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Stock valuation along a Markov chain

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Abstract

A novel dividend valuation model is put forward by using a Markov chain. The valuation procedure turns out to be very simple, since it requires the solution of a system of linear equations. The dividend valuation model is in accordance with the empirical evidence whereby dividend-price ratios can change as time proceeds. Moreover, it offers fresh insights into previous dividend valuation models.

Keywords: Finance; Valuation; Markov processes

1. Introduction

The dividend valuation model [1] is a basic tool in financial analysis to calculate the value of a firm. It takes the form of a series and usually relies on a simplifying assumption about the dividend dynamics. The seminal and most popular one is that by Gordon and Shapiro [2], whereby dividends grow at a constant rate. However, as remarked by Miller and Modigliani [3, p. 421], this case is more of a theoretical interest than an operational use. Other assumptions and the resulting valuation models are surveyed in Farrell [4, Chapters 6,7].

A more recent simplifying assumption extending that by Gordon and Shapiro [2] is made by Hurley and Johnson [5,6] and Yao [7], who represent the

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dynamics of the dividend growth rate as a sequence of independent, identically distributed, discrete random variables. When empirically tested by their proponents, the resulting valuation models fit actual prices, thus displaying explanatory power.

In this paper, a general treatment is provided to the valuation problem in which the dividend growth rate is a discrete variable. To do so, a state of a stationary Markov chain is attached to each feasible value of the dividend growth rate. On deriving an existence condition, the valuation problem is turned into a system of linear equations, with the unknowns being price–dividend ratios, each corresponding to a different state of the Markov chain. This procedure holds also when the above-mentioned assumption of independent, identically distributed, random variables is relaxed. No sum of a series has to be calculated. Thus, considering a more general valuation setting results in a simpler solution procedure. Moreover, fresh insights are taken into the valuation models by Hurley and Johnson [5,6] and Yao [7].

As long as the dividend growth rates are independent, identically distributed, discrete random variables as in Hurley and Johnson [5,6] and Yao [7], the same dividend–price ratio is attached to each state of the Markov chain. In contrast, when such an assumption is relaxed, a different dividend-price ratio is attached to each state of the Markov chain. Since actual dividend-price ratios are known to vary through time, our model is likely to come closer to reality.

The paper is organized as follows. The valuation problem is stated in Section 2 and dealt with analytically in Section 3. Discussion and conclusion take place in Section 4.

2. Problem statement

Consider an efficient stock market, i.e. a market in which stock prices reflect all available information [8]. Suppose that investors have homogeneous expectations. Let t be the time at which valuation takes place. Moreover, let k be a time index, p represent a stock price, d denote the corresponding dividend, r be equal to one plus the required rate of return. For the market to be efficient, price p must obey the equation

$$p_k = \frac{E_k[d_{k+1} + p_{k+1}]}{r}, \tag{1}$$

where E_k denotes a conditional expected value. As shown by Samuelson [9], when it exists, the solution to Eq. (1) is given by

$$p_t = \sum_{i=1}^{+\infty} \frac{E_t[d_{t+i}]}{r^i}. \tag{2}$$

Remark 1. When deriving (2), one has to assume that

$$\lim_{i\to+\infty} \frac{E_t[p_{t+i}]}{r^i} = 0. \tag{3}$$

As shown by Blanchard and Watson [10], if the asymptotic condition (3) is not imposed, there can exist other solutions to Eq. (1), embodying a speculative bubble. However, those solutions are not considered in the sequel.

Remark 2. According to the empirical evidence, surveyed for instance, by Campbell et al. [11, Chapter 7], the required rate of return varies over time. Unfortunately, for the time being this seems to be much easier to ascertain *ex-post* than to model *ex-ante*. This is why discounting at a constant rate is a usual practice in financial analysis, as reported by Farrell [4, Chapters 6 and 7]. According to financial theory, the use of a constant rate can be in contrast with the stock market equilibrium, as explained in Blanchard and Fischer [12, p. 294]. However, if investors follow a buy and hold policy, discounting at a constant rate can be consistent with the stock market equilibrium, as shown in Buzzacchi and Ghezzi [13].

In our setting, dividend d obeys the difference equation

$$d_{k+1} = g_{k+1} d_k \quad k = t, t+1, \ldots, \tag{4}$$

where the growth factor g can take either of the two values g^1, g^2 with $g^1, g^2 \geqslant 0$. The dynamics of g takes place along a stationary two state Markov chain (see [14]), where π_{ij} is the conditional probability of $g_{k+1} = g^j$ whenever $g_k = g^i$, with $\pi_{i1} + \pi_{i2} = 1$. Reference to a two state Markov chain is made to enhance comprehension; the extension to an n state Markov chain is straightforward and made in the sequel.

The probabilities x_k^i that state i will occur at time k obey the difference equations

$$\begin{aligned} x_{k+1}^1 &= \pi_{11} x_k^1 + \pi_{21} x_k^2, \\ x_{k+1}^2 &= \pi_{12} x_k^1 + \pi_{22} x_k^2 \quad k = t, t+1, \ldots, \end{aligned} \tag{5}$$

where the initial condition is either $x_t^1 = 1, x_t^2 = 0$ when $g_t = g^1$ or $x_t^1 = 0, x_t^2 = 1$ when $g_t = g^2$. Since

$$E_t[d_{t+i}] = E_t\left[\prod_{j=1}^{i} g_{t+j}\right] d_t$$

each initial condition results in a different set of expectations.

The valuation problem under scrutiny is to ascertain how (2) specializes owing to (4). This is shown in next section, where the case of an n state Markov chain is examined as well.

3. Problem solution

Remark 3. Using (4) within (2) yields

$$p_t = \sum_{i=1}^{+\infty} \frac{E_t\left[\prod_{j=1}^{i} g_{t+j} d_t\right]}{r^i} = \left(\sum_{i=1}^{+\infty} \frac{E_t\left[\prod_{j=1}^{i} g_{t+j}\right]}{r^i}\right) d_t.$$

Since the expected values in the former factor do depend only on g_t, the valuation formula (2) takes the form $p_t = \varphi(g_t)d_t$. Since the Markov chain has only two states, g_t and hence $\varphi(g_t)$ can take only two values, namely $\varphi(g^1)$ and $\varphi(g^2)$.

Consider a two state Markov chain and let

$$\bar{g} = \max(\pi_{11}g^1 + \pi_{12}g^2; \pi_{21}g^1 + \pi_{22}g^2) \tag{6}$$

be the largest one step conditional expectation on the dividend growth rate.

Proposition 1. *If $\bar{g} < r$, the series* (2) *converges and meets the asymptotic condition* (3).

Proof. First we prove that (6) entails

$$E_t\left[\prod_{j=1}^{i} g_{t+j}\right] \leqslant (\bar{g})^i.$$

In doing so, attention is confined to the case $i = 2$; since g_k is a Markov random variable, the extension to the case $i > 2$ is straightforward. Recall that $E_t[g_{t+1}g_{t+2}] = E_t[g_{t+1}E_{t+1}[g_{t+2}]]$ and observe that (6) entails $E_{t+1}[g_{t+2}] \leqslant \bar{g}$. As a consequence, $E_t[g_{t+1}E_{t+1}[g_{t+2}]] \leqslant E_t[g_{t+1}\bar{g}]$. Since (6) entails $E_t[g_{t+1}] \leqslant \bar{g}$ as well, we have $E_t[g_{t+1}E_{t+1}[g_{t+2}]] \leqslant (\bar{g})^2$.Now we prove that if $\bar{g} < r$, the series (2) converges. The inequality

$$E_t\left[\prod_{j=1}^{i} g_{t+j}\right] \leqslant (\bar{g})^i$$

implies that

$$0 \leqslant \sum_{i=1}^{+\infty} \frac{E_t\left[\prod_{j=1}^{i} g_{t+j}\right]}{r^i} \leqslant \sum_{i=1}^{+\infty} \left(\frac{\bar{g}}{r}\right)^i.$$

Since $\bar{g} < r$, $\sum_{i=1}^{+\infty}(\bar{g}/r)^i$ is convergent; since

$$E_t\left[\prod_{j=1}^{i} g_{t+j}\right]\Big/ r^i$$

is positive, the series (2) is convergent as well.

Finally, we prove that if the series (2) converges, the asymptotic condition (3) is met. Remark 3 implies that $p_{t+i} = \varphi(g_{t+i})d_{t+i}$. Let $\bar{\varphi} = \max(\varphi(g^1), \varphi(g^2))$. Then we have $0 \leqslant E_t[p_{t+i}] \leqslant \bar{\varphi}E_t[d_{t+i}]$, which can be rewritten as $0 \leqslant (E_t[p_{t+i}]/r^i) \leqslant \bar{\varphi}(E_t[d_{t+i}]/r^i)$. Since (2) converges, we have

$$\lim_{i\to+\infty} \frac{E_t[d_{t+i}]}{r^i} = 0$$

and hence

$$\lim_{i\to+\infty} \frac{E_t[p_{t+i}]}{r^i} = 0. \qquad \square$$

Proposition 2. *If $\bar{g} < r$, the pair $(\varphi(g^1), \varphi(g^2))$ is the unique and nonnegative solution to the linear system*

$$\begin{aligned} \varphi(g^1) &= \pi_{11}\frac{\varphi(g^1)g^1 + g^1}{r} + \pi_{12}\frac{\varphi(g^2)g^2 + g^2}{r}, \\ \varphi(g^2) &= \pi_{21}\frac{\varphi(g^1)g^1 + g^1}{r} + \pi_{22}\frac{\varphi(g^2)g^2 + g^2}{r}. \end{aligned} \tag{7}$$

Proof. Substituting $p_t = \varphi(g_t)d_t$ and (4) into (1) with $g_t = g^1$ yields the former of Eq. (7), while substituting $p_t = \varphi(g_t)d_t$ and (4) into (1) with $g_t = g^2$ yields the latter of Eq. (7). Since the series (2) must converge owing to Proposition 1, the linear system (7) must possess a unique solution $(\varphi(g^1), \varphi(g^2))$, which is non-negative owing to the assumption $g^1, g^2 \geqslant 0$. $\square$

For any given d_t the forecasts $E_t[d_{t+i}]$ made when $g_t = g^1$ are in general other than those made when $g_t = g^2$. Proposition 2 takes this into account and states that different price–dividend ratios $\varphi(g_t)$ have to be attached to the stock when $g_t = g^1$ and $g_t = g^2$. As we will argue later, this captures an important feature of the working of a stock market.

Now simplify notation as follows: $\pi_{11} = \pi_1$, $\pi_{12} = 1 - \pi_1$, $\pi_{21} = 1 - \pi_2$, $\pi_{22} = \pi_2$. According to Proposition 1, if $\bar{g} = \max(\pi_1 g^1 + (1-\pi_1)g^2; (1-\pi_2)g^1 + \pi_2 g^2) < r$, the series (2) is convergent. It is readily ascertained that the unique and nonnegative solution to system (7) is

$$\varphi(g^1) = \frac{r(\pi_1 g^1 + (1-\pi_1)g^2) + (1-\pi_1-\pi_2)g^1 g^2}{r(r-\pi_1 g^1 - \pi_2 g^2) - (1-\pi_1-\pi_2)g^1 g^2},$$
$$\varphi(g^2) = \frac{r((1-\pi_2)g^1 + \pi_2 g^2) + (1-\pi_1-\pi_2)g^1 g^2}{r(r-\pi_1 g^1 - \pi_2 g^2) - (1-\pi_1-\pi_2)g^1 g^2}.$$

Remark 4. Note that $\varphi(g^1) < \varphi(g^2)$ iff $\pi_1 g^1 + (1-\pi_1)g^2 < (1-\pi_2)g^1 + \pi_2 g^2$. In other words, the largest price–dividend ratio is attached to the state displaying the largest one step expectation on the dividend growth rate. However, it can be numerically ascertained that such a result does not carry over to a setting in which the growth factor g can take one out of $n > 2$ values.

Assume now as in Hurley and Johnson [5,6] that for any given d_t the forecasts $E_t[d_{t+i}]$ are the same irrespective of whether $g_t = g^1$ or $g_t = g^2$. We have accordingly $\pi_1 = \pi$ and $\pi_2 = 1-\pi$ so that the solution to system (7) becomes

$$\varphi(g^1) = \varphi(g^2) = \frac{\pi g^1 + (1-\pi)g^2}{r - \pi g^1 - (1-\pi)g^2},$$

thus implying that the same dividend–price ratio is attached to each state of the Markov chain. *Mutatis mutandis* the same feature is shared by the dividend valuation models by Yao [7] and Hurley and Johnson [5,6].

Let us turn our attention to the general case of an n state Markov chain where the growth factor g can take one out of the n values $g^1, g^2, \ldots, g^n$ with $g^1, g^2, \ldots, g^n \geqslant 0$. Let π_{ij} be the probability of $g_{k+1} = g^j$ whenever $g_k = g^i$ with $\sum_{j=1}^{n} \pi_{ij} = 1$. Eqs. (5) and (6) can then be rewritten as:

$$x^j_{k+1} = \sum_{i=1}^{n} \pi_{ij} x^i_k, \quad j = 1, 2, \ldots, n, \tag{8}$$

$$\bar{g} = \max\left(\sum_{j=1}^{n} \pi_{1j} g^j; \sum_{j=1}^{n} \pi_{2j} g^j; \ldots; \sum_{j=1}^{n} \pi_{nj} g^j\right). \tag{9}$$

Proposition 3. *If $\bar{g} < r$ with $\bar{g}$ given by* (9), *the series* (2) *converges. Moreover, the valuation formula* (2) *takes the form $p_t = \varphi(g_t)d_t$, where $(\varphi(g^1), \varphi(g^2), \ldots, \varphi(g^n))$ is the unique and nonnegative solution to the linear system*

$$\varphi(g^i) = \sum_{j=1}^{n} \pi_{ij} \frac{\varphi(g^j)g^j + g^j}{r} \quad i = 1, 2, \ldots, n. \tag{10}$$

Proof. Remark 3 and Proposition 1 can be restated with trivial changes. As a consequence, Proposition 2 can be restated too. Substituting $p_t = \varphi(g_t)d_t$ and

(4) into (1) with $g_t = g^i$ yields the linear system (10). Since the series (2) must converge, the linear system (10) must possess a unique solution $(\varphi(g^1), \varphi(g^2), \dots, \varphi(g^n))$, which is nonnegative owing to the assumption $g^1, g^2, \dots, g^n \geqslant 0$. $\square$

The financial implications of Proposition 2 carry over to Proposition 3. For any given given d_t, the forecasts $E_t[d_{t+i}]$ made when $g_t = g^i$ are in general other than those made when $g_t \neq g^i$. As a consequence, each possible value of g_t commands a different price–dividend ratio $\varphi(g_t)$.

In view of Proposition 3 the case of bankruptcy can be easily dealt with. Consider a three state Markov chain, where $g^1 = 0$, $g^2 > 0$, $g^3 > 0$ and $\pi_{11} = 1$, $\pi_{12} = \pi_{13} = 0$, $\pi_{22} = \pi_{32} = \pi_2$, $\pi_{23} = \pi_{33} = \pi_3$, $\pi_{21} = \pi_{31} = 1 - \pi_2 - \pi_3 > 0$. In other words, for any given d_t the forecasts $E_t[d_{t+i}]$ are the same irrespective of whether $g_t = g^2$ or $g_t = g^3$. In contrast, if state 1 occurs, the firm goes bankrupt without paying any dividend. Let

$$\bar{x}_i = \lim_{k \to +\infty} x_k^i$$

be the asymptotic probability that state i will occur. Eq. (8) imply that the asymptotic probabilities are $\bar{x}_1 = 1, \bar{x}_2 = \bar{x}_3 = 0$. In other words, state 1 is absorbing and the firm is doomed to fail. According to Proposition 3, if $\bar{g} = \pi_2 g^2 + \pi_3 g^3 < r$, the series (2) is convergent. It is readily ascertained that the unique and nonnegative solution to system (10) is

$$\varphi(0) = 0, \quad \varphi(g^2) = \varphi(g^3) = \frac{\pi_2 g^2 + \pi_3 g^3}{r - \pi_2 g^2 - \pi_3 g^3}.$$

If $g_t = g^2$ or $g_t = g^3$, then $1/(1 - \pi_2 - \pi_3)$ is the mean time before bankruptcy occurs (see, e.g., Theorem 2 in Luenberger [15, p. 240]). This case is dealt with by Hurley and Johnson [5,6] and extended by Yao [7]. Our analysis complements both works and points out by using a Markov chain that bankruptcy occurs with probability one. Note that the mean time before bankruptcy occurs can be a helpful reference when estimating the probabilities π_{ij}.

4. Discussion and conclusion

According to the valuation models by Gordon and Shapiro [2], Hurley and Johnson [5,6] and Yao [7] each stock is assigned a peculiar dividend–price ratio, which cannot vary as time proceeds. This is so, because the forecasts on each dividend growth rate are made once and for all. As a consequence, time cannot convey information and update such forecasts, thus calling for changes in the dividend–price ratios.

However, actual dividend–price ratios can change through time, as shown, for instance, in the work by Bernstein [16], which is concerned with the dynamics of the American stock market over the last century.

Changes in the dividend–price ratios can stem from changes in the forecasts or the required rates of return. Our valuation setting extends the settings by Gordon and Shapiro [2], Hurley and Johnson [5,6] and Yao [7] by placing more emphasis on how forecasts are updated. It is based on a stationary Markov chain, where each state corresponds to a feasible value of a dividend growth rate. In the most general case, future prospects are state dependent so that forecasts are implicitly updated when moving from a state to another. This explains why a different dividend–price ratio is associated to each state by the resulting valuation model. If a handy existence condition is met, the dividend–price ratios can be conveniently calculated by solving a system of linear equations.

In our opinion, our valuation model enables the reader to grasp the rationale underlying a stock market without any involvement in misleading analytical subtleties. Discrete random variables and linear equations are both easy to handle. Moreover, our valuation model is amenable to the operational use. Needless to say, if reliable estimates of the transition probabilities are to be obtained, the range of the dividend growth rate has to include few representative values. In other words, the Markov chain has to include few states. Finally, our valuation model provides some fresh insights into the analyses by Hurley and Johnson [5,6] and Yao [7]. More specifically, it makes clear that bankruptcy cannot be escaped in the long run and that the mean time to bankruptcy can be readily computed. This provides a helpful check when estimating the transition probabilities.

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