Mechanizing a Theory of Concurrent ML

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1 Introduction

For this master's thesis, I have developed a formal semantics of a concurrent language, an initial formal analysis, along with related theorems and formal proofs. The language under analysis is a very simplified version of *Concurrent ML* [18]. The formal analysis recasts an analysis with informal proofs developed by Reppy and Xiao [17]. It categorizes communication described by programs into simple topologies. One description of topologies is static; that is, it describes topologies in terms of the finite structure of programs. Another description is dynamic; that is, it describes topologies in terms of running a program for an arbitrary number of steps. The main formal theorem states that the static analysis is sound with respect to the dynamic analysis. Two versions of the static analysis have been developed so far; one with lower precision, and one with higher precision. The higher precision analysis is closer to the work by Reppy and Xiao, but contains many more details making it more challenging to prove formally than the lower precision analysis. The proofs for the soundness theorems of the lower precision analysis have been mechanically verified using Isabelle/HOL [16], while the higher precision analysis is currently under development. Indeed, one of the motivations for implementing the analysis in a mechanical setting is to enable gradual extension of analysis and language without introducing uncaught bugs in the definitions or proofs. The definitions used in this formal theory differ significantly from that of Reppy and Xiao, in order to aid formal reasoning. Thus, recasting Reppy and Xiao's work was far more nuanced than a straightforward syntactic transliteration. Although the definitions are structurally quite different, their philosophical equivalence is hopefully apparent. In this formal theory, the dynamic semantics of Concurrent ML consists of a CEK machine [5]. The static semantics consists of a 0CFA control-flow analysis [19], defined in terms of constraints [14].

1.1 Concurrent ML

In programing languages, concurrency is a program structuring technique that allows evaluation steps to hop back and forth between disjoint syntactic structures within a program. It is useful when conceptually distinct tasks need to overlap in time, but are easier to understand if they are written as distinct structures within the program. Concurrent languages may also allow the evaluation order between steps of terms to be nondeterministic. If it's not necessary for tasks to be ordered in a precise way, then it may be better to allow a static or dynamic scheduler pick the most efficient execution order. A common use case for concurrent languages is for programs that interact with humans, in which a program has to process various requests while remaining responsive to subsequent user inputs, and it must continually provide the user feedback with latest information it has processed.

Concurrent ML is a particularly elegant concurrent programing language. It features threads, which are pieces of code allowed to have a wide range of evaluation orders relative to code encapsulated in other threads. Its synchronization mechanism can mandate the execution order between parts of separate threads. It is often the case that synchronization is necessary when data is shared. Thus, in *Concurrent ML*, synchronization is inherent in communication. Addi-

tional threads can be spawned in order to share data asynchronously.

Threads communicate by having shared access to a common channel. A channel can be used to either send data or receive data. When a thread sends on a channel, another thread must receive on the same channel before the sending thread can continue. Likewise, when a thread receives on a channel, another thread must send on the same channel before the receiving thread can continue.

```
type thread_id
val spawn : (unit -> unit) -> thread_id

type 'a chan
val channel : unit -> 'a chan
val recv : 'a chan -> 'a
val send : ('a chan * 'a) -> unit
```

A given channel can have any arbitrary number of threads sending or receiving data on it over the course of the program's execution. A simple example, derived from Reppy's book *Concurrent Programing in ML*, illustrates these essential features.

The implementation of `Serv` defines a server that holds a number in its state. When a client gives the server a number `v`, the server gives back the number in its state, and updates its state with the number `v`. The next client request will get the number `v`, and so on. Essentially, a request and reply is equivalent to reading and writing a mutable cell in isolation. The function `make` makes a new server, by creating a new channel `reqCh`, and a loop `loop` which listens for requests. The loop expects the request to be composed of a number `v` and a channel `replCh`. It sends its current state's number on `replCh` and update's the loop's state with the request's number `v`, by calling the loop with a new that number. The server is created with a new thread with the initial state `0` by calling `spawn (fn () => loop 0)`. The request channel is returned as the handle to the server. The function `call` makes a request to the passed in server server `server` with a number `v` and returns a number from the server. Internally, it extracts the request channel `reqCh` from the server handle and creates a new channel `replCh`. It makes a request to the server with the number `v` and the reply channel `replCh` by calling `send (reqCh, (v, replCh))`. Then it receives the reply with the new number by calling `recv replCh`.

```
signature SERV =
sig
  type serv
  val make : unit -> serv
  val call : serv * int -> int
end

structure Serv : SERV =
struct
  datatype serv = S of (int * int chan) channel
```

```
  fun make () =
  let
    val reqCh = channel ()
    fun loop state =
    let
      val (v, replCh) = recv reqCh
      val () = send (replCh, state)
    in
      loop v
    end
    val () = spawn (fn () => loop 0)
  in
    S reqCh
  end

  fun call (server, v) =
  let
    val S reqCh = server
    val replCh = channel ()
    val () = send (reqCh, (v, replCh))
  in
    recv replCh
  end

end
```

Concurrent ML actually allows for events other than sending and receiving to occur during synchronization. in fact, the synchronization mechanism is decoupled from events, like sending and receiving, much in the same way that function application is decoupled from function function. Sending and receiving events are represented by `sendEvt` and `recvEvt` and synchronization is represented by `sync`.

```
type 'a event
val sync : 'a event -> 'a

val recvEvt : 'a chan -> 'a event
val sendEvt : 'a channel * 'a -> unit event

fun send (ch, v) = sync (sendEvt (ch, v))
fun recv v = sync (recvEvt v)
```

An advantageous consequence of decoupling synchronization from events, is that events can be combined with other events via event combinators, and synchronized on exactly once. One such event combinator is `choose`, which constructs a new event consisting of two constituent events, such that when sycnronized on, exactly one of the two events may take effect. There are many other useful combinators, such as the `wrap` and `guard` combinators designed by Reppy[8].

Additionally, Donnelly and Fluet extended *Concurrent ML* with the `thenEvt` combinator described in their work on transactional events [4]. Transactional events enable more robust structuring of programs by allowing non-isolated code to be turned into isolated code via the `thenEvt` combinator, instead of duplicating code with the addition of stronger isolation. When the event constructed by the `thenEvt` combinator is is synchronized on, either all of its constituent events and functions evaluate in isolation, or none evaluates.

```
val choose : 'a event * 'a event -> 'a event
val thenEvt : 'a event * ('a -> 'b event) -> 'b event
```

1.2 Isabelle/HOL

An interactive theorem proving assistant, or proof assistant, is a machine that helps its user specify propositions and prove theorems. Like typical programming systems, it checks that propositions are lexically correct, syntatically correct, and even logically correct according to its type system. To determine if a proposition is valid, a proof assistant often requires the user to supply a proof. If the proof assistant verified that the user's proof is correct, then the proposition may be considered a theorem. In addition to checking the correctness of proofs and propositions, the proof assistant also assists the user in constructing proofs.

Isabelle/HOL is a popular proof assistant that assists in specifying and proving properties formulated in Higher Order Logic (HOL). Predicates and terms may be higher order. That is, they may take other predicates or functions as arguments and return predicates or functions as arguments. In HOL, a predicate is actually just a function that returns a value of type boolean, and a proposition is simply a term of type boolean. Terms are deemed logical according to a system of simple types, similar to that of Standard ML. Isabelle/HOL excels at assisting with proving propositions with numerous details details.

A proof is just a sequence of manipulations from existing theorems and axioms to the proposition to be proved. Typically, unassisted informal proofs, are written in a declarative form. The goal is stated at the top, and then an axiom or theorem is stated. Subsequently derived theorems are stated beneath, with or without the aspects of the manipulation explained. For instance, say you want to prove `P1 ∨ P2 → Q`. You might begin by stating the axiom `P1 ∨ P2 ⊢ P1 ∨ P2`, commonly written as **`assume`** `P1 ∨ P2`. You can also state `P1 ⊢ P1` and `P2 ⊢ P2`. Since these are inteded to form complementary cases of `P1 ∨ P2`, they are written as **`case`** `P1: ...` **`case`** `P2`. Perhaps you know **`theorem`** `A: ⊢ P1 → Q`. Using modus ponens, you derive `P1 ⊢ Q`. Let's say you also know **`theorem`** `B: ⊢ P2 → Q`. Using modus ponens with theorem B, you derive `P2 ⊢ Q`. Your realize that these two theorems may be combined into `P1 ∨ P2 ⊢ Q`, which reduces to `⊢ P1 ∨ P2 → Q`.

```
theorem P1 ∨ P2 → Q
proof
```

```
assume P1 ∨ P2:
  case P1:
    have ⊢ P1 → Q by A
    have ⊢ Q by modus ponens
  case P2:
    have ⊢ P2 → Q by B
    have ⊢ Q by modus ponens
  have P1 ⊢ Q, P2 ⊢ Q
  have ⊢ Q by disjunction elimination
have P1 ∨ P2 ⊢ Q
have ⊢ P1 ∨ P2 → Q by implication introduction

qed
```

For proving simple propositions, it may be easy to conjure up the theorems and axioms needed to combine and manipulate into the goal. However, for proving complex or unfamiliar propositions, it may be less clear. In addition to the declarative forward proof style, Isabelle/HOL also supports an imperative backwards proof style, by starting with the goal and breaking it into simpler and simpler subgoals until all subgoals are manipulated into axioms. The backwards style of reasoning, allows the proof writer to focus on the manipulation rule, rather than remembering and gathering up all the theorems that are necessary to combine to reach the goal. The interface of the interactive theorem prover displays the subgoals after applying each manipulation rule.

```
theorem P1 ∨ P2 → Q
goal: ⊢ P1 ∨ P2 → Q
apply (rule impI)
goal: P1 ∨ P2 ⊢ Q
apply (erule disjE)
goal: P1 ⊢ Q
goal: P2 ⊢ Q
apply (insert A)
goal: P1, P1 → Q ⊢ Q
goal: P2 ⊢ Q
apply (erule mp)
goal: P1 ⊢ P1
goal: P2 ⊢ Q
apply assumption
goal: P2 ⊢ Q
apply (insert B)
goal: P2, P2 → Q ⊢ Q
apply (erule mp)
goal: P2 ⊢ P2
apply assumption
done
```

Note that the syntax used in Isabelle/HOL for writing proofs and propositions differs from that of the examples shown here. To make these examples easier to understand, I have chosen to

use a syntax that is more typical of mathematical logic and language theory literature.

In addition to creating theorems by proving propositions, Isabelle/HOL also allows creating theorems by defining predicates. This feature is critical for constructing inductive propositions that may hold over an infinite range of values. Infinite ranges may be defined as inductive datatypes, similar to Standard ML.

For instance, you can define the infinite set of natural numbers (with zero) as an inductive datatype, and the binary relation that a natural number is less than or equal to another as an inductive predicate.

```
datatype nat = Z | S nat

predicate lte: nat -> nat -> bool
where only
  eq: ⊢ lte n n
* lt: lte n1 n2 ⊢ lte n1 (S n2)
```

Additionally, you can define the infinite set of lists inductively, and the higher order predicte that checks sortedness using a supplied binary relation.

```
datatype 'a list = Nil | Cons 'a ('a list)

predicate sorted: ('a -> 'a -> bool) -> 'a list -> bool
where only
  nil: ⊢ sorted r Nil
* uni: ⊢ sorted r (Cons x Nil)
* cons: r x y, sorted p (Cons y ys) ⊢ sorted r (Cons x (Cons y ys))
```

Each case of the the predicate definition is considered a theorem, and you have the option to give it a name. Additionaly, these are only cases that can hold for the predicate. Therefore, the predicate applied to some free variables is equivalent to the disjunction of all the cases, with the variables equal to their respective patterns in each case. This kind of theorem, and other similar theorems for inversion and induction are created with the each predicate definition.

```
⊢ sorted r xs ≡
  (xs = Nil)
∨ (xs = (Cons x Nil))
∨ (xs = (Cons x (Cons y ys)) ∧ r x y ∧ sorted r (Cons y ys))
```

By composing these definitions, you can state and prove a list of natural numbers is sorted in non-decreasing order.

```
theorem sorted lte (Cons (Z) (Cons (S Z) (Cons (S Z) (Cons (S (S (S Z))) Nil))))
apply (rule cons)
goal: ⊢ lte Z (S Z)
goal: ⊢ sorted lte (Cons (S Z) (Cons (S Z) (Cons (S (S (S Z))) Nil)))
apply (rule lt)
goal: ⊢ lte Z Z
```

```
goal: ⊢ sorted lte (Cons (S Z) (Cons (S Z) (Cons (S (S (S Z))) Nil)))
apply (rule eq)
goal: ⊢ sorted lte (Cons (S Z) (Cons (S Z) (Cons (S (S (S Z))) Nil)))
apply (rule cons)
goal: ⊢ lte (S Z) (S Z)
goal: ⊢ sorted lte (Cons (S Z) (Cons (S (S (S Z))) Nil))
apply (rule eq)
goal: ⊢ sorted lte (Cons (S Z) (Cons (S (S (S Z))) Nil))
apply (rule cons)
goal: ⊢ lte (S Z) (S (S (S Z)))
goal: ⊢ sorted lte (Cons (S (S (S Z))) Nil)
apply (rule lt)
goal: ⊢ lte (S Z) (S (S Z))
goal: ⊢ sorted lte (Cons (S (S (S Z))) Nil)
apply (rule lt)
goal: ⊢ lte (S Z) (S Z)
goal: ⊢ sorted lte (Cons (S (S (S Z))) Nil)
apply (rule eq)
goal: ⊢ sorted lte (Cons (S (S (S Z))) Nil)
apply (rule uni)
done
```

The learning curve for using proof assistants in general and Isabelle/HOL in particular is very steep. Nevertheless, once the user has gained some fluency, there are great benefits for certain kinds of projects. In a theory of Concurrent ML, with various propositions about evaluation and communication, there are many tedious details that must be specified. Furthermore, due to there being greater complexity with greater number of language features, or greater precision of propositions, it is very useful to start simple features and propositions to create a minimal viable theory, and then incrementally increase complexity and modify proofs. The automatic checking of propositions and proofs finding errors buried in numerous tedious details. Additionally, it eases the process incremental extension by pinpointing where the proofs and propositions break as features and complexity is added.

2 Synchronization

Synchronization of sending threads and receiving threads requires determining which threads should wait, and which threads should be dispatched. The greater the information needed to determine this scheduling, the higher the performance penalty. A uniprocessor implementation of synchronization can have very little penalty. Since only one thread can make progress at a time, only one thread requests synchronization at a time, meaning the scheduler won't waste steps checking for threads competing for the same synchronization opportunity, before dispatching. A multiprocessor implementation, on the other hand, must consider that competing threads may exists, therefore perform additional checks. Additionally, there may be overhead in sharing data

between processors due to memory hierarchy designs [10].

One way to lower synchronization and communication costs is to use specialized implementations for channels that never have more than one thread ever sending or receiving on them. These specialized implementations would avoid unnecessary checks for competing threads. Concurrent ML does not feature multiple kinds of channels distinguished by their communication topologies, i.e. the number of threads that may end up sending or receiving on the channels. However, channels can be classified into various topologies simply by counting the number of threads per channel during the execution of a program. A many-to-many channel has any number of sending threads and receiving threads; a one-to-many channel has at most one sending thread and any number of receiving threads; a many-to-one channel has any number of sending threads and at most one receiving thread; a one-to-one channel has one or none of each; a one-shot channel has exactly one sending attempt.

The following reimplementation of `Serv` is annotated to indicate the communication topologies derived from its usage. Since there are four threads that make calls to the server, the server's particular `reqCh` has four senders. Servers are created with only one thread listening for requests, so the `reqCh` of this server has just one receiver. So the server's `reqCh` is classified as many-to-one. Each application of `call` creates a distinct new channel `replCh` for receiving data. The function `call` receives on the channel once and the server sends on the channel once, so each instance of `replCh` is one-shot.

```
structure Serv : SERV =
struct

  datatype serv = S of (int * int chan) channel

  fun make () =
  let
    val reqCh = ManyToOne.channel ()
    fun loop state =
    let
      val (v, replCh) = ManyToOne.recv reqCh
      val () = OneShot.send (replCh, state)
    in
      loop v
    end
    val () = spawn (fn () => loop 0)
  in
    S reqCh
  end

  fun call (server, v) =
  let
    val S reqCh = server
    val replCh = OneShot.channel ()
```

```
    val () = ManyToOne.send (reqCh, (v, replCh))
  in
    OneShot.recv replCh
  end

end

val server = Serv.make ()
val () = spawn (fn () => Serv.call (server, 35))
val () =
  (spawn fn () =>
    Serv.call (server, 12);
    Serv.call (server, 13)
  )
val () = spawn fn () => Serv.call (server, 81)
val () = spawn (fn () => Serv.call (server, 44))
```

Some hypothetical implementations of specialized and generic Concurrent ML illustrate opportunities for cheaper synchronization. These implementaitons use feasible low-level thread-centric features such as wait and poll. The thread-centric approach allows us to focus on optimizations common to many implementations by decoupling the implementation of communication features from thread scheduling and management. However, a lower level view or scheduler-centric view of synchronization might offer more opportunites for optimization.

In a language with low-level support for concurrency, Concurrent ML could be implemented as a library, which is the case for SML/NJ [3] and MLton [21]. It could also be implemented by a compiler and runtime or interpreter. Thus, the implementations shown here can be viewed either as a library or as an intermediate representation within a compiler or interpreter presented with concrete syntax.

```
signature CHANNEL =
sig
  type 'a channel
  val channel : unit -> 'a chan
  val send : 'a channel * 'a -> unit
  val recv : 'a chan -> 'a
end
```

The benefits of specialization would be much more significant in multiprocessor implementations than in uniprocessor implementations. A uniprocessor implementation could avoid overhead caused by contention to acquire locks, by coupling the implementation of channels with scheduling and only scheduling the sending and receiving operations when no other pending operations have yet to start or have already finished. Reppy's implementation of Concurrent ML uses SML/NJ's first class continuations to implement scheduling and communication as one with very low overhead. in contrast, a multiprocessor implementation would allow threads to run on different processors for increased parallelism, therefore it would not be able to mandate

when threads attempt synchronization relative to others without losing the parallel advantage. The cost of trying to achieve parallelism is increased overhead due to contention over acquiring synchronization rights.

2.1 Many-to-many Synchronization

A channel can be in one of three states. Either some threads are trying to send on it, some threads are trying to receive on it, or no threads are trying to send or receive on it. Additionally a channel is composed of a mutex lock, so that sending and receiving operations can yield to each other when updating the channel state. When multiple threads are trying to send on a channel, the channel is associated with a queue consisting of messages to be sent, along with conditions waited on by sending threads. When multiple threads are trying to receive on a channel, the channel is associated with a queue consisting of initially empty cells that are accessible by receiving threads and conditions waited on by the receiving threads. The channel content holds one of three three potential states and their associated content of queues and conditions. The channel is composed of the channel content and also a mutex lock that regulates access to the channel content.

The sending operation acquires the channel's lock to ensure that it updates the channel based on its current state. If the channel is in the receiving state, i.e. there are threads trying to receive from the channel, then the sending operation dequeues an item from the state's associated queue. The item consists of a condition waited on by a receiving thread and an empty cell that can be accessed by the receiving thread. The sending operation deposits the message in the cell and signals on the receiving state's condition. Then, if there are no further receiving threads waiting, it updates the channel's state to inactive; otherwise, it leaves the state in the receiving state. Next, it releases the lock, signals on the receiving state's condition and returns the unit value.

If there are no threads receiving on the channel, the sending operation updates the channel state to the sending state, and enqueues a condition and the message. It releases the lock and waits on the enqueued condition. Once a receiving thread signals on the same condition, the sending operation returns with the unit value.

The receiving operation acquires the channel's lock to ensure that it updates the channel based on its current state. If there are threads sending on the channel, the receiving operation dequeues an item from the sending state's associated queue. The item consists of a condition waited on by a sending thread along with a message. The receiving operation signals on the sending state's condition. If there are no further sending threads waiting, it updates the channel's state to inactive; otherwise, it leaves the state in the sending state. Next, it releases the lock and returns the message from the sending state. If there are no sending threads on the channel, the receiving operation updates the channel state to the receiving state, and enqueues a new condition `recvCond` and an empty cell. It releases the lock and waits on the its condition `recvCond`. Once a sending thread signals on its condition, the receiving operation returns with the value deposited in its cell.

```
structure ManyToManyChan : CHANNEL =
```

```
struct
  type message_queue = 'a option ref queue

  datatype 'a chan_content =
    Send of (condition * 'a) queue
  | Recv of (condition * 'a option ref) queue
  | Inac

  datatype 'a channel =
    Chn of 'a chan_content ref * mutex_lock

  fun channel () = Chn (ref Inac, mutexLock ())

  fun send (Chn (ctntRef, lock)) m =
    acquire lock;
    (case !ctntRef of
      Recv q =>
      let
        val (recvCond, msgCell) = dequeue q
        val () = msgCell := Some m
        val () = if (isEmpty q) then ctntRef := Inac else ()
      in
        release lock; signal recvCond
      end
    | Send q =>
      let
        val sendCond = condition ()
        val () = enqueue (q, (sendCond, m))
      in
        release lock; wait sendCond
      end
    | Inac =>
      let
        val sendCond = condition () in
        val () = ctntRef := Send (queue [(sendCond, m)])
      in
        release lock; wait sendCond
      end
    )

  fun recv (Chn (ctntRef, lock)) =
    acquire lock;
    (case !ctntRef of
      Send q =>
      let
        val (sendCond, m) = dequeue q
```

```
          val () =
          case (isEmpty q) of
            true => ctntRef := Inac
          | false => ()

        in
          release lock; signal sendCond; m
        end
      | Recv q =>
        let
          val recvCond = condition ()
          val msgCell = ref NONE
          val () = enqueue (q, (recvCond, msgCell))
          val () = release lock; wait recvCond
        in
          valOf (!msgCell)
        end
      | Inac =>
        let
          val recvCond = condition ()
          val msgCell = ref NONE
          val () = ctntRef := Recv (queue [(recvCond, msgCell)])
          val () = release lock; wait recvCond
        in
          valOf (!msgCell)
        end
      )

  end
```

2.2 One-to-many Synchronization

Implementation of one-to-many channels, compared to that of many-to-many channels, requires fewer steps to synchronize and can execute more steps outside of critical regions, which reduces contention for locks. A channel is composed of a lock and one of three possible states, as is the case for many-to-many channels. However, the state of a thread trying to send only needs to be associated with one condition and one message, rather than a queue.

The sending operation starts by creating a condition `sendCond`, then checks if the channel's state is inactive and tries to use the compare-and-swap operator to transactionally update the state of the channel to a sending state. If successful, it simply waits on its condition `sendCond`. After the receiving thread signal on `sendCond`, the sending operation returns the unit value. If the transactional update fails and the state is that of threads trying to receive on the channel, then the sending operation acquires the lock, then dequeues an item from the associated queue where

the item consists of a receiving condition `recvCond`, and a cell for depositing the message to the receiving thread. If there are no further items on the queue, the sending operation updates the state to inactive; otherwise, it leaves the state in the receiving state. Next, it releases the lock it, then signals on the receiving condition and returns the unit value.

The lock is acquired after the state is determined to be that of threads trying to receive, since the expectation is that the current thread is the only one that tries to update the channel from that state. If the communication classification analysis were incorrect and there were actually multiple threads that could call the sending operation, then there might be data races. Likewise, due to the expectation of a single thread sending on the channel, the sending operation will never witness the state in the sending state, which would mean another thread is in the process of sending a message.

The receiving operation acquires the lock and checks the state of the channel, just like the receiving operation for many-to-many channels. If the channel is in a state where there is no sending thread waiting, then it updates the state to receiving, behaving the same as the receiving operation of many-to-many channels. If there is already a sending thread waiting, then it updates the state to inactive and releases the lock. Then it signals on the sending state's condition and returns the message held in the sending state.

```
structure OneToManyChan : CHANNEL =
struct

  datatype 'a chan_content =
    Send of condition * 'a
  | Recv of (condition * 'a option ref) queue
  | Inac

  datatype 'a channel =
    Chn of 'a chan_content ref * mutex_lock

  fun channel () = Chn (ref Inac, mutexLock ())

  fun send (Chn (ctntRef, lock)) m =
  let
    val sendCond = condition ()
  in
  case (cas (ctntRef, Inac, Send (sendCond, m))) of
    Inac =>
      (* ctntRef is already set to sending state by cas *)
      wait sendCond
  | Recv q =>
    let
      (*
        the current thread is the only one that
```

```
        updates from this state
      *)
      val () = acquire lock
      val (recvCond, msgCell) = dequeue q
      val () = msgCell := SOME m
      val () =
      case (isEmpty q) of
        true => ctntRef := Inac
      | false => ()
    in
      release lock; signal (recvCond)
    end
  | Send _ => raise NeverHappens
  end

  fun recv (Chn (ctntRef, lock)) =
    acquire lock;
    (case !ctntRef of
      Inac =>
      let
        val recvCond = condition ()
        val msgCell = ref NONE
        val () = ctntRef := Recv (queue [(recvCond, msgCell)])
        val () = release lock; wait recvCond
      in
        valOf (!msgCell)
      end
    | Recv q =>
      let
        val recvCond = condition ()
        val msgCell = ref NONE
        val () = enqueue (q, (recvCond, msgCell))
        val () = release lock; wait recvCond
      in
        valOf (!msgCell)
      end
    | Send (sendCond, m) =>
        ctntRef := Inac;
        release lock;
        signal sendCond;
        m
    )

end
```

2.3 Many-to-one Synchronization

The implementation of many-to-one channels is very similar to that of one-to-many channels.

```
structure ManyToOneChan : CHANNEL =
struct

  datatype 'a chan_content =
    Send of (condition * 'a) queue
  | Recv of condition * 'a option ref
  | Inac

  datatype 'a channel =
    Chn of 'a chan_content ref * mutex_lock

  fun channel () = Chn (ref Inac, mutexLock ())

  fun send (Chn (ctntRef, lock)) m =
    acquire lock;
    (case !ctntRef of
      Recv (recvCond, msgCell) =>
        msgCell := SOME m; ctntRef := Inac;
        release lock; signal recvCond
    | Send q =>
      let
        val sendCond = condition ()
        val () = enqueue (q, (sendCond, m))
      in
        release lock; wait sendCond
      end
    | Inac =>
      let
        val sendCond = condition ()
        val () = ctntRef := Send (queue [(sendCond, m)])
      in
        release lock; wait sendCond
      end
    )

  fun recv (Chn (ctntRef, lock)) =
  let
    val recvCond = condition ()
    val msgCell = ref NONE
  in
  case cas (ctntRef, Inac, Recv (recvCond, msgCell)) of
    Inac =>
      (* ctntRef is already set to receiving state by cas *)
```

```
      wait recvCond; valOf (!msgCell)
  | Send q =>
    let
      (*
        the current thread is the only one that
        updates the state from this state
      *)
       val () = acquire lock
       val (sendCond, m) = dequeue q
       val () =
       case (isEmpty q) of
         true => ctntRef := Inac
       | false => ()
    in
      release lock;
      signal sendCond;
      m
    end
  | Recv _ => raise NeverHappens
  end

end
```

2.4 One-to-one Synchronization

A one-to-one channel can also be in one of three possible states, but there is no associated lock. Additionally, none of the states is associated a queue. Instead, the potential states are that of a thread trying to send, with a condition and a message, that of a thread trying to receive with a condition and an empty cell, or the inactive state.

The sending operation creates a condition `sendCond` and checks if the channel's state is inactive and tries to use the compare-and-swap operator to transactionally update the state of the channel to a sending state. If successful, it simply waits on it condition `sendCond`, then returns the unit value. If the transactional update fails and the state is a receiving state, then it deposits the message in the receiving state's associated cell, updates the channel state to inactive, then signals on the receiving state's condition and returns the unit value. If the communication analysis for the channel is truly one-to-one, then no other thread will be trying to update the state while in the receiving state, so no locks are necessary. Additionally, if the channel is truly one-on-one, the sending operation will never witness a preexisting sending state since it is running on the one and only sending thread.

The receiving operation creates a condition `recvCond` and an empty cell, then checks if the channel's state is inactive and tries to use the compare-and-swap operator to transactionally update the state of the channel to the receiving state. If successful, it simply waits on its condition `recvCond`. If the transactional update fails and the state is a sending state, then it updates the

channel state to inactive, then signals on the sending state's condition and returns the message held in the sending state. If the communication analysis for the channel is truly one-to-one, then no other thread will be trying trying to send, so no locks are necessary. Additionally, if the channel is truly one-to-one, the receiving operation will never witness a preexisting receiving state since it is running on the one and only receiving thread.

```
structure OneToOneChan : CHANNEL =
struct

  datatype 'a chan_content =
    Send of condition * 'a
  | Recv of condition * 'a option ref
  | Inac

  datatype 'a channel = Chn of 'a chan_content ref

  fun channel () = Chn (ref Inac)

  fun send (Chn ctntRef) m =
  let
    val sendCond = condition ()
  in
  case (cas (ctntRef, Inac, Send (sendCond, m))) of
    Inac =>
      (* ctntRef is already set to sending state by cas *)
      wait sendCond
  | Recv (recvCond, msgCell) =>
      (*
        the current thread is the only one that
        accesses ctntRef for this state
      *)
      msgCell := SOME m; ctntRef := Inac;
      signal recvCond
  | Send _ => raise NeverHappens
  end


  fun recv (Chn ctntRef) =
  let
    val recvCond = condition ()
    val msgCell = ref NONE
  in
  case (cas (ctntRef, Inac, Recv (recvCond, msgCell))) of
    Inac =>
      (* ctntRef is already set to receiving state by cas *)
```

```
      wait recvCond; valOf (!msgCell)
  | Send (sendCond, m) =>
      (*
        the current thread is the only one that
        accesses ctntRef for this state
      *)
      ctntRef := Inac;
      signal sendCond;
      m
  | Recv _ => raise NeverHappens
  end

end
```

2.5 One-shot Synchronization

A one-shot channel consists of the same possible states as a one-to-one channel, but is additionally associated with a mutex lock, to account for the fact that multiple threads may try to receive on the channel, even though only at most one message is ever sent.

The sending operation is like that of one-to-one channels, except that if the state is a receiving state, it simply deposits the message and signals on the receiving state's condition, without updating the channel's state to inactive, which would be unnecessary, since no further attempts to send are expected.

The receiving operation creates a condition `recvCond` and an empty cell, then checks if the channel's state is inactive and tries to use the compare-and-swap operator to transactionally update the state of the channel to the receiving state. If successful, it simply waits on its condition `recvCond`, then returns the message deposited in its cell. If the transactional update fails and the state is a sending state, then it acquires the lock, signals on the state's associated condition and returns the message held in the sending state. It never releases the lock, blocking any additional attempts to receive, which is fine if there is truly at most one message ever sent on the channel. If the state is a receiving state, then the receiving operation attempts to acquire the lock, but it will never actually acquire it since the thread associated with the receiving state will never release it.

```
structure OneShotChan : CHANNEL =
struct

  datatype 'a chan_content =
    Send of condition * 'a
  | Recv of condition * 'a option ref
  | Inac

  datatype 'a channel = Chn of 'a chan_content ref * mutex_lock

  fun channel () = Chn (ref Inac, lock ())
```

```
  fun send (Chn (ctntRef, lock)) m =
  let
    val sendCond = condition ()
  in
  case (ctntRef, Inac, Send (sendCond, m)) of
    Inac =>
      (* ctntRef is already set to sending state by cas *)
      wait sendCond
  | Recv (recvCond, msgCell) =>
      msgCell := SOME m; signal recvCond
  | Send _ => raise NeverHappens
  end


  fun recv (Chn (ctntRef, lock)) =
  let
    val recvCond = condition ()
    val msgCell = ref NONE
  in
  case (ctntRef, Inac, Recv (recvCond, msgCell)) of
    Inac =>
      (* ctntRef is already set to receiving state by cas*)
      wait recvCond; valOf (!msgCell)
  | Send (sendCond, m) =>
      acquire lock; signal sendCond;
      (* never releases lock, so blocks others forever *)
      m
  | Recv _ =>
      acquire lock;
      (* never able to acquire lock, so blocked forever *)
      raise NeverHappens
  end

end
```

2.6 One-shot-to-one Synchronization

An even more restrictive version of a channel with at most one send could be used if it's determined that the number of receiving threads is at most one, such as `replCh` in the server example. The one-shot-to-one channel is composed of a possibly empty message cell, a condition for a sending thread to wait on, and a condition for a receiving thread to wait on.

The sending operation deposits the message in the cell, signals on the channel's condition `recvCond`, waits on the condition `sendCond`, and then returns the unit value. The receiving operation waits on `recvCond`, then signals on `sendCond`, then returns the deposited message.

```
structure OneShotToOneChan : CHANNEL =
struct

  datatype 'a channel =
    Chn of condition * condition * 'a option ref

  fun channel () =
    Chn (condition (), condition (), ref NONE)

  fun send (Chn (sendCond, recvCond, msgCell)) m =
    msgCell := SOME m; signal recvCond;
    wait sendCond

  fun recv (Chn (sendCond, recvCond, msgCell)) =
    wait recvCond; signal sendCond;
    valOf (!msgCell)

end
```

2.7 Discussion

The example implementations of generic synchronization and specialized synchronization suggest that cost savings of specialized implementation are significant. For example, if you know that a channel has at most one sending thread and one receiving thread, then you will lower synchronization costs by using an implementation that is specialized for one-to-one communication. To be certain that the new program with the specialized implementation behaves the same as the original program with the generic implementation, you need to be certain of three basic properties: that the specialized program behaves the same given one-to-one communication; that you have a procedure to determine the one-to-one communication classification, and that the relation between the procedure's input program and output classification upper bound is sound with respect to the semantics of the program.

Spending your energy to determine the topologies for each unique program and then verifying them for each program would be exhausting. Instead, you would probably rather have a generic procedure that can compute communication topologies for any program in a language, along with a proof that the procedure is sound with respect to the programing language.

This work discusses proofs that a static analysis to determine communication topologies is sound with respect to the dynamic semantics. Additionally, it would be important to have proofs that the above specialized implementations are equivalent to the many-to-many implementation under the assumption of particular communication topologies.

3 Formal Theory

The definitions and theorems of this work were constructed in the formal language of Isabelle/HOL, to enable mechanical verification of the correctness of the proofs. However, the formal logic used for presentation in this paper has been somewhat modified from Isabelle's syntax. To aid the development of formal proofs, the analyses are stated using relational specifications. The definitions of static relations are syntax-directed, which provides strong evidence of computability. For a static relation, the proof that it holds for a program is defined to depend on the syntactic form of the program. The syntax is defined inductively, with syntactic forms constructed of substructures of the self-similar and different forms. Likewise, for the definitions of static relations to conform to the syntactic structure, the static relations are defined to hold by structural induction on the program. Therefore, for any program, the static relation is decidable. Additionally, for any given program, there should be instances of the other parameters that satisfy the static relation, in which case there is an algorithm to compute sufficient parameters from a program, by following the basic structure of the proof of the static relation.

This work does not contain formal proofs that the specialized implementations are behaviorally equivalent to a generic implementation, but the example implementations should provide good evidence for that.

A static analysis that describes communication topologies of channels has practical benefits in at least two ways. It can highlight which channels are candidates for optimized implementations of communication; or in a language extension allowing the specification of specialized channels, it can conservatively verify their correct usage. Without a static analysis to check the usage of the special channels, one could inadvertently use a one-shot channel for a channel that has multiple senders, thus violating the intended semantics.

The utility of the static analysis additionally depends on it being precise, sound, and computable. The analysis is precise iff there exist programs about which the analysis describes information that is not directly observable. The analysis is sound iff the information it describes about a program is the same or less precise than the dynamic semantics of the program. The analysis is computable iff there exists an algorithm that determines all the values described by the analysis on any input program.

Analyses can be described in a variety of ways. A computable algorithm that take programs as input and produces information about the behavior as output is ideal for automation. A non-algorithmic relation (i.e. a relation expressed in a language without an inherent evaluator from some parameters to the others), stated in terms of programs and execution information, may be more suitable for clarity of meaning and correctness with respect to the operational semantics. However, a non-algorithmic relation can be translated into an algorithm. One rather mechanical method essentially involves specifying a reasoner associated with the relation. First, the reasoner generates a comprehensive set of data structures representing constraints from the relation's description, then the reasoner the constraints.

For a subset of Concurrent ML without event combinators, Reppy and Xiao developed an efficient algorithm that determines for each channel, all possible threads that send and receive on

it. The algorithm depends on each atom operation in the program being labeled with a program step. A sequence of program steps ordered in a valid execution sequence forms a control path. Distinction between threads in a program can be inferred from whether or not their control paths diverge.

The algorithm proceeds in multiple steps that produce intermediate data structures, used for efficient lookup in the subsequent steps. It starts with a control-flow analysis that results in multiple mappings. One mapping is from names to abstract values that may be bound to the names. Another mapping is from channel-bound names to abstract values that are sent on the respective channels. Another is from function-bound names to abstract values that are the result of respective function applications. It constructs a control-flow graph with possible paths for conditional tests and thread spawning determined directly from the atoms used in the program. Relying on information from the mappings to abstract values, it constructs the possible paths of execution via function application and channel communication. It uses the graph for live variable analysis of channels, which limits the scope for the remaining analysis. Using the spawn and application edges of the control-flow graph, the algorithm then performs a data-flow analysis to determine a mapping from program steps to all possible control paths leading into the respective program steps. Using the CFA's mappings to abstract values, the algorithm determines the program steps for sends and receives per channel name. Then it uses the mapping to control paths to determine all control paths that send or receive on each channel, from which it classifies channels as one-shot, one-to-one, many-to-one, one-to-many, or many-to-many.

The information at each program step is derived from control structures in the program, which dictate how information flows between program steps. Some uses of control structures are literally represented in the syntax, such as the sequencing of namings and assignments in the previous examples. Other uses of control structures may be indirectly represented through names. Function application is a control structure that allows a calling piece of code to flow into a function function's code. Function functions can be named, which allows multiple pieces of code to all flow into into the same section of code. The name adds an additional step in to uncover control structures, and determine data flow. Additionally, in languages with higher order functions and recursion, such as those in the Lisp and ML families, it may be impossible to exactly determine all the function functions that terms resolve to. However, a control flow analysis can reveal a good approximation of the control structures and values that have been obfuscated by higher order function functions. Uncovering the control structures depends on resolving terms to values, and resolving terms to values depends on on uncovering the control structures. The mutual dependency means that control flow analysis is a form of static semantics that describes abstract evaluations of programs. in this work, control flow analysis is used for tracking certain kinds of values, like channels and events, in addition to constructing precise data flow analysis.

3.1 Syntax

The syntax used in this formal theory contains a very small subset of *Concurrent ML*'s features. The features include recursive function function with application, left and right construction with

pattern matching, pair construction with first and second selections, sending and receiving event construction with synchronization, channel creation, thread spawning, and the unit literal. The syntax is defined in a way to make it possible to relate the dynamic semantics of programs to the syntax programs. The syntax is defined in administrative normal form (ANF) [6], in which every term is bound to a name. Furthermore, terms only accept names in place of eagerly evaluated inputs.

Restricting the grammar to ANF allows the operational semantics to maintain graph information by associating values with succint names. Maintaining the values' ties to the syntax, simplifies proofs of soundness, since they must relate dynamic evaluation information to static information based on the syntax.

Additionally, ANF melds nicely with the semantincs of control paths, which succintly identify the evaluation taken to reach some intermediate result. Instead of relying on additional meta-syntax to associate atom operations with identifiers, the analysis can simply use the required names of ANF syntax to identify locations in the program.

The ANF syntax is impractical for a programer to write, yet it is still practical for a language under automated analysis since there is a straightforward procedure to transform more user-friendly syntax into ANF.

```
datatype name = Nm string

datatype term =
  Bind name complex term
| Rslt name

and complex =
  Unt
| MkChn
| Atom atom
| Spwn term
| Sync name
| Fst name
| Snd name
| Case name name term name term
| App name name

and atom =
  SendEvt name name
| RecvEvt name
| Pair name name
| Lft name
| Rht name
| Fun name name term
```

3.2 Dynamic Semantics

The dynamic semantics describes how programs evaluate to values. A history of execution is represented as a list of steps, where a step is a binding name or resulting name of a term, paired with a mode of control indicating flows by sequencing, spawning, calling, or returning. Channels have no literal representation, but each time a channel is created, it is uniquely identified by the history of the execution up until the step of creation. Atomic terms are not simplified. Instead, atoms are evaluated to closures consisting of the atom syntax, along with an environment that maps its constituent names to their values.

In order to relate the static analyses to the operational semantics, I borrowed Reppy and Xiao's strategy of stepping between sets of execution paths and their associated terms.

The semantics are defined as a CEK machine, rather than a substitution based operational semantics. By avoiding simplification of terms in the operational semantics, it is possible to relate the abstract evaluations of the static semantics to the evaluations produced by the dynamic semantics, which in turn is relied on to prove soundness of the static semantics.

```
datatype dynamic_step =
  DNxt name
| DSpwn name
| DCll name
| DRtn name

type dynamic_path = dynamic_step list

datatype chan =
  Chan dynamic_path name

datatype dynamic_value =
  VUnt
| VChn chan
| VAtm atom (name -> dynamic_value option)

type environment =
  name -> dynamic_value option
```

The evaluation of some complex terms results in sequencing, meaning there is no coordination with other threads, and there is no need to save terms on the continuation stack for later evaluation. These terms are the unit literal, atoms, pairs, and first and second selections. The evaluation depends only on the syntax and an environment for looking up the values of names within the syntax. Additionally, all these terms evaluate to values in a single step.

```
predicate seqEval:
  complex -> environment -> dynamic_value -> bool
where only
  ∀ env .
    seqEval Unt env VUnt
```

```
* ∀ a env .
    seqEval (Atom a) env (VAtm a env)
* ∀ env n_p n1 n2 env_p v.
    if
      env n_p = Some (VAtm (Pair n1 n2) env_p),
      env_p n1 = Some v
    then
      seqEval (Fst n_p) env v
* ∀ env n_p n1 n2 env_p v .
    if
      env n_p = Some (VAtm (Pair n1 n2) env_p),
      env_p n2 = Some v
    then
      seqEval (Snd n_p) env v
```

The evaluation of a complex term for application or conditional testing results in flowing by calling. A calling flows is characterized by the need to save a subterm in the continuation stack for later evaluation. The evaluation depends on the syntax and an environment for looking up the values of names within the syntax. A term is evaluated to a subterm, and a new environment that will later be used in the evaluation of the subterm. For conditional testing, either the left or the right term is called, and the environment is updated with the corresponding name mapped to the value extracted from the pattern. For application, the term inside of an applied function is called, and the environment is updated with the function's parameter mapped to the application's argument. The environment is also updated with the recursive name mapped to the same applied function.

```
predicate callEval: complex -> env -> term -> env -> bool
where only
  ∀ env n_s n_c env_s v n_l t_l n_r t_r .
    if
      env n_s = Some (VAtm (Lft n_c) env_s),
      env_s n_c = Some v
    then
      callEval (Case n_s n_l t_l n_r t_r) env t_l (env(n_l -> v))
* ∀ env n_s n_c env_s v n_l t_l n_r t_r .
    if
      env n_s = Some (VAtm (Rht n_c) env_s),
      env_s n_c = Some v
    then
      callEval (Case n_s n_l t_l n_r t_r) env t_r (env(n_r -> v))
* ∀ env n_f n_f' n_p t_b env_f n_a v .
    if
      env n_f = Some (VAtm (Fun n_f' n_p t_b) env_f),
      env n_a = Some v
    then
      callEval
```

```
(App n_f n_a) env t_b
(env_f(
  n_f' -> (VAtm (Fun n_f' n_p t_b) env_f),
  n_p -> v
))
```

The continuation stack maintains a record of terms that should be evaluated once a corresponding called branch of the evaluation has returned. Each continuation in the stack consists of a term, the environment for resolving the term's names, an unresolved name, to be resolved when the corrsponding branch returns. The initial state of execution consists of a program, an empty environment, and an empty stack of continuations. With each sequential step, the program is reduced to a subterm, and the environment is updated with the name bound to the value of the term. Each time a embedded term is sidestepped to evaluate a dependent term, a continuation is formed around it and pushed onto a stack of continuations. A continuation is popped off the stack when a state's program is reduces to a result program. A pool of states keeps track of all the states that have been reached through the evaluation of an initial program. Each state is indexed by the dynamic path taken to reach it. A pool's leaf path indicates a state that has yet to be evaluated. Additionally, The communication between threads is also recorded as a set of correspondences consisting of the path to the sending state, the path to the receiving state, and the channel used for communication.

```
datatype contin = Ctn name program env

type stack = contin list

datatype state =
  Stt program env stack

type pool =
  dynamic_path -> state option

predicate leaf: pool -> dynamic_path -> bool
where only
  ∀ pool path stt .
    if
      pool path = Some stt,
      (∄ path' stt' .
        pool path' = Some stt',
        strictPrefix path path'
        then
      leaf pool path

type corresp = dynamic_path * chan * dynamic_path

type communication = corresp set
```

The evaluation of a program may involve evaluation of multiple threads concurrently and also communication between threads. Since pools contain multiple states and paths, they can accommodate multiple threads as well. A single evaluation step depends on one pool and evaluates to a new pool based on one or more states in that pool. The initial pool for a program contains just one state indexed by an empty path. The state contains the program, an empty environment, and an empty stack. The pool will grow strictly larger with each evaluation step, maintaining a full history. Each step adds new states and paths extended from previous ones, and each step in the path indicates the mode of flow to take to reach the state. Only states indexed by leaf paths are used to evaluate to the next pool.

A sequencing evaluation step of a program picks a leaf state and relies on sequential evaluation of its top term. It updates the state's environment with the value of the term, leaves the stack unchanged, and reduces the program to the next embedded term. A calling evaluation step relies on the calling evaluation of a state's top term. The binding name, embedded term, environment are pushed onto the stack, and the new state gets its program and environment from the evaluation of the term.

For the evaluation a leaf path steping to a result program, a continuation is popped of the stack, the new state's program is taken from the continuation, and the new state's environment is taken from the continuation and modified with the result value.

In the case of channel creation, the evaluation updates the state's environment with the value of a channel consisting of the path leading to its creation; it leaves the stack unchanged and reduces the program to the next embedded term.

In the case of spawning, the evaluation is updated with two new paths extending the leaf path. For one, the leaf path is extended with a sequential program step whose state has the next term and the environment updated with the unit value bound to the bind name, and the original continuation stack. For the other, the leaf path extended with an program step indicating a spawning flow. Its state has the spawned term, the original environment, and an empty continuation stack. The evaluation updates the state's environment with the unit value, leaves the stack unchanged, and reduces the program to the next embedded term. Additionally, it generates another state consisting of the spawning term's child program, the same environment unchanged, and an empty stack.

In the case where two leaf paths in the pool correspond to synchronization on the same channel, and one synchronizes on a send event and the other synchronizes on a receive event, then evaluation updates the pool with two new paths and corresponding states. It updates the send event's state with its embedded term, the environment updates with the unit value, and the stack unchanged. It updates the receive event's state with its embedded term, the environment updates with the sent value, and the stack unchanged.

Additionally, the communication is updated with the sending and receiving paths, and the channel that the synchronization used for communication.

```
predicate dynamicEval:
  pool -> communication -> pool -> communication -> bool:
where only
```

```
  ∀ pool path n env n_k t_k env_k stack' v comm .
    if
      leaf pool path,
      pool path = Some (Stt (Rslt n) env ((Ctn n_k t_k env_k) # stack')),
      env n = Some v
    then
      dynamicEval
        pool comm
        (pool(
          path @ [DRtn n] ->
            (Stt t_k env_k(n_k -> v) stack')
        ))
        comm
* ∀ pool path n c t' env stack v .
    if
      leaf pool path,
      pool path = Some (Stt (Bind n c t') env stack),
      seqEval c env v
    then
      dynamicEval
        pool comm
        (pool(
          path @ [DNxt n] -> (Stt t' (env(n -> v)) stack)
        ))
        comm
* ∀ pool path n c t' env stack t_c env_c comm .
    if
      leaf pool path,
      pool path = Some (Stt (Bind n c t') env stack),
      callEval c env t_c env_c
    then
      dynamicEval
        pool comm
        (pool(
          path @ [DCll n] -> (Stt t_c env_c ((Ctn n t' env) # stack
        ))
        comm
* ∀ pool path n t' env stack .
    if
      leaf pool path,
      pool path = Some (Stt (Bind n MkChn t') env stack)
    then
      dynamicEval
        pool comm
        (pool(
          path @ [DNxt n] ->
```

```
            (Stt t' (env(n -> (VChn (Chan path x)))) stack)
        ))
        comm
* ∀ pool path n t_c t' env stack comm .
    if
      leaf pool path,
      pool path = Some (Stt (Bind n (Spwn t_c) t') env stack)
    then
      dynamicEval
      pool comm
      (pool(
        path @ [DNxt n] -> (Stt t' (env(n -> VUnt)) stack),
        path @ [DSpwn n] -> (Stt t_c env [])
      ))
      comm
* ∀ pool path_s path n_s n_se t_s env_s stack_s n_sc n_m
    env_se path_r n_r n_re t_r env_r stack_r n_rc env_re c comm .
    if
      leaf pool path_s,
      pool path_s = Some
        (Stt (Bind n_s (Sync n_se) t_s) env_s stack_s),
      env_s n_se = Some
        (VAtm (SendEvt n_sc n_m) env_se),
      leaf pool path_r,
      pool path_r = Some
        (Stt (Bind n_r (Sync n_re) t_r) env_r stack_r),
      env_r n_re = Some
        (VAtm (RecvEvt n_rc) env_re),
      env_se n_sc = Some (VChn c),
      env_re n_rc = Some (VChn c),
      env_se n_m = Some v_m
    then
      dynamicEval
        pool comm
        (pool(
          path_s @ [DNxt n_s] -> (Stt t_s (env_s(n_s -> VUnt)) stack_s),
          path_r @ [DNxt n_r] -> (Stt t_r (env_r(n_r -> v_m)) stack_r)
        ))
        (comm ∪ {(path_s, c, path_r)})
```

3.3 Dynamic Communication

The dynamic one shot classification describes pools where there is only one dynamic path that synchronizes and sends on a given channel. Whether or not two attempts to synchronize on a channel are competitive can be determined by looking at the paths of the pool. If two paths are

ordered, that is, one is the prefix of the other or vice versa, then necessarily occur in sequence, so the shorter path synchronizes before the longer path. Two paths may be competitive only if they are unordered. The dynamic many-to-one classification means that there is no competition on the receiving end of a channel; any two paths that synchronize to receive on a channel are ordered. The dynamic one-to-many classification means that there is no competition on the sending end of a channel; any two paths that synchronize to send on a channel are ordered. The dynamic one-to-one classification means that there is no competition on either the receiving or the sending ends of a channel; any two paths that synchronize on a channel are necessarily ordered for either end of the channel.

```
predicate isSendPath: pool -> chan -> dynamic_path -> bool
where only
  ∀ pool path n n_e t' env stack n_sc n_m env_e c.
    if
      pool path = Some (Stt (Bind n (Sync n_e) t') env stack),
      env n_e = Some (VAtm (SendEvt n_sc n_m) env_e),
      env_e n_sc = Some (VChn c)
    then
      isSendPath pool c path

predicate isRecvPath: pool -> chan -> dynamic_path -> bool
where only
  ∀ pool path n n_e t' env stack n_rc env_e c .
    then
      pool path = Some (Stt (Bind n (Sync n_e) t') env stack),
      env n_e = Some (VAtm (RecvEvt n_rc) env_e),
      env_e n_rc = Some (VChn c)
    then
      isRecvPath pool c path

predicate forEveryTwo: ('a -> bool) -> ('a -> 'a -> bool) -> bool
where only
  ∀ t r .
    if
      ∀ path1 path2 .
        if t path1, t path2 then r path1 path2
    then
      forEveryTwo t r

predicate ordered: 'a list -> 'a list -> bool
where only
  ∀ path1 path2 .
    if prefix path1 path2 then ordered path1 path2
* ∀ path2 path1 .
    if prefix path2 path1 then ordered path1 path2
```

```
predicate oneShot: pool -> chan -> bool
where only
  ∀ pool c .
    if
      forEveryTwo (isSendPath pool c) (op =)
    then
      oneShot pool c

predicate oneToMany: pool -> chan -> bool
where only
  ∀ pool c .
    if
      forEveryTwo (isSendPath pool c) ordered
    then
      oneToMany pool c

predicate manyToOne: pool -> chan -> bool
where only
  ∀ pool c .
    if
      forEveryTwo (isRecvPath pool c) ordered
    then
      manyToOne pool c

predicate oneToOne: pool -> chan -> bool
where only
  ∀ pool c.
    if
      oneToMany pool c,
      manyToOne pool c
    then
      oneToOne pool c
```

3.4 Static Semantics

The static semantics describes an estimation of intermediate static values and embedded terms that might result from running a program. Although the estimations are imprecise with respect to the dynamic semantics, they are certainly accurate, which is confirmed by the formal proofs of soundness. The static semantics enable deduction of static information about channels and events, which is crucial for statically deducing information about synchronization on channels and communication classification. The static values consist of the static unit value, static channels, and static atom values. The static unit value is no less precise than the dynamic unit value, but static channels and static atom values are imprecise versions of their dynamic counterparts. The static channel is identified only by the name it binds to at creation time, rather than the

full path that leads up to its creation. A static atom value is simply an atomic term without an environment for looking up its named arguments. The static environment contains the internal evaluation results by associating names to multiple potential static values. Thus, in addition to some static values being imprecise, the results of evaluation may be decrease precision even further by containinng multiple potential static values. In order to find the return value of a program term, it is useful to fetch the name embedded within its eventual result term, which is formally defined by `resultName`.

```
datatype static_value =
  SChn name
| SUnt
| SAtm atom

type static_value_map =
  name -> static_value set

fun resultName of term -> name:
where only
  ∀ n .
    resultName (Rslt n) = x
* ∀ n c t' .
    resultName (Bind n c t') = (resultName t)
```

The static evaluation is a control flow analysis (0CFA) that describes a relation between a program term and two static environments. The first static environment contains binding names associated with the evaluations of terms that are bound to those names in the program. The second static environment contains names of channels associated with values that might be sent over channels identified by those names.

The definition of static evaluation is syntax-directed, meaning the form of the syntax determines the proof for static evaluation. Additionally, the proof of a static evaluation is definied to be strucutrally inductive following the self-similar structure of the syntax. Thus, it should be possible to decide if a static evaluation holds by unraveling the program term into smaller and smaller terms, until reaching a term without any smaller terms. Additionally, for any given program, there should be instances of static environments, such that the static evaluation holds, in which case there is likely an algorithm to compute the static environments from a program, by following the basic structure of the definitional proof of static evaluation. This certainly appears likely, but it has not been formally proven in this work.

The static evaluation relation is defined in a single definition. The definition is fairly uniform and mimics the structure of the syntax. The static evaluation for each syntactic form is very similar. For instance, if a term has an embedded term, the static evaluation of the term is defined by the static evaluation of its embedded term, whether the orginal term is a spawning term, a function term, or a conditional test term. In constrast, in the definition of dynamic evaluation, the evaluation of certain syntactic forms is more similar to some forms than others. Conditional test terms are evaluated similarly to application terms. They both require save some terms on

the continuation stack, while evaluating other embedded terms. Function terms are dynamically evaluated similarly to other atomic terms. Additionally, the static evaluation has only a single global environment for looking of values of names in the whole program, whereas the dynamic evaluation associates local environemnts with different terms, in the program, allowing the same names to resolve to different values depending on the context. Therefore, the static evaluation is less precise.

```
predicate staticEval:
  static_value_map -> static_value_map -> term -> bool
where only
  ∀ staticEnv staticComm n .
    staticEval staticEnv staticComm (Rslt n)
* ∀ staticEnv n staticComm t' .
    if
      SUnt ∈ staticEnv n,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n Unt t')
* ∀ n staticEnv staticComm t' .
    if
      (SChn x) ∈ staticEnv n,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n MkChn t')
* ∀ n_c n_m staticEnv n staticComm t' .
    if
      (SAtm (SendEvt n_c n_m)) ∈ staticEnv n,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Atom (SendEvt n_c n_m)) t')
* ∀ n_c staticEnv n staticComm t' .
    if
      (SAtm (RecvEvt n_c)) ∈ staticEnv n,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Atom (RecvEvt n_c)) t')
* ∀ n1 n2 staticEnv n staticComm t' .
    if
      (SAtm (Pair n1 n2)) ∈ staticEnv n,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Atom (Pair n1 n2)) e)
* ∀ n_s staticEnv n staticComm t' .
    if
      (SAtm(Lft n_s)) ∈ staticEnv n,
      staticEval staticEnv staticComm t'
```

```
    then
      staticEval staticEnv staticComm (Bind n (Atom (Lft n_s)) t')
* ∀ n_s staticEnv n staticComm t' .
    if
      (SAtm(Rht n_s)) ∈ staticEnv n,
      staticEval staticEnv staticComm t
    then
      staticEval staticEnv staticComm (Bind n (Atom (Rht n_s)) t')
* ∀ n_f n_t t_b staticEnv staticComm n t' .
    if
      (SAtm (Fun n_f n_t t_b)) ∈ staticEnv f,
      staticEval staticEnv staticComm t_b,
      (SAtm (Fun n_f n_t t_b)) ∈ staticEnv n,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Atom (Fun n_f n_t t_b)) t')
* ∀ n_f n_t t_b staticEnv staticComm n t' .
    if
      SUnt ∈ staticEnv n,
      staticEval staticEnv staticComm t_c,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Spwn t_c) t')
* ∀ staticEnv n_e n staticComm  t'.
    if
      ∀ n_sc n_m n_c .
        if
          (SAtm (SendEvt n_sc n_m)) ∈ staticEnv n_e,
          SChn n_c ∈ staticEnv n_sc
        then
          SUnt ∈ staticEnv n, staticEnv n_m ⊆ staticComm n_c),
      ∀ n_rc n_c .
        if
          (SAtm (RecvEvt n_rc)) ∈ staticEnv n_e,
          SChn n_c ∈ staticEnv n_rc,
        then
          staticComm n_c ⊆ staticEnv n),
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Sync n_e) t')
* ∀ staticEnv n_t n staticComm t' .
    if
      ∀ n1 n2 . if (SAtm (Pair n1 n2)) ∈ staticEnv n_t then
        staticEnv n1 ⊆ staticEnv n),
      staticEval staticEnv staticComm t'
    then
```

```
      staticEval staticEnv staticComm (Bind n (Fst n_t) t')
* ∀ staticEnv n_t n staticComm t' .
    if
      ∀ n1 n2 . if (SAtm (Pair n1 n2)) ∈ staticEnv n_t then
        staticEnv n2 ⊆ staticEnv n),
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Snd n_t) t')
* ∀ staticEnv n_s n_l t_l n staticComm n_r t_r t' .
    if
      ∀ n_c . if (SAtm (Lft n_c)) ∈ staticEnv n_s then
        staticEnv n_c ⊆ staticEnv n_l,
        staticEnv (resultName t_l) ⊆ staticEnv n,
      staticEval staticEnv staticComm t_l,
      ∀ n_c . if (SAtm (Rht n_c)) ∈ staticEnv n_s then
        staticEnv n_c ⊆ staticEnv n_r,
        staticEnv (resultName t_r) ⊆ staticEnv n,
      staticEval staticEnv staticComm t_r,
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (Case n_s n_l t_l n_r t_r) t')
* ∀ staticEnv n_f n_a n staticComm t' .
    if
      ∀ n_f' n_t t_b . if (SAtm (Fun n_f' n_t t_b)) ∈ staticEnv n_f then
        staticEnv n_a ⊆ staticEnv n_t,
        staticEnv (resultName t_b) ⊆ staticEnv n),
      staticEval staticEnv staticComm t'
    then
      staticEval staticEnv staticComm (Bind n (App n_f n_a) t')
```

It is straightforward to follow the rules of static evaluation in order to build up functions mapping names to static values for the static environment and the static communication. Recasting the example server implementation into the ANF syntax is demonstrates this informal procedure.

```
bind u1 = unt
bind r1 = rht u1
bind l1 = lft r1
bind l2 = lft l1

bind mksr = fun _ x2 =>
(
  bind k1 = mkChn
  bind srv = fun srv' x3 =>
  (
    bind e1 = recvEvt k1
    bind p1 = sync e1
    bind v1 = fst p1
```

```
    bind k2 = snd p1
    bind e2 = sendEvt k2 x3
    bind z5 = sync e2
    bind z6 = srv' v1
    bind u4 = unt
    rslt u4
    bind z7 = spawn
    (
      bind z8 = srv r1
      bind u5 = unt
      rslt u5
      rslt k1
    )
  )
)

bind rqst = fun _ x4 =>
(
  bind k3 = fst x4
  bind v2 = snd x4
  bind k4 = mkChn
  bind p2 = pair v2 k4
  bind e3 = sendEvt k3 p2
  bind z9 = sync e3
  bind e4 = recvEvt k4
  bind v3 = sync e4
  rslt v3
)

bind srvr = mksr u1
bind z10 = spawn
(
  bind p3 = pair srvr l1
  bind z11 = rqst p3
  rslt z11
)
bind p4 = pair srvr l2
bind z12 = rqst p4
rslt z12
```

Let's see how an informal procedure can produce the static environments by following the structure of the definitional proof structure of static evaluation. We start at the top of the program and pick a rule from the definition of static evaluation that might hold true for the current syntactic form. Then we choose the smallest environment that satifies that rule's conditions. In the server implementation, the program starts with `bind u1 = unt in ...`, which only unifies with the rule concluding with `staticEval staticEnv staticComm (Bind n Unt ...)`, with

`n = (Nm "u1")`. The conditions for that rule require `SUnt ∈ staticEnv (Nm "u1"), staticEval staticEnv staticComm ...`. We choose the smallest static environment `staticEnv`, for which `SUnt in staticEnv (Nm "u1")` holds, and that happens to be `λ n . if n = (Nm "u1") then {SUnt} else {}`. Since there's no condition that directly states what's required of the static communication, we can simply choose an empty environment to start with. The second condition is static evaluation on a smaller term, which indicates that we should repeat this procedure again for the remainder of the program, incrementally adding more static values for each binding name in the program. We continually repeat this procedure from the top of the program until there's nothing more we can add to the static environments. The rule for syncrhonization is the only rule in which there are conditions on the static communication environment. So we will only add to the static communication environment when we encounter syncrhonization terms. The following static environemnts result from following this informal procedure on the example ANF server implementation. To make the presentation clear, the syntactic sugar `(r1 -> {rht u1}, ...)` is used to mean `λ n . if n = (Nm "r1") then {SAtm (Rht (Nm "u1"))} else ... else {}`. The representation of static values closely resembles the concrete syntax for complex terms.

```
val serverStaticEnv: name -> static_value set =
(
  u1 -> {unt},
  r1 -> {rht u1},
  l1 -> {lft r1},
  l2 -> {lft l1},
  mksr -> {fun _ x2 => ...},
  x2 -> {unt},
  k1 -> {chn k1},
  srv -> {fun srv' x3 => ...},
  srv' -> {fun srv' x3 => ...},
  x3 -> {rht u1, lft r1, lft l1},
  e1 -> {recvEvt k1},
  p1 -> {pair v2 k4},
  v1 -> {lft r1, lft l1},
  k2 -> {chn k4},
  e2 -> {sendEvt k2 x3},
  z5 -> {unt},
  z6 -> {unt},
  u4 -> {unt},
  z7 -> {unt},
  z8 -> {unt},
  u5 -> {unt},
  rqst -> {fun _ x4 => ...},
  x4 -> {pair srvr l1, pair srvr l2},
  k3 -> {chn k1},
  v2 -> {lft r1, lft l1},
  k4 -> {chn k4},
  p2 -> {pair v2 k4},
```

```
  e3 -> {sendEvt k3 p2},
  z9 -> {unt},
  e4 -> {recvEvt k4},
  v3 -> {rht u1, lft r1, lft r2},
  srvr -> {chn k1},
  z10 -> {unt},
  p3 -> {pair srvr l1},
  z11 -> {rht u1, lft r2},
  p4 -> {pair srvr l2},
  z12 -> {rht u1, lft l1}
)

val serverStaticComm: name -> static_value set =
(
  k1 -> {pair v2 k4},
  k4 -> {rht u1, lft l1, lft l2}
)
```

The static reachability describes terms that might be reachable from larger terms, during dynamic evaluation. A sound approximation for dynamically reachable terms are terms that are transitively embedded within larger terms. A term is statically reachable from itself, and an initial term can statically reach any term that its embedded terms can statically reach.

```
predicate staticReachable: term -> term -> bool
where only
  ∀ t .
    staticReachable t t
* ∀ t_c t_z n t' .
    if
      staticReachable t_c t_z
    then
      staticReachable (Bind n (Spwn t_c) t') t_z
* ∀ t_l t_z n n_s n_l n_r t_r t' .
    if
      staticReachable t_l t_z
    then
      staticReachable (Bind n (Case n_s n_l t_l n_r t_r) t') t_z
* ∀ t_r t_z n n_s n_l t_l n_r t' .
    if
      staticReachable t_r t_z
    then
      staticReachable (Bind n (Case n_s n_l t_l n_r t_r) t') t_z
* ∀ t_b t_z n n_f n_t t_b t' .
    if
      staticReachable t_b t_z
    then
      staticReachable (Bind n (Atom (Fun n_f n_t t_b)) t') t_z
```

```
* ∀ t' t_z n c .
    if
      staticReachable t' t_z
    then
      staticReachable (Bind n c t') t_z
```

3.5 Static Communication

To describe communication statically, it is helpful to identify each term with a short description. The term identifier of a binding term is the binding name, and indication of its use in a binding. The term identifier of a result term is the embedded name, and indication of its use in a result.

```
datatype termId =
  NBnd name
| NRslt var

fun termId: term -> termId
where only
  ∀ n c t' .
    termId (Bind n c t') = NBnd n
* ∀ n .
    termId (Rslt n) = NRslt n

type term_id_map = termId -> name set
```

The static communication describes a sound approximation of the static paths that communicate on static channels. The static sending identifier classification means that a term identifier might represent a sycnronization to send on a given static channel. The static receiving identifier classification means that a term identifier might represent a synchronization to receive on a given abstract channel.

```
predicate staticSendId:
  static_value_map -> term -> name -> termId -> bool
where only
  ∀ t0 n n_e t' n_sc n_m staticEnv n_c .
    if
      staticReachable t0 (Bind n (Sync n_e) t'),
      (SAtm (SendEvt n_sc n_m)) ⊆ staticEnv n_e,
      (SChn n_c) ∈ staticEnv n_sc
    then
      staticSendId staticEnv t0 n_c (NBnd n)

predicate staticRecvId:
  static_value_map -> term -> name -> termId -> bool
where only
  ∀ t0 n n_e t' n_rc staticEnv n_c .
```

```
if
  staticReachable t0 (Bind n (Sync n_e) t'),
  (SAtm (RecvEvt n_rc)) ∈ staticEnv n_e,
  (SChn n_c) ∈ staticEnv n_rc
then
  staticRecvId staticEnv t0 n_c (NBnd n)
```

In the server implementation, the static channel identified by the name `k1` is waited on by the server. It has one receiving identifier in the server function at identifier **`bind`** `p1` and a sending identifier in the request function at identifier **`bind`** `z9`. The channel idetifiend by the name `k4` is sent with a client's request for the server to reply on. It has a receiving identifier in the request function at **`bind`** `v3` and a sending identifier in the server function at **`bind`** `z5`.

Reppy and Xiao's work relies on detecting the liveness of channels in order to gain higher precision in the static classification of communication. Since formal proofs are inherently complicated with numerous details, it was easier to first formally prove soundness for a version without the added complication of considering liveness of channel.

The definitions are purposely structured to allow adding live channel analysis to the definition fairly straightforward with just a few alterations. Section ? expands on these alterations and outlines a strategy that is likely to result in formal proofs of soundness, although the actual formal proof of the version with live channel analysis is not yet complete.

For the lower precision version without the liveness of channels, there are four modes, indicating how the term identifier flows to the term identifier of one of its embedded terms. The modes of flow are sequencing, calling, spawning, and returning. A flow is a triplet of a term identifier, a mode of flow, and a term identifier of a embedded term. A static step is just a term identifier along with the mode it uses to flow to its embedded term. A static path is a list of static steps.

```
datatype mode =
  MNxt
| MSpwn
| MCll
| MRtn

type flow = termId * mode * termId

type graph = flow set

type static_step = termId * mode

type static_path = static_step list
```

The evaluation of a term results in the flow to new terms, via sequencing, calling, returning, or spawning. These flows are represented concretely by the term identifiers of the starting and ending terms and a mode to represent the nature of the flow. The static acceptance by flows describes a set of flows consisting of all the flows that could be traversed during a program's

evaluation. It depends on the static environment for name bindings. For a result term, there are no demands on the flow graph. For all bind terms, except those binding to case matching and function application, the sequential flow from the top term to the sequenced term might accept the term, and the accepting flows are also the accepting flows for the sequenced term. For binding to function function, the accepting flows are also accepting flows for the inner term of the function function. For binding to spawning, the spawning flow from the top term to the spawned term might be traversed, and the accepting flows are also accepting flows for the spawned term.

In the case of conditional testing, the calling flow from the conditional testing term to its left case's embedded term might accept the conditional testing term, and the calling flow from the a term to the right case's embedded term might accept the conditional testing term. The returning flow from the result of the left case's embedded term to the sequenced embedded term might accept the result term, and the returning flow from the result of the right embedded term to the sequenced embedded term might also accept the result term. Additionally, the accepting flows for a term are also accepting flows for its left case's embedded term, right case's embedded term, and the sequenced embedded term.

In the case of application, if the applied name is actually bound to a function function, then a calling flow from the application term to the function's embedded term might accept the term, and the returning flow from the result of the function function to the sequenced embedded term might accept the term. Additionally, the accepting flows for the application term are also accepting flows for the sequenced embedded term.

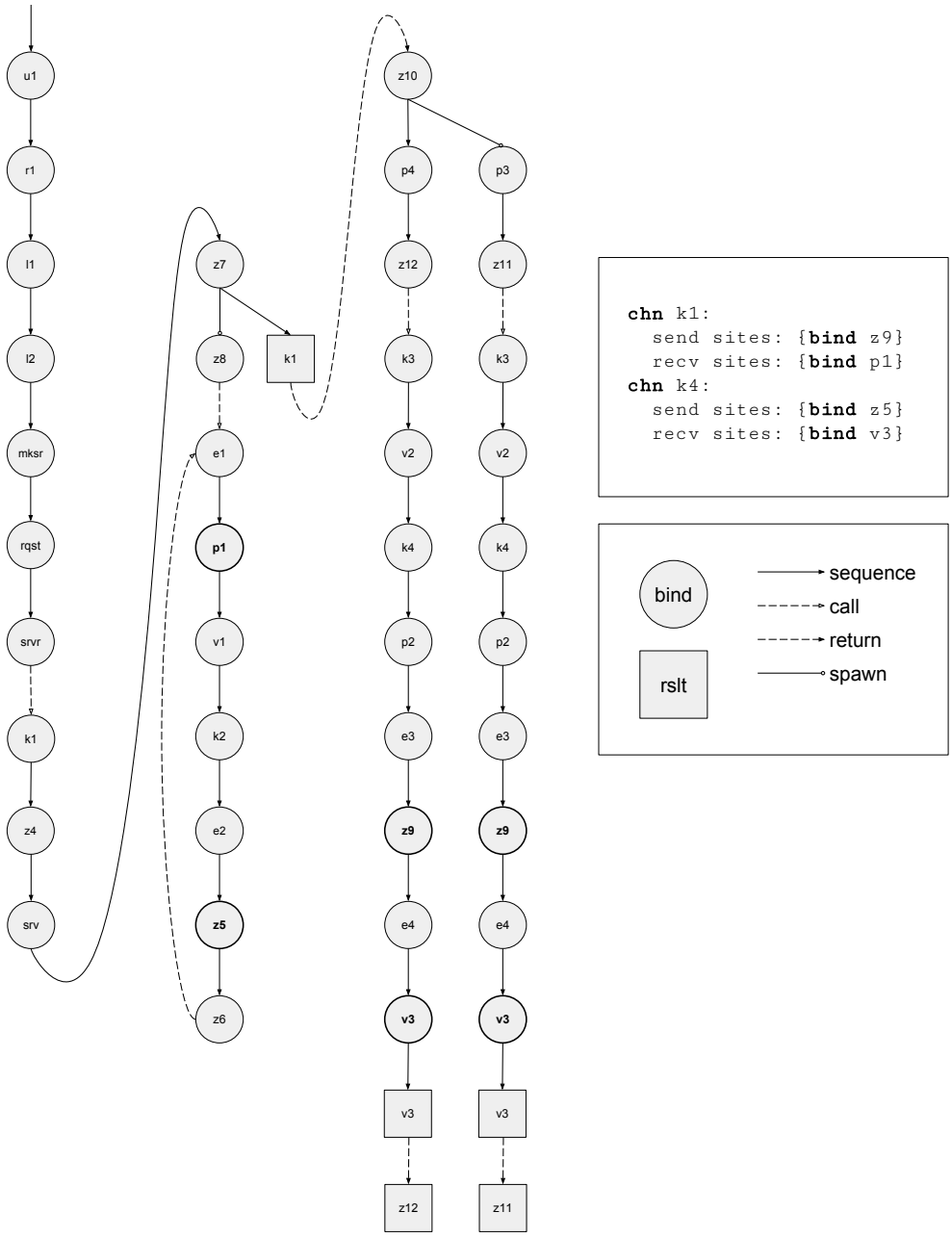
```
predicate staticFlowsAccept:
  static_value_map -> graph -> term -> bool
where only
  ∀ staticEnv graph n .
    staticFlowsAccept staticEnv graph (Rslt n)
* ∀ n t' graph staticEnv  .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n Unt t')
* ∀ n t' graph staticEnv  .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n MkChn t')
* ∀ n t' graph staticEnv  n_c n_m .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept
```

```
        staticEnv graph
        (Bind n (Atom (SendEvt n_c n_m)) t')
* ∀ n t' graph staticEnv n_c .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (RecvEvt n_c)) t')
* ∀ n t' graph staticEnv n1 n2 .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (Pair n1 n2)) t')
* ∀ n t' graph staticEnv n_s .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (Lft n_s)) t')
* ∀ n t' graph staticEnv n_s .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (Rht n_s)) t')
* ∀ n t' graph staticEnv t_b n_f n_t .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t',
      staticFlowsAccept staticEnv graph t_b
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (Fun n_f n_t t_b)) t')
* ∀ n t' t_c graph staticEnv.
    if
      {(NBnd n, MNxt, termId t'),
        (NBnd n, MSpwn, termId t_c)} ⊆ graph,
      staticFlowsAccept staticEnv graph t_c,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Spwn t_c) t')
* ∀ n t' graph staticEnv n_se .
    if
      (NBnd n, MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
```

```
      staticFlowsAccept staticEnv graph (Bind n (Sync n_se) t')
* ∀ n t' graph staticEnv n_t .
    if
      (NBnd n, MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t',
    then
      staticFlowsAccept staticEnv graph (Bind n (Fst n_t) t')
* ∀ n t' graph staticEnv n_t .
    if
      (NBnd n, MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t',
    then
      staticFlowsAccept staticEnv graph (Bind n (Snd n_t) t')
* ∀ n t_l t_r t' graph staticEnv n_s .
    if
      {
        (NBnd n, MCll, termId t_l),
        (NBnd n, MCll, termId t_r),
        (NRslt (resultName t_l), MRtn, termId t'),
        (NRslt (resultName t_r), MRtn, termId t')
      } ⊆ graph,
      staticFlowsAccept staticEnv graph t_l,
      staticFlowsAccept staticEnv graph t_r,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Case n_s n_l t_l n_r t_r) t')
* ∀  staticEnv n_f n t' n n_a .
    if
      ∀ n_f' n_t t_b . if (SAtm (Fun n_f' n_t t_b)) ∈ staticEnv n_f then
        {
          (NBnd n, MCll, termId t_b),
          (NRslt (resultName t_b), MRtn, termId t')
        } ⊆ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (App n_f n_a) t')
```

The smallest graph of flows that accepts a progam is finite. Additionally, the static acceptance relation is syntax-directed, which offers guidance towards computing the graph from a program. Thus, to statically determining communication classifications, it should be possible to compute the two shortest paths that send or receive on the same channel.

The server implementation represented as a graph illustrates how static acceptance by flows can interpret a graph from a dynamic program.



The static traceability means that a static path with a given starting step, and ending condition, can be traced by traversing the flows in a graph. The empty path is statically traceable if the starting step meets the ending condition. Otherwise, a path is statically traceable if the last static

step corresponds to a flow that meets the ending condition, and the longest strict prefix of the path is statically traceable.

```
predicate staticTraceable:
  flow set -> termId -> (termId -> bool) -> static_path -> bool
where only
  ∀ start graph isEnd .
    if
      isEnd start
    then
      staticTraceable graph start isEnd []
* ∀  graph star middle path isEnd end mode .
    if
      staticTraceable graph start (λ l . l = middle) path,
      isEnd end,
      (middle, mode, end) ∈ graph
    then
      staticTraceable graph start isEnd (path @ [(middle, mode)])
```

In the graph of the server implementation, there are two paths each corresponding to its own thread that lead to sending on static channel **chn** `k1` and a potentially infinite number of paths that lead to receiving on channel **chn** `k1`, but all on the same thread. There are an infinite number of paths that lead to sending on static channel **chn** `k4`, and two paths that lead to receiving on static channel **chn** `k4`. This is certainly imprecise, as the static **chn** `k4` corresponds to multiple distinct dynamic channels, each with just one sender and one receiver. The higher precision analysis discussed in section ? addresses this issue.

The static inclusion means that two static paths might be traced in the same run of a program. Ordered paths might be inclusive, and also a path that diverges from another at a spawn flow might be inclusive. This concept is useful for achieving greater precision, since if two paths cannot occur in the same run of a program, only one needs to be counted towards the communication classification.

```
predicate staticInclusive: static_path -> static_path -> bool
where only
  ∀ path1 path2 .
    if
      prefix path1 path2
    then
      staticInclusive path1 path2
* ∀ path2 path1 .
    if
      prefix path2 path1
    then
      staticInclusive path1 path2
* ∀ path n path1 path2 .
    staticInclusive
```

```
      (path @ (NBnd n, MSpwn) # path1)
      (path @ (NBnd n, MNxt) # path2)
* ∀ path n path1 path2 .
    staticInclusive
      (path @ (NBnd n, MNxt) # path1)
      (path @ (NBnd n, MSpwn) # path2)
```

The singularness means that two paths are the same or only of them can occur in a given run of a program. The noncompetitiveness means states that two paths can't compete during a run of a program, since they are ordered or cannot occur in the same run of a program.

```
predicate singular: static_path -> static_path -> bool
where only
  ∀ path .
    singular path path
* ∀ path1 path2 .
    if
      not (staticInclusive path1 path2)
    then
      singular path1 path2

predicate noncompetitive: static_path -> static_path -> bool
where only
  ∀ path1 path2 .
    if
      ordered path1 path2
    then
      noncompetitive path1 path2
* ∀ path1 path2 .
    if
      not (staticInclusive path1 path2)
    then
      noncompetitive path1 path2
```

The static one-shot classification means that there is at most one attempt to synchronize to send on a static channel in any run of a given program.

```
predicate staticOneShot: static_value_map -> term -> name -> bool
where only
  ∀ graph t staticEnv n_c .
    if
      forEveryTwo
        (staticTraceable graph
          (termId t) (staticSendId staticEnv t n_c))
        singular,
      staticFlowsAccept staticEnv graph t
    then
      staticOneShot graph t n_c
```

The static one-to-one classification means that there is at most one thread that attempts to send and at most one thread that attempts to receive on a given static channel for any time during a run of a given program.

```
predicate staticOneToOne: static_value_map -> term -> name -> bool
where only
  ∀ graph t staticEnv n_c .
    if
      forEveryTwo
        (staticTraceable graph
          (termId t) (staticSendId staticEnv t n_c))
        noncompetitive,
      forEveryTwo
        (staticTraceable graph
          (termId t) (staticRecvId staticEnv t n_c))
        noncompetitive,
      staticFlowsAccept staticEnv graph t
    then
      staticOneToOne staticEnv t n_c
```

The static one-to-many classification means that there is at most one thread that attempts to send on a given static channel at any time during a run of a given program, but there may be many threads that attempt to receive on the channel.

```
predicate staticOneToMany: static_value_map -> term -> name -> bool
where only
  ∀ graph t staticEnv n_c .
    if
      forEveryTwo
        (staticTraceable graph
          (termId t) (staticSendId staticEnv t n_c))
        noncompetitive,
      staticFlowsAccept staticEnv graph t
    then
      staticOneToMany staticEnv t n_c
```

The static many-to-one predicate means that there may be many threads that attempt to send on a static channel, but there is at most one thread that attempts to receive on the channel for any time during a run of a given program.

```
predicate staticManyToOne: static_value_map -> term -> name -> bool
where only
  ∀ graph t staticEnv n_c .
    if
      forEveryTwo
        (staticTraceable graph
          (termId t) (staticRecvId staticEnv t n_c))
        noncompetitive,
```

```
    staticFlowsAccept staticEnv graph t
  then
    staticManyToOne staticEnv t n_c
```

3.6 Formal Reasoning

Reppy and Xiao informally prove soundness of their analysis by showing that their static analysis determines that more than one thread sends (or receives) on a channel if the execution allows more than one to send (or receive) on that channel. The proof of soundness depends on the ability to relate the execution of a program to the static analysis of a program. The static analysis describes threads in terms of control paths, since it can only describe threads in terms of statically available information. Thus, in order to describe the relationship between the threads of the static analysis and the operational semantics, the operational semantics is defined as stepping between sets of control paths paired with terms. Divergent control paths are added whenever a new thread is spawned.

The semantics and analysis must contain many details. To ensure the correctness of proofs, it is necessary to check that there are no subtle errors in either the definitions or proofs. Proofs in general require many subtle manipulations of symbols. The difference between a false statement and a true statement can often be difficult to spot, since the two may be very similar lexically. However, a mechanical proof checker, such as that of Isabelle, has no difficulty discerning between valid and invalid derivations. Mechanical checking of proofs can notify users of errors in the proofs or definitions far better and faster than manual checking. This work has greatly benefited from Isabelle's proof checker in order to correctly define the language semantics, control flow analysis, communication analysis, and other helpful definitions. For instance, some bugs in the definintions were found trying to prove soundness. The proof checker would not accept the proof unless I provided facts that should be false, indicating that the definitions did not state my intentions. After correcting the errors in the definitions, the proof was completed such that the proof checker was satisfied.

The reasoning involved in proving the soundness of each communication classification is based around breaking the goal into simpler subgoals, and generalizing assumptions to create useful induction hypotheses. It is often useful to create helper definitions that can be deduced from premises of the theorem being proved and enable general reasoning across arbitrary programs. A frequent pattern is to define predicates in terms of semantic structures, like the environment, stack, and pool, and deduce the instantiation of these predicates on the initial program state.

Some aspects of the generalized predicate definitions exist simply to prove that they imply instantiations of the original program based predicates. However, the generalized definitions exist in order to allow direct access to properties that would otherwise be deeply nested in an inductive structure and inaccessible by a predictable number of logical steps for an arbitrary program.

One of the most difficult aspects of formal reasoning is in developing adequate definitions. It is often possible to define a single semantics in multiple ways. For instance, the sortedness of a list could be defined in terms of the sortedness of its tail or in terms of the sortedness of its longest

strict prefix. To prove theorems relating sortedness to other relations, it may be important that the other relations are inductively defined on the same subpart of the list. Some relations may only be definable on the tail, while others can be defined only on the strict prefix. in such cases, it is necessary to define sortedness in two ways, and prove their equivalence, in order to prove theorems relating to less flexible relations.

```
predicate sortedLeft: nat list -> bool
where only
  sortedLeft []
* ∀ x .
    sortedLeft [x]
* ∀ x y zs .
    if
      n ≤ y,
      sortedLeft (y # zs)
    then
      sortedLeft (x # y # zs)

predicate sortedRight: nat list -> bool
where only
  sortedRight []
* ∀ z .
    sortedRight [z]
* ∀ xs y z .
    if
      sortedRight (xs @ [y]),
      y ≤ z
    then
      sortedRight (xs @ [y] @ [z])

lemma sortedEquiv:
  ∀ xs . sortedLeft xs ≡ sortedRight xs
```

3.7 Soundness

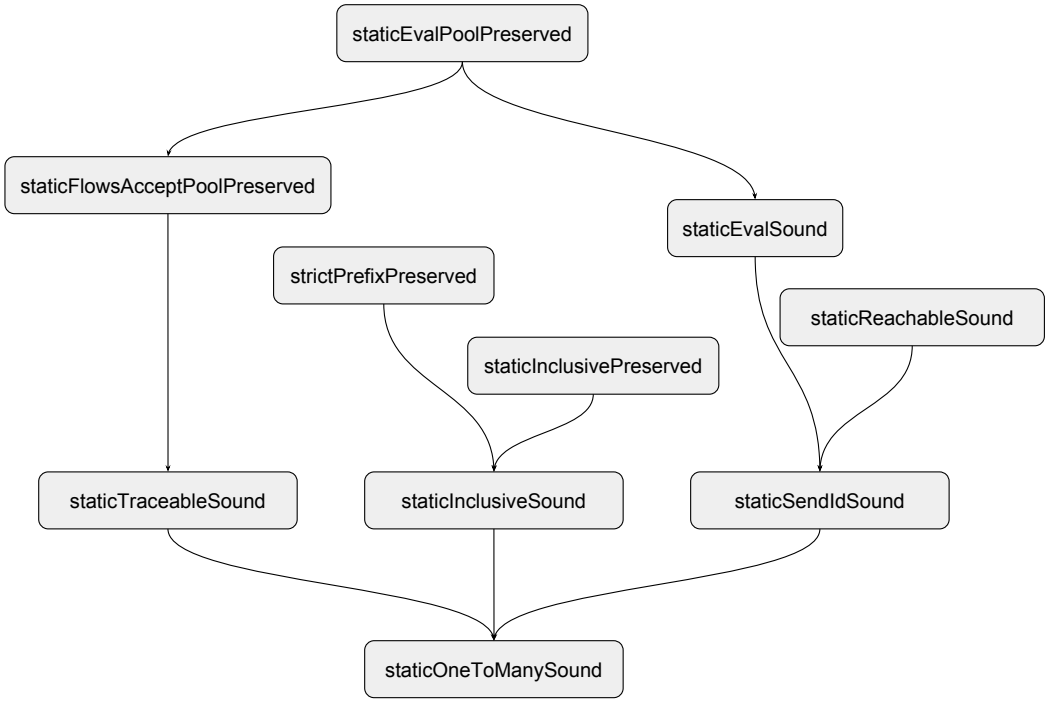
The theorem for soundness of static one-shot classification states that if a static channel is statically classified as one-shot for a given program and static environment consistent with the program, then any corresponding dynamic channel is classified as one-shot over any pool that results from running the program. The theorem for soundness of static one-to-many classification states that if a static channel is statically classified as one-to-many for a given program and static environment consistent with that program, then any corresponding dynamic channel is classified as one-to-many over any pool that results from running the program. The theorems for soundness of many-to-one classification and one-to-one classification follow the same pattern.

```
theorem staticOneShotSound:
```

```
  ∀ staticEnv staticComm t0 n_c pool comm path_c .
    if
      staticEval staticEnv staticComm t0,
      staticOneShot staticEnv t0 n_c,
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm
    then
      oneShot pool (Chan path_c n_c)

theorem staticOneToManySound:
  ∀ staticEnv staticComm t0 n_c pool comm path_c.
    if
      staticEval staticEnv staticComm t0,
      staticOneToMany staticEnv t0 n_c,
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm
    then
      oneToMany pool (Chan path_c n_c)

theorem staticManyToOneSound:
  ∀ staticEnv staticComm t0 n_c pool comm path_c.
    if
      staticEval staticEnv staticComm t0,
      staticManyToOne staticEnv t0 n_c,
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm
    then
      manyToOne pool (Chan path_c n_c)

theorem staticOneToOneSound:
  ∀ staticEnv staticComm t0 n_c pool comm path_c.
    if
      staticEval staticEnv, staticComm t0,
      staticOneToOne staticEnv t0 n_c,
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm
    then
      oneToOne pool (Chan path_c n_c)
```

The formal proofs of soundness of each static classification follow a similar structure. Let's examine in some detail the formal proof of soudness of static one-to-many classification, by unwinding the theorem into the lemmas that it follows from.

The following diagram illustrates the key dependencies of the theorems and lemmas in used in the derivations.



The soundness of static one-to-many classification is proved by a few simpler lemmas and the definitions of static and dynamic one-to-many classification. There is an isomorphic correspondence between the paths of dynamic evaluation and the paths of static evaluation, by definition. The three main lemmas state the soundness of the static traceability, the soundness of the static inclusiveness, and the soundness of a program step not being a static sending identifier. These lemmas depend on a correspondence between static paths and dynamic paths, which is bijective for the lower precision analysis. The lemma for soundness of static inclusiveness states that any two dynamic paths traced by running a program correspond to statically inclusive static paths. It follows from a straightforward case analysis of static inclusivity. The lemma for soundness of static traceability states that for any dynamic path traced by running a program, there is a corresponding static path that is statically traceable. The lemma for soundness of a program step not being a static sending identifier states that running a program reaches a synchronization on a sending event, then that synchronization is statically identified as a sending identifier by its term identifier.

```
predicate pathsCorrespond: dynamic_path -> static_path -> bool
where only
  pathsCorrespond [] []
* ∀ path staticPath n .
```

```
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DNxt n])
        (staticPath @ [(NBnd n, MNxt)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DSpwn n])
        (staticPath @ [(NBnd n, MSpwn)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DCll n])
        (staticPath @ [(NBnd n, MCll)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DRtn n])
        (staticPath @ [(NRslt n, MRtn)])

lemma staticTraceableSound:
  ∀ t0 pool comm path n c t' env stack staticEnv staticComm graph isEnd .
    if
      star dynamicEval ([[] -> (Stt t0 [->] [])], {}) (pool, comm),
      pool path = Some (Stt (Bind n c t') env stack),
      staticEval staticEnv staticComm t0,
      staticFlowsAccept staticEnv graph t0,
      isEnd (NBnd n)
    then
      exists staticPath .
        pathsCorrespond path staticPath,
        staticTraceable graph (termId t0) isEnd staticPath

lemma staticInclusiveSound:
  ∀ t0 pool comm path1 stt1 path2 stt2 staticPath1 staticPath2 .
    if
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm
      pool path1 = Some stt1,
```

```
      pool path2 = Some stt2,
      pathsCorrespond path1 staticPath1,
      pathsCorrespond path2 staticPath2
    then
      staticInclusive staticPath1 staticPath2

lemma staticSendIdSound:
  ∀ t0 pool comm path n n_e t' env stack n_sc n_m env' path_c n_c .
    if
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm,
      pool path = Some (Stt (Bind n (Sync n_e) t') env stack),
      env n_e = Some (VAtm (SendEvt n_sc n_m) env'),
      env' n_sc = Some (VChn (Chan path_c n_c)),
      staticEval staticEnv staticComm t0
    then
      staticSendId staticEnv t0 n_c (NBnd n)
```

The completeness of static traceability is proved by generalizing static acceptance by flows and static evaluation over pools, such that information about a step in the program can be deduced by a fixed number of logical steps regardless of the location of the program step or the size of the program. Without such generalization, it would be possible to prove soundness for a fixed program, but not any arbitrary program.

The generalization of static acceptance by flows is comprised of static acceptance by flows over values, static acceptance by flows over environments, static acceptance by flows over stacks, and static acceptance by flows over pools. In most cases, it simply states that a embedded term of some semantic element is also statically accepting. The exception is in the case of static acceptance by flows over a non-empty stack, where there is an additional condition that the flow from a result identifier to the term identifier of the continuation program exists in the graph. This information is consistent with static acceptance by flows over programs, but provides direct information about a flow in the graph, which would otherwise only be deducible by a varying number of logical steps depending on the program.

```
predicate staticFlowsAcceptValue:
  static_value_map -> graph -> dynamic_value -> bool
where only
  ∀ staticEnv graph .
    staticFlowsAcceptValue staticEnv graph VUnt
* ∀ staticEnv graph n_c .
    staticFlowsAcceptValue staticEnv graph (VChn n_c)
* ∀ staticEnv graph env n_c n_m.
    if
      staticFlowsAcceptEnv staticEnv graph env
    then
      staticFlowsAcceptVal
        staticEnv graph (VAtm (SendEvt n_c n_m) env)
```

```
* ∀ staticEnv graph env n_c.
    if
      staticFlowsAcceptEnv staticEnv graph env
    then
      staticFlowsAcceptVal
        staticEnv graph (VAtm (RecvEvt n_c) env)
* ∀ staticEnv graph env n_p .
    if
      staticFlowsAcceptEnv staticEnv graph env
    then
      staticFlowsAcceptVal
        staticEnv graph (VAtm (Lft n_p) env)
* ∀ staticEnv graph env n_p .
    if
      staticFlowsAcceptEnv staticEnv graph env
    then
      staticFlowsAcceptVal
        staticEnv graph (VAtm (Rht n_p) env)
* ∀ staticEnv graph t_b env n_f n_p .
    if
      staticFlowsAccept staticEnv graph t_b,
      staticFlowsAcceptEnv staticEnv graph env
    then
      staticFlowsAcceptVal
        staticEnv graph (VAtm (Fun n_f n_p t_b) env)
* ∀ staticEnv graph env .
    if
      staticFlowsAcceptEnv staticEnv graph env
    then
      staticFlowsAcceptVal
      staticEnv graph (VAtm (Pair n1 n2) env)

predicate staticFlowsAcceptEnv:
  static_value_map -> graph -> env -> bool:
where only
  ∀ staticEnv graph env .
    if
      ∀ n v . if env n = Some v then
        staticFlowsAcceptValue staticEnv graph v
    then
      staticFlowsAcceptEnv staticEnv graph env

predicate staticFlowsAcceptStack:
  static_value_map -> graph -> name -> stack -> bool
where only
  ∀ staticEnv graph y .
```

```
    staticFlowsAcceptStack staticEnv graph y []
* ∀ y t graph staticEnv graph env stack n env .
    if
      {(NRslt y, MRtn, termId e)} ⊆ graph,
      staticFlowsAccept staticEnv graph t,
      staticFlowsAcceptEnv staticEnv graph env,
      staticFlowsAcceptStack staticEnv graph (resultName t) stack
    then
      staticFlowsAcceptStack staticEnv graph y ((Ctn n t env) # stack)

predicate staticFlowsAcceptPool of
  static_value_map -> graph -> pool -> bool
where only
  ∀ staticEnv graph pool .
    if
      ∀ path t env stack .
        if
          env path = Some (Stt t env stack)
        then
          staticFlowsAccept staticEnv graph t,
          staticFlowsAcceptEnv staticEnv graph env,
          staticFlowsAcceptStack staticEnv graph (resultName t) stack
      )
    then
      staticFlowsAcceptPool staticEnv graph pool
```

The flows described by the various versions of static acceptance by flows depend on static environments in order to look up the control flow in the case where the term is a function. The static environment results from the static evaluation of the program that is dynamically evaluated. Thus, generalized versions of static evaluation enable further deduction about flows. As with the generalized versions of static acceptance by flows, the generalized versions of static evaluation are designed to preserve static environments across dynamic evaluations of pools. They also provide direct access to binding information from names to static values in a fixed number of logical steps. Static evaluation of programs correlates program syntax to static values, but the generalized static evaluations correlate dynamic semantic structures, like, value, environments, and stacks, to static values. The function function relates dynamic values to static values and helps the larger goal of relating dynamic semantic elements to static values and static environments.

```
fun abstract: dynamic_value -> static_value
where only
  abstract VUnt = SUnt
* ∀ path n .
    abstract (VChn (Chan path x)) = SChn x)
* ∀ atom env .
    abstract (VAtm atom env) = SAtm atom)
```

```
predicate staticEvalValue:
  static_value_map -> abstract_comm -> dynamic_value -> bool
where only
  ∀ staticEnv staticComm .
    staticEvalValue staticEnv staticComm VUnit)
* ∀ staticEnv staticComm c .
    staticEvalValue staticEnv staticComm (VChn c))
* ∀ staticEnv staticComm env n_c n_m .
    if
      staticEvalEnv staticEnv staticComm env
    then
      staticEvalValue staticEnv staticComm
        (VAtm (SendEvt n_c n_m) env))
* ∀ staticEnv staticComm env n_c  .
    if
      staticEvalEnv staticEnv staticComm env
    then
      staticEvalValue staticEnv staticComm
        (VAtm (RecvEvt n_c) env))
* ∀ staticEnv staticComm env n_p  .
    if
      staticEvalEnv staticEnv staticComm env
    then
      staticEvalValue staticEnv staticComm
        (VAtm (Lft n_p) env))
* ∀ staticEnv staticComm env n_p  .
    if
      staticEvalEnv staticEnv staticComm env
    then
      staticEvalValue staticEnv staticComm
        (VAtm (Rht n_p) env))
* ∀ n_f n_p t_b staticEnv staticComm env .
    if
      {SAtm (Fun n_f n_p t_b)} ⊆ staticEnv f,
      staticEval staticEnv staticComm t_b,
      staticEvalEnv staticEnv staticComm env
    then
      staticEvalValue staticEnv staticComm
        (VAtm (Fun n_f n_p t_b) env))
* ∀ staticEnv staticComm env n1 n2 .
    if
      staticEvalEnv staticEnv staticComm env
    then
      staticEvalValue staticEnv staticComm
        (VAtm (Pair n1 n2) env))
```

```
predicate staticEvalEnv:
  static_value_map -> static_value_map -> env -> bool
where only
  ∀ staticEnv staticComm env .
    if
      ∀ n v . if env n = Some v then
        {abstract v} ⊆ staticEnv n,
        staticEvalValue staticEnv staticComm v)
    then
      staticEvalEnv staticEnv staticComm env)

predicate staticEvalStack of
  static_value_map -> static_value_map ->
  static_value set -> stack -> bool
where only
  ∀ staticEnv staticComm staticVals .
    staticEvalStack staticEnv staticComm staticVals [])
* ∀ staticVals staticEnv staticComm .
    if
      staticVals ⊆ staticEnv n,
      staticEval staticEnv staticComm t,
      staticEvalEnv staticEnv staticComm env,
      staticEvalStack staticEnv staticComm staticEnv (resultName t) stack
    then
      staticEvalStack staticEnv staticComm staticVals ((Ctn n t env) # stack))

predicate staticEvalPool of
  static_value_map -> static_value_map -> pool -> bool
where only
  ∀ staticEnv staticComm pool .
    if
      ∀ path t env stack .
        if
          pool path = Some (Stt t env stack)
        then
          staticEval staticEnv staticComm t,
          staticEvalEnv staticEnv staticComm env,
          staticEvalStack staticEnv staticComm staticEnv (resultName t) stack)
    then
      staticEvalPool staticEnv staticComm pool)
```

A variant of star that inducts toward the left of the transitive connection is helpful for relating dynamic traceability to static traceability, since it mirrors the direction that way paths grow, which influenced the choice of induction in the definition of static traceability.

```
predicate starLeft: ('a -> 'a -> bool) -> 'a -> 'a -> bool
```

```
  where only
    ∀ r z z .
      starLeft r z z
  * ∀ r x y z .
      if
        starLeft r x y, r y z
      then
        starLeft r x z

  lemma starImpliesStarLeft:
    ∀ r x y .
      if
        star r x z
      then
        starLeft r x z

  lemma starLeftTrans:
    ∀ r x y z .
      if
        starLeft r x y,
        starLeft r y z
      then
        starLeft r x z
```

The lemma for the completeness of static traceability follows from the generalized definitions of static acceptance by flows, the definition of static traceability, and the preservation of static acceptance by flows across multiples steps of evaluation.

```
  lemma staticFlowsAcceptPoolPreserved:
    ∀ t0 pool comm staticEnv staticComm graph .
      if
        star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm,
        staticEval staticEnv staticComm t0,
        staticFlowsAcceptPool staticEnv graph [[] -> (Stt t0 [->] [])]
      then
        staticFlowsAcceptPool staticEnv graph pool
```

The preservation of static acceptance by flows over pools is proved by the equivalence between star and its leftward variant, and induction on the leftward variant. The preservation of static evalatuation of pools over multioples steps is also relied upon.

```
  lemma staticEvalPoolPreserved:
    ∀ pool comm pool' comm' staticEnv staticComm .
      if
        star dynamicEval pool comm pool' comm'
        staticEvalPool staticEnv staticComm pool
      then
        staticEvalPool staticEnv staticComm pool'
```

The completeness of static inclusiveness is derived from various lemmas that preserve relations from pairs of dynamic paths to pairs of corresponding static paths. Some of these, among many others, are the preservation of the strict prefix relation from static to dynamic paths, and the preservation of static inclusiveness over extension of static paths.

```
lemma strictPrefixPreservedCorresp:
  ∀ staticPathath1 staticPath2 dynamicPath1 dynamicPath2 .
    if
      strictPrefix staticPathath1 staticPath2,
      pathsCorrespond dynamicPath1 staticPath1,
      pathsCorrespond dynamicPath2 staticPath2
    then
      strictPrefix dynamicPath1 dynamicPath2


lemma staticInclusivePreservedUnorderedExtension:
  ∀ path1 path2 l1 l2 .
    if
      staticInclusive path1 path2,
      not (prefix path1 path2),
      not (prefix path2 path1),
    then
      staticInclusive (path1 @ [l1]) (path2 @ [l2])
```

These various preservation lemmas are derived from the basic properites of lists and straight forward properities of path correspondence, such as commutitivity, as well as foundational principles like induction of corresponding paths.

The lemma for completeness of sending identifier classification `staticSendIdSound` is proved using the lemma for soundness of static evaluation for synchronization of a send event, and the lemma for soudness of static evaluation. Since only sending identifiers are relevant the completeness of static reachability is used to ensure that the static step is indeed a sending identifier.

```
lemma sendChanStaticEvalSound:
  ∀ t0 pool comm staticEnv staticComm path
    n n_e t' env stack n_sc n_m env_e path_c n_c .
    if
      star dynamicEval [[] -> (Stt t0 [->] [])], {} pool comm,
      staticEval staticEnv staticComm t0,
      pool path = Some (Stt (Bind n (Sync n_e) t') env stack),
      env n_e = Some (VAtm (SendEvt n_sc n_m) env_e),
      env_e n_sc = Some (VChn (Chan path_c n_c))
    then
      SChn n_c ∈ staticEnv n_sc

lemma staticEvalSound:
  ∀ t0 pool comm staticEnv staticComm path t env stack n v .
    if
```

```
    staticEval staticEnv staticComm t0,
    star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm,
    pool path = Some (Stt t env stack),
    env n = Some v
  then
    abstract v ∈ staticEnv n

lemma staticReachableSound:
  ∀ t0 pool comm staticEnv staticComm path t env stack .
    if
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm,
      pool path = Some (Stt t env stack)
    then
      staticReachable t0 t
```

Both the soundness of static evaluation on the syncrhonization of a send event, and the soundness of static evaluation follow from the preservation of static evaluation over multiple steps of dynamic evaluation.

The lemma for completeness of static reachability relies on the a reformulation of static reachability that defined by proofs that induct on a larger term containing the reachable term. This definition is useful for forward derivations of reachability relations, however it doesn't offer much guidance for deciding reachability. In constrast, the definition of the original static reachability relation is syntax-directed in order to portray a clear connection to a computable algorithm that can determine the reachable term from an initial program. However, to show that an term is reachable from the initial program, it is necessary to show that each intermediate term is reachable from the initial term. Thus, the induction needs to enable unraveling the goals from the end to the beginning of the program, maintaining the initial program state in context for each subgoal.

```
predicate staticReachableForward: term -> term -> bool
where only
  ∀ t0 .
    staticReachableForward t0 t0
* ∀ t0 n t_c t' .
    if
      staticReachableForward t0 (Bind n (Spwn t_c) t')
    then
      staticReachableForward t0 t_c
* ∀ t0 n n_s n_l t_l n_r t_r t' .
    if
      staticReachableForward t0 (Bind n (Case n_s n_l t_l n_r t_r) t')
    then
      staticReachableForward t0 t_l
* ∀ t0 n n_s n_l t_l n_r t_r t' .
    if
      staticReachableForward t0 (Bind n (Case n_s n_l t_l n_r t_r) t')
    then
```

```
      staticReachableForward t0 t_r
* ∀ t0 n n_f n_p t_b t' .
    if
      staticReachableForward t0 (Bind n (Atom (Fun n_f n_p t_b)) t')
    then
      staticReachableForward t0 t_b
* ∀ t0 n n_f n_p t_b t' .
    if
      staticReachableForward t0 (Bind n c t')
    then
      staticReachableForward t0 t'

predicate staticReachableAtom: term -> atom -> bool
where only
  ∀ t0 n_c n_m .
    staticReachableAtom t0 (SendEvt n_c n_m)
* ∀ t0 n_c .
    staticReachableAtom t0 (RecvEvt n_c)
* ∀ t0 n1 n2 .
    staticReachableAtom t0 (Pair n1 n2)
* ∀ t0 n_l .
    staticReachableAtom t0 (Lft n_l)
* ∀ t0 n_r
    staticReachableAtom t0 (Rht n_r)
* ∀ t0 t_b n_f n_p t_b .
    if
      staticReachableForward t0 t_b
    then
      staticReachableAtom t0 (Fun n_f n_p t_b)

predicate staticReachable_val: term -> dynamic_value -> bool
where only
  ∀ t0 .
    staticReachableValue t0 VUnt
* ∀ t0 c .
    staticReachableValue t0 (VChn c)
* ∀ t0 t env .
    if
      staticReachableAtom t0 t,
      staticReachableEnv t0 env
    then
      staticReachableValue t0 (VAtm t env)

predicate staticReachable_env: term -> env -> bool
where only
  ∀ t0 env
```

```
    if
      ∀ n v . if env n = Some v then
          staticReachableValue t0 v
    then
      staticReachableEnv t0 env

predicate staticReachableStack: term -> stack -> bool
where only
  ∀ t0 .
    staticReachableStack t0 [])
* ∀ t0 t_k env_k stack' .
    if
      staticReachableForward t0 t_k,
      staticReachableEnv t0 env_k,
      staticReachableStack t0 stack'
    then
      staticReachableStack t0 ((Ctn n_k t_k env_k) # stack')

predicate staticReachablePool: term -> pool -> bool
where only
  ∀ t0 pool .
    then
      ∀ path t env stack . if pool path = Some (Stt t env stack) then
        staticReachableForward t0 t,
        staticReachableEnv t0 env,
        staticReachableStack t0 stack
        then
      staticReachablePool t0 pool
```

The completeness of static reachability follows the definitions a generalized form of completeness over pools.

```
lemma staticReachablePoolSound:
  ∀ t0 pool .
    if
      star dynamicEval [[] -> (Stt t0 [->] [])], {} pool comm
    then
      staticReachablePool t0 pool
```

The completeness over pools follows from the lemma that the forward static reachability implies the rightward (and syntax-directed) static reachability, and the equivalence between star and the forward star. It relies on induction of the forward star and constructs the static reachability proposition using the forward definition.

```
lemma staticReachableForwardImpliesstaticReachable:
  ∀ t0 t.
    if
      staticReachableForward t0 t
```

```
    then
      staticReachable t0 t

lemma staticReachableTrans:
  ∀ t1 t2 t3 .
    if
      staticReachable t1 t2,
      staticReachable t2 t3
    then
      staticReachable t1 t3
```

The lemma that the forward variant of static reachability implies the syntax-directed static reachability follows from induction on the forward static reachability and the transitivity of static reachability, which follows from induction on static reachability.

4 Higher Precision Communication

In many programs, like in the server example, channels are created within function functions. The function functions may be applied multiple times, creating multiple distinct channels with each application. It may be that each channel is used just once and then discarded. However, the static analysis just described would identify all the distinct channels by the same name, since each distinct channel is created by the same piece of syntax. Thus, it would classify those channels as being used more than once.

It is possible to be more precise by trimming the program under analysis down to just the part where the static channel is live. The static channel cannot be live between the last use of a dynamic channel and the creation of a new dynamic channel with the same name. Thus, each truncated program would have just one dynamic channel corresponding to the static channel under analysis.

A trimmed graph structure of graph is used static analysis for this higher precision analysis, which can better differentiate between distinct channels. A trimmed graph is specialized for a particular dynamic channel. From the creation step, it must contain transitive flows to all the program steps where the channel is live. It should also be as small as possible, for higher precision.

In the whole graph used in the previous analysis, a spawning flow connects a child thread to the rest of the program. For a trimmed graph, it may be clear the channel of interest is not created until after the spawn step, so there is not need to include the spawning flow. However, later on in the program it may become apparent that the channel of interest is sent via another channel to that spawned thread. Since there is no spawning flow already connecting that thread to the trimmed graph, a flow with a sending mode is used between the sending identifier and the receiving identifier of synchronization. Modes for typical control flow of sequencing, calling, returning, and spawning are also included flows.

```
datatype mode =
  MNxt
```

```
| MSpwn
| ESend name
| MCll
| MRtn

type flow = termId * mode * termId

type static_step = termId * mode

type staticPath = static_step list
```

We use a slightly modified version of the server implementation to demonstrate some key concepts of the higher precision analysis. An additional loop function `lp` has been added to the server implementation. The loop basically just wastes steps, but it is used to demontrate how liveness analysis treats functions that don't contain any channel of interest.

```
bind u1 = unt
bind r1 = rht u1
bind l1 = lft r1
bind l2 = lft l1

bind lp = fun lp' x1 =>
(
  bind z1 = case x1 of
    lft y1 => bind z2 = lp' y1 z2
  | rht y2 => bind u2 = unt rslt u2
  bind u3 = unt
  rslt u3
)

bind mksr = fun _ x2 =>
(
  bind k1 = mkChn
  bind z4 = lp l2
  bind srv = fun srv' x3 =>
  (
    bind e1 = recvEvt k1
    bind p1 = sync e1
    bind v1 = fst p1
    bind k2 = snd p1
    bind e2 = sendEvt k2 x3
    bind z5 = sync e2
    bind z6 = srv' v1
    bind u4 = unt
    rslt u4
    bind z7 = spawn
    (
```

```
      bind z8 = srv r1
      bind u5 = unt
      rslt u5
      rslt k1
    )
  )
)

bind rqst = fun _ x4 =>
(
  bind k3 = fst x4
  bind v2 = snd x4
  bind k4 = mkChn
  bind p2 = pair v2 k4
  bind e3 = sendEvt k3 p2
  bind z9 = sync e3
  bind e4 = recvEvt k4
  bind v3 = sync e4
  rslt v3
)

bind srvr = mksr u1
bind z10 = spawn
(
  bind p3 = pair srvr l1
  bind z11 = rqst p3
  rslt z11
)
bind p4 = pair srvr l2
bind z12 = rqst p4
rslt z12
```

The static acceptance by flows for higher precision is similar to that of the lower precision analysis. However, it must additionally consider flows with the sending mode.

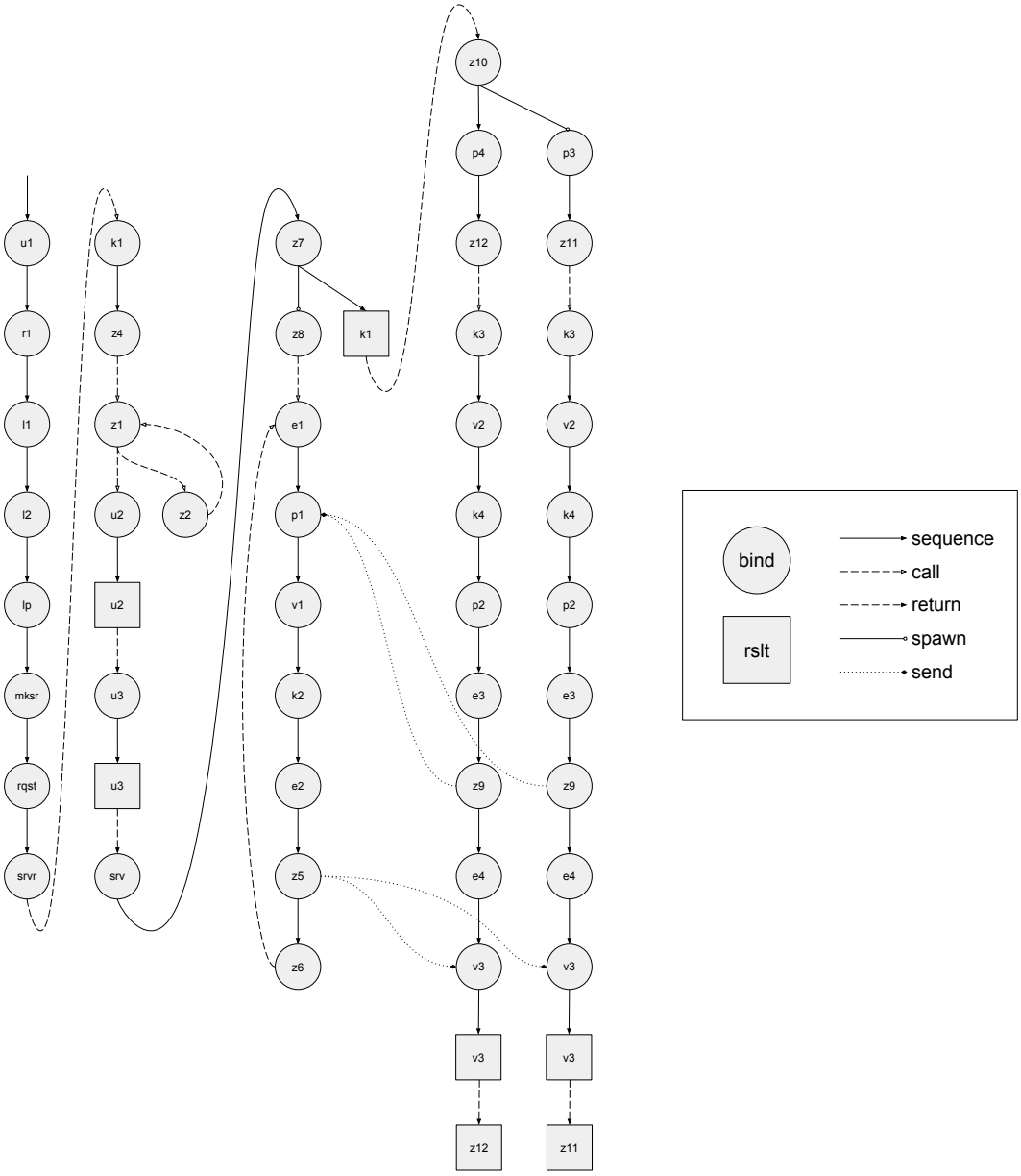
```
predicate staticFlowsAccept:
  static_value_map -> graph -> term -> bool
where only
  ∀ staticEnv graph n .
    staticFlowsAccept staticEnv graph (Rslt n)
* ∀ n t' graph staticEnv .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n Unt t')
* ∀ n t' graph staticEnv  .
```

```
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n MkChn t')
* ∀ n t' graph staticEnv  n_c n_m .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept
        staticEnv graph
        (Bind n (Atom (SendEvt n_c n_m)) t')
* ∀ n t' graph staticEnv n_c .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (RecvEvt n_c)) t')
* ∀ n t' graph staticEnv n1 n2 .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (Pair n1 n2)) t')
* ∀ n t' graph staticEnv n_s .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
     staticFlowsAccept staticEnv graph (Bind n (Atom (Lft n_s)) t')
* ∀ n t' graph staticEnv n_s .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (Rht n_s)) t')
* ∀ n t' graph staticEnv t_b n_f n_p .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t',
      staticFlowsAccept staticEnv graph t_b
    then
      staticFlowsAccept staticEnv graph (Bind n (Atom (Fun n_f n_p t_b)) t')
* ∀ n t' t_c graph staticEnv.
    if
```

```
      {(NBnd n, MNxt, termId t'),
        (NBnd n, MSpwn, termId t_c)} ⊆ graph,
      staticFlowsAccept staticEnv graph t_c,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Spwn t_c) t')
* ∀ n t' graph staticEnv n_se .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
    * ∀ n_sc n_m n_c n_y.
        if
          (SAtm (SendEvt n_sc n_m)) ∈ staticEnv nSE,
          (SChn n_c) ∈ staticEnv n_sc,
          staticRecvId staticEnv t' n_c (NBnd n_y)
        then
          (NBnd n, ESend n_se, NBnd n_y) ∈ graph),
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Sync n_se) t')
* ∀ n t' graph staticEnv n_p .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t',
    then
      staticFlowsAccept staticEnv graph (Bind n (Fst n_p) t')
* ∀ n t' graph staticEnv n_p .
    if
      (NBnd n , MNxt, termId t') ∈ graph,
      staticFlowsAccept staticEnv graph t',
    then
      staticFlowsAccept staticEnv graph (Bind n (Snd n_p) t')
* ∀ n t_l t_r t' graph staticEnv n_s .
    if
      {
        (NBnd n, MCll, termId t_l),
        (NBnd n, MCll, termId t_r),
        (NRslt (resultName t_l), MRtn, termId t'),
        (NRslt (resultName t_r), MRtn, termId t')
      } ⊆ graph,
      staticFlowsAccept staticEnv graph t_l,
      staticFlowsAccept staticEnv graph t_r,
      staticFlowsAccept staticEnv graph t'
    then
      staticFlowsAccept staticEnv graph (Bind n (Case n_s n_l t_l n_r t_r) t')
* ∀  staticEnv n_f n t' n n_a .
    if
```

```
  ∀ n_f' n_p t_b . if (SAtm (Fun n_f' n_p t_b)) ∈ staticEnv n_f then
    {
      (NBnd n, MCll, termId t_b),
      (NRslt (resultName t_b), MRtn, termId t')
    } ⊆ graph),
  staticFlowsAccept staticEnv graph t'
then
  staticFlowsAccept staticEnv graph (Bind n (App n_f n_a) t')
```

The server implementation represented as a graph illustrates how static acceptance by flows can interpret a program as a flow graph.



For the liveness of channel analysis, it is necessary to track any name built on a channel. A name is build on a channel if the name binds to a static value containing a channel of interest, or a static value that contains names built on the channel. In the case where the tracked name possibly binds to a function, for the name to be considered built on a channel, the channel simply needs to be live in the body of the function. This condition is represented formally as the requirement

that there is a name, such that it is a free variable in the function, and it's built on the channel.

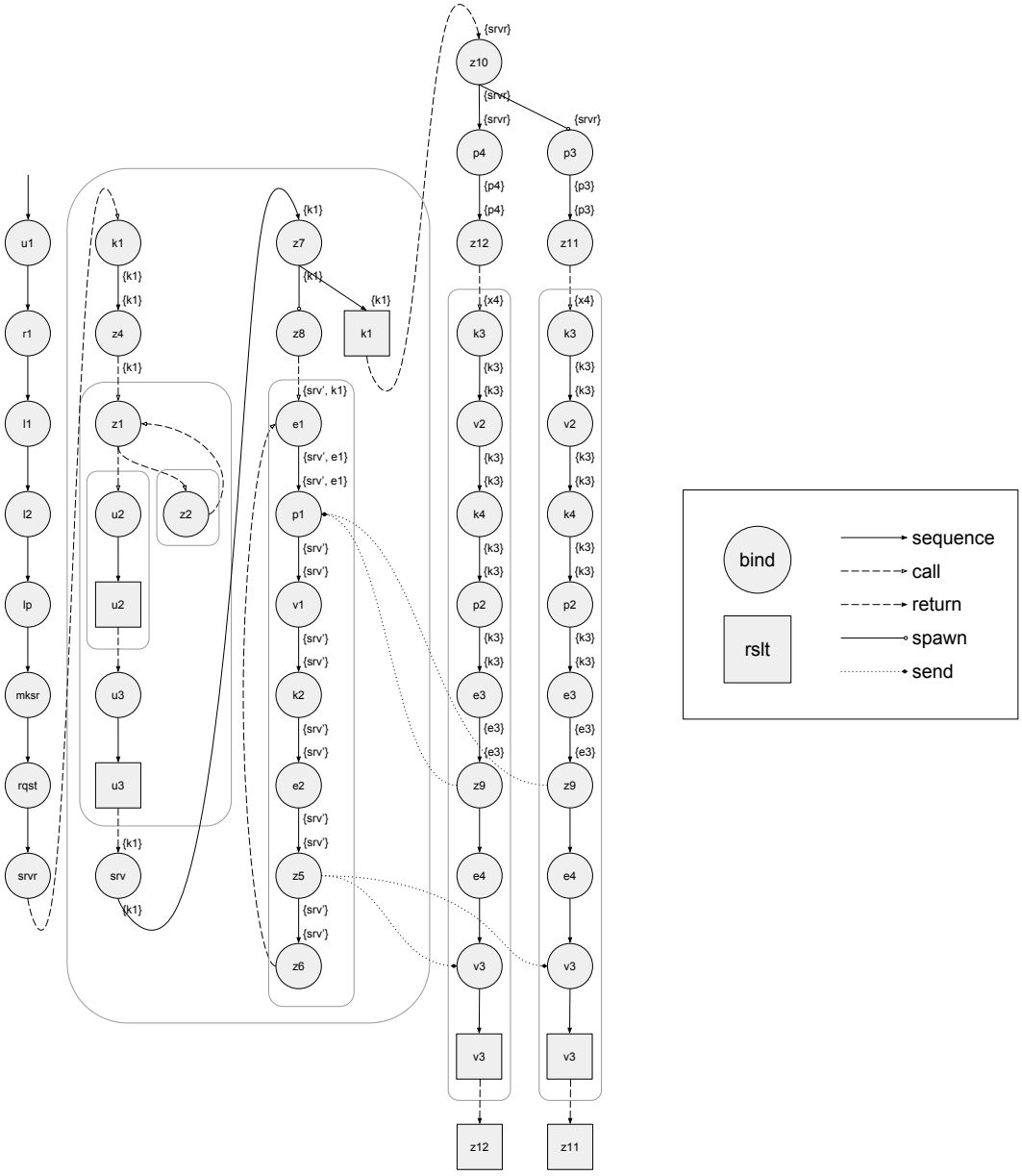
```
fun freeVarsAtom of atom -> name set:
where only
  ∀ n_c n_m .
    freeVarsAtom (SendEvt n_c n_m) = {n_c, n_m}
* ∀ n_c .
    freeVarsAtom (RecvEvt n_c) = {n_c}
* ∀ n1 n2 .
    freeVarsAtom (Pair n1 n2) = {n1, n2}
* ∀ n .
    freeVarsAtom (Lft n) = {n}
* ∀ n .
    freeVarsAtom (Rht n) = {n}
* ∀ n_f n_p t_b .
    freeVarsAtom (Fun n_f n_p t_b) = freeVarsTerm t_b \ {n_f, n_p}

and freeVarsComplex of complex -> name set:
  freeVarsComplex Unt = {}
* freeVarsComplex MkChn = {}
* ∀ atom .
    freeVarsComplex (Atom atom) = freeVarsAtom atom
* ∀ t .
    freeVarsComplex (Spwn t) = freeVarsTerm t
* ∀ n .
    freeVarsComplex (Sync n) = {n}
* ∀ n .
    freeVarsComplex (Fst n) = {n}
* ∀ n .
    freeVarsComplex (Snd n) = {n},

* ∀ n_s n_l t_l n_r t_r .
    freeVarsComplex (Case n_s n_l t_l n_r t_r) =
      {n_s} ∪ freeVarsTerm t_l ∪ freeVarsTerm t_r \ {n_l, n_r}
* ∀ n_f n_a .
    freeVarsComplex (App n_f n_a) = {n_f, n_a},

and freeVarsTerm of term -> name set:
where only
  ∀ n c t .
    freeVarsTerm (Bind n c t) = freeVarsComplex c ∪ freeVarsTerm t \ {n}
* ∀ n .
    freeVarsTerm (Rslt n) = {n}

predicate staticBuiltOnChan: static_value_map -> name -> name -> bool
where only
```

```
  ∀ n_c staticEnv n .
    if
      SChn n_c ∈ staticEnv n
    then
      staticBuiltOnChan staticEnv n_c n
* ∀ n_sc n_m staticEnv n n_c .
    if
      (SAtm (SendEvt n_sc n_m)) ∈ staticEnv n,
      (staticBuiltOnChan staticEnv n_c n_sc or
        staticBuiltOnChan staticEnv n_c n_m)
    then
      staticBuiltOnChan staticEnv n_c n
* ∀ n_rc staticEnv n n_c .
    if
      (SAtm (RecvEvt n_rc)) ∈ staticEnv n,
      staticBuiltOnChan staticEnv n_c n_rc
    then
      staticBuiltOnChan staticEnv n_c n
* ∀ n1 n2 staticEnv n n_c .
    if
      (SAtm (Pair n1 n2)) ∈ staticEnv n,
      (staticBuiltOnChan staticEnv n_c n1 or
        staticBuiltOnChan staticEnv n_c n2)
    then
      staticBuiltOnChan staticEnv n_c n
* ∀ n_a staticEnv n n_c .
    if
      (SAtm (Lft n_a)) ∈ staticEnv n,
      staticBuiltOnChan staticEnv n_c n_a
    then
      staticBuiltOnChan staticEnv n_c n
* ∀ n_a staticEnv n n_c .
    if
      (SAtm (Rht n_a)) ∈ staticEnv n,
      staticBuiltOnChan staticEnv n_c n_a
    then
      staticBuiltOnChan staticEnv n_c n
* ∀ n_f n_p t_b n_fv .
    if
      (SAtm (Fun n_f n_p t_b)) ∈ staticEnv n,
      n_fv ∈ freeVarsAtom (Fun n_f n_p t_b),
      staticBuiltOnChan staticEnv n_fv n
    then
      staticBuiltOnChan staticEnv n_c n
```

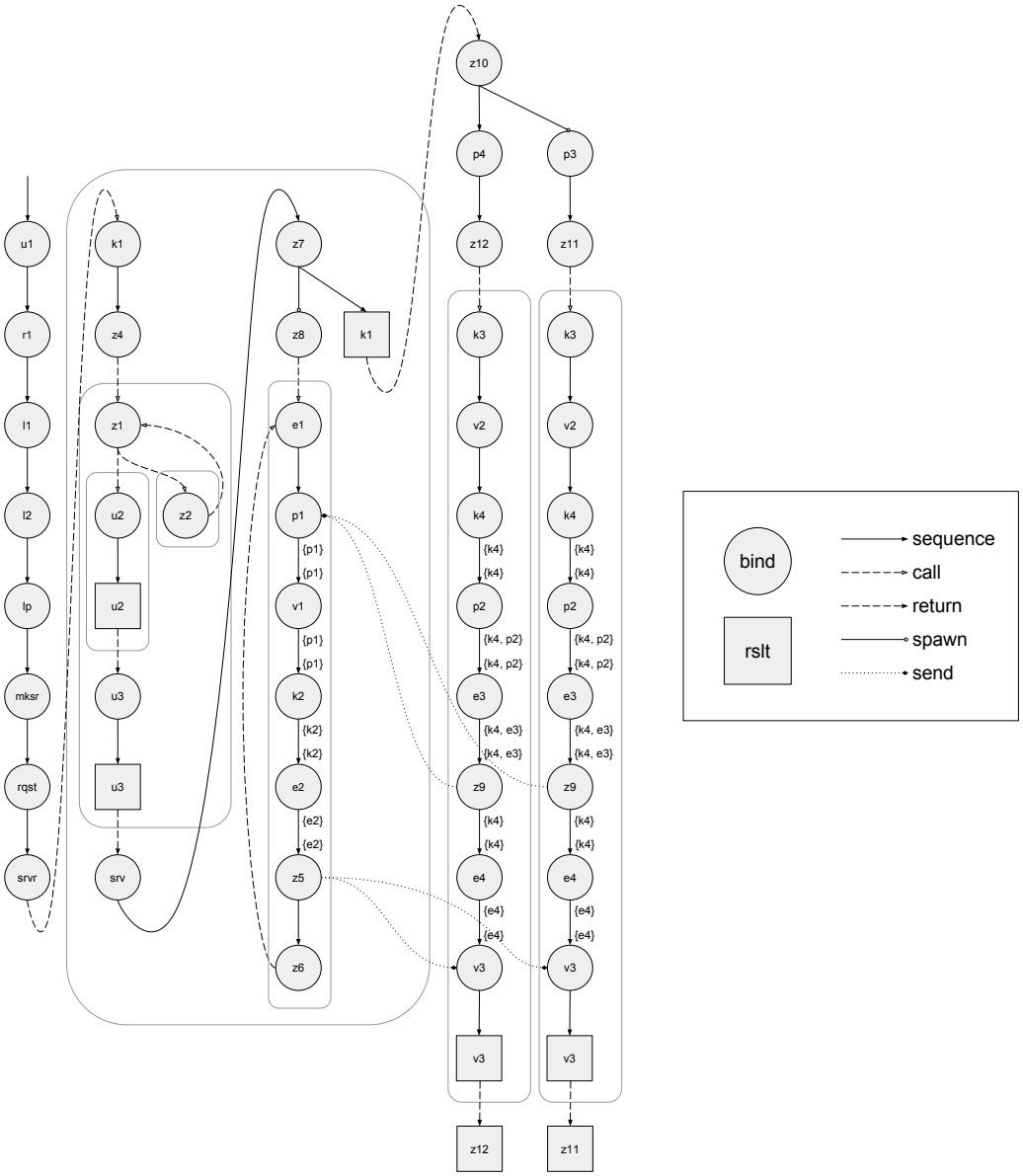
The static liveness of a channel describes entry functions and exit functions. The entry func-

tion maps a term identifier to a set of names built on the given channel, if those names are live at the entry of that term identifier. The exit function maps a term identifier to a set of names built on the given channel, if those names are live at the exit of that term identifier.

The following diagram illustrates the entry and exit sets of each term id for channel `k1` in the server example. Each entry set appears right above its related term identifier, and each exit setappears right below its related term identifier.



The following diagram illustrates the entry and exit sets of each term id for channel k4 in the server example. Each entry set appears right above its related term identifier, and each exit setappears right below its related term identifier.




```
predicate staticLiveChan:
  static_value_map -> term_id_map ->
  term_id_map -> name -> term -> bool
```

```
where only
  ∀ staticEnv entr n_c n_y exit .
    if
      if (staticBuiltOnChan staticEnv n_c n_y) then
        {n_y} ⊆ entr (NRslt n_y)
    then
      staticLiveChan staticEnv entr exit n_c (Rslt n_y)
* ∀ exit n entr t' staticEnv n_c .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c (Bind n Unt t')
* ∀ exit n entr t' staticEnv n_c .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c (Bind n MkChn t')
* ∀ exit n entr staticEnv n_c n_sc n_m t' n_c .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_sc then
        {n_sc} ⊆ entr (NBnd n)),
      (if staticBuiltOnChan staticEnv n_c n_m then
        {n_m} ⊆ entr (NBnd n)),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c
        (Bind n (Atom (SendEvt n_sc n_m)) t')
* ∀ exit n entr staticEnv n_c n_r n_rc.
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_r then
        {n_r} ⊆ entr (NBnd n)),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c
        (Bind n (Atom (RecvEvt n_rc)) t')
* ∀ exit n entr staticEnv t_c n1 n2 t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
```

```
      (if staticBuiltOnChan staticEnv n_c n1 then
        {n1} ⊆ entr (NBnd n)),
      (if staticBuiltOnChan staticEnv n_c n2 then
        {n2} ⊆ entr (NBnd n)),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c (Bind n (Atom (Pair n1 n2)) t'))
* ∀ exit n entr staticEnv n_c n_a t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_a then
        {n_a} ⊆ entr (NBnd n)),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c (Bind n (Atom (Lft n_a)) t'))
* ∀ exit n entr staticEnv n_c n_a t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_a then
        {n_a} ⊆ entr (NBnd n))
      entr (termId e) ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t
    then
      staticLiveChan staticEnv entr exit n_c (Bind n (Atom (Rht n_a)) e)
* ∀  exit n entr t_b n_p n staticEnv n_c t' n_f .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (entr (termId t_b) \ {n_p}) ⊆ entr (NBnd n),
      staticLiveChan staticEnv entr exit n_c t_b,
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c
        (Bind n (Atom (Fun n_f n_p t_b)) t')
* ∀ exit n entr t' t_c n_c staticEnv .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      entr (termId t') ⊆ exit (NBnd n),
      entr (termId t_c) ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t_c,
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c
        (Bind n (Spwn t_c) t')
```

```
* ∀ exit n entr staticEnv n_c n_e t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_e then
        {n_e} ⊆ entr (NBnd n)),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t',
    then
      staticLiveChan staticEnv entr exit n_c
        (Bind n (Sync n_e) t')
* ∀ exit n entr staticEnv n_c n_a t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_a then
        {n_a} ⊆ entr (NBnd n)),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c (Bind n (Fst n_a) t')
* ∀ exit n entr staticEnv n_c n_a t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_a then
        {n_a} ⊆ entr (NBnd n)),
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c
        (Bind n (Snd n_a) t')
* ∀ exit n entr t_l n_l t_r n_r staticEnv n_c n_s t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (entr (termId t_l) \ {n_l}) ⊆ entr (NBnd n),
      (entr (termId t_r) \ {n_r}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_s then
        {n_s} ⊆ entr (NBnd n)),
      staticLiveChan staticEnv entr exit n_c t_l,
      staticLiveChan staticEnv entr exit n_c t_r,
      entr (termId t') ⊆ exit (NBnd n),
      staticLiveChan staticEnv entr exit n_c t'
    then
      staticLiveChan staticEnv entr exit n_c (Bind n (Case n_s n_l t_l n_r t_r) t')
* ∀ exit n entr staticEnv n_c n_a n_f t' .
    if
      (exit (NBnd n) \ {n}) ⊆ entr (NBnd n),
      (if staticBuiltOnChan staticEnv n_c n_a then
```

```
      {n_a} ⊆ entr (NBnd n)),
    (if staticBuiltOnChan staticEnv n_c n_f then
      {n_f} ⊆ entr (NBnd n)),
    entr (termId t') ⊆ exit (NBnd n),
    staticLiveChan staticEnv entr exit n_c t'
  then
    staticLiveChan staticEnv entr exit n_c (Bind n (App n_f n_a) t')
```

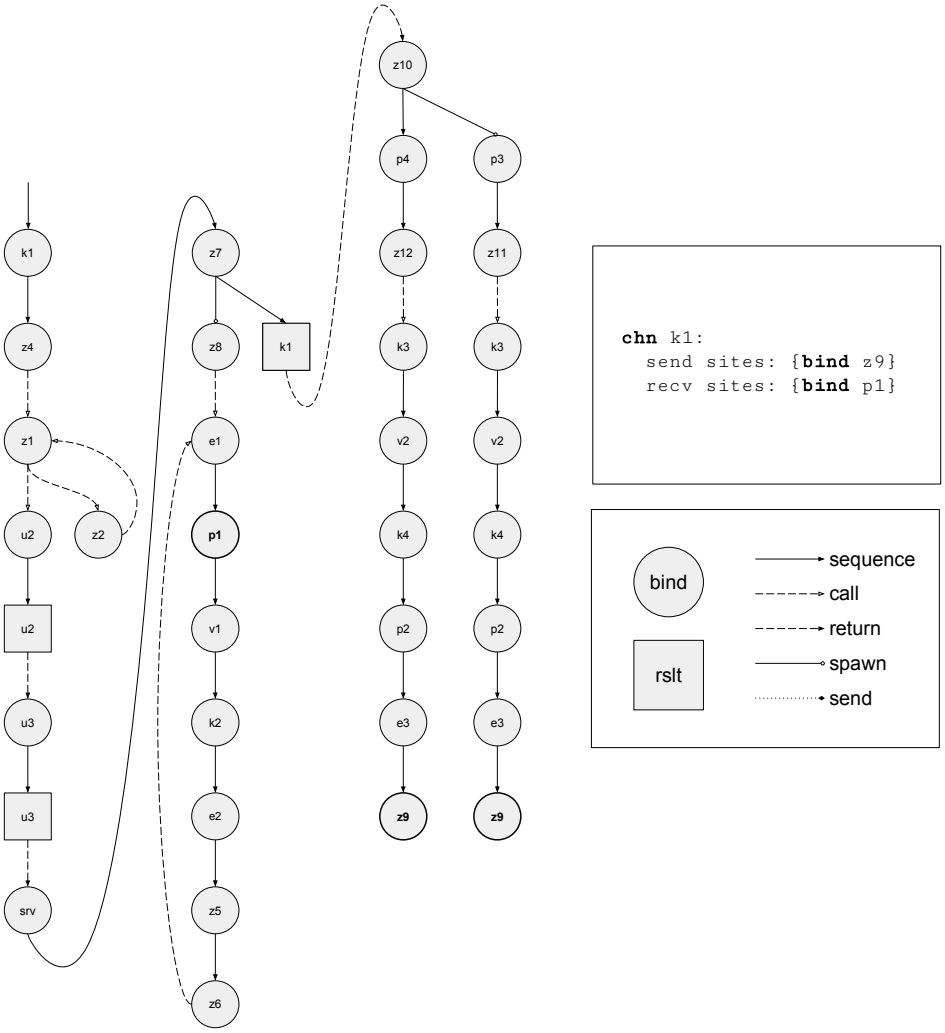
The static liveness of a flow checks if a flow exists in a whole graph, and if it meets certain criteria with respect to the entry and exit liveness functions. No channel is considered to be live in the body of the loop at **bind** `lp`. However, a channel may be live before the loop is called and after the loop returns. In such a case, the live flow is retained from the caller to within the loop function's body, even though the steps in the loop may not be live according to the entry function in the static channel liveness.

```
predicate staticLiveFlow:
  graph -> term_id_map -> term_id_map -> flow -> bool
where only
  ∀ l l' graph exit entr .
    if
      (l, MNxt, l') ∈ graph,
      not (exit l = {}),
      not (entr l' = {})
    then
      staticLiveFlow graph entr exit (l, MNxt, l')
* ∀ l l' graph exit entr .
    if
      (l, MSpwn, l') ∈ graph,
      not (exit l = {}),
      not (entr l' = {})
    then
      staticLiveFlow graph entr exit (l, MSpwn, l')
* ∀ l l' graph exit entr .
    if
      (l, MCll, l') ∈ graph,
      (not (exit l = {})) ∨ (not (entr l' = {}))
    then
      staticLiveFlow graph entr exit (l, MCll, l')
* ∀ l l' graph entr exit .
    if
      (l, MRtn, l') ∈ graph,
      not (entr l' = {})
    then
      staticLiveFlow graph entr exit (l, MRtn, l')
* ∀ n_s n_e n_r graph entr exit .
    if
```

```
  ((NBnd n_s), ESend n_e, (NBnd n_r)) ∈ graph,
  {n_e} ⊆ (entr (NBnd n_s))
then
  staticLiveFlow graph entr exit
  ((NBnd n_s), ESend n_e, (NBnd n_r))
```

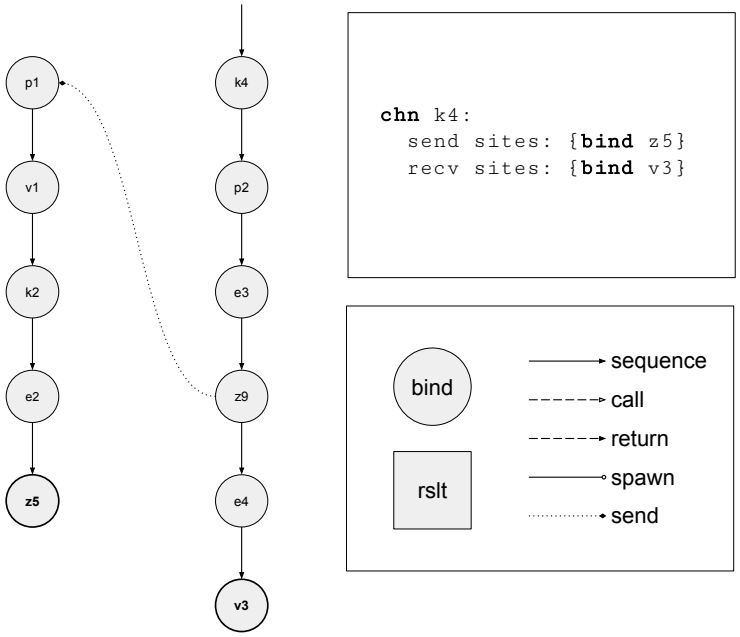
The static traceability for the higher precision analysis states that an entire static path can be traced through some graph and is live with respect to some entry and exit functions.

The following diagram illustrates the graph of the server example, containing only live flows for channel `k1`.



The following diagram illustrates the graph of the server example, containing only live flows

for channel k4.




```
predicate staticTraceable:
  static_value_map -> graph -> term_id_map -> term_id_map ->
  termId -> (termId -> bool) -> static_path -> bool
where only
  ∀ isEnd start staticEnv graph entr exit .
    if
      isEnd start
    then
      staticTraceable graph entr exit start isEnd []
 * ∀ graph entr exit start middle path isEnd mode.
    if
      staticTraceable graph entr exit start (λ l . l = middle) path,
      (isEnd end),
      staticLiveFlow graph entr exit (middle, mode, end)
    then
      staticTraceable graph entr exit start isEnd (path @ [(middle, mode)]
```

As with the lower precision analysis, the higher precision analysis relies on recognizing whether or not two paths can actually occur within in a single run of a program. The static inclusiveness states which paths might occur within the same run of the program. In contrast to the analogous definition for the lower precision alysis, the higher precision definition needs to consider paths containing the sending mode. As mentioned earlier, the path from the synchro-

nization on sending to the synchronization on receiving is necessary to ensure that all uses of a channel are reachable from the channel's creation identifier. The singularness means that only one of the two given paths can occur in a run of program. The noncompetitveness means that the two given paths do not compete in any run of a program.

```
predicate staticInclusive: static_path -> static_path -> bool
where only
  ∀ path1 path2 .
    if
      prefix path1 path2 ∨ path2 path1
    then
      staticInclusive path1 path2
* ∀ path n path1 path2 .
    staticInclusive
      (path @ (NBnd n, MSpwn) # path1)
      (path @ (NBnd n, MNxt) # path2)
* ∀ path n path1 path2 .
    staticInclusive
      (path @ (NBnd n, MNxt) # path1)
      (path @ (NBnd n, MSpwn) # path2)
* ∀ path n path1 path2 .
    staticInclusive
      (path @ (NBnd n, ESend xE) # path1)
      (path @ (NBnd n, MNxt) # path2)
* ∀ path n path1 path2 .
    staticInclusive
      (path @ (NBnd n, MNxt) # path1)
      (path @ (NBnd n, ESend xE) # path2)

predicate singular: static_path -> static_path -> bool
where only
  ∀ path .
    singular path path
* ∀ path1 path2 .
    if
      not (staticInclusive path1 path2)
    then
      singular path1 path2

predicate noncompetitive: static_path -> static_path -> bool
where only
  ∀ path1 path2 .
    if
      ordered path1 path2
    then
      noncompetitive path1 path2
```

```
* ∀ path1 path2 .
    if
      not (staticInclusive path1 path2)
    then
      noncompetitive path1 path2
```

The communication classifications are described using the liveness properties, but are otherwise similar to the lower precision classifications.

```
predicate staticOneShot: static_value_map -> term -> name -> bool
where only
  ∀ graph entr exit n_c staticEnv t .
    if
      forEveryTwo
        (staticTraceable graph entr exit
          (NBnd n_c) (staticSendId staticEnv t n_c))
        singular,
      staticLiveChan staticEnv entr exit n_c t,
      staticFlowsAccept staticEnv graph t
    then
      staticOneShot V t n_c)

predicate staticOneToOne: static_value_map -> term -> name -> bool
where only
  ∀ graph entr exit n_c staticEnv t .
    if
      forEveryTwo
        (staticTraceable graph
          entr exit (NBnd n_c) (staticSendId staticEnv t n_c))
        noncompetitive,
      forEveryTwo
        (staticTraceable graph
          entr exit (NBnd n_c) (staticRecvId staticEnv t n_c))
        noncompetitive,
        staticLiveChan staticEnv entr exit n_c t,
        staticFlowsAccept staticEnv graph t
    then
      staticOneToOne staticEnv t n_c)

predicate staticOneToMany: static_value_map -> term -> name -> bool
where only
  ∀ graph entr exit n_c staticEnv t .
    if
      forEveryTwo
        (staticTraceable graph
          entr exit (NBnd n_c) (staticSendId staticEnv t n_c))
```

```
        noncompetitive,
      staticLiveChan staticEnv entr exit n_c t,
      staticFlowsAccept staticEnv graph t
    then
      staticOneToMany staticEnv t n_c)

predicate staticManyToOne: static_value_map -> term -> name -> bool
where only
  ∀ graph entr exit n_c staticEnv t .
    if
      forEveryTwo
        (staticTraceable graph
          entr exit (NBnd n_c) (staticRecvId staticEnv t n_c))
        noncompetitive,
      staticLiveChan staticEnv entr exit n_c t,
      staticFlowsAccept staticEnv graph t
    then
      staticManyToOne staticEnv t n_c)
```

4.1 Higher Precision Soundness Proof Strategy

To prove soundness of the communication classification, it should be possible to use previous techniques of generalizing propositions over pools and other semantic components, along with finding equivalent representations of propositions that vary in the inductive subcomponent. One thing that will make carrying out the formal proof particularly tricky is that dynamic paths in the dynamic semantics need to correspond to static paths from the trimmed graphs, which might also contain sending flows, instead of the dynamic paths spawning flows. The correspondence between these dynamic paths and static paths is not bijective, as it is for the lower precision analysis. However, finding a satisfactory correspondence for each dynamic and static path is critical for proving soundness.

Essentially, it will be necessary to show that static properties that hold for some static path are preserved for corresponding dyamic paths. However, in the higher precision analysis these paths correspond modulo the channel of interest. Let's outline the derivation of soundness of one-shot classification.

```
theorem staticOneShotSound:
∀ t0 pool comm staticEnv staticComm n_c path_c .
  if
    star dynamicEval [[] -> (Stt t0 [->] []] {} pool comm,
    staticEval staticEnv staticComm t0,
    staticOneShot staticEnv t0 n_c
  then
    oneShot pool (Chan path_c n_c)
```

The theorem for soundness of one-shot classification depends on correlating dynamic paths with static paths.

```
predicate pathsCorrespond: dynamic_path -> static_path -> bool
where only
  pathsCorrespond [] []
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DNxt n])
        (staticPath @ [(NBnd n, MNxt)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DSpwn n])
        (staticPath @ [(NBnd n, MSpwn)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DCll n])
        (staticPath @ [(NBnd n, MCll)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DRtn n])
      (staticPath @ [(NRslt n, MRtn)])

predicate pathsCorrespondModChan:
  pool -> communication -> chan -> dynamic_path -> static_path -> bool
where only
  ∀ pool path_c n_c path_sfx stt staticPath comm .
    if
      pool (path_c @ (DNxt n_c) # path_sfx) = Some stt,
      pathsCorrespond ((DNxt n_c) # path_sfx) staticPath
    then
      pathsCorrespondModChan
        (pool, comm) (Chan path_c n_c)
        (path_c @ (DNxt n_c) # path_sfx) staticPath
```

```
* ∀ pool path_r n_r path_sfx stt  path_s n_s n_se t_sy env_sy stack_sy
    n_re t_ry env_ry stack_ry c_c comm c staticPath_re staticPath_sfx .
    if
      pool (path_r @ (DNxt n_r) # path_sfx) = Some stt,
      pool path_s = Some (Stt (Bind n_s (Sync n_se) t_sy) env_sy stack_sy),
      pool path_r = Some (Stt (Bind n_r (Sync n_re) t_ry) env_ry stack_ry),
      {(path_s, c_c, path_r)} ⊆ comm,
      dynamicBuiltOnChanVar env_ry c n_r,
      pathsCorrespondModChan pool comm c path_s staticPath_pfx,
      pathsCorrespond path_sfx staticPath_sfx
    then
      pathsCorrespondModChan pool comm c
        (path_r @ (DNxt n_r) # path_sfx)
        (staticPath_pfx @ (NBnd n_s, ESend n_se)
          # (NBnd n_r, MNxt) # staticPath_sfx)
```

Additionally the soundness theorem follows from the completeness of static traceability, the completeness of static inclusiveness, and the completeness of a sending identifier classification. The reasoning about the sending identifier is identical to that of the lower precision analysis, but the reasoning for the former two is significantly more complicated and not yet completed. The complication arises from the correlation between dynamic paths and static paths. The proofs depend on finding a static path that depends on a given dynamic path. in the lower precision analysis the correlation was straightforward. There was only one possible static path to choose for it to correlate with the given dynamic path. in the higher precision analysis, the relationship between the two kinds of paths is not so simple, and finding a description of the static path that correlates with the dynamic path is much more challenging.

```
predicate pathsCorrespond: dynamic_path -> static_path -> bool
where only
  pathsCorrespond [] []
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DNxt n])
        (staticPath @ [(NBnd n, MNxt)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DSpwn n])
        (staticPath @ [(NBnd n, MSpwn)])
* ∀ path staticPath n .
    if
```

```
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DCll n])
      (staticPath @ [(NBnd n, MCll)])
* ∀ path staticPath n .
    if
      pathsCorrespond path staticPath
    then
      pathsCorrespond
        (path @ [DRtn n])
      (staticPath @ [(NRslt n, MRtn)])

predicate pathsCorrespondModChan:
  pool -> communication -> chan -> dynamic_path -> static_path -> bool
where only
  ∀ pool path_c n_c path_sfx stt staticPath comm .
    if
      pool (path_c @ (DNxt n_c) # path_sfx) = Some stt,
      pathsCorrespond ((DNxt n_c) # path_sfx) staticPath
    then
      pathsCorrespondModChan
        (pool, comm) (Chan path_c n_c)
        (path_c @ (DNxt n_c) # path_sfx) staticPath
* ∀ pool path_r n_r path_sfx stt path_s n_s n_se t_sy env_sy stack_sy
    n_re t_ry env_ry stack_ry c_c comm c staticPath_re staticPath_sfx .
    if
      pool (path_r @ (DNxt n_r) # path_sfx) = Some stt,
      pool path_s = Some (Stt (Bind n_s (Sync n_se) t_sy) env_sy stack_sy),
      pool path_r = Some (Stt (Bind n_r (Sync n_re) t_ry) env_ry stack_ry),
      {(path_s, c_c, path_r)} ⊆ comm,
      dynamicBuiltOnChanVar env_ry c n_r,
      pathsCorrespondModChan pool comm c path_s staticPath_pfx,
      pathsCorrespond path_sfx staticPath_sfx
    then
      pathsCorrespondModChan pool comm c
        (path_r @ (DNxt n_r) # path_sfx)
        (staticPath_pfx @ (NBnd n_s, ESend n_se)
          # (NBnd n_r, MNxt) # staticPath_sfx)

lemma staticTraceableSound:
  ∀ t0 pool comm path n c t' env stack staticEnv staticComm
    entr exit n_c graph isEnd path_c  .
    if
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm,
      pool path = Some (Stt (Bind n c t') env stack),
```

```
      staticEval staticEnv staticComm t0,
      staticLiveChan staticEnv entr exit n_c t0,
      staticFlowsAccept staticEnv graph t0,
      isEnd (NBnd n)
    then
      (exists staticPath .
        pathsCorrespondModChan pool comm (Chan path_c n_c) path staticPath,
        staticTraceable graph entr exit (NBnd n_c) isEnd staticPath)

lemma staticInclusiveSound:
  ∀ t0 pool comm staticEnv entr exit n_c graph staticComm
    path1 stt1 path_c staticPath1 path2 stt2 staticPath2 .
    if
      star dynamicEval [[] -> (Stt t0 [->] [])] {} pool comm,
      staticLiveChan staticEnv entr exit n_c t0,
      staticFlowsAccept staticEnv graph t0,
      staticEval staticEnv staticComm t0,
      pool path1 = Some stt1,
      pathsCorrespondModChan pool comm (Chan path_c n_c) path1 staticPath1,
      staticTraceable graph entr exit
        (NBnd n_c) (staticSendId staticEnv t0 n_c) staticPath1,
      pool path2 = Some stt2,
      pathsCorrespondModChan pool comm (Chan path_c n_c) path2 staticPath2,
      staticTraceable graph entr exit
        (NBnd n_c) (staticSendId staticEnv t0 n_c) staticPath2
    then
      staticInclusive staticPath1 staticPath2
```

5 Related Work

There has been much research on both dynamic and static analysis of concurrent languages. The formal communication classification analysis and soundness proofs in this work are based on the analysis and proofs of *Specialization of CML message-passing primitives* by Reppy and Xiao [17]. The mechanization of concurrency analyses is prevalent, and mechanization is typically the main goal when developing the analyses. Examples include type checkers in compilers, model checking tools for concurrent models, such as Lustre [9] and Kind [20], and also verification libraries in proof assistants, such as Affeldt et al's Coq library [2]. These systems can verify certain properties of concurrent programs or models, but they don't make any guarantees about the analysis itself. Rather than focus on mechanizing the analysis, this work has focused on mechanizing the theory of analyses for concurrent languages, i.e. the meta-theory of concurrency. There have been a number of works on the meta-theory of Concurrent ML, such as the work of Reppy and Xiao, Nielson et al [15], Kobayashi et al [11], and Gasser et al [7]. There has been relatively little work to mechanize theories of Concurrent ML; however, there has been much work in the mechanization

of the theories of π-calculus [13], such as the work by Gay [8] and Melham [12].

6 Future Work

The formal syntax, semantics, and communication analysis of this work form the basis of a framework for studying concurrency functions, synchronization mechanisms, and their applications. These language features enable the construction of reactive programs, which have separation of parts that are conceptually distinct, yet still depend on each other.

This work has kicked off the framework with a formal communication analysis that has practical applications in aiding optimizations for parallel computation. In the future, additional analyses could be built on the existing semantics, in order to verify the correctness of language extensions or optimizations. Extending the semantics to handle event combinators for choosing events, sequencencing events, guardining events, among others, would be an important next step.

Concurreny is a double edged sword. Without specification of ordering, programs may desribe their behavior more clearly or allow parallelism for faster execution. On the other hand, unspecified orderings may also lead to nondetermistic behavior, which may be not be wanted. To gain the benefits of concurrency without its hinderance, the language could be extended with syntax to identify blocks of code that are required to be deterministic, along with a corresponding static analysis that checks if such code is actually deterministic. The determinism analysis could rely on the static communication analysis to ensure that all syncronized receiving events receive from at most one channel, that channel is sent on by at most one thread, and that thread is also deterministic.

Other analyses could aid optimizations for incremental computation [1]. One possible optimzation could transform a program into one that checks for altered dependencies and only recomputes the data that depends on altered dependencies.

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