1.

Z_{21} forms a group with the modulo addition operation

- Closure: Clearly, adding one element in the set to another element in the set, the output would still be the element in the set.

- Associativity:

We know: $[(w + x) + y]\ mod\ n = [w + (x + y)]\ mod\ n$, in this case, n=21, w, x and y are the element in the set.

- Identity element: 0 is the identity element

- Inverse element

Every element in the set has additive inverse, the additive inverse of 0 is 0.
For element that is not 0 in the set, its additive inverse is 21-n.

Since all four properties are satisfied with addition operation, Z_{21} forms a group with the modulo addition operation.

Z_{21} does not form a group with the modulo multiplication operation, because not every element in the set has a multiplicative inverse, for example, the element 0 in the set does not have a multiplicative inverse, which makes it not a group with multiplication operation.

2.

The identity element is 0, because 0 is the divisor for all integers, so for any integer a, gcd(a,0) =a.

However, we are not able to find any pairs (a,b) in the set such that gcd(a,b)=0, because we know 0 cannot be divided by any integer, meaning that for any integer a, $\frac{a}{0}$ is invalid, hence the greatest common divisor cannot be 0, which leads to the non-existence of inverse element of each element in the set. Therefore, this set is not a group.

3.

$\gcd(21609, 18432)$
$= \gcd(18432, 3177)$
$= \gcd(3177, 2547)$
$= \gcd(2547, 630)$
$= \gcd(630, 27)$
$= \gcd(27, 9)$
$= \gcd(9, 0)$

Therefore, $\gcd(21609, 18432) = 9$

4.

$\gcd(24, 35)$		
$= \gcd(35, 24)$	residue	$24 = 1 \times 24 + 0 \times 35$
$= \gcd(24, 11)$	residue	$11 = -1 \times 24 + 1 \times 35$
$= \gcd(11, 2)$	residue	$2 = 1 \times 24 - 2 \times 11$
		$= 1 \times 24 - 2 \times (-1 \times 24 + 1 \times 35)$
		$= 1 \times 24 + 2 \times 24 - 2 \times 35$
		$= 3 \times 24 - 2 \times 35$
$= \gcd(2, 1)$	residue	$1 = 1 \times 11 - 5 \times 2$
		$= (35 - 24) - 5 \times (3 \times 24 - 2 \times 35)$
		$= 1 \times 35 - 1 \times 24 - 15 \times 24 + 10 \times 35$
		$= -16 \times 24 + 11 \times 35$
		$= 19 \times 24 + 11 \times 35$

Therefore, the multiplicative inverse of 24 in Z_{35} is 19.

5.

(a)

$$6x \equiv 3(mod\ 23)$$

6 is relatively prime to 23, so there is multiplicative inverse for 6 in Z_{23}, finding it using Extended Euclid's algorithm:

$\gcd(23, 6)$		
$= \gcd(6, 5)$	residue	$5 = 1 \times 23 - 3 \times 6$
$= \gcd(5, 1)$	residue	$1 = 1 \times 6 - 1 \times 5$
		$= 1 \times 6 - 1 \times (1 \times 23 - 3 \times 6)$
		$= 1 \times 6 - 1 \times 23 + 3 \times 6$
		$= 4 \times 6 - 1 \times 23$

Which means 6^{-1} in Z_{23} is 4, so

$$x = (3 \times 6^{-1})\ mod\ 23 = 12$$

(b)

$$7x \equiv 11(mod\ 13)$$

7 is relatively prime to 13, so there is multiplicative inverse for 7 in Z_{13}, finding it using Extended Euclid's algorithm:

gcd $(13, 7)$

$= $ gcd $(7, 6)$ | residue $6 = 1 \times 13 - 1 \times 7$

$= $ gcd $(6, 1)$ | residue $1 = 1 \times 7 - 1 \times 6$

| $= 1 \times 7 - 1 \times (1 \times 13 - 1 \times 7)$

| $= 1 \times 7 - 1 \times 13 + 1 \times 7$

| $= 2 \times 7 - 1 \times 13$

Which means 7^{-1} in Z_{13} is 2, so

$$x = (11 \times 7^{-1})\ mod\ 13 = 9$$

(c)

$$5x \equiv 7(mod\ 11)$$

5 is relatively prime to 11, so there is multiplicative inverse for 5 in Z_{11}, finding it using Extended Euclid's algorithm:

gcd $(11, 5)$

$= $ gcd $(5, 1)$ | residue $1 = 1 \times 11 - 2 \times 5$

| $= 1 \times 11 + 9 \times 5$

Which means 5^{-1} in Z_{11} is 9, so

$$x = (7 \times 5^{-1})\ mod\ 11 = 8$$