Bachelor of Sciences Honours in Surveying Sciences - 2024
22GES

Lecture Notes on

FC11221 - Calculus

Dr. M.K.Abeyratne
Senior Lecturer in Mathematics

1 SOME ELEMENTS OF LOGIC

In this section we will discuss in an informal way some notions of logic and their importance of mathematical proofs and some topics on set theory.

All Mathematical results are obtained by logical deductions from previously obtained results which are now accepted as true, or from the axioms which have been assumed to be true. Therefore, the logic plays an important role in Mathematical analysis. In this chapter we are concerned on two concepts of elementary logic namely, propositions and quantifiers and some related results.

1.1 Propositions (Sentences)

In this section, we will look at propositions (sentences), their truth or falsity and the way of combining or connecting two or more sentences to produce new sentences.

A proposition (or sentence) is a statement (or expression) which is either true or false (but not both). The proposition "$3 + 5 = 8$" is true, while the proposition "π is rational" is false. However, it is not the task of logic to decide whether a given statement is true or false. Let us look at the following expressions:

Read this lecture notes carefully

$1 + 1 = 2$

$2 + 2 = 3$

$x + 1 = 2$

First expression is not a proposition because it is not a declarative sentence. Second and third expressions are propositions and they take "True" and "False" as truth values respectively. Forth expression is different from first three and it is not a proposition because, it is neither true or false since the variable x has not been assigned values. Since there are expressions which are propositions under our definition we discuss ways to connect propositions to form new propositions.

The first logic symbols we use in this notes are listed in the following table:

Symbol	Read as
$\sim$	NOT
$\wedge$	AND
$\vee$	OR
$\Longrightarrow$	IMPLIES
$\Longleftrightarrow$	IF AND ONLY IF

The above symbols are use to build new statements (propositions) using old/existing statements and thus these are called logical operators. Moreover, the operator $\sim$ is known as a unitary operator since it operates on one statement. Other four operators are often called binary (logical) operators, connectives since the connect in general two statements.

Definition 1.1 *(NOT) If P is a statement then the negation of P is denoted by $\sim P$ and read as not P. This means that if P is true the $\sim P$ is false and if P is false then $\sim P$ is true.*

If we use the letter T to denote "true" and the letter "F" to denote "false", this result can be represented by the following truth table.

P	$\sim P$
T	F
F	T

For example, if we define a statement P : *I will attend in the calculus lecture*, then the resultant statement for $\sim P$ is simply $\sim P$: *I will not attend in the calculus lecture* or $\sim P$: *it is not the case that I will attend in the calculus lecture*. Moreover $\sim\sim P$ is same as P.

Example 1.1 *The negation of the proposition $1 + 3 = 4$ is $1 + 3 \neq 4$.*

To define the binary logical operators (Logical connectives), let P and Q be two propositions.

Definition 1.2 *(CONJUNCTION) We say that the proposition $P \wedge Q$ (or P and Q) is true, if the two propositions P and Q both are true,and is false otherwise.*

The corresponding truth table is as follows:

P	P	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.3 *(DISJUNCTION) We say that the proposition $P \vee Q$ (or P or Q) is true, if at least one of two propositions P and Q is true, and is false otherwise.*

The corresponding truth table is as follows:

P	P	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.2
The proposition "$1+4=5$ *and* $4+9=13$*" is true.*
The proposition "$1+4=6$ *or* $4+9=12$*" is false.*
The proposition "$1+4=6$ *or* $4+9=13$*" is true.*

Definition 1.4 *(CONDITIONAL) We say that the proposition* $P \Longrightarrow Q$ *(or if* P *then* Q*, or* P *implies* Q*) is true, if the sentence* P *is false or if the sentence* Q *is true or true both and is false otherwise.*

It should be noted that the proposition $P \Longrightarrow Q$ is false only if p is true and q is false. To understand this, note that if we draw a false conclusion from a true assumption, then our argument must be faulty. On the other hand, if our assumption is false and the conclusion is true, the argument may still be acceptable. The corresponding truth table is as follows:

P	P	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.3

- *Let* $P: 2+2=2$*, and* $Q: 1+3=5$*. Then the proposition* $P \Longrightarrow Q$ *(or if* P *then* Q*) is true, because* P *is false and* Q *is also false.*
- $P: 2+2=4$*,* $Q: 1+3=5$ *then* $p \Longrightarrow Q$ *is false*
- $P: 2+2 \neq 4$*,* $Q: 1+3=5$ *then* $P \Longrightarrow Q$ *is true.*

Definition 1.5 *(BICONDITIONAL) We say that the proposition* $P \Longleftrightarrow Q$ *(read as* P *if and only if* Q*) is true, if the two propositions* P*,* Q *both are true or both are false, and is false otherwise.*

The corresponding truth table is as follows:

P	P	$P \Longleftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

It is remarked here that the proposition $P \Longleftrightarrow Q$ is the same as the proposition $(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$.

Example 1.4

- *Let* $P: 2+2=4$*, and* $Q: \pi$ *is irrational. Then the proposition* $P \Longleftrightarrow Q$ *is true.*

- $P : 2 + 2 = 4$, $Q : 1 + 3 = 5$ *then* $P \Longleftrightarrow Q$ *is false*
- $P : 2 + 2 \neq 4$, $Q : 1 + 3 = 5$ *then* $P \Longleftrightarrow Q$ *is true.*

The above five definitions can be summarized in the following table.

P	Q	$P \wedge Q$	$P \vee Q$	$\sim P$	$P \Longrightarrow Q$	$P \Longleftrightarrow Q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

1.2 Tautologies and Contradictions

Definition 1.6 *A compound proposition that is always true, no matter what the truth values of the propositions that occur in, is called a* **tautology** *(i.e. a tautology is a proposition which is always true) A compound proposition that is always false, no matter what the truth values of the propositions that occur in, is called a* **contradiction** *(i.e. a contradiction is a proposition which is always false).*

For example, $P \wedge \sim P$ is a contradiction and $P \vee \sim P$ is a tautology.

1.3 Logically Equivalence

Definition 1.7 *We say that a proposition P is logically equivalent to a proposition Q if and only if $P \longleftrightarrow Q$ is a tautology. We denote it by $P \equiv Q$.*

For example, $P \Longrightarrow Q$ is logically equivalent to $\sim Q \Longrightarrow \sim P$. The latter is known as the contrapositive of the former. (There are some related implications that can be formed from $P \Longrightarrow Q$). The proposition $P \Longrightarrow Q$ is not logically equivalent to $Q \Longrightarrow P$, where the latter one is known as the converse of the former. Moreover, the proposition $\sim P \Longrightarrow \sim Q$ is the inverse of $P \Longrightarrow Q$ and they are not equivalent to each other..

1.4 Precedence of logical operators

Operator	Precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\Longrightarrow$	4
$\Longleftrightarrow$	5

Examples: $\sim P \wedge Q \equiv (\sim P) \wedge Q$ and $P \wedge Q \vee R \equiv (P \wedge Q) \vee R$

Theorem 1.1 *Let P, Q, R be propositions. Then the followings hold.*

1. *Distributive law :*

   (a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

   (b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

2. *De Morgan's law :*

   (a) $\sim (P \wedge Q) \equiv (\sim P \vee \sim Q)$

   (b) $\sim (P \vee Q) \equiv (\sim P \wedge \sim Q)$

3. *Associative law :*

   (a) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R.$

   (b) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R.$

Exercise 1.1 *Construct truth tables for the following statements*

1. $(\sim P) \iff (\sim Q)$ *(Compare with $P \iff Q$)*
2. $[P \vee (\sim Q)] \implies P$
3. $\sim [P \wedge (Q \vee R)$ *(consider all possible combinations of truth values for P,Q and R).*

Exercise 1.2 *For the following expressions, decide which are tautologies,which are contradictions and which are neither.Try to decide using intuition and then check with truth tables.*

1. $P \implies P$
2. $P \iff \sim P$
3. $P \implies (\sim P)$
4. $P \wedge \sim P \implies Q$
5. $P \wedge Q \implies P$
6. $P \vee Q \implies P$
7. $P \implies (P \wedge Q)$
8. $P \implies (P \vee Q)$

Exercise 1.3 *Is the statements $P \implies Q$ and $\sim P \vee Q$ logically equivalent ? Justify your answer.*

1.5 Propositional Functions

In many instances we have propositions like **x** is even which contains one or more variables. We shall call them propositional functions. It is clear that the expression **x** is even is true for only certain values of x and is false for others. In this point various questions may arise:

1. What values of x do we permit ?
2. Is the statement true for all such values of x in question ?
3. Is the statement true for some such values of x in question ?

To answer the first question we need the notion of a universe (domain of x). At this point we have to consider sets. An important thing about a set is what it contains (what are the elements of a set ?). If P is a set and x is an element of P, then we write $x \in P$. A set is usually described by the following two ways:

1. by enumeration. e.g. $\{1, 2, 3\}$ denotes the set consisting of the elements $1, 2$ and 3.
2. by a defining property $p(x)$. $P = \{x \mid x \in U, p(x) \text{ is true }\}$ or simply $P = \{x \mid p(x)\}$.

The set with no elements is called empty set and denoted by Φ.

Example 1.5

a) $\mathbb{N} = \{1, 2, 3, \dots\}$ = *set of natural numbers.*

b) $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ = *set of integers.*

c) $P = \{x \mid x \in \mathbb{Z}, -3 < x < 2\} = \{-2, -1, 0, 1\}$.

1.6 Mathematical Statements

In a mathematical proof or discussion one makes various assertions often called statements or sentences. For example

1. $(x + y)^2 = x^2 + 2xy + y^2$
2. $3x^2 + 2x + -1 = 0$
3. if n, $n \geq 3$ is an integer then $a^n + b^n = c^n$ has no positive integer solution. 0erivative of a function x^2 is $2x$

Remark that, since all mathematical statements have precise meaning the above (1) may be an of the precise statement *for all real numbers x and y*,$(x + y)^2 = x^2 + 2xy + y^2$. We discuss such statement in the following section.

1.7 Quantifier Logic

Let us now recall our propositional function $P(x)$: x is even, and suppose that we have restricted ourselves the variable x to lie in the set of all integers ($\mathbb{Z}$). Then the propositional function $P(x)$ is true only for some $x \in \mathbb{Z}$. Therefore the proposition **for some** $x \in \mathbb{Z}, P(x)$ is true. Moreover, **for all** $x \in \mathbb{Z}, P(x)$ is false. Therefore, we have two cases *for some* x, $P(x)$ and *for all* x, $P(x)$. Here we consider in general, a propositional function $P(x)$, where the variable x lies in a clearly stated set (which is sometimes called domain or universe of discourse).

Definition 1.8

1. *The expression* **for all** *(or for every or for each or for any) is called the* **universal qualifier** *and is denoted as* $\forall$
2. *The expression* **there exists** *(or there is or there is atleast one or there are some) is called the* **existential qualifier** *and is denoted as* $\exists$
3. *The* **uniqueness quantifier** *is read as bf unique and is denoted by* !.

Remark 1.1 *The symbols* $\forall x$ *(for all* x*) and* $\exists x$ *(for some* x *or there exists* x*) are called the* **universal quantifier** *and* **existential quantifier***, respectively. Note that the variable* x *is a "dummy variable", and thus there is no difference between, for example,* $\forall x P(x)$ *and* $\forall y P(y)$*.*

Example 1.6 *Let* S *be a set and* $P(x)$ *be a statement about* x*.*

- $(\forall x \in S)P(x)$ *i.e. for all* $x \in S$, $P(x)$ *is true.*
- $(\exists x \in S)P(x)$ *i.e. there exists an* $x \in S$ *such that* $P(x)$ *is true.*
- $(\exists! x \in S)P(x)$ *i.e. there exists a unique (exactly one)* $x \in S$ *such that* $P(x)$ *is true.*

Example 1.7 *Let* S *be a set and* $P(x)$ *be a statement about* x*.*

- $(\forall x \in \mathbb{R})\ (x + x = 2x)$
- $(\exists x \in \mathbb{R})\ (x + 2 > 3)$
- $(\exists! x \in \mathbb{R})\ (x + 2 = 3)$

Example 1.8 *The following all have the same meaning*

- *For all* x *and for all* y, $(x + y)^2 = x^2 + 2xy + y^2$
- *For all* x *and* y, $(x + y)^2 = x^2 + 2xy + y^2$
- *For each* x *and for each* y, $(x + y)^2 = x^2 + 2xy + y^2$
- $\forall x \forall y$, $(x + y)^2 = x^2 + 2xy + y^2$

1.8 Order of Quantifiers

The order in which quantifiers occur is often critical. For example consider the statements

$$\forall x \exists y\ (x < y) \tag{1}$$
$$\exists y \forall x\ (x < y) \tag{2}$$

We read these statements (1) and (2) as for all x there exists y such that $x < y$ and there exists y such that for all x, $x < y$ respectively. Suppose that x and y are real numbers (i.e. they are in the real line).

Let x be an arbitrary real number. They, clearly $x < x + 1$, and so if we choose $y = x + 1$, $x < y$ is true. i.e (1) is true. Now let y be an arbitrary number,. Then clearly $y + 1 < y$ is false. Hence $\forall x(x < y)$ is false for the value $x = y + 1$. Therefore (2) is false.

Example 1.9 *(Goldbach Conjecture) Every even natural number greater than* 2 *is the some of two primes. This can be written, in logical notation mentioned above,as*

$$\forall n \in \mathbb{N} \setminus \{1\}, \exists p, q, prime\ 2n = p + q$$

which means for all $n \in \mathbb{N}$ *which is greater than* 1*, there exist prime numbers* p *and* q *such that* $p + q = 2n$*.*

Note that it is not yet known whether this is true or false (an unsolved problem in mathematics).

Example 1.10 *Let* B *be the set of male undergraduate students in the second year batch of B.Sc. Honours in Surveying Science programme and* G *be the set of female undergraduate students in the first year batch of B.Sc. Honours in Surveying Science programme. The Statement* $L(x, y)$ *is defined as* x *likes* y*, where* $x \in B$ *and* $y \in G$*. Observe the meanings of* $\forall x L(x, y)$ *and* $\exists y \forall x L(x, y)$*. Check whether they are same or not.*

1.9 Negation of quantifiers

Now we are concerned on how to develop a rule for negating proposition with quantifiers which is one of the most important aspects of mathematical analysis. (This will be frequently used in this course). Let us say, for instance, that "you all are following geography as a subject". Symbolically we can formulate it as follows: $P(x)$: x is following geography as a subject. Then $P(x)$. Naturally, you will disagree with $\forall x P 9 x)$ and some of you will complain. So it is natural to aspect the negation of the proposition "all are following Geography as a subject" is the proposition "some of you are not following Geography as a subject" or symbolically the negation of $\forall x P(x)$ is $\exists \sim P(x)$. This negation can also be read as there exists atleast one x such that $\sim P(x)$.

1. The negation of $\forall x P(x)$, (i.e. the statement $\sim (\forall x P(x))$ is equivalent to $\exists x \sim P(x)$. Likewise the negation of $\forall x \in \mathbb{R} P(x)$ is equivalent to $\exists x \in \mathbb{R}\ (\sim P(x))$

2. The negation of $\exists x P(x)$, (i.e. the statement $\sim (\exists x P(x))$ is equivalent to $\forall \sim P(x)$. Likewise the negation of $\exists x \in \mathbb{R} P(x)$ is equivalent to $\forall x \in \mathbb{R}\ (\sim P(x))$

3. The negation of $\forall x \exists y P(x,y)$, is equivalent to $\exists x \forall y \sim P(x,y)$ and the negation of $\exists x \forall y P(x,y)$, is equivalent to $\forall x \exists y \sim P(x,y)$

Example 1.11 *The negation of Goldbach Conjecture is, in logical notation,*

$$\exists n \in \mathbb{N}\ \{1\}, \forall p, q \; prime \;\; 2n \neq p + q$$

Example 1.12 *Let a be a fixed real number. Then the negation of $x \in \mathbb{R}\ (x > a)$ is $\forall x \in \mathbb{R}\ (x \leq a)$*

Exercise 1.4 *Negate the following logical statements*

1. $\exists x \in S(P(x) \implies Q(x))$
2. $\exists x \in S(P(x) \wedge Q(x))$

Exercise 1.5 *For each of the following logical statements, write a negation of the statement and decide whether which is true the original statement or its negation*

1. $\exists x \in \mathbb{R}\ (x^2 < 0)$
2. $\forall x \in \mathbb{R}\ (|-x| = x)$
3. $\forall x \in \mathbb{R} \exists y \in \mathbb{R}\ (y = 2x + 1)$

Exercise 1.6 *Consider the statement $\forall x, y \in \mathbb{R}\ (x < y \implies x^2 < y^2)$.*

1. *Write the negation of the statement.*
2. *In fact it is the negation that is true. Explain why.*

Exercise 1.7 *Consider the statement, for any $\epsilon > 0$ there exists a natural number N such that $|a_n - l| < \epsilon$ for all $n \geq N$. Where a_n is the n^{th} term of a sequence of real numbers $\{a_n\}$ and l is the finite limit. Write down the statement symbolically and find its negation.*

1.10 Proofs

A mathematical proof of a theorem is a sequence of assertions (mathematical statements) of which the last assertion is the desired conclusion. Each assertion

- is an axioms or previously proved theorem OR
- is an assumption stated in the theorem OR
- follows from earlier assertion in the proof (in an appropriate way).

1.10.1 Proofs of Statements Involving Logical Connectives

1.10.2 Proofs of Statements Involving $p \wedge q$ and $p \vee q$

To prove a theorem whose conclusion is of the form p and q we have to show that both p is true and q is true. To prove a theorem whose conclusion is of the form p or q we have to show that atleast one of the statement p or q is true. Three different ways to doing this are

- Assume p is true and use this to show q is true
- Assume q is false and use this to show p is true
- Assume p and $q$are both false and obtain a contradiction.

1.10.3 Proofs of Statements Involving p implies q

To prove a theorem of the form p **implies** q we can proceed one of the following ways

- Assume p is true and use this to show q is true
- Assume q is false and use this to show p is false (contrapositive)
- Assume p is true and q is false and use this to obtain a contradiction

1.10.4 Proofs of Statements Involving p iff q

To prove a theorem of the form p **if and only if** q we usually show that p implies q and show that q implies p.

1.10.5 Proofs of Statements Involving "There Exists"

In order to prove a theorem whose conclusion is of the form **there exists** x **such that** $P(x)$, we use one of the following methods:

- show that for a certain explicit value of x the statement $P(x)$ is true.
- use an indirect argument to show that some x with property $P(x)$ does exist.

- assume $P(x)$ is false for all x and obtain a contradiction.

As examples, consider the proofs of the following (simple) theorems/statements.

Theorem 1.2 *There exists a real number x such that $P(x) = x^5 - 5x - 7 = 0$*

Using a suitable technique we can find an explicit real value of x such that $P(x)$ is true. That is, we find a root of the equation.
Alternatively, as an indirect method, we can get the existence of such real x indirectly. To do this let $f(x) = x^5 - 5x - 7$. We observe that $f(1) < 0$ and f(2)¿0. Since f is continuous another theorem (intermediate value theorem) gives that there exists real x such that $f(x) = 0$ which completes the proof. Finally, it is remarked that one may assume that $P(x)$ is false and deduce a contradiction.

1.10.6 Proofs of Statements Involving "For All"

In order to prove a theorem whose conclusion is of the form **for all** x **such that** $P(x)$, we may use the following method. Choose an arbitrary x and deduce a contradiction from $\sim P(x)$.

Theorem 1.3 *Prove that for any positive real x, $x^5 + x^3 + x > 0$.*

Direct Proof:
Use the property that $x > 0$ and consider the term $x^5 + x^3 + x$ and obtain that this term is positive.

$$\begin{aligned} x^5 + x^3 + x &= x(x^4 + x^2 + 1) \\ &= (> 0)((> 0) + (> 0) + (> 0)) \\ &> 0 \end{aligned}$$

Theorem 1.4 *Prove that if $x^5 + x^3 + x > 0$ then $x > 0$.*

Indirect Proof- 01:
Let $P(x) : x^5 + x^3 + x > 0$ and $Q(x) : x > 0$. Now we assume that P is true and Q is false and make a contradiction. i.e. Suppose that $x^5 + x^3 + x > 0$ and $x \leq 0$. Consider two cases $x = 0$ and $x < 0$ separately.
When $x = 0$, $x^5 + x^3 + x = 0$ This is a contradiction since P is true.
When $x < 0$, we have $x^5 < 0$, $x^3 < 0$ and $x < 0$. Therefore $x^5 + x^3 + x < 0$ This is a contradiction. Therefore, our assumption that $x \leq 0$ is wrong. Hence $x > 0$ which completes the proof.

Indirect Proof- 02:
We use the method of contrapositive. That is we prove that $\sim Q \Longrightarrow \sim P$
Suppose that $x \leq 0$. Then $x = 0$ or $x < 0$.
When $x = 0$, $x^5 + x^3 + x = 0$
When $x < 0$, then as before $x^5 + x^3 + x < 0$ In both cases $x^5 + x^3 + x \leq 0$, That is $\sim Q$,

which completes the proof.

To disprove a statement we commonly use the method of counterexample. Consider the following proof of the statement that if n is a positive integer then n^2+n+1 is prime. Then we observe the term for $n = 1, 2, 3$. In all cases we get prime number. But $n = 4$ can be considered as the counterexample since 4 is a positive integer but $4^2 + 4 + 1 = 21$ is not prime.

We say that a number q is rational if there is two integers m, n such that $q = m/n$, where $n \neq 0$ and there is no common factors between m and n other than 1. i.e $(m, n) = 1$ or m and n are relatively prime. The set of rational numbers is denoted by $\mathbb{Q}$.

$$\mathbb{Q} = \{q \mid q = \frac{m}{n},\ \ m, n \in \mathbb{Z},\ (m, n) = 1\}.$$

The real numbers which are not rational are called irrational numbers.

Theorem 1.5 $\sqrt{2}$ *is irrational.*

Proof (By contradiction
We assume that the number $\sqrt{2}$ is rational and create a contradiction.

From the definition of irrational numbers this theorem can also be interpreted as follows: *for all integers m, n, $\frac{m}{n} \neq \sqrt{2}$.* We prove this statement.

Proof: Suppose that $\sqrt{2}$ is rational. Then there exists integers m, n such that $\dfrac{m}{n} = \sqrt{2}$, where m and n has no common factors.

We have $\dfrac{m}{n} = \sqrt{2}$ and hence $m^2 = 2n^2$. It gives us m^2 is an even number. Therefore, m is even.
Let $m = 2k$, where k is an integer.
Then $m^2 = (2k)^2 = 2n^2$ and hence we have $n^2 = 2k^2$.
It follows that n^2 is an even number and so n.
Therefore, we have concluded that both m and n are even numbers. Thus they have common factor 2.
This is a contradiction with the assumption of rational numbers. $\sqrt{2}$ is irrational.