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Efficient Replication of Factor Returns

To appear in the Journal of Portfolio Management

June 2009

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Summary

We present alternative methods for constructing factor-mimicking portfolios in practice. We illustrate how portfolios with a limited number of assets and relatively low turnover can be used to track pure factor returns. These portfolios provide an effective instrument to support the practice of investment management. We illustrate how they can be used to hedge out unintended factor exposures of a passive benchmark, thus facilitating the optimal management of beta exposure. We illustrate how they can be used to hedge out unintended factor exposures of an active strategy, thus isolating pure alpha and facilitating the management of alternative sources of alpha.

Introduction

The empirical evidence of Chen, Roll, and Ross (1986) and Fama and French (1993) suggests that asset dynamics are characterized best by a multifactor representation of asset returns. As in Sharpe (1963), "beta" now captures multiple sources of systematic risk; exposures to each systematic risk are compensated by their respective risk premia. Passive investment management essentially aims to optimize exposures to various sources of beta. Active investment management aims to optimize sources of "alpha" that are unrelated to systematic beta risks. In either context, investors can capture the premium or hedge the risk associated with a particular beta factor through factor-mimicking portfolios. These portfolios have unit exposure to a target factor, zero exposure to other factors, and minimum portfolio risk. The ability to transact in factor-mimicking portfolios may thus provide an efficient enhancement to the practice of investment management.

In this paper, we examine different methods for constructing factor-mimicking portfolios. In particular, given a factor-based model of risk and return, we consider two types of factor replication. "Full replication" uses the underlying information embedded in the model to re-engineer the targeted factor return, while "optimized replication" uses constrained mean-variance optimization with appropriately chosen expected asset returns and asset covariance matrix. In addition to imposing factor exposure constraints, in order to make factor portfolios easier to implement in practice, we examine methods that impose limits on turnover, asset holdings, and the number of assets in the portfolio. This analysis is applied in the context of the Barra Global Equity Model (GEM). We present evidence that the momentum and value factors can be replicated efficiently. Our tracking portfolios capture the risk and return properties of these factors for the period analyzed (Jan 1998 – Jun 2008) as well as periods of extreme market turbulence.

Portfolios that replicate factor returns efficiently can be utilized potentially to enhance the risk and return profile of passive and active investment strategies, as illustrated by the evidence we present. We show that a manager who had attempted to track the MSCI World Small Cap Index could have enhanced the risk-return profile of this passive strategy by hedging the factor exposures unrelated to "size" over the period 31 Dec 2003 to 31 Dec 2008. We also consider a perfect-foresight active stock-picking strategy that is long the top-50 performing US stocks and short the bottom-50 performing US stocks. In this context, factor-mimicking portfolios can be utilized to hedge out the unintended factor exposures, leaving the manager with pure stock-selection "alpha." Using the Barra USE3 model, we illustrate how the risk and return profile of this active investment strategy could have been enhanced by neutralizing some of the large factor exposures for the period 31 Dec 1997 to 30 Jun 2008.

This paper is thus presented in two sections. In section one, we provide analytical considerations in the construction of factor-mimicking portfolios, along with empirical evidence. In section two, we illustrate passive and active management applications of factor-mimicking portfolios.

1. Factor-Mimicking Portfolios — Analytics

Full Replication

We consider the problem of constructing factor-mimicking portfolios corresponding to the factors of a fundamental multifactor model described by the following equations[2]:

$$r = Xf + e \tag{1}$$

$$V = XFX' + D \tag{2}$$

These equations decompose return and risk into a systematic component (respectively *Xf* and *XFX'*) and a specific component (respectively *e* and *D*). Factor models often are estimated through weighted cross-sectional regression. In this regression, observations typically are weighted by the square root of market cap or by the inverse of specific volatility. This is done to ensure that the estimated factor returns are not unduly influenced by very small or very volatile assets[3]. In this case, we can compute the weights of factor-mimicking portfolios directly from the factor return estimation regression[4] below, where *W* represents the weight applied in the OLS estimation:

$$r = Xf + e \Rightarrow f = (X'WX)^{-1} X'Wr \tag{3}$$

The last equation computes factor returns *f* as a weighted average of asset returns *r*. In this equation, the rows of matrix $(X'WX)^{-1}X'W$ correspond to the weights of the factor-mimicking portfolios. The main advantage of this method is that the resulting portfolios replicate exactly the factor returns estimated by the multifactor model. However, an important drawback of this method is that it does not necessarily lead to factor-mimicking portfolios with minimum ex-ante risk; in practice, this is an important consideration, as the error term may represent systematic risks that are not captured by the empirical factor model. Furthermore, the resulting portfolios have long or short positions in all the assets in the estimation universe, and these positions can change significantly from one period to the next. Therefore, these theoretical portfolios that replicate exactly the factor returns from multifactor models may be difficult or costly to implement in practice.

Optimized Replication

We can specify the factor-mimicking portfolio construction problem in general terms as follows. Given a factor model, we would like to construct portfolios that have maximum exposure to a target factor, zero exposure to all other factors, and minimum portfolio risk. We can express this problem as a general mean-variance optimization problem:

$$\max_h \left\{ h' X_\alpha - \frac{1}{2} \lambda h' V h \right\} \tag{4}$$

$$s.t. \quad h' X_\sigma = 0 \tag{5}$$

Here, X_α and X_σ represent exposures to the target factor and to all other factors. This constrained optimization problem can be solved analytically using the method of Lagrange multipliers. Optimal portfolio weights are given by the following expression:

$$h^* = \frac{1}{\lambda} V^{-1} \left[X_\alpha - X_\sigma \left(X_\sigma' V^{-1} X_\sigma \right)^{-1} \left(X_\sigma' V^{-1} X_\alpha \right) \right] \tag{6}$$

One interesting observation here is that if we substitute V^{-1} in expression (6) with the weights W used in the factor return estimation regression, then the factor portfolios we obtain through expression (6) are exactly the same as the full replication portfolios obtained through expression (3)[5]. Therefore, the full replication method can be viewed as a special case of the general optimization framework (4)-(6). Another interesting observation is that, as we maximize exposure to a target factor and constrain exposure to all other risk factors, we are effectively minimizing specific risk; therefore, we could substitute the total risk matrix V with the specific risk matrix D in expression (4). In the Appendix, we show analytically that using specific risk instead of total risk in the optimization leads to the same optimal portfolios, up to a scaling parameter.

In general, optimized replication leads to portfolios that replicate closely (but not exactly) the factor return estimated by the model. In the special case where the inverse of specific variances were used as weights in the model cross-sectional regression, the optimized replication method captures exactly the factor return estimated by the model and leads to the same factor-mimicking portfolios as the full replication method.

Adding Investability Constraints

Full replication and optimized replication portfolios have long or short positions in all the assets in the underlying model estimation universe; liquidity considerations or the availability of shorts may impact the ability to implement such a strategy in practice. In addition, positions can change significantly from one period to the next, possibly resulting in high portfolio turnover and significant transactions costs.

To make factor portfolios easier to implement and manage, investors may wish to impose additional constraints on these portfolios. For example, internal risk management controls or regulatory requirements may impose limits on the leverage of the long/short factor-replicating portfolio[6]. Similar institutional requirements may impose constraints on the amount of capital that can be allocated to a single asset or a group of assets[7] in the portfolio. Also, low liquidity, limited borrowing availability, increased operational complexity, and transaction cost considerations[8] make it cumbersome to manage portfolios that require long or short positions across many assets, especially medium- and small-capitalization assets. As a result, investors may wish to impose constraints on the turnover, maximum asset weight, and number of assets in the factor-replicating portfolio. Our general optimization framework for constructing factor-mimicking portfolios can be extended to include different types of constraints and different benchmarks.

Constructing Factor Portfolios Using Active Risk Optimization

So far, we expressed the factor replication problem as a total-risk optimization problem:

$$\max_{h}\left\{h' X_{\alpha} - \frac{1}{2}\lambda h' V h\right\} \tag{7}$$

However, in certain applications, minimizing tracking error relative to the target factor return is more important than minimizing the total risk of the factor portfolio. In this case, using h_F to represent the full replication portfolio weights, we can express the factor replication problem as an active-risk optimization problem:

$$\max_{h}\left\{h' X_{\alpha} - \frac{1}{2}\lambda (h-h_F)' V (h-h_F)\right\} \tag{8}$$

These two optimization approaches for constructing factor portfolios (total-risk optimization, active-risk optimization) may be appropriate for different applications. In general, investors can use factor portfolios to capture systematic returns or to hedge common factor risks. In the former application, investors wishing to capture risk premia may place more emphasis on minimizing the total risk of the factor portfolio, in which case the total-risk optimization approach would be more appropriate. On the other hand, for investors using factor portfolios to hedge their exposure to common factor risk, the active-risk approach may be more appropriate, as it ensures closer tracking and therefore better hedging of the targeted factor.

Empirical Results

We use data from the Barra Global Equity Model (GEM) to analyze five different factor replication methods. First, we compare unconstrained full-replication and optimized-replication portfolios. Then, we examine three constrained-replication methods, with constraints on turnover and on the maximum number of assets in the replicating portfolio.

We test these five replication methods over the period Jan 1998 to Jun 2008. At the beginning of each month, we form long/short portfolios targeting the value factor and the momentum factor, respectively. The weights of the full-replication and optimized-replication portfolios are computed using the analytical formulas presented in the full-replication and optimized-replication sections. The weights of the other replication portfolios are computed numerically[9]. The exposures and variance-covariance forecasts that are used to construct the factor portfolios are given at the beginning of each month and are based solely on data available at the beginning of each month.

Exhibit 1

Portfolio Statistics	Full Replication	Total Risk Optimisation	Total Risk Optimisation 10% Turnover	Active Risk Optimisation 10% Turnover	Active Risk Optimisation 10% Turnover 400 Assets
GEM Value Factor Portfolios					
Average Monthly Return	0.25%	0.27%	0.25%	0.21%	0.21%
Annualised Total Risk	2.46%	1.98%	1.97%	2.42%	2.45%
Realised Sharpe Ratio	1.233	1.667	1.546	1.039	1.029
Annualised Tracking Error*	0.00%	1.12%	1.13%	0.33%	0.73%
Average Specific Risk Forecast	1.17%	0.82%	0.84%	1.18%	1.31%
Average Total Risk Forecast	2.53%	2.38%	2.38%	2.53%	2.59%
Portfolio Leverage	111.0%	106.1%	107.9%	112.0%	89.7%
Monthly One-Way Turnover	15.2%	14.9%	9.9%	8.9%	10.1%
GEM Momentum Factor Portfolios					
Average Monthly Return	0.08%	0.05%	0.12%	0.13%	0.13%
Annualised Total Risk	4.92%	4.19%	4.22%	4.90%	4.99%
Realised Sharpe Ratio	0.188	0.147	0.334	0.329	0.315
Annualised Tracking Error*	0.00%	1.53%	1.62%	0.86%	1.07%
Average Specific Risk Forecast	1.23%	0.84%	0.95%	1.32%	1.46%
Average Total Risk Forecast	4.41%	4.31%	4.34%	4.44%	4.49%
Portfolio Leverage	100.2%	107.5%	114.8%	109.8%	87.4%
Monthly One-Way Turnover	22.8%	24.7%	9.9%	9.9%	10.7%

*Relative to Full Replication

Exhibit 1 presents the performance of the five replicating portfolios. Total-risk methods achieved lower predicted, as well as realized, portfolio risk, leading to a higher realized Sharpe ratio. For example, value portfolios based on total-risk optimization had realized volatility of 1.97% and 1.98%, compared with realized volatility of 2.42% and 2.45% for value portfolios based on active-risk optimization. On the other hand, active-risk methods experienced lower realized tracking error compared to total-risk methods. For example, momentum portfolios based on total-risk optimization experienced tracking error of 1.53% and 1.62%, compared with tracking error of 0.86% and 1.07% for momentum portfolios based on active-risk optimization. Momentum portfolios were generally more volatile and had higher tracking error and higher turnover compared to value portfolios. Specifically, momentum portfolios had average monthly one-way turnover of around 24%, compared to around 15% for value portfolios based on the same methods. Nonetheless, even when we allow for a very conservative transaction cost estimate of 25 bps, the replicating portfolios with constrained turnover broadly succeed in capturing the risk-adjusted performance (net of costs) of the target factor.

Though the first two sample moments of the target factor appear to be captured by our factor-mimicking portfolios, it is also important to examine tracking performance during periods of high volatility or extreme market conditions. This is relevant particularly when these portfolios are utilized as hedging instruments. Exhibit 2 reports the performance of the five replication methods during periods of extreme market conditions. These results highlight that the hedging portfolios, especially those based on active risk optimization, track factor returns relatively well even during the observed periods of extreme market turmoil.[10]

Exhibit 2

Period	Event	Full Replication*	Total Risk Optimisation**	Total Risk Optimisation 10% Turnover**	Active Risk Optimisation 10% Turnover**	Active Risk Optimisation 10% Turnover 400 Assets**
			GEM Value Factor Portfolios			
Oct-97	Asian Crisis	1.46%	-0.17%	-0.19%	-0.08%	-0.18%
Sep-98	LTCM	-1.07%	0.56%	0.54%	-0.06%	-0.22%
Apr-00	TMT Bubble	1.39%	-0.45%	-0.46%	-0.06%	-0.33%
Oct-01	WTC Attack	-0.42%	-0.52%	-0.86%	-0.36%	-0.29%
Nov-02	Enron	0.89%	0.17%	0.15%	-0.02%	0.23%
Apr-03	Iraq War	0.85%	-0.18%	-0.11%	0.02%	0.14%
Sep-07	Quant "Meltdown"	-1.31%	0.37%	0.37%	0.00%	0.16%
			GEM Momentum Factor Portfolios			
Sep-97	Asian Crisis	1.74%	0.14%	0.24%	0.09%	0.08%
Oct-98	LTCM	-3.59%	0.67%	1.20%	0.42%	0.19%
Mar-00	TMT Bubble	-4.39%	1.84%	1.72%	0.14%	0.58%
Oct-01	WTC Attack	-0.71%	0.59%	0.60%	0.66%	0.80%
Nov-02	Enron	-4.15%	1.00%	1.11%	0.25%	0.28%
Apr-03	Iraq War	-3.41%	0.27%	0.38%	0.26%	0.14%
Sep-07	Quant "Meltdown"	1.30%	0.05%	0.26%	0.12%	0.21%

*Absolute return
**Return relative to Full Replication

Factor portfolio leverage (gross exposure to net asset value) ranged between 89.7% and 112.0% for the value portfolios, and between 87.4% and 114.8% for the momentum portfolios. Thus, during the observed period, these factor-mimicking portfolios did not require high leverage in their implementation.[11]

2. Factor-Mimicking Portfolios — Applications

Passive Investment Strategies

A simple way to capture factor returns would be to build a portfolio with stocks that have high exposure to a particular factor. For example, a value index, such as the MSCI World Value Index, that contains only stocks that have been screened on different valuation ratios could be used as a proxy to capture the value risk premium. Also, a small cap index, such as the MSCI World Small Cap Index, could be used as a proxy to capture the size risk premium. However, this simple approach based on screening typically leads to portfolios that, in addition to the target factor, have significant exposures to other factors. Exhibit 3 provides a snapshot of the factor exposures of these passive benchmarks. Indeed, a passive allocation to the MSCI World Value Index would result in small but nonzero exposures to other factors and high exposure to the Financial sector. Similarly, a passive allocation to the MSCI World Small Cap Index would result in significant positive exposure to the volatility risk factor and somewhat smaller negative exposure to the value and momentum factors. Plan sponsors may wish to recognize these exposures explicitly when building their passive, optimal allocations to beta factors.

Exhibit 3

Exposures	MSCI World Style Indices		Barra GEM Factor Portfolios			
as of 31/12/07	Value	Small Cap	Size	Momentum	Value	Volatility
GEM Factor Exposures* (standard deviations)						
Size Factor	0.24	***-3.00***	***1.00***	0.00	0.00	0.00
Momentum Factor	-0.27	-0.17	0.00	***1.00***	0.00	0.00
Value Factor	***0.56***	-0.22	0.00	0.00	***1.00***	0.00
Volatility Factor	-0.29	1.19	0.00	0.00	0.00	***1.00***
GICS Sector Exposures** (%)						
Consumer Discretionary	-1.53	5.48	0.00	0.00	0.00	0.00
Consumer Staples	-2.63	-4.38	0.00	0.00	0.00	0.00
Energy	4.09	-3.73	0.00	0.00	0.00	0.00
Financials	14.30	-3.69	0.00	0.00	0.00	0.00
Health Care	-1.72	0.68	0.00	0.00	0.00	0.00
Industrials	-3.11	7.02	0.00	0.00	0.00	0.00
Information Technology	-9.67	1.66	0.00	0.00	0.00	0.00
Materials	-3.57	1.81	0.00	0.00	0.00	0.00
Telecomm Services	2.45	-3.71	0.00	0.00	0.00	0.00
Utilities	1.38	-1.13	0.00	0.00	0.00	0.00

* Factor exposures greater than 0.2 standard deviation are typically considered significant exposures
** Sector exposure relative to MSCI World for Style Indices, relative to cash for Barra Factor Portfolios

Factor-mimicking portfolios can be used as an overlay to benchmarks to isolate the risk premia investors aim to capture. In Exhibit 4, we provide a snapshot of the risk characteristics of the MSCI World Small Cap Index benchmark. We then provide the risk characteristics of this benchmark after sequentially hedging each of the unintended factor exposures using full replication. The final, hedged portfolio retains some of the key risk characteristics of the benchmark: identical country, industry, and currency risk, and similar total risk; however, by mitigating the value, success, and volatility style factors, we obtain a portfolio that targets the global size premium.

Exhibit 4

Barra GEM Model Ex-Ante Risk Analysis Data as of 31/12/2007	MSCI World Small Cap	Momentum Factor Portfolio	Value Factor Portfolio	Volatility Factor Portfolio	MSCI SC + Value Hedge	MSCI SC + Val Hedge + Mom Hedge	MSCI SC + Value Hedge + Mom Hedge + Vola Hedge
Risk Factor Exposures							
Size Factor	-3.00	0.00	0.00	0.00	-3.00	-3.00	-3.00
Momentum Factor	***-0.17***	1.00	0.00	0.00	-0.17	0.00	***0.00***
Value Factor	***-0.22***	0.00	1.00	0.00	0.00	0.00	***0.00***
Volatility Factor	***1.19***	0.00	0.00	1.00	1.19	1.19	***0.00***
Forecast Risk Decomposition							
Risk Indices Total	7.19%	3.84%	2.19%	2.75%	7.25%	7.05%	7.12%
Size Factor	7.11%	0.00%	0.00%	0.00%	7.11%	7.11%	7.12%
Momentum Factor	***0.64%***	3.84%	0.00%	0.00%	0.64%	0.00%	***0.00%***
Value Factor	***0.49%***	0.00%	2.19%	0.00%	0.00%	0.00%	***0.00%***
Volatility Factor	***3.28%***	0.00%	0.00%	2.75%	3.28%	3.28%	***0.00%***
Industry Risk	1.27%	0.00%	0.00%	0.00%	1.27%	1.27%	1.27%
Country Risk	14.49%	0.00%	0.00%	0.00%	14.49%	14.49%	14.49%
Currency Risk	2.95%	0.00%	0.00%	0.00%	2.95%	2.95%	2.95%
Specific Risk	0.64%	0.95%	0.84%	0.77%	0.66%	0.68%	1.10%
Total Risk	17.60%	3.98%	2.32%	2.83%	17.61%	17.46%	17.40%

Exhibits 3 and 4 provide snapshots of risk. We thus considered the following experiment: what if we hedged the unintended style exposures to the MSCI World Small Cap Index over time? Since the analysis is directed at passive investing, we utilize full replication on a monthly basis for the period 31 Dec 2003 to 31 Dec 2008. The results were somewhat startling; over the period of analysis, the MSCI World Small Cap Index returned 2.54% p.a. with a volatility of 17.9%, whereas the hedged MSCI World Small Cap Index returned 6.25% p.a. with a volatility of 17.6%. The outperformance of the hedged benchmark is attributable largely to eliminating exposure to the volatility risk factor, which had negative return. Though this outperformance cannot be guaranteed into the future, our experiment illustrates the importance of identifying and isolating the relevant beta premium when constructing optimal, passive allocations.

Active Investment Strategies

Fundamental active investment strategies are often characterized as being “top down” or “bottom up.” Bottom-up strategies emphasize security selection. Analysts will rate the relative attractiveness of individual companies on the basis of balance sheet fundamentals, expected future cash flows, and the quality of management in the context of the current environment. Portfolio managers of bottom-up investment processes build their active exposures by going long the top analyst recommendations and going short the “dogs” (or stocks that are expected to underperform), while trying to maintain balanced or minimal exposures to sectors and investment styles.

Exhibit 5

USE3 Risk Factors	Volatility	Momentum	Trading Activity	Earnings Yield	Value	Earnings Variability	Leverage
Factor Exposures of Long-Short Perfect Foresight Portfolios							
Average Exposure	-0.49	1.19	-0.08	0.04	-0.50	-0.17	-0.11
Average Positive Exposure	0.75	1.38	0.51	0.51	0.35	0.40	0.30
Num of Months with Positive Exposure	26	112	52	65	30	45	52
Average Negative Exposure	-0.82	-0.39	-0.50	-0.45	-0.76	-0.49	-0.40
Num of Months with Negative Exposure	100	14	74	61	96	81	74
Factor Risk and Return							
Realized Factor Return (%, p.a.)	1.60	0.55	-0.85	4.10	-0.21	-0.70	-1.28
Average Forecast Risk (%, p.a.)	6.97	5.16	2.56	3.02	1.52	1.68	1.49

Factor-mimicking portfolios can be use to mitigate style exposures, enabling portfolio managers to capture pure “alpha.” To illustrate this concept, we consider a hypothetical strategy where the investment manager has perfect foresight over a 12-month forecast horizon spanning US stocks. To exploit these views, he constructs an “absolute return” investment strategy going long the 50 best “expected” performing stocks and shorting the 50 worst “expected” performers, on an equal-weighted basis.

Although these portfolios will deliver large positive returns (by construction), they may be exposed to large systematic risks. In Exhibit 5, we provide summary statistics on the “style” exposures of these portfolios for the period 31 Dec 1997 to 30 Jun 2008 using the Barra USE3 model. As shown in the table, the resulting portfolios have nonzero exposure to systematic risk factors. For instance, the long-short strategy maintains an average short exposure to the Volatility factor in 100 of the 126 months of the analysis. This would have detracted from risk-adjusted performance, as this factor earned about 160 bps/annum and contributed to systematic portfolio risk as the average factor volatility was about 7% over this period.

Exhibit 6

Perfect Foresight Portfolio Performance 31/12/97 - 30/06/08	Unhedged Perfect Foresight Portfolio	Full Replication Overlay	Total Risk Optimization 10% Turnover Overlay	Active Risk Optimization 10% Turnover Overlay	Active Risk Optimization 10% Turnover 400 Assets Overlay
Long-Short Perfect Foresight Portfolio Performance					
Average Return to Realised Risk Ratio	0.337	0.519	0.490	0.493	0.481
% Gain in Return to Realised Risk Ratio	-	53.8	45.3	46.2	42.5
Average Return to Forecast Risk Ratio	0.526	0.730	0.705	0.705	0.677
% Gain in Return to Forecast Risk Ratio	-	39.0	34.2	34.1	28.8
Realised Return (Monthly, %)	9.34	8.79	8.59	8.61	8.49
Realised Risk (Annualised, %)	27.69	16.94	17.52	17.46	17.67
Average Forecast Risk (Annualised, %)	17.77	12.03	12.18	12.22	12.55
Average Common Factor Risk (Annualized, %)	16.08	9.60	9.60	9.60	9.63
Average Specific Risk (Annualized, %)	7.02	7.07	7.34	7.40	7.90

Our portfolio manager might ask whether the return from his strategy originated from stock-specific alpha — reflecting the stock-selection skill of his analysts — or from alpha derived from exposure to systematic risk. Alternatively, he might ask whether the risk-adjusted performance of the strategy would be enhanced by hedging out the “unwanted” or “incidental” factor risk. In Exhibit 5, we noted that portfolios had significant exposure to Volatility, Momentum, and Trading Activity. These three factors were also among the most volatile during this period. Unintended exposure to these factors may reduce the risk-adjusted performance of the underlying stock-picking strategy by eroding the alpha and increasing the volatility of the portfolio.

We thus consider the following experiment: at the beginning of each month, the manager will hedge away the unintended systematic factor exposures to Volatility, Momentum, and Trading Activity using factor-overlay portfolios that neutralize the portfolio exposure to these factors. Clearly, one might hedge each of the systematic exposures, and this will be reviewed in future research; our results, however, will suffice to illustrate the potential benefits of hedging common-factor risk. Our hypothetical implementation is carried out with tracking portfolios that hold 400 assets and that have a maximum monthly turnover of 10%, as this may reflect best how the factor-mimicking portfolios might be implemented in practice.

Exhibit 6 shows that the hedged perfect-foresight portfolios significantly outperform the unhedged portfolios in terms of percentage gain in risk-adjusted performance by about 40%. While the return of the hedged portfolios is slightly decreased, this is more than compensated with a substantial reduction in total risk from 18% to 12%, mainly coming from reduced common-factor risk, while specific risk is only slightly increased, if changed at all. Moreover, when monitoring performance at the end of each calendar year, Exhibit 7 shows that the hedged portfolios outperform the perfect-foresight strategy every year during the backtest. Interestingly, the gains in yearly cumulative returns to risk seem to be higher in periods of increased market volatility.

Exhibit 7

Year	Full Replication Overlay	Total Risk Optimization 10% Turnover Overlay	Active Risk Optimization 10% Turnover Overlay	Active Risk Optimization 10% Turnover 400 Assets Overlay	S&P 500 Average Monthly Forecast Risk (Annualized,%)
1998	20.07	6.53	6.21	3.23	20.12
1999	18.04	13.97	18.10	13.64	18.19
2000	21.94	11.10	11.74	-3.17	22.46
2001	71.82	53.28	59.33	51.70	21.59
2002	99.58	97.45	94.88	97.07	26.24
2003	45.03	35.97	36.11	31.12	17.29
2004	32.61	27.65	26.03	25.39	11.23
2005	16.32	14.85	13.19	5.84	10.36
2006	28.10	28.03	26.18	23.88	10.16
2007	14.28	15.34	14.77	13.97	16.14
2008**	32.61	34.06	28.93	15.94	21.39

*Percentage change in return to forecast risk ratio of hedged long-short portfolios relative to unhedged long-short portfolios

**Performance computed over period 31/12/07 - 30/06/08

Indeed, when market volatility peaked in 2002, the unhedged portfolio saw a significant portion of its risk coming from Volatility and Momentum, with average contributions over the year to total risk from these two factors being 58% and 35%, respectively. Hedging away exposure to these two factors significantly improved the portfolio's risk-adjusted performance in 2002. In fact, the substantial improvement in risk-adjusted performance from hedging common-factor risk in 2002 can be attributed largely to the months of October and especially November of that year.

Exhibit 8

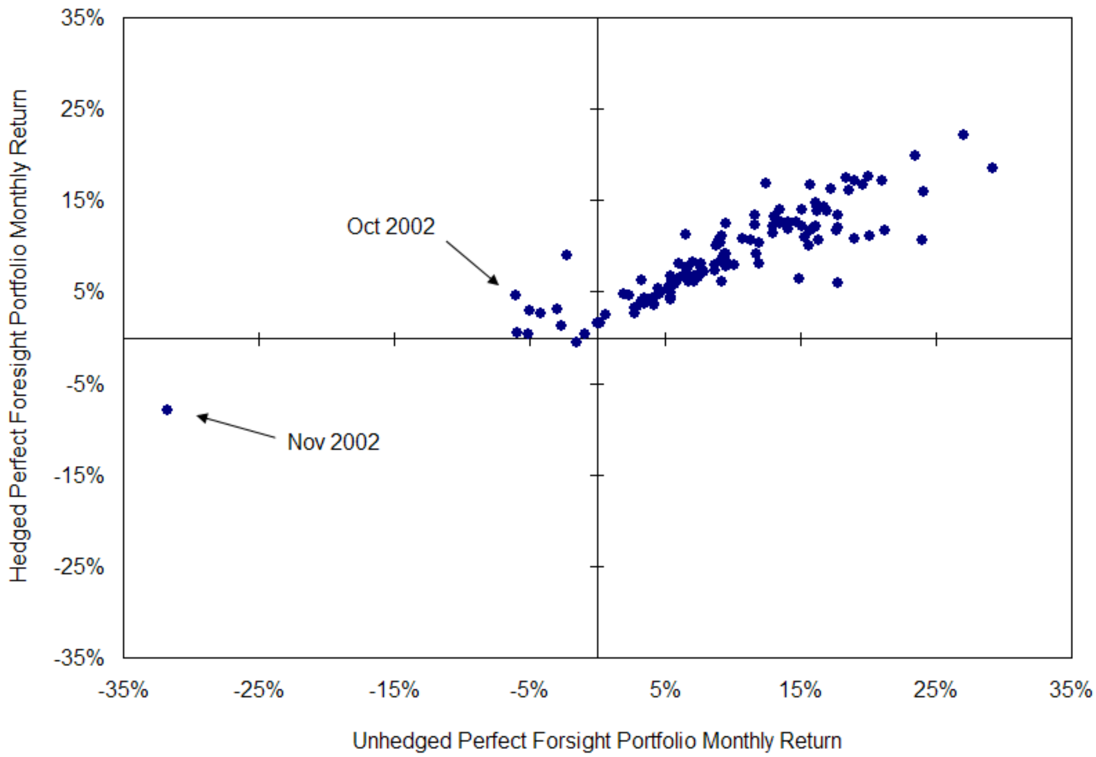

Exhibit 8 plots unhedged versus hedged portfolio returns over the entire backtesting period. This exhibit shows, despite having perfect foresight over 12-month horizons, that the manager of the unhedged long/short portfolio would still have experienced 11 different months of negative monthly returns and a maximum drawdown of approximately -35% during the period Oct - Nov 2002. Even our "super manager" may not survive losing a third of his assets over a two-month period! On the other hand, the "prudent manager" who used factor overlays to hedge her portfolio's exposure to the most volatile sources of common factor risk would have experienced only a -5% drawdown over the same two-month period. The option-like payoff structure of the hedged portfolio illustrated in this diagram is clearly most interesting.

Conclusions

We have presented alternative methods for constructing factor-replicating portfolios. We addressed the issue of implementation cost by recognizing that portfolio turnover (and therefore transaction costs) is an important parameter to control in the construction of constrained-factor portfolios. Importantly, we demonstrated that constrained-factor portfolios, with a limited number of assets and relatively low turnover, track pure factor returns reasonably well and therefore can serve as an investment instrument for factor-based hedging or to obtain beta exposure to a particular factor.

We illustrate how factor-mimicking portfolios can be applied to passive and active investment strategies. Factor-mimicking portfolios can be utilized to hedge out the unintended factor exposures of conventional benchmarks aimed at targeting a particular beta factor; this enables plan sponsors better to manage their optimal allocations to beta factor risks. Factor-mimicking portfolios can be utilized to hedge out the style exposures of active stock-picking strategies; this enables the active manager to capture pure “alpha.” In brief, the practice of active and passive investment management can be enhanced via factor-mimicking portfolios.

Appendix

In this appendix, we show that specific-risk weighted and total-risk weighted factor portfolios are the same, up to a scaling parameter. Also, we show how factor portfolio weights can be computed either through GLS regression or constrained optimization.

Defining Performance Criteria

To assess the performance of optimized replication portfolios, we need to define appropriate performance criteria reflecting the objectives of the factor replication problem, namely, maximum exposure to the targeted factor and minimum portfolio risk. This reasoning leads us to consider two reward-to-variability performance criteria, similar to Sharpe (1975), that reflect these objectives. The two performance criteria we consider are the ratio of portfolio exposure to the targeted factor divided by specific risk (J_S) and the ratio of portfolio exposure to the targeted factor divided by total risk (J_T):

$$J_S(h) = \frac{h' X_\alpha}{\left(h' D h\right)^{1/2}} \qquad J_T(h) = \frac{h' X_\alpha}{\left(h' V h\right)^{1/2}} \tag{A1.1}$$

The Specific-Risk Approach

First, we consider the specific-risk performance criterion J_S. The standard approach for constructing portfolios that maximize this criterion and have zero exposure to the remaining risk factors involves constrained long/short optimization. In this optimization, asset exposures to the targeted factor play the role of expected returns, while constraints are imposed to control portfolio exposure to the remaining risk factors. Using our general framework, the weights that maximize criterion J_S are given by the following expression:

$$h_S^* = \frac{1}{\lambda} D^{-1}\alpha = \frac{1}{\lambda} D^{-1}\left[X_\alpha - X_\sigma\left(X_\sigma' D^{-1} X_\sigma\right)^{-1}\left(X_\sigma' D^{-1} X_\alpha\right)\right] \tag{A1.2}$$

The Total-Risk Approach

Next, we consider the problem of maximising the total-risk performance criterion J_T, subject to having zero exposure to all other factors. As in the specific-risk case, this problem can be solved through constrained optimization that maximizes portfolio exposure to the targeted factor for a given level of total risk, subject to constraints on all remaining risk factors. The weights that maximize criterion J_S are as follows:

$$h_T^* = \frac{1}{\lambda} V^{-1}\alpha = \frac{1}{\lambda} V^{-1}\left[X_\alpha - X_\sigma\left(X_\sigma' V^{-1} X_\sigma\right)^{-1}\left(X_\sigma' V^{-1} X_\alpha\right)\right] \tag{A1.3}$$

Reconciling the Specific-Risk and Total-Risk Problems

By construction, the common-factor risk σ_{CF} of factor-replicating portfolios depends only on the exposure x_α of these portfolios to the targeted factor and the risk σ_α of the targeted factor. In other words, factor-replicating portfolios are not exposed to common-factor risk due to the variance and covariance of factors other than the targeted factor. As a result, we can express the common factor risk σ_{CF} of factor replicating portfolios as follows:

$$\sigma_{CF} = x_\alpha \sigma_\alpha \tag{A1.4}$$

This expression enables us to rewrite the total-risk performance criterion J_T as follows:

$$J_T(h) = \frac{h' X_\alpha}{(h'Vh)^{1/2}} = \frac{x_\alpha}{\left(\sigma_{CF}^2 + h' Dh\right)^{1/2}} = \frac{1}{\left(\frac{x_\alpha^2 \sigma_\alpha^2}{x_a^2} + \frac{h' Dh}{x_a^2}\right)^{1/2}} = \frac{1}{\left(\sigma_\alpha^2 + \frac{1}{J_S^2}\right)^{1/2}} \tag{A1.5}$$

The last equation demonstrates that performance criteria J_S and J_T, constrained to $h'X_\sigma=0$, are maximized by the same optimal portfolio, up to a scaling factor. In other words:

$$h_S^* = \gamma\, h_T^* \tag{A1.6}$$

Equivalence between Constrained Optimization and GLS Regression

Factor returns estimated through the GLS regression (1) with weights W^{-1} are given by:

$$\hat{f} = (X'W^{-1}X)^{-1} X'W^{-1}r \tag{A1.7}$$

The component of factor returns f_α corresponding to the target factor can be estimated through a two-step regression process. In the first step, we regress target-factor exposures X_α on all other factor exposures X_σ using the same weighting matrix W^{-1}:

$$X_\alpha = X_\sigma b + \alpha \tag{A1.8}$$

The residuals from this weighted-least-squares regression can be estimated as follows:

$$\hat{\alpha} = X_\alpha - X_\sigma \hat{b} = X_\alpha - X_\sigma (X_\sigma' W^{-1} X_\sigma)^{-1} \left(X_\sigma' W^{-1} X_\alpha\right) \tag{A1.9}$$

In the second step, we regress asset returns on the residuals from the first-step regression:

$$r = \hat{\alpha}\, f_\alpha + \varepsilon \tag{A1.10}$$

The estimated target-factor-return component can then be written as follows:

$$\hat{f}_\alpha = (\hat{\alpha}' W^{-1} \hat{\alpha})^{-1} \hat{\alpha}' W^{-1} r = h_\alpha' r \tag{A1.11}$$

We can see that the factor-mimicking portfolio weights h_α are also the solution to the constrained-optimization problem (4)-(5), with a scaling parameter λ equal to $\alpha' W^{-1} \alpha$:

$$h_\alpha = \frac{1}{\lambda} W^{-1} \left[X_\alpha - X_\sigma \left(X_\sigma' W^{-1} X_\sigma\right)^{-1} \left(X_\sigma' W^{-1} X_\alpha\right)\right] = (\hat{\alpha}' W^{-1} \hat{\alpha})^{-1} W^{-1} \hat{\alpha} \tag{A1.12}$$

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Endnotes

[1] Dimitris Melas is Executive Director and Raghu Suryanarayanan is Senior Associate at MSCI Barra in London. This paper was completed while Stefano Cavaglia was employed as Head of Quantitative Strategies at UBS O'Connor LLC; the views presented in this research are those of the author and not those of UBS O'Connor LLC.
[2] In equation (1), r is the $n \times 1$ vector of asset excess returns, X is the $n \times k$ matrix containing asset exposures to fundamental factors, f is the $k \times 1$ vector of factor returns, e is the $n \times 1$ vector of specific returns, n is the number of assets in the estimation universe, and k is the number of factors in the model. In equation (2), F is the $k \times k$ factor return covariance matrix, D is the $n \times n$ specific return covariance matrix, and V is the $n \times n$ asset return covariance matrix. Asset exposures to fundamental factors are represented typically by dummy variables for industry factors and cross-sectional z-scores for style factors, for example, size, value, growth, momentum. The specific-return covariance matrix D is usually assumed to be diagonal. However, some off-diagonal elements may be different from zero, for example, elements corresponding to securities issued by the same company. In our analysis. we do not need to impose any limiting assumptions on the structure of the specific-return covariance matrix.
[3] From a theoretical perspective, regression coefficients estimated through unweighted-least-squares regression are BLUE (best linear unbiased estimators) if the errors are uncorrelated with each other and with the independent variables and have equal variance. However, if the errors have different variance, then weighted-least-squares regression using the inverse of the error variances as weights leads to regression coefficients that are BLUE.
[4] For further details on the full replication approach, see Grinold and Khan [1995], page 74.
[5] For a suitably chosen value of the scaling parameter λ.
[6] Institutional constraints limit portfolio leverage for certain regulated mutual funds to 2:1. For detailed analysis of the impact of leverage constraints on portfolio efficiency, see Melas and Suryanarayanan [2008].
[7] For example, the 10/40 rule under the UCITS regulation in Europe restricts the amount of capital that a fund can allocate to a single asset to less than 10% and the total amount allocated to assets above 5% to less than 40% of the net asset value of the portfolio.
[8] In addition to standard transaction costs (commission and market impact) that are a linear or power function of traded volume, there are other transaction costs, for example, ticket costs and custody fees, that are a function of the number of securities traded or held in the portfolio.
[9] In the numerical simulations, we used Matlab and the Barra Open Optimizer.
[10] The statistics reported in Exhibit 2 highlight the performance of different replication methods over monthly investment horizons. However, in certain hedging applications, investors may also wish to ensure that the hedging portfolios over shorter horizons track the underlying factor returns well. In order to assess tracking error over daily investment horizons, we use data from the Barra US Equity Model (USE3) to assess the daily performance of different factor-replicating methods during the recent market turmoil of August 2007. The analysis, which is available on request from the authors, suggests that our broad conclusions remain unchanged at the daily frequency.
[11] It may seem surprising at first sight that in all cases leverage fluctuates around 100%. This is due to the fact that factor portfolios maintain unit exposure to the target factor. In order to illustrate this point analytically, we assume that all assets have the same specific risk. Then optimal weights of otherwise unconstrained optimized replication portfolios may be expressed as follows:

$$h_i^* = \frac{1}{\lambda}\frac{\alpha_i}{\delta^2} \qquad (9)$$

Here, δ is the specific risk of asset i, and α_i is the residual from the regression of target factor exposures on exposures to all the other factors. So, the scaling factor λ is given by:

$$\sum_{i=1}^{n} h_i^* X_{\alpha,i} = 1 \Rightarrow \lambda = \sum_{i=1}^{n} \frac{\alpha_i X_{\alpha,i}}{\delta^2} \qquad (10)$$

Then, for large number of assets n, we can compute factor portfolio leverage as follows:

$$L = \sum_{i=1}^{n} \left| h_i^* \right| = \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{|\alpha_i|}{\delta^2}}{\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha_i X_{\alpha,i}}{\delta^2}} \approx \frac{E[|\alpha|]}{E[\alpha^2]} \approx \frac{\sigma\sqrt{\frac{2}{\pi}}}{\sigma^2} = \frac{2}{\sigma}\sqrt{\frac{1}{2\pi}} \qquad (11)$$

Here, we assumed that the residuals are approximately normally distributed across assets, with zero mean and σ standard deviation. Empirically, we find that σ is equal to 0.78 on average for the Value and Momentum factors, which leads to a leverage of 102%.

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