

Chapter 6

Limits of sequences

6.1 Convergence and limit laws

Definition 6.1.1 (Distance between two real numbers).

Given two real numbers x and y , we define their distance $d(x, y)$ to be $d(x, y) := |x - y|$.

Definition 6.1.2 (ε -close real numbers).

Let $\varepsilon > 0$ be a real number. We say that two real numbers x, y are ε -close iff we have $d(y, x) \leq \varepsilon$.

Definition 6.1.3 (Cauchy sequences of reals).

Let $\varepsilon > 0$ be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers starting at some integer index N is said to be ε -steady iff a_j and a_k are ε -close for every $j, k \geq N$. A sequence $(a_n)_{n=m}^{\infty}$ starting at some integer index m is said to be eventually ε -steady iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε -steady. We say that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence iff it is eventually ε -steady for every $\varepsilon > 0$.

Proposition 6.1.4.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers starting at some integer index m . Then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Definition 5.1.8 if and only if it is a Cauchy sequence in the sense of Definition 6.1.3.

Definition 6.1.5 (Convergence of sequences).

Let $\varepsilon > 0$ be a real number, and let L be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be ε -close to L iff a_n is ε -close to L for every $n \geq N$, i.e., we have $|a_n - L| \leq \varepsilon$ for every $n \geq N$. We say that a sequence $(a_n)_{n=m}^{\infty}$ is eventually ε -close to L iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε -close to L . We say that a sequence $(a_n)_{n=m}^{\infty}$ converges to L iff it is eventually ε -close to L for every real $\varepsilon > 0$.

Proposition 6.1.7 (Uniqueness of limits).

Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m , and let $L = L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L' .

Definition 6.1.8 (Limits of sequences).

If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L , we say that $(a_n)_{n=m}^{\infty}$ is convergent and that its limit is L ; we write

$$L = \lim_{n \rightarrow \infty} a_n$$

to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L , we say that the sequence $(a_n)_{n=m}^{\infty}$ is divergent and we leave $\lim_{n \rightarrow \infty} a_n$ undefined.

Proposition 6.1.11.

We have $\lim_{n \rightarrow \infty} 1/n = 0$.

Proposition 6.1.12 (Convergent sequences are Cauchy).

Suppose that $(a_n)_{n=m}^{\infty}$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^{\infty}$ is also a Cauchy sequence.

Proposition 6.1.15 (Formal limits are genuine limits).

Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=1}^{\infty}$ converges to $\text{LIM}_{n \rightarrow \infty} a_n$, i.e.,

$$\text{LIM}_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n.$$

Definition 6.1.16 (Bounded sequences).

A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is bounded by a real number M iff we have $|a_n| \leq M$ for all $n \geq m$. We say that $(a_n)_{n=m}^{\infty}$ is bounded iff it is bounded by M for some real number $M > 0$.

Corollary 6.1.17.

Every convergent sequence of real numbers is bounded.

Theorem 6.1.19 (Limit Laws).

Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n \rightarrow \infty} a_n$ and $y := \lim_{n \rightarrow \infty} b_n$.

- (a) The sequence $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$; in other words,

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n.$$

- (b) The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy ; in other words,

$$\lim_{n \rightarrow \infty} (a_n b_n) = (\lim_{n \rightarrow \infty} a_n)(\lim_{n \rightarrow \infty} b_n).$$

- (c) For any real number c , the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx ; in other words,

$$\lim_{n \rightarrow \infty} (ca_n) = c \lim_{n \rightarrow \infty} a_n.$$

- (d) The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$; in other words,

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n.$$

- (e) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1} ; in other words,

$$\lim_{n \rightarrow \infty} b_n^{-1} = (\lim_{n \rightarrow \infty} b_n)^{-1}.$$

- (f) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y ; in other words,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}.$$

(g) The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$; in other words,

$$\lim_{n \rightarrow \infty} \max(a_n, b_n) = \max(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n).$$

(h) The sequence $(\min(a_n, b_n))_{n=m}^{\infty}$ converges to $\min(x, y)$; in other words,

$$\lim_{n \rightarrow \infty} \min(a_n, b_n) = \min(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n).$$

Exercise 6.1.1.

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, such that $a_{n+1} > a_n$ for each natural number n . Prove that whenever n and m are natural numbers such that $m > n$, then we have $a_m > a_n$.

Proof. Induct on p to show that $a_{n+p} > a_n$ for all positive integers p . When $p = 1$, as stated in the problem, we have $a_{n+p} = a_{n+1} > a_n$. Suppose inductively $a_{n+p} > a_n$ for positive integer p . Then $a_{n+p+1} > a_{n+p} > a_n$. This closes the induction. Since $m > n$, $m - n$ is a positive integer. Therefore, $a_{n+(m-n)} = a_m > a_n$. \square

Exercise 6.1.2.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let L be a real number. Show that $(a_n)_{n=m}^{\infty}$ converges to L if and only if, given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$.

Proof. Suppose $(a_n)_{n=m}^{\infty}$ converges to L . By definition, $(a_n)_{n=m}^{\infty}$ is eventually ε -close to L for every real $\varepsilon > 0$. Therefore, for any real $\varepsilon > 0$, there exists $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$.

Suppose given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$. Then any real $\varepsilon > 0$, there exists an $N \geq m$ such that $(a_n)_{n=m}^{\infty}$ is eventually ε -close to L . Therefore, by definition, $(a_n)_{n=m}^{\infty}$ converges to L . \square

Exercise 6.1.3.

Let $(a_n)_{n=m}^{\infty}$ be sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{n=m'}^{\infty}$ converges to c .

Proof. Suppose $(a_n)_{n=m}^{\infty}$ converges to c . Then for any arbitrary $\varepsilon > 0$, there exists $N_{\varepsilon} \geq m$ such that $|a_n - c| \leq \varepsilon$. So for any arbitrary $\varepsilon > 0$, we can find $M_{\varepsilon} = \max(N_{\varepsilon}, m')$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq M_{\varepsilon} \geq m'$. Therefore, $(a_n)_{n=m}^{\infty}$ converges to c .

Suppose $(a_n)_{n=m'}^{\infty}$ converges to c . For any arbitrary $\varepsilon > 0$, there exists $N_{\varepsilon} \geq m'$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq N_{\varepsilon}$, since $m' \geq m$, we have $N_{\varepsilon} \geq m$. Therefore, for any arbitrary $\varepsilon > 0$, we can find $N_{\varepsilon} \geq m$ such that $|a_n - c| \leq \varepsilon$ for all $n \geq N_{\varepsilon}$.

Thus, $(a_n)_{n=m}^{\infty}$ converges to $c \iff (a_n)_{n=m'}^{\infty}$ converges to c . \square

Exercise 6.1.4.

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $k \geq 0$ be a non-negative integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_{n+k})_{n=m}^{\infty}$ converges to c .

Proof. We can rewrite $(a_{n+k})_{n=m}^{\infty} = (a_n)_{m+k}^{\infty}$ since both of them are equal to the infinite sequence $a_{m+k}, a_{m+k+1}, \dots$. Since $k \geq 0$, $m+k \geq m$. Therefore, by Exercise 6.1.3, we have $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{m+k}^{\infty} = (a_{n+k})_{n=m}^{\infty}$ converges to c . \square

Exercise 6.1.5.

Prove proposition 6.1.12.

Proof. Suppose $\lim_{n \rightarrow \infty} (a_n)_{n=m}^{\infty} = L$. By definition, for all real $\varepsilon/2 > 0$, there exists $N_{\varepsilon} \geq m$ such that $|a_i - L| \leq \varepsilon/2$ and $|a_j - L| = |L - a_j| \leq \varepsilon/2$ for all $i, j \geq N_{\varepsilon}$. Then for all real $\varepsilon > 0$, let $N = N_{\varepsilon} \geq m$, we have $|a_i - a_j| \leq |a_i - L| + |L - a_j| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n, m \geq N$. Therefore, $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence. \square

Exercise 6.1.6.

Prove Proposition 6.1.15.

Proof. Write $L := \text{LIM}_{n \rightarrow \infty} a_n$. We want to show that $L = \lim_{n \rightarrow \infty} a_n$. Suppose L and $(a_n)_{n=m}^{\infty}$ are not eventually ε -close. Consider an arbitrary $\varepsilon/2 > 0$, since $(a_n)_{n=m}^{\infty}$ is Cauchy, there exists $N \geq m$ such that $|a_i - a_j| \leq \varepsilon/2$ for all $i, j \geq N$. For this N , since L and $(a_n)_{n=m}^{\infty}$ are not eventually ε -close, there exists $i \geq N$ such that $|a_i - L| > \varepsilon$. And since this $i \geq N$, we have $|a_i - a_j| \leq \varepsilon/2$ for all $j \geq i$. Since $|a_i - L| > \varepsilon$, we have

$$a_i > L + \varepsilon \text{ or } a_i < L - \varepsilon.$$

If $a_i > L + \varepsilon$, since $|a_i - a_j| \leq \varepsilon/2 \implies a_i - \varepsilon/2 \leq a_j \leq a_i + \varepsilon/2$, we have $L + \varepsilon/2 < a_i - \varepsilon/2 \leq a_j$. If $a_i < L - \varepsilon$, since $a_i - \varepsilon/2 \leq a_j \leq a_i + \varepsilon/2$, we have $a_j \leq a_i + \varepsilon/2 < L - \varepsilon/2$. Therefore, we either have $a_j < L - \varepsilon/2$ or $a_j > L + \varepsilon/2$ for all $j \geq i$.

If $a_j < L - \varepsilon/2$ for all $j \geq i$, by Exercise 5.4.8, we have $\text{LIM}_{n \rightarrow \infty} a_n \leq L - \varepsilon/2 < L$. If $a_j > L + \varepsilon/2$ for all $j \geq i$, by Exercise 5.4.8, we have $\text{LIM}_{n \rightarrow \infty} a_n \geq L + \varepsilon/2 > L$. Then we have either $\text{LIM}_{n \rightarrow \infty} a_n > L$ or $\text{LIM}_{n \rightarrow \infty} a_n < L$, and it contradicts the fact that $\text{LIM}_{n \rightarrow \infty} a_n = L$. Thus, L and $(a_n)_{n=m}^{\infty}$ are eventually ε -close, hence $L = \lim_{n \rightarrow \infty} a_n$. \square

Exercise 6.1.7.

Show that Definition 6.1.16 is consistent with Definition 5.1.12.

Proof. We would like to show that if $(a_n)_{n=1}^{\infty}$ is bounded by M then $(a_n)_{n=m}^{\infty}$ is bounded by M . If $(a_n)_{n=1}^{\infty}$ is bounded by M , by Definition 5.1.12, $|a_i| \leq M$ for all $i \geq 1$. Then for all $i \geq m \geq 1$, we have $|a_i| \leq M$. Therefore, $(a_n)_{n=m}^{\infty}$ is bounded by M . \square

Exercise 6.1.8.

Prove Theorem 6.1.19.

- (a) *Proof.* Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=m}^{\infty}$ converges to x , there exists $N_1 \geq m$ such that $|a_n - x| \leq \varepsilon/2$ for all $n \geq N_1$. Since $(b_n)_{n=m}^{\infty}$ converges to y , there exists $N_2 \geq m$ such that $|b_n - y| \leq \varepsilon/2$ for all $n \geq N_2$. Let $N = \max(N_1, N_2)$, we have $|(a_n + b_n) - (x + y)| = |(a_n - x) + (b_n - y)| \leq |a_n - x| + |b_n - y| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n \geq N$. Therefore, $\lim_{a_n+b_n} = x + y$. \square
- (b) *Proof.* Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=m}^{\infty}$ converges to x , $(a_n)_{n=m}^{\infty}$ is bounded by a positive number M_1 . There exists $N_1 \geq m$ such that $|a_n - x| \leq \varepsilon/2M_1$ for all $n \geq N_1$. We can easily know that there exists a positive number M_2 such that $M_2 \geq |x|$. Since $(b_n)_{n=m}^{\infty}$ converges to y , there exists N_2 such that $|b_n - y| \leq \varepsilon/2M_2$ for all $n \geq N_2$. Let $N = \max(N_1, N_2)$, then

$$\begin{aligned}
|a_n b_n - xy| &= |a_n b_n - x b_n + x b_n - xy| \\
&\leq |b_n| \cdot |a_n - x| + |x| \cdot |b_n - y| \\
&\leq M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} (a_n b_n) = (\lim_{n \rightarrow \infty} a_n)(\lim_{n \rightarrow \infty} b_n)$. \square

- (c) *Proof.* Let $(b_n)_{n=m}^{\infty} = c, c, c, \dots$, by (b), we have $\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} (c a_n) = c \lim_{n \rightarrow \infty} (a_n)$. \square
- (d) *Proof.* By (c), $\lim_{n \rightarrow \infty} (-b_n) = -\lim_{n \rightarrow \infty} b_n$. By (a), we have $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} (a_n + (-b_n)) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} (-b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} (b_n) = x - y$. \square
- (e) *Proof.* Since $(b_n)_{n=m}^{\infty}$ converges to y , there exists $N \geq m$ such that $|b_n - y| \leq |y|/2$ for all $n \geq N$. Then $y - |y|/2 \leq b_n \leq y + |y|/2$. If $y < 0$, $b_n \leq y + |y|/2 = y - y/2 = y/2 = -|y|/2$. If $y > 0$, $b_n \geq y - y/2 = |y|/2$. Therefore, $|b_n| \geq |y|/2$ for $n \geq N$. Let $c = \min(|b_m|, \dots, |b_{N-1}|, |y|/2)$, we have $|b_i| \leq c$ for all $i \geq m$. So $(b_n)_{n=m}^{\infty}$ is a Cauchy sequence bounded away from zero, hence $(b_n^{-1})_{n=m}^{\infty}$ is also a Cauchy sequence.

Consider an arbitrary $\varepsilon > 0$. Since $|b_n| \geq c$ for all $n \geq m$, we have $1/|b_n| \leq 1/c$ for all $n \geq m$. Since $(b_n)_{n=m}^\infty$ converges to y , there exists $N \geq m$ such that $|b_n - y| \leq c|y|\varepsilon$. Then

$$\begin{aligned} \left| \frac{1}{b_n} - \frac{1}{y} \right| &= \left| \frac{b_n - y}{b_n \cdot y} \right| \\ &= \frac{1}{|b_n|} \cdot \frac{1}{|y|} \cdot |b_n - y| \\ &\leq \frac{1}{c} \cdot \frac{1}{|y|} \cdot c|y|\varepsilon \\ &= \varepsilon. \end{aligned}$$

Thus, $(b_n^{-1})_{n=m}^\infty$ converges to $1/y$. \square

(f) *Proof.* By (b), $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = (\lim_{n \rightarrow \infty} a_n)(\lim_{n \rightarrow \infty} \frac{1}{b_n})$. By (e), $\lim_{n \rightarrow \infty} \frac{1}{b_n} = 1/y$. Therefore, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = x \cdot \frac{1}{y} = \frac{x}{y}$. \square

(g) *Proof.* Suppose $x \geq y$. Consider an arbitrary $\varepsilon > 0$. Since $(a_n)_{n=m}^\infty$ converges to x , there exists N_1 such that $|a_n - x| \leq (x - y)/2$ for $n \geq N_1$. Then $-(x - y)/2 \leq a_n - x \leq (x - y)/2$, hence $a_n \geq (x + y)/2$. Since $(b_n)_{n=m}^\infty$ converges to y , there exists N_2 such that $|b_n - y| \leq (x - y)/2$ for $n \geq N_2$. Then $b_n \leq (x + y)/2$. Let $N = \max(N_1, N_2)$, we have $a_n \geq (x + y)/2 \geq b_n$ for all $n \geq N$. So when $n \geq N$, $\max(a_n, b_n) = a_n$. Since $(a_n)_{n=m}^\infty$ converges to x , there exists $M_1 \geq m$ such that $|a_n - x| \leq \varepsilon$. Let $M = \max(N, M_1)$, we have $|\max(a_n, b_n) - \max(x, y)| = |a_n - x| \leq \varepsilon$ for all $n \geq M$.

Similarly, we can show that if $y > x$, there exists $M \geq m$ such that $|\max(a_n, b_n) - \max(x, y)| = |b_n - y| \leq \varepsilon$ for all $n \geq M$. Thus, $(\max(a_n, b_n))_{n=m}^\infty$ converges to $\max(x, y)$. \square

(h) *Proof.* By (b) and (g), we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \min(a_n, b_n) &= - \lim_{n \rightarrow \infty} \max(-a_n, -b_n) \\
&= - \max\left(\lim_{n \rightarrow \infty} (-a_n), \lim_{n \rightarrow \infty} (-b_n)\right) \\
&= - \max\left(- \lim_{n \rightarrow \infty} (a_n), - \lim_{n \rightarrow \infty} (b_n)\right) \\
&= \min\left(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n\right).
\end{aligned}$$

□

Exercise 6.1.9.

Explain why Theorem 6.1.19 (f) fails when the limit of the denominator is 0.

Proof. 0 does not have a reciprocal. A counterexample is $a_n = 1$, $b_n = 1/n$. Then $(a_n/b_n)_{n=m}^{\infty} = (n)_{n=m}^{\infty}$ is not convergent and hence does not have a limit. □

Exercise 6.1.10.

Show that the concept of equivalent Cauchy sequence, as defined in Definition 5.2.6, does not change if ε is required to be positive real instead of positive rational.

Proof. For all real $\varepsilon > 0$, suppose $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε -close. Since ε is also a rational number, we have $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε -close for every rational $\varepsilon > 0$. Suppose $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε -close for every rational $\varepsilon > 0$. Consider an arbitrary real $\varepsilon > 0$. Since there exists a rational number such that $0 < \varepsilon' < \varepsilon$. Since $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε' -close, there exists $N \geq m$ such that $|a_n - b_n| \leq \varepsilon' < \varepsilon$ for all $n \geq N$. Therefore, $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε -close for all real $\varepsilon > 0$. □