ECI Internal Training
First Session

November 2025

1 Introduction

- Asset pricing = present value of future cash flows.
- Two building blocks:
  - Time: discounting, intertemporal substitution.
  - Risk: states of the world, contingent claims.
- Use expected utility: $E[u(c)]$.

2 Microfoundations of Asset Pricing Theory

2.1 Few Recap on Optimization

Unconstrained optimization: $\max_x F(x)$**, FOC** $F'(x^*) = 0$**, SOC** $F''(x^*) < 0$**.**

- Function:
$$F(x) = -\frac{1}{2}(x-5)^2.$$
- First derivative:
$$F'(x) = -(x-5).$$
- Solve FOC:
$$F'(x^*) = 0 \quad \Rightarrow \quad x^* = 5.$$
- Second derivative:
$$F''(x) = -1 < 0,$$
so the SOC confirms a strict maximum at $x = 5$.

Concavity: $F''(x) < 0$ for all $x \Rightarrow$ FOC is necessary and sufficient.

- Function:
$$F(x) = -\frac{1}{2}(x-5)^2.$$
- Second derivative:
$$F''(x) = -1 < 0 \quad \forall x.$$
- Since F is globally concave, *any* solution to $F'(x) = 0$ is automatically the global maximizer.

Constrained problem: $\max_x F(x)$ s.t. $c \geq G(x)$.

- non-binding:
$$\max_x -\frac{1}{2}(x-5)^2 \quad \text{s.t.} \quad 7 \geq x.$$
- Unconstrained optimum: $x^* = 5$.
- Check constraint: $7 > 5 \rightarrow$ constraint non-binding $\rightarrow$ optimal solution unchanged: $x^* = 5$.

Lagrangian: $L(x, \lambda) = F(x) + \lambda(c - G(x))$.

- Lagrangian:
$$L(x, \lambda) = -\frac{1}{2}(x-5)^2 + \lambda(7-x).$$
- First-order condition:
$$\frac{\partial L}{\partial x} = -(x-5) - \lambda = 0.$$
- Complementary slackness:
$$\lambda(7-x) = 0.$$
- Non-binding case: $x = 5$, $\lambda = 0$ satisfies both.

KKT conditions.

- Maximization:
$$\max_x -\frac{1}{2}(x-5)^2 \quad \text{s.t.} \quad 4 \geq x.$$
- Lagrangian:
$$L(x, \lambda) = -\frac{1}{2}(x-5)^2 + \lambda(4-x).$$
- First-order condition:
$$-(x-5) - \lambda = 0.$$

- Complementary slackness:
$$\lambda(4 - x) = 0.$$
- Since unconstrained optimum $x = 5$ is infeasible, constraint must bind: $x^* = 4$.
- Solve for λ:
$$-(4-5) - \lambda = 0 \quad \Rightarrow \quad 1 - \lambda = 0 \quad \Rightarrow \quad \lambda = 1.$$
- Final solution:
$$x^* = 4, \qquad \lambda^* = 1.$$

2.2 The Time Dimension

Utility: $u(c_0) + \beta u(c_1)$**.**

- As in the slides, reinterpret the two goods as:
$$c_0 = \text{consumption today}, \qquad c_1 = \text{consumption next year}.$$
- Fisher's two–period intertemporal utility:
$$U = u(c_0) + \beta u(c_1),$$
where β is the discount factor (slide 68).
- Concavity of u implies a preference for smooth consumption (slide 69).

Budget constraints.

- Today's tradeoff (slide 71):
$$Y_0 \geq c_0 + s,$$
where s is saving (or borrowing if $s < 0$).
- Next year's constraint (slide 71):
$$Y_1 + (1 + r)s \geq c_1.$$
- Saving increases next year's consumption; borrowing reduces it.

Lifetime budget constraint.

- Divide next year's constraint by $(1+r)$ (slide 72):

$$\frac{Y_1}{1+r} + s \geq \frac{c_1}{1+r}.$$

- Combine with the first-period constraint $Y_0 \geq c_0 + s$:

$$Y_0 + \frac{Y_1}{1+r} \geq c_0 + \frac{c_1}{1+r}.$$

- This is the *lifetime* budget constraint derived on slide 73: the present value of consumption cannot exceed the present value of income.

Intertemporal marginal rate of substitution.

- Lagrangian for the time-allocation problem (slide 77):

$$L = u(c_0) + \beta u(c_1) + \lambda\left(Y_0 + \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r}\right).$$

- First-order conditions (slide 78):

$$u'(c_0) - \lambda = 0, \qquad \beta u'(c_1) - \lambda\left(\frac{1}{1+r}\right) = 0.$$

- Eliminate λ using $\lambda = u'(c_0)$:

$$\beta u'(c_1) = \frac{u'(c_0)}{1+r}.$$

- Rearrange to obtain the intertemporal MRS (slide 78):

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r.$$

- Interpretation: MRS (marginal willingness to give up c_1 for c_0) equals the relative price of consumption across periods.

2.3 The Risk Dimension

Setup: Utility under uncertainty.

- As in slides (86–91), income next year is state-dependent:

$$Y_1^G \quad \text{(good state)}, \qquad Y_1^B \quad \text{(bad state)}, \qquad Y_1^G > Y_1^B.$$

- Probabilities:
$$\pi = P(G), \qquad 1-\pi = P(B).$$
- Consumption choices:
$$c_0, \quad c_1^G, \quad c_1^B.$$
- Expected utility (slide 91):
$$U = u(c_0) + \beta\big[\pi u(c_1^G) + (1-\pi)u(c_1^B)\big].$$
- Concavity of u captures risk aversion and desire for smoothness across states (slide 94).

Budget constraints and contingent claims.

- The consumer can buy contingent claims (slides 95–101):
$$q_G: \text{ price of claim paying 1 in good state}, \qquad q_B: \text{ price of claim paying 1 in bad state}.$$
- Purchases:
$$s_G,\, s_B \quad \text{(negative = shorting)}.$$
- Budget today (slide 101):
$$Y_0 \geq c_0 + q_G s_G + q_B s_B.$$
- State-by-state constraints (slide 102):
$$Y_1^G + s_G \geq c_1^G, \qquad Y_1^B + s_B \geq c_1^B.$$

Deriving the lifetime budget constraint.

- Multiply the good-state constraint by q_G and the bad-state constraint by q_B (slide 103):
$$q_G Y_1^G + q_G s_G \geq q_G c_1^G,$$
$$q_B Y_1^B + q_B s_B \geq q_B c_1^B.$$
- Add these and combine with today's constraint:
$$Y_0 + q_G Y_1^G + q_B Y_1^B \geq c_0 + q_G c_1^G + q_B c_1^B.$$
- This is the lifetime Arrow–Debreu budget constraint (slide 103).

FOCs and marginal rates of substitution.

- Lagrangian (slide 105):

$$L = u(c_0)+\beta\pi u(c_1^G)+\beta(1-\pi)u(c_1^B)+\lambda\big(Y_0 + q_G Y_1^G + q_B Y_1^B - c_0 - q_G c_1^G - q_B c_1^B\big)\,.$$

- First-order conditions (slide 105):

$$u'(c_0) - \lambda = 0,$$

$$\beta\pi u'(c_1^G) - \lambda q_G = 0,$$
$$\beta(1-\pi)u'(c_1^B) - \lambda q_B = 0.$$

- Eliminate λ to get the pricing relations (slide 106):

$$\frac{u'(c_0)}{\beta\pi u'(c_1^G)} = \frac{1}{q_G}, \qquad \frac{u'(c_0)}{\beta(1-\pi)u'(c_1^B)} = \frac{1}{q_B}.$$

- State-relative price condition (slide 106):

$$\frac{\pi u'(c_1^G)}{(1-\pi)u'(c_1^B)} = \frac{q_G}{q_B}.$$

Interpreting contingent claims as building blocks.

- A stock with dividends (d_G, d_B) is equivalent to a bundle of contingent claims (slides 108–112):

$$q_{\text{stock}} = q_G d_G + q_B d_B.$$

- A risk-free bond paying 1 in every state (slide 114):

$$q_{\text{bond}} = q_G + q_B = \frac{1}{1+r}.$$

- So the risk-free rate is (slide 114):

$$1 + r = \frac{1}{q_G + q_B}.$$

Recovering contingent claim prices from traded assets.

- Using a stock (d_G, d_B) and a bond, we can solve for q_G, q_B via replication (slides 117–122).

- To replicate a good-state claim: solve

$$sd_G + b = 1, \qquad sd_B + b = 0.$$

- Solution (slide 118):

$$s = \frac{1}{d_G - d_B}, \qquad b = \frac{-d_B}{d_G - d_B}.$$

- Hence (slide 119):

$$q_G = q_{\text{stock}}\, s + q_{\text{bond}}\, b.$$

- Similarly for the bad-state claim (slide 122):

$$q_B = \frac{d_G q_{\text{bond}} - q_{\text{stock}}}{d_G - d_B}.$$

2.4 General Equilibrium

Pareto optimality condition.

- In the Edgeworth box (slides 126–132), two consumers trade goods a and b.
- A feasible allocation is Pareto optimal when no reallocation can make one consumer better off without making the other worse off.
- Geometrically: indifference curves are tangent (slide 131).
- Mathematically (slide 132):

$$MRS^1_{a,b} = MRS^2_{a,b}.$$

- This states that the two consumers' marginal rates of substitution must coincide.

Social planner conditions.

- Utilities from slides (133–135):

$$U_1 = u(c^1_a) + \alpha u(c^1_b), \qquad U_2 = v(c^2_a) + \beta v(c^2_b).$$

- Resource constraints:

$$Y_a \geq c^1_a + c^2_a, \qquad Y_b \geq c^1_b + c^2_b.$$

- Social planner maximizes weighted sum (slide 134):

$$\theta U_1 + (1-\theta) U_2.$$

- Lagrangian (slide 135):

$$L = \theta[u(c^1_a) + \alpha u(c^1_b)] + (1-\theta)[v(c^2_a) + \beta v(c^2_b)] + \lambda_a(Y_a - c^1_a - c^2_a) + \lambda_b(Y_b - c^1_b - c^2_b).$$

- First-order conditions (slide 135):

$$\theta u'(c_a^1) = \lambda_a, \qquad \theta\alpha u'(c_b^1) = \lambda_b,$$
$$(1-\theta)v'(c_a^2) = \lambda_a, \qquad (1-\theta)\beta v'(c_b^2) = \lambda_b.$$

- Divide each consumer's a–equation by the b–equation (slide 136):

$$\frac{u'(c_a^1)}{\alpha u'(c_b^1)} = \frac{v'(c_a^2)}{\beta v'(c_b^2)}.$$

- This is exactly the Pareto condition:

$$MRS_{a,b}^1 = MRS_{a,b}^2.$$

Competitive equilibrium condition.

- Consumers maximize utility subject to market prices (slides 138–142).
- Consumer 1's Lagrangian (slide 139):

$$L_1 = u(c_a^1) + \alpha u(c_b^1) + \lambda_1\big(p_a Y_a^1 + p_b Y_b^1 - p_a c_a^1 - p_b c_b^1\big).$$

- FOCs:

$$u'(c_a^1) = \lambda_1 p_a, \qquad \alpha u'(c_b^1) = \lambda_1 p_b.$$

- Therefore (slide 139):

$$\frac{u'(c_a^1)}{\alpha u'(c_b^1)} = \frac{p_a}{p_b}.$$

- Similarly for consumer 2 (slide 141):

$$\frac{v'(c_a^2)}{\beta v'(c_b^2)} = \frac{p_a}{p_b}.$$

- Hence in any competitive equilibrium (slide 142):

$$MRS_{a,b}^1 = \frac{p_a}{p_b} = MRS_{a,b}^2.$$

Welfare theorems.

- Slide 145 formalizes the main results:
  - **First Welfare Theorem:** Any competitive equilibrium allocation satisfies the PO condition:

    $$CE \;\Rightarrow\; PO.$$

  - **Second Welfare Theorem:** Any Pareto optimal allocation can be supported as a competitive equilibrium with appropriate prices:

    $$PO \;\Rightarrow\; CE \text{ for some prices.}$$

3 Overview of Asset Pricing Theory

3.1 Pricing Safe Cash Flows

Discount bond pricing.

- A T-year discount bond pays \$1 at maturity (slide 3).
- If its price today is P_T, the gross return is:

$$1 + r_T = \left(\frac{1}{P_T}\right)^{1/T}.$$

- Therefore (slide 3):

$$P_T = \frac{1}{(1 + r_T)^T}.$$

- Interpretation: price = present discounted value of a certain future payment.

Coupon bond pricing.

- Coupon bond pays C each year for T years and face value F at maturity (slide 5).
- Decomposed as portfolio of discount bonds (slide 6).
- No-arbitrage implies (slide 8):

$$P_T^C = \frac{C}{1 + r_1} + \frac{C}{(1 + r_2)^2} + \cdots + \frac{C}{(1 + r_T)^T} + \frac{F}{(1 + r_T)^T}.$$

- The yield to maturity r solves (slide 9):

$$P_T^C = \sum_{t=1}^{T} \frac{C}{(1 + r)^t} + \frac{F}{(1 + r)^T}.$$

General stream of safe cash flows.

- For riskless cash flows $C_1, \ldots, C_T$ (slide 11):

$$P = \sum_{t=1}^{T} C_t P_t.$$

- Since $P_t = \frac{1}{(1+r_t)^t}$ (slide 12):

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + r_t)^t}.$$

3.2 Pricing Risky Cash Flows

Expected value approach.

- Random payoff $\tilde{C}_t$ with possible outcomes $\{C_{t,i}\}$ and probabilities $\{\pi_i\}$ (slide 15):
$$E(\tilde{C}_t) = \sum_i \pi_i C_{t,i}.$$
- Risk-adjusted discounting (slide 16):
$$P_t^A = \frac{E(\tilde{C}_t)}{(1 + r_t + \psi_t)^t},$$
or equivalently
$$P_t^A = \frac{E(\tilde{C}_t) - \Psi_t}{(1 + r_t)^t}.$$
- CAPM, CCAPM, APT provide formulas for ψ_t or Ψ_t (slide 17).

Arrow–Debreu (contingent claim) pricing.

- Decompose payoff into state components (slide 18):
$$\tilde{C}_t = (C_{t,1}, \ldots, C_{t,n}).$$
- With contingent claim prices $q_{t,i}$ (slide 19):
$$P_t^A = \sum_{i=1}^{n} q_{t,i} C_{t,i}.$$
- Interpretation: the risky asset = portfolio of state-contingent discount bonds.

Martingale / risk-neutral pricing.

- "Distort" probabilities: replace π_i by $\hat{\pi}_i$ (slide 20).
- Compute distorted expectation:
$$\hat{E}(C_t) = \sum_i \hat{\pi}_i C_{t,i}.$$
- Price using risk-free discounting:
$$P_t^A = \frac{\hat{E}(C_t)}{(1 + r_t)^t}.$$
- This is the basis of martingale pricing and risk-neutral valuation.

3.3 Two Perspectives on Asset Pricing

No-arbitrage perspective.

- Takes some prices as given and derives others (slide 21).
- Examples in the slides:
  - Coupon bond = portfolio of discount bonds.
  - Risky payoff = portfolio of contingent claims.
- Requires only that arbitrage opportunities do not exist.

Equilibrium perspective.

- Uses microeconomic foundations: consumers maximize expected utility; markets clear (slide 21).
- Determines all prices simultaneously.
- Links asset prices to fundamentals such as marginal rates of substitution.

Unified view.

- Slide 24 summary:

Model	Equilibrium	No-Arbitrage
Risk Premia	CAPM, CCAPM	APT
Contingent Claims	A–D	A–D
Distorted Probabilities	Martingale	

- All models aim to compute

$$P_0 = \frac{\text{risk-adjusted payoff}}{\text{risk-free discount factor}}.$$