Sustainable Investing Theory

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Contents

Introduction

Chapter 1

Capital Asset Pricing Model (CAPM)

PLACEHOLDER

Figure 1.1: Efficient Frontier

Chapter 2

ESG Preferences

2.1 Expected Utility and Optimal Portfolio

2.1.1 Setting the Investor's Expected Utility

Let's assume a single period model, from $t = 0$ to $t = 1$. We have N stocks.

We have a $N \times 1$ vector of returns $\tilde{r}_1$ at period 1, assumed to be normally distributed:

$$\tilde{r}_1 = \mu + \tilde{\epsilon}_1 \tag{2.1}$$

with μ the equilibrium expected excess returns and $\tilde{\epsilon}_1$ the random component of the returns $\tilde{\epsilon}_1 \sim N(0, \Sigma)$.

The investor i has an exponential CARA utility function, with $\tilde{W}_{1,i}$ the wealth at period 1, and X_i the $N \times 1$ vector of portfolio weights.

$$V(\tilde{W}_{1,i}, X_i) = -\exp\left(-A_i \tilde{W}_{1,i} - b_i^T X_i\right) \tag{2.2}$$

with A_i agent's absolute risk aversion, b_i an $N \times 1$ vector of nonpecuniary benefits.

$$b_i = d_i g \tag{2.3}$$

with g an $N \times 1$ vector and $d_i \geq 0$ a scalar measuring the agent's taste for the nonpecuniary benefits.

The expectation of agent i's in period 0 are:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = E_0(-\exp\left(-A_i \tilde{W}_{1,i} - b_i^T X_i\right)) \tag{2.4}$$

We can replace $\tilde{W}_{1,i}$ by the relation $\tilde{W}_{1,i} = W_{0,i}(1 + r_f + X_i^T \tilde{r}_1)$ and define $a_i := A_i W_{0,i}$. The idea is to make out from the expectation the terms that we know about (in period 0), and reexpress the terms within the expectation as a function of the portfolio weights X_i. The last two steps use the fact that $\tilde{r}_1$ is normally distributed with mean μ and variance Σ.

$$
\begin{aligned}
E_0(V(\tilde{W}_{1,i}, X_i)) &= E_0(-\exp(-A_i W_{0,i}(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\
&= E_0(-\exp(-a_i(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\
&= E_0(-\exp(-a_i(1 + r_f) - a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\
&= -\exp(-a_i(1 + r_f)) E_0(-\exp(-a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\
&= -\exp(-a_i(1 + r_f)) E_0(-\exp(-a_i X_i^T (\tilde{r}_1 + \frac{b_i}{a_i}))) \\
&= -\exp(-a_i(1 + r_f)) \exp(-a_i X_i^T (E_0(\tilde{r}_1) + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \mathrm{Var}(\tilde{r}_1) X_i) \\
&= -\exp(-a_i(1 + r_f)) \exp(-a_i X_i^T (\mu + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \Sigma X_i)
\end{aligned}
\tag{2.5}
$$

2.1.2 Solving for the Investor's Optimal Portfolio

The investors choose their optimal portfolios at time 0. The optimal portfolio X_i is the one that maximizes the expected utility. To find it, we differentiate the expected utility with respect to X_i and set it to zero, to obtain the first-order condition.

We are going to do it step by step:

1. Combine the Exponential Terms:

$$
E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp(-a_i(1 + r_f) - a_i X_i^T (\mu + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \Sigma X_i) \tag{2.6}
$$

and let $f(X_i)$ be the exponent:

$$
E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp f(X_i) \tag{2.7}
$$

2. Differentiate $f(X_i)$ with respect to X_i. We have the chain rule:

$$
\frac{\partial h}{\partial X_i} = \frac{\partial h}{\partial f} \frac{\partial f}{\partial X_i} \tag{2.8}
$$

If $h = -\exp(f)$, then $\frac{\partial h}{\partial f} = -\exp(f)$. Therefore we have:

$$\frac{\partial h}{\partial X_i} = -\exp(f)\frac{\partial f}{\partial X_i} \tag{2.9}$$

To tackle the derivative of $f(X_i)$, we use two rules. First $\frac{\partial x^T b}{\partial x} = b$ and $\frac{\partial x^T Ax}{\partial x} = 2Ax$ if A is symmetric. We have:

$$\frac{\partial f}{\partial X_i} = -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i \tag{2.10}$$

Combining:

$$\frac{\partial h}{\partial X_i} = -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) \tag{2.11}$$

3. Set the derivative to zero:

$$\begin{aligned} -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) &= 0 \\ -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i &= 0 \end{aligned} \tag{2.12}$$

where the exponential term is always positive, so we can drop it.

4. Rearrange and solve for X_i:

$$\begin{aligned} a_i^2 \Sigma X_i &= a_i(\mu + \frac{b_i}{a_i}) \\ a_i \Sigma X_i &= \mu + \frac{b_i}{a_i} \\ \Sigma X_i &= \frac{1}{a_i}(\mu + \frac{b_i}{a_i}) \\ X_i &= \frac{1}{a_i}\Sigma^{-1}(\mu + \frac{b_i}{a_i}) \end{aligned} \tag{2.13}$$

For the sake of simplicity, we assume that $a_i = a$ for all investors. We now have:

$$\begin{aligned} X_i &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{b_i}{a}) \\ &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a} g) \end{aligned} \tag{2.14}$$

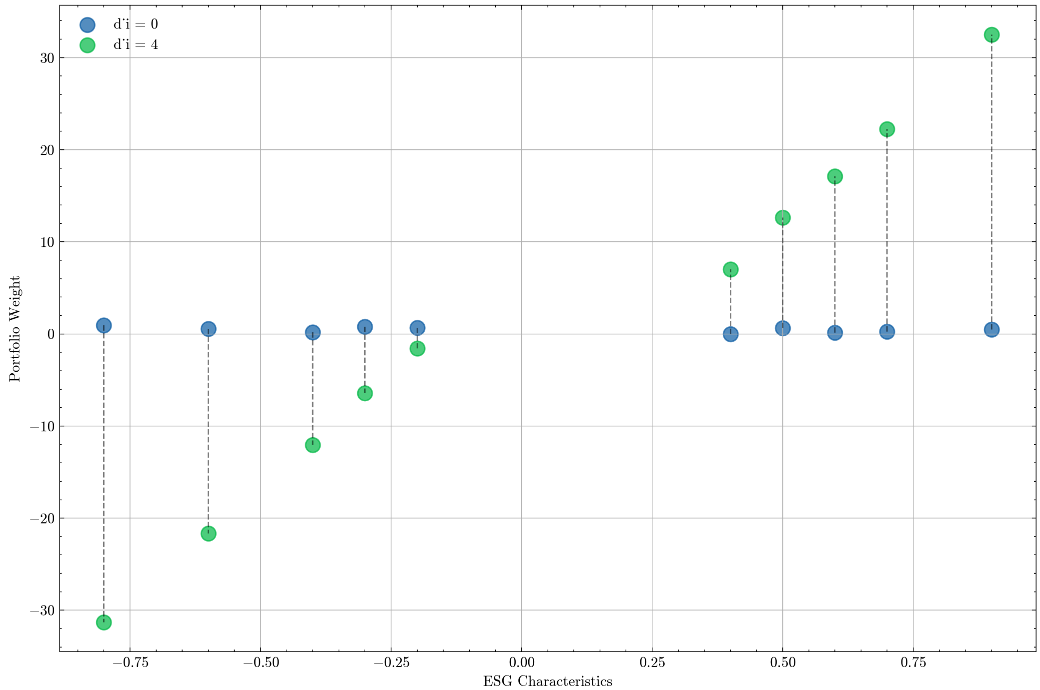

Figure 2.1: Portfolio Weights vs ESG Preferences

Therefore, the optimal portfolio differs across investors due to the ESG characteristics g of the stocks and the investors' taste for nonpecuniary benefits d_i.

2.2 Heterogeneous Investors and Expected Returns

2.2.1 Heterogeneous Market

The nth element of investor i's portfolio weight vector X_i is:

$$X_{i,n} = \frac{W_{0,i,n}}{W_{0,i}} \tag{2.15}$$

with $W_{0,i,n}$ the wealth invested in stock n by investor i at time 0.

The total wealth invested in stock n at time 0 is:

$$W_{0,n} := \int_i W_{0,i,n} di \tag{2.16}$$

The nth element of the market-weight vector w_m is:

$$w_{m,n} = \frac{W_{0,n}}{W_0} \tag{2.17}$$

We can now express $W_{0,n}$ in terms of individual investors' wealths by using the definition of $W_{0,n}$:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i,n} di \tag{2.18}$$

We now that $W_{0,i,n} = W_{0,i} X_{i,n}$, so we can rewrite the equation:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i} X_{i,n} di \tag{2.19}$$

Defining $\omega_i = \frac{W_{0,i}}{W_0}$, we have:

$$\begin{aligned} w_{m,n} &= \int_i \frac{W_{0,i}}{W_0} X_{i,n} di \\ &= \int_i \omega_i X_{i,n} di \end{aligned} \tag{2.20}$$

We can now plug in X_i to obtain w_m the vector of market weights:

$$
\begin{aligned}
w_m &= \int_i \omega_i X_i di \\
&= \int_i \omega_i \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a}g)_n di \\
&= \frac{1}{a}\sigma^{-1}\mu(\int_i \omega_i di) + \frac{1}{a^2}\Sigma^{-1}g(\int_i \omega_i d_i di)
\end{aligned} \tag{2.21}
$$

We have $\int_i \omega_i di = 1$ and we define $\bar{d} := \int_i d_i di \geq 0$, the wealth-weighted mean of ESG tastes d_i across agents. Therefore:

$$
w_m = \frac{1}{a}\Sigma^{-1}\mu + \frac{1}{a^2}\Sigma^{-1}g\bar{d} \tag{2.22}
$$

2.2.2 Expected Returns

Starting from the the vector of market weights w_m, we now can solve for μ the vector of expected returns. We have:

$$
\begin{aligned}
w_m &= \frac{1}{a}\Sigma^{-1}\mu + \frac{1}{a^2}\Sigma^{-1}g\bar{d} \\
aw_m &= \Sigma^{-1}\mu + \frac{1}{a}\Sigma^{-1}g\bar{d} \\
aw_m - \frac{1}{a}\Sigma^{-1}g\bar{d} &= \Sigma^{-1}\mu \\
\Sigma(aw_m - \frac{1}{a}\Sigma^{-1}g\bar{d}) &= \mu \\
\mu &= a\Sigma w_m - \frac{1}{a}\Sigma\Sigma^{-1}g\bar{d} \\
\mu &= a\Sigma w_m - \frac{1}{a}g\bar{d}
\end{aligned} \tag{2.23}
$$

Multiplying by w_m, we find the market equity premium $\mu_m = w_m^T\mu$:

$$
\begin{aligned}
\mu_m &= aw_m^T\Sigma w_m - \frac{\bar{d}}{a}w_m^T g \\
&= a\sigma_m^2 - \frac{\bar{d}}{a}w_m^T g
\end{aligned} \tag{2.24}
$$

where $\sigma_m^2 = w_m^T\Sigma w_m$ is the market return variance.

PLACEHOLDER

Figure 2.2: μ_m and $w_m^T g$ relationship.

The equity premium μ_m depends on the average of ESG tastes, $\bar{d}$, through the "greeness" of the market portfolio $w_m^T g$. If the market is net green (i.e., $w_m^T g > 0$), then stronger ESG tastes (higher $\bar{d}$) lead to lower equity premium.

Conversely, if the market is net "brown" ($w_m^T g < 0$), then stronger ESG tastes lead to higher equity premium as investors demand compensation for holding brown stocks.

2.2.3 Expected Excess Returns

Average Expected Excess Returns

For simplicity, we assume that the market portfolio is ESG-neutral:

$$w_m^T g = 0 \tag{2.25}$$

which implies that the equity premium is:

$$\mu_m = a\sigma_m^2 \tag{2.26}$$

that is, independent of the average ESG tastes $\bar{d}$.

From the last equation, we note that $a = \frac{\mu_m}{\sigma_m^2}$, then the expected excess returns can be reexpressed as:

$$\begin{aligned} \mu &= a\Sigma w_m - \frac{1}{a} g\bar{d} \\ &= \frac{\mu_m}{\sigma_m^2}\Sigma w_m - \frac{1}{a} g\bar{d} \\ &= \mu_m \beta_m - \frac{1}{a} g\bar{d} \end{aligned} \tag{2.27}$$

where we have used the fact that the vector of market betas is $\beta_m = \frac{\Sigma w_m}{\sigma_m^2}$.

This gives the first proposition of the model:

Proposition 1. *Expected excess returns in equilibirium are given by:*

$$\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g \tag{2.28}$$

PLACEHOLDER

Figure 2.3: α_n relationship with g_n

The expected excess returns deviate from their CAPM values due to ESG tastes for holding green stocks.

Corrolary 1. *If $\bar{d} > 0$, the expected return on stock n is decreasing in g_n.*

Given their ESG tastes, agents are willing to pay more for greener firms, then lowering the firms' expected returns.

Corrolary 2. *Because the vector of stocks' CAPM alphas is defined as $\alpha := \mu - \mu_m \beta_m$, we have:*

$$\alpha_n = -\frac{\bar{d}}{a} g_n \tag{2.29}$$

If $\bar{d} > 0$, green stocks have negative alphas, and brown stocks have positive alphas. Greener stocks have lower alphas.

Investor i's Excess Returns Mean and Variance

Investor i's expected excess return is given by:

$$E(\tilde{r}_{1,i}) = X_i^T \mu \tag{2.30}$$

We know that $\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g$ from the Proposition 1:

$$E(\tilde{r}_{1,i}) = X_i^T (\mu_m \beta_m - \frac{\bar{d}}{a} g) \tag{2.31}$$

We can express X_i in terms of w_m by susbtracting the expression w_m from the expression of X_i. Recall the assumption that $a_i = a$ and distribute:

$$\begin{aligned}
E(\tilde{r}_{1,i}) &= (w_m^T + \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a} g) - \frac{1}{a}\Sigma^{-1}\mu - \frac{\bar{d}}{a^2}\Sigma^{-1} g)(\mu_m \beta_m - \frac{\bar{d}}{a} g) \\
&= (w_m^T + \frac{1}{a}\Sigma^{-1}\mu - \frac{1}{a}\Sigma^{-1}\mu + \frac{d_i}{a^2}\Sigma^{-1} g - \frac{\bar{d}}{a^2}\Sigma^{-1} g)(\mu_m \beta_m - \frac{\bar{d}}{a} g) \\
&= (w_m^T + \frac{d_i - \bar{d}}{a^2}\Sigma^{-1} g)(\mu_m \beta_m - \frac{\bar{d}}{a} g)
\end{aligned} \tag{2.32}$$

Rewriting $d_i - \bar{d} = \delta_i$, recalling that $\beta_m = (\frac{1}{\sigma_m^2})\Sigma w_m$ and distribute:

$$\begin{aligned} E(\tilde{r}_{1,i}) &= (w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g)(\frac{\mu_m}{\sigma_m^2}\Sigma w_m - \frac{\bar{d}}{a}g) \\ &= w_m^T\frac{\mu_m}{\sigma_m^2}\Sigma w_m - w_m^T\frac{\bar{d}}{a}g + \frac{\delta_i\mu_m}{a^2\sigma_m^2}\Sigma^{-1}\Sigma g^T w_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \\ &= w_m^T\frac{\mu_m}{\sigma_m^2}\Sigma w_m - w_m^T\frac{\bar{d}}{a}g + \frac{\delta_i\mu_m}{a^2\sigma_m^2}g^T w_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \end{aligned} \tag{2.33}$$

We now that $w_m^T\Sigma w_m = \sigma_m^2$, so we have:

$$E(\tilde{r}_{1,i}) = \mu_m - w_m^T\frac{\bar{d}}{a}g + \frac{\delta_i\mu_m}{a^2\sigma_m^2}g^T w_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \tag{2.34}$$

Recalling the assumption that $w_m^T g = 0$, we finally have:

$$E(\tilde{r}_{1,i}) = \mu_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \tag{2.35}$$

Proposition 2. *The mean of the excess return on investor i's portfolio is given by:*

$$E(\tilde{r}_{1,i}) = \mu_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \tag{2.36}$$

Investor i with $\delta_i > 0$ accepts below-market expected returns in exchange for satisfying their stronger tastes for holding green stocks. Conversely, and as a result, investor i with $\delta_i < 0$ enjoys above-market expected returns.

The variance of the excess return on investor i's portfolio is:

$$\mathrm{Var}(\tilde{r}_{1,i}) = X_i^T\Sigma X_i \tag{2.37}$$

Again, we can express X_i in terms of w_m by susbtracting the expression w_m from the expression of X_i, then distribute:

$$\begin{aligned} \mathrm{Var}(\tilde{r}_{1,i}) &= (w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g)\Sigma(w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g) \\ &= w_m^T\Sigma w_m + w_m^T\Sigma\frac{\delta_i}{a^2}\Sigma^{-1}g + w_m^T\Sigma\frac{\delta_i}{a^2}\Sigma^{-1}g + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}\Sigma\Sigma^{-1}g \\ &= w_m^T\Sigma w_m + w_m^T\frac{\delta_i}{a^2}g + w_m^T\frac{\delta_i}{a^2}g + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}g \end{aligned} \tag{2.38}$$

Finally, we recall that $w_m^T \Sigma w_m = \sigma_m^2$ and the assumption that $w_m^T g = 0$, then we have:

$$\mathrm{Var}(\tilde{r}_{1,i}) = \sigma_m^2 + \frac{\delta_i^2}{a^4} g^T \Sigma^{-1} g \tag{2.39}$$

Proposition 3. *The variance of the excess return on investor i's portfolio is given by:*

$$\mathrm{Var}(\tilde{r}_{1,i}) = \sigma_m^2 + \frac{\delta_i^2}{a^4} g^T \Sigma^{-1} g \tag{2.40}$$

In departing from the market portfolio, all agents with $\delta_i \neq 0$ incur higher volatility than that of the market portfolio.

2.2.4 Investor's Utility in Equilibrium

The lower expected returns earned by ESG-oriented investors do not imply that these agents are unhappy. Indeed, the more an investor's ESG preferences d_i differ from the average in either direction, the more ESG preferences contribute to the investor's utility. To see this, we start again from the investor's expected utility:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp\left(-a_i(1+r_f)\right)\exp\left(-a_i X_i^T(\mu + \frac{b_i}{a_i}) + \frac{1}{2}a_i^2 X_i^T \Sigma X_i\right) \tag{2.41}$$

In the second exponent term, we know from the equation of the investor's expected excess returns that (and recalling the assumption $a_i = a$):

$$\begin{aligned} -a_i X_i^T \mu &= -a(\mu_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g) \\ &= -a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g \end{aligned} \tag{2.42}$$

We have the term $-aiX_i^T \frac{b_i}{a_i} = -X_i^T b_i$, where we again can express X_i in terms of w_m and recall that $b_i = d_i g$ and the assumption that $w_m^T g = 0$:

$$\begin{aligned}
-X_i^T b_i &= -X_i^T d_i g \\
&= -(w_m^T + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) d_i g \\
&= -w_m^T d_i g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g \\
&= -\frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g
\end{aligned} \tag{2.43}$$

And we have finally the term $\frac{1}{2}a_i^2 X_i^T \Sigma X_i$, where we recognize $X_i^T \Sigma X_i$ that we have found earlier:

$$\begin{aligned}
\frac{1}{2}a_i^2 X_i^T \Sigma X_i = \frac{1}{2}a_i^2 (w_m^T + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) \Sigma (w_m + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) \\
= \frac{a^2}{2}(\sigma_m^2 + \frac{\delta_i^2}{a^4} g^T \Sigma^{-1} g) \\
= \frac{a^2}{2}\sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g
\end{aligned} \tag{2.44}$$

Adding the three terms together, we have:

$$\begin{aligned}
-a_i X^T \mu - X_i^T b_i + (\frac{a^2}{2}) X_i^T \Sigma X_i \\
= -a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g + \frac{a^2}{2}\sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g
\end{aligned} \tag{2.45}$$

We can factorize with $\frac{1}{a^2}$ and $g^T \Sigma g$:

$$\begin{aligned}
-a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g + \frac{a^2}{2}\sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \\
= -a\mu_m + \frac{a^2}{2}\sigma_m^2 + \frac{1}{a^2}(\delta_i \bar{d} - \delta_i d_i + \frac{\delta_i^2}{2}) g^T \Sigma g
\end{aligned} \tag{2.46}$$

with $\delta_i \bar{d} - d_i \delta_i = (d_i - \bar{d})\delta_i = \delta_i^2$ and factorizing with $-a$ we have:

$$-a\mu_m + \frac{a^2}{2}\sigma_m^2 + \frac{1}{a^2}(\delta_i \bar{d} - \delta_i d_i + \frac{\delta_i^2}{2}) g^T \Sigma g = -a(\mu_m + \frac{a}{2}\sigma_m^2) - \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \tag{2.47}$$

Substituting this into the utility function we have:

PLACEHOLDER

Figure 2.4: Investor's Utility with ESG Preferences

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp\left(-a(1+r_f)\right)\exp\left(-a(\mu_m + \frac{a}{2}\sigma_m^2) - \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g\right) \tag{2.48}$$

We can separate the terms related with δ_i:

$$\begin{aligned} E_0(V(\tilde{W}_{1,i}, X_i)) &= \left(-\exp\left(-a(1+r_f)\right)\exp\left(-a(\mu_m + \frac{a}{2}\sigma_m^2)\right)\right)\exp\left(-\frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g\right) \\ &= \bar{V}\exp\left(-\frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g\right) \end{aligned} \tag{2.49}$$

If the investor's ESG preferences are on the average, then $\delta_i = 0$ and the investor's utility is $\bar{V}$. The expected utility is increasing in δ_i^2, so the more an agent's ESG preferences differ from the average in either direction, the more ESG preferences contributes to the agent's utility.

2.3 ESG Portfolio

2.3.1 Portfolio Tilts

2.3.2 Factor Pricing with the ESG Portfolio

Chapter 3

Climate Risk