# Sustainable Investing Theory

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### 2.1.1 Setting the Investor's Expected Utility

Let's assume a single period model, from $t = 0$ to $t = 1$. We have $N$ stocks.

We have a $N \times 1$ vector of returns $\tilde{r}_1$ at period 1, assumed to be normally distributed:

$$\tilde{r}_1 = \mu + \tilde{\epsilon}_1 \tag{2.1}$$

with $\mu$ the equilibrium expected excess returns and $\tilde{\epsilon}_1$ the random component of the returns $\tilde{\epsilon}_1 \sim N(0, \Sigma)$.

The investor $i$ has an exponential CARA utility function, with $\tilde{W}_{1,i}$ the wealth at period 1, and $X_i$ the $N \times 1$ vector of portfolio weights.

$$V(\tilde{W}_{1,i}, X_i) = -\exp\left(-A_i \tilde{W}_{1,i} - b_i^T X_i\right) \tag{2.2}$$

with $A_i$ agent's absolute risk aversion, $b_i$ an $N \times 1$ vector of nonpecuniary benefits.

$$b_i = d_i g \tag{2.3}$$

with $g$ an $N \times 1$ vector and $d_i \geq 0$ a scalar measuring the agent's taste for the nonpecuniary benefits.

The expectation of agent $i$'s in period 0 are:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = E_0(-\exp\left(-A_i \tilde{W}_{1,i} - b_i^T X_i\right)) \tag{2.4}$$

We can replace $\tilde{W}_{1,i}$ by the relation $\tilde{W}_{1,i} = W_{0,i}(1 + r_f + X_i^T \tilde{r}_1)$ and define $a_i := A_i W_{0,i}$. The idea is to make out from the expectation the terms that we know about (in period 0), and reexpress the terms within the expectation as a function of the portfolio weights $X_i$. The last two steps use the fact that $\tilde{r}_1$ is normally distributed with mean $\mu$ and variance $\Sigma$.

$$\begin{aligned}
E_0(V(\tilde{W}_{1,i}, X_i)) &= E_0(-\exp(-A_i W_{0,i}(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\
&= E_0(-\exp(-a_i(1 + r_f + X_i^T \tilde{r}_1) - b_i^T X_i)) \\
&= E_0(-\exp(-a_i(1 + r_f) - a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\
&= -\exp(-a_i(1 + r_f)) E_0(-\exp(-a_i X_i^T \tilde{r}_1 - b_i^T X_i)) \\
&= -\exp(-a_i(1 + r_f)) E_0(-\exp(-a_i X_i^T (\tilde{r}_1 + \frac{b_i}{a_i}))) \\
&= -\exp(-a_i(1 + r_f)) \exp(-a_i X_i^T (E_0(\tilde{r}_1) + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \mathrm{Var}(\tilde{r}_1) X_i) \\
&= -\exp(-a_i(1 + r_f)) \exp(-a_i X_i^T (\mu + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \Sigma X_i)
\end{aligned} \tag{2.5}$$

### 2.1.2 Solving for the Investor's Optimal Portfolio

The investors choose their optimal portfolios at time 0. The optimal portfolio $X_i$ is the one that maximizes the expected utility. To find it, we differentiate the expected utility with respect to $X_i$ and set it to zero, to obtain the first-order condition.

We are going to do it step by step:

1. Combine the Exponential Terms:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp(-a_i(1 + r_f) - a_i X_i^T (\mu + \frac{b_i}{a_i}) + \frac{1}{2} a_i^2 X_i^T \Sigma X_i) \tag{2.6}$$

and let $f(X_i)$ be the exponent:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp f(X_i) \tag{2.7}$$

2. Differentiate $f(X_i)$ with respect to $X_i$. We have the chain rule:

$$\frac{\partial h}{\partial X_i} = \frac{\partial h}{\partial f} \frac{\partial f}{\partial X_i} \tag{2.8}$$

If $h = -\exp(f)$, then $\frac{\partial h}{\partial f} = -\exp(f)$. Therefore we have:

$$\frac{\partial h}{\partial X_i} = -\exp(f)\frac{\partial f}{\partial X_i} \tag{2.9}$$

To tackle the derivative of $f(X_i)$, we use two rules. First $\frac{\partial x^T b}{\partial x} = b$ and $\frac{\partial x^T A x}{\partial x} = 2Ax$ if $A$ is symmetric. We have:

$$\frac{\partial f}{\partial X_i} = -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i \tag{2.10}$$

Combining:

$$\frac{\partial h}{\partial X_i} = -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) \tag{2.11}$$

3. Set the derivative to zero:

$$\begin{aligned} -\exp(f)(-a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i) = 0 \\ -a_i(\mu + \frac{b_i}{a_i}) + a_i^2 \Sigma X_i = 0 \end{aligned} \tag{2.12}$$

where the exponential term is always positive, so we can drop it.

4. Rearrange and solve for $X_i$:

$$\begin{aligned} a_i^2 \Sigma X_i &= a_i(\mu + \frac{b_i}{a_i}) \\ a_i \Sigma X_i &= \mu + \frac{b_i}{a_i} \\ \Sigma X_i &= \frac{1}{a_i}(\mu + \frac{b_i}{a_i}) \\ X_i &= \frac{1}{a_i}\Sigma^{-1}(\mu + \frac{b_i}{a_i}) \end{aligned} \tag{2.13}$$

For the sake of simplicity, we assume that $a_i = a$ for all investors. We now have:

$$\begin{aligned} X_i &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{b_i}{a}) \\ &= \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a}g) \end{aligned} \tag{2.14}$$

PLACEHOLDER

Figure 2.1: Efficient Frontier with ESG Preferences

Therefore, the optimal portfolio differs across investors due to the ESG characteristics $g$ of the stocks and the investors' taste for nonpecuniary benefits $d_i$.

## 2.2 Heterogeneous Investors and Expected Returns

### 2.2.1 Heterogeneous Market

The $n$th element of investor $i$'s portfolio weight vector $X_i$ is:

$$X_{i,n} = \frac{W_{0,i,n}}{W_{0,i}} \tag{2.15}$$

with $W_{0,i,n}$ the wealth invested in stock $n$ by investor $i$ at time 0.
The total wealth invested in stock $n$ at time 0 is:

$$W_{0,n} := \int_i W_{0,i,n} di \tag{2.16}$$

The $n$th element of the market-weight vector $w_m$ is:

$$w_{m,n} = \frac{W_{0,n}}{W_0} \tag{2.17}$$

We can now express $W_{0,n}$ in terms of individual investors' wealths by using the definition of $W_{0,n}$:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i,n} di \tag{2.18}$$

We now that $W_{0,i,n} = W_{0,i} X_{i,n}$, so we can rewrite the equation:

$$w_{m,n} = \frac{1}{W_0} \int_i W_{0,i} X_{i,n} di \tag{2.19}$$

Defining $\omega_i = \frac{W_{0,i}}{W_0}$, we have:

$$\begin{aligned} w_{m,n} &= \int_i \frac{W_{0,i}}{W_0} X_{i,n} di \\ &= \int_i \omega_i X_{i,n} di \end{aligned} \tag{2.20}$$

We can now plug in $X_i$ to obtain $w_m$ the vector of market weights:

$$\begin{aligned} w_m &= \int_i \omega_i X_i di \\ &= \int_i \omega_i \frac{1}{a} \Sigma^{-1} (\mu + \frac{d_i}{a} g)_n di \\ &= \frac{1}{a} \sigma^{-1} \mu (\int_i \omega_i di) + \frac{1}{a^2} \Sigma^{-1} g (\int_i \omega_i d_i di) \end{aligned} \tag{2.21}$$

We have $\int_i \omega_i di = 1$ and we define $\bar{d} := \int_i d_i di \geq 0$, the wealth-weighted mean of ESG tastes $d_i$ across agents. Therefore:

$$w_m = \frac{1}{a} \Sigma^{-1} \mu + \frac{1}{a^2} \Sigma^{-1} g \bar{d} \tag{2.22}$$

### 2.2.2 Expected Returns

Starting from the the vector of market weights $w_m$, we now can solve for $\mu$ the vector of expected returns. We have:

$$\begin{aligned} w_m &= \frac{1}{a} \Sigma^{-1} \mu + \frac{1}{a^2} \Sigma^{-1} g \bar{d} \\ a w_m &= \Sigma^{-1} \mu + \frac{1}{a} \Sigma^{-1} g \bar{d} \\ a w_m - \frac{1}{a} \Sigma^{-1} g \bar{d} &= \Sigma^{-1} \mu \\ \Sigma (a w_m - \frac{1}{a} \Sigma^{-1} g \bar{d}) &= \mu \\ \mu = a \Sigma w_m &- \frac{1}{a} \Sigma \Sigma^{-1} g \bar{d} \\ \mu = a \Sigma w_m &- \frac{1}{a} g \bar{d} \end{aligned} \tag{2.23}$$

Multiplying by $w_m$, we find the market equity premium $\mu_m = w_m^T \mu$:

PLACEHOLDER

Figure 2.2: $\mu_m$ and $w_m^T g$ relationship.

$$\begin{aligned}\mu_m &= a w_m^T \Sigma w_m - \frac{\bar{d}}{a} w_m^T g \\ &= a\sigma_m^2 - \frac{\bar{d}}{a} w_m^T g\end{aligned} \tag{2.24}$$

where $\sigma_m^2 = w_m^T \Sigma w_m$ is the market return variance.

The equity premium $\mu_m$ depends on the average of ESG tastes, $\bar{d}$, through the "greeness" of the market portfolio $w_m^T g$. If the market is net green (i.e., $w_m^T g > 0$), then stronger ESG tastes (higher $\bar{d}$) lead to lower equity premium.

Conversely, if the market is net "brown" ($w_m^T g < 0$), then stronger ESG tastes lead to higher equity premium as investors demand compensation for holding brown stocks.

### 2.2.3 Expected Excess Returns

#### Average Expected Excess Returns

For simplicity, we assume that the market portfolio is ESG-neutral:

$$w_m^T g = 0 \tag{2.25}$$

which implies that the equity premium is:

$$\mu_m = a\sigma_m^2 \tag{2.26}$$

that is, independent of the average ESG tastes $\bar{d}$.

From the last equation, we note that $a = \frac{\mu_m}{\sigma_m^2}$, then the expected excess returns can be reexpressed as:

$$\begin{aligned}\mu &= a\Sigma w_m - \frac{1}{a} g\bar{d} \\ &= \frac{\mu_m}{\sigma_m^2}\Sigma w_m - \frac{1}{a} g\bar{d} \\ &= \mu_m \beta_m - \frac{1}{a} g\bar{d}\end{aligned} \tag{2.27}$$

PLACEHOLDER

Figure 2.3: $\alpha_n$ relationship with $g_n$

where we have used the fact that the vector of market betas is $\beta_m = \frac{\Sigma w_m}{\sigma_m^2}$. This gives the first proposition of the model:

**Proposition 1.** *Expected excess returns in equilibirium are given by:*

$$\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g \tag{2.28}$$

*The expected excess returns deviate from their CAPM values due to ESG tastes for holding green stocks.*

**Corrolary 1.** *If $\bar{d} > 0$, the expected return on stock $n$ is decreasing in $g_n$.*

*Given their ESG tastes, agents are willing to pay more for greener firms, then lowering the firms' expected returns.*

**Corrolary 2.** *Because the vector of stocks' CAPM alphas is defined as $\alpha := \mu - \mu_m \beta_m$, we have:*

$$\alpha_n = -\frac{\bar{d}}{a} g_n \tag{2.29}$$

*If $\bar{d} > 0$, green stocks have negative alphas, and brown stocks have positive alphas. Greener stocks have lower alphas.*

### Investor $i$'s Excess Returns Mean and Variance

Investor $i$'s expected excess return is given by:

$$E(\tilde{r}_{1,i}) = X_i^T \mu \tag{2.30}$$

We know that $\mu = \mu_m \beta_m - \frac{\bar{d}}{a} g$ from the Proposition 1:

$$E(\tilde{r}_{1,i}) = X_i^T (\mu_m \beta_m - \frac{\bar{d}}{a} g) \tag{2.31}$$

We can express $X_i$ in terms of $w_m$ by susbtracting the expression $w_m$ from the expression of $X_i$. Recall the assumption that $a_i = a$ and distribute:

$$\begin{aligned}E(\tilde{r}_{1,i}) &= (w_m^T + \frac{1}{a}\Sigma^{-1}(\mu + \frac{d_i}{a}g) - \frac{1}{a}\Sigma^{-1}\mu - \frac{\bar{d}}{a^2}\Sigma^{-1}g)(\mu_m\beta_m - \frac{\bar{d}}{a}g) \\ &= (w_m^T + \frac{1}{a}\Sigma^{-1}\mu - \frac{1}{a}\Sigma^{-1}\mu + \frac{d_i}{a^2}\Sigma^{-1}g - \frac{\bar{d}}{a^2}\Sigma^{-1}g)(\mu_m\beta_m - \frac{\bar{d}}{a}g) \\ &= (w_m^T + \frac{d_i - \bar{d}}{a^2}\Sigma^{-1}g)(\mu_m\beta_m - \frac{\bar{d}}{a}g)\end{aligned} \tag{2.32}$$

Rewriting $d_i - \bar{d} = \delta_i$, recalling that $\beta_m = (\frac{1}{\sigma_m^2})\Sigma w_m$ and distribute:

$$\begin{aligned}E(\tilde{r}_{1,i}) &= (w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g)(\frac{\mu_m}{\sigma_m^2}\Sigma w_m - \frac{\bar{d}}{a}g) \\ &= w_m^T\frac{\mu_m}{\sigma_m^2}\Sigma w_m - w_m^T\frac{\bar{d}}{a}g + \frac{\delta_i\mu_m}{a^2\sigma_m^2}\Sigma^{-1}\Sigma g^T w_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \\ &= w_m^T\frac{\mu_m}{\sigma_m^2}\Sigma w_m - w_m^T\frac{\bar{d}}{a}g + \frac{\delta_i\mu_m}{a^2\sigma_m^2}g^T w_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g\end{aligned} \tag{2.33}$$

We now that $w_m^T\Sigma w_m = \sigma_m^2$, so we have:

$$E(\tilde{r}_{1,i}) = \mu_m - w_m^T\frac{\bar{d}}{a}g + \frac{\delta_i\mu_m}{a^2\sigma_m^2}g^T w_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \tag{2.34}$$

Recalling the assumption that $w_m^T g = 0$, we finally have:

$$E(\tilde{r}_{1,i}) = \mu_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \tag{2.35}$$

**Proposition 2.** *The mean of the excess return on investor i's portfolio is given by:*

$$E(\tilde{r}_{1,i}) = \mu_m - \frac{\delta_i\bar{d}}{a^3}g^T\Sigma g \tag{2.36}$$

*Investor i with $\delta_i > 0$ accepts below-market expected returns in exchange for satisfying their stronger tastes for holding green stocks. Conversely, and as a result, investor i with $\delta_i < 0$ enjoys above-market expected returns.*

The variance of the excess return on investor $i$'s portfolio is:

$$\mathrm{Var}(\tilde{r}_{1,i}) = X_i^T\Sigma X_i \tag{2.37}$$

Again, we can express $X_i$ in terms of $w_m$ by susbtracting the expression $w_m$ from the expression of $X_i$, then distribute:

$$\begin{aligned} \mathrm{Var}(\tilde{r}_{1,i}) &= (w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g)\Sigma(w_m^T + \frac{\delta_i}{a^2}\Sigma^{-1}g) \\ &= w_m^T\Sigma w_m + w_m^T\Sigma\frac{\delta_i}{a^2}\Sigma^{-1}g + w_m^T\Sigma\frac{\delta_i}{a^2}\Sigma^{-1}g + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}\Sigma\Sigma^{-1}g \\ &= w_m^T\Sigma w_m + w_m^T\frac{\delta_i}{a^2}g + w_m^T\frac{\delta_i}{a^2}g + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}g \end{aligned} \tag{2.38}$$

Finally, we recall that $w_m^T\Sigma w_m = \sigma_m^2$ and the assumption that $w_m^T g = 0$, then we have:

$$\mathrm{Var}(\tilde{r}_{1,i}) = \sigma_m^2 + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}g \tag{2.39}$$

**Proposition 3.** *The variance of the excess return on investor i's portfolio is given by:*

$$\mathrm{Var}(\tilde{r}_{1,i}) = \sigma_m^2 + \frac{\delta_i^2}{a^4}g^T\Sigma^{-1}g \tag{2.40}$$

*In departing from the market portfolio, all agents with $\delta_i \neq 0$ incur higher volatility than that of the market portfolio.*

### 2.2.4 Investor's Utility in Equilibrium

The lower expected returns earned by ESG-oriented investors do not imply that these agents are unhappy. Indeed, the more an investor's ESG preferences $d_i$ differ from the average in either direction, the more ESG preferences contribute to the investor's utility. To see this, we start again from the investor's expected utility:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp\left(-a_i(1+r_f)\right)\exp\left(-a_iX_i^T(\mu + \frac{b_i}{a_i}) + \frac{1}{2}a_i^2X_i^T\Sigma X_i\right) \tag{2.41}$$

In the second exponent term, we know from the equation of the investor's expected excess returns that (and recalling the assumption $a_i = a$):

$$\begin{aligned} -a_i X_i^T \mu &= -a(\mu_m - \frac{\delta_i \bar{d}}{a^3} g^T \Sigma g) \\ &= -a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g \end{aligned} \tag{2.42}$$

We have the term $-aiX_i^T \frac{b_i}{a_i} = -X_i^T b_i$, where we again can express $X_i$ in terms of $w_m$ and recall that $b_i = d_i g$ and the assumption that $w_m^T g = 0$:

$$\begin{aligned} -X_i^T b_i &= -X_i^T d_i g \\ &= -(w_m^T + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) d_i g \\ &= -w_m^T d_i g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g \\ &= -\frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g \end{aligned} \tag{2.43}$$

And we have finally the term $\frac{1}{2} a_i^2 X_i^T \Sigma X_i$, where we recognize $X_i^T \Sigma X_i$ that we have found earlier:

$$\begin{aligned} \frac{1}{2} a_i^2 X_i^T \Sigma X_i = \frac{1}{2} a_i^2 (w_m^T + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) \Sigma (w_m + \frac{\delta_i}{a^2} g^T \Sigma^{-1}) \\ = \frac{a^2}{2} (\sigma_m^2 + \frac{\delta_i^2}{a^4} g^T \Sigma^{-1} g) \\ = \frac{a^2}{2} \sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \end{aligned} \tag{2.44}$$

Adding the three terms together, we have:

$$\begin{aligned} -a_i X^T \mu - X_i^T b_i + (\frac{a^2}{2}) X_i^T \Sigma X_i \\ = -a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g + \frac{a^2}{2} \sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \end{aligned} \tag{2.45}$$

We can factorize with $\frac{1}{a^2}$ and $g^T \Sigma g$:

$$\begin{aligned} -a\mu_m + \frac{\delta_i \bar{d}}{a^2} g^T \Sigma g - \frac{\delta_i}{a^2} g^T \Sigma^{-1} d_i g + \frac{a^2}{2} \sigma_m^2 + \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \\ = -a\mu_m + \frac{a^2}{2} \sigma_m^2 + \frac{1}{a^2} (\delta_i \bar{d} - \delta_i d_i + \frac{\delta_i^2}{2}) g^T \Sigma g \end{aligned} \tag{2.46}$$

with $\delta_i \bar{d} - d_i \delta_i = (d_i - \bar{d})\delta_i = \delta_i^2$ and factorizing with $-a$ we have:

$$-a\mu_m + \frac{a^2}{2}\sigma_m^2 + \frac{1}{a^2}(\delta_i \bar{d} - \delta_i d_i + \frac{\delta_i^2}{2}) g^T \Sigma g = -a(\mu_m + \frac{a}{2}\sigma_m^2) - \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g \tag{2.47}$$

Substituting this into the utility function we have:

$$E_0(V(\tilde{W}_{1,i}, X_i)) = -\exp\left(-a(1 + r_f)\right)\exp\left(-a(\mu_m + \frac{a}{2}\sigma_m^2) - \frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g\right) \tag{2.48}$$

We can separate the terms related with $\delta_i$:

$$\begin{aligned} E_0(V(\tilde{W}_{1,i}, X_i)) &= \left(-\exp\left(-a(1 + r_f)\right)\exp\left(-a(\mu_m + \frac{a}{2}\sigma_m^2)\right)\right)\exp\left(-\frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g\right) \\ &= \bar{V}\exp\left(-\frac{\delta_i^2}{2a^2} g^T \Sigma^{-1} g\right) \end{aligned} \tag{2.49}$$

If the investor's ESG preferences are on the average, then $\delta_i = 0$ and the investor's utility is $\bar{V}$. The expected utility is increasing in $\delta_i^2$, so the more an agent's ESG preferences differ from the average in either direction, the more ESG preferences contributes to the agent's utility.

## 2.3 ESG Portfolio

### 2.3.1 Portfolio Tilts

### 2.3.2 Factor Pricing with the ESG Portfolio

# Chapter 3

# Climate Risk