu)$, $w \in L^1(\nu)$. Let $\vec{\mu} = (\mu_1, ..., \mu_L)$ satisfy $\sum_{l=1}^{L} \mu_l = \mu$. We can generalize the subproblem in (1.4) into the following

$$P_c(\vec{\mu}) := \sup_{\pi \in \mathcal{M}(\mu,\nu)} \sum_{l=1}^{L} \mathbb{E}_{\gamma_l}[c_l], \quad \text{where} \quad \gamma_l = \pi_x \otimes \mu_l. \tag{2.2}$$

Financially, the option holder has the right of selecting one of the payouts c_l, which can be randomized depending on x, as described by the probability vector $\big(\frac{d\mu_l}{d\mu}(x)\big)$.

The problem (2.2) has a natural dual problem. To describe, let $s_l : \mathbb{R} \to [0,1]$, $\sum_{l=1}^{L} s_l(x) = 1$, be a version of the densities $\frac{d\mu_l}{d\mu}$. Then (2.2) can be rewritten as

$$P_c(\vec{\mu}) = \sup_{\pi \in \mathcal{M}(\mu,\nu)} \mathbb{E}_\pi[\textstyle\sum_l s_l c_l], \tag{2.3}$$

where $s_l c_l(x,y) = s_l(x) c_l(x,y)$. Regarding $\sum_l s_l c_l$ as the cost function, a well known dual problem of (2.3) is given by

$$\inf_{(\varphi,\psi,\theta)} \mu(\varphi) + \nu(\psi), \tag{2.4}$$

where $\varphi \in C(\mathbb{R}) \cap L^1(\mu)$, $\psi \in C(\mathbb{R}) \cap L^1(\nu)$, $\theta \in C_b(\mathbb{R})$, that satisfies $\sum_l s_l c_l(x,y) \leq \varphi(x) + \psi(y) + \theta(x)(y-x)$, where $C(\mathcal{X})$, $C_b(\mathcal{X})$ denotes the space of all continuous (or continuous and bounded) functions on $\mathcal{X}$. Then it is known that the values (2.3) and (2.4) coincide (see [5,6].) Nonetheless, the dual formulation (2.4) turns out to be not so helpful for our needs, prompting us to consider the following alternative. Define Ψ_c to be the space of functions $(\vec{\varphi}, \psi, \vec{\theta}) = (\varphi_1, ..., \varphi_L, \psi, \theta_1, ..., \theta_L)$ such that $\varphi_l \in C(\mathbb{R}) \cap L^1(\mu)$, $\psi \in C(\mathbb{R}) \cap L^1(\nu)$, $\theta_l \in C_b(\mathbb{R})$, and that

$$c_l(x,y) \leq \varphi_l(x) + \psi(y) + \theta_l(x)(y-x) \quad \text{for all } l = 1, ..., L \text{ and } (x,y) \in \mathbb{R}^2. \tag{2.5}$$

We then consider the following problem

$$D_c(\vec{\mu}) := \inf_{(\vec{\varphi},\psi,\vec{\theta}) \in \Psi_c} \nu(\psi) + \sum_l \mu_l(\varphi_l). \tag{2.6}$$

The inequality $P_c(\vec{\mu}) \leq D_c(\vec{\mu})$ is immediate by integrating (2.5) by γ_l and summing up in l. We will show the following duality result for the problem (2.2).

Proposition 2.1. *$P_c(\vec{\mu}) = D_c(\vec{\mu})$.*

We turn to the second duality result. As noted, the domain of the problem (1.2), in terms of the variable (γ_1, γ_2), is nonconvex. This leads us to consider a relaxed problem for (1.2); see also [1] for related results. Let $\mathcal{M} := \cup_{\mu \preceq_c \nu} \mathcal{M}(\mu,\nu)$, that is, $\mathcal{M}$ is the set of all martingale transports between some probability marginals in convex order, hence $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^2)$. Let $\overline{\mathcal{M}}$ be the set of all martingale transports with arbitrary nonnegative finite total mass, that is, $\gamma \in \overline{\mathcal{M}}$ if $\gamma \equiv 0$ or $\gamma/||\gamma|| \in \mathcal{M}$ where $||\gamma|| = \int_{\mathbb{R}^2} \gamma(dx,dy) \in (0,\infty)$ denotes the total mass. Define

$$\mathcal{M}_L(\mu,\nu) := \left\{ \vec{\gamma} = (\gamma_1, ..., \gamma_L) \;\middle|\; \sum_{l=1}^{L} \gamma_l \in \mathcal{M}(\mu,\nu) \text{ and } \gamma_l \in \overline{\mathcal{M}} \text{ for all } l = 1, ..., L. \right\}$$

$\mathcal{M}_L(\mu,\nu)$ is clearly convex. Now we define the relaxed problem

$$\overline{P}_c := \sup_{\vec{\gamma}\in\mathcal{M}_L(\mu,\nu)} \sum_{l=1}^{L} \mathbb{E}_{\gamma_l}[c_l]. \tag{2.7}$$

The difference is that in (1.2) (with the generalized cost (2.1)), $\{\gamma_l\}_l$ are assumed to have the same kernel π_x inherited from a model $\pi \in \mathcal{M}(\mu,\nu)$, whereas in (2.7), this restriction is relaxed. Both problems satisfy the condition $\sum_l \gamma_l \in \mathcal{M}(\mu,\nu)$. Hence, $P_c \le \overline{P}_c$. We will show $P_c = \overline{P}_c$. This can be best seen via duality associated to (2.7). Define $\overline{\Psi}_c$ to be the space of functions $(\varphi,\psi,\vec{\theta}) = (\varphi,\psi,\theta_1,...,\theta_L)$ such that $\varphi \in C(\mathbb{R})\cap L^1(\mu)$, $\psi \in C(\mathbb{R})\cap L^1(\nu)$, $\theta_l \in C_b(\mathbb{R})$, satisfying

$$c_l(x,y) \le \varphi(x) + \psi(y) + \theta_l(x)(y-x) \quad \text{for all } l = 1,...,L \text{ and } (x,y)\in\mathbb{R}^2. \tag{2.8}$$

The difference is that in (2.5) the functions φ_l can be distinct, whereas in (2.8), there is only one φ. The dual problem to (2.7) is now given by

$$\overline{D}_c := \inf_{(\varphi,\psi,\vec{\theta})\in\overline{\Psi}_c} \mu(\varphi) + \nu(\psi). \tag{2.9}$$

The second duality result is the following; see also [1].

Proposition 2.2. *$P_c = \overline{P}_c = \overline{D}_c$.*

For the financial meaning of the dual problems in terms of American option superhedging, we refer to [1,3,14,15,17]. The final element required to prove Theorem 1.1 is the dual attainment result, which asserts that there is an appropriate solution to the dual problem (2.9). For $\xi \in \mathcal{P}(\mathbb{R})$ with finite first moment, its potential function is defined by $u_\xi(x) := \int |x-y| d\xi(y)$. Then we say that a pair of probabilities $\mu \preceq_c \nu$ in convex order is irreducible if the set $I := \{x\in\mathbb{R} \,|\, u_\mu(x) < u_\nu(x)\}$ is a connected (open) interval containing the full mass of μ, i.e., $\mu(I) = \mu(\mathbb{R})$.

Proposition 2.3. *Let $\mu \preceq_c \nu$ be an irreducible pair of marginals in $\mathcal{P}(\mathbb{R})$. Assume* [**A**]. *Then there exists a* dual optimizer $(\varphi,\psi,\vec{\theta})$, $\varphi,\psi : \mathbb{R}\to\mathbb{R}\cup\{+\infty\}$, $\theta_l : \mathbb{R}\to\mathbb{R}$, *that satisfies* (2.8) *tightly in the following pathwise sense (but needs not be in $\overline{\Psi}_c$):*

$$c_l(x,y) = \varphi(x) + \psi(y) + \theta_l(x)(y-x) \quad \gamma_l - a.e., \quad \textit{for all } l = 1,...,L \tag{2.10}$$

for every solution $\vec{\gamma} = (\gamma_1,...,\gamma_L)$ to the problem (2.7)*, and hence,* (1.2).

We emphasize that $(\varphi,\psi,\vec{\theta})$ may not be in $\overline{\Psi}_c$ but are only measurable, with φ,ψ real-valued μ,ν-a.s., respectively. They need not be integrable nor continuous.

3. Proofs

Proof of Proposition 2.1. Let $\mathcal{N}$ be the set of all nonnegative measures on $\mathbb{R}^2$ (that do not need to be martingales.) For $\gamma \in \mathcal{N}$, let γ^X, γ^Y denote its marginal on the x, y-coordinate respectively. Below, $\gamma_l = \pi_x \otimes \mu_l$, $s_l = s_l(x)$ is a nonnegative Borel function on $\mathbb{R}$, and $(\vec{\varphi}, \psi, \vec{\theta}) \in \Psi_c$. We claim that the following equalities hold:

$$
\begin{aligned}
P_c(\vec{\mu}) &= \sup_{\pi \in \mathcal{M}(\mu,\nu)} \textstyle\sum_{l=1}^L \gamma_l(c_l) \\
&= \sup_{\pi\in\mathcal{N}} \sup_{s_1,\dots,s_L} \inf_{(\vec{\varphi},\psi,\vec{\theta})} \textstyle\sum_l s_l\pi(c_l) + \sum_l (\mu_l - (s_l\pi)^X)(\varphi_l) + (\nu - \sum_l (s_l\pi)^Y)(\psi) - \sum_l s_l\pi(\theta_l(x)(y-x)) \\
&= \sup_{\pi\in\mathcal{N}} \inf_{(\vec{\varphi},\psi,\vec{\theta})} \sup_{s_1,\dots,s_L} \textstyle\sum_l \mu_l(\varphi_l) + \nu(\psi) + \sum_l s_l\pi(c_l(x,y) - \varphi_l(x) - \psi(y) - \theta_l(x)(y-x)) \\
&= \sup_{\pi\in\mathcal{N}} \inf_{c_l(x,y)\le\varphi_l(x)+\psi(y)+\theta_l(x)(y-x)\,\forall l \text{ on } \operatorname{spt}(\pi)} \textstyle\sum_l \mu_l(\varphi_l) + \nu(\psi) \\
&= \inf_{c_l(x,y)\le\varphi_l(x)+\psi(y)+\theta_l(x)(y-x)\,\forall l} \textstyle\sum_l \mu_l(\varphi_l) + \nu(\psi) = D_c(\vec{\mu}).
\end{aligned}
$$

The derivation of the equalities is fairly standard: the second equality holds because the infimum achieves $-\infty$ as soon as $(s_l\pi)^X \neq \mu_l$, $\sum_l (s_l\pi)^Y \neq \nu$, or $s_l\pi \neq \overline{\mathcal{M}}$, thus, the collection of measures $(s_l\pi)_l$ must observe the constraint imposed on $(\gamma_l)_l$ in the problem $P_c(\vec{\mu})$. The third equality is based on a standard minimax theorem, which asserts that the equality holds when the sup and inf are swapped. Because the objective function is bilinear, i.e., linear in each variable ($(s_l)_l$ and $(\vec{\varphi}, \psi, \vec{\theta})$), the minimax theorem holds in this case and we omit the detail. The fourth equality is because, if $c_l(x,y) - \varphi_l(x) - \psi(y) - \theta_l(x)(y-x) > 0$ for some $(x, y) \in \operatorname{spt}(\pi)$, one can select s_l such that the last supremum in the third line achieves $+\infty$, which is not helpful to achieve the middle infimum. Finally, the supremum in the fourth line is obviously achieved for those π with $\operatorname{spt}(\pi) = \mathbb{R}^2$, completing the proof. $\square$

Proof of Proposition 2.2. Let $(\varphi, \psi, \vec{\theta}) \in \overline{\Psi}_c$. The following equalities hold:

$$
\begin{aligned}
\overline{P}_c &= \sup_{\vec{\gamma}\in\mathcal{M}_L(\mu,\nu)} \sum_{l=1}^L \mathbb{E}_{\gamma_l}[c_l] \\
&= \sup_{\gamma_l\in\mathcal{N}\,\forall l} \inf_{(\varphi,\psi,\vec{\theta})} \textstyle\sum_l \gamma_l(c_l) + (\mu - \sum_l \gamma_l^X)(\varphi) + (\nu - \sum_l \gamma_l^Y)(\psi) - \sum_l \gamma_l(\theta_l(x)(y-x)) \\
&= \inf_{(\varphi,\psi,\vec{\theta})} \sup_{\gamma_l\in\mathcal{N}\,\forall l} \mu(\varphi) + \nu(\psi) + \textstyle\sum_l \gamma_l(c_l(x,y) - \varphi(x) - \psi(y) - \theta_l(x)(y-x)) \\
&= \inf_{c_l(x,y)\le\varphi(x)+\psi(y)+\theta_l(x)(y-x)\,\forall l} \mu(\varphi) + \nu(\psi) = \overline{D}_c.
\end{aligned}
$$

The derivation is simpler than that of Proposition 2.1 and we omit the detail. Let $\Xi = \{\vec{\mu} = (\mu_1, ..., \mu_L) \,|\, \mu_l$ is a nonnegative measure on $\mathbb{R}$, and $\sum_l \mu_l = \mu\}$. Below, $\overline{\varphi} := \max(\varphi_1, \varphi_2, ..., \varphi_L)$. Now to show $P_c = \overline{P}_c$, we can proceed

$$\begin{aligned} P_c &= \sup_{\vec{\mu} \in \Xi} P_c(\vec{\mu}) = \sup_{\vec{\mu} \in \Xi} D_c(\vec{\mu}) \\ &= \sup_{\vec{\mu} \in \Xi} \inf_{(\vec{\varphi}, \psi, \vec{\theta}) \in \Psi_c} \nu(\psi) + \sum_l \mu_l(\varphi_l) \\ &= \inf_{(\vec{\varphi}, \psi, \vec{\theta}) \in \Psi_c} \sup_{\vec{\mu} \in \Xi} \nu(\psi) + \sum_l \mu_l(\varphi_l) \\ &= \inf_{(\vec{\varphi}, \psi, \vec{\theta}) \in \Psi_c} \nu(\psi) + \mu(\overline{\varphi}) \\ &= \inf_{(\varphi, \psi, \vec{\theta}) \in \overline{\Psi}_c} \nu(\psi) + \mu(\varphi) = \overline{D}_c = \overline{P}_c, \end{aligned}$$

where it is clear that $\sup_{\vec{\mu} \in \Xi} \sum_l \mu_l(\varphi_l) = \mu(\overline{\varphi})$, so is the equality in the last line. $\square$

Proof of Proposition 2.3. The proof mostly consist of extending the ideas in [5, 6] to the vectorial cost (2.1). We will follow the five steps illustrated in [16], thereby omitting some details here but referring to the corresponding steps in [16].

Step 1. $\sum_{l=1}^L |c_l(x, y)| \le v(x) + w(y)$ for some continuous functions $v \in L^1(\mu)$, $w \in L^1(\nu)$. A dual optimizer exists for $\vec{c}$ iff so does for $\tilde{c} := (c_l(x, y) + v(x) + w(y))_l$. Thus by replacing $\vec{c} = (c_1, ..., c_L)$ with $\tilde{c}$, from now on we assume $c_l \ge 0$ for all l.

As $\overline{P}_c = \overline{D}_c \in \mathbb{R}$, we can find an approximating dual optimizer $(\varphi_n, \psi_n, \theta_{l,n}) \in \overline{\Psi}_c$, $n \in \mathbb{N}$, such that the following duality holds (for all $l = 1, ..., L$):

$$\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y - x) \ge c_l(x, y) \ge 0, \tag{3.1}$$

$$\mu(\varphi_n) + \nu(\psi_n) \searrow \overline{P}_c \quad \text{as} \quad n \to \infty. \tag{3.2}$$

Define $f_n = -\varphi_n$, $h_{l,n} = -\theta_{l,n}$, so that (3.1) becomes

$$f_n(x) + h_{l,n}(x)(y - x) \le \psi_n(y) - c_l(x, y) \le \psi_n(y). \tag{3.3}$$

Define the convex functions

$$\chi_{l,n}(y) := \sup_{x \in \mathbb{R}} f_n(x) + h_{l,n}(x)(y - x), \quad \chi_n := \sup_{l=1,...,L} \chi_{l,n}. \tag{3.4}$$

Notice $\chi_{l,n}(y) \ge f_n(y) + h_{l,n}(y)(y - y) = f_n(y)$ for all $y \in \mathbb{R}$. Hence,

$$f_n \le \chi_n \le \psi_n \quad \text{for all } n. \tag{3.5}$$

By (3.2), this yields the uniform integral bound

$$\int \chi_n \, d(\nu - \mu) \leq \nu(\psi_n) - \mu(f_n) \leq C \text{ for all } l = 1, ..., L \text{ and } n \in \mathbb{N}. \tag{3.6}$$

Using (3.6) and the assumption that (μ, ν) is irreducible, a local uniform boundedness of $\{\chi_n\}_n$ can be obtained (cf. Step 1 in the proof of [16, Theorem 1.2]): there exists an increasing sequence of compact intervals $J_k := [c_k, d_k]$ and constants $M_k \geq 0$ for each $k \in \mathbb{N}$, such that $\cup_{k=1}^{\infty} J_k = J$, and

$$0 \leq \sup_n \chi_n \leq M_k \text{ in } J_k. \tag{3.7}$$

Step 2. Given any approximating dual optimizer $(\varphi_n, \psi_n, \theta_{l,n})$ satisfying (3.2), (3.3), the goal is to suitably modify it and deduce pointwise convergence of φ_n, ψ_n to some functions φ, ψ μ, ν-a.s. as $n \to \infty$, repectively, where $\varphi, \psi \in \mathbb{R} \cup \{+\infty\}$ is μ, ν-a.s. finite. From convexity of χ_n with $\mu \preceq_c \nu$, we deduce, for all n,

$$C \geq \nu(\psi_n) - \mu(f_n) \geq \nu(\chi_n) - \mu(f_n) \geq \mu(\chi_n) - \mu(f_n) = ||\chi_n - f_n||_{L^1(\mu)}, \tag{3.8}$$

Meanwhile, (3.3) gives $f_n(x) + h_{l,n}(x)(y - x) - \psi_n(y) \leq -c_l(x, y) \leq 0$, hence

$$f_n(x) + h_{l,n}(x)(y - x) - \psi_n(y) \leq \chi_n(y) - \psi_n(y) \leq 0.$$

Integrating by any $\pi \in \mathcal{M}(\mu, \nu)$ implies

$$||\psi_n - \chi_n||_{L^1(\nu)} \leq \nu(\psi_n) - \mu(f_n) \leq C \text{ for all } n. \tag{3.9}$$

These uniform L^1 bounds, combined with the local uniform bound (3.7) and Komlós compactness theorem, can imply the desired almost sure convergence of $\{\varphi_n\}$ and $\{\psi_n\}$ as presented in [6] and in Step 2 in the proof of [16, Theorem 1.2], thus we omit the detail here. Also, by following Step 3 in the same proof, one can deduce the following pointwise convergence of χ_n to a convex function χ

$$\lim_{n \to \infty} \chi_n(y) = \chi(y) \in \mathbb{R} \text{ for every } y \in J. \tag{3.10}$$

Step 3. We have obtained the almost sure limit functions φ, ψ, with $f := -\varphi$. We may define $\varphi := +\infty$ on a μ-null set which includes $\mathbb{R} \setminus I$, and $\psi := +\infty$ on a ν-null set which includes $\mathbb{R} \setminus J$, so that they are defined everywhere on $\mathbb{R}$. We will show there exists a function $\theta_l : \mathbb{R} \to \mathbb{R}$, with $h_l := -\theta_l$, $l = 1, ..., L$, such that

$$\varphi(x) + \psi(y) + \theta_l(x)(y - x) \geq c_l(x, y). \tag{3.11}$$

For any function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ which is bounded below by an affine function, let $\mathrm{conv}[f] : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ denote the lower semi-continuous convex envelope of f, that is the supremum of all affine functions λ satisfying $\lambda \le f$ (If there is no such λ, let $\mathrm{conv}[f] \equiv -\infty$.) Set $H_{l,n}(x,y) := \mathrm{conv}[\psi_n(\,\cdot\,) - c_l(x,\,\cdot\,)](y)$. By (3.3),

$$f_n(x) + h_{l,n}(x)(y-x) \le H_{l,n}(x,y) \le \psi_n(y) - c_l(x,y), \tag{3.12}$$

because the left hand side is affine in y. Letting $y = x$ gives $f_n(x) \le H_{l,n}(x,x)$.

Next, since the lim sup of convex functions is convex, we have

$$\begin{aligned}\limsup_{n\to\infty} H_{l,n}(x,y) &\le \mathrm{conv}[\limsup_{n\to\infty} \big(\psi_n(\,\cdot\,) - c_l(x,\,\cdot\,)\big)](y)\\ &\le \mathrm{conv}[\psi(\,\cdot\,) - c_l(x,\,\cdot\,)](y) =: H_l(x,y).\end{aligned}$$

Then by the convergence $f_n \to f$ and the definition of $H_l(x,y)$, we get

$$f(x) \le H_l(x,x), \text{ and } H_l(x,y) \le \psi(y) - c_l(x,y).$$

Set $A := \{x \in I \mid \lim_{n\to\infty} f_n(x) = f(x) \in \mathbb{R}\}$, so that $\mu(A) = 1$. Since $y \mapsto H_l(x,y)$ is continuous in J for every $x \in A$ due to the convexity of $y \mapsto H_l(x,y)$ and ν-a.s. finiteness of ψ, the subdifferential $\partial H_l(x,\,\cdot\,)(y)$ is nonempty, convex and compact for every $y \in I = \mathrm{int}(J)$. This allows us to choose a measurable function $h_l : A \to \mathbb{R}$ satisfying $h_l(x) \in \partial H_l(x,\,\cdot\,)(x)$. Such choice yields (3.11) as follows:

$$f(x) + h_l(x)(y-x) \le H_l(x,x) + h_l(x)(y-x) \le H_l(x,y) \le \psi(y) - c_l(x,y).$$

We may define $h_l \equiv 0$ on $\mathbb{R} \setminus A$, noting that $f := -\infty$ on $\mathbb{R} \setminus A$.

Step 4. We will show that for any functions $\theta_l : \mathbb{R} \to \mathbb{R}$, $l = 1, ..., L$ that satisfies (3.11) (whose existence was shown in the previous step), and for any maximizer $\vec{\gamma}^* = (\gamma_1^*, ..., \gamma_L^*) \in \mathcal{M}_L(\mu,\nu)$ for the problem (2.7), it holds

$$\varphi(x) + \psi(y) + \theta_l(x)(y-x) = c_l(x,y) \quad \gamma_l^* - a.e. \text{ for all } l = 1, ..., L. \tag{3.13}$$

For any $\vec{\gamma} = (\gamma_1, ..., \gamma_L) \in \mathcal{M}_L(\mu,\nu)$, Assumption **[A]** yields $c_l \in L^1(\gamma_l)$. We claim

$$\begin{aligned}&\liminf_{n\to\infty} \sum_{l=1}^{L} \int \big(\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y-x)\big) d\gamma_l \\ &\qquad \ge \sum_{l=1}^{L} \int \big(\varphi(x) + \psi(y) + \theta_l(x)(y-x)\big) d\gamma_l \ \text{ for every } l.\end{aligned} \tag{3.14}$$

To see how the claim implies (3.13), let $\vec{\gamma}^*$ be any maximizer for (2.7). Then

$$\begin{aligned}
\overline{P}_c &= \lim_{n\to\infty} \sum_{l=1}^{L} \int \big(\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y-x)\big) d\gamma_l^* \\
&\geq \sum_{l=1}^{L} \liminf_{n\to\infty} \int \big(\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y-x)\big) d\gamma_l^* \\
&\geq \sum_{l=1}^{L} \int \big(\varphi(x) + \psi(y) + \theta_l(x)(y-x)\big) d\gamma_l^* \\
&\geq \sum_{l=1}^{L} \int c_l(x,y)\, d\gamma_l^* = \overline{P}_c,
\end{aligned}$$

hence equality holds throughout. Notice this yields (3.13), hence the proposition.

To prove (3.14), fix any $\vec{\gamma} = (\gamma_1, ..., \gamma_L) \in \mathcal{M}_L(\mu,\nu)$. The nonnegativity (3.1) gives $\gamma_l^X(\varphi_n) + \gamma_l^Y(\psi_n) \geq 0$, and (3.2) gives $\sum_{l=1}^{L}(\gamma_l^X(\varphi_n) + \gamma_l^Y(\psi_n)) = \mu(\varphi_n) + \nu(\psi_n) \searrow \overline{P}_c$. This implies the sequence $\{\gamma_l^X(\varphi_n) + \gamma_l^Y(\psi_n)\}_n$ is bounded for all l. With this and (3.5), as in Step 2 (but $\gamma_l^X \preceq_c \gamma_l^Y$ instead of $\mu \preceq_c \nu$), we deduce

$$\sup_n ||\chi_n + \varphi_n||_{L^1(\gamma_l^X)} < \infty, \quad \sup_n ||\psi_n - \chi_n||_{L^1(\gamma_l^Y)} < \infty, \quad \text{for all } l.$$

From this, since $\varphi_n \to \varphi$, $\psi_n \to \psi$, $\chi_n \to \chi$, by Fatou's lemma, we get

$$\chi + \varphi \in L^1(\gamma_l^X), \quad \psi - \chi \in L^1(\gamma_l^Y),$$

$$\liminf_{n\to\infty} \int (\chi_n + \varphi_n)\, d\gamma_l^X \geq \int (\chi + \varphi) d\gamma_l^X, \ \liminf_{n\to\infty} \int (\psi_n - \chi_n)\, d\gamma_l^Y \geq \int (\psi - \chi) d\gamma_l^Y.$$

This allows us to proceed

$$\begin{aligned}
&\liminf_{n\to\infty} \int \big(\varphi_n(x) + \psi_n(y) + \theta_{l,n}(x)(y-x)\big) d\gamma_l \\
&= \liminf_{n\to\infty} \int \big(\varphi_n(x) + \chi_n(x) - \chi_n(y) + \psi_n(y) - \chi_n(x) + \chi_n(y) + \theta_{l,n}(x)(y-x)\big) d\gamma_l \\
&\geq \int (\chi + \varphi) d\gamma_l^X + \int (\psi - \chi) d\gamma_l^Y + \liminf_{n\to\infty} \int \big(\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y-x)\big) d\gamma_l.
\end{aligned}$$

To handle the last term, disintegrate $\gamma_l = (\gamma_l)_x \otimes \gamma_l^X$, and let $\xi_n : I \to \mathbb{R}$ be a sequence of functions satisfying $\xi_n(x) \in \partial\chi_n(x)$. This allows us to proceed

$$\begin{aligned}
&\int \big(\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y-x)\big)d\gamma_l \\
&= \iint \big(\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y-x)\big)(\gamma_l)_x(dy)\gamma_l^X(dx) \\
&= \iint \big(\chi_n(y) - \chi_n(x) + \xi_n(x)(y-x)\big)(\gamma_l)_x(dy)\gamma_l^X(dx),
\end{aligned}$$

because $\int \theta_{l,n}(x)(y-x)(\gamma_l)_x(dy) = \int \xi_n(x)(y-x)(\gamma_l)_x(dy) = 0$. Notice that the last integrand is nonnegative. Thus by repeated Fatou's lemma, we deduce

$$\begin{aligned}
&\liminf_{n\to\infty} \int \big(\chi_n(y) - \chi_n(x) + \theta_{l,n}(x)(y-x)\big)d\gamma_l \\
&\geq \int \liminf_{n\to\infty} \bigg(\int \big(\chi_n(y) - \chi_n(x) + \xi_n(x)(y-x)\big)(\gamma_l)_x(dy)\bigg)\gamma_l^X(dx) \\
&\geq \int \bigg(\int \big(\chi(y) - \chi(x) + \xi(x)(y-x)\big)(\gamma_l)_x(dy)\bigg)\gamma_l^X(dx),
\end{aligned}$$

for some $\xi(x) \in \partial\chi(x)$ which is a limit point of the bounded sequence $\{\xi_n(x)\}_n$. Finally, in the last line, the inner integral equals

$$\int \big(\chi(y) - \chi(x) + \theta_l(x)(y-x)\big)(\gamma_l)_x(dy).$$

This proves the claim, hence the proposition. □

We are prepared to prove Theorem 1.1.

Proof of Theorem 1.1. Fix any optimal pair (π, μ_1) for the problem (1.2), and let $\gamma_l = \pi_x \otimes \mu_l$, $l = 1, 2$, with $\mu_2 = \mu - \mu_1$ and the kernel $\{\pi_x\}_x$ inherited from π. We understand $c_1(x,y) = c_1(x)$ in the proof. Let us first assume that $\mu \preceq_c \nu$ is irreducible. Then by Proposition 2.3, with $f = -\varphi$ and $h_l = -\theta_l$, we have

$$f(x) + h_l(x)(y-x) + c_l(x,y) \leq \psi(y) \quad \text{for each } l = 1,2 \text{ and } (x,y) \in \mathbb{R}^2, \tag{3.15}$$

$$f(x) + h_l(x)(y-x) + c_l(x,y) = \psi(y) \quad \gamma_l - a.e.\,(x,y) \text{ for each } l = 1,2. \tag{3.16}$$

Now, saying that an American option holder randomizes her exercise between c_1, c_2 is equivalent to saying that the common mass of μ_1, μ_2 (written as $\mu_1 \wedge \mu_2$) is nonzero. The common mass of μ_1, μ_2 is defined by the largest measure $\rho = \mu_1 \wedge \mu_2$

satisfying $\rho \le \mu_1$ and $\rho \le \mu_2$. Since γ_1 and γ_2 have the same kernel, (3.16) implies

$$f(x) + h_l(x)(y - x) + c_l(x, y) = \psi(y) \quad \pi_x \otimes \rho - a.e.\,(x, y) \text{ for } l = 1, 2. \tag{3.17}$$

Observe that ψ can be taken as $\psi := \max(\psi_1, \psi_2)$, where

$$\psi_l(y) := \sup_x f(x) + h_l(x)(y - x) + c_l(x, y),$$

and consequently, ψ_1, ψ_2, ψ are all convex since c_2 is convex in y (while c_1 is independent of y.) Now the idea is to differentiate (3.17) by y for ν-a.e. y, which is enabled by the fact that ψ is differentiable ν-a.s., since ν is assumed to be absolutely continuous with respect to Lebesgue. By the differentiation combined with the first-order optimality condition from (3.15), (3.16) for each $l = 1, 2$, we deduce

$$h_1(x) = \psi'(y) = h_2(x) + (c_2)_y(x, y) \quad \pi_x \otimes \rho - a.e.\,(x, y), \tag{3.18}$$

where $(c_2)_y$ denotes the partial derivative of c_2 by y, noting that (3.15), (3.16) implies $(c_2)_y(x, y)$ exists γ_2-a.e., since ψ is differentiable ν-a.e..

Now since $c_1 = c_1(x)$, the left hand side of (3.15) is linear in y when $l = 1$, while ψ is convex. With this, the first equality in (3.18) implies that for ρ-a.e. x, ψ is linear in the smallest interval containing $\mathrm{spt}(\pi_x)$ which contains x. Hence,

$$\psi'(y) = \psi'(x) \quad \pi_x \otimes \rho - a.e.\,(x, y). \tag{3.19}$$

The second equality in (3.18) thus becomes

$$(c_2)_y(x, y) = \psi'(x) - h_2(x) \quad \pi_x \otimes \rho - a.e.\,(x, y). \tag{3.20}$$

Because c_2 is assumed to be strictly convex in y, the solution y to (3.20) must be unique, and hence, $y = x$ since π_x has its barycenter at x. We conclude

$$\pi_x = \delta_x \quad \rho - a.e.\, x, \tag{3.21}$$

where $\delta_x \in \mathcal{P}(\mathbb{R})$ is the Dirac mass at x. (3.17) then yields

$$c_1(x) = c_2(x, x) \quad \rho - a.e.\, x. \tag{3.22}$$

Now if $c_1(x) \neq c_2(x, x)$ μ-a.s., then (3.22) implies $\rho \equiv 0$, yielding $\mu_1 \perp \mu - \mu_1$ for any optimal pair (π, μ_1). This proves the theorem when $\mu \preceq_c \nu$ is irreducible.

For general $\mu \preceq_c \nu$, it is well known that any convex-ordered pair (μ, ν) can be decomposed as at most countably many irreducible pairs, and the decomposition is uniquely determined by the potential functions u_μ, u_ν. More precisely, we have: [6, Proposition 2.3] Let $(I_k)_{1 \le k \le N}$ be the open components of the open set $\{u_\mu < u_\nu\}$ in $\mathbb{R}$, where $N \in \mathbb{N} \cup \{+\infty\}$. Let $I_0 = \mathbb{R} \setminus \cup_{k \ge 1} I_k$ and $\mu_k = \mu\big|_{I_k}$ for $k \ge 0$, so that $\mu = \sum_{k \ge 0} \mu_k$. There exists a unique decomposition $\nu = \sum_{k \ge 0} \nu_k$ such that

$$\mu_0 = \nu_0, \text{ and } (\mu_k, \nu_k) \text{ is irreducible for } k \ge 1 \text{ with } \mu_k(I_k) = \mu_k(\mathbb{R}).$$

Moreover, any $\pi \in \mathcal{M}(\mu, \nu)$ admits a unique decomposition $\pi = \sum_{k \ge 0} \pi_k$ such that $\pi_k \in \mathcal{M}(\mu_k, \nu_k)$ for all $k \ge 0$.

Here, π_0 must be the identity transport, i.e., $(\pi_0)_x = \delta_x$, since it is a martingale transport between the same marginal. Since the theorem has already been proven for the irreducible pairs (μ_k, ν_k), $k \ge 1$, we only need to prove it for the identity transport π_0. In this case, $\int c_2(x,y)(\pi_0)_x(dy) = c_2(x,x)$, yielding that it is optimal to exercise c_1 when $c_1(x) > c_2(x,x)$, while it is optimal to exercise c_2 when $c_1(x) < c_2(x,x)$. The assumption $c_1(x) \ne c_2(x,x)$ μ-a.s. therefore completes the proof. $\square$

References

[1] A. Aksamit, S. Deng, J. Obłój and X. Tan. The robust pricing–hedging duality for American options in discrete time financial markets. *Mathematical Finance*, Jul;29(3) (2019) 861–897.

[2] E. Bayraktar, A.M.G. Cox and Y. Stoev. Martingale Optimal Transport with Stopping. *SIAM Journal on Control and Optimization*, 56.1 (2018) 417–433.

[3] E. Bayraktar, Y.J. Huang and Z. Zhou. On hedging American options under model uncertainty. *SIAM J. Financ. Math.*, 6 (2015) 425–447.

[4] M. Beiglböck, A.M.G. Cox, and M. Huesmann. Optimal transport and Skorokhod embedding. *Invent. Math.*, 208 (2017) 327–400.

[5] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.*, 44(1) (2016) 42–106.

[6] M. Beiglböck, M. Nutz and N. Touzi. Complete duality for martingale optimal transport on the line. *Ann. Probab.*, 45(5) (2017) 3038–3074.

[7] P. Cheridito, M. Kiiski, D.J. Prömel and H.M. Soner. Martingale Optimal Transport Duality. *Mathematische Annalen*, 379 (2021) 1685–1712.

[8] M.H.A. Davis. Model-Free Methods in Valuation and Hedging of Derivative Securities. *The Handbook of Post Crisis Financial Modeling*, 2016.

[9] Y. Dolinsky and H.M. Soner. Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Relat. Fields.*, 160 (2014) 391–427.

[10] A. Galichon, P. Henry-Labordère, and N. Touzi. A Stochastic Control Approach to No-Arbitrage Bounds Given Marginals, with an Application to Lookback Options. *Annals of Applied Probability.*, Volume 24, Number 1 (2014) 312–336.

[11] P. Henry-Labordère. Model-free Hedging: A Martingale Optimal Transport Viewpoint. *Chapman & Hall/CRC Financ. Math. Ser.* CRC Press, Boca Raton, FL, 2017.

[12] P. Henry-Labordère and N. Touzi. An explicit martingale version of the one-dimensional Brenier theorem. *Finance Stoch.*, 20 (2016) 635–668.

[13] D. Hobson. The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, Springer, Berlin (2011) 267–318.

[14] D. Hobson and A. Neuberger. Model uncertainty and the pricing of American options. *Finance Stoch.*, 21 (2017) 285–329.

[15] D. Hobson and D. Norgilas. Robust bounds for the American put. *Finance and Stochastics*, 2019 Apr 15;23(2) (2019) 359–395.

[16] T. Lim. Geometry of vectorial martingale optimal transportations and duality. *Mathematical Programming*, https://doi.org/10.1007/s10107-023-01954-4 (2023)

[17] A. Neuberger. Bounds on the American option. Preprint (2007). Available online at `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=966333`

[18] J. Obłój. The Skorokhod embedding problem and its offspring. *Probability Surveys*, 1 (2004) 321–392.

[19] F. Santambrogio. Optimal transport for applied mathematicians. Calculus of variations, PDEs, and modeling. *Progress in Nonlinear Differential Equations and their Applications*, 87 Birkhuser/Springer, Cham, (2015)

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