Climate Risk Hedging

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Risk Premium

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1.2.1 Expected Utility and Optimal Portfolio

Setting the Investor's Expected Utility

Let's assume a single period model, from $t = 0$ to $t = 1$. We have N stocks.

We have a $N \times 1$ vector of returns $\tilde{r}_1$ at period 1, assumed to be normally distributed:

$$\tilde{r}_1 = \mu + \tilde{\epsilon}_1 \tag{1.1}$$

with μ the equilibrium expected excess returns and $\tilde{\epsilon}_1$ the random component of the returns $\tilde{\epsilon}_1 \sim N(0, \Sigma)$.

The investor i has an exponential CARA utility function, with $\tilde{W}_{1,i}$ the wealth at period 1, and X_i the $N \times 1$ vector of portfolio weights.

$$V(\tilde{W}_{1,i}, X_i) = -\exp\left(-A_i \tilde{W}_{1,i} - b_i^T X_i\right) \tag{1.2}$$

with A_i agent's absolute risk aversion, b_i an $N \times 1$ vector of nonpecuniary benefits.

PLACEHOLDER

Figure 1.1: Efficient Frontier

$$b_i = d_i g \tag{1.3}$$

with g an $N \times 1$ vector and $d_i \geq 0$ a scalar measuring the agent's taste for the nonpecuniary benefits.

The expectation of agent i's in period 0 are:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = E_0(-\exp(-A_i \tilde{W}_{1,i} - b_i^T X_i)) \tag{1.4}$$

We can replace $\tilde{W}_{1,i}$ by the relation $\tilde{W}_{1,i} = W_{0,i}(1 + r_f + X_i^T \tilde{r}_1)$ and define $a_i := A_i W_{0,i}$. The idea is to make out from the expectation the terms that we know about (in period 0), and reexpress the terms within the expectation as a function of the portfolio weights X_i. The last two steps use the fact that $\tilde{r}_1$ is normally distributed with mean μ and variance Σ.

$$\begin{aligned} E_0(V(\tilde{W}_{1,i}, X_i)) &= E_0(-\exp(-A_i W_{0,i}(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\ &= E_0(-\exp(-a_i(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\ &= E_0(-\exp(-a_i(1 + r_f) - a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\ &= -\exp(-a_i(1 + r_f)) E_0(-\exp(-a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\ &= -\exp(-a_i(1 + r_f)) E_0(-\exp(-a_i X_i^T (\tilde{r}_1 + \frac{b_i}{a_i}))) \\ &= -\exp(-a_i(1 + r_f)) \exp(-a_i X_i^T (E_0(\tilde{r}_1) + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \mathrm{Var}(\tilde{r}_1) X_i) \\ &= -\exp(-a_i(1 + r_f)) \exp(-a_i X_i^T (\mu + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \Sigma X_i) \end{aligned} \tag{1.5}$$

Solving for the Investor's Optimal Portfolio

The investors choose their optimal portfolios at time 0. The optimal portfolio X_i is the one that maximizes the expected utility. To find it, we differentiate the expected utility with respect to X_i and set it to zero, to obtain the first-order condition.

We are going to do it step by step:

1. Combine the Exponential Terms:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp(-a_i(1 + r_f) - a_i X_i^T (\mu + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \Sigma X_i) \tag{1.6}$$

and let $f(X_i)$ be the exponent:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp f(X_i) \tag{1.7}$$

2. Differentiate $f(X_i)$ with respect to X_i. We have the chain rule:

$$\frac{\partial h}{\partial X_i} = \frac{\partial h}{\partial f}\frac{\partial f}{\partial X_i} \tag{1.8}$$

If $h = -\exp(f)$, then $\frac{\partial h}{\partial f} = -\exp(f)$. Therefore we have:

$$\frac{\partial h}{\partial X_i} = -\exp(f)\frac{\partial f}{\partial X_i} \tag{1.9}$$

To tackle the derivative of $f(X_i)$, we use two rules. First $\frac{\partial x^T b}{\partial x} = b$ and $\frac{\partial x^T Ax}{\partial x} = 2Ax$ if A is symmetric. We have:

$$\frac{\partial f}{\partial X_i} = -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i \tag{1.10}$$

Combining:

$$\frac{\partial h}{\partial X_i} = -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) \tag{1.11}$$

3. Set the derivative to zero:

$$\begin{aligned} -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) &= 0 \\ -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i &= 0 \end{aligned} \tag{1.12}$$

where the exponential term is always positive, so we can drop it.

4. Rearrange and solve for X_i:

$$\begin{aligned} a_i^2 \Sigma X_i &= a_i(\mu + \frac{b_i}{a_i}) \\ a_i \Sigma X_i &= \mu + \frac{b_i}{a_i} \\ \Sigma X_i &= \frac{1}{a_i}(\mu + \frac{b_i}{a_i}) \\ X_i &= \frac{1}{a_i}\Sigma^{-1}(\mu + \frac{b_i}{a_i}) \end{aligned} \tag{1.13}$$

PLACEHOLDER

Figure 1.2: Efficient Frontier with ESG Preferences

For the sake of simplicity, we assume that $a_i = a$ for all investors. We now have:

$$\begin{aligned} X_i &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{b_i}{a}) \\ &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a}g) \end{aligned} \tag{1.14}$$

Therefore, the optimal portfolio differs across investors due to the ESG characteristics g of the stocks and the investors' taste for nonpecuniary benefits d_i.

1.2.2 Heterogeneous Investors and Expected Returns

Heterogeneous Market

The nth element of investor i's portfolio weight vector X_i is:

$$X_{i,n} = \frac{W_{0,i,n}}{W_{0,i}} \tag{1.15}$$

with $W_{0,i,n}$ the wealth invested in stock n by investor i at time 0.
The total wealth invested in stock n at time 0 is:

$$W_{0,n} := \int_i W_{0,i,n} di \tag{1.16}$$

The nth element of the market-weight vector w_m is:

$$w_{m,n} = \frac{W_{0,n}}{W_0} \tag{1.17}$$

We can now express $W_{0,n}$ in terms of individual investors' wealths by using the definition of $W_{0,n}$:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i,n} di \tag{1.18}$$

We now that $W_{0,i,n} = W_{0,i}X_{i,n}$, so we can rewrite the equation:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i} X_{i,n} di \tag{1.19}$$

Defining $\omega_i = \frac{W_{0,i}}{W_0}$, we have:

$$\begin{aligned} w_{m,n} &= \int_i \frac{W_{0,i}}{W_0} X_{i,n} di \\ &= \int_i \omega_i X_{i,n} di \end{aligned} \tag{1.20}$$

We can now plug in X_i to obtain w_m the vector of market weights:

$$\begin{aligned} w_m &= \int_i \omega_i X_i di \\ &= \int_i \omega_i \frac{1}{a} \Sigma^{-1} (\mu + \frac{d_i}{a} g)_n di \\ &= \frac{1}{a} \sigma^{-1} \mu (\int_i \omega_i di) + \frac{1}{a^2} \Sigma^{-1} g (\int_i \omega_i d_i di) \end{aligned} \tag{1.21}$$

We have $\int_i \omega_i di = 1$ and we define $\bar{d} := \int_i d_i di \geq 0$, the wealth-weighted mean of ESG tastes d_i across agents. Therefore:

$$w_m = \frac{1}{a} \Sigma^{-1} \mu + \frac{1}{a^2} \Sigma^{-1} g \bar{d} \tag{1.22}$$

Expected Returns

Starting from the the vector of market weights w_m, we now can solve for μ the vector of expected returns. We have:

$$
\begin{aligned}
w_m &= \frac{1}{a}\Sigma^{-1}\mu + \frac{1}{a^2}\Sigma^{-1}g\bar{d} \\
aw_m &= \Sigma^{-1}\mu + \frac{1}{a}\Sigma^{-1}g\bar{d} \\
aw_m - \frac{1}{a}\Sigma^{-1}g\bar{d} &= \Sigma^{-1}\mu \\
\Sigma(aw_m - \frac{1}{a}\Sigma^{-1}g\bar{d}) &= \mu \\
\mu &= a\Sigma w_m - \frac{1}{a}\Sigma\Sigma^{-1}g\bar{d} \\
\mu &= a\Sigma w_m - \frac{1}{a}g\bar{d}
\end{aligned}
\tag{1.23}
$$

Multiplying by w_m, we find the market equity premium $\mu_m = w_m^T\mu$:

$$
\begin{aligned}
\mu_m &= aw_m^T\Sigma w_m - \frac{\bar{d}}{a}w_m^T g \\
&= a\sigma_m^2 - \frac{\bar{d}}{a}w_m^T g
\end{aligned}
\tag{1.24}
$$

where $\sigma_m^2 = w_m^T\Sigma w_m$ is the market return variance.

Expected Excess Returns

1.2.3 ESG Portfolio

Portfolio Tilts

Factor Pricing with the ESG Portfolio

1.3 Climate Risk

Chapter 2

Sources of Risk

2.1 Market Risk

2.2 ESG Factor Risk

2.3 Climate Risk