Chapter 6

Limits of sequences

6.1 Convergence and limit laws

Definition 6.1.1 (Distance between two real numbers).

Given two real numbers x and y, we define their distance $d(x, y)$ to be $d(x, y) := |x-y|$.

Definition 6.1.2 (ε-close real numbers).

Let $\varepsilon > 0$ be a real number. We say that two real numbers x, y are ε-close iff we have $d(y, x) \leq \varepsilon$.

Definition 6.1.3 (Cauchy sequences of reals).

Let $\varepsilon > 0$ be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers starting at some integer index N is said to be ε-steady iff a_j and a_k are ε-close for every $j, k \geq N$. A sequence $(a_n)_{n=m}^{\infty}$ starting at some integer index m is said to be eventually ε-steady iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-steady. We say that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence iff it is eventually ε-steady for every $\varepsilon > 0$.

Proposition 6.1.4.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers starting at some integer index m. Then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Definition 5.1.8 if and only if it is a Cauchy sequence in the sense of Definition 6.1.3.

Definition 6.1.5 (Convergence of sequences).

Let $\varepsilon > 0$ be a real number, and let L be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be ε-close to L iff a_n is ε-close to L for every $n \geq N$, i.e., we have $|a_n - L| \leq \varepsilon$ for every $n \geq N$. We say that a sequence $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-close to L. We say that a sequence $(a_n)_{n=m}^{\infty}$ converges to L iff it is eventually ε-close to L for every real $\varepsilon > 0$.

Proposition 6.1.7 (Uniqueness of limits).

Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m, and let $L = L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L'.

Definition 6.1.8 (Limits of sequences).

If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say that $(a_n)_{n=m}^{\infty}$ is convergent and that its limit is L; we write

$$L = \lim_{n\to\infty} a_n$$

to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L, we say that the sequence $(a_n)_{n=m}^{\infty}$ is divergent and we leave $\lim_{n\to\infty} a_n$ undefined.

Proposition 6.1.11.

We have $\lim_{n\to\infty} 1/n = 0$.

Proposition 6.1.12 (Convergent sequences are Cauchy).

Suppose that $(a_n)_{n=m}^{\infty}$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^{\infty}$ is also a Cauchy sequence.

Proposition 6.1.15 (Formal limits are genuine limits).

Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=1}^{\infty}$ converges to $\mathrm{LIM}_{n\to\infty} a_n$, i.e.,

$$\mathrm{LIM}_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

Definition 6.1.16 (Bounded sequences).

A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is bounded by a real number M iff we have $|a_n| \le M$ for all $n \ge m$. We say that $(a_n)_{n=m}^{\infty}$ is bounded iff it is bounded by M for some real number $M > 0$.

Corollary 6.1.17.

Every convergent sequence of real numbers is bounded.

Theorem 6.1.19 (Limit Laws).

Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.

(a) The sequence $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$; in other words,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

(b) The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy; in other words,

$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

(c) For any real number c, the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx; in other words,

$$\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n.$$

(d) The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$; in other words,

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

(e) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1}; in other words,

$$\lim_{n\to\infty} b_n^{-1} = (\lim_{n\to\infty} b_n)^{-1}.$$

(f) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y; in other words,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

(g) The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$; in other words,

$$\lim_{n\to\infty} \max(a_n, b_n) = \max(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n).$$

(h) The sequence $(\min(a_n, b_n))_{n=m}^{\infty}$ converges to $\min(x, y)$; in other words,

$$\lim_{n\to\infty} \min(a_n, b_n) = \min(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n).$$

Exercise 6.1.1.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, such that $a_{n+1} > a_n$ for each natural number n. Prove that whenever n and m are natural numbers such that $m > n$, then we have $a_m > a_n$.

Proof. Induct on p to show that $a_{n+p} > a_n$ for all positive integers p. When $p = 1$, as stated in the problem, we have $a_{n+p} = a_{n+1} > a_n$. Suppose inductively $a_{n+p} > a_n$ for positive integer p. Then $a_{n+p+1} > a_{n+p} > a_n$. This closes the induction. Since $m > n$, $m - n$ is a positive integer. Therefore, $a_{n+(m-n)} = a_m > a_n$. □

Exercise 6.1.2.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let L be a real number. Show that $(a_n)_{n=m}^{\infty}$ converges to L if and only if, given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$.

Proof. Suppose $(a_n)_{n=m}^{\infty}$ converges to L. By definition, $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L for every real $\varepsilon > 0$. Therefore, for any real $\varepsilon > 0$, there exists $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$.

Suppose given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$. Then any real $\varepsilon > 0$, there exists an $N \geq m$ such that $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L. Therefore, by definition, $(a_n)_{n=m}^{\infty}$ converges to L. □

Exercise 6.1.3.

Let $(a_n)_{n=m}^{\infty}$ be sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{n=m'}^{\infty}$ converges to c.

Proof. Suppose $(a_n)_{n=m}^{\infty}$ converges to c. Then for any arbitrary $\varepsilon > 0$, there exists $N_\varepsilon \geq m$ such that $|a_n - c| \leq \varepsilon$. So for any arbitrary $\varepsilon > 0$, we can find $M_\varepsilon = \max(N_\varepsilon, m')$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq M_\varepsilon \geq m'$. Therefore, $(a_n)_{n=m}^{\infty}$ converges to c.

Suppose $(a_n)_{n=m'}^{\infty}$ converges to c. For any arbitrary $\varepsilon > 0$, there exists $N_\varepsilon \geq m'$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq N_\varepsilon$, since $m' \geq m$, we have $N_\varepsilon \geq m$. Therefore, for any arbitrary $\varepsilon > 0$, we can find $N_\varepsilon \geq m$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq N_\varepsilon$.

Thus, $(a_n)_{n=m}^{\infty}$ converges to $c \iff (a_n)_{n=m}^{\infty}$ converges to c. $\square$

Exercise 6.1.4.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $k \geq 0$ be a non-negative integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_{n+k})_{n=m}^{\infty}$ converges to c.

Proof. We can rewrite $(a_{n+k})_{n=m}^{\infty} = (a_n)_{m+k}^{\infty}$ since both of them are equal to the infinite sequence $a_{m+k}, a_{m+k+1}, \ldots$. Since $k \geq 0$, $m + k \geq m$. Therefore, by Exercise 6.1.4, we have $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{m+k}^{\infty} = (a_{n+k})_{n=m}^{\infty}$ converges to c. $\square$

Exercise 6.1.5.

Prove proposition 6.1.12.

Proof. Suppose $\lim_{n\to\infty} (a_n)_{n=m}^{\infty} = L$. By definition, for all real $\varepsilon/2 > 0$, there exists $N_\varepsilon \geq m$ such that $|a_i - L| \leq \varepsilon/2$ and $|a_j - L| = |L - a_j| \leq \varepsilon/2$ for all $i, j \geq N_\varepsilon$. Then for all real $\varepsilon > 0$, let $N = N_\varepsilon \geq m$, we have $|a_i - a_j| \leq |a_i - L| + |L - a_j| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n, m \geq N$. Therefore, $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence. $\square$

Exercise 6.1.6.

Prove Proposition 6.1.15.

Proof. Write $L :=\mathrm{LIM}_{n\to\infty}a_n$. We want to show that $L = \lim_{n\to\infty} a_n$. Suppose L and $(a_n)_{n=m}^{\infty}$ are not eventually ε-close. Consider an arbitrary $\varepsilon/2 > 0$, since $(a_n)_{n=m}^{\infty}$ is Cauchy, there exists $N \geq m$ such that $|a_i - a_j| \leq \varepsilon/2$ for all $i, j \geq N$. For this N, since L and $(a_n)_{n=m}^{\infty}$ are not eventually ε-close, there exists $i \geq N$ such that $|a_i - L| > \varepsilon$. And since this $i \geq N$, we have $|a_i - a_j| \leq \varepsilon/2$ for all $j \geq i$. Since $|a_i - L| > \varepsilon$, we have

$$a_i > L + \varepsilon \text{ or } a_i < L - \varepsilon.$$

If $a_i > L + \varepsilon$, since $|a_i - a_j| \leq \varepsilon/2 \implies a_i - \varepsilon/2 \leq a_j \leq a_i + \varepsilon/2$, we have $L + \varepsilon/2 < a_i - \varepsilon/2 \leq a_j$. If $a_i < L - \varepsilon$, since $a_i - \varepsilon/2 \leq a_j \leq a_i + \varepsilon/2$, we have $a_j \leq a_i + \varepsilon/2 < L - \varepsilon/2$. Therefore, we either have $a_j < L - \varepsilon/2$ or $a_j > L + \varepsilon/2$ for all $j \geq i$.

If $a_j < L - \varepsilon/2$ for all $j \geq i$, by Exericse 5.4.8, we have $\mathrm{LIM}_{n\to\infty}a_n \leq L - \varepsilon/2 < L$. If $a_j > L + \varepsilon/2$ for all $j \geq i$, by Exericse 5.4.8, we have $\mathrm{LIM}_{n\to\infty}a_n \geq L + \varepsilon/2 > L$. Then we have either $\mathrm{LIM}_{n\to\infty}a_n > L$ or $\mathrm{LIM}_{n\to\infty}a_n < L$, and it contradicts the fact that $\mathrm{LIM}_{n\to\infty}a_n = L$. Thus, L and $(a_n)_{n=m}^{\infty}$ are eventually ε-close, hence $L = \lim_{n\to\infty} a_n$. □

Exercise 6.1.7.

Show that Definition 6.1.16 is consistent with Definition 5.1.12.

Proof. We would like to show that if $(a_n)_{n=1}^{\infty}$ is bounded by M then $(a_n)_{n=m}^{\infty}$ is bounded by M. If $(a_n)_{n=1}^{\infty}$ is bounded by M, by Definition 5.1.12, $|a_i| \leq M$ for all $i \geq 1$. Then for all $i \geq m \geq 1$, we have $|a_i| \leq M$. Therefore, $(a_n)_{n=m}^{\infty}$ is bounded by M. □

Exercise 6.1.8.

Prove Theorem 6.1.19.

(a) *Proof.* Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=m}^{\infty}$ converges to x, there exists $N_1 \geq m$ such that $|a_n - x| \leq \varepsilon/2$ for all $n \geq N_1$. Since $(b_n)_{n=m}^{\infty}$ converges to x, there exists $N_2 \geq m$ such that $|b_n - x| \leq \varepsilon/2$ for all $n \geq N_2$. Let $N = \max(N_1, N_2)$, we have $|(a_n + b_n) - (x + y)| = |(a_n - x) + (b_n - y)| \leq |a_n - x| + |b_n - y| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n \geq N$. Therefore, $\lim\limits_{a_n+b_n} = x + y$. □

(b) *Proof.* Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=m}^{\infty}$ converges to x, $(a_n)_{n=m}^{\infty}$ is bounded by a positive number M_1. There exists $N_1 \geq m$ such that $|a_n - x| \leq \varepsilon/2M_1$ for all $n \geq N_1$. We can easily know that there exists a positive number M_2 such that $M_2 \geq |x|$. Since $(b_n)_{n=m}^{\infty}$ converges to y, there exists N_2 such that $|b_n - y| \leq \varepsilon/2M_2$ for all $n \geq N_2$. Let $N = \max(N_1, N_2)$, then

$$\begin{aligned}
|a_n b_n - xy| &= |a_n b_n - x b_n + x b_n - xy| \\
&\leq |b_n| \cdot |a_n - x| + |x| \cdot |b_n - y| \\
&\leq M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}(a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$. □

(c) *Proof.* Let $(b_n)_{n=m}^{\infty} = c, c, c, \ldots$, by (b), we have $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty}(c a_n) = c \lim\limits_{n\to\infty}(a_n)$. □

(d) *Proof.* By (c), $\lim\limits_{n\to\infty}(-b_n) = -\lim\limits_{n\to\infty} b_n$. By (a), we have $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty}(a_n + (-b_n)) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty}(-b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty}(b_n) = x - y$. □

(e) *Proof.* Since $(b_n)_{n=m}^{\infty}$ converges to y, there exists $N \geq m$ such that $|b_n - y| \leq |y|/2$ for all $n \geq N$. Then $y - |y|/2 \leq b_n \leq y + |y|/2$. If $y < 0$, $b_n \leq y + |y|/2 = y - y/2 = y/2 = -|y|/2$. If $y > 0$, $b_n \geq y - y/2 = |y|/2$. Therefore, $|b_n| \geq |y|/2$ for $n \geq N$. Let $c = \min(|b_m|, \ldots, |b_{N-1}|, |y|/2)$, we have $|b_i| \leq c$ for all $i \geq m$. So $(b_n)_{n=m}^{\infty}$ is a Cauchy sequence bounded away from zero, hence $(b_n^{-1})_{n=m}^{\infty}$ is also a Cauchy sequence.

Consider an arbitrary $\varepsilon > 0$. Since $|b_n| \geq c$ for all $n \geq m$, we have $1/|b_n| \leq 1/c$ for all $n \geq m$. Since $(b_n)_{n=m}^{\infty}$ converges to y, there exists $N \geq m$ such that $|b_n - y| \leq c|y|\varepsilon$. Then

$$\begin{aligned}
\left|\frac{1}{b_n} - \frac{1}{y}\right| &= \left|\frac{b_n - y}{b_n \cdot y}\right| \\
&= \frac{1}{|b_n|} \cdot \frac{1}{|y|} \cdot |b_n - y| \\
&\leq \frac{1}{c} \cdot \frac{1}{|y|} \cdot c|y|\varepsilon \\
&= \varepsilon.
\end{aligned}$$

Thus, $(b_n^{-1})_{n=m}^{\infty}$ converges to $1/y$. □

(f) *Proof.* By (b), $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} \frac{1}{b_n})$. By (e), $\lim\limits_{n\to\infty} \frac{1}{b_n} = 1/y$. Therefore, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = x \cdot \frac{1}{y} = \frac{x}{y}$. □

(g) *Proof.* Suppose $x \geq y$. Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=m}^{\infty}$ converges to x, there exists N_1 such that $|a_n - x| \leq (x-y)/2$ for $n \geq N_1$. Then $-(x-y)/2 \leq a_n - x \leq (x-y)/2$, hence $a_n \geq (x+y)/2$. Since $(b_n)_{n=m}^{\infty}$ converges to y, there exists N_2 such that $|b_n - y| \leq (x-y)/2$ for $n \geq N_2$. Then $b_n \leq (x+y)/2$. Let $N = \max(N_1, N_2)$, we have $a_n \geq (x+y)/2 \geq b_n$ for all $n \geq N$. So when $n \geq N$, $\max(a_n, b_n) = a_n$. Since $(a_n)_{n=m}^{\infty}$ converges to x, there exists $M_1 \geq m$ such that $|a_n - x| \leq \varepsilon$. Let $M = \max(N, M_1)$, we have $|\max(a_n, b_n) - \max(x, y)| = |a_n - x| \leq \varepsilon$ for all $n \geq M$.

Similarly, we can show that if $y > x$, there exists $M \geq m$ such that $|\max(a_n, b_n) - \max(x, y)| = |b_n - y| \leq \varepsilon$ for all $n \geq M$. Thus, $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$. □

(h) *Proof.* By (b) and (g), we have

$$\begin{aligned}\lim_{n\to\infty} \min(a_n, b_n) &= -\lim_{n\to\infty} \max(-a_n, -b_n)\\ &= -\max(\lim_{n\to\infty}(-a_n), \lim_{n\to\infty}(-b_n))\\ &= -\max(-\lim_{n\to\infty}(a_n), -\lim_{n\to\infty}(b_n))\\ &= \min(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n).\end{aligned}$$

□

Exercise 6.1.9.

Explain why Theorem 6.1.19 (f) fails when the limit of the denominator is 0.

Proof. 0 does not have a reciprocal. A counterexample is $a_n = 1$, $b_n = 1/n$. Then $(a_n/b_n)_{n=m}^{\infty} = (n)_{n=m}^{\infty}$ is not convergent and hence does not have a limit. □

Exercise 6.1.10.

Show that the concept of equivalent Cauchy sequence, as defined in Definition 5.2.6, does not change if ε is required to be positive real instead of positive rational.

Proof. For all real $\varepsilon > 0$, suppose $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close. Since ε is also a rational number, we have $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for every rational $\varepsilon > 0$. Suppose $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for every rational $\varepsilon > 0$. Consider an arbitrary real $\varepsilon > 0$. Since there exists a rational number such that $0 < \varepsilon' < \varepsilon$. Since $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε'-close, there exists $N \geq m$ such that $|a_n - b_n| \leq \varepsilon' < \varepsilon$ for all $n \geq N$. Therefore, $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for all real $\varepsilon > 0$. □

6.2 The Extended real number system

Definition 6.2.1 (Extended real number system).

The extended real number system $\mathbf{R}^*$ is the real line $\mathbf{R}$ with two additional elements attached, called $+\infty$ and $-\infty$. These elements are distinct from each other and also distinct from every real number. An extended real number x is called finited iff it is a real number, and infinite iff it is equal to $+\infty$ and $-\infty$.

Definition 6.2.2 (Negation of extended reals).

The operation of negation $x \mapsto -x$ on $\mathbf{R}$, we now extend to $\mathbf{R}^*$ by defining $-(+\infty) := -\infty$ and $-(-\infty) := +\infty$.

Definition 6.2.3 (Ordering of extended reals).

Let x and y be extended real numbers. We say that $x \geq y$, i.e., x is less than or equal to y, iff one of the following three statements is true:

(a) x and y are real numbers, and $x \geq y$ as real numbers.

(b) $y = +\infty$.

(c) $x = -\infty$.

We say that $x < y$ if we have $x \leq y$ and $x \neq y$. We sometimes write $x < y$ as $y > x$, and $x \leq y$ as $y \geq x$.

Proposition 6.2.5.

Let x, y, z be extended real numbers. Then the following statements are true:

(a) (Reflexivity) We have $x \leq x$.

(b) (Trichotomy) Exactly one of the statements $x < y$, $x = y$, or $x > y$ is true.

(c) (Transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.

(d) (Negation reverses order) If $x \leq y$, then $-y \leq -x$.

Definition 6.2.6 (Supremum of sets of extended reals).

Let E be a subset of $\mathbf{R}^*$. Then we define the supremum $\sup(E)$ or least upper bound of E by the following rule.

(a) If E is contained in $\mathbf{R}$ (i.e., $+\infty$ and $-\infty$ are not elements of E), then we let $\sup(E)$ be as defined in Definition 5.5.10.

(b) If E contains $+\infty$, then we set $\sup(E) := +\infty$.

(c) If E does not contain $+\infty$ but does contain $-\infty$, then we set $\sup(E) := \sup(E\backslash\{-\infty\})$ (which is a subset of $\mathbf{R}$ and thus falls under case (a)).

We also define the infimum $\inf(E)$ of E (also known as the greatest lower bound of E) by the formula

$$\inf(E) := -\sup(-E)$$

where $-E$ is the set $-E := \{-x : x \in E\}$.

Theorem 6.2.11.

Let E be a subset of $\mathbf{R}^*$. Then the following statements are true.

(a) For every $x \in E$ we have $x \leq \sup(E)$ and $x \geq \inf(E)$.

(b) Suppose that $M \in \mathbf{R}^*$ is an upper bound for E, i.e., $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.

(c) Suppose that $M \in \mathbf{R}^*$ is a lower bound for E, i.e., $x \geq M$ for all $x \in E$. Then we have $\inf(E) \geq M$.

Exercise 6.2.1.

Prove Proposition 6.2.5.

(a) *Proof.* If x is a real number, since $x = x$, we have $x \leq x$. If $x = +\infty$, by Definition 6.2.3, we have $x \leq x$. If $x = -\infty$, by Definition 6.2.3, we have $x \leq x$. Therefore, $x \leq x$ for extended real x. $\square$

(b) *Proof.* x and y are both real numbers. By Proposition 5.4.7, the statement is true.

$x = +\infty$, $y = +\infty$. By Definition 6.2.3, we have $x \leq y$. Since $x = y$, $x \not< y$ and $y \not< x$. Similarly, we can show that if $x = -\infty$ and $y = -\infty$, we have $x = y$ but $x \not< y$ and $y \not< x$.

$x = +\infty$, $y = -\infty$. Since $x = +\infty$, by Definition 6.2.3, we have $y \leq x$. As $x \neq y$. $y < x$. Since $y \neq +\infty$ and $x \neq -\infty$, $x \neq y$ does not hold. Thus, $x \not< y$.

x is real ($x = +\infty$), $y = +\infty$ (y is real). Since $y = +\infty$, by Definition 6.2.3, $x \leq y$. Since $x \neq +\infty$, $x \neq$ and hence $x < y$. Also by Definition 6.2.3, $y \leq x$ does not hold, hence $y \not< x$. The cases of $x = -\infty$ (x is real), y is real ($y = -\infty$) can be shown in a similar way. □

(c) *Proof.* x, y, z are real. Can be derived from the transitivity of real numbers.

$x = -\infty$. By Definition 6.2.3, $x \leq z$.

$y = -\infty$. By Definition 6.2.3, we have $x = -\infty$, hence $x \leq z$.

$z = -\infty$. By Definition 6.2.3, $y = -\infty$. Since $x \leq y$, $x = -\infty$. Hence, $x \leq z$.

$x = +\infty$. Since $x \leq y$, by Definition 6.2.3, we have $y = +\infty$. Since $y \leq z$, by Definition 6.2.3, we have $z = +\infty$. Hence, $x \leq z$.

$y = +\infty$. Since $y \leq z$, by Definition 6.2.3, we have $z = +\infty$. Hence, $x \leq z$.

$z = +\infty$. By Definition 6.2.3, $x \leq z$. □

(d) *Proof.* x, y are real. Since $x \leq y$, $y - x \geq 0$. Then $-y - (-x) \leq 0$. Therefore, $-y \leq -x$.

$y = +\infty$. Then $-y = -\infty$, by Definition 6.2.3, $-y \leq -x$.

$x = -\infty$. Then $-x = +\infty$, by Definition 6.2.3, $-y \leq -x$. □

Exercise 6.2.2.

Prove Theorem 6.2.11.

(a) *Proof.* $E \subseteq \mathbf{R}$. By the definition of supremum, infimum, and the least upper bound, we have $x \leq \sup(E)$ and $x \geq \inf(E)$.

$+\infty \in E$, $-\infty \notin E$. By Definition 6.2.6, $\sup(E) = +\infty$, and by Definition 6.2.3, $x \leq \sup(E)$. In this case, $\inf(E) = \inf(E \backslash \{+\infty\})$. For all $x \neq +\infty$, it

follows the case we have proved above ($E \subseteq \mathbf{R}$). For $x = +\infty$, by Definition 6.2.3, $x \geq \inf(E)$.
$-\infty \in E$, $+\infty \in E$. Similar to the case above.
$+\infty \in E$, $-\infty \in E$. Since $+\infty \in E$, $x \leq \sup(E) = +\infty$. In this case, $\inf(E) = -\infty$. By Definition 6.2.3, $x \geq \inf(E)$. $\square$

(b) *Proof.* $E \subseteq \mathbf{R}$. We have $x \leq \sup(E) \leq M$.
$+\infty \in E$. Since $+\infty \leq M$, $M = +\infty$. By Definition 6.2.3, $\sup(E) \leq M = +\infty$.
$-\infty \in E$, $+\infty \notin E$. $\sup(E) = \sup(E \backslash \{-\infty\})$, then $\sup(E) = \sup(E \backslash \{-\infty\}) \leq M$. $\square$

(c) *Proof.* For all $x \in E$, $M \leq x$. Then $-M \geq -x$ for all $-x \in -E$, so $-M \geq \sup(-E)$. Therefore, $M \leq -\sup(-E) = \inf(-E)$. $\square$

6.3 Suprema and Infima of sequences

Definition 6.3.1 (Sup and inf of sequences).

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then we define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n : n \geq m\}$, and $\inf(a_n)_{n=m}^{\infty}$ to the infimum of the same set $\{a_n : n \geq m\}$.

Proposition 6.3.6 (Least upper bound property).

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \mathbf{R}^*$ is an upper bound for a_n (i.e., $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.

Proposition 6.3.8 (Monotone bounded sequences converge).

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \mathbf{R}$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq m$). Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \le M.$$

Proposition 6.3.10

Let $0 < x < 1$. Then we have $\lim_{n\to\infty} x^n = 0$.

Exercise 6.3.1.

Verify the claim in Example 6.3.4.

Proof. $\sup(a_n)_{n=1}^{\infty} = 1$. Since $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ for all $n \ge 1$, we have $a_n \le a_1 = 1$ for all $n \ge 1$. Therefore, $M = 1$ is an upper bound for $(a_n)_{n=m}^{\infty}$. For any $\varepsilon > 0$, suppose $1 - \varepsilon$ is an upper bound of $(a_n)_{n=m}^{\infty}$. Then $a_1 = 1 > 1 - \varepsilon$, a contradiction. Thus, $\sup(a_n)_{n=1}^{\infty} = 1$.
$\inf(a_n)_{n=1}^{\infty} = 0$. Since $a_n = 1/n > 0$ for all $n \ge 1$, 0 is a lower bound for $(a_n)_{n=1}^{\infty}$. Suppose there exists an $\varepsilon > 0$ such that ε is a lower bound for $(a_n)_{n=1}^{\infty}$. Let $N = \lceil \frac{1}{\varepsilon} \rceil + 1 > \frac{1}{\varepsilon}$, then $0 < \frac{1}{N} = \frac{1}{\lceil \frac{1}{\varepsilon} \rceil + 1} < \varepsilon$, a contraction. Thus, $\inf(a_n)_{n=1}^{\infty} = 0$. $\square$

Exercise 6.3.2.

Prove Proposition 6.3.6.

Proof. Consider the set $E = \{a_n : n \ge m\}$. Since $a_n \le M$ for all $n \ge m$, M is an upper bound for E. By Theorem 6.2.11, we have $x = \sup(E) \le M$. Suppose there exists $y < x$ such that for all $n \ge m$ we have $a_n \le y < x$. Then by definition, since $y \ge a_n$ for all $n \ge m$, y is an upper bound for E. This contradicts the fact that x is the least upper bound for E. Therefore, there exists at least one $n \ge m$ for which $y < a_n \le x$. $\square$

Exercise 6.3.3.

Prove Proposition 6.3.8.

Proof. Since $(a_n)_{n=m}^{\infty}$ has a finite upper bound, $\sup(a_n)_{n=m}^{\infty}$ is a real number. Denote $\sup(a_n)_{n=m}^{\infty}$ by x. We want to show that $\lim\limits_{n\to\infty}(a_n)_{n=m}^{\infty} = x$. Consider an arbitrary real $\varepsilon > 0$, we have $y = x - \varepsilon < x$. By Proposition 6.3.6, there exists an $N \geq m$ for which $y < a_N \leq x$. Since a_n is increasing, for all $n \geq N \geq m$, we have $y < a_n \leq x$. In other words, for all $\varepsilon > 0$, we can find an integer $N \geq m$ such that $|x - a_n| = x - a_n \leq |x - y| = \varepsilon$ for all $n \geq N$. Therefore, $(a_n)_{n=m}^{\infty}$ converges to $\sup(a_n)_{n=m}^{\infty}$. By Proposition 6.3.6, $\sup(a_n)_{n=m}^{\infty} \leq M$. Thus, $\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \leq M$. □

Exercise 6.3.4.

Explain why Proposition 6.3.10 fails when $x > 1$. In fact, show that the sequence $(x^n)_{n=1}^{\infty}$ diverges when $x > 1$.

Proof. We would like to show that $(x^n)_{n=1}^{\infty}$ diverges when $x > 1$. Suppose $(x^n)_{n=1}^{\infty}$ converges to a real number L. Since $0 < 1/x < 1$, by Proposition 6.3.10, we have $\lim_{n\to\infty}(1/x)^n = 0$. Then $\lim_{n\to\infty}(1/x)^n x^n = \lim_{n\to\infty}(1/x)^n \cdot \lim_{n\to\infty} x^n = 0 \cdot L = 0$. By identity, we have $\lim_{n\to\infty}(1/x)^n x^n = \lim_{n\to\infty} 1 = 1$. (contradiction) Thus, $(x^n)_{n=1}^{\infty}$ diverges when $x > 1$.

In Example 1.2.3, it asserts that x^n is convergent without specifying the range of x. If $x > 1$, the sequence does not converge, which would cause the proof to be faulty. □