

# Formal Theory of Communication Topology in Concurrent ML

Thomas Logan

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# 1 Mathematical Artifacts

$$f(x) = x^2$$

```

1 type thread_id
2 val spawn: (unit -> unit) -> thread_id
3
4 type 'a chan
5 val channel : unit -> 'a chan
6 val recv : 'a chan -> 'a
7 val send : ('a chan * 'a) -> unit
8
9
10
11
12 signature SERV = sig
13     type serv
14     val make : unit -> serv
15     val call : serv * int -> int
16 end
17
18 structure Serv : SERV = struct
19     datatype serv = S of (int * int chan) chan
20
21     fun make () = let
22         val reqChn = channel ()
23         fun loop state = let
24             val (v, replCh) = recv reqChn in
25             send (replCh, state);
26             loop v end in
27         spawn (fn () => loop 0);
28         S reqChn end
29
30     fun call (server, v) = let
31         val S reqChn = server
32         val replChn = channel () in
33         send (reqCh, (v, replCh));
34         recv replChn end end
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11   val thenEvt: 'a event * ('a -> 'b event) -> 'b event
12
13
1   val server = Serv.make ()
2   val _ = spawn (fn () => Serv.call (server, 35))
3   val _ = spawn (fn () =>
4       Serv.call (server, 12);
5       Serv.call (server, 13))
6   val _ = spawn (fn () => Serv.call (server, 81))
7   val _ = spawn (fn () => Serv.call (server, 44))
8
1   structure Serv : SERV = struct
2     datatype serv = S of (int * int chan) chan
3
4     fun make () = let
5
6       val reqChn = FanIn.channel()
7
8       fun loop state = let
9         val (v, replCh) = FanIn.recv reqChn in
10        OneShot.send (replCh, state);
11        loop v end in
12
13      spawn (fn () => loop 0);
14      S reqChn end
15
16    fun call (server, v) = let
17      val S reqChn = server
18      val replChn = OneShot.channel () in
19      FanIn.send (reqCh, (v, replCh));
20      OneShot.recv replChn end
21
22    end
23
1   let
2     val w = 4
3     val x = ref 1
4     val y = ref 2
5     val z = (!x + 1) + (!y + 2) + (w - 3)
6     val w = 1 in
7     y := 0;
8     (!y + 2) - (!x + 1) * (w - 3) end
9
1   let
2     val x = 1
3     val y = 2

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4      val z = ref (4 * 73)
5      val x = 4 in
6      z := 1;
7      x * !z end
8
1
2      let
3          val f = fn x => x 1
4          val g = fn y => y + 2
5          val h = fn z => z + 3 in
6          (f g) + (f h) end
7
8
1
2      datatype 'a list = Nil | Cons 'a ('a list)
3
4      inductive sorted ::
5          ('a => 'a => bool) =>
6          'a list => bool where
7          Nil : sorted P Nil |
8          Single : sorted P (Cons x Nil) |
9          Cons :
10             P x y ==>
11             sorted P (Cons y ys) ==>
12             sorted P (Cons x (Cons y ys))
13
1
2      datatype nat = Z | S nat
3
4      inductive lte :: nat => nat => bool where
5          Eq : lte n n |
6          Lt : lte n1 n2 ==> lte n1 (S n2)
7
8      theorem "
9          sorted lte
10             (Cons (Z) (Cons (S Z)
11                 (Cons (S Z) (Cons
12                     (S (S (S Z))) Nil))))"
13      apply (rule Cons)
14      apply (rule Lt)
15      apply (rule Eq)
16      apply (rule Cons)
17      apply (rule Eq)
18      apply (rule Cons)
19      apply (rule Lt)
20      apply (rule Lt)
21      apply (rule Eq)
22      apply (rule Single)

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```

22     done
23

```

```

1
2  definition True :: bool where
3      True  $\equiv$  (( $\lambda x :: \text{bool}.$  x) = ( $\lambda x.$  x))
4
5  definition False :: bool where
6      False  $\equiv$  ( $\forall P.$  P)
7
8

```

```

1
2  signature CHAN = sig
3      type 'a chan
4      val channel: unit -> 'a chan
5      val send: 'a chan * 'a -> unit
6      val recv: 'a chan -> 'a
7  end
8

```

```

1
2  structure ManyToManyChan : CHAN = struct
3      type message_queue = 'a option ref queue
4
5      datatype 'a chan_content =
6          Send of (condition * 'a) queue |
7          Recv of (condition * 'a option ref) queue |
8          Inac
9
10     datatype 'a chan =
11         Chn of 'a chan_content ref * mutex_lock
12
13     fun channel () = Chn (ref Inac, mutexLock ())
14
15     fun send (Chn (conRef, lock)) m =
16         acquire lock;
17         (case !conRef of
18             Recv q => let
19                 val (recvCond, mopRef) = dequeue q in
20                 mopRef := Some m;
21                 if (isEmpty q) then conRef := Inac else ();
22                 release lock; signal recvCond; () end |
23             Send q => let
24                 val sendCond = condition () in
25                 enqueue (q, (sendCond, m));
26                 release lock; wait sendCond; () end |
27             Inac => let
28                 val sendCond = condition () in
29                 conRef := Send (queue [(sendCond, m)]);

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30         release lock; wait sendCond; () end)
31
32 fun recv (Chn (conRef, lock)) =
33     acquire lock;
34     (case !conRef of
35         Send q => let
36             val (sendCond, m) = dequeue q in
37             if (isEmpty q) then
38                 conRef := Inac
39             else
40                 ();
41             release lock; signal sendCond; m end |
42         Recv q => let
43             val recvCond = condition ()
44             val mopRef = ref None in
45             enqueue (q, (recvCond, mopRef));
46             release lock; wait recvCond;
47             valOf (!mopRef) end |
48         Inac => let
49             val recvCond = condition ()
50             val mopRef = ref None in
51             conRef := Recv (queue [(recvCond, mopRef)]);
52             release lock; wait recvCond;
53             valOf (!mopRef) end)
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55 end
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22         acquire lock;
23         (let
24             val (recvCond, mopRef) = dequeue q in
25             mopRef := Some m;
26             if (isEmpty q) then conRef := Inac else ();
27             release lock; signal (recvCond);
28             () end) |
29         Send _ => raise NeverHappens end
30
31     fun recv (Chn (conRef, lock)) =
32         acquire lock;
33         (case !conRef of
34             Inac => let
35                 val recvCond = condition ()
36                 val mopRef = ref None in
37                 conRef := Recv (queue [(recvCond, mopRef)]);
38                 release lock; wait recvCond;
39                 valOf (!mopRef) end |
40             Recv q => let
41                 val recvCond = condition ()
42                 val mopRef = ref None in
43                 enqueue (q, (recvCond, mopRef));
44                 release lock; wait recvCond;
45                 valOf (!mopRef) end |
46             Send (sendCond, m) =>
47                 conRef := Inac;
48                 release lock;
49                 signal sendCond;
50                 m end)
51
52     end
53
54
55 1     structure FanInChan : CHAN = struct
56 2
57 3     datatype 'a chan_content =
58 4         Send of (condition * 'a) queue |
59 5         Recv of condition * 'a option ref |
60 6         Inac
61 7
62 8     datatype 'a chan =
63 9         Chn of 'a chan_content ref * mutex_lock
64 10
65 11     fun channel () = Chn (ref Inac, mutexLock ())
66 12
67 13     fun send (Chn (conRef, lock)) m =
68 14         acquire lock;
69 15         case !conRef of
70 16             Recv (recvCond, mopRef) =>
71 17                 mopRef := Some m; conRef := Inac;

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18         release lock; signal recvCond;
19         () |
20     Send q => let
21         val sendCond = condition () in
22         enqueue (q, (sendCond, m));
23         release lock; wait sendCond;
24         () end |
25     Inac => let
26         val sendCond = condition () in
27         conRef := Send (queue [(sendCond, m)])
28         release lock; wait sendCond; () end
29
30 fun recv (Chn (conRef, lock)) = let
31     val recvCond = condition ()
32     val mopRef = ref None in
33     case cas (conRef, Inac, Recv (recvCond, mopRef)) of
34         Inac => (* conRef already set *)
35             wait recvCond; valOf (!mopRef) |
36         Send q =>
37             (* the current thread is the only one
38              * that updates the state from this state *)
39             acquire lock;
40             (let
41                 val (sendCond, m) = dequeue q in
42                 if (isEmpty q) then conRef := Inac else ();
43                 release lock; signal sendCond; m end) |
44         Recv _ => raise NeverHappens end end
45
46

```

```

1
2 structure OneToOneChan : CHAN = struct
3
4     datatype 'a chan_content =
5         Send of condition * 'a |
6         Recv of condition * 'a option ref |
7         Inac
8
9     datatype 'a chan = Chn of 'a chan_content ref
10
11 fun channel () = Chn (ref Inac)
12
13 fun send (Chn conRef) m = let
14     val sendCond = condition () in
15     case cas (conRef, Inac, Send (sendCond, m)) of
16         Inac =>
17             (* conRef already set to Send *)
18             wait sendCond; () |
19         Recv (recvCond, mopRef) =>
20             (* the current thread is the only one

```



```

21      -* that accesses conRef for this state *
22      mopRef := Some m; conRef := Inac;
23      signal recvCond; () |
24      Send _ => raise NeverHappens end end
25
26
27 fun recv (Chn conRef) = let
28   val recvCond = condition ();
29   val mopRef = ref None in
30   case cas (conRef, Inac, Recv (recvCond, mopRef)) of
31     Inac => (* conRef already set to Recv*)
32     wait recvCond; valOf (!mopRef) |
33     Send (sendCond, m) =>
34       (* the current thread is the only one
35        -* that accesses conRef for this state *)
36       conRef := Inac; signal sendCond; m |
37     Recv _ => raise NeverHappens end end
38
39 end
40
1  structure OneShotChan : CHAN = struct
2
3  datatype 'a chan_content =
4    Send of condition * 'a |
5    Recv of condition * 'a option ref |
6    Inac
7
8  datatype 'a chan = Chn of 'a chan_content ref *
mutex_lock
9
10 fun channel () = Chn (ref Inac, lock ())
11
12 fun send (Chn (conRef, lock)) m = let
13   val sendCond = condition () in
14   case (conRef, Inac, Send (sendCond, m)) of
15     Inac =>
16       (* conRef already set to Send*)
17       wait sendCond; () |
18     Recv (recvCond, mopRef) =>
19       mopRef := Some m; signal recvCond;
20       () |
21     Send _ => raise NeverHappens end end
22
23
24 fun recv (Chn (conRef, lock)) = let
25   val recvCond = condition ()
26   val mopRef = ref None in
27   case (conRef, Inac, Recv (recvCond, mopRef)) of
28     Inac =>

```

```

29      (* conRef already set to Recv*)
30      wait recvCond; valOf (!mopRef) |
31      Send (sendCond, m) =>
32          acquire lock; signal sendCond;
33          (* never releases lock;
34          -* blocks others forever *)
35          m |
36      Recv _ =>
37          acquire lock;
38          (* never able to acquire lock;
39          -* blocked forever *)
40          raise NeverHappens end end
41
42 end
43
1  structure OneShotToOneChan : CHAN = struct
2
3      datatype 'a chan =
4          Chn of condition * condition * 'a option ref
5
6      fun channel () =
7          Chn (condition (), condition (), ref None)
8
9      fun send (Chn (sendCond, recvCond, mopRef)) m =
10          mopRef := Some m; signal recvCond;
11          wait sendCond; ()
12
13      fun recv (Chn (sendCond, recvCond, mopRef)) =
14          wait recvCond; signal sendCond;
15          valOf (!mopRef)
16
17      end
18
1
2      datatype var = Var string
3
4      datatype exp =
5          Let var boundexp exp |
6          Rslt var
7
8      boundexp =
9          Unit |
10         MkChn |
11         Prim prim |
12         Spwn exp |
13         Sync var |
14         Fst var |
15         Snd var |

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16     Case var var exp var exp |
17     App var var and
18
19   prim =
20     SendEvt var var |
21     RecvEvt var |
22     Pair var var |
23     Lft var |
24     Rht var |
25     Abs var var ex
26
27
1   datatype ctrl_label =
2     LNxt var | LSpwn var | LCall var | LRtn var
3
4   type_synonym ctrl_path = (ctrl_label list)
5
6   datatype chan = Chn ctrl_path var
7
8   datatype val =
9     VUnit | VChn chan | VClsr prim (var  $\rightarrow$  val)
10
11   datatype ctn = Ctn var exp (var  $\rightarrow$  val)
12
13   datatype state = Stt exp (var  $\rightarrow$  val) (ctn list)
14
15
1
2   inductive seq_step ::
3     bind * (var  $\rightarrow$  val))  $\Rightarrow$  val  $\Rightarrow$  bool where
4     LetUnit:
5       seq_step (Unit, env) VUnit |
6     LetPrim:
7       seq_step (Prim p, env) (VClsr p env) |
8     LetFst:
9       env xp = Some (VClsr (Pair x1 x2) envp)  $\Rightarrow$ 
10       envp x1 = Some v  $\Rightarrow$ 
11       seq_step (Fst xp, env) v |
12     LetSnd:
13       env xp = Some (VClsr (Pair x1 x2) envp)  $\Rightarrow$ 
14       envp x2 = Some v  $\Rightarrow$ 
15       seq_step (Snd xp, env) v
16
17   inductive seq_step_up ::
18     bind * (var  $\rightarrow$  val))  $\Rightarrow$  exp * val_env  $\Rightarrow$  bool where
19     LetCaseLft:
20       env xs = Some (VClsr (Lft x1') envl)  $\Rightarrow$ 
21       envl x1' = Some vl  $\Rightarrow$ 

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22     seq_step_up
23       (Case xs xl el xr er, env)
24       (el, env(xl ↦ vl)) |
25   LetCaseRht:
26     env xs = Some (VClsr (Rht xr') envr) ⇒
27     envr xr' = Some vr ⇒
28     seq_step_up
29       (Case xs xl el xr er, env)
30       (er, env(xr ↦ vr)) |
31   LetApp:
32     env f = Some (VClsr (Abs fp xp el) envl) ⇒
33     env xa = Some va ⇒
34     seq_step_up
35       (App f xa, env)
36       (el, envl(
37         fp ↦ (VClsr (Abs fp xp el) envl),
38         xp ↦ va))
39
40
41   type_synonym cmmn_set = (ctrl_path * chan * ctrl_path)
42   set
43
44   type_synonym trace_pool = ctrl_path → state
45
46   inductive leaf ::
47     trace_pool ⇒ ctrl_path ⇒ bool where
48     Intro:
49       trpl pi ≠ None ⇒
50       (∄ pi' . trpl pi' ≠ None ∧ strict_prefix pi pi') ⇒
51       leaf trpl pi
52
53   inductive concur_step ::
54     trace_pool * cmmn_set ⇒
55     trace_pool * cmmn_set ⇒
56     bool where
57     Seq_Ststep_Down:
58       leaf trpl pi ⇒
59       trpl pi = Some
60       (Stt (Rslt x) env
61         ((Ctn xk ek envk) # k)) ⇒
62       env x = Some v ⇒
63       concur_step
64       (trpl, ys)
65       (trpl(pi @ [LRtn xk] ↦
66         (Stt ek (envk(xk ↦ v)) k)), ys) |
67     Seq_Step:
68       leaf trpl pi ⇒
69       trpl pi = Some
70       (Stt (Let x b e) env k) ⇒
71       seq_step (b, env) v ⇒

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71     concur_step
72     (trpl, ys)
73     (trpl(pi @ [LNxt x] ↦
74       (Stt e (env(x ↦ v)) k), ys) |
75 Seq_Step_Up:
76   leaf trpl pi ⇒
77   trpl pi = Some
78     (Stt (Let x b e) env k) ⇒
79   seq_step_up (b, env) (e', env') ⇒
80   concur_step
81     (trpl, ys)
82     (trpl(pi @ [LCall x] ↦
83       (Stt e' env'
84         ((Ctn x e env) # k))), ys) |
85 LetMkCh:
86   leaf trpl pi ⇒
87   trpl pi = Some (Stt (Let x MkChn e) env k) ⇒
88   concur_step
89     (trpl, ys)
90     (trpl(pi @ [LNxt x] ↦
91       (Stt e (env(x ↦ (VChn (Chn pi x)))) k))), ys) |
92 LetSpwn:
93   leaf trpl pi ⇒
94   trpl pi = Some
95     (Stt (Let x (Spwn ec) e) env k) ⇒
96   concur_step
97     (trpl, ys)
98     (trpl(
99       pi @ [LNxt x] ↦
100       (St e (env(x ↦ VUnit)) k),
101       pi @ [LSpwn x] ↦
102       (St ec env []), ys) |
103 LetSync:
104   leaf trpl pis ⇒
105   trpl pis = Some
106     (Stt (Let xs (Sync xse) es) envs ks) ⇒
107   envs xse = Some
108     (VClsr (SendEvt xsc xm) envse) ⇒
109   leaf trpl pir ⇒
110   trpl pir = Some
111     (Stt (Let xr (Sync xre) er) envr kr) ⇒
112   envr xre = Some
113     (VClsr (RecvEvt xrc) envre) ⇒
114   envse xsc = Some (VChn c) ⇒
115   envre xrc = Some (VChn c) ⇒
116   envse xm = Some vm ⇒
117   concur_step
118     (trpl, ys)
119     (trpl(
120       pis @ [LNxt xs] ↦

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121         (Stt es (envs(xs ↦ VUnit)) ks),
122         pir @ [LNxt xr] ↦
123         (Stt er (envr(xr ↦ vm)) kr)),
124         ys ∪ {(pis, c, pir)})
125
126
1
inductive star :: ('a ⇒ 'a ⇒ bool) ⇒ 'a ⇒ 'a ⇒ bool
2   for r where
3     refl: star r x x |
4     step: r x y ⇒ star r y z ⇒ star r x z
5

```