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Stochastic dividend discount model: covariance of random stock prices

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Abstract

The price of common stocks, defined as the sum of all future discounted dividends, is at the heart of both the dividend discount models (DDM) and the stochastic DDM (SDDM). Gordon and Shapiro (Manag Sci 3:102–110 1956) assume a deterministic and constant dividends' growth rate, whereas Hurley and Johnson (Financ Anal J 4:50–54 1994, J Portf Manag 25(1)27–31 1998) and Yao (J Portf Manag 23(4)99–103 1997) introduce randomness by letting the growth rate be a finite-state random variable and random dividends behave in a Markovian fashion. In this second case expected stock price is determined, but what if higher-order moments are needed? In order to address a number of financial topics, the present contribution presents an explicit formula for the covariance between (possibly) correlated stock prices.

Keywords Stock valuation · Dividend discount model · Markov chain · Financial risk management

JEL Classification G11 · G12 · G32

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1 Introduction

Dividend discount models (DDM in the following) are the first attempt to find a financially correct pricing formula for common stocks. DDM date back to the seminal works of Williams (1938) and Gordon and Shapiro (1956) where, in an entirely deterministic setting, the price of a common stock is obtained by discounting all future dividends per share by means of a rate reflecting the riskiness of a company. A first attempt to relax the assumption of a non-stochastic framework is due to Hurley and Johnson (1994, 1998), and Yao (1997). These authors introduce randomness, paving the road to Stochastic DDM (SDDM for short), by assuming that dividends' growth rate is described by a finite-state random variable.

As a further improvement, Hurley (2013) assumes a continuous random variable for the growth rates, and D'Amico (2013) evaluates a stock when the dividend growth rate follows a finite state discrete time semi-Markov chain. These contributions aim at obtaining expected values of stock prices.

However, when dealing with practical financial problems such as, for instance, portfolio selection and risk management, expected values are not enough. A more reliable analysis requires, along means which measure performance, at least standard deviations that - as explained by Markowitz - represent risk (1952). Ingersoll (1987) exemplifies a portfolio analysis considering also skewness. It is then obvious that higher moments for prices and stock returns are required to make models as close as possible to markets real behaviour. A first answer to this issue in the SDDM framework has been provided by Agosto and Moretto (2015), who propose a closed-form expression for the variance of random stock prices. This piece of information allows, for instance, to find the *value-at-risk* of a stock under the well known delta-normal approach. Under the assumption of a dividend growth rate following a discrete time semi-Markov chain, D'Amico (2017) provides sufficient conditions under which expected prices and variances are finite, and equations describing the first and second order price-dividend ratios. Nevertheless, in some instances, such as portfolio selection, variance has to be paired with covariance. To fill this gap, the current paper provides an explicit formula for the covariance between random stock prices with (possibly) correlated random growth rates. A closed-formula for the covariance permits also to study its (monotonic) dependence on the covariance between growth rates.

The structure of paper is the following: Section 2 describes DDM and SDDM; Section 3 presents the formula for the covariance between stock prices; Section 4 applies this formula to the portfolio selection problem and illustrates a numerical example using real market data; Section 5 concludes.

2 From DDM to SDDM

Let d_j be the dividends per share paid by some company at time $j = 1, 2, \ldots$, and d_0 the current ones. The assumption that dividends grow at the constant geometric rate

$g > -1$ leads to $d_j = d_0(1+g)^j$. According to DDM, the current stock price is the sum of all dividends per share discounted at the rate $k > g$:

$$P_0 = \sum_{j=1}^{+\infty} \frac{d_j}{(1+k)^j} = \frac{d_0(1+g)}{k-g}.$$

Going further, Hurley and Johnson (1998) let the growth rate be the finite-state random variable

$$\tilde{g} = \begin{cases} \text{growth rates} & g_1, \ \dots, \ g_n \\ \text{probabilities} & p_1, \ \dots, \ p_n, \end{cases} \tag{1}$$

where $-1 < g_1 < \cdots < g_n$, $p_i = \mathbb{P}\left[\tilde{g} = g_i\right] > 0$, $i = 1, \dots, n$, and $\sum_{i=1}^{n} p_i = 1$. Let $\bar{g}$ and $\mathrm{Var}\left[\tilde{g}\right]$ be, respectively, its mean and variance. The sequence of random dividends $\tilde{d}_j$ is Markovian, since it satisfies the recursive equation $\tilde{d}_{j+1} = \tilde{d}_j\,(1+\tilde{g})$. According to the SDDM, the current stock price is described by the random variable

$$\tilde{P}_0 = \sum_{j=1}^{+\infty} \frac{\tilde{d}_j}{(1+k)^j}. \tag{2}$$

If $k > \bar{g}$ then the expected value of $\tilde{P}_0$ is

$$\bar{P}_0 = \frac{d_0(1+\bar{g})}{k-\bar{g}}. \tag{3}$$

Agosto and Moretto (2015) determine the variance of the stock price:

$$\mathrm{Var}\left[\tilde{P}_0\right] = \frac{\mathrm{Var}\left[\tilde{g}\right]}{(1+k)^2 - (1+\bar{g})^2 - \mathrm{Var}\left[\tilde{g}\right]} \times \frac{(1+k)^2}{(1+\bar{g})^2} \times \bar{P}_0^2. \tag{4}$$

$\mathrm{Var}\left[\tilde{P}_0\right]$ exists and is non-negative if $\mathrm{Var}\left[\tilde{g}\right] < (1+k)^2 - (1+\bar{g})^2$. In the following section this formula is generalized by that of the covariance between two stock prices.

3 Enlarging the scene: a formula for the covariance between stock prices

Consider two companies, say A and B, with random growth rates $\tilde{g}_m$, $m = A, B$, represented by the following random variables:

$$\tilde{g}_m = \begin{cases} \text{growth rates} & g_{m1}, \ \dots, \ g_{mn} \\ \text{probabilities} & p_{m1}, \ \dots, \ p_{mn}, \end{cases} \tag{5}$$

being, as in (1), $-1 < g_{m1} < g_{m2} < \ldots < g_{mn}$. Their joint distribution is

$$\pi_{cd} = \mathbb{P}\left[(\tilde{g}_A = g_{Ac}) \cap (\tilde{g}_B = g_{Bd})\right] \geq 0 \qquad \text{with} \qquad \sum_{c=1}^{n}\sum_{d=1}^{n} \pi_{cd} = 1,$$

for $c, d = 1, 2, \ldots, n$. Let $\bar{g}_m$, $\mathrm{Var}\left[\tilde{g}_m\right]$, and $\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]$ be, respectively, the mean and the variance of $\tilde{g}_m$, and the covariance between $\tilde{g}_A$ and $\tilde{g}_B$. Assume also that $k_m > \bar{g}_m$, being k_m the discount rate for company $m = A, B$. According to (2) and (3), the current stock price of company m and its expected value are, respectively,

$$\tilde{P}_{m0} = \sum_{j=1}^{+\infty} \frac{\tilde{d}_{mj}}{(1+k_m)^j} \qquad \text{and} \qquad \bar{P}_{m0} = \frac{d_{m0}(1+\bar{g}_m)}{k_m - \bar{g}_m},$$

being d_{m0} their current dividend.

Proposition 1 *Assume that $\bar{g}_m < k_m$, $m = A, B$, and*

$$\left|(1+\bar{g}_A)(1+\bar{g}_B) + Cov\left[\tilde{g}_A, \tilde{g}_B\right]\right| < (1+k_A)(1+k_B). \tag{6}$$

Then the covariance between stock prices of A and B at time 0 is finite and has the explicit formula

$$\begin{aligned} &Cov\left[\tilde{P}_{A0}, \tilde{P}_{B0}\right] \\ &= \frac{Cov\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+k_A)(1+k_B) - (1+\bar{g}_A)(1+\bar{g}_B) - Cov\left[\tilde{g}_A, \tilde{g}_B\right]} \times \frac{(1+k_A)(1+k_B)}{(1+\bar{g}_A)(1+\bar{g}_B)} \times \bar{P}_{A0}\bar{P}_{B0}. \end{aligned} \tag{7}$$

Proof The calculation of the covariance between $\tilde{P}_{A0}$ and $\tilde{P}_{B0}$ requires to compute the expected value

$$\mathbb{E}\left[\tilde{P}_{A0}\tilde{P}_{B0}\right] = \sum_{j=1}^{+\infty}\sum_{p=1}^{+\infty} \frac{\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right]}{(1+k_A)^j(1+k_B)^p}. \tag{8}$$

To determine $\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right]$, two cases need to be considered, namely, $j \le p$ and $j > p$. If $j \le p$ let

$$d\left(s_{11}, s_{12}, s_{13}, \ldots, s_{nn}\right) = d_{A0}d_{B0} \prod_{c,d=1}^{n} (1+g_{Ac})^{s_{cd}} (1+g_{Bd})^{s_{cd}}$$

denote the possible outcomes of time j product $\tilde{d}_{Aj}\tilde{d}_{Bj}$, when $\tilde{d}_{Aj}$ has grown in j steps s_{cd} times at rate g_{Ac} while, at the same time, $\tilde{d}_{Bj}$ has grown in j steps s_{cd} times at rate g_{Bd}, with $\sum_{c,d=1}^{n} s_{cd} = j$. Probabilities of such outcomes are

$$\binom{j}{s_{11}, s_{12}, s_{13} \ldots, s_{nn}} \pi_{cd}^{s_{cd}}. \tag{9}$$

On the other hand, let $z_d = \sum_{c=1}^{n} s_{cd}$ be the number of times in which dividend $\tilde{d}_{Bj}$ has grown at rate g_{Bd} in the first j steps, regardless of the behavior of $\tilde{d}_{Aj}$. If, overall, $\tilde{d}_{Bp}$ grows w_d times at rate g_{Bd}, from $j+1$ to p it can grow at the same rate at most $r_d = \max\{w_d - z_d, 0\}$ times. This allows to define a further random variable $\tilde{d}_{Bjp}$

that represents the behavior of B's dividends between $j+1$ and p. Possible outcomes of $\tilde{d}_{Bjp}$ are

$$d\left(r_1,\ldots,r_n\right)=\prod_{d=1}^{n}\left(1+g_{Bd}\right)^{r_d},$$

with $\sum_{d=1}^{n} r_d = p - j$. The corresponding probabilities are

$$\binom{p-j}{r_1,\ldots,r_n} p_{Bd}^{r_d}.$$

The Markovian structure of the dividends' sequence leads to

$$\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right]=d_{A0}d_{B0}\times\underbrace{\left(\sum_{s_{11}+\ldots+s_{nn}=j}\prod_{c,d=1}^{n}\left(1+g_{Ac}\right)^{s_{cd}}\left(1+g_{Bd}\right)^{s_{cd}}\binom{j}{s_{11},\ldots,s_{nn}}\pi_{cd}^{s_{cd}}\right)}_{(*)}$$
$$\times\underbrace{\left(\sum_{r_1+\ldots+r_n=p-j}\prod_{d=1}^{n}\left(1+g_{Bd}\right)^{r_d}\binom{p-j}{r_1,\ldots,r_n}p_{Bd}^{r_d}\right)}_{(**)}. \tag{10}$$

The other case, $j > p$, is analogous to the preceding if two changes are done. The first regards sum $(*)$ in (10), where p replaces j in $s_{11}+\ldots+s_{nn}=j$. The second change relates to sum $(**)$ in (10), where g_{Bd} and p_{Bd} are replaced by g_{Ac} and p_{Ac}, and r_d is replaced by r_c, which is defined analogously to r_d. Standard algebra yields

$$\begin{aligned}\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right]&=d_{A0}d_{B0}\left(\sum_{c=1}^{n}\sum_{d=1}^{n}(1+g_{Ac})(1+g_{Bd})\pi_{cd}\right)^{j}\left(\sum_{d=1}^{n}(1+g_{Bd})p_{Bd}\right)^{p-j}\\&=d_{A0}d_{B0}\left[(1+\bar{g}_A)(1+\bar{g}_B)+\mathrm{Cov}[\tilde{g}_A,\tilde{g}_B]\right]^{j}(1+\bar{g}_B)^{p-j},\end{aligned}$$

for $j \le p$ whereas, if $j > p$, then

$$\begin{aligned}\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right]&=d_{A0}d_{B0}\left(\sum_{c=1}^{n}\sum_{d=1}^{n}(1+g_{Ac})(1+g_{Bd})\pi_{cd}\right)^{p}\left(\sum_{c=1}^{n}(1+g_{Ac})p_{Ac}\right)^{j-p}\\&=d_{A0}d_{B0}\left[(1+\bar{g}_A)(1+\bar{g}_B)+\mathrm{Cov}[\tilde{g}_A,\tilde{g}_B]\right]^{p}(1+\bar{g}_A)^{j-p}.\end{aligned}$$

If

$$G_m=\bar{g}_m+\frac{\mathrm{Cov}\left[\tilde{g}_m,\tilde{g}_l\right]}{1+\bar{g}_l},\quad m,l=A,B,\ m\neq l, \tag{11}$$

is seen as a risk-adjusted growth rate, then the following expression

$$(1+G_m)(1+\bar{g}_l)=(1+\bar{g}_A)(1+\bar{g}_B)+\mathrm{Cov}\left[\tilde{g}_A,\tilde{g}_B\right],\quad m,l=A,B,\ m\neq l,$$

holds, yielding

$$\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right] = \begin{cases} d_{A0}d_{B0}\left(1+G_A\right)^j\left(1+\bar{g}_B\right)^p & j \le p \\ d_{A0}d_{B0}\left(1+G_B\right)^p\left(1+\bar{g}_A\right)^j & j > p. \end{cases} \tag{12}$$

Finally, substituting (12) in (8), leads to simple (but tedious) calculations that can be found in Appendix A. At last, formula (7) results once condition (6) holds. See Appendix A for more details. □

Clearly, when $A = B$ formula (4) is recovered. Note that condition (6) ensures that the sign of the covariance between stock prices is the same of that of the covariance between growth rates. Moreover, covariance between stock prices is zero if dividends' growth rates are not correlated, and monotonically increases in the covariance between growth rates; finally, it diverges whenever the latter approaches $(1+k_A)(1+k_B) - (1+\bar{g}_A)(1+\bar{g}_B)$. Formula (7) turns out to be useful in many applications, one of which is described in Section 4.

Proposition 2 *Assume that $\bar{g}_m < k_m$, $m = A, B$, and*

$$\left|(1+\bar{g}_A)(1+\bar{g}_B) + Cov\left[\tilde{g}_A, \tilde{g}_B\right]\right| < \prod_{m=A,B}(1+k_m). \tag{13}$$

Then the covariance between stock prices of A and B at time t is finite and has the explicit formula

$$Cov\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right] = \bar{P}_{A0}\bar{P}_{B0}(1+\bar{g}_A)^t(1+\bar{g}_B)^t \times \left(\frac{(1+k_A)(1+k_B)-(1+\bar{g}_A)(1+\bar{g}_B)}{(1+k_A)(1+k_B)-(1+\bar{g}_A)(1+\bar{g}_B)-Cov\left[\tilde{g}_A, \tilde{g}_B\right]}\left(1+\frac{Cov\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+\bar{g}_A)(1+\bar{g}_B)}\right)^{t+1} - 1\right). \tag{14}$$

Remark 1 Clearly, for $t = 0$ we recover formula (7) from (14). Further some simple algebra yields

$$\mathrm{Cov}\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right] = \mathrm{Cov}\left[\tilde{P}_{A0}, \tilde{P}_{B0}\right] \times \frac{(1+\bar{g}_A)^t(1+\bar{g}_B)^t}{(1+k_A)(1+k_B)\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]} \times \left(((1+k_A)(1+k_B)-(1+\bar{g}_A)(1+\bar{g}_B))\left(\left(1+\frac{\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+\bar{g}_A)(1+\bar{g}_B)}\right)^{t+1} - 1\right) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right), \tag{15}$$

where we point out the relation between prices covariances at time 0 and t.

Observe that the sign of covariances between stock prices (which refers to a macro phenomenon) depends on the sign of covariances between growth rates (which refers to a micro phenomenon). In particular, for each t,

1. $\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right] > 0$ implies $\mathrm{Cov}\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right] > 0$,
2. $\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right] = 0$ implies $\mathrm{Cov}\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right] = 0$,

3. $\text{Cov}\left[\tilde{g}_A, \tilde{g}_B\right] < 0$ implies $\text{Cov}\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right] < 0$ only in the case
$$1 + \frac{\text{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+\bar{g}_A)(1+\bar{g}_B)} > 0, \tag{16}$$
otherwise the sign of $\text{Cov}\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right]$ is not uniquely determinable.

This also implies that the ratio
$$\frac{\text{Cov}\left[\tilde{P}_{A,t+1}, \tilde{P}_{B,t+1}\right]}{\text{Cov}\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right]}$$
is

1. greater than 1 whenever $\text{Cov}\left[\tilde{g}_A, \tilde{g}_B\right] > 0$, and
2. positive but smaller than 1 whenever $\text{Cov}\left[\tilde{g}_A, \tilde{g}_B\right] < 0$ and (16) holds;

that is, in both cases $\text{Cov}\left[\tilde{P}_{At}, \tilde{P}_{Bt}\right]$ is increasing in time. In particular, in the first case it diverges as $t \to +\infty$, whereas in the second one tends to zero. We conclude this section by noting an interesting issue that, from our point of view, deserves future investigation: the possibility of accommodating expression (15) to some well-known econometric models, such as, for instance, GARCH. This involves a non-straightforward rewriting of (15) in a number of meaningful components, and the analysis of the role of each of them.

4 Covariance in practice: optimal portfolio choice

The main goal of portfolio theory is to identify the "best" investment strategy in terms of the efficient combination of stocks that maximizes some utility function. A formula for the covariance between stock prices at each time t allows to determine a time-dependent multi-stage portfolio rule that takes into account portfolio re-balancing thus hedging against possible changes in the financial market conditions.

To give an idea of how the formula presented in this paper could be useful in a portfolio choice, we fit the theoretical results of SDDM into the standard Markowitz (1952) model, where random prices in $t = 1$ are transformed into the random one-period rates of return from 0 to 1:
$$\tilde{r}_{m0} = \frac{\tilde{P}_{m1} - P_{m0}}{P_{m0}}, \quad m = A, B, \tag{17}$$
and where P_{m0} is the actual observed market price of company m. In Subsection 4.1 such price will be replaced by $\bar{P}_{m0}$, that is, the theoretical SDDM expected price. Observing (17), the random behavior of $\tilde{r}_{m0}$ turns out being implied by $\tilde{P}_{m1}$; in general terms, when some random behavior of $\tilde{P}_{m,t+1}$ is assumed, the corresponding time t random one-period rate of return is accordingly defined. Key values in a

standard mean-variance analysis are not only expected values and variances of $\tilde{r}_{A0}$ and $\tilde{r}_{B0}$, but also their covariance:

$$\bar{r}_m = \frac{\bar{P}_{m1} - P_{m0}}{P_{m0}}, \qquad \mathrm{Var}\left[\tilde{r}_m\right] = \frac{\mathrm{Var}\left[\tilde{P}_{m1}\right]}{P_{m0}^2}, \qquad \mathrm{Cov}\left[\tilde{r}_A, \tilde{r}_B\right] = \frac{\mathrm{Cov}\left[\tilde{P}_{A1}, \tilde{P}_{B1}\right]}{P_{A0} P_{B0}}.$$

As shown before, formulas about current stock prices can be easily generalized to each future date; in particular,

$$\bar{P}_{m1} = (1 + \bar{g}_m)\, \bar{P}_{m0}$$

(see Appendix B), whereas $\mathrm{Cov}\left[\tilde{P}_{A1}, \tilde{P}_{B1}\right]$ is obtained by substituting $t = 1$ into (14) or (15), and $\mathrm{Var}\left[\tilde{P}_{m1}\right]$ is easily recovered by taking $\mathrm{Cov}\left[\tilde{P}_{A1}, \tilde{P}_{B1}\right]$ and substituting m to both A and B.

Getting back to the issue at hand, the random return of portfolio $\mathbf{x} = [x_A, x_B]$ investing in A and B, under the condition $x_A + x_B = 1$, is denoted by $\tilde{r}(\mathbf{x}) = \tilde{r}_A x_A + \tilde{r}_B x_B$. Assuming a quadratic utility function $u\,(y) = y - 0.5\alpha y^2$, $\alpha > 0$, $y < 1/\alpha$, the expected utility of $\tilde{r}(\mathbf{x})$ is given by

$$\mathbb{E}\left[u\,(\tilde{r}(\mathbf{x}))\right] = \bar{r}(\mathbf{x}) - 0.5\alpha \left(\mathrm{Var}\left[\tilde{r}(\mathbf{x})\right] + \bar{r}^2(\mathbf{x})\right),$$

where $\bar{r}(\mathbf{x}) = \bar{r}_A x_A + \bar{r}_B x_B$ and

$$\mathrm{Var}\left[\tilde{r}(\mathbf{x})\right] = \mathrm{Var}\left[\tilde{r}_A\right] x_A^2 + 2\mathrm{Cov}\left[\tilde{r}_A, \tilde{r}_B\right] x_A x_B + \mathrm{Var}\left[\tilde{r}_B\right] x_B^2.$$

Using standard Lagrangian method, the unique optimal portfolio $\mathbf{x}^* = \left[x_A^*, x_B^*\right]$ that maximizes $\mathbb{E}\left[u\,(\tilde{r}(\mathbf{x}))\right]$ is such that

$$x_A^* = \frac{(\bar{r}_A - \bar{r}_B)\,(1/\alpha - \bar{r}_B) + \mathrm{Var}\left[\tilde{r}_B\right] - \mathrm{Cov}\left[\tilde{r}_A, \tilde{r}_B\right]}{\mathrm{Var}\left[\tilde{r}_A\right] - 2\mathrm{Cov}\left[\tilde{r}_A, \tilde{r}_B\right] + \mathrm{Var}\left[\tilde{r}_B\right] + (\bar{r}_A - \bar{r}_B)^2} \tag{18}$$

$\mathbf{x}^*$ is then evaluated using the (explicit) expressions of $\bar{r}_m$, $\mathrm{Var}\left[\tilde{r}_m\right]$, and $\mathrm{Cov}\left[\tilde{r}_A, \tilde{r}_B\right]$.

4.1 An econometric analysis

In order to empirically test the theoretical result presented in the previous sections, real market data (source: Bloomberg) of E.ON (A) and Saint-Gobain (B) are exploited. These two companies are, as of December 31, 2016, included in the Eurostoxx 50 market index, which encompasses the 50 European companies with the largest capitalization. The reason for choosing these companies is that the historical series of their dividends is the largest available, with data ranging from 1989 to 2016 for a total of 28 dividends. Further, these data satisfy the condition for which denominator of (4) is strictly positive. Table 1 displays dividends paid in 1989 and 2016, the minimum and maximum yearly dividends growth rate in the time series as well as the geometric mean and median of growth rates.

Any pricing formula can be used in two ways: the first one requires, as inputs, all the parameters in the expression and determines a theoretical value that is as much accurate as the formula is able to capture the real phenomenon. The second one

Table 1 Dividends times series: 1989-2016

Company	d_{1989}	d_{2016}	Min	Max	Geom. Mean	Median
E.ON	0.293	0.5	−0.4545	0.2239	0.02	0.09091
Saint-Gobain	0.664	1.24	−0.4631	0.25	0.0234	0.0303

instead (see for instance Koutmos (2015) for the equity prices case) exploits the market price and allows to extract some unobservable parameter as, for instance, implied volatility in option prices. As the random nature of the dividends' growth rate covers the pivot role allowing to extend standard DDM in a stochastic fashion, here discount rates k for both companies are obtained using the standard Capital Asset Pricing Model (CAPM) approach. This should lead to an equilibrium pricing formula that permits to identify possible mispriced stocks; on the other hand, implied discount rates will capture the market as it is instead of as it should be.

For each company the following linear regression model is estimated:

$$r_i - R_F = \alpha + \beta_i \left(R_M - R_F \right),$$

where r_i is the company i's stock return, R_F is the risk-free rate while R_M (i.e. the market portfolio return of the CAPM) is the Eurostoxx 50 index return. In order to estimate the slope of the regression lines, that is Sharpe's β's, weekly returns from January 2012 to December 2016 (261 observations) are considered. Using the returns observed in the last five years of the dividends time series allows the estimated discount rates not to be influenced by values very distant in time. The results are shown in Tables 2 and 3, where parameter estimates, t-statistics and p-values are displayed. As, using a 5% level of significance, the constant parameter α turns out not to be different from zero in either regression, the model applied in order to determine the discount rate for company i is the following:

$$k_i = R_F + \beta_i \left(R_M - R_F \right),$$

where β_i is the slope parameter estimated for company i. It is then necessary to determine the value of the risk-free rate and expected market return. The proxy used for the risk-less rate is $R_F = 0.5\%$. This choice is justified by the fact that, on one hand, the EUREPO rate index has been discontinued at the beginning of 2015 while, on the other, European interest rates have reached very low levels in the last years. As a proxy for the expected market return R_M, the logarithmic yearly mean of the Eurostoxx 50 index (i.e. $R_M = 6.905\%$) from January 2012 to December 2016 is used. The risk-adjusted discount rates are therefore $k_A = 6.631\%$ and $k_B = 7.943\%$.

Table 2 Linear regression output - Eurostoxx 50 vs E.ON: $R^2 = 0.3741$

Parameter	Estimate	$t-$stat	$P-$value
α	−0.0032	−1.6591	0.0983
β	0.9571	12.4647	2.8504×10^{-28}

Table 3 Linear regression output - Eurostoxx 50 vs Saint-Gobain: $R^2 = 0.5639$

Parameter	Estimate	$t-$stat	$P-$value
α	0.0007	0.4341	0.6646
β	1.1621	18.3010	1.3869×10^{-48}

The limited number of available dividends suggests to consider, for both companies, only two alternative outcomes (i.e g_1 and g_2) for the dividends' rate of growth. Such values have been obtained in two different ways:

1. g_1 and g_2 are the geometric mean of the growth rates that happen to be below or above the geometric mean, reported in Table 1, and
2. g_1 and g_2 are the median values (i.e. the first and third quartiles) of growth rates that are below or above their median, reported in Table 1.

Historical probabilities π_{cd} are the ratios between the number of years in which the dividends' rates of growth have jointly been either below or above one of the two thresholds above and the number of observations. Tables 4 and 5 report joint and marginal probabilities, along with the values for growth rates, obtained in the two ways just described.

As $\mathrm{Var}\left[\tilde{g}_A\right] = 0.02431$ and $\mathrm{Var}\left[\tilde{g}_B\right] = 0.01447$, plugging available data into formulæ (3) and (4) yields the following values: $\bar{P}_{A0} = 11.01$ and $\bar{P}_{B0} = 22.65$, $\mathrm{Var}\left[\tilde{P}_{A0}\right] = 44.60$ and $\mathrm{Var}\left[\tilde{P}_{B0}\right] = 79.86$, where $\bar{P}_{i0}$ and $\mathrm{Var}\left[\tilde{P}_{i0}\right]$ are, respectively, the i-th company stock price expected value and variance at 31 December 2016. Further, in the geometric mean case, covariance and correlation between the rates of growth are, respectively, 0.00043 and 0.18232; such values become 0.00036 and 0.12179 when medians are considered. The reduction in correlation observed in the second case can be, at least partially, explained by the fact that dividends' rates of growth have reached, in 2007 and 2008, the outlier values of 21.818% and 22.388% for E.ON and 25% and 20.588% for Saint-Gobain. Finally, plugging all data in (7), the covariance between stock prices of the two companies are 1.11872 when geometric average growth rates are considered and 0.93951 if, instead, medians are used.

These data are now used to perform the mean-variance analysis introduced at the beginning of this Section. The vector of expected returns and the matrix of variances and covariances are, respectively,

$$\begin{pmatrix} \bar{r}_A \\ \bar{r}_B \end{pmatrix} = \begin{pmatrix} 2\% \\ 2.34\% \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{Var}\left[\tilde{r}_A\right] & \mathrm{Cov}\left[\tilde{r}_A, \tilde{r}_B\right] \\ \mathrm{Cov}\left[\tilde{r}_A, \tilde{r}_B\right] & \mathrm{Var}\left[\tilde{r}_B\right] \end{pmatrix} = \begin{pmatrix} 0.4155 & 0.0051 \\ 0.0051 & 0.1798 \end{pmatrix}.$$

Table 4 E.ON vs Saint-Gobain joint probabilities (geometric mean case)

	$g_{B1} = -2.627\%$	$g_{B2} = 5.100\%$	
$g_{A1} = -5.019\%$	$\pi_{11} = 0.25926$	$\pi_{12} = 0.18519$	$p_{A1} = 0.44444$
$g_{A2} = 7.390\%$	$\pi_{21} = 0.22222$	$\pi_{22} = 0.33333$	$p_{A2} = 0.55556$
	$p_{B1} = 0.48148$	$p_{B2} = 0.51852$	

Table 5 E.ON vs Saint-Gobain joint probabilities (median case)

	$g_{B1} = 0.000\%$	$g_{B2} = 8.688\%$	
$g_{A1} = 0.000\%$	$\pi_{11} = 0.28$	$\pi_{12} = 0.24$	$p_{A1} = 0.52$
$g_{A2} = 13.810\%$	$\pi_{21} = 0.2$	$\pi_{22} = 0.28$	$p_{A2} = 0.48$
	$p_{B1} = 0.48$	$p_{B2} = 0.52$	

The minimum variance portfolio is $\mathbf{x}_{\min} = [29.85\%, 70.15\%]$, with $\bar{r}_{\min} = 2.24\%$ and $\mathrm{Var}\left[\tilde{r}_{\min}\right] = 0.1276$. According to (18) the composition of portfolio $\mathbf{x}^*$ depends on the parameter α: Fig. 1 shows the monotonic relationship between risk aversion and the quantity invested in E.ON. As α grows, risk aversion increases so that the fraction of wealth invested in E.ON increases until it reaches its minimum portfolio's level $\mathbf{x}_{\min}$. This occurs because even if SG dominates, in terms of mean-variance, E.ON (that is: $\bar{r}_A < \bar{r}_B$ and, simultaneously, $\mathrm{Var}\left[\tilde{r}_A\right] > \mathrm{Var}\left[\tilde{r}_B\right]$), a combination of the two stocks (with a covariance very close to 0) can reduce risk an investor should bear in investing in only one asset. The higher the risk aversion is the closer to the minimum variance portfolio the optimal investment choice is: a proper portion of E.ON stock leads to diversification.

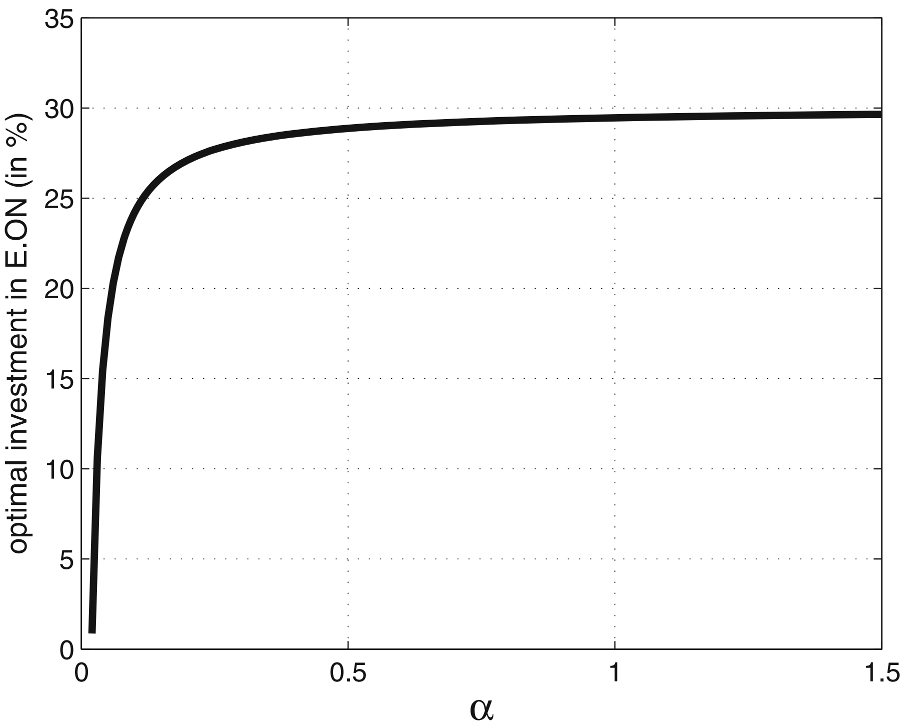

Fig. 1 Optimal investment in E.ON against α

5 Concluding remarks

In this article a formula for the covariance between stock prices that behave according to the SDDM is presented. Its closed-form expression permits to describe its dependence on the covariance of the dividends' growth rates, which, in turn, depends on their joint distribution. Having a formula for both the variance and covariance of SDDM stock prices allows to complete a set of tools to investigate, and likely extend, existing results in corporate finance, such as the analysis of exchange ratio determination in merging agreements, as well as in financial mathematics (Value-at-Risk and Expected Shortfall risk measures for portfolios containing stocks whose prices behave according to SDDM).

On top of this, the formula is grounded on joint probabilities, giving room for the introduction of copulæ that, loosely speaking, allow to model a vast range of (not necessarily linear) links between the marginal behavior of two stocks.

Another possible extension is considering future dividends as the product of two random variables: the earnings-per-share and the distribution ratio. This permits to incorporate dividend policies that change over time. Aside, one of the critiques usually brought about DDM is that it cannot evaluate companies that do not pay dividends (as, for instance, Google-Alphabet Inc.). These companies will eventually stop being fast-growing and will begin paying off dividends. Assuming that the distribution ratio will end being null at a future random time expressed in terms of a jump process will incorporate non-paying dividends companies into the SDDM framework.

As a final remark it is worth mentioning that Polk et al. (2006) exploit Gordon (1962) to establish a relationship between the extra return of some stock with the difference between its dividend yield and the risk-less rate and the expected value of the dividends' growth rate. The SDDM allows to pair such expression with a similar one connecting the stock price variance to variances of both the dividend yield and the rate of growth, paving the road to an econometric test for the SDDM.

Appendix: A

The analytic expression for

$$\mathbb{E}\left[\tilde{P}_{A0}\tilde{P}_{B0}\right]=\sum_{j=1}^{+\infty}\sum_{p=1}^{+\infty}\frac{\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right]}{(1+k_A)^j(1+k_B)^p}$$

is obtained recalling (12). It results that

$$\mathbb{E}\left[\tilde{P}_{A0}\tilde{P}_{B0}\right]=d_{A0}d_{B0}\sum_{p=1}^{+\infty}\left(\underbrace{\sum_{j=1}^{p}\frac{(1+G_A)^j(1+\bar{g}_B)^p}{(1+k_A)^j(1+k_B)^p}}_{\#1}+\underbrace{\sum_{j=p+1}^{+\infty}\frac{(1+\bar{g}_A)^j(1+G_B)^p}{(1+k_A)^j(1+k_B)^p}}_{\#2}\right). \tag{19}$$

For ease of notation, let, for $i = A, B$,

$$\gamma_{\bar{g}_i} = \frac{1+\bar{g}_i}{1+k_i} \qquad \text{and} \qquad \gamma_{G_i} = \frac{1+G_i}{1+k_i}.$$

Sum #1 becomes

$$\gamma_{\bar{g}_B}^p \sum_{j=1}^{p} \gamma_{G_A}^j = \frac{\gamma_{G_A}}{1-\gamma_{G_A}} \left(1-\gamma_{G_A}^p\right) \gamma_{\bar{g}_B}^p,$$

while sum #2, that converges to a positive value as soon as $\bar{g}_A < k_A$, reduces to

$$\gamma_{G_B}^p \sum_{j=p+1}^{+\infty} \gamma_{\bar{g}_A}^j = \frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}} \left(\gamma_{G_B}\gamma_{\bar{g}_A}\right)^p.$$

Observing that

$$\gamma_{G_B}\gamma_{\bar{g}_A} = \frac{(1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+k_A)(1+k_B)} = \gamma_{G_A}\gamma_{\bar{g}_B},$$

sum (19) results being

$$\begin{aligned}
\mathbb{E}\left[\tilde{P}_{A0}\tilde{P}_{B0}\right] &= d_{A0}d_{B0} \sum_{p=1}^{+\infty} \left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}} \gamma_{\bar{g}_B}^p - \frac{\gamma_{G_A}}{1-\gamma_{G_A}} \left(\gamma_{G_A}\gamma_{\bar{g}_B}\right)^p + \frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}} \left(\gamma_{G_B}\gamma_{\bar{g}_A}\right)^p \right) \\
&= d_{A0}d_{B0} \left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}} \sum_{p=1}^{+\infty} \gamma_{\bar{g}_B}^p + \left(\frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}} - \frac{\gamma_{G_A}}{1-\gamma_{G_A}} \right) \sum_{p=1}^{+\infty} \left(\gamma_{G_B}\gamma_{\bar{g}_A}\right)^p \right).
\end{aligned}$$

Now, $\sum_{p=1}^{+\infty} \gamma_{\bar{g}_B}^p$ converges to

$$\frac{\gamma_{\bar{g}_B}}{1-\gamma_{\bar{g}_B}} = \frac{1+\bar{g}_B}{k_B-\bar{g}_B} > 0$$

if $\bar{g}_B < k_B$, whereas $\sum_{p=1}^{+\infty} \left(\gamma_{G_B}\gamma_{\bar{g}_A}\right)^p$ converges to

$$\frac{\gamma_{G_A}\gamma_{\bar{g}_B}}{1-\gamma_{G_A}\gamma_{\bar{g}_B}} = \frac{(1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+k_A)(1+k_B) - (1+\bar{g}_A)(1+\bar{g}_B) - \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}$$

if

$$\left|(1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right| < (1+k_A)(1+k_B).$$

Moreover, observe that

$$\bar{P}_{m0} = \frac{d_{m0}\gamma_{\bar{g}_m}}{1-\gamma_{\bar{g}_m}}, \qquad m = A, B.$$

Covariance between $\tilde{P}_{A0}$ and $\tilde{P}_{B0}$ is, then,

$$\begin{aligned}
&\operatorname{Cov}\left[\tilde{P}_{A0}\tilde{P}_{B0}\right] \\
&= d_{A0}d_{B0}\left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}}\frac{\gamma_{\bar{g}_B}}{1-\gamma_{\bar{g}_B}} + \left(\frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}} - \frac{\gamma_{G_A}}{1-\gamma_{G_A}}\right)\frac{\gamma_{G_A}\gamma_{\bar{g}_B}}{1-\gamma_{G_A}\gamma_{\bar{g}_B}} - \frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}}\frac{\gamma_{\bar{g}_B}}{1-\gamma_{\bar{g}_B}}\right) \\
&= d_{A0}d_{B0}\left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}} - \frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}}\right)\left(\frac{\gamma_{\bar{g}_B}}{1-\gamma_{\bar{g}_B}} - \frac{\gamma_{G_A}\gamma_{\bar{g}_B}}{1-\gamma_{G_A}\gamma_{\bar{g}_B}}\right) \\
&= d_{A0}d_{B0}\left(\frac{\gamma_{G_A}-\gamma_{\bar{g}_A}}{\left(1-\gamma_{G_A}\right)\left(1-\gamma_{\bar{g}_A}\right)}\right)\left(\frac{\gamma_{\bar{g}_B}\left(1-\gamma_{G_A}\right)}{\left(1-\gamma_{\bar{g}_B}\right)\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)}\right) \\
&= \left(\frac{d_{A0}\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}}\right)\left(\frac{d_{B0}\gamma_{\bar{g}_B}}{1-\gamma_{\bar{g}_B}}\right)\left(\frac{\gamma_{G_A}-\gamma_{\bar{g}_A}}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)}\right) = \frac{\bar{P}_{A0}\bar{P}_{B0}\left(\gamma_{G_A}-\gamma_{\bar{g}_A}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)}.
\end{aligned} \tag{20}$$

Now, recall the definition of γ_{G_A}, $\gamma_{\bar{g}_A}$, $\gamma_{\bar{g}_B}$, and G_A. Then,

$$\gamma_{G_A} - \gamma_{\bar{g}_A} = \frac{\operatorname{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+k_A)(1+\bar{g}_B)},$$

and

$$\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right) = \frac{(1+\bar{g}_A)\left((1+k_A)(1+k_B) - (1+\bar{g}_A)(1+\bar{g}_B) - \operatorname{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right)}{(1+k_A)^2(1+k_B)}$$

Substituting these expressions in (20),

$$\begin{aligned}
&\operatorname{Cov}\left[\tilde{P}_{A0}\tilde{P}_{B0}\right] \\
&= \bar{P}_{A0}\bar{P}_{B0} \times \frac{\operatorname{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+k_A)(1+\bar{g}_B)} \times \frac{(1+k_A)^2(1+k_B)}{(1+\bar{g}_A)\left((1+k_A)(1+k_B) - (1+\bar{g}_A)(1+\bar{g}_B) - \operatorname{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right)} \\
&= \bar{P}_{A0}\bar{P}_{B0} \times \frac{(1+k_A)(1+k_B)}{(1+\bar{g}_A)(1+\bar{g}_B)} \times \frac{\operatorname{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+k_A)(1+k_B) - (1+\bar{g}_A)(1+\bar{g}_B) - \operatorname{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}.
\end{aligned}$$

This proves the claimed formula.

Appendix: B

The analytic expression for

$$\mathbb{E}\left[\tilde{P}_{At}\tilde{P}_{Bt}\right] = \sum_{j=t+1}^{+\infty}\sum_{p=t+1}^{+\infty} \frac{\mathbb{E}\left[\tilde{d}_{Aj}\tilde{d}_{Bp}\right]}{(1+k_A)^{j-t}(1+k_B)^{p-t}}$$

is obtained analogously to that of $\mathbb{E}\left[\tilde{P}_{A0}\tilde{P}_{B0}\right]$ by means of the substitutions $u := j - t$ and $v := p - t$, and recalling that

$$\mathbb{E}\left[\tilde{d}_{A,u+t}\tilde{d}_{B,v+t}\right] = \begin{cases} d_{A0}d_{B0}\left(1+G_A\right)^{u+t}\left(1+\bar{g}_B\right)^{v+t} & u \le v \\ d_{A0}d_{B0}\left(1+G_B\right)^{v+t}\left(1+\bar{g}_A\right)^{u+t} & u > v. \end{cases}$$

Therefore

$$\mathbb{E}\left[\tilde{P}_{At}\tilde{P}_{Bt}\right] = d_{A0}d_{B0}\sum_{v=1}^{+\infty}\left(\underbrace{\sum_{u=1}^{v}\frac{(1+G_A)^{u+t}(1+\bar{g}_B)^{v+t}}{(1+k_A)^u(1+k_B)^v}}_{\#1} + \underbrace{\sum_{u=v+1}^{+\infty}\frac{(1+\bar{g}_A)^{u+t}(1+G_B)^{v+t}}{(1+k_A)^u(1+k_B)^v}}_{\#2}\right). \tag{21}$$

Under the assumptions of Proposition 1, sum #1 becomes

$$(1+G_A)^t(1+\bar{g}_B)^t\gamma_{\bar{g}_B}^v\sum_{u=1}^{v}\gamma_{G_A}^u = (1+G_A)^t(1+\bar{g}_B)^t\frac{\gamma_{G_A}}{1-\gamma_{G_A}}\left(1-\gamma_{G_A}^v\right)\gamma_{\bar{g}_B}^v,$$

while sum #2 reduces to

$$(1+G_B)^t(1+\bar{g}_A)^t\gamma_{G_B}^v\sum_{u=v+1}^{+\infty}\gamma_{\bar{g}_A}^u = (1+G_B)^t(1+\bar{g}_A)^t\frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}}\left(\gamma_{G_B}\gamma_{\bar{g}_A}\right)^v.$$

Recalling that

$$(1+G_A)(1+\bar{g}_B) = (1+G_B)(1+\bar{g}_A) = (1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right],$$

and that $\gamma_{G_B}\gamma_{\bar{g}_A} = \gamma_{G_A}\gamma_{\bar{g}_B}$, sum (21) results being

$$\begin{aligned}
&\mathbb{E}\left[\tilde{P}_{At}\tilde{P}_{Bt}\right] \\
&= f(t)\sum_{v=1}^{+\infty}\left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}}\gamma_{\bar{g}_B}^v - \frac{\gamma_{G_A}}{1-\gamma_{G_A}}\left(\gamma_{G_A}\gamma_{\bar{g}_B}\right)^v + \frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}}\left(\gamma_{G_B}\gamma_{\bar{g}_A}\right)^v\right) \\
&= f(t)\left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}}\sum_{v=1}^{+\infty}\gamma_{\bar{g}_B}^v + \left(\frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}} - \frac{\gamma_{G_A}}{1-\gamma_{G_A}}\right)\sum_{v=1}^{+\infty}\left(\gamma_{G_A}\gamma_{\bar{g}_B}\right)^v\right) \\
&= f(t)\left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}}\frac{\gamma_{\bar{g}_B}}{1-\gamma_{\bar{g}_B}} + \left(\frac{\gamma_{\bar{g}_A}}{1-\gamma_{\bar{g}_A}} - \frac{\gamma_{G_A}}{1-\gamma_{G_A}}\right)\frac{\gamma_{G_A}\gamma_{\bar{g}_B}}{1-\gamma_{G_A}\gamma_{\bar{g}_B}}\right),
\end{aligned}$$

where

$$f(t) = d_{A0}d_{B0}\left((1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right)^t.$$

Observing that

$$f(t) = \bar{P}_{A0}\bar{P}_{B0}\left((1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right)^t\frac{1-\gamma_{\bar{g}_A}}{\gamma_{\bar{g}_A}}\frac{1-\gamma_{\bar{g}_B}}{\gamma_{\bar{g}_B}},$$

one can write

$$\mathbb{E}\left[\tilde{P}_{At}\tilde{P}_{Bt}\right] = g(t)\left(\frac{\gamma_{G_A}}{1-\gamma_{G_A}}\frac{1-\gamma_{\bar{g}_A}}{\gamma_{\bar{g}_A}} + \left(1 - \frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_A}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\right)}\right)\frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_B}\right)}{1-\gamma_{G_A}\gamma_{\bar{g}_B}}\right),$$

where

$$g(t) = \bar{P}_{A0}\bar{P}_{B0}\left((1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right)^t .$$

Trivial algebra proves that

$$\frac{\gamma_{G_A}}{1-\gamma_{G_A}}\frac{1-\gamma_{\bar{g}_A}}{\gamma_{\bar{g}_A}} + \left(1 - \frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_A}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\right)}\right)\frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_B}\right)}{1-\gamma_{G_A}\gamma_{\bar{g}_B}} = \frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_A}\gamma_{\bar{g}_B}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)},$$

so that

$$\mathbb{E}\left[\tilde{P}_{At}\tilde{P}_{Bt}\right] = \bar{P}_{A0}\bar{P}_{B0}\frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_A}\gamma_{\bar{g}_B}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)}\left((1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right)^t .$$

Moreover, for $m = A, B$ we have

$$\bar{P}_{mt} = \mathbb{E}\left[\sum_{j=t+1}^{+\infty}\frac{\tilde{d}_{mj}}{(1+k_m)^{j-t}}\right] = \mathbb{E}\left[\sum_{u=1}^{+\infty}\frac{\tilde{d}_{m,u+t}}{(1+k_m)^u}\right] = \bar{P}_{m0}\left(1+\bar{g}_m\right)^t ,$$

and covariance between $\tilde{P}_{At}$ and $\tilde{P}_{Bt}$ is

$$\begin{aligned}
&\mathrm{Cov}\left[\tilde{P}_{At}\tilde{P}_{Bt}\right] \\
&= \bar{P}_{A0}\bar{P}_{B0}\frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_A}\gamma_{\bar{g}_B}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)}\left((1+\bar{g}_A)(1+\bar{g}_B) + \mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]\right)^t - \bar{P}_{A0}\bar{P}_{B0}\left(1+\bar{g}_A\right)^t\left(1+\bar{g}_B\right)^t \\
&= \bar{P}_{A0}\bar{P}_{B0}\left(1+\bar{g}_A\right)^t\left(1+\bar{g}_B\right)^t\left(\frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_A}\gamma_{\bar{g}_B}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)}\left(1+\frac{\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+\bar{g}_A)(1+\bar{g}_B)}\right)^t - 1\right).
\end{aligned}$$

Recalling the definitions of γ_{G_A}, $\gamma_{\bar{g}_A}$, $\gamma_{\bar{g}_B}$, and G_A,

$$\frac{\gamma_{G_A}\left(1-\gamma_{\bar{g}_A}\gamma_{\bar{g}_B}\right)}{\gamma_{\bar{g}_A}\left(1-\gamma_{G_A}\gamma_{\bar{g}_B}\right)} = \left(1+\frac{\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]}{(1+\bar{g}_A)(1+\bar{g}_B)}\right)\frac{(1+k_A)(1+k_B)-(1+\bar{g}_A)(1+\bar{g}_B)}{(1+k_A)(1+k_B)-(1+\bar{g}_A)(1+\bar{g}_B)-\mathrm{Cov}\left[\tilde{g}_A, \tilde{g}_B\right]},$$

and the claimed formula holds.

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