Sustainable Investing Theory

Thomas Lorans

June 20, 2024

Contents

Introduction

Chapter 1

ESG Preferences

1.1 Expected Utility and Optimal Portfolio

1.1.1 Setting the Investor's Expected Utility

Let's assume a single period model, from $t = 0$ to $t = 1$. We have N stocks.

We have a $N \times 1$ vector of returns $\tilde{r}_1$ at period 1, assumed to be normally distributed:

$$\tilde{r}_1 = \mu + \tilde{\epsilon}_1 \tag{1.1}$$

with μ the equilibrium expected excess returns and $\tilde{\epsilon}_1$ the random component of the returns $\tilde{\epsilon}_1 \sim N(0, \Sigma)$.

The investor i has an exponential CARA utility function, with $\tilde{W}_{1,i}$ the wealth at period 1, and X_i the $N \times 1$ vector of portfolio weights.

$$V(\tilde{W}_{1,i}, X_i) = -\exp\left(-A_i\tilde{W}_{1,i} - b_i^T X_i\right) \tag{1.2}$$

with A_i agent's absolute risk aversion, b_i an $N \times 1$ vector of nonpecuniary benefits.

$$b_i = d_i g \tag{1.3}$$

with g an $N \times 1$ vector and $d_i \geq 0$ a scalar measuring the agent's taste for the nonpecuniary benefits.

The expectation of agent i's in period 0 are:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = E_0(-\exp\left(-A_i\tilde{W}_{1,i} - b_i^T X_i\right)) \tag{1.4}$$

We can replace $\tilde{W}_{1,i}$ by the relation $\tilde{W}_{1,i} = W_{0,i}(1 + r_f + X_i^T \tilde{r}_1)$ and define $a_i := A_i W_{0,i}$. The idea is to make out from the expectation the terms that we know about (in period 0), and reexpress the terms within the expectation as a function of the portfolio weights X_i. The last two steps use the fact that $\tilde{r}_1$ is normally distributed with mean μ and variance Σ.

$$\begin{aligned}
E_0(V(\tilde{W}_{1,i}, X_i)) &= E_0(-\exp(-A_i W_{0,i}(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\
&= E_0(-\exp(-a_i(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\
&= E_0(-\exp(-a_i(1 + r_f) - a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\
&= -\exp(-a_i(1 + r_f))E_0(-\exp(-a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\
&= -\exp(-a_i(1 + r_f))E_0(-\exp(-a_i X_i^T(\tilde{r}_1 + \frac{b_i}{a_i}))) \\
&= -\exp(-a_i(1 + r_f))\exp(-a_i X_i^T(E_0(\tilde{r}_1) + \frac{b_i}{a_i}) + \frac{1}{2}a_i^2 X_i^T \mathrm{Var}(\tilde{r}_1) X_i) \\
&= -\exp(-a_i(1 + r_f))\exp(-a_i X_i^T(\mu + \frac{b_i}{a_i}) + \frac{1}{2}a_i^2 X_i^T \Sigma X_i)
\end{aligned} \tag{1.5}$$

1.1.2 Solving for the Investor's Optimal Portfolio

The investors choose their optimal portfolios at time 0. The optimal portfolio X_i is the one that maximizes the expected utility. To find it, we differentiate the expected utility with respect to X_i and set it to zero, to obtain the first-order condition.

We are going to do it step by step:

1. Combine the Exponential Terms:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp(-a_i(1 + r_f) - a_i X_i^T(\mu + \frac{b_i}{a_i}) + \frac{1}{2}a_i^2 X_i^T \Sigma X_i) \tag{1.6}$$

and let $f(X_i)$ be the exponent:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp f(X_i) \tag{1.7}$$

2. Differentiate $f(X_i)$ with respect to X_i. We have the chain rule:

$$\frac{\partial h}{\partial X_i} = \frac{\partial h}{\partial f}\frac{\partial f}{\partial X_i} \tag{1.8}$$

If $h = -\exp(f)$, then $\frac{\partial h}{\partial f} = -\exp(f)$. Therefore we have:

$$\frac{\partial h}{\partial X_i} = -\exp(f)\frac{\partial f}{\partial X_i} \tag{1.9}$$

To tackle the derivative of $f(X_i)$, we use two rules. First $\frac{\partial x^T b}{\partial x} = b$ and $\frac{\partial x^T Ax}{\partial x} = 2Ax$ if A is symmetric. We have:

$$\frac{\partial f}{\partial X_i} = -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i \tag{1.10}$$

Combining:

$$\frac{\partial h}{\partial X_i} = -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) \tag{1.11}$$

3. Set the derivative to zero:

$$\begin{aligned} -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) &= 0 \\ -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i &= 0 \end{aligned} \tag{1.12}$$

where the exponential term is always positive, so we can drop it.

4. Rearrange and solve for X_i:

$$\begin{aligned} a_i^2 \Sigma X_i &= a_i(\mu + \frac{b_i}{a_i}) \\ a_i \Sigma X_i &= \mu + \frac{b_i}{a_i} \\ \Sigma X_i &= \frac{1}{a_i}(\mu + \frac{b_i}{a_i}) \\ X_i &= \frac{1}{a_i}\Sigma^{-1}(\mu + \frac{b_i}{a_i}) \end{aligned} \tag{1.13}$$

For the sake of simplicity, we assume that $a_i = a$ for all investors. We now have:

$$\begin{aligned} X_i &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{b_i}{a}) \\ &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a}g) \end{aligned} \tag{1.14}$$

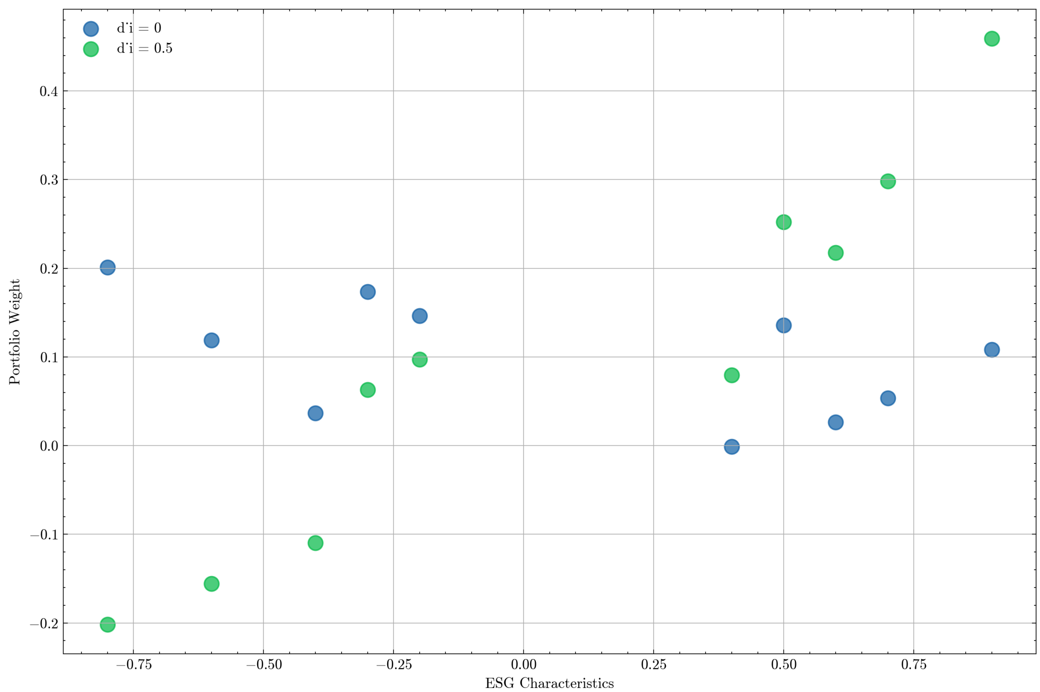

Figure 1.1: Portfolio Weights vs ESG Preferences

Therefore, the optimal portfolio differs across investors due to the ESG characteristics g of the stocks and the investors' taste for nonpecuniary benefits d_i.

1.2 Heterogeneous Investors and Expected Returns

1.2.1 Heterogeneous Market and Market Portfolio Weights

The nth element of investor i's portfolio weight vector X_i is:

$$X_{i,n} = \frac{W_{0,i,n}}{W_{0,i}} \tag{1.15}$$

with $W_{0,i,n}$ the wealth invested in stock n by investor i at time 0.
The total wealth invested in stock n at time 0 is:

$$W_{0,n} := \int_i W_{0,i,n} di \tag{1.16}$$

The nth element of the market-weight vector w_m is:

$$w_{m,n} = \frac{W_{0,n}}{W_0} \tag{1.17}$$

We can now express $W_{0,n}$ in terms of individual investors' wealths by using the definition of $W_{0,n}$:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i,n} di \tag{1.18}$$

We now that $W_{0,i,n} = W_{0,i} X_{i,n}$, so we can rewrite the equation:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i} X_{i,n} di \tag{1.19}$$

Defining $\omega_i = \frac{W_{0,i}}{W_0}$, we have:

$$\begin{aligned} w_{m,n} &= \int_i \frac{W_{0,i}}{W_0} X_{i,n} di \\ &= \int_i \omega_i X_{i,n} di \end{aligned} \tag{1.20}$$

We can now plug in X_i to obtain w_m the vector of market weights:

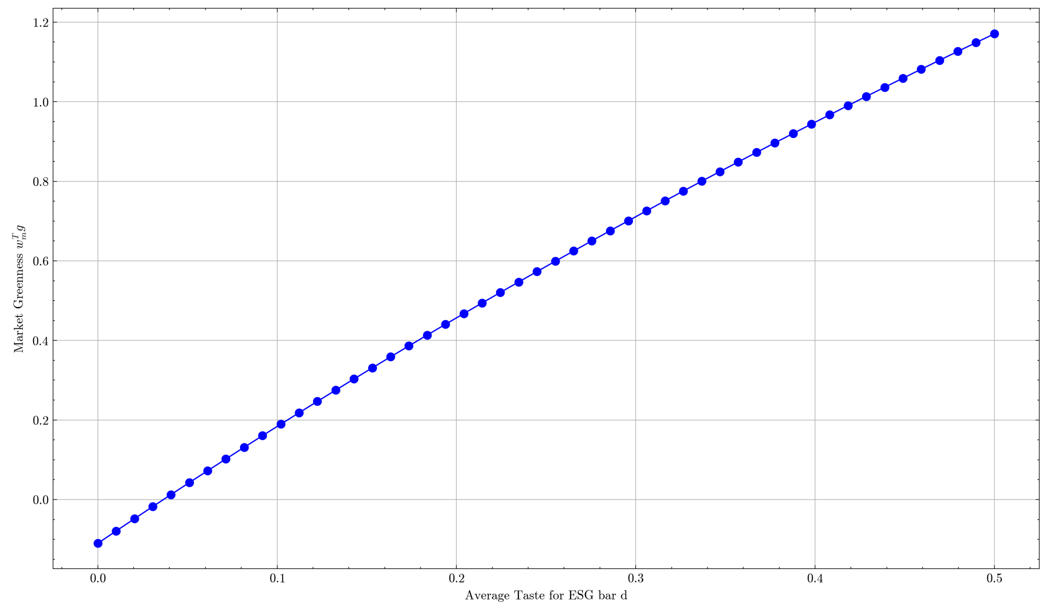

Figure 1.2: Relationship between Market Greenness and Average Taste for ESG

$$
\begin{aligned}
w_m &= \int_i \omega_i X_i di \\
&= \int_i \omega_i \frac{1}{a} \Sigma^{-1} (\mu + \frac{d_i}{a} g)_n di \\
&= \frac{1}{a} \sigma^{-1} \mu (\int_i \omega_i di) + \frac{1}{a^2} \Sigma^{-1} g (\int_i \omega_i d_i di)
\end{aligned}
\tag{1.21}
$$

We have $\int_i \omega_i di = 1$ and we define $\bar{d} := \int_i d_i di \geq 0$, the wealth-weighted mean of ESG tastes d_i across agents. Therefore, the market portfolio weights are:

$$
w_m = \frac{1}{a} \Sigma^{-1} \mu + \frac{1}{a^2} \Sigma^{-1} g \bar{d} \tag{1.22}
$$

This equation is the same as the one found for the investor's optimal portfolio weights, but with the average ESG tastes $\bar{d}$ instead of individual tastes d_i.

1.2.2 Expected Returns

Starting from the the vector of market weights w_m, we now can solve for μ the vector of expected returns. We have:

$$\begin{aligned} w_m &= \frac{1}{a}\Sigma^{-1}\mu + \frac{1}{a^2}\Sigma^{-1}g\bar{d} \\ aw_m &= \Sigma^{-1}\mu + \frac{1}{a}\Sigma^{-1}g\bar{d} \\ aw_m - \frac{1}{a}\Sigma^{-1}g\bar{d} &= \Sigma^{-1}\mu \\ \Sigma(aw_m - \frac{1}{a}\Sigma^{-1}g\bar{d}) &= \mu \\ \mu &= a\Sigma w_m - \frac{1}{a}\Sigma\Sigma^{-1}g\bar{d} \\ \mu &= a\Sigma w_m - \frac{1}{a}g\bar{d} \end{aligned} \tag{1.23}$$

Multiplying by w_m, we find the market equity premium $\mu_m = w_m^T\mu$:

$$\begin{aligned} \mu_m &= aw_m^T\Sigma w_m - \frac{\bar{d}}{a}w_m^Tg \\ &= a\sigma_m^2 - \frac{\bar{d}}{a}w_m^Tg \end{aligned} \tag{1.24}$$

where $\sigma_m^2 = w_m^T\Sigma w_m$ is the market return variance.

The equity premium μ_m depends on the average of ESG tastes, $\bar{d}$, through the "greeness" of the market portfolio w_m^Tg. If the market is net green (i.e., $w_m^Tg > 0$), then stronger ESG tastes (higher $\bar{d}$) lead to lower equity premium.

Conversely, if the market is net "brown" ($w_m^Tg < 0$), then stronger ESG tastes lead to higher equity premium as investors demand compensation for holding brown stocks.

1.2.3 Expected Excess Returns

Average Expected Excess Returns

For simplicity, we assume that the market portfolio is ESG-neutral:

$$w_m^Tg = 0 \tag{1.25}$$

which implies that the equity premium is:

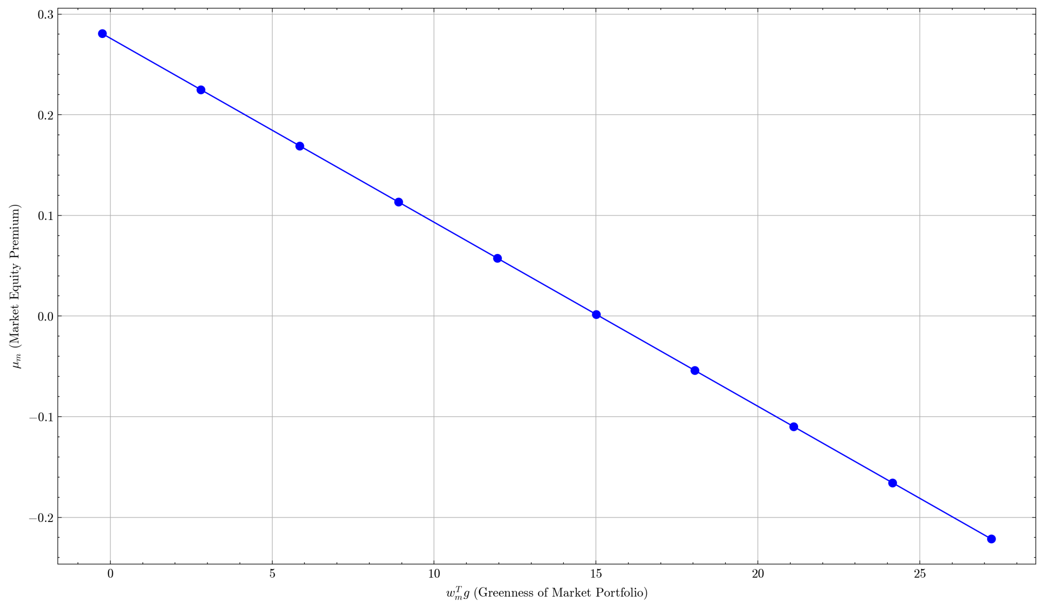

Figure 1.3: Market Equity Premium vs Market Greeness

$$\mu_m = a\sigma_m^2 \tag{1.26}$$

that is, independent of the average ESG tastes $\bar{d}$.

From the last equation, we note that $a = \frac{\mu_m}{\sigma_m^2}$, then the expected excess returns can be reexpressed as:

$$\begin{aligned} \mu &= a\Sigma w_m - \frac{1}{a} g\bar{d} \\ &= \frac{\mu_m}{\sigma_m^2}\Sigma w_m - \frac{1}{a} g\bar{d} \\ &= \mu_m \beta_m - \frac{1}{a} g\bar{d} \end{aligned} \tag{1.27}$$

where we have used the fact that the vector of market betas is $\beta_m = \frac{\Sigma w_m}{\sigma_m^2}$. This gives the first proposition of the model:

Proposition 1. *Expected excess returns in equilibirium are given by:*

$$\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g \tag{1.28}$$

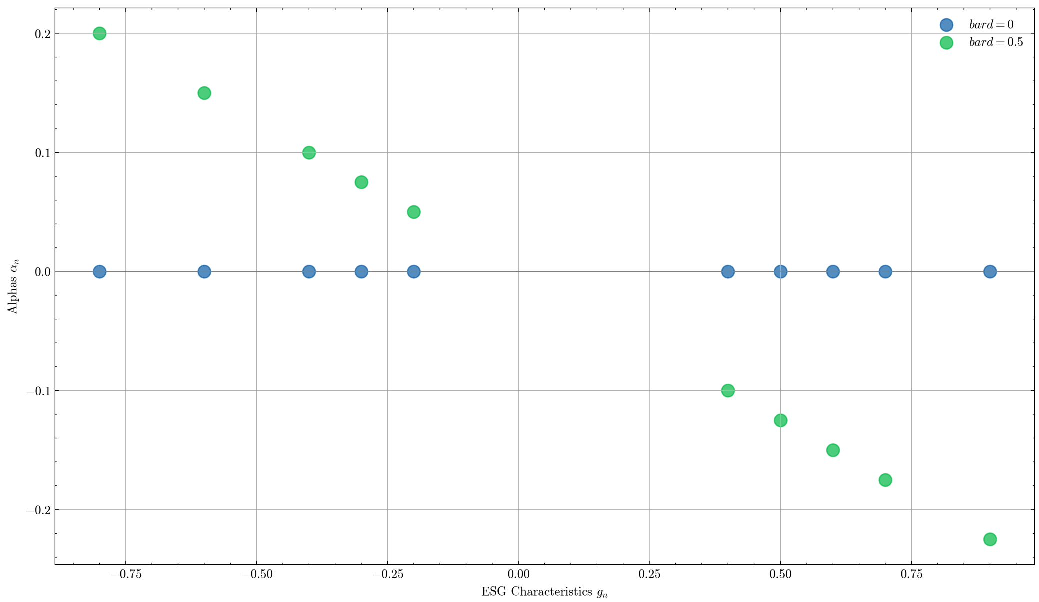

Figure 1.4: α_n relationship with g_n

The expected excess returns deviate from their CAPM values due to ESG tastes for holding green stocks.

Corrolary 1. *If $\bar{d} > 0$, the expected return on stock n is decreasing in g_n.*

Given their ESG tastes, agents are willing to pay more for greener firms, then lowering the firms' expected returns.

Corrolary 2. *Because the vector of stocks' CAPM alphas is defined as $\alpha := \mu - \mu_m \beta_m$, we have:*

$$\alpha_n = -\frac{\bar{d}}{a} g_n \tag{1.29}$$

If $\bar{d} > 0$, green stocks have negative alphas, and brown stocks have positive alphas. Greener stocks have lower alphas.

Investor i's Excess Returns Mean and Variance

Investor i's expected excess return is given by:

$$E(\tilde{r}_{1,i}) = X_i^T \mu \tag{1.30}$$

We know that $\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g$ from the Proposition 1:

$$E(\tilde{r}_{1,i}) = X_i^T (\mu_m \beta_m - \frac{\bar{d}}{a} g) \tag{1.31}$$

We can express X_i in terms of w_m by susbtracting the expression w_m from the expression of X_i. Recall the assumption that $a_i = a$ and distribute:

$$\begin{aligned} E(\tilde{r}_{1,i}) &= (w_m^T + \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a} g) - \frac{1}{a}\Sigma^{-1}\mu - \frac{\bar{d}}{a^2}\Sigma^{-1} g)(\mu_m \beta_m - \frac{\bar{d}}{a} g) \\ &= (w_m^T + \frac{1}{a}\Sigma^{-1}\mu - \frac{1}{a}\Sigma^{-1}\mu + \frac{d_i}{a^2}\Sigma^{-1} g - \frac{\bar{d}}{a^2}\Sigma^{-1} g)(\mu_m \beta_m - \frac{\bar{d}}{a} g) \\ &= (w_m^T + \frac{d_i - \bar{d}}{a^2}\Sigma^{-1} g)(\mu_m \beta_m - \frac{\bar{d}}{a} g) \end{aligned} \tag{1.32}$$

Rewriting $d_i - \bar{d} = \delta_i$, recalling that $\beta_m = (\frac{1}{\sigma_m^2})\Sigma w_m$ and distribute:

$$\begin{aligned} E(\tilde{r}_{1,i}) &= (w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1} g)(\frac{\mu_m}{\sigma_m^2}\Sigma w_m - \frac{\bar{d}}{a} g) \\ &= w_m^T \frac{\mu_m}{\sigma_m^2}\Sigma w_m - w_m^T \frac{\bar{d}}{a} g + \frac{\delta_i \mu_m}{a^2 \sigma_m^2}\Sigma^{-1}\Sigma g^T w_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g \\ &= w_m^T \frac{\mu_m}{\sigma_m^2}\Sigma w_m - w_m^T \frac{\bar{d}}{a} g + \frac{\delta_i \mu_m}{a^2 \sigma_m^2} g^T w_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g \end{aligned} \tag{1.33}$$

We now that $w_m^T \Sigma w_m = \sigma_m^2$, so we have:

$$E(\tilde{r}_{1,i}) = \mu_m - w_m^T \frac{\bar{d}}{a} g + \frac{\delta_i \mu_m}{a^2 \sigma_m^2} g^T w_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g \tag{1.34}$$

Recalling the assumption that $w_m^T g = 0$, we finally have:

$$E(\tilde{r}_{1,i}) = \mu_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g \tag{1.35}$$

Proposition 2. *The mean of the excess return on investor i's portfolio is given by:*

$$E(\tilde{r}_{1,i}) = \mu_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g \tag{1.36}$$

Investor i with $\delta_i > 0$ accepts below-market expected returns in exchange for satisfying their stronger tastes for holding green stocks. Conversely, and as a result, investor i with $\delta_i < 0$ enjoys above-market expected returns.

The variance of the excess return on investor i's portfolio is:

$$\mathrm{Var}(\tilde{r}_{1,i}) = X_i^T \Sigma X_i \tag{1.37}$$

Again, we can express X_i in terms of w_m by susbtracting the expression w_m from the expression of X_i, then distribute:

$$\begin{aligned}\mathrm{Var}(\tilde{r}_{1,i}) &= (w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g)\Sigma(w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g) \\ = w_m^T\Sigma w_m + w_m^T\Sigma\frac{\delta_i}{a^2}\Sigma^{-1}g &+ w_m^T\Sigma\frac{\delta_i}{a^2}\Sigma^{-1}g + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}\Sigma\Sigma^{-1}g \\ &= w_m^T\Sigma w_m + w_m^T\frac{\delta_i}{a^2}g + w_m^T\frac{\delta_i}{a^2}g + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}g\end{aligned} \tag{1.38}$$

Finally, we recall that $w_m^T\Sigma w_m = \sigma_m^2$ and the assumption that $w_m^T g = 0$, then we have:

$$\mathrm{Var}(\tilde{r}_{1,i}) = \sigma_m^2 + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}g \tag{1.39}$$

Proposition 3. *The variance of the excess return on investor i's portfolio is given by:*

$$\mathrm{Var}(\tilde{r}_{1,i}) = \sigma_m^2 + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}g \tag{1.40}$$

In departing from the market portfolio, all agents with $\delta_i \neq 0$ incur higher volatility than that of the market portfolio.

1.2.4 Investor's Utility in Equilibrium

The lower expected returns earned by ESG-oriented investors do not imply that these agents are unhappy. Indeed, the more an investor's ESG preferences d_i differ from the average in either direction, the more ESG preferences contribute to the investor's utility. To see this, we start again from the investor's expected utility:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp\left(-a_i(1+r_f)\right)\exp\left(-a_i X_i^T(\mu + \frac{b_i}{a_i}) + \frac{1}{2}a_i^2 X_i^T \Sigma X_i\right) \tag{1.41}$$

In the second exponent term, we know from the equation of the investor's expected excess returns that (and recalling the assumption $a_i = a$):

$$\begin{aligned} -a_i X_i^T \mu &= -a(\mu_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g) \\ &= -a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g \end{aligned} \tag{1.42}$$

We have the term $-aiX_i^T \frac{b_i}{a_i} = -X_i^T b_i$, where we again can express X_i in terms of w_m and recall that $b_i = d_i g$ and the assumption that $w_m^T g = 0$:

$$\begin{aligned} -X_i^T b_i &= -X_i^T d_i g \\ &= -(w_m^T + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) d_i g \\ &= -w_m^T d_i g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g \\ &= -\frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g \end{aligned} \tag{1.43}$$

And we have finally the term $\frac{1}{2}a_i^2 X_i^T \Sigma X_i$, where we recognize $X_i^T \Sigma X_i$ that we have found earlier:

$$\begin{aligned} \frac{1}{2}a_i^2 X_i^T \Sigma X_i &= \frac{1}{2}a_i^2(w_m^T + \frac{\delta_i}{a^2} g^T \Sigma^{-1})\Sigma(w_m + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) \\ &= \frac{a^2}{2}(\sigma_m^2 + \frac{\delta_i^2}{a^4} g^T \Sigma^{-1} g) \\ &= \frac{a^2}{2}\sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \end{aligned} \tag{1.44}$$

Adding the three terms together, we have:

$$\begin{aligned} & -a_i X^T \mu - X_i^T b_i + (\frac{a^2}{2}) X_i^T \Sigma X_i \\ &= -a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g + \frac{a^2}{2}\sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \end{aligned} \tag{1.45}$$

PLACEHOLDER

Figure 1.5: Investor's Utility with ESG Preferences

We can factorize with $\frac{1}{a^2}$ and $g^T\Sigma g$:

$$\begin{aligned}-a\mu_m + \frac{\delta_i\bar{d}}{a^2}g^T\Sigma g - \frac{\delta_i}{a^2}g^T\Sigma^{-1}d_i g + \frac{a^2}{2}\sigma_m^2 + \frac{\delta_i^2}{2a^2}g^T\Sigma^{-1}g \\ = -a\mu_m + \frac{a^2}{2}\sigma_m^2 + \frac{1}{a^2}(\delta_i\bar{d} - \delta_i d_i + \frac{\delta_i^2}{2})g^T\Sigma g\end{aligned} \tag{1.46}$$

with $\delta_i\bar{d} - d_i\delta_i = (d_i - \bar{d})\delta_i = \delta_i^2$ and factorizing with $-a$ we have:

$$-a\mu_m + \frac{a^2}{2}\sigma_m^2 + \frac{1}{a^2}(\delta_i\bar{d} - \delta_i d_i + \frac{\delta_i^2}{2})g^T\Sigma g = -a(\mu_m + \frac{a}{2}\sigma_m^2) - \frac{\delta_i^2}{2a^2}g^T\Sigma^{-1}g \tag{1.47}$$

Substituting this into the utility function we have:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp(-a(1+r_f))\exp(-a(\mu_m + \frac{a}{2}\sigma_m^2) - \frac{\delta_i^2}{2a^2}g^T\Sigma^{-1}g) \tag{1.48}$$

We can separate the terms related with δ_i:

$$\begin{aligned}E_0(V(\tilde{W}_{1,i}, X_i)) = (-\exp(-a(1+r_f))\exp(-a(\mu_m + \frac{a}{2}\sigma_m^2)))\exp(-\frac{\delta_i^2}{2a^2}g^T\Sigma^{-1}g) \\ = \bar{V}\exp(-\frac{\delta_i^2}{2a^2}g^T\Sigma^{-1}g)\end{aligned} \tag{1.49}$$

If the investor's ESG preferences are on the average, then $\delta_i = 0$ and the investor's utility is $\bar{V}$. The expected utility is increasing in δ_i^2, so the more an agent's ESG preferences differ from the average in either direction, the more ESG preferences contributes to the agent's utility.

1.3 ESG Portfolio

1.3.1 Portfolio Tilts

We want to reexpress the investor's optimal portfolio weights X_i in terms of the ESG characteristics g.

Plugging excess returns $\mu = a\Sigma w_m - \frac{\bar{d}}{a}$ into the investor's optimal portfolio weights $X_i = \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a}g)$, we get the portfolio weights X_i as a function of the ESG characteristics g and the investor's taste for ESG benefits d_i:

$$\begin{aligned}
X_i &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a}g) \\
&= \frac{1}{a}\Sigma^{-1}\mu + \frac{1}{a^2}\Sigma^{-1}gd_i \\
&= \frac{1}{a}\Sigma^{-1}(a\Sigma w_m - \frac{\bar{d}}{a}g) + \frac{1}{a^2}\Sigma^{-1}gd_i \\
&= \frac{1}{a}a\Sigma^{-1}\Sigma w_m - \frac{\bar{d}}{a^2}\Sigma^{-1}g + \frac{1}{a^2}\Sigma^{-1}gd_i \\
&= w_m - \frac{\bar{d}}{a^2}\Sigma^{-1}g + \frac{d_i}{a^2}\Sigma^{-1}g \\
&= w_m + \frac{d_i - \bar{d}}{a^2}\Sigma^{-1}g \\
&= w_m + \frac{\delta_i}{a^2}\Sigma^{-1}g
\end{aligned} \tag{1.50}$$

Therefore, we have a new proposition:

Proposition 4. *Investor i's optimal portfolio weights on the N stocks are given by:*

$$X_i = w_m + \frac{\delta_i}{a^2}\Sigma^{-1}g \tag{1.51}$$

This proposition implies three-fund separation, as each investor's portfolio can be implemented with three assets: (i) the risk-free asset, (ii) the market portfolio, and (iii) the ESG portfolio. The ESG portfolio weights are proportional to $\Sigma^{-1}g$. The fraction of an investor i's wealth in the risk-free asset, $1 - \mathbf{1}^T X_i = -(\delta_i/a^2)\mathbf{1}^T\Sigma^{-1}g$, can be positive or negative. The investor's remaining wealth is invested in stocks. Specifically, the investor allocates a fraction ϕ_i of her remaining wealth to the ESG portfolio, and a fraction $1-\phi_i$ to the market portfolio.

To see this, we note that the $N \times 1$ vector of weights within investor i's stock portfolio w_i is X_i normalized by the sum of its elements:

$$\begin{aligned} w_i &= X_i/\mathbf{1}^T X_i \\ &= \frac{w_m + \frac{\delta_i}{a^2}\Sigma^{-1}g}{\mathbf{1}^T(w_m + \frac{\delta_i}{a^2}\Sigma^{-1}g)} \end{aligned} \tag{1.52}$$

We can expand the denominator:

$$\begin{aligned} \mathbf{1}^T(w_m + \frac{\delta_i}{a^2}\Sigma^{-1}g) &= \mathbf{1}^T w_m + \frac{\delta_i}{a^2}\mathbf{1}^T\Sigma^{-1}g \\ &= 1 + \frac{\delta_i}{a^2}\mathbf{1}^T\Sigma^{-1}g \end{aligned} \tag{1.53}$$

because $\mathbf{1}^T w_m = 1$. Substitute back into the normalization formula and separate the terms in the numerator:

$$\begin{aligned} w_i &= \frac{w_m + \frac{\delta_i}{a^2}\Sigma^{-1}g}{1 + \frac{\delta_i}{a^2}\mathbf{1}^T\Sigma^{-1}g} \\ &= \frac{w_m}{1 + \frac{\delta_i}{a^2}\mathbf{1}^T\Sigma^{-1}g} + \frac{\frac{\delta_i}{a^2}\Sigma^{-1}g}{1 + \frac{\delta_i}{a^2}\mathbf{1}^T\Sigma^{-1}g} \end{aligned} \tag{1.54}$$

Using the identity $\frac{1}{1+x} = 1 - \frac{x}{1+x}$, with $x = \frac{\delta_i}{a^2}\mathbf{1}^T\Sigma^{-1}g$, we can rewrite the first term:

$$\frac{w_m}{1 + \frac{\delta_i}{a^2}\mathbf{1}^T\Sigma^{-1}g} = w_m(1 - \frac{\delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}{1 + \delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}) \tag{1.55}$$

We put it back into the formula for w_i:

$$w_i = w_m(1 - \frac{\delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}{1 + \delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}) + \frac{\delta_i/a^2\Sigma^{-1}g}{1 + \delta_i/a^2\mathbf{1}^T\Sigma^{-1}g} \tag{1.56}$$

$\Sigma^{-1}g$ must be normalized to sum to 1:

$$w_g = \frac{1}{\mathbf{1}^T\Sigma^{-1}g}\Sigma^{-1}g \tag{1.57}$$

So we can rewrite the second term as:

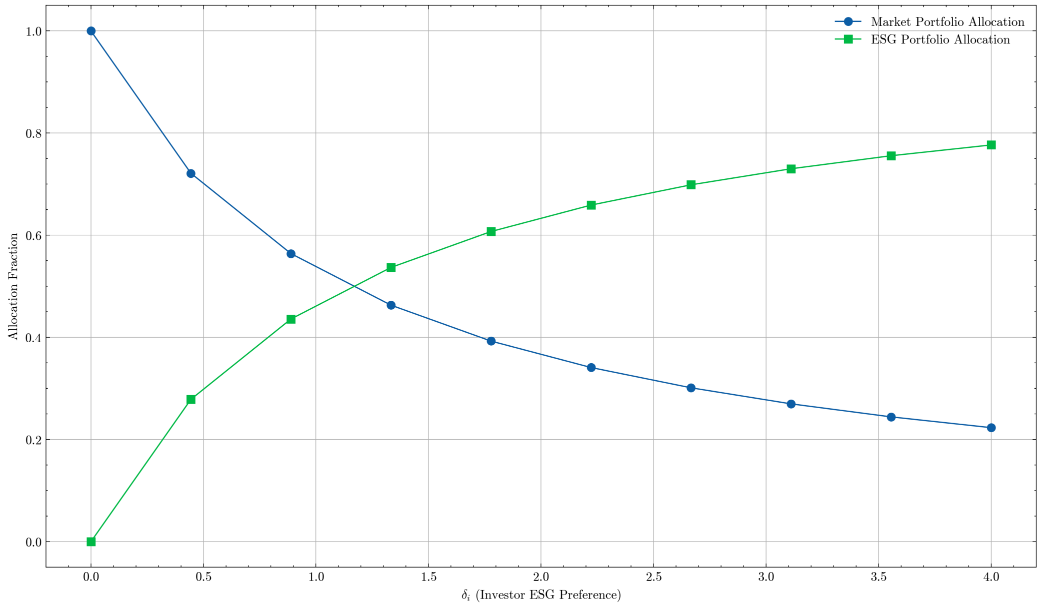

Figure 1.6: Allocation Between Market and ESG Portfolio According to δ_i, with $\delta_i \geq 0$

$$\begin{aligned} w_i = w_m(1 - \frac{\delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}{1+\delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}) + \frac{\delta_i/a^2}{1+\delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}w_g \\ = w_m\phi_i + w_g(1-\phi_i) \end{aligned} \tag{1.58}$$

with $\phi_i = \frac{\delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}{1+\delta_i/a^2\mathbf{1}^T\Sigma^{-1}g}$.

In the special case where $\mathbf{1}^T\Sigma^{-1}g = 0$, no investor holds the risk-free asset, and the ESG portfolio is a zero-cost position with:

$$w_g = \Sigma^{-1}g \tag{1.59}$$

so that:

$$w_i = w_m + \phi_i\Sigma^{-1}g \tag{1.60}$$

where $\phi_i = \delta_i/a^2$.

We denote the ESG portfolio greeness as:

$$g_g = w_g^T g \tag{1.61}$$

From the equation above, g_g is nonzero as long as $g \neq 0$. Also, g_g is negative if $\mathbf{1}^T\Sigma^{-1}g < 0$, but it is otherwise positive. We see that ϕ_i has the same sign as the product of δ_i and g_g, if the denominator of ϕ_i is positive (that is, if investor i invests a positive fraction of her wealth in stocks, so that $\mathbf{1}^T X_i > 0$).

Therefore, for an investor with positive wealth in stocks and $\delta_i > 0$, ϕ_i is positive (negative) if g_g is positive (negative). That is, such an investor tilts away from the market portfolio in the direction of greeness, that is she tilts towards the ESG portfolio when it is green and away from it when it is brown. Applying our previous result, the ESG portfolio CAPM alpha is given by:

$$\alpha_g = -\frac{\bar{d}}{a} g_g \tag{1.62}$$

whose sign is opposite to that of g_g. Therefore, investors with positive (negative) value of δ_i have ESG portfolios tilts that produce negative (positive) alphas for their overall portfolios.

The ESG tilt is zero (ie, $\phi_i = 0$) for investors with average ESG tastes ($\delta_i = 0$). Those investors hold the market portfolio. In contrast, investors who are indifferent to ESG ($d_i = 0$, thus $\delta_i < 0$) tilt away from the market portfolio. In a world with ESG concerns, investors indifferent to ESG should tilt away from the market portfolio. Otherwise they are not optimizing. The market portfolio is optimal for investors with average ESG tastes.

If all agents have identical ESG concerns, so that $\delta_i = 0$ for all i, then there is zero ESG tilt for each investor. We thus have the following corrolary:

Corrolary 3. *If there is no dispertion in ESG tastes across investors, then all investors hold the market portfolio.*

All investors hold the market portfolio when none of them have ESG concerns, as in the standard CAPM. All agents also hold the market portfolio when all of them have the same ESG concerns. The reason is that stock prices then fully adjust to reflect those tastes, again making the market everybody's optimal choice. Dispersion in ESG tastes is necessary for an ESG industry to exist.

1.3.2 Factor Pricing with the ESG Portfolio

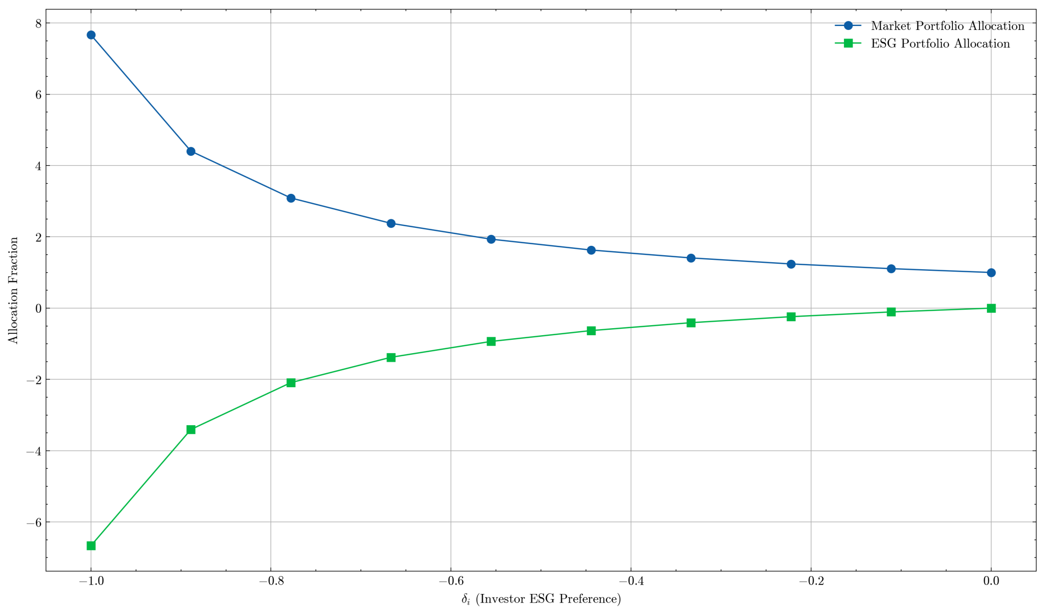

Figure 1.7: Allocation Between Market and ESG Portfolio According to δ_i, with $\delta_i < 0$

Chapter 2

Climate Risk

2.1 Expected Utility and Optimal Portfolio

As with ESG preferences only, we start by setting up the utility function of an investor who cares about climate risk, and then derive the optimal portfolio

2.1.1 Investor's Expected Utility

Let $\tilde{C}_1$ denote climate at time 1, which is unknown at time 0. The investor utility function is now:

$$V(\tilde{W}_{1,i}, X_i, \tilde{C}_1) = -\exp\left(-A_i \tilde{W}_{1,i} - b_i^T X_i - c_i \tilde{C}_1\right) \tag{2.1}$$

where c_i is the investor's climate risk sensitivity.

Taking the expectation of the utility function from period 0, we get:

$$E_0(V(\tilde{W}_{1,i}, X_i, \tilde{C}_1)) = E_0(-\exp\left(-A_i W_{0,i} - b_i^T X_i - c_i \tilde{C}_1\right)) \tag{2.2}$$

Again, we can replace $\tilde{W}_{1,i}$ with the relation $\tilde{W}_{1,i} = W_{0,i}(1 + r_f + X_i^T \tilde{r}_1)$ and define $a_i := A_i W_{0,i}$. We still want to make out from the expectation the terms that we know about in period 0, and reexpress the terms with the expectation as a function of the portfolio weights X_i.

$$
\begin{aligned}
E_0(V(\tilde{W}_{1,i}, X_i, \tilde{C}_1)) &= E_0(-\exp(-A_i W_{0,i} - b_i^T X_i - c_i \tilde{C}_1)) \\
&= E_0(-\exp(-a_i(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i - c_i \tilde{C}_1)) \\
&= -\exp(-a_i(1 + r_f))E_0(-\exp(-a_i X_i^T \tilde{r}_1 - b_i^T X_i - c_i \tilde{C}_1)) \\
&= -\exp(-a_i(1 + r_f))E_0(-\exp(-a_i X_i^T (\tilde{r}_1 + \frac{b_i}{a_i}) - c_i \tilde{C}_1)) \\
&= -\exp(-a_i(1 + r_f)) - \exp(a_i X_i^T (E_0(\tilde{r}_1) + \frac{b_i}{a_i})+ \\
&\quad \frac{1}{2} a_i^2 X_i^T \mathrm{Var}(\tilde{\epsilon}_1) X_i + a_i c_i X_i^T \mathrm{Cov}(\tilde{\epsilon}_1, \tilde{C}_1) + \frac{1}{2} c_i^2 \mathrm{Var}(\tilde{C}_1)) \\
&= -\exp(-a_i(1 + r_f)) - \exp(-a_i X_i^T (\mu + \frac{b_i}{a_i})+ \\
&\quad \frac{1}{2} a_i^2 X_i^T \Sigma X_i + a_i c_i X_i^T \sigma_{\tilde{\epsilon}_1, \tilde{C}_1} + \frac{1}{2} c_i^2 \sigma^2_{\tilde{C}_1})
\end{aligned}
\tag{2.3}
$$

where $\sigma_{\tilde{\epsilon}_1, \tilde{C}_1} = \mathrm{Cov}(\tilde{\epsilon}_1, \tilde{C}_1)$.

2.1.2 Optimal Portfolio

Again, the investor i seeks to maximize its expected utility, by choosing the optimal portfolio weights X_i at time 0. We need to find the first order conditions for the optimization problem.

We are going to follow the same steps as in the previous chapter.

1. We combine the exponential terms:

$$
\begin{aligned}
E_0(V(\tilde{W}_1, X_i, \tilde{C}_1)) = -\exp(-a_i(1 + r_f) - a_i X_i^T (\mu + \frac{b_i}{a_i})+ \\
\frac{1}{2} a_i^2 X_i^T \Sigma X_i + a_i c_i X_i^T \sigma_{\tilde{\epsilon}_1, \tilde{C}_1} + \frac{1}{2} c_i^2 \sigma^2_{\tilde{C}_1})
\end{aligned}
\tag{2.4}
$$

   and let $f(X_i)$ denotes the exponent:

$$
E_0(V(\tilde{W}_1, X_i, \tilde{C}_1)) = -\exp(f(X_i)) \tag{2.5}
$$

2. To differentiate $f(X_i)$ with respect to X_i, we use the chain rule $\frac{\partial h}{\partial X_i} = \frac{\partial h}{\partial f} \frac{\partial f}{\partial X_i}$. If $h = -\exp(f)$, then $\frac{\partial h}{\partial f} = -\exp(f)$. Thus:

$$
\frac{\partial h}{\partial X_i} = -\exp(f) \frac{\partial f}{\partial X_i} \tag{2.6}
$$

3. We can again use the rules that $\frac{\partial x^T b}{\partial x} = b$ and $\frac{\partial x^T A x}{\partial x} = 2Ax$:

$$\frac{\partial f}{\partial X_i} = -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i + a_i c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1} \tag{2.7}$$

Combining:

$$\frac{\partial h}{\partial X_i} = -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i + a_i c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1}) \tag{2.8}$$

4. We set the derivative to zero:

$$\begin{aligned} -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i + a_i c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1}) = 0 \\ -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i + a_i c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1} = 0 \end{aligned} \tag{2.9}$$

because the exponential term is always positive.

5. We solve for X_i:

$$\begin{aligned} a_i^2 \Sigma X_i &= a_i(\mu + \frac{b_i}{a_i} - c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1}) \\ a_i \Sigma X_i &= \mu + \frac{b_i}{a_i} - c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1} \\ \Sigma X_i &= \frac{1}{a_i}(\mu + \frac{b_i}{a_i} - c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1}) \\ X_i &= \frac{1}{a_i}\Sigma^{-1}(\mu + \frac{b_i}{a_i} - c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1}) \end{aligned} \tag{2.10}$$

Again, we will assume that $a_i = a$ for all investors:

$$X_i = \frac{1}{a}\Sigma^{-1}(\mu + \frac{b_i}{a} - c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1}) \tag{2.11}$$

2.2 Heterogeneous Climate Risk Expectations, Market Portfolio and Expected Returns

2.2.1 Heterogeneous Climate Risk Expectations and Market Portfolio

We follow the same process than in the previous chapter, now including differences in expectations c_i about climate risk $\tilde{C}_1$.

The nth elements of investor i's portfolio weight vector X_i is still:

$$X_{i,n} = \frac{W_{0,i,n}}{W_{0,i}} \tag{2.12}$$

The total wealth invested in stock n at time 0:

$$W_{0,i,n} := \int_i W_{0,i,n} di \tag{2.13}$$

The nth element of the market portfolio weight vector w_m is:

$$w_{m,n} = \frac{W_{0,m,n}}{W_{0,m}} \tag{2.14}$$

We reexpress $W_{0,n}$ in terms of individual investors' wealth by using the definition of $W_{0,n}$:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i,n} di \tag{2.15}$$

with $W_{0,i,n} = W_{0,i} X_{i,n}$, we can rewrite the equation:

$$\begin{aligned} w_{m,n} &= \frac{1}{W_0} \int_i W_{0,i} X_{i,n} di \\ &= \int_i \frac{W_{0,i}}{W_0} X_{i,n} di \\ &= \int_i \omega_i X_{i,n} di \end{aligned} \tag{2.16}$$

We now plug the optimal portfolio weights X_i we have found in the previous section into the equation above to obtain the market weights w_m:

$$\begin{aligned} w_{m,n} &= \int_i \omega_i \frac{1}{a} \Sigma^{-1} (\mu + \frac{b_i}{a} - c_i \sigma_{\tilde{\epsilon}_1, \tilde{C}_1}) di \\ &= \frac{1}{a} \Sigma^{-1} \mu (\int_i \omega_i di) + \frac{1}{a^2} \Sigma^{-1} g (\int_i \omega_i d_i di) - \frac{1}{a} \Sigma^{-1} \sigma_{\tilde{\epsilon}_1, \tilde{C}_1} (\int_i \omega_i c_i di) \end{aligned} \tag{2.17}$$

We have $\int_i \omega_i di = 1$ and $\int_i \omega_i c_i di := \bar{c} \geq 0$, the wealth-weighted average expectation about climate risk across investors. The market portfolio weights are:

$$
\begin{aligned}
w_m &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{g}{a}\bar{d} - \bar{c}\sigma_{\tilde{\epsilon}_1,\tilde{C}_1}) \\
&= \frac{1}{a}\Sigma^{-1}\mu + \frac{g}{a^2}\Sigma^{-1}\bar{d} - \frac{1}{a}\Sigma^{-1}\bar{c}\sigma_{\tilde{\epsilon}_1,\tilde{C}_1}
\end{aligned}
\tag{2.18}
$$

2.2.2 Market Portfolio Expected Returns

Starting from the vector of marekt weights w_m, we now can solver for μ the vector of expected returns:

$$
\begin{aligned}
w_m &= \frac{1}{a}\Sigma^{-1}\mu + \frac{g}{a^2}\Sigma^{-1}\bar{d} - \frac{1}{a}\Sigma^{-1}\bar{c}\sigma_{\tilde{\epsilon}_1,\tilde{C}_1} \\
aw_m &= \Sigma^{-1}\mu + \frac{g}{a}\Sigma^{-1}\bar{d} - \Sigma^{-1}\bar{c}\sigma_{\tilde{\epsilon}_1,\tilde{C}_1} \\
aw_m - \frac{g}{a}\Sigma^{-1}\bar{d} + \Sigma^{-1}\bar{c}\sigma_{\tilde{\epsilon}_1,\tilde{C}_1} &= \Sigma^{-1}\mu \\
\Sigma(aw_m - \frac{g}{a}\bar{d} + \bar{c}\sigma_{\tilde{\epsilon}_1,\tilde{C}_1}) &= \mu \\
\mu &= a\Sigma w_m - \frac{g}{a}\Sigma\Sigma^{-1}\bar{d} + \bar{c}\Sigma\sigma_{\tilde{\epsilon}_1,\tilde{C}_1} \\
\mu &= a\Sigma w_m - \frac{g}{a}\bar{d} + \bar{c}\Sigma\sigma_{\tilde{\epsilon}_1,\tilde{C}_1}
\end{aligned}
\tag{2.19}
$$

Multiplying by w_m, we find the market equity premium ($\mu_m = w_m^T\mu$):

$$
\begin{aligned}
\mu_m &= aw_m^T\Sigma w_m - \frac{g}{a}w_m^T\bar{d} + \bar{c}w_m^T\Sigma\sigma_{\tilde{\epsilon}_1,\tilde{C}_1} \\
&= aw_m^T\Sigma w_m + \bar{c}w_m^T\Sigma\sigma_{\tilde{\epsilon}_1,\tilde{C}_1} \\
&= a\sigma_m^2 + \bar{c}\sigma_{mC}
\end{aligned}
\tag{2.20}
$$

where we still maintain the assumption of an ESG-neutral market portfolio ($w_m^T g = 0$), and we have the market portfolio variance $\sigma_m^2 = w_m^T\Sigma w_m$ and $w_m^T\sigma_{\tilde{\epsilon}_1,\tilde{C}_1} = \mathrm{Cov}(\tilde{\epsilon}_1,\tilde{C}_1) = \sigma_{mC}$.

2.2.3 Expected Returns with Climate Risk

We use the last equation in the previous section to solve for a:

$$
\begin{aligned}
\mu_m &= a\sigma_m^2 + \bar{c}\sigma_{mC} \\
a &= \frac{\mu_m - \bar{c}\sigma_{mC}}{\sigma_m^2}
\end{aligned}
\tag{2.21}
$$

Then, the expected excess returns can be reexpressed as:

$$\begin{aligned} \mu &= a\Sigma w_m - \frac{g}{a}\bar{d} + \bar{c}\sigma_{mC} \\ &= \frac{\mu_m - \bar{c}\sigma_{mC}}{\sigma_m^2}\Sigma w_m - \frac{g}{a}\bar{d} + \bar{c}\sigma_{mC} \end{aligned} \tag{2.22}$$

We know that $\frac{1}{\sigma_m^2}\Sigma w_m = \beta_m$, the market beta:

$$\begin{aligned} \mu &= \frac{\mu_m - \bar{c}\sigma_{mC}}{\sigma_m^2}\Sigma w_m - \frac{g}{a}\bar{d} + \bar{c}\sigma_{mC} \\ &= (\mu_m - \bar{c}\sigma_{mC})\beta_m - \frac{g}{a}\bar{d} + \bar{c}\sigma_{mC} \\ &= \mu_m\beta_m - \bar{c}\sigma_{mC}\beta_m - \frac{g}{a}\bar{d} + \bar{c}\sigma_{mC} \\ &= \mu_m\beta_m - \frac{g}{a}\bar{d} + \bar{c}(\sigma_{mC} - \beta_m\sigma_{mC}) \end{aligned} \tag{2.23}$$

We know that $\beta_m = (\frac{1}{\sigma_m^2}\sigma_{\tilde{\epsilon}_1,m})$:

$$\begin{aligned} \mu &= \mu_m\beta_m - \frac{g}{a}\bar{d} + \bar{c}(\sigma_{mC} - \beta_m\sigma_{mC}) \\ &= \mu_m\beta_m - \frac{g}{a}\bar{d} + \bar{c}(\sigma_{mC} - \frac{1}{\sigma_m^2}\sigma_{\tilde{\epsilon}_1,m}\sigma_{m,C}) \end{aligned} \tag{2.24}$$

In the multivariate regression of $\tilde{\epsilon}_1$ on $\tilde{\epsilon}_m$ and $\tilde{C}_1$, the slope coefficients are given by:

$$\begin{aligned} &\begin{bmatrix} \sigma_{\tilde{\epsilon}_1,m} & \sigma_{\tilde{\epsilon}_1,C} \end{bmatrix} \begin{bmatrix} \sigma_m^2 & \sigma_{m,C} \\ \sigma_{m,C} & \sigma_C^2 \end{bmatrix}^{-1} \\ &= \frac{1}{\sigma_m^2\sigma_C^2 - \sigma_{mC}^2} \begin{bmatrix} \sigma_C^2\sigma_{\tilde{\epsilon}_1,m} - \sigma_{mC}\sigma_{\tilde{\epsilon}_1,C} & \sigma_m^2\sigma_{\tilde{\epsilon}_1,C} - \sigma_{mC}\sigma_{\tilde{\epsilon}_1,m} \end{bmatrix} \end{aligned} \tag{2.25}$$

So the second column (the coefficient of $\tilde{C}_1$) is:

$$\psi = \frac{1}{\sigma_m^2\sigma_C^2 - \sigma_{mC}^2}(\sigma_m^2\sigma_{\tilde{\epsilon}_1,C} - \sigma_{mC}\sigma_{\tilde{\epsilon}_1,m}) \tag{2.26}$$

We can use ψ to rewrite the expected returns:

$$
\begin{aligned}
\mu &= \mu_m \beta_m - \frac{g}{a}\bar{d} + \bar{c}(\sigma_{mC} - \frac{1}{\sigma_m^2}\sigma_{\tilde{\epsilon}_1,m}\sigma_{m,C}) \\
&= \mu_m \beta_m - \frac{g}{a}\bar{d} + \bar{c}\frac{\sigma_m^2\sigma_C^2 - \sigma_{mC}^2}{\sigma_m^2}\psi \qquad (2.27) \\
&= \mu_m \beta_m - \frac{\bar{d}}{a}g + \bar{c}(1 - \rho_{mC}^2)\psi
\end{aligned}
$$

recalling that $\sigma_C = 1$.

Expected returns depend on climate betas, ψ, which represent firms' exposures to non-market climate risk. A firm's climate beta is its loading on $\tilde{C}_1$ after controlling for the market return.

2.3 Utility in Equilibrium with Heterogenous Climate Risk Expectations

2.4 Climate Risk Hedging Portfolio