PROOF REPAIR

TALIA RINGER

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2021

Reading Committee:
Dan Grossman, Chair
Zachary Tatlock
Rastislav Bodik

Program Authorized to Offer Degree:
Computer Science & Engineering

ABSTRACT

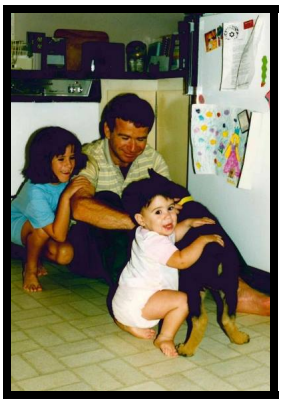
PROOF REPAIR

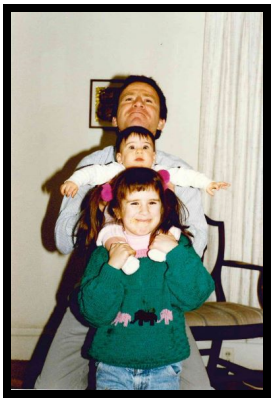
Talia Ringer

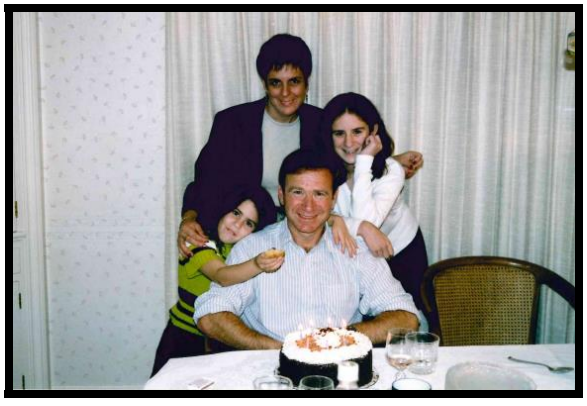
Chair of the Supervisory Committee:
Dan Grossman
Computer Science & Engineering

Abstract will go here.

To my family.







I love all of you.

CONTENTS

1 INTRODUCTION 3
- 1.1 Thesis 4
- 1.2 Approach 5
- 1.3 Results 6
- 1.4 Reading Guide 7

2 MOTIVATING PROOF REPAIR 9
- 2.1 Proof Development 10
- 2.2 Proof Maintenance 20
- 2.3 Proof Repair 24

3 PROOF REPAIR BY EXAMPLE 27
- 3.1 Motivating Example 28
- 3.2 Approach 29
- 3.3 Differencing 36
- 3.4 Transformation 40
- 3.5 Implementation 44
- 3.6 Results 53
- 3.7 Conclusion 59

4 PROOF REPAIR ACROSS TYPE EQUIVALENCES 61
- 4.1 Motivating Example 62
- 4.2 Approach 64
- 4.3 Differencing 70
- 4.4 Transformation 83
- 4.5 Implementation 87
- 4.6 Results 95
- 4.7 Conclusion 102

5 RELATED WORK 105
- 5.1 Proof Engineering 105
- 5.2 Program Repair 106
- 5.3 More Inspiration 108

6 CONCLUSIONS & FUTURE WORK 111

ACKNOWLEDGMENTS

I've always believed the acknowledgments section to be one of the most important parts of a paper. But there's never enough room to thank everyone I want to thank. Now that I have the chance—where do I begin?

We got other wonderful feedback on the paper from Cyril Cohen, Tej Chajed, Ben Delaware, Jacob Van Geffen, Janno, James Wilcox, Chandrakana Nandi, Martin Kellogg, Audrey Seo, James Decker, and Ben Kushigian. And we got wonderful feedback on e-graph integration for future work from Max Willsey, Chandrakana Nandi, Remy Wang, Zach Tatlock, Bas Spitters, Steven Lyubomirsky, Andrew Liu, Mike He, Ben Kushigian, Gus Smith, and Bill Zorn. The Coq developers have for years given us frequent and efficient feedback on plugin APIs for tool implementation. Merge in more PUMPKIN Pi thank yous. Merge in survey paper thank yous. Michael Shulman (feels like univalence, like categorical coherence, like an endofunctor). Michelle Lee.

Dan Grossman, Jeff Foster, Zach Tatlock, Derek Dreyer, Alexandra Silva, the Coq community (Emilio J. Gallego Arias, Enrico Tassi, Gaëtan Gilbert, Maxime Dénès, Matthieu Sozeau, Vincent Laporte, Théo Zimmermann, Jason Gross, Nicolas Tabareau, Cyril Cohen, Pierre-Marie Pédrot, Yves Bertot, Tej Chajed, Ben Delaware, Janno), coauthors, Valentin Robert, my family, PLSE lab (especially Chandrakana Nandi oh my gosh), James Wilcox, Jasper Hugunin, Marisa Kirisame, Jacob Van Geffen, Martin Kellogg, Audrey Seo, James Decker, Ben Kushigian, Gus Smith, Max Willsey, Zach Tatlock, Steven Lyubomirsky, Andrew Liu, Mike He, Ben Kushigian, Bill Zorn, Anders Mörtberg, Conor McBride, Carlo Angiuli, Bas Spitters, UCSD Programming Systems group, Misha, PL Twitter, Roy, Vikram, Alex Polozov, Esther, Ellie, Mer, students, Qi, Saba.

1

INTRODUCTION

What would it take to empower programmers of all skill levels across all domains to formally prove the absence of costly or dangerous bugs in software systems—that is, to formally *verify* them?

Verification has already come a long way toward this since its inception, especially when it comes to the scale of systems that can be verified. The seL4 [27] verified operating system (OS) microkernel, for example, is the effort of a team of proof engineers spanning more than a million lines of proof, costing over 20 person-years. Given a famous 1977 critique of verification [17] (emphasis mine):

> *A sufficiently fanatical researcher* might be willing to devote *two or three years* to verifying a significant piece of software if he could be assured that the software would remain stable.

I could argue that, over 40 years, either verification has become easier, or researchers have become more fanatical. Unfortunately, not all has changed (emphasis still mine):

> But real-life programs need to be maintained and modified. There is *no reason to believe* that verifying a modified program is any easier than verifying the original the first time around.

As we will soon see, this remains so difficult that sometimes, even experts give up in the face of change.

This thesis aims to change that by taking advantage of a missed opportunity: tools for developing verified systems have no understanding of how these systems evolve over time, so they miss out on crucial information. This thesis introduces a new class of verification tools called *proof repair* tools. Proof repair tools understand how software systems evolve, and use the crucial information that evolution carries to automatically evolve proofs about those systems. This gives us reason to believe.

1.1 THESIS

Proof repair falls under the umbrella of *proof engineering*: the technologies that make it easier to develop and maintain verified systems. Much like software engineering scales programming to large systems, so proof engineering scales verification to large systems. In recent years, proof engineers have verified OS microkernels [27, 26], machine learning systems [**?**], distributed systems [**?**], constraint solvers [**?**], web browser kernels [**?**], compilers [29, 30], file systems [**?**], and even a quantum optimizer [**?**]. As we will soon see, practitioners have found these verified systems to be more robust and secure in deployment.

Proof engineering focuses in particular on verified systems that have been developed using special tools called *proof assistants* or interactive theorem provers (ITPs). Examples of proof assistants include Coq [14], Isabelle/HOL [24], HOL Light [22], and Lean [2]. The proof assistant that I focus on in this thesis will be the Coq proof assistant. A discussion of how this work carries over to other proof assistants is in Section 5.

To develop a verified system using a proof assistant like Coq, the proof engineer does three things:

1. implements a program using a functional programming language,
2. specifies what it means for the program to be correct, and
3. proves that the program satisfies the specification.

This proof assistant then automatically checks this proof with a small trusted part of its system [7, 8]. If the proof is correct, then the program satisfies its specification—it is *verified*.

Proof repair automatically fixes broken proofs in response to changes in programs and specifications. For example, a proof engineer who optimizes an algorithm may change the program, but not the specification; a proof engineer who adapts an OS to new hardware may change both. Even a small change to a program or specification can break many proofs, especially in large systems. Changing a verified library, for example, can break proofs about programs that depend on that library—and those breaking changes can be outside of the proof engineers' control.

Proof repair views these broken proofs as bugs that a tool can patch. In doing so, it shows that there *is* reason to believe that verifying a modified system should often, in practical use cases, be easier than verifying the original the first time around, even when proof engineers do not follow good development processes, or when change occurs outside of proof engineers' control. More formally:

> **Thesis**: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and

Figure 1: TODO

apply to fix other proofs broken by the same change. A tool that automates this can save work for proof engineers relative to reference manual repairs in practical use cases.

1.2 APPROACH

The way that proof engineers typically write proofs can obfuscate the information that changes in programs, specifications, and proofs carry. The typical proof engineering workflow in Coq is interactive: The proof engineer passes Coq high-level search procedures called *tactics* (like `induction`), and Coq responds to each tactic by refining the current goal to some subgoal (like the proof obligation for the base case). This loop of tactics and goals continues until no goals remain, at which point the proof engineer has constructed a high-level sequence of tactics called a *proof script*. To check the proof, the proof assistant compiles it down to a low-level representation called a *proof term*, then checks that the proof term has the expected type. Figure 1 illustrates this workflow.

The high-level language of tactics can abstract away important details that a proof repair tool needs, but the low-level language of proof terms can be brittle and challenging to work with. Crucially, though, the low-level language comes with lots of structure and strong guarantees. My approach to proof repair works in the low-level language to take advantage of that. It then builds back up to the high-level language in the end.

By working at the low-level language, it is able to systematically and with strong guarantees extract and generalize the information that breaking changes carry, then apply those changes to fix other proofs broken by the same change. But by later building up to the high-level language, it can in the end produce proofs that integrate more naturally with proof engineering workflows.

This works using a combination of semantic differencing and program transformations in this low-level language. In particular, it uses a semantic differencing algorithm to extract information from a breaking change in a program, specification, or proof. It then combines that with program transformations to generalize and, in some cases, apply that information to fix other proofs broken by the same change. In the end, it uses a prototype decompiler to get from the low-level language back up to the high-level language, so that proof engineers can continue to work in that language going forward.

1.3 RESULTS

The technical results of this thesis are threefold:

1. the **design** of differencing algorithms & program transformations for proof repair,
2. an **implementation** of a proof repair tool suite, and
3. **case studies** to evaluate the tool suite on real proof repair scenarios.

Viewing the thesis statement as a theorem, the proof is as follows:

> **Thesis Proof**: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and apply to fix other proofs broken by the same change (by **design** and **implementation**). A tool that automates this can save work for proof engineers relative to reference manual repairs in practical use cases (by **case studies**).

DESIGN The design describes semantic differencing algorithms to extract information from breaking changes in verified systems, along with proof term transformations to generalize and, in some cases, apply the information to fix proofs broken by the change. The semantic differencing algorithms compare the old and new versions of a changed term or type, and from that find a diff that describes that information corresponding to that change; the transformations then use that diff to transform some term to a more general fix. The details vary by the class of change supported. These design is guided heavily by foundational developments in depenedent type theory; the theory is sprinkle throughout as appropriate. More details including limitations are in the corresponding chapters.

IMPLEMENTATION The implementation shows that in fact *a tool* can extract and generalize the information that changes carry, and then apply that information to fix other proofs broken by the same change. This implementation comes in the form of a proof repair tool suite for Coq called PUMPKIN PATCH (Proof Updater Mechanically Passing Knowledge Into New Proofs, Assisting the Coq Hacker). PUMPKIN PATCH implements two kinds of proof repair: proof repair by example (Chapter 3) and proof repair across type equivalences (Chapter 4). Notably, since all repairs that PUMPKIN PATCH produces are checked Coq in the end, PUMPKIN PATCH does not extend *Trusted Computing Base* (TCB): the set of unverified components that the correctnes of the proof development depends on [**?**]. In total, PUMPKIN PATCH is about 15000 lines of code implemented in OCaml. These 15000 lines of code consist of three plugins and a library, which together bridge the gap

between the theory supported by design and the practical proof repair needed for the case studies. Toward that end, five notable features include:

1. a preprocessing tool to support features in the implementation language missing from the theory,
2. a prototype decompiler from proof terms to proof scripts for better workflow integration,
3. optimizations for efficiency,
4. meaningful error messages for usability, and
5. additional automation for applying patches.

More details and other features are in the corresponding chapters.

CASE STUDIES The case studies show that PUMPKIN PATCH can save work for proof engineers relative to reference manual repairs in practical use cases. (This paragraph is not done yet, but not really needed to understand flow.)

1.4 READING GUIDE

(This subsection is not done yet, but not really needed to understand what I'm going for, since it's pretty low-level.)

How to read this thesis

Mapping of papers to chapters (conclusion paves path to the next era of verification)

Authorship statements for included paper materials, to credit coauthors. Discussion of "we" versus "I" or something, and use of author names. Description of who they are and how we collaborated. Indexed for links: *RanDair Porter*, *Nathaniel Yazdani* (Nate), and so on.

Maybe pronouns too.

Expected reader background & where to find more info

2

MOTIVATING PROOF REPAIR

This thesis describes techniques and tools for automatically repairing broken proofs in a proof assistant. It focuses in particular on proofs about formally verified programs, though many of the techniques and tools carry over to mathematical proofs as well. Formal verification of a program can improve actual and perceived reliability. It can help the programmer think about the desired and actual behavior of the program, perhaps finding and fixing bugs in the process [37]. It can make explicit which parts of the system are trusted, and further decrease the burden of trust as more of the system is verified.

One noteworthy program verification success story is the CompCert [29, 30] verified optimizing C compiler. Both the back-end and front-end compilation passes of CompCert have been verified, ensuring the correctness of their composition [25]. CompCert has stood up to the trials of human trust: it has been used, for example, to compile code for safety-critical flight control software [18]. It has also stood up to rigorous testing: while the test generation tool Csmith [51] found 79 bugs in GCC and 202 bugs in LLVM, it was unable to find any bugs in the verified parts of CompCert.

CompCert, however, was not a simple endeavor: the original development comprised of approximately 35000 lines of Coq code; functionality accounted for only 13% of this, while specifications and proofs accounted for the other 87%. This is not unusual for large proof developments. The initial correctness proofs for an OS microkernel, for example, consisted of 480000 lines of specifications and proofs [26]. Proof engineering technologies make it possible to develop verified systems at this scale. See Ringer and friends 2019 [42] for a comprehensive overview of proof engineering.

Proof repair—the focus of this thesis—is a new proof engineering technology that focuses in particular on minimizing the burden of change as verified systems evolve over time. But for the sake of this chapter, I motivate proof repair not on a large verified system like a C compiler or an OS microkernel. Instead, I motivate it on a simple proof development: a list zip function accompanied by a formal proof that it preserves the lengths of its inputs. This is a small example, but it is worth noting that large proof developments like compilers and

microkernels are often made up of many of these small examples built on top of each other.

The proof assistant that I motivate this on is Coq, since this is the proof assistant for which PUMPKIN PATCH is implemented. I motivate this using the small list zip example in three parts:

1. the workflow of and theory beneath *proof development* (Section 2.1),
2. some challenges of and approaches to *proof maintenance* (Section 2.2), and
3. the motivation for and approach to *proof repair* that follow (Section 2.3).

I will refer to the example and theory introduced in this chapter in later chapters, so it is good to at least skim this chapter regardless of Coq experience.

2.1 PROOF DEVELOPMENT

Before I motivate proof maintenance and repair, it helps to understand proof development in Coq to begin with. In the introduction, I briefly explained the workflow for using Coq to develop a verified system:

> The proof engineer does three things:
>
> 1. implements a program using a functional programming language,
> 2. specifies what it means for the program to be correct, and
> 3. proves that the program satisfies the specification.

That functional programming language is a rich functional programming language called *Gallina*. It is possile to use Gallina to write the program, the specification, and the proof—but writing the proof in Gallina can be challenging. Instead, proof engineers typically use Gallina to write only the program and specification, and write the proof interactively. I alluded to this in the introduction as well, when I explained the typical proof development workflow in Coq:

> The proof engineer passes Coq high-level search procedures called *tactics* (like `induction`), and Coq responds to each tactic by refining the current goal to some subgoal (like the proof obligation for the base case). This loop of tactics and goals continues until no goals remain, at which point the proof engineer has constructed a high-level sequence of tactics called a *proof script*. To check the proof,

```
zip {T1} {T2} (l1 : list T1) (l2 : list T2) : list (T1 * T2) :=
  match l1, l2 with
  | nil, _ => nil
  | _, nil => nil
  | cons t1 tl1, cons t2 tl2 => cons (t1, t2) (zip tl1 tl2)
  end.
```

Figure 2: The list `zip` function, taken from an existing tool [**?**].

> the proof assistant compiles it down to a low-level representation called a *proof term*, then checks that the proof term has the expected type.

The low-level language of proof terms in Coq is Gallina—the same rich functional programming language proof engineers use to write programs and specifications. The high-level language of proof scripts in Coq is a language called *Ltac* that we will soon see.

In this thesis, I will not teach you all of Coq. Good sources for learning more about Coq include the books Certified Programming with Dependent Types [11] and Software Foundations [40], and the survey paper by Ringer and friends 2019 [42]. What I will do is motivate this workflow on an example (Section 2.1.1) and explain the theory beneath (Section 2.1.2).

2.1.1 *The Workflow*

For a moment, let us assume some primitives from the Coq standard library: the type `nat` of natural numbers, the type `list` of polymorphic lists, and the `length` function that computes the length of a list as a natural number. We start by writing the list `zip` program, then specify what it means to preserve its length, and then finally write an interactive proof that shows that specification actually holds. Coq checks this proof and lets us now that our proof is correct, so our `zip` function is verified.

PROGRAM The list `zip` function is in Figure 2. It takes as arguments two lists $\texttt{l}_1$ and $\texttt{l}_2$ of possibly different types $\texttt{T}_1$ and $\texttt{T}_2$, and zips them together into a list of pairs $(\texttt{T}_1 \texttt{ * } \texttt{T}_2)$. For example, if the input lists are:

```
(* [1; 2; 3; 4] *)
l1 := cons 1 (cons 2 (cons 3 (cons 4 nil))).

(* ["x"; "y"; "z"] *)
l2 := cons "x" (cons "y" (cons "z" nil)).
```

then `zip` applied to those two lists returns:

```
(* [(1, "x"); (2, "y"); (3, "z")] *)
cons (1, "x") (cons (2, "y") (cons (3, "z") nil)).
```

```
Theorem zip_preserves_length :
  ∀ {T1} {T2} (l1 : list T1) (l2
      : list T2),
    length l1 = length l2 →
    length (zip l1 l2) = length
        l1.
```

```
Theorem zip_preserves_length :
  ∀ {T1} {T2} (l1 : list T1) (l2
      : list T2),
    length (zip l1 l2) = min (
        length l1) (length l2).
```

Figure 3: Two possible specifications of a proof that `zip` preserves the length of the input lists.

It is worth noting that the implementation of `zip` has to make some decision with what to do with the extra `4` at the end—that is, how `zip` behaves when the input lists are different lengths. The decision that this implementation makes is to just ignore those extra elements.

Otherwise, the implementation is fairly standard. If l_1 is `nil` (first case), or if l_2 is `nil` (second case), then `zip` returns `nil`. Otherwise, `zip` combines the first two elements of each list into a pair ((t_1, t_2)), then sticks that in front of (using `cons`) the result of recursively calling `zip` on the tails of each list (`zip` tl_1 tl_2).

SPECIFICATION Once we have written our `zip` function, we can then specify what we want to prove about it: that the `zip` function preserves the lengths of the inputs l_1 and l_2. We do this by defining a type `zip_preserves_length` (Figure 3, left), which in Coq we state as a `Theorem`. This theorem takes advantage of Gallina's rich type system to quantify over all possible input lists l_1 and l_2. It says that if the lengths of the inputs are the same, then the length of the output is the same as the lengths of the inputs. Our proof will soon show that this type is inhabited, and so this statement is true.

It is worth noting that this step of choosing a specification is a bit of an art—we have some freedom when we choose our specification. We could just as well have chosen a different version of `zip_preserves_length` (Figure 3, right) that states that the length of the output is the *minimum* of the lengths of the inputs (using `min` from the Coq standard library). This is also true for our `zip` implementation, and in fact it is stronger—it implies the original theorem as well. But regardless of which version we choose, we then get to the fun part of actually writing our proof.

PROOF As I mentioned earlier, it is possible to write proofs directly in Gallina—but this can be difficult. Instead, it is more common to write proofs interactively using the tactic language Ltac. Each tactic in Ltac is effectively a search procedure for a proof term, given the context and goals at each step of the proof. The way that this works is, after we state the theorem that we want to prove:

```
Theorem zip_preserves_length :
  ∀ {T1} {T2} (l1 : list T1) (l2 : list T2),
```

```
  length l1 = length l2 →
  length (zip l1 l2) = length l1.
```

we then add one more word:

```
Proof.
```

then step down past that word inside of an IDE. The IDE then drops into an interactive proof mode. In that proof mode, it tracks the context of the proof so far, along with the goal we want to prove. After each tactic we type and step past, Coq responds by refining the goal into some subgoal and updating the context. We continue this until no goals remain. The survey paper [42] has a good overview of the tactic language in Coq and in other proof assistants, plus different interfaces and IDEs for writing proofs interactively and screenshots of them in action.

In this case, after stepping past `Proof` in our IDE, our initial context (above the line) is empty, and our initial goal (below the line) is the original theorem:

```
______________________________________(1/1)
∀ {T1} {T2} (l1 : list T1) (l2 : list T2),
  length l1 = length l2 →
  length (zip l1 l2) = length l1.
```

We can start this proof with the introduction tactic `intros`:

```
  intros T1 T2 l1.
```

This is essentially the equivalent of the natural language proof strategy "assume arbitrary `T1`, `T2`, and `l1`." That is, it moves the universally quantified arguments from our goal into our context:

```
T1 : Type
T2 : Type
l1 : list T1
______________________________________(1/1)
∀ (l2 : list T2),
  length l1 = length l2 →
  length (zip l1 l2) = length l1.
```

From this state, we can induct over the input list (choosing names for variables Coq introduces in the inductive case):

```
  induction l1 as [|t1 tl1 IHtl1].
```

This breaks into two subgoals and subcontexts: one for the base case and one for the inductive case. The base case:

```
T1 : Type
T2 : Type
______________________________________(1/2)
∀ l2 : list T2,
  length nil = length l3 →
  length (zip nil l2) = length nil.
```

holds by reflexivity, which the `auto` tactic takes care of.

In the inductive case:

```
T1 : Type
T2 : Type
t1 : T1
tl1 : list T1
IHtl1 : ∀ l2 : list T2,
        length tl1 = length l2 →
        length (zip tl1 l2) = length tl1
______________________________________(2/2)
∀ l2 : list T2,
  length (cons t1 tl1) = length l2 →
  length (zip (cons t1 tl1) l2) = length (cons t1 tl1).
```

we again use `intros` and `induction`, this time to induct over l_2. This again produces two subgoals: one for the base case and one for the inductive case. The base case has an absurd hypothesis, which we introduce as `H` and then use `auto` to show our conclusion holds: The inductive case holds by simplification and rewriting by the inductive hypothesis $IHtl_1$.

After this, no goals remain, so our proof is done; we can write `Defined`. What happens when we write `Defined` is that Coq compiles that proof script down to a proof term. It then checks the type of that term and ensures that it is exactly the theorem we have stated. Since it is, Coq lets us know that our proof is correct, so our `zip` function is verified.

Figure 4 shows the resulting proof script for this theorem (top), along with the corresponding proof term (bottom). As we can see, the proof term is quite complicated—I will explain what it means soon, in Section 2.1.2. The important thing to note for now is that the details of this low-leve proof term do not matter to us, since we can write the high-level proof script on the top instead. Even though this proof script is still a bit manual for the sake of demonstration, it is much simpler than the low-level proof term.

Writing proofs using tactics does indeed proof development easier than writing raw proof terms. But these highly structured proof terms carry a lot of information that is lost at the level of tactics. It is exactly that rich structure—the type theory beneath Gallina—that makes a principled approach to proof repair possible.

2.1.2 *The Theory Beneath*

Now that we have written a small proof development in Coq, let us take a step back and look at the theoretical foundations that make this possible. While proof scripts help humans write proofs, it is thanks to the proof terms these compile down to that Coq is able to check our proof in the end. These proof terms are in a rich functional programming language called Gallina (Section 2.1.2.1). Gallina implements a rich type theory called the *Calculus of Inductive Constructions* (Section 2.1.2.2). This rich type theory makes it possible to write programs,

```
Theorem zip_preserves_length :
  ∀ {T1} {T2} (l1 : list T1) (l2 : list T2),
    length l1 = length l2 →
    length (zip l1 l2) = length l1.
Proof.
  intros T1 T2 l1. induction l1[1] as [|t1 tl1 IHtl1].
  - auto.[2]
  - intros l2. induction l2[3] as [|t2 tl2 IHtl2].
    + intros H. auto.[4]
    + intros H. simpl.  rewrite IHtl1; auto.[5]
Defined.

zip_preserves_length :
  ∀ {T1} {T2} (l1 : list T1) (l2 : list T2),
    length l1 = length l2 →
    length (zip l1 l2) = length l1
:=
fun (T1 T2 : Type) (l1 : list T1) (l2 : list T2) =>
  list_rect[1] (fun (l1 : list T1) => ...)
    (fun (l2 :  list T2) _=> eq_refl)[2]
    (fun (t1 : T1) (tl1 : list T1) (IHtl1 : ...) (l2 : list T2) =>
      list_rect[3] (fun (l2 : list T2) => ...)
        (fun (H : ...) => eq_sym H)[4]
        (fun (t2 : T2) (tl2 : list T2) (IHtl2 : ...) =>
          fun (H : ...) => eq_ind_r ... eq_refl (IHtl1 ...)[5])
        l2[3])
  l1[1]
  l2.
```

Figure 4: A proof script (top) and corresponding proof term (bottom) in Coq that shows that the list zip function preserves its length. Some details of the proof term are omitted for simplicity. Corresponding parts of the proof are highlighted in the same color and annotated with the same number; the rest is boilerplate.

```
Inductive nat :=
| O : nat
| S : nat → nat.
```

```
nat_rect :
  ∀ (P : nat → Type),
    P O → (* base case *)
    (∀ (n : nat),
      (* inductive case *)
      P n → P (S n)) →
    ∀ (n : nat), P n.
```

Figure 5: The type of natural numbers `nat` in Coq defined inductively by its two constructors (left), and the type of the corresponding eliminator or induction principle `nat_rect` that Coq generates (right).

specifications, and proofs in Coq, and have a small part of Coq check those proofs in the end.

2.1.2.1 *Gallina*

Gallina is a rich functional programming language with support for writing proofs. To see the power of Gallina, let us dissect our proof that `zip` preserves its length. I mentioned earlier that the `nat` and `list` datatypes, and the `length` function itself, can all be found inside of the Coq standard library. But looking into these datatypes and this function already points to many useful features of Gallina.

Each of `nat` and `list` in Gallina is an *inductive type*: it is defined by its *constructors* (ways of constructing a term with that type). A `nat` (Figure 5, left), for example, is either `O` or the successor `S` of another `nat`; these are the two constructors of `nat`.

Every inductive type in Gallina comes equipped with an *eliminator* or induction principle that the proof engineer can use to write functions and proofs about the datatype. For example, the eliminator for `nat` (Figure 5, right) is the standard induction principle for natural numbers, which Coq calls `nat_rect`. This eliminator states that a statement `P` (called the inductive *motive*) about the natural numbers holds for every number if it holds for `O` in the base case and, in the inductive case, assuming it holds for some `n`, it also holds for the successor `S n`.

A `list` (Figure 6, left) is similar to a `nat`, but with two differences: `list` is polymorphic over some type `T` (so we can have a list of natural numbers, for example, written `list nat`), and the second constructor adds a new element of the type `T` to the front of the list. Otherwise, `list` also has two constructors, `nil` and `cons`, where `nil` represents the empty list, and `cons` sticks a new element in front of any existing list. Similarly, the eliminator for `list` (Figure 6, right) looks like the eliminator for `nat` but with an argument corresponding to the parameter `T` over which `list` is polymorphic, and with an additional argument corresponding to the new element in the inductive case.

```
Inductive list {T : Type} :=
| nil : list T
| cons :
    T → list T → list T.
```

```
list_rect :
  ∀ {T : Type} (P : list T → Type
      ),
    P nil → (* base case *)
    (∀ (t : T) (tl : list T),
      (* inductive case *)
      P tl → P (cons t tl)) →
    ∀ (l : list T), P l.
```

Figure 6: The type of polymorphic lists `list` in Coq defined inductively by its two constructors (left), and the type of the corresponding eliminator or induction principle `list_rect` that Coq generates (right). The curly brace notation means that the type parameter `T` is implicit in applications.

```
Fixpoint length {T} l :=
  match l with
  | nil => 0
  | cons t tl =>
     S (length tl)
  end.
```

```
Definition length {T} l :=
  list_rect T (fun _ => nat)
    0
    (fun t tl (length_tl : nat) =>
      S length_tl)
    l.
```

Figure 7: The list `length` function, defined both by pattern matching and recursion (left) and using the eliminator `list_rect` (right).

One interesting thing about the types of these eliminators `list_rect` and `nat_rect` include universal quantification over all inputs, written ∀. Gallina's type system is expressive enough to include universal quantification over inputs, as we will soon see.

We can use these eliminators to write functions and proofs, like the `length` function (Figure 7, right). More standard is to write functions using pattern matching and guarded recursion, like the `length` function from the Coq standard library (Figure 7, left). Both of these two functions behave the same way, but the function on the left is perhaps a bit easier to understand from a traditional programming background: the `length` of the empty list `nil` is `0`, and the length of any other list is just the successor (`S`) of the result of recursively calling `length` on everything but the first element of the list. Indeed, `list_rect`—like all eliminators in Coq—is just a constant that refers to a function itself defined using pattern matching and guarded recursion. In fact, eliminators are equally expressive to pattern matching and guarded recursion [**?**].

For the sake of this thesis, however, I will assume *primitive eliminators*: eliminators that are a part of the core syntax and theory itself, and that do not reduce to terms that use pattern matching and guarded recursion. Likewise, when I show Gallina code, from this point for-

```
zip {T1} {T2} (l1 : list T1) (l2 : list T2) : list (T1 * T2) :=
  list_rect (fun _ : list T1 => list T2 → list (T1 * T2))
    (fun _ => nil)
    (fun t1 tl1 (zip_tl1 : list T2 → list (T1 * T2)) l2 =>
      list_rect (fun _ : list T2 => list (T1 * T2))
        nil
        (fun t2 tl2 (_ : list (T1 * T2)) =>
          cons (t1, t2) (zip_tl1 tl2))
        l2)
    l1
    l2.
```

Figure 8: The list `zip` function, taken from an existing tool [**?**] and translated to use eliminators.

ward, I will favor functions that apply eliminators rather than pattern matching, like the `length` function from Figure 7 on the right.

To handle practical code that uses pattern matching and guarded recursion, I preprocesss the code using a tool by my coauthor Nate Yazdani (more about this later). Figure 8, for example, shows the `zip` function after running this tool. In the rest of the paper, I skip this preprocessing step in examples, but I describe it more in the implementation section later.

With that in mind, we can now look back at the proof script and corresponding proof term in Figure 4. There is a correpsondence between the proof script and the proof term, highlighted in the same color: The proof term is a function from the context to a body that proves the goal type. Every call to `induction` in the proof script shows up as an application of the `list` eliminator `list_rect`, with the cases corresponding to the appropriate arguments of the eliminator. The first call to `auto` compiles down to `eq_refl`, the constructor for the inductive equality type. The second call to `auto` compiles down to `eq_sym`, the proof of symmetry of equality in the Coq standard library. Rewrites compile down to applications of an eliminator over the inductive equality type called `eq_ind_r`.

Still, the proof terms can be difficult for a proof engineer to write and understand. In Section 4.5, I will introduce a prototype decompiler from proof terms back up to proof scripts that makes it possible for Pumpkin Patch to work over highly structured Gallina terms, but produce proof scripts that the proof engineer can use going forward.

2.1.2.2 *Calculus of Inductive Constructions*

The type theory that Gallina implements is CIC_ω, or the Calculus of Inductive Constructions. CIC_ω is based on the Calculus of Constructions (CoC), a variant of the lambda calculus with polymorphism (types that dependent on types) and dependent types (types that depend on terms) [15]. CoC with an infinite universe hierarchy is called CoC_ω.

$\langle i \rangle \in \mathbb{N}$, $\langle v \rangle \in \text{Vars}$, $\langle s \rangle \in \{ \text{Prop}, \text{Set}, \text{Type}\langle i \rangle \}$

$\langle t \rangle ::= \langle v \rangle \mid \langle s \rangle \mid \Pi\ (\langle v \rangle : \langle t \rangle)\ .\ \langle t \rangle \mid \lambda\ (\langle v \rangle : \langle t \rangle)\ .\ \langle t \rangle \mid \langle t \rangle\ \langle t \rangle$

Figure 9: Syntax for CoC_ω with (from left to right) variables, sorts, dependent types, functions, and application.

$\langle t \rangle ::= \ldots \mid \text{Ind}\ (\langle v \rangle : \langle t \rangle)\{\langle t \rangle, \ldots, \langle t \rangle\} \mid \text{Constr}\ (\langle i \rangle, \langle t \rangle) \mid \text{Elim}(\langle t \rangle, \langle t \rangle)\{\langle t \rangle, \ldots, \langle t \rangle\}$

Figure 10: CIC_ω is CoC_ω with inductive types, inductive constructors, and primitive eliminators.

The syntax for CoC_ω is in Figure 9. Note that whereas in Gallina we represent universal quantification over terms or types with $\forall$, here we represent it with Π, as is standard.

The syntax for CIC_ω is in Figure 10), building on syntax from an existing paper [48]; the typing rules are standard and omitted. CIC_ω extends CoC_ω with inductive types [16]. As in Gallina, inductive types are defined by their constructors and eliminators. Consider the inductive type `nat` of unary natural numbers that we saw in Figure 5, this time in CIC_ω:

```
Ind (nat : Set) {
  nat,
  nat → nat
}
```

where the `O` constructor type is the zeroth element in the list, and the `S` constructor type is the first element. Accordingly, the terms:

```
Constr (O, nat)
```

and:

```
Constr (1, nat)
```

refer to the constructors `O` and `S`, respectively.

As in Gallina, `nat` comes associated with an eliminator. Unlike in Gallina, here we truly assume primitive eliminators—that these eliminators do not reduce at all. Instead, we represent them explicitly with the `Elim` construct. Thus, to eliminate over a natural number `n` with motive `P`, we write:

```
Elim (n, P) {
  f_O,
  f_S
}
```

where functions:

```
f_O : P O
```

and:

```
f_S : Π (n : nat) . Π (IHn : P n) . P (S n)
```

prove the base and inductive cases, respectively. When `n`, `P`, $\texttt{f}_O$, and $\texttt{f}_S$ are arbitrary, this statement has the same type as `nat_rect` in Gallina.

CONVENTIONS Throughout this thesis, I assume an inductive type Σ with constructor $\exists$ and projections π_l and π_r, and an equality type `=` with constructor `eq_refl`. I use $\vec{t}$ and $\{t_1, \ldots, t_n\}$ to denote lists of terms. I often present algorithms over CIC_ω relationally, using a set of judgments; to turn these relations into algorithms, prioritize the rules by running the derivations in order, falling back to the original term when no rules match. The default rule for a list of terms is to run the derivation on each element of the list individually.

FROM CIC_ω BACK TO GALLINA Gallina implements CIC_ω, but with a few important differences. More information is on the website, but two differences are relevant to repair: The first is that Gallina lacks primitive eliminators, as I mentioned earlier. The second notable difference is that Gallina has constants that define terms—later on, this will help with building optimizations for repair tools.

Otherwise, a proof repair tool for Gallina can harness the power of CIC_ω. This type theory is fairly simple, but Π makes it possible to quantify over both terms and types, so that we can state powerful theorems and prove that they hold. Inductive types make it possible to write proofs by induction. Both of these constructs mean that terms in Gallina are extremely structured, and as we will soon see, that structure makes a proof repair tool's job much easier.

FROM GALLINA BACK TO COQ But this structure can be difficult for proof engineers to work with, which is why proof engineers typically rely on the tactics we saw in Section 2.1.1. Tactics more generally are a form of *proof automation*, and this proof automation makes it much simpler to develop proofs to begin with. But it turns out this proof automation is a bit naive when it comes to *maintaining* proofs as programs and specifications change over time. Proof repair is a new form of proof automation for maintaining proofs: it uses the rich type information carried by proof terms to automatically fix broken proofs in response to change.

2.2 PROOF MAINTENANCE

What does it mean to *maintain* a verified system? Like all software systems, verified systems evolve over time. The difference is that, for verified systems, the proofs must evolve alongside the rest of the system (Section 2.2.1). Proof engineers typically use development processes to make proofs less likely to break in the face of these changes (Section 2.2.2). Still, even with these development processes, breaking changes happen all the time, even for experts (Section 2.2.3).

All of this points to a need for change-aware proof automation—that is, proof repair.

2.2.1 *Breaking Changes*

At its core, a verified system has three parts, corresponding to the workflow from Section 2.1:

1. programs,
2. specifications, and
3. proofs.

As verified systems evolve over time, both programs and specifications can change. Either of these changes can break existing proofs.

Consider the example from Section **??**. We had two choices for the specification of `zip_preserves_length`. We chose the weaker specification on the left of Figure 3. This gives us some freedom in how we implement our `zip` function. At some point, we may wish to change `zip`, and update our proof so that it still holds. Alternatively, we may wish to port our development to use the stronger specification on the right of Figure 3 We may even wish to use a datatype more expressive than `list`, as we will see in Section **??**. Any of these changes can break proofs in our proof development.

CHANGING OUR PROGRAM Our specification of `zip_preserves_length` gives us some freedom to change how our `zip` function from Figure 2 behaves on edge cases, when the lengths of input lists are not equal. Suppose we change our `zip` function to always return `nil` in those cases, by just returning the old behavior when the lengths are equal, and otherwise returning `nil`. To do this, we rename our old `zip` function to be `zip_same_length`. We then define a new `zip` function that breaks into those two cases, calling `zip_same_length` when the lengths are equal, and otherwise returning `nil`:

```
zip {T1} {T2} (l1 : list T1) (l2 : list T2) : list (T1 * T2) :=
  sumbool_rect (fun _ => list (T1 * T2))
    (fun (_ : length l1 = length l2) =>
      zip_same_length l1 l2)
    (fun (_ : length l1 <> length l2) =>
      nil)
    (eq_dec (length l1) (length l2)).
```

where `sumbool_rect` is an eliminator that lets us break into these two cases, and `eq_dec` says that equality is decidable over natural numbers (that is, any two numbers are either equal or not equal).

Our theorem `zip_preserves_length` still holds, but after changing our program, the *proof* that it holds breaks. We can fix it by adding the highlighted tactics:

```
Proof.
  intros.  unfold zip.
  induction (eq_dec (length l1) (length l2)); try contradiction.
  simpl.  revert a.  revert H. revert l2.
  induction l1 as [|t1 tl1 IHtl1].
  - auto.
  - intros l2. induction l2 as [|t2 tl2 IHtl2].
    + intros H. auto.
    + intros H. simpl. rewrite IHtl1; auto.
Defined.
```

If we have many proofs about `zip`, they may break in similar ways, and require similar patchwork.

CHANGING OUR SPECIFICATION Suppose we had instead chosen the stronger specficiation on the right of Figure 3, and kept our `zip` function the same. We can update our proof accordingly, but after changing this specification, other proofs may break. For example, if we had written a lemma for the `cons` case:

```
Lemma zip_preserves_length_cons :
  ∀ {T1 : Type} {T2 : Type} (l1 : list T1) (l2 : list T2) (t1 : T
      1) (t2 : T2),
    length l1 = length l2 →
    length (zip (cons t1 l1) (cons t2 l2)) = S (length l1).
Proof.
  intros T1 T2 l1 l2 t1 t2 H.
  simpl. f_equal.
  rewrite zip_preserves_length; auto.
Defined.
```

that followed by `zip_preserves_length`, then after the change, this proof would break.

We would have two choices to fix it. Either we could leave our specification alone, and fix our proof. In that case, the proof would look like this instead (with the difference highlighted):

```
Proof.
  intros T1 T2 l1 l2 t1 t2 H.
  simpl. f_equal.
  rewrite ← min_id.  rewrite H at 2.
  apply zip_preserves_length; auto.
Defined.
```

The extra tactics correspond to an extra proof obligation: we must now show that `length l1 = min (length l1) (length l2)`. This holds by the lemma `min_id` from the Coq standard library, combined with the hypothesis that says that `length l1= length l2`.

Alternatively, we could strengthen the specification of that lemma as well, and leave the proof alone:

```
Lemma zip_preserves_length_cons :
  ∀ {T1 : Type} {T2 : Type} (l1 : list T1) (l2 : list T2) (t1 : T
      1) (t2 : T2),
    length (zip (cons t1 l1) (cons t2 l2)) = S (min (length l1) (
        length l2)).
```

```
Proof.
  intros T₁ T₂ l₁ l₂ t₁ t₂.
  simpl. f_equal.
  apply zip_preserves_length_alt; auto.
Defined.
```

But this could continue to break other downstream proofs that depend on `zip_preserves_length_cons`, causing a cascading effect of change.

2.2.2 *Building Robust Proofs*

(This subsection is not done yet, but not really needed to understand flow of paper.)

There are a lot of development processes people use to make proofs less likely to break to begin with (survey paper).

Two examples I like: First, affinity lemmas and reference to `zip` example. Second, better tactics like from CPDT and how could help is in `zip` example.

Quick summary of some other ideas from survey paper. For more, see the survey paper.

2.2.3 *Even Experts are Human*

Even with good development processes, proof engineers change programs and specifications all the time—and this does break proofs, even for experts. To find evidence of this in the real world, I built a Coq plugin called REPLica (REPL Instrumentation for Coq Analysis) that listens to the Read Eval Print Loop (REPL)—a simple loop that all user interaction with Coq passes through—to collect data that the proof engineer sends to Coq during development. I used REPLica to collect a month's worth of granular data on the proof developments of 8 intermediate to expert Coq users. I visualized and analyzed this data to classify hundreds changes to programs and specifications, and fixes to broken proofs. The resulting data, analyses, and proof repair benchmarks are publicly available with the proof engineers' consent.[1]

Changes to programs and specifications were often formulaic and repetitive. For example, Figure 11 shows an example change by an expert proof engineer. In this change, the proof engineer wraps two arguments into a single application of `Val` in three different hypotheses of a lemma. This change did not occur in isolation: the proof engineer patched 10 other definitions or lemmas in similarly, wrapping arguments into an application of `Val`.

Changes to programs and specifications did break proofs, even for expert proof engineers. The proof engineers most often (75% of the time) fixed broken proofs by stepping up above those proofs in the UI

1 http://github.com/uwplse/analytics-data

```
 Lemma proc_rspec_crash_refines_op T (p : proc C_Op T)
   (rec : proc C_Op unit) spec (op : A_Op T) :
   (forall sA sC,
-    absr sA sC tt -> proc_rspec c_sem p rec (refine_spec spec sA)) ->
-   (forall sA sC, absr sA sC tt -> (spec sA).(pre)) ->
+    absr sA (Val sC tt) -> proc_rspec c_sem p rec (refine_spec spec sA)) ->
+   (forall sA sC, absr sA (Val sC tt) -> (spec sA).(pre)) ->
    (forall sA sC sA' v,
-    absr sA' sC tt ->
+    absr sA' (Val sC tt) ->
     (spec sA).(post) sA' v -> (op_spec a_sem op sA).(post) sA' v) ->
    (forall sA sC sA' v,
-    absr sA sC tt ->
+    absr sA (Val sC tt) ->
     (spec sA).(alternate) sA' v -> (op_spec a_sem op sA).(alternate) sA' v) ->
   crash_refines absr c_sem p rec (a_sem.(step) op)
     (a_sem.(crash_step) + (a_sem.(step) op;; a_sem.(crash_step))).
```

Figure 11: Patches to a lemma by an expert proof engineer.

and fixing something else, like a specification. That is, development and maintenance were in reality tightly coupled.

But sometimes, proof engineers did not successfully fix proofs broken by changes in programs and specifications. For example, for the change in Figure 11, the expert proof engineer admitted or aborted (that is, gave up on) the proofs of four of the five broken lemmas after this change. In other words, right now, even experts sometimes just give up in the face of change.

(The rest of this subsection is not done yet, but not really needed to understand flow.) And it's an extra big problem when you have a large development and the changes are outside of your control.

Hence Social Processes.

Why automation breaks, even with good development processes. In other words, even experts are human. And automation doesn't understand how things change, so can't help the human out. But proof repair—smarter proof automation—can.

2.3 PROOF REPAIR

(Still outline text, since I'm getting too antsy and feel like I want to move to the technical chapters again for a bit, but I'm going to explain the parallel structure of the technical chapters at the bottom of this section, so that you can understand what I'll be attempting and how this will flow in.)

Name inspired by program repair, but quite different as we'll soon see.

Recall thesis: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and apply to fix other proofs broken by the same change. A tool that automates this can

save work for proof engineers relative to reference manual repairs in practical use cases.

Proof repair accomplishes this using a combination of differencing and program transformations.

Differencing extracts the information from the change in program, specification, or proof.

The transformations then generalize that information to a more general fix for other proofs broken by the same change.

The details of applying the fix vary by the kind of fix, as we'll soon see.

Crucially, all of this happens over the proof terms in this rich language we saw in the Development section. This is kind of the key insight that makes it all work.

This is great because this language gives us so much information and certainty. This helps us with two of the biggest challenges from program repair. (generals related work—but can just refer to the related work section for more information here)

But it's also challenging because this language is so unforgiving. Plus, in the end, we need these tactic proofs, not just proof terms. So we can't just reuse program repair tools. (generals related work)

So next two chapters will show two tools in my tool suite that work this way, how they handle these challenges, and how they save work. They will have parallel organization (informal for now but hopefully gives you a sense of what I'm thinking):

1. **Motivating Example**: an example change that motivates the class of repairs the tool supports.

2. **Approach**: a specification for the repair tool, including what kind of changes it supports, where it looks to extract & generalize changes, what the extracted & generalized change actually is, and (for the second tool) how it applies those changes. expected inputs and outputs, plus fit to thesis frame.

3. **Differencing**: design of differencing algorithms.

4. **Transformation**: design of proof term transformations.

5. **Implementation**: how this is actually implemented for Coq.

6. **Results**: proof that this can save work relative to reference manual repairs.

7. **Conclusion**: conclusion, limitations, lessons.

3

PROOF REPAIR BY EXAMPLE

The first tool in the Pumpkin Patch plugin suite is the original namesake Pumpkin Patch plugin, which I implemented in 2018 as a prototype to show that proof repair was possible. To prevent confusion, when I refer to the Pumpkin Patch prototype and not to the plugin suite as a whole, I will abbreviate it as Pumpkin. As Pumpkin is a prototype, it includes only preliminary automation for *applying* patches, and supports changes that are limited in scope in a way that is fundamental to the approach. The results for Pumpkin are also preliminary, and the implementation does not yet integrate smoothly into proof engineering workflows. The Pumpkin Pi extension that I will show you in Chapter 4 will address all of these limitations.

Pumpkin implements an *example-based* approach to proof repair in response to breaking changes in the content of programs and specifications, so called because of its resemblence to programming by example in the domain of program synthesis [20]. In this approach, the proof engineer provides an *example* of how to adapt a proof to a breaking change. A tool then generalizes the example adaptation into a *reusable patch* that the proof engineer can use to fix other proofs broken by that change. In this way, example-based proof repair is a new form of proof automation that accounts for how breaking changes in programs and specifications are reflected in repairs to the proofs they break.

In other words, in the frame of the thesis, example-based proof repair extracts information from changes in proofs, then generalizes it to information corresponding to changes in the programs and specifications that broke those proofs to begin with (Section 3.2). This extraction and generalization works at the level of proof terms, through a combination of a novel semantic differencing algorithm over proof terms (Section 3.3) and a suite of semantics-aware proof term transformations (Section 3.4). Pumpkin automates this process (Section 3.5). Case studies show retroactively that Pumpkin could have saved work for proof engineers on major proof developments (Section 3.6).

```
Definition IZR (z:Z) : R :=
  match z with
  | Z0 => 0
  | Zpos n => INR (Pos.to_nat n)
  | Zneg n => - INR (Pos.to_nat n)
  end.
```

```
Definition IZR (z:Z) : R :=
  match z with
  | Z0 => 0
  | Zpos n => IPR n
  | Zneg n => - IPR n
  end.
```

Figure 12: Old (left) and new (right) definitions of IZR in Coq. The old definition applies injection from naturals to reals and conversion of positives to naturals; the new definition applies injection from positives to reals.

3.1 MOTIVATING EXAMPLE

Traditional proof automation considers only the current state of programs, specifications, and proofs. This is a missed opportunity: verified software systems are rarely static. Like unverified systems, verified systems evolve over time.

With traditional proof automation, the burden of change largely falls on proof engineers. Proof repair by example shows that this does not have to be true. It is a form of proof automation that views programs, specifications, and proofs as fluid entities. When a program or specification changes and this breaks many proofs, it extracts information found in the difference between the old and new versions of a single patched proof, and generalizes that to a *reusable patch* that can fix other proofs broken by the same change.

WITHOUT PROOF REPAIR Experienced proof engineers use design principles and custom tactics to make proofs resilient to change. These techniques are useful for large proof developments, but they place the burden of change on the proof engineer. This can be problematic when change occurs outside of the proof engineers's control.

Consider a commit from the Coq 8.7 release [35]. This commit redefined injection from integers to reals (Figure 12). This change broke 18 proofs in the standard library.

The Coq standard library developer who committed the change fixed most of the broken proofs, but failed to fix some of them. The developer then made an additional 12 commits to address the change in `coq-contribs`, a regression suite of projects that the Coq standard library developers maintain as Coq versions change. Many of these changes were simple. For example, the developer wrote a lemma that describes the change:

```
Lemma INR_IPR : ∀p, INR (Pos.to_nat p) = IPR p.
```

The developer then used this lemma to fix broken proofs within the standard library. For example, the proof of the lemma `plus_negative_positive` broke on this line:

```
rewrite Pos2Nat.inj_sub by trivial.
```

It succeeded with the lemma:

```
rewrite ← 3!INR_IPR, Pos2Nat.inj_sub by trivial.✓
```

These changes were outside-facing: proof engineers had to make similar changes to their own proofs when they updated their developments from Coq 8.6 to Coq 8.7. The Coq standard library developer could have updated some tactics to account for this, but it would have been impossible to account for every tactic that proof engineers could use. Furthermore, while the library developer responsible for the changes knows about the lemma that describes the change, the proof engineer does not. The proof engineer must determine how the definition has changed and how to address the change, perhaps by reading documentation or by talking to the Coq standard library developers.

WITH PROOF REPAIR When a proof engineer updates the Coq standard library, a proof repair tool can determine that the definition has changed, then analyze changes in the standard library and in `coq-contribs` that resulted from the change in definition (in this case, rewriting by the lemma). It can extract a reusable patch from those changes, which it can automatically apply within broken user proofs. The proof engineer never has to consider how the definition has changed.

3.2 APPROACH

In the example from Section 3.1, the change in `IZR` broke many proofs. The example patch to a single proof (the proof of `plus_negative_positive`) carried enough information (the rewrite by `INR_IPR`) to fix the other broken proofs. More generally, example-based proof repair takes advantage of the fact that an example patch to a broken proof can carry enough information to fix other proofs broken by the same change.

PUMPKIN implements a prototype of this. To use PUMPKIN (Section 3.2.1), the proof engineer modifies a single proof script to provide an *example* of how to adapt a proof to a change. PUMPKIN extracts that information into a *patch candidate*—a function that describes the change in the example patched proof, but that is localized to the context of the example, and not yet enough to fix other proofs broken by the change. PUMPKIN then generalizes that candidate into a *reusable patch*: a function that can be used to fix other broken proofs broken by the same change, which PUMPKIN defines as a Coq term. In other words, looking back to the thesis statement, the information shows up in the difference between versions of the example patched proof. PUMPKIN can extract, generalize, and in some cases apply that information.

The PUMPKIN prototype focuses on finding reusable patches to proofs in response to certain changes in the content of programs and specifications (Section 3.2.2). It does this using a combination of semantic differencing and proof term transformations: Differencing

Figure 13: find_patch(old_proof, new_proof)

diff types of old_proof and new_proof for goals
diff terms old_proof and new_proof for candidates
if there are candidates **then**
  transform candidates
  if there are patches **then return** patches
return failure

Figure 14: Search procedure for a reusable proof patch in PUMPKIN.

(Section 3.2.3 looks at the difference between versions of the example patched proof for this information, and finds the candidate. Then, proof term transformations (Section **??**) modify that candidate to produce the reusable proof patch. All of this happens over proof terms in Gallina, since tactics may hide necessary information as I will soon show. PUMPKIN has only preliminary support for proof script integration and patch application (see Section 3.5), though I will address this limitation with the PUMPKIN Pi extension in Chapter 4.

3.2.1 *Workflow: Repair by Example*

The interface to PUMPKIN is exposed to the proof engineer as a *command*. Commands in Coq are similar to tactics, except that they can occur outside of the context of proofs, and define new terms. In this case, PUMPKIN extends Coq with a new command called `Patch Proof`, with the syntax:

```
Patch Proof old_proof new_proof as patch_name.
```

where `old_proof` and `new_proof` are the old and new versions of the example patched proof, and `patch_name` is the desired name of the reusable proof patch.[1] This invokes the PUMPKIN plugin, which searches for a reusable proof patch and defines it as a new term if successful. All terms that PUMPKIN defines are type checked in the end, so PUMPKIN does not extend the TCB.

When the proof engineer calls `Patch Proof`, this invokes the proof patch search procedure in Figure 14. The search procedure starts by differencing the *types* of `old_proof` and `new_proof` (that is, the theorems they prove). The result that it finds is the *goal type*: the type that the reusable proof patch should have. It then differences the *terms* `old_proof` and `new_proof` directly to identify candidate proof patches, which are themselves proof terms. Finally, it transforms those proof patches directly into a reusable patch. If it finds a patch with the goal type, it succeeds and defines it.

Consider, for example, the change from the theorem `old` to the slightly stronger theorem `new` in Figure 15. Changing `old` to `new` can

1 Section 3.5 describes an alternative interface for PUMPKIN with Git integration.

```
1  Theorem old: ∀ (n m p : nat),
     n <= m -> m <= p ->
2    n <= p + 1.
                        (* P p *)
3  Proof.
4    intros. induction H0.
5    - auto with arith.
6    - constructor. auto.
7  Qed.
8
9  fun (n m p : nat) (H : n <= m
    ) (H0 : m <= p) =>
10   le_ind
11     m
                                (*
    m *)
12     (fun p0 => n <= p0 + 1)
        (* P *)
13     (le_plus_trans n m 1 H)
        (* : P m *)
14     (fun (m0 : nat) (_ : m <=
     m0) (IHle : n <= m0 + 1) =>
15       le_S n (m0 + 1) IHle)
16     p
                                (*
    p *)
17     H0
```

```
1  Theorem new: ∀ (n m p : nat
    ), n <= m -> m <= p ->
2    n <= p.
    (* P' p *)
3  Proof.
4    intros. induction H0.
5    - auto with arith.
6    - constructor. auto.
7  Qed.
8
9  fun (n m p : nat) (H : n <=
    m) (H0 : m <= p) =>
10   le_ind
11     m
    (* m *)
12     (fun p0 => n <= p0)
    (* P' *)
13     H
    (* : P' m *)
14     (fun (m0 : nat) (_ : m
    <= m0) (IHle : n <= m0) =>
15       le_S n m0 IHle)
16     p
    (* p *)
17     H0
```

Figure 15: Two proofs with different conclusions (top) and the corresponding proof terms (bottom) with relevant type information. We highlight the change in theorem conclusion and the difference in terms that corresponds to a patch.

break proofs that used to successfully apply `old`, so that a proof like this:

```
Proof.
  ...
  apply old.✓
  ...
Defined.
```

fails after migrating to `new`:

```
Proof.
  ...
  apply new.✗
  ...
Defined.
```

When we call:

```
Patch Proof old new as patch.
```

PUMPKIN invokes the search procedure, which differences `old` and `new` to infer the goal type for the patch. In this case, it infers the following goal:

```
∀ n m p, n <= m → m <= p → n <= p → n <= p + 1
```

which takes us from the conclusion of `new` back to the conclusion of `old`. It then differences the terms `old` and `new` to identify candidate proof patches (Section 3.2.3), then tranforms those candidates to a reusable proof patch with that type (Section 3.2.4), which it defines as a new constant `patch`. This is something that we can use to fix other proofs broken by this change, either by applying it with traditional proof automation:

```
Proof.
  ...
  apply patch. apply new.✓
  ...
Defined.
```

or by using the automation in Section 3.5.

3.2.2 *Scope: Changes in Content*

The search procedure in Figure 14 searches for patches to proofs broken by changes in the *content* of programs and specifications. For example, PUMPKIN can support the change in Figure 15, since content (the conclusion of the theorem) changes, but all else remains identical. In general, the PUMPKIN prototype does not support any changes that add, remove, or rearrange any hypotheses. Chapter 4 introduces an extension to PUMPKIN that supports a broad class of changes in datatypes that may change in those ways.

The search procedure can be configured to different classes of change in the content of programs and specifications. Thus, before running the search procedure, PUMPKIN infers a *configuration* from the example change. This configuration customizes the highlighted lines

for an entire class of changes: it determines what to diff on lines 1 and 2, and what transformations to run to achieve what goal on line 4.

Figure 15 used the configuration for a change in the conclusion of a theorem. Given two such proofs:

```
∀ x, H x → P x
∀ x, H x → P' x
```

PUMPKIN searches for a patch with this goal type:

```
∀ x, H x → P' x → P x
```

In total, the PUMPKIN prototype currently implements six configurations. Section 3.5.1.1 explains these configurations, and Section 3.6 describes real-world examples that demonstrate more configurations.

3.2.3 *Differencing: Candidates from Examples*

Differencing operates over terms and types. Differencing tactics would be insufficient, since tactics and hints may mask patches. For example, for the change in Figure 15, the tactics are identical, even though the proof term changes.[2] Differencing instead looks at the change in terms to extract the patch candidates.

In the end, differencing identifies the semantic difference between the old and new versions of the proof terms for the example patched proof. The semantic difference is the difference between two terms that corresponds to the difference between their types. I will explain this more in Section 3.3.

The details of the semantic difference and where differencing looks to find it vary by configuration. Consider a simplified version of the example in Figure 15, using the configuration for changes in conclusions:

```
1: diff theorem conclusions of old_proof and new_proof for goals
2: diff function bodies of old_proof and new_proof for candidates
3: if there are candidates then
4:     transform candidates
```

Rather than look at the entire example, let us look for now at just the base case (line 13):

```
old_proof := le_plus_trans n m 1 H : n <= m + 1
new_proof := H : n <= m
```

The semantic differencing component first identifies the difference in their types (lines 11 and 12), here:

```
n <= m → n <= m + 1
```

This is the *candidate* goal type. It then finds a difference in terms that has that type (line 13):

```
fun (H : n <= m) => le_plus_trans n m 1 H
```

2 Since this is a simple example, replaying an existing tactic happens to work. There are additional examples in the repository (`Cex.v`).

This is the *candidate* for a reusable patch. This candidate is close, but it is not yet a reusable patch. In particular, this candidate maps base case to base case (it is applied to `m`); the patch should map conclusion to conclusion (it should be applied to `p`). This is where the proof term transformations will come in.

SUMMARY In summary, differencing has the following specification:

- **Inputs**: `old_proof`, `new_proof`, a configuration `config`, and a final goal type `goal`, assuming:
  - the change from `old_proof` to `new_proof` is in the class of changes supported by `config`.
- **Outputs**: a list of terms `candidates` of patch candidates, and a candidate goal type `candidate_goal`, guaranteeing:
  - each term in `candidates` has type `candidate_goal`.

PUMPKIN infers the configuration type and the final goal from the change itself, so the proof engineer does not have to provide this information. PUMPKIN could in theory infer the wrong configuration or the wrong goal type, but this would not sacrifice soundness—it would mean only that the patch procedure would either fail to produce a patch, or produce a patch that is not useful in the end. All terms that PUMPKIN produces are well-typed.

3.2.4 *Transformations: Patches from Candidates*

Differencing produces patch candidates that are localized to a particular context according to the inferred goal for that change, but does not yet generalize to other contexts. The transformations are what take each candidate and tries to modify it to produce a term that *does* generalize to other contexts. If it succeeds, it has found a *reusable patch*.

Consider once more the example in Figure 15. The candidate patch that differencing found:

```
candidate := fun (H : n <= m) => le_plus_trans n m 1 H.
```

has this type:

```
candidate : n <= m → n <= m + 1
```

The reusable patch that PUMPKIN is looking for, however, should have this type:

```
∀ n m p, n <= m → m <= p → n <= p → n <= p + 1
```

as this is the goal that PUMPKIN inferred for this configuration. The transformations that PUMPKIN runs will attempt to transform the candidate into a patch with that type.

The details of which transformations to run vary by configuration. There are four transformations that turn patch candidates into reusable proof patches:

1. *patch specialization* to arguments,
2. *patch generalization* of arguments or functions,
3. *patch inversion* to reverse a patch, and
4. *lemma factoring* to break a term into parts.

Each configuration chooses among these transformations strategically based on the structure of the proof term.

For Figure 15, we can instantiate *transform* in the configuration with two transformations:

```
1: diff conclusions of the theorems of old_proof and new_proof for goals
2: diff bodies of the proof terms for candidates
3: if there are candidates then
4:     generalize and then specialize candidates
```

That is, first, PUMPKIN *generalizes* the candidate by `m` (line 11), which lifts it out of the base case:

```
fun n0 n m p H0 H1 =>
  (fun (H : n <= n0) => le_plus_trans n n0 1 H)
: ∀ n0 n m p,
    n <= m →
    m <= p →
    n <= n0 →
    n <= n0 + 1.
```

PUMPKIN then *specializes* this generalized candidate to `p` (line 16), the argument to the conclusion of `le_ind`. This produces a patch:

```
patch n m p H0 H1 :=
  (fun (H : n <= p) => le_plus_trans n p 1 H)
: ∀ n m p, n <= m → m <= p → n <= p -> n <= p + 1
```

which has the goal type.

This simple example uses only two transformations. The other transformations help turn candidates into patches in similar ways, all guided by the structure of the proof term. I will describe these transformations more in Section 3.4, and present real-world examples that demonstrate more configurations in Section 3.6.

SUMMARY In summary, the transformations together have the following specification:

- **Inputs**: the inputs and outputs of differencing, assuming:
  - the assumptions and guarantees from differencing hold.
- **Outputs**: a term `patch` that is the reusable proof patch, guaranteeing:
  - `patch` has the inferred final `goal` type for the change.

When these transformations fail, or when the list of candidates that differencing returns is empty, PUMPKIN simply fails to return a patch. As with differencing, it is possible that a mistake in the implementation of a given configuration leads to a final `goal` type is not useful to the proof engineer, but this cannot soundness: every `patch` REPLICA produces is well-typed.

3.3 DIFFERENCING

Differencing is aware of and guided by the semantics of Coq's rich proof term language Gallina—that is, it is a *semantic differencing* algorithm. This means that differencing can take advantage of the structure and information carried in every proof term, thanks to Gallina's rich type theory CIC$_\omega$. The rich structure of terms helps guide differencing for each configuration, while the rich information in their types helps ensure correctness in the end.

Consider once again the example from Figure 15, but this time not just the base case. Both versions of the proof are inductive proofs using the same induction principle, with slightly different motives. Accordingly, differencing knows that there are two places to look for candidates, namely the base case (line 13) and the inductive case (line 14). Differencing breaks each inductive proof into these cases, then recursively calls itself for each case. In the base case, it finds the candidate from Section 3.2.3. Since this candidate has the desired type for the configuration specialized to the base case, differencing knows it has successfully found a candidate.

The rich type information proof terms carry helps prevent exploration of syntactic differences that are not meaningful. For example, in the inductive case of the proof term from Figure 15 (line 14), the inductive hypothesis `IHle` changes:

```
... (IHle : n <= m0 + 1) ...
... (IHle : n <= m0) ...
```

Notably, though, the type of `IHle` changes for *any* two inductive proofs over `le` with different conclusions. A syntactic differencing component may identify this change as a candidate. My semantic differencing algorithms know that they can ignore this change. This section describes the design (Section 3.3.1) and limitations (Section 3.3.2) of these algorithms. Section 3.5.1.2 describes the implementation in PUMPKIN.

3.3.1 *Design*

Differencing recurses over the structure of two terms t_A and t_B in a common environment Γ. When it recurses, it extends Γ with common assumptions, then differences subterms. In each case, it carries a goal type G, and returns a list of patch candidates $\vec{t}$ that each have that

goal type. That is, we can view it as a judgment $\Gamma \vdash (t_a,\ t_b,\ G) \Downarrow_d \vec{t}$, where in the end, for every t in $\vec{t}$, $\Gamma \vdash t : G$.

The details of this vary by the structure of the term and the configuration, with different heuristics corresponding to different subterm-configuration combinations. For historical reasons, as PUMPKIN was a prototype and differencing of proof terms was novel at the time, these heuristics are not formalized. Here I describe the design of some of the heuristics in CIC_ω. Section 3.5.1.2 describes additional features needed for implementation in Gallina, and the PUMPKIN Pi extension in Chapter 4 formalizes a more elegant differencing algorithm building on some of the insights from the PUMPKIN differencing prototype.

IDENTITY The simplest patch is the identity patch. When two terms are definitionally equal, differencing infers that the goal is identity, and returns a singleton list containing only the identity function instantiated to the appropriate type.

APPLICATION When one proof term is a function application, for example:

$\Gamma \vdash (f\ t_a,\ t_b,\ G) \Downarrow_d \vec{t}$

differencing checks to see if t_b is in $f\ t_a$. That is, it searches for a subterm of $f\ t_a$ that is definitionally equal to t_b. This is how differencing can identify the candidate for the base case of Figure 15 (line 13). It is also a core building block that other differencing heuristics rely on.

When both proof terms are function applications, and the above heuristic fails:

$\Gamma \vdash (f_a\ t_a,\ f_b\ t_b,\ G) \Downarrow_d \vec{t}$

differencing may recurse into both the functions and the arguments, search for patches, and then compose the results. How to compose those results varies by configuration.

FUNCTIONS The treatment of functions depends on whether a hypothesis or a conclusion has changed. When recursing into the body of two functions, each with a hypothesis of the same type:

$\Gamma \vdash (\lambda(t_a : T).b_a, \lambda(t_b : T).b_b,\ G) \Downarrow_d \vec{t}$

differencing assumes that the conclusion has changed. That is, it assumes that t_a and t_b are the same, adds one of them to a common environment, and differences the body:

$\Gamma,\ t_a : T \vdash (b_a,\ b_b[t_a/t_b]) \Downarrow_d \vec{b}$

It then filters those candidates $\vec{b}$ to only those with an adjusted goal type $G\ t_a$, then wraps each candidate b in $\vec{b}$ in a function in the end:

$\lambda(t_a : T).b$

with type G.

When a hypothesis type has changed:

$\Gamma \vdash (\lambda(t_a : T_a).b_a, \lambda(t_b : T_b).b_b,\ G) \Downarrow_d \vec{t}$

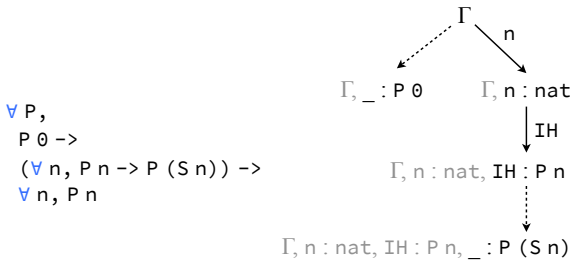


Figure 16: The type of (left) and tree for (right) the eliminator of `nat`. The solid edges represent hypotheses, and the dotted edges represent the proof obligations for each case in an inductive proof.

differencing acts similarly, but it substitutes the changed hypothesis type in the body in order to recurse into a well-typed environment. It also has some additional logic to remove hypothesis that need not show up in the goal type.

ELIMINATORS The semantic differencing algorithms views inductive types as *trees* that represent their eliminators. In these trees, every node is a type context, and every edge is an extension to that type context with a new term.[3] Correspondingly, type differencing (to identify goal types) compares nodes, and term differencing (to find candidates) compares edges.

The key benefit to this model is that it provides a natural way to express inductive proofs, so that differencing can efficiently identify good candidates. Consider, for example, searching for a patch between conclusions of two inductive proofs of theorems about the natural numbers:

$Elim(nat,\ P)\ \{f_O,\ f_S\}$
$Elim(nat,\ Q)\ \{g_O,\ g_S\}$

with goal type:

$Q \to P$

Differencing looks in both the base case and in the inductive case for candidates. In each case, differencing diffs the terms in the dotted edges of the tree for the eliminator of `nat` (Figure 16) to try to find a term that maps between conclusions of that case:

$\Gamma \vdash (f_O,\ g_O,\ Q\ O \to P\ O) \Downarrow_d \vec{t_O}$
$\Gamma \vdash (f_S,\ g_S,\ \Pi(n : nat).Q\ (S\ n) \to P\ (S\ n)) \Downarrow_d \vec{t_S}$

In the inductive case, differencing also knows that the change in the type of the inductive hypothesis is not semantically relevant (it occurs for any change in the inductive motive). Furthermore, it knows that the inductive hypotheis cannot show up in the patch itself, since the

3 These trees are inspired by categorical models of dependent type theory [21].

goal type does not reference the inductive hypothesis, so it attempts to remove any occurrences of the inductive hypothesis in any candidate.

When differencing finds a candidate, it knows Q and P as well as the arguments O or $S\ n$. This makes it simple for PUMPKIN to later query the transformations for the final patch, with type $Q \to P$.

3.3.2 *Limitations*

This section describes a few fundamental limitations of differencing in PUMPKIN, and whether they are addressed in the later PUMPKIN Pi extension. Limitations that are due to the choice of implementation strategy are in Section 3.5.1.2.

HEURISTICS Differencing in PUMPKIN is heuristic-based. This is to some degree inevitable, as the space of possible changes is infinite. In particular, it is the cartesian product of the space of every possible old term and type, combined with every possible new term and type. Still, this is partly historical: for the PUMPKIN prototype, the heuristics were designed bottom-up, and were in many cases too narrow or redundant. The extension to differencing in PUMPKIN Pi not only supports a large class of changes not supported by PUMPKIN, but also abstracts many more details of heuristics.

```
fun n m p (H : n <= m) (H0 : m
    <= p) =>
  le_S n p (* ... proof of
     stronger lemma *)
: ∀ n m p, n <= m -> m <= p ->
    n <= S p
```

```
fun n m p (H : n <= m) (H0 : m
    <= p) =>
  le_plus_trans n p 1 (* ...
     proof of stronger lemma
     *)
: ∀ n m p, n <= m -> m <= p ->
    n <= p + 1
```

Figure 17: Two proof terms `old` (left) and `new` (right) that contain the same proof of a stronger lemma.

ISOLATING CHANGES Differencing supports only incremental changes. Proof engineers, in contrast, often make multiple changes to a verification project in the same commit. The challenge for differencing is to break down large, composite changes into small, isolated changes, then use the appropriate heuristics. Differencing cannot do this yet, and the later PUMPKIN Pi cannot do this either. However, the user study made it clear that in real life, this would be very useful during development, since proof engineers make much more granular changes in development time, so this is less of an issue than I had originally expected it to be. Still, change isolation would help with extracting repair benchmarks from artifacts, supporting library and version updates, and integrating with CI systems. One idea for this is to draw on work in change and dependency management [23, 5, 10]

to identify changes, then use the factoring component to break those changes into smaller parts.

3.4 TRANSFORMATION

The proof term transformations together transform a patch candidate into a reusable proof patch. At a high level, these transformations adapt the candidate to the context of the goal type that PUMPKIN infers. As with differencing, the transformations are aware of and guided by the semantics of Gallina's type theory CIC_ω. This section describes the design 3.4.1 and limitations 3.4.2 of these transformations. Section 3.5.1.3 describes the implementation in PUMPKIN.

3.4.1 *Design*

The transformations together recurse over the structure of each term in the list of candidates $\vec{t}$ in an environment Γ, and adapt that candidate to some new context in a goal-direct manner. In the end, if successful, they produce a reusable proof patch p with type G, where G is the inferred goal type. That is, we can view the high-level composition of transformations as a single judgment $\Gamma \vdash (\vec{t}, G) \Downarrow_t p$, where in the end, $\Gamma \vdash p : G$.

The details vary by transformation, and the details of which transformations run at all and in what order to reach the goal vary by configuration. For historical reasons, as PUMPKIN was a prototype, these tranformations are not formalized. Here I describe at a high level the design of each of the four transformations in CIC_ω. Section 3.5.1.3 describes additional features needed for implementation in Gallina, and the PUMPKIN Pi extension in Chapter 4 formalizes a more elegant proof term transformation building on some of the insights from the PUMPKIN transformation prototypes.

SPECIALIZATION Sometimes, patch candidates are too general. Specialization takes a candidate that is too general, and specializes it to specific arguments as determined by the difference in terms. To find a patch for Figure 15, for example, PUMPKIN specialized the patch candidate to `p` to produce the final patch.

Specialization takes a single patch candidate, some arguments, and a reduction strategy, and returns a new candidate. It first applies the function to the argument, then applies the reduction strategy on the result. The default reducer, for example, uses $\beta\iota$-reduction in Coq [12]. Section 3.5.1.3 decribes other reducers. The only requirement for a reducer is that the end result should be definitionally equal to the original.

Depending on the configuration and the step in the process, the transformed candidate may be the reusable patch, or it may just be an

intermediate candidate. It is the job of the patch finding procedure to provide both the candidate and the arguments, and to determine which transformation to run next, if applicable.

GENERALIZATION In other cases, a patch candidate is too specific. Generalization takes a candidate that is too specific and generalizes it. We saw this for the example in Figure 15 as well: to go from the candidate that PUMPKIN found in the base case to the eventual reusable patch, PUMPKIN generalized the candidate by the argument `m` (before applying specialization).

There are two kinds of generalization. The first generalizes candidates that map between types that share a common argument, like:

$Q\ t\ \rightarrow P\ t$

to abstract the common argument:

$\Pi(\texttt{t0} : T),\ Q\ \texttt{t0} \rightarrow P\ \texttt{t0}$

where T is the type of t. The second generalizes candidates that map between types that share a common function, like:

$P\ t'\ \rightarrow P\ t$

to abstract the common function:

$\Pi(\texttt{P0} : T),\ \texttt{P0}t' \rightarrow \texttt{P0}t$

where T is the type of $P0$.

Generalization takes a patch candidate, the goal type, and the function arguments or function to abstract. It first wraps the candidate inside of a lambda from the type of the term to abstract. Then, it substitutes terms inside the body with the abstract term. It continues to do this until there is nothing left to generalize, then filters results by the goal type. Consider, for example, abstracting this candidate by `m`:

$\lambda(H : n <= m).\text{le\_plus\_trans}\ n\ m\ (SO)\ H$
$: n <= m \rightarrow n <= \text{plus}\ m\ 1$

where `<=` is an inductive type, and `le_plus_trans` and `plus` are both functions in the current context. The first step wraps this in a lambda from some `nat`, the type of `m`:

$\lambda(\texttt{n0} : nat).\lambda(H : n <= m).\text{le\_plus\_trans}\ n\ m\ (SO)\ H)$
$: \Pi(\texttt{n0} : nat), n <= m \rightarrow n <= \text{plus}\ m\ 1$

The second step substitutes `n0` for `m`:

$\lambda(\texttt{n0} : nat).\lambda(H : n <= \texttt{n0}).\text{le\_plus\_trans}\ n\ \texttt{n0}\ (SO)\ H)$
$: \Pi(\texttt{n0} : nat), n <= \texttt{n0} \rightarrow n <= \text{plus}\ \texttt{n0}\ 1$

In general, generalization is a kind of anti-unification problem, as the terms and types may be reduced, so that the common function or argument does not appear explicitly. This is of course undecidable. This poses a challenge for abstraction in the second step—substitution.

To handle this challenge, generalization uses a list of *abstraction strategies* to determine what subterms to substitute. In this case, the simplest strategy works: the tool replaces all terms that are convertible to the concrete argument `m` with the abstract argument `n0`, which

produces a single candidate. Type checking this candidate confirms that it is a patch. In some cases, the simplest strategy is not sufficient, even when it is possible to abstract the term. Section 3.5.1.3 describes a sample of other strategies.

It is the job of the patch finding procedure to provide the candidate and the terms to abstract. In addition, each configuration includes a list of strategies. The configuration for changes in conclusions, for example, starts with the simplest strategy, and moves on to more complex strategies only if that strategy fails. This design makes abstraction simple to extend with new strategies and simple to call with different strategies for different configurations, or even as an optimization for the proof engineer.

INVERSION Sometimes, when two types are propositionally equal, candidate patches may appear in the wrong direction. For example, consider two list lemmas (this example is in Coq rather than CIC$_\omega$ for simplicity):

```
old : ∀ l' l, length (l' ++ l) = length l' + length l
new : ∀ l' l, length (l' ++ l) = length l' + length (rev l)
```

If PUMPKIN searches the difference in proofs of these lemmas for a patch from the conclusion of `new` to the conclusion of `old`, it may find a candidate *backwards*:

```
candidate l' l (H : old l' l) :=
  eq_rect_r ... (rev_length l)
: ∀ l' l, old l' l → new l' l
```

The component can invert this to get the patch:

```
patch l' l (H : new l' l) :=
  eq_rect_r ... (eq_sym (rev_length l))
: ∀ l' l, new l' l → old l' l
```

We can then use this patch to port proofs. For example, if we add this patch to a hint database [1], we can port this proof:

```
Theorem app_rev_len : ∀ l l',
  length (rev (l' ++ l)) = length (rev l) + length (rev l').
Proof.
  intros. rewrite rev_app_distr. apply old.✓
Defined.
```

to this proof:

```
Theorem app_rev_len : ∀ l l',
  length (rev (l' ++ l)) = length (rev l) + length (rev l').
Proof.
  intros. rewrite rev_app_distr. apply new.✓
Defined.
```

Rewrites like `candidate` are *invertible*: We can invert any rewrite in one direction by rewriting in the opposite direction. In contrast, it is not possible to invert the patch PUMPKIN found for Figure 15.

When a candidate is invertible, patch inversion exploits symmetry to try to reverse the conclusions of a candidate patch. It first factors the candidate using the factoring component, then calls the primitive inversion function on each factor, then finally folds the resulting list

in reverse. The primitive inversion function exploits symmetry. For example, equality is symmetric, so the component can invert any application of the equality eliminators. I will explain this more in Section 3.5.1.2.

FACTORING The other transformations sometimes need help breaking a large function into smaller subterms. This can break other problems, like abstraction, into smaller subproblems. It is also necessary to invert certain terms, since the inverse of:

$$g \circ f \\ : \; X \to Z$$

where:

$$g : Y \to Z \\ f : X \to Y$$

for arbitrary non-dependent types X, Y, and Z, is of course:

$$f^{-1} \circ g^{-1} \\ : \; Z \to X$$

This shows up often for inverting sequences of rewrites, as I will show in Section 3.5.1.2. To invert a term like this, PUMPKIN identifies the factors `[f; g]`, inverts each factor to `[f`$^{-1}$`; g`$^{-1}$`]`, then folds and applies the inverse factors in the opposite direction.

The factoring transformation looks within a term for its factors. For the term above, it returns both factors: `f` and `g`. In this case, factoring takes the composite term and `X` as arguments. It first searches as deep as possible for a term of type `X` $\to$ `Y` for some `Y`. If it finds such a term, then it recursively searches for a term with type `Y` $\to$ `Z`. It maintains all possible paths of factors along the way, and it discards any paths that cannot reach `Z`.

3.4.2 *Limitations*

This section describes a few fundamental limitations of the transformations in PUMPKIN, and whether they are addressed in the later PUMPKIN Pi extension. Limitations that are due to the choice of implementation strategy are in Section 3.5.1.3.

UNDECIDABILITY (Needs rewording but draft text, whatever.) Not reduced and so on. Abstracting candidates is not always possible; abstraction will necessarily be a collection of heuristics. Inversion will necessarily sometimes fail, since not all terms are invertible. (Needs a sentence about SOTA.)

DEPENDENT FACTORING The current factoring algorithm can handle paths with more than two factors, but it fails when `Y` depends on `X`. Other components may benefit from dependent factoring; we leave this to future work. (Needs a sentence about SOTA.)

MODELING DIVERSE PROOF STYLES (Needs adjustment to current frame.) Coq programmers use diverse proof styles; the ideal tool should support many different styles. Proofs about decidable domains that apply the term `dec_not_not` pose difficulties for abstraction and inversion; the ideal tool should support these. PUMPKIN has limited support for changes in hypotheses, fixpoints, constructors, pattern matching, and nested induction; the ideal tool should implement these features. (Needs a sentence about SOTA.)

3.5 IMPLEMENTATION

The source code of the PUMPKIN PATCH proof repair plugin suite is on Github.[4] It includes the source code of the PUMPKIN prototype plugin, extended with a number of new features, including the more advanced PUMPKIN Pi plugin that I will present in Chapter 4. The latest version supports Coq 8.8, with Coq 8.9.1 support in a branch.

This section decribes the implementation of the PUMPKIN prototype (Section 3.5.1), along with some new features that go beyond the original prototype and help with workflow integration (Section 3.5.2), plus an evaluation of the boundaries of the PUMPKIN prototype that still stand (Section 4.6.2). The interested reader can follow along in the repository.

3.5.1 *Tool Details*

The implementation (Section 3.5.1.1) of the procedure from Figure 14 in Section 3.2.1 starts with a preprocessing step which compiles the proof terms to trees (like the tree in Figure 16).[5] The implementation always maintains pointers to easily switch between the tree and AST representations of the terms. Differencing (Section 3.5.1.2 operates over trees, while the transformations (Section 3.5.1.3) operate directly over the terms those trees represent. PUMPKIN has no impact on the trusted computing base (Section 3.5.1.4).

3.5.1.1 *The Procedure*

The PUMPKIN prototype exposes the patch finding procedure (`patcher.ml4`) to users through the Coq command `Patch Proof`. After compiling to trees (`evaluation.ml`, PUMPKIN automatically infers which configuration to use for the procedure from the example change. For example,

4 `http://github.com/uwplse/PUMPKIN-PATCH`

5 Representing proof terms directly as trees in the implementation rather than just in the theory is one of my biggest regrets three years later, and I am actively working on removing this representation from the implementation. It is very useful for understanding how differencing works over inductive types, but it adds clutter to the implementation that make it difficult to maintain. PUMPKIN Pi represents proof terms as terms, and avoids this representation.

to find a patch for the case study we will see in Section 3.6.1, I used this command:

```
Patch Proof Old.unsigned_range unsigned_range as patch.
```

PUMPKIN analyzed both versions of `unsigned_range` and determined that a constructor of the `int` type changed (Figure 19), so it initialized the configuration for changes in constructors.

Internally, PUMPKIN represents configurations as sets of options, which it passes to the procedure. The procedure uses these options to determine how to compose components (for example, whether to abstract candidates) and how to customize components (for example, whether semantic differencing should look for an intermediate lemma). In total, the PUMPKIN prototype currently implements six configurations. The first five correspond to changes in:

1. conclusions of theorems,
2. hypotheses of theorems,
3. dependent arguments to constructors of inductive types,
4. conclusions of constructors of inductive types, and
5. cases of fixpoints.

The final configuration is useful for proof optimization (Section 3.5.2). The support for these changes is limited in expressiveness in power; more information on limitations in scope can be found in the repository. Extending PUMPKIN with new configurations amounts to extending key functions in the implementation with a case corresponding to the new configuration.

3.5.1.2 *Differencing*

As noted in Section 3.3, differencing (`differencing.ml`) operates over trees. Differencing uses the structure of these trees to prioritize semantically relevant differences. At the lowest level, it calls a primitive differencing function which checks if it can substitute one term within another term to find a function between their types.

The differencing component is *lazy*: it compiles terms into trees one step at a time. It then *expands* each tree as needed to find candidates (`expansion.ml`). For example, differencing two functions for a patch between conclusions:

```
fun (t : T) => b
fun (t' : T) => b'
```

Differencing introduces a single term of type `T` to a common environment, then expands and recursively diffs the bodies `b` and `b'` in that environment.

3.5.1.3 *Transformation*

PUMPKIN implements the four transformations from Section 3.4: specialization, generalization, inversion, and factoring. These transformations operate directly over terms in Gallina. The PUMPKIN procedure chooses among them by configuration, and in the end checks the type of the patch to ensure that is has the goal type. PUMPKIN also determines what arguments to pass to each transformation based on the configuration. As a bonus, PUMPKIN exposes commands that correspond to each of these transformations (`patcher.ml4`), so that proof engineers can call them outside of the proof patching procedure.

SPECIALIZATION Specialization (`specialize.ml`) takes a patch candidate and some arguments, all of which are Gallina terms. It also takes a custom reducer (`reducers.ml`) as a higher-order function that reducers a Coq term. It applies the candidate function to the arguments, then reduces the result using the supplied reducer.

The default specializer $\beta\iota\zeta$-reduces [12] the result using Coq's `Reduction.nf_betaiotazeta` function. Other reducers include one that does not reduce at all, one that removes unecessary applications of the identity function, one that does weak head reduction, one that completely normalizes terms, one that removes unused hypotheses, and one that applies δ reduction. There are also higher-order combinators for reducers, including chain reduction with errors, chain reduction without errors, and reduction over the types of supplied terms. This makes it possible for configurations to highly customize specialization, and further ends up being a useful programming paradigm for developing the PUMPKIN Pi extension in Chapter 4. I expose these reducers and combinators in the Coq plugin library.

GENERALIZATION Generalization (`abstraction.ml`) takes a patch candidate, the goal type, and the arguments or function to abstract, all as Gallina terms. It also takes a list of abstraction strategies to determine what subterms to substitute. After abstracting the candidate to a lambda term, it substitutes subterms inside of the body with the abstract term using the supplied list of abstraction straegies, in order.

The simplest strategy replaces all terms convertible to a particular concrete argument with the supplied abstract argument. In some cases, this strategy is not sufficient. It may be possible to produce a patch only by generalizing *some* of the subterms convertible to the argument or function (I will show an example of this in Section 3.5.3.2), or the term may not contain any subterms convertible to the argument or function at all.

I implement several strategies to account for this. The combinations strategy, for example, tries all combinations of substituting only some of the convertible subterms with the abstract argument. The pattern-based strategy substitutes subterms that match a certain pattern with

a term that corresponds to that pattern. Some strategies reduce before abstracting, and some do not. I expose these strategies in the PUMPKIN OCaml API (`abstracters.mli`).

INVERSION Inversion (`inverting.ml`) takes as input a patch candidate as a Gallina proof term, and tries to reverse the conclusion of its type. It works by first factoring the candidate, then exploiting symmetry properties to invert those factors, then finally composing the inverted factors in the opposite order.

For example, the equality eliminator in Gallina is `eq_rect`–the `rewrite` tactic in Ltac often compiles down to an application of `eq_rect` Since equality is symmetric, the Coq standard library also comes equipped with an inverse function `eq_rect_r`, related to `eq_rect` by symmetry of equality:

```
eq_rect_r A x P (H : P x) y (H0 : y = x) :=
  eq_rect x (fun y0 : A => P y0) H y (eq_sym H0)
```

When inversion encounters a single application of `eq_rect` in one of its factors, it reverses it by applying symmetry of equality, effectively producing an application of `eq_rect_r`. The opposite direction works similarly.

If inversion does not recognize any type symmetry properties it can exploit in a factor, it strategically swaps subterms in the factor and type checks the result to see if it is an inverse. This essentially amounts to an ad hoc attempt to discover symmetry properties in the factor.

FACTORING Factoring (`factoring.ml`) takes as input a proof term in Gallina, and attempts to break it into factors. When it succeeds, it returns the factors as a list of terms; otherwise, it returns the empty list.

Factoring works by searching with a term for factors. Consider factoring a sequence of rewrites:

```
t (a b c d : nat) (H : a = b) (H0 : b = c) (H1 : c = d) : a =
    d :=
  eq_rect_r
    (fun (a0 : nat) => a0 = d)
    (eq_rect_r (fun (b0 : nat) => b0 = d) H1 H0)
    H.
```

into two independent rewrites:

```
f (a b c d : nat) (H : a = b) (H0 : b = c) (H1 : c = d) : b =
    d :=
  eq_rect_r (fun (b0 : nat) => b0 = d) H1 H0.

g (a b c d : nat) (H : a = b) (H0 : b = c) (H1 : b = d) : a =
    d :=
  eq_rect_r (fun (a0 : nat) => a0 = d) H1 H.

t (a b c d : nat) (H : a = b) (H0 : b = c) (H1 : c = d) : a =
    d :=
```

```
g a b c d H H0 (f a b c d H H0 H1).
```

To discover `f` and `g`, factoring starts by assuming all of the hypotheses of `t`, then searching as deep as possible within the conclusion of `t` for a term of type `Y` for some `Y` (here, the conclusion of `f`, with type `b = d`). It then assumes `Y`, and recursively factors the term it gets from substituting in that hypothesis for `f` (here, it assumes `b = d`, and substitutes to derive the term `g`). It repeats this until it is able to reach the conclusion type (here `a = d`, the type of the conclusion of `g`), at which point it has found the only possible path of factors, and it is done. It returns these factors as Gallina terms.

3.5.1.4 *Trusted Computing Base*

A common concern for proof developments is an increase in the trusted computing base (TCB). PUMPKIN is implemented as a Coq plugin. Coq type checks all terms that plugins produce. Since PUMPKIN does not modify the type checker, it cannot produce an ill typed term. PUMPKIN also does not add any axioms, and so does not increase the TCB.

3.5.2 *Extensions*

Since releasing the PUMPKIN prototype, I have extended PUMPKIN with many features for better integration into proof engineering workflows. This section summarizes three: Git integration, support for applying patches, and proof optmization. I detail tactic generation and more when I discuss the PUMPKIN Pi extension to PUMPKIN PATCH in Chapter 4.

GIT INTEGRATION PUMPKIN PATCH exposes a Git interface to PUMPKIN called PUMPKIN-git [43]. PUMPKIN-git makes it possible to call PUMPKIN's `Patch Proof` command by command line over Git commits. To call PUMPKIN-git, the proof engineer simply runs the command:

```
pumpkin-git example_proof file.v -rev rev
```

This will search for a patch that corresponds to the change in `example_proof` in `file.v` compared to the the revision `rev` of the local repository. It will then prompt the proof engineer with the patch it finds, and either overwrite the file (with consent) or otherwise save the results to a temporary file. There are many options to control the behavior of PUMPKIN-git, all of which can be found in the repository.

PATCH APPLICATION For the PUMPKIN prototype, the differencing algorithm and proof term transformations extract and generalize information form example patched proofs in the form of reusable patches. The prototype does not yet support applying the patches that it finds automatically. Since implementing the prototype, I have extended

Pumpkin-git with preliminary support for patch application via hint generation. The interface is:

```
pumpkin-git example_proof_id file.v -rev rev -hint
```

This places the generated patch in a hint database in Coq. Coq applies hints in its hint databases automatically, in some cases taking care of the changes at the proof script level that the proof engineer would have to make to use these patches.

Since the Pumpkin prototype, I have also extended Pumpkin Patch with the Pumpkin Pi plugin. The Pumpkin Pi plugin does applies the changes that it finds in all cases. Furthermore, it exposes a decompiler from proof terms back up to tactic scripts—this decompiler is exposed to the Pumpkin plugin as well, so that proof engineers can decompile patches back to tactics by calling the `Decompile` command. I will discuss this and more in Chapter 4.

PROOF OPTIMIZATION Five of the implemented configurations correspond to changes, but the sixth configuration is special: it corresponds to the absence of change. This makes it possible to reuse all of the Pumpkin infrascture to optimize proofs to automatically remove extra calls to induction. See `Optimization.v` for more information.

3.5.3 *Testing Boundaries*

```
fun n m p (H : n <= m) (H0 : m
    <= p) =>
  le_S n p (* ... proof of
     stronger lemma *)
: ∀ n m p, n <= m -> m <= p ->
    n <= S p
```

```
fun n m p (H : n <= m) (H0 : m
    <= p) =>
  le_plus_trans n p 1 (* ...
     proof of stronger lemma
     *)
: ∀ n m p, n <= m -> m <= p ->
    n <= p + 1
```

Figure 18: Two proof terms `old` (left) and `new` (right) that contain the same proof of a stronger lemma.

In this section, I explore the boundary between what the semantic differencing and transformation implementations in the Pumpkin prototype can and cannot handle. It is precisely this boundary that informs how to improve the implementations.

To evaluate this boundary, I tested the Pumpkin prototype on a suite of 50 pairs of proofs (Section 3.5.3.1). I designed 11 of these pairs to succeed, then modified their proofs to produce the remaining 39 pairs that try to stress the core functionality of the tool.

I learned the following from the pairs that tested Pumpkin's limitations:

1. **The failed pairs drive improvements.**
   Pumpkin failed on 17 of 50 pairs. These pairs tell inform how

to improve differencing and transformations in the future. (Section 3.5.3.2)

2. **The pairs unearth abstraction strategies.**
   PUMPKIN produced an exponential number of candidates in 5 of 50 pairs. New abstraction strategies can dramatically reduce the number of candidates. (Section 3.5.3.2)

3. **Pumpkin was fast, and it can be faster.**
   The slowest successful patch took 48 ms. The slowest failure took 7 ms. Simple changes can make PUMPKIN more efficient. (Section 3.5.3.3)

3.5.3.1 *Patch Generation Suite*

I wrote a suite[6] of 50 pairs of proofs. I wrote these proofs myself since there was no existing benchmark suite to work with. I used the following methodology:

1. Choose theorems `old` and `new`.
2. Write similar inductive proofs of `old` and `new`.
3. Modify the proof of `old` to produce more pairs.
4. Search for patches from `new` to `old`.
5. If possible, search for patches from `old` to `new`.

In total, I chose 11 pairs of theorems `old` and `new`, and I wrote 50 pairs of proofs of those theorems.

My goal was to determine what changes to proofs stress the components and how to use that information to drive improvements. I focused on differences in conclusions, the most supported configuration at the time. Since PUMPKIN operates over terms, I removed redundant proof terms, even if they were produced by different tactics. I controlled the first pair of proofs of each pair of theorems for features I had not yet implemented at the time, like nested induction, changes in hypotheses, and abstracting `omega` terms. These features sometimes showed up in later proofs (for example, after moving a rewrite); I kept these proofs in the suite, since isolated changes to supported proofs that introduce unsupported features can inform future improvements.

3.5.3.2 *Three Challenges*

PUMPKIN found patches for 33 of the 50 pairs. 28 of the 33 successes did not stress PUMPKIN at all: PUMPKIN found the correct candidate immediately and was able to abstract it in one try. The pairs that

6 `http://github.com/uwplse/PUMPKIN-PATCH/blob/cpp18/plugin/coq/Variants.v`

Pumpkin failed to patch and the successful pairs that stressed abstraction reveal key information about how to improve differencing and the transformations. I walk through three examples below. Future tools can use these as challenge problems to improve upon Pumpkin.

A CHALLENGE FOR DIFFERENCING For one pair of proofs of theorems with propositionally equal conclusions (Figure 18), differencing failed to find candidates in either direction. These proofs both contain the same proof of a stronger lemma; Pumpkin found patches from this lemma to both `old` and `new`, but it was unable to find a patch between `old` and `new`. A patch may show up deep in the difference between `le_plus_trans` and `le_S`, but even if we δ-reduce (unfold the definition of [12]) `le_plus_trans`, this is not obvious:

```
le_plus_trans n m p (H : n <= m) :=
  (fun lemma : m <= m + p =>
    trans_contra_inv_impl_morphism
      PreOrder_Transitive
      (m + p)
      m
      lemma)
  (le_add_r m p)
  H.
```

This points to two difficulties in finding patches: Knowing when to δ-reduce terms is difficult; exploring the appropriate time for reduction may produce patches for pairs that Pumpkin currently cannot patch (the Pumpkin Pi extension in Chapter 4 is deliberately strategic about δ reduction, building on this insight). Furthermore, finding patches is more challenging when neither theorem has a conclusion that is as strong as possible.

A CHALLENGE FOR INVERSION For one pair of proofs with propositionally equal conclusions, Pumpkin found a patch in one direction, but failed to invert it:

```
fun n m p (_ : n <= m) (_ : m <= p) (H1 : n <= p) =>
  gt_le_S n (S p) (le_lt_n_Sm n p H1)
: ∀ n m p, n <= m → m <= p → n <= p → S n <= S p.
```

Inversion was unable to invert this term, even though an inverse does exist. To invert this, inversion needs to know to δ-reduce `gt_le_S`:

```
gt_le_S n m :=
  (fun (H : ∀ n0 m0, n0 < m0 -> S n0 <= m0) => H n m)
  ...
: ∀ n m, n < m → S n <= m.
```

It then needs to swap the hypothesis with the conclusion in `H` to produce the inverse:

```
gt_le_S⁻¹ n m :=
  (fun (H : ∀ n0 m0, S n0 <= m0 -> n0 < m0) => H n m)
  ...
 : ∀ n m, S n <= m → n < m
```

Inversion currently swaps subterms when it is not aware of any symmetry properties about the inductive type. However, it does not know

when to δ-reduce function definitions (again, something the PUMPKIN Pi extension in Chapter 4 is deliberately strategic about). Furthermore, there are many possible subterms to swap; for inversion to know to only swap the subterms of `H`, it must have a better understanding of the structure of the term. Both of these are ways to improve inversion.

A CHALLENGE FOR GENERALIZATION Generalization produced an exponential number of candidates when abstracting a patch candidate with this type:

```
∀ n n0,
  (fun m => n <= max m n0) n →
  (fun m => n <= max n0 m) n
```

The goal was to generalize by `n` and produce a patch with this type:

```
∀ m0 n n0,
  n <= max m0 n0 →
  n <= max n0 m0
```

The difficulty was in determining which occurrences of `n` to generalize. The component needed to generalize only the highlighted occurrences:

```
fun n n0 (H0 : n <= max n0 n) =>
  @eq_ind_r
    nat
    (max n0 n)
    (fun n1 => n <= n1)
    H0
    (max n n0)
    (max_comm n n0)
```

The simplest abstraction strategy failed, and a more complex strategy succeeded only after producing exponentially many candidates. While this did not have a significant impact on time, this makes a good case for abstraction strategies that are semantics-aware. In this case, we know from the type of the candidate and the type of `eq_ind_r` that these two hypothesis types are equivalent (similarly for the conclusion types):

```
(fun m => n <= max m n0) n
(fun n1 => n <= n1) (max n0 n)
```

The tool could search recursively for patches to find two patches that bridge the two equivalent types:

```
p1 := fun n => max n0 n
p2 := fun n => max n n0
```

Then the candidate type is exactly this:

```
∀ n n0,
  (fun n1 => n <= n1) (p2 n) ->
  (fun n1 => n <= n1) (p1 n)
```

Abstraction should thus abstract the highlighted subterms and the terms that have types constrained by those subterms. This would produce a patch in one candidate:

```
fun m0 n n0 (H0 : n <= max n0 m0) =>
  @eq_ind_r
    nat
    (max n0 m0)          (* p1 m0 *)
```

```
(fun n1 => n <= n1) (* P *)
H0                  (* : P (p1 m0) *)
(max m0 n0)         (* p2 m0 *)
(max_comm m0 n0)    (* : p1 m0 = p2 m0 *)
```

This same strategy would find a patch for one of the pairs that PUMPKIN failed to abstract. This is a natural future direction for abstraction.

3.5.3.3 *Performance*

PUMPKIN performed well for all pairs. The slowest success took 48 ms.[7] When PUMPKIN failed, it failed fast. The slowest failure took 7 ms. While I found this promising, proof terms were small ($\leq$ 67 LOC); the evaluation for the PUMPKIN Pi extension in Chapter 4 will show how PUMPKIN PATCH can support large proof developments in an order of seconds.

3.6 RESULTS

To show how PUMPKIN could save work for proof engineers on real proof developments, I used the PUMPKIN prototype to emulate three motivating scenarios from real-world code:

1. **Updating definitions** within a project (CompCert, Section 3.6.1)
2. **Porting definitions** between libraries (Software Foundations, Section 3.6.2)
3. **Updating proof assistant versions** (Coq Standard Library, Section 3.6.3)

The code I chose for these scenarios demonstrated different classes of changes. For each case, I describe how PUMPKIN configures the procedure to use differencing and transformations for that class of changes. My experiences with these scenarios suggest that patches are useful and that both differencing and the transformations are effective and flexible.

IDENTIFYING CHANGES I identified Git commits from popular Coq projects that demonstrated each scenario. These commits updated proofs in response to breaking changes. I emulated each scenario as follows:

1. *Replay* an example proof update for PUMPKIN.
2. *Search* the example for a patch using PUMPKIN.
3. *Apply* the patch to fix a different broken proof.

7 i7-4790K, at 4.00 GHz, 32 GB RAM

```
Record int : Type :=
  mkint { intval: Z; intrange
      : 0 <= intval < modulus
       }.
```

```
Record int : Type :=
  mkint { intval: Z; intrange
      : -1 < intval < modulus
       }.
```

Figure 19: Old (left) and new (right) definitions of `int` in CompCert.

```
Fixpoint bin_to_nat (b : bin) :
    nat :=
  match b with
  | B0 => 0
  | B2 b' => 2 * (bin_to_nat b
      ')
  | B21 b' => 1 + 2 * (
      bin_to_nat b')
  end.
```

```
Fixpoint bin_to_nat (b : bin) :
    nat :=
  match b with
  | B0 => 0
  | B2 b' => (bin_to_nat b') +
      (bin_to_nat b')
  | B21 b' => S ((bin_to_nat b
      ') + (bin_to_nat b'))
  end.
```

Figure 20: Definitions of `bin_to_nat` for Users A (left) and B (right).

My goal was to simulate incremental use of a repair tool, at the level of a small change or a commit that follows best practices. I favored commits with changes that I could isolate. When isolating examples for PUMPKIN, I replayed changes from the bottom up, as if I was making the changes myself. This means that I did not always make the same change as the user. For example, the real change from Section 3.6.1 updated multiple definitions; I updated only one.

PUMPKIN does not yet handle structural changes like adding constructors or parameters, so I focused on changes that preserve structure, like modifying constructors. Chapter 4 describes an extension to PUMPKIN PATCH that supports changes in structure.

3.6.1 *Updating Definitions*

Coq programmers sometimes make changes to definitions that break proofs within the same project. To emulate this use case, I identified a CompCert commit [31] with a breaking change to `int` (Figure 19). I used PUMPKIN to find a patch that corresponds to the change in `int`. The patch PUMPKIN found fixed broken inductive proofs.

REPLAY I used the proof of `unsigned_range` as the example for PUMPKIN. The proof failed with the new `int`:

```
Theorem unsigned_range:
  ∀(i : int), 0 <= unsigned i < modulus.
Proof.
  intros i. induction i using int_ind; auto.
```
✗

I replayed the change to `unsigned_range`:

```
  intros i. induction i using int_ind. simpl.  omega.
```
✓

SEARCH I used PUMPKIN to search the example for a patch that corresponds to the change in `int`. It found a patch with this type:

```
∀ z : Z, -1 < z < modulus -> 0 <= z < modulus
```

APPLY After changing the definition of `int`, the proof of the theorem `repr_unsigned` failed on the last tactic:

```
Theorem repr_unsigned:
  ∀(i : int), repr (unsigned i) = i.
Proof.
  ... apply Zmod_small; auto.✗
```

Manually trying `omega`—the tactic which helped us in the proof of `unsigned_range`—did not succeed. I added the patch that PUMPKIN found to a hint database. The proof of the theorem `repr_unsigned` then went through:

```
... apply Zmod_small; auto.✓
```

3.6.1.1 *Configuration*

This scenario used the configuration for changes in constructors of an inductive type. Given such a change:

```
Inductive T := ... | C : ... → H → T
Inductive T' := ... | C : ... → H' → T'
```

PUMPKIN searches in the difference between two inductive proofs of theorems:

```
∀ (t : T), P t
∀ (t : T'), P t
```

for an isomorphism[8] between the constructors:

```
... → H → H'
... → H' → H
```

The proof engineer can apply these patches within the inductive case that corresponds to the constructor `C` to fix other broken proofs that induct over the changed type. PUMPKIN uses this configuration for changes in constructors:

1: *diff* inductive constructors for goals
2: *diff* and *transform* to recursively search for changes in conclusions of the corresponding case of the proof
3: **if** there are candidates **then**
4: try to *invert* the patch to find an isomorphism

3.6.2 *Porting Definitions*

Proof engineers sometimes port theorems and proofs to use definitions from different libraries. To simulate this, I used PUMPKIN to port two solutions [3, 6] to an exercise in Software Foundations to each use the other solution's definition of the fixpoint `bin_to_nat` (Figure 20). I demonstrate one direction; the opposite was similar.

8 If PUMPKIN finds just one implication, it returns that.

REPLAY I used the proof of `bin_to_nat_pres_incr` from User A as the example for PUMPKIN. User A cut an inline lemma in an inductive case and proved it using a rewrite:

```
assert (∀a, S (a + S (a + 0)) = S (S (a + (a + 0)))).
- ... rewrite ← plus_n_O. rewrite → plus_comm.
```

When I ported User A's solution to use User B's definition of `bin_to_nat`, the application of this inline lemma failed. I changed the conclusion of the inline lemma and removed the corresponding rewrite:

```
assert (∀a, S (a + S a) = S (S (a + a))).
- ... rewrite → plus_comm.
```

SEARCH I used PUMPKIN to search in the difference between the old and new versions of the example patched proof for a patch that corresponds to the change in `bin_to_nat`. It found an isomorphism:

```
∀P b, P (bin_to_nat b) → P (bin_to_nat b + 0)
∀P b, P (bin_to_nat b + 0) → P (bin_to_nat b)
```

APPLY After porting to User B's definition, a rewrite in the proof of the theorem `normalize_correctness` failed:

```
Theorem normalize_correctness:
  ∀b, nat_to_bin (bin_to_nat b) = normalize b.
Proof.
  ... rewrite → plus_0_r.✗
```

Attempting the obvious patch from the difference in tactics—rewriting by `plus_n_O`—failed. Applying the patch that PUMPKIN found fixed the broken proof:

```
... apply patch_inv. rewrite → plus_0_r.✓
```

In this case, since I ported User A's definition to a simpler definition,[9] PUMPKIN found a patch that was not the most natural patch. The natural patch would be to remove the rewrite, just as I removed a different rewrite from the example proof. This did not occur when I ported User B's definition, which suggests that in the future, a patch finding tool may help inform novice users which definition is simpler: It can factor the proof, then inform the user if two factors are inverses. Tactic-level changes do not provide enough information to determine this; the tool must have a semantic understanding of the terms. My Magic tutorial plugin [10] implements a prototype of the functionality needed for this, based on lessons from this case study.

3.6.2.1 *Configuration*

This scenario used the configuration for changes in cases of a fixpoint. Given such a change:

```
Fixpoint f ... := ... | g x
Fixpoint f' ... := ... | g x'
```

9 User A uses `*`; User B uses `+`. For arbitrary `n`, the term `2 * n` reduces to `n + (n + 0)`, which does not reduce any further.

10 `https://github.com/uwplse/magic`

```
Definition divide p q := ∃
    r, p * r = q.
```

```
Definition divide p q := ∃
    r, q = r * p.
```

Figure 21: Old (left) and new (right) definitions of `divide` in Coq.

PUMPKIN searches in the difference between two versions of proofs of the theorems:

```
∀ ..., P (f ...)
∀ ..., P (f' ...)
```

for an isomorphism that corresponds to the change:

```
∀P, P x → P x'
∀P, P x' → P x
```

The proof engineer can apply these patches to fix other broken proofs about the fixpoint.

The key feature that differentiates these from the patches we have encountered so far is that these patches hold for *all* `P`; for changes in fixpoint cases, the procedure abstracts candidates by `P`, not by its arguments. PUMPKIN uses this configuration for changes in fixpoint cases:

1: *diff* fixpoint cases for goals
2: *diff* and *transform* to recursively search within an intermediate lemma for a change in conclusions
3: **if** there are candidates **then**
4: *specialize* and *factor* the candidate
abstract the factors by functions
try to *invert* the patch to find an isomorphism

For the prototype, I require the user to cut the intermediate lemma explicitly and to pass its type and arguments. In the future, an improved semantic differencing component can infer both the intermediate lemma and the arguments: it can search within the proof for some proof of a function that is applied to the fixpoint.

3.6.3 *Updating Proof Assistant Versions*

Coq sometimes makes changes to its standard library that break backwards compatibility. To test the plausibility of using a patch finding tool for proof assistant version updates, I identified a breaking change in the Coq standard library [32]. The commit changed the definition of `divide` prior to the Coq 8.4 release (Figure 21). The change broke 46 proofs in the standard library. I used PUMPKIN to find an isomorphism that corresponds to the change in `divide`. The isomorphism PUMPKIN found fixed broken proofs.

REPLAY I used the proof of `mod_divide` as the example for PUMPKIN. The proof broke with the new `divide`:

```
Theorem mod_divide:
  ∀ a b, b~=0 → (a mod b == 0 ↔ (divide b a)).
```

```
Proof.
  ... rewrite (div_mod a b Hb) at 2.✗
```

I replayed changes to `mod_divide`:

```
  ... rewrite mul_comm.  symmetry.
  rewrite (div_mod a b Hb) at 2.✓
```

SEARCH I used PUMPKIN to search within the example patched proof for a patch that corresponds to the change in `divide`. It found an isomorphism:

```
∀r p q, p * r = q → q = r * p
∀r p q, q = r * p → p * r = q
```

APPLY The proof of the theorem `Zmod_divides` broke after rewriting by the changed theorem `mod_divide`:

```
Theorem Zmod_divides:
  ∀a b, b<>0 → (a mod b = 0 ↔ ∃c, a = b * c).
Proof.
  ... split; intros (c,Hc); exists c; auto.✗
```

Adding the patches PUMPKIN found to a hint database made the proof go through:

```
  ... split; intros (c,Hc); exists c; auto.✓
```

3.6.3.1 *Configuration*

This scenario used the configuration for changes in dependent arguments to constructors. PUMPKIN searches within the difference between two versions of a proof that apply the same constructor to different dependent arguments:

```
... (C (P x)) ...
... (C (P' x)) ...
```

for an isomorphism between the arguments:

```
∀ x, P x -> P' x
∀ x, P' x -> P x
```

The proof engineer can apply these patches to patch proofs that apply the constructor (in this case study, to fix broken proofs that instantiate `divide` with some specific `r`).

So far, we have encountered changes of this form as arguments to an induction principle; in this case, the change is an argument to a constructor. A patch between arguments to an induction principle maps directly between conclusions of the new and old theorem without induction; a patch between constructors does not. For example, for `divide`, we can find a patch with this form:

```
∀x, P x → P' x
```

However, without using the induction principle for `exists`, we can't use that patch to prove this:

```
(∃x, P x) → (∃ x, P' x)
```

This changes the goal type that semantic differencing determines.

PUMPKIN uses this configuration for changes in constructor arguments:

1: *diff* constructor arguments for goals
2: *diff* and *transform* to recursively search within those arguments for changes in conclusions
3: **if** there are candidates **then**
4: *abstract* the candidate
factor and try to *invert* the patch to find an isomorphism

For the prototype, the model of constructors for the semantic differencing component is limited, so PUMPKIN asks the user to provide the type of the change in argument (to guide line 2). Extending semantic differencing may help remove this restriction.

3.7 CONCLUSION

Recall that this thesis set out to prove that:

> **Thesis**: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and apply to fix other proofs broken by the same change. A tool that automates this can save work for proof engineers relative to reference manual repairs in practical use cases.

Or, in other words, there *is* reason to believe that verifying a modified system should often, in practical use cases, be easier than verifying the original the first time around, even when proof engineers do not follow good development processes, or when change occurs outside of proof engineers' control.

With PUMPKIN, it is fair to say:

> **Thesis So Far**: Some changes in programs and specifications, and the corresponding changes in proofs broken by those changes carry information that a tool can extract, generalize, and sometimes apply to fix other proofs broken by the same change (Sections 3.2, 3.3, and 3.4). A tool that automates this (Section 3.5) could have saved work for proof engineers relative to reference manual repairs in a few practical use cases (Section 3.6).

This is progress, but it is not so satisfying. As I have shown you throughout this chapter, the PUMPKIN prototype is too limited in both theory and implementation. Most notably, the PUMPKIN prototype has limited support for patch application and supports a narrow class of changes. And as I mentioned earlier, without considering the extension from the next chapter, the PUMPKIN prototype does not include any support for tactics beyond the use of hints.

The next chapter will address these limitations. In doing so, it will show how the thesis holds on a broad class of changes, with more principled and better integrated support for patch application and tactic generation. It will show how all of this helps proof engineers in the real world—not just retroactively, but in real time.

4

PROOF REPAIR ACROSS TYPE EQUIVALENCES

This chapter presents the Pumpkin Pi extension to the Pumpkin Patch proof repair plugin suite for Coq 8.8 that is available on Github.[1] Pumpkin Pi is a plugin that adds support for proof repair across a broad class of changes in datatypes called *type equivalences* (Section 4.2.2), thereby supporting a large class of practical repair scenarios. Proof repair across type equivalences with Pumpkin Pi makes progress on two challenges that the Pumpkin prototype had left open:

1. Pumpkin supported a very limited classes of changes in datatypes, namely those that do not change structure. Similar tools developed since still supported only a predefined set of changes [44, 50]. As my user study showed, none of these were informed by the current needs of proof engineers.

2. Pumpkin had only preliminary integration with typical proof engineering workflows. Similar tools developed since likewise lacked support for workflow integration [42, 44], and in some cases imposed additional proof obligations like proving relations corresponding to changes [47].

ADDRESSING CHALLENGE 1: FLEXIBLE TYPE SUPPORT The case studies in Section 4.6—summarized in Table 1 on page 96—show that Pumpkin Pi is flexible enough to support a wide range of proof engineering use cases. In general, Pumpkin Pi can support any change described by an equivalence, though it takes the equivalence in a deconstructed form—the *configuration* for Pumpkin Pi. The configuration expresses to the proof term transformation how to translate functions and proofs defined over the old version of a type to refer only to the new version, and how to do so in a way that does not break definitional equality. The proof engineer can write this configuration in Coq and feed it to REPLica (*manual configuration* in Table 1), configuring REPLica to support the change.

ADDRESSING CHALLENGE 2: WORKFLOW INTEGRATION Research on workflow integration for proof repair tools is in its infancy. Pump-

1 I annotate each claim to which code is relevant with a circled number like ①. These circled numbers are links to code, and are detailed in `GUIDE.md`.

```
Inductive list (T : Type) :
    Type :=
| nil : list T
| cons : T → list T → list
    T.
```

```
Inductive list (T : Type) :
    Type :=
| cons : T → list T → list
    T
| nil : list T.
```

Figure 22: A change from the old version of `list` (left) to the new version of `list` (right). Recall that `list` is an inductive datatype that is either empty (the `nil` constructor), or the result of placing an element in front of another `list` (the `cons` constructor). The change swaps these two constructors (orange).

KIN Pi is built with workflow integration in mind. For example, PUMPKIN Pi is the only proof repair tool I am aware of that produces suggested proof scripts (rather than proof terms) for repaired proofs, a challenge highlighted in the previous chapter, in other existing proof repair work [44], and in my survey of proof engineering [42]. In addition, PUMPKIN Pi implements search procedures that automatically discover configurations and prove the equivalences they induce for four different classes of changes (*automatic configuration* in Table 1), decreasing the burden of proof obligations imposed on the proof engineer. My partnership with an industrial proof engineer has informed other changes to further improve workflow integration (Sections 4.5 and 4.6).

BRINGING IT TOGETHER In the frame of the thesis, proof repair across type equivalences is a new form of proof automation that extracts general information from breaking changes in the datatypes that programs, specifications, and proofs refer to, then applies that information to fix any program, specification, or proof broken by that change (Section 4.2). This extraction, generalization, and application works at the level of proof terms, through a combination of novel semantic differencing algorithms over datatypes (Section 4.3) and a configurable proof term transformation (Section 4.4). PUMPKIN Pi automates this process, with additional support for manual configuration by proof engineers and for integration with typical proof engineering workflows (Section 4.5). Case studies show that PUMPKIN Pi can save and in several cases *has* saved work for proof engineers on major proof developments and on changes that matter (Section 4.6).

4.1 MOTIVATING EXAMPLE

Consider a simple example of using PUMPKIN Pi: repairing proofs after swapping the two constructors of the `list` datatype (Figure 22). This is inspired by a similar change from a user study of proof engineers

(Section 4.6). Even such a simple change can cause trouble, as in this proof from the Coq standard library (comments ours for clarity):[2]

```
Lemma rev_app_distr {A} :
  ∀ (x y : list A), rev (x ++ y) = rev y ++ rev x.
Proof. (* by induction over x and y *)
  induction x as [| a l IHl].
  (* x nil:  *) induction y as [| a l IHl].
  (* y nil:  *) simpl. auto.
  (* y cons *) simpl. rewrite app_nil_r; auto.
  (* both cons:  *) intro y. simpl.
  rewrite (IHl y). rewrite app_assoc; trivial.
Defined.
```

This lemma says that appending (++) two lists and reversing (`rev`) the result behaves the same as appending the reverse of the second list onto the reverse of the first list. The proof script works by induction over the input lists `x` and `y`: In the base case for both `x` and `y`, the result holds by reflexivity. In the base case for `x` and the inductive case for `y`, the result follows from the existing lemma `app_nil_r`. Finally, in the inductive case for both `x` and `y`, the result follows by the inductive hypothesis and the existing lemma `app_assoc`.

When we change the `list` type, this proof no longer works. To repair this proof with PUMPKIN Pi, we run this command:

```
Repair Old.list New.list in rev_app_distr.
```

assuming the old and new list types from Figure 22 are in modules `Old` and `New`. This suggests a proof script that succeeds (in light blue to denote PUMPKIN Pi produces it automatically):

```
Proof. (* by induction over x and y *)
  intros x. induction x as [a l IHl| ]; intro y0.
  - (* both cons:  *) simpl. rewrite IHl. simpl.
    rewrite app_assoc. auto.
  - (* x nil:  *) induction y0 as [a l H| ].
    + (* y cons:  *) simpl. rewrite app_nil_r. auto.
    + (* y nil:  *) auto.
Defined.
```

where the dependencies (`rev`, `++`, `app_assoc`, and `app_nil_r`) have also been updated automatically ①. If we would like, we can manually modify this to something that more closely matches the style of the original proof script:

```
Proof. (* by induction over x and y *)
  induction x as [a l IHl|].
  (* both cons:  *) intro y. simpl.
  rewrite (IHl y). rewrite app_assoc; trivial.
  (* x nil:  *) induction y as [a l IHl|].
  (* y cons:  *) simpl. rewrite app_nil_r; auto.
  (* y nil:  *) simpl. auto.
Defined.
```

We can even repair the entire list module from the Coq standard library all at once by running the `Repair module` command ①. When we are done, we can get rid of `Old.list`.

The key to success is taking advantage of Coq's structured proof term language: Recall that Coq compiles every proof script to a proof

2 We use induction instead of pattern matching.

term in the rich functional programming language Gallina—PUMPKIN Pi repairs that term. PUMPKIN Pi then decompiles the repaired proof term (with optional hints from the original proof script) back to a suggested proof script that the proof engineer can maintain.

In contrast, updating the poorly structured proof script directly would not be straightforward. Even for the simple proof script above, grouping tactics by line, there are $6! = 720$ permutations of this proof script. It is not clear which lines to swap since these tactics do not have a semantics beyond the searches their evaluation performs. Furthermore, just swapping lines is not enough: even for such a simple change, we must also swap arguments, so `induction x as [| a l IHl ]` becomes `induction x as [a l IHl|]`. Valentin Robert's thesis [44] describes the challenges of repairing tactics in detail. PUMPKIN Pi's approach circumvents this challenge.

4.2 APPROACH

PUMPKIN Pi can do much more than permute constructors. Given an equivalence between types A and B, PUMPKIN Pi repairs functions and proofs defined over A to instead refer to B. It does this in a way that allows for removing references to A, which is essential for proof repair, since A may be an old version of an updated type.

The proof engineer can use PUMPKIN Pi (Section 4.5.2) to automatically patch broken proofs in response to a broad class of changes in datatypes. PUMPKIN Pi in particular repairs proofs in response to changes in types that correspond to *type equivalences* [49], or pairs of functions that map between two types (possibly with some additional information) and are mutual inverses (Section 4.2.2).[3] In other words, looking back to the thesis statement, the information shows up in the difference between versions of the changed datatype. With automatic configuration, REPLICA can extract and generalize that information to a type equivalence, then apply it to fix other broken proofs. With manual configuration, the human extracts and generalizes the information corresponding to the change, but PUMPKIN Pi can still apply it to fix other broken proofs.

Like the original PUMPKIN prototype, it also does this using a combination of differencing and proof term transformations. The corresponding differencing algorithms (Section 4.2.3) run in response to a breaking change in a datatype that corresponds to a type equivalence. When they succeed, the diff that they find is that type equivalence. The proof engineer can also pass the type equivalence to PUMPKIN Pi directly, effectively doing differencing by hand. In either case, the proof term transformation (Section 4.2.4) then transforms a proof term defined over the old version of the dataype directly to a proof term defined over the new version of the datatype. PUMPKIN Pi further

3 The adjoint follows, and PUMPKIN Pi includes machinery to prove it (10) (23).

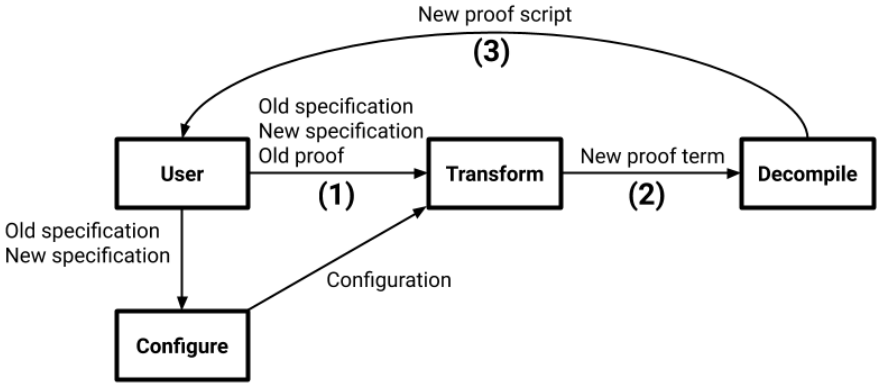


Figure 23: The workflow for PUMPKIN Pi.

supports proof script integration and other features for better workflow integration (Section **??**), with real payoffs for proof engineers (Section 4.6).

4.2.1 *Workflow: Configure, Transform, Decompile*

As with PUMPKIN, the interface to PUMPKIN Pi is exposed to the proof engineer as a command. PUMPKIN Pi extends PUMPKIN PATCH with a new command called `Repair`, with the syntax:

```
Repair old_type new_type in old_proof.
```

where `old_type` and `new_type` are the old and new versions of the changed datatype, and `old_proof` is the old version of the proof broken by that change in datatype. This invokes the PUMPKIN Pi plugin, which updates the old version of the proof to some new version of the proof defined over the new version of the datatype, then defines it as a new proof term and suggests a corresponding new proof script if successful. All terms that PUMPKIN Pi defines are type checked in the end, so PUMPKIN Pi does not extend the TCB.

Figure 23 shows how this comes together when the proof engineer invokes PUMPKIN Pi:

1. The proof engineer **Configure**s PUMPKIN Pi, either manually or automatically.
2. The configured **Transform** transforms the old proof term into the new proof term.
3. **Decompile** suggests a new proof script.

There are currently four search procedures for automatic configuration implemented in PUMPKIN Pi (see Table 1 on page 96). Manual configuration makes it possible for the proof engineer to configure the transformation to any equivalence, even without a search procedure.

The breaking change to Figure 22 in Section **??**, for example, used automatic configuration. When we invoked PUMPKIN Pi:

```
swap T (l : Old.list T) : New.
    list T :=
  Old.list_rect T (fun (l :
      Old.list T) => New.list T
      )
    New.nil
    (fun t _ (IHl : New.list T)
        => New.cons T t IHl)
    l.
```

```
swap^-1 T (l : New.list T) :
    Old.list T :=
  New.list_rect T (fun (l :
      New.list T) => Old.list T
      )
    (fun t _ (IHl : Old.list T)
        => Old.cons T t IHl)
    Old.nil
    l.
```

```
Lemma section: ∀ T (l : Old.
    list T),
  swap^-1 T (swap T l) = l.
Proof.
  intros T l. symmetry.
    induction l as [ |a l0 H
    ].
  - auto.
  - simpl. rewrite ← H. auto.
Qed.
```

```
Lemma retraction: ∀ T (l : New.
    list T),
  swap T (swap^-1 T l) = l.
Proof.
  intros T l. symmetry.
    induction l as [t l0 H|
    ].
  - simpl. rewrite ← H. auto.
  - auto.
Qed.
```

Figure 24: Two functions between `Old.list` and `New.list` (top) that form an equivalence (bottom).

```
Inductive I :=
| A : I
| B : I.
```

```
Inductive J :=
| makeJ : bool → J.
```

Figure 25: The old type `I` (left) is either `A` or `B`. The new type `J` (right) is `I` with `A` and `B` factored out to `bool` (orange).

```
Repair Old.list New.list in rev_app_distr.
```

it invoked a search procedure that differenced `Old.list` and `New.list`, then transformed `rev_app_distr` to use `New.list` in place of `Old.list`. In the end, it suggested a proof script to us that we could use going forward.

4.2.2 *Scope: Type Equivalences*

Pumpkin Pi automatically repairs proofs in response to changes in types that correspond to type equivalences. When a type equivalence between types A and B exists, those types are *equivalent* (denoted $A \simeq B$). Figure 24 shows a type equivalence between the two versions of `list` from Figure 22 that Pumpkin Pi discovered and proved automatically ①.

To give some intuition for what kinds of changes can be described by equivalences, I preview two changes below. See Table 1 on page 96 for more examples.

FACTORING OUT CONSTRUCTORS Consider changing the type `I` to the type `J` in Figure 25. `J` can be viewed as `I` with its two constructors `A` and `B` pulled out to a new argument of type `bool` for a single constructor. With Pumpkin Pi, the proof engineer can repair functions

```
Inductive list (T : Type) :
    Type :=
| nil : list T
| cons : T → list T → list T.
```

```
Inductive vector (T : Type) : nat →
    Type :=
| nil : vector T 0
| cons : T → ∀ (n : nat), vector T
    n → vector T (S n).
```

Figure 26: A vector (bottom) is a list (top) indexed by its length (orange). Vectors effectively make it possible to enforce length invariants about lists at compile time.

and proofs about `I` to instead use `J`, as long as she configures PUMPKIN Pi to describe which constructor of `I` maps to `true` and which maps to `false`. This information about constructor mappings induces an equivalence `I ≃ J` across which PUMPKIN Pi repairs functions and proofs. File ② shows an example of this, mapping `A` to `true` and `B` to false, and repairing proofs of De Morgan's laws.

ADDING A DEPENDENT INDEX At first glance, the word *equivalence* may seem to imply that PUMPKIN Pi can support only changes in which the proof engineer does not add or remove information. But equivalences are more powerful than they may seem. Consider, for example, changing a list to a length-indexed vector (Figure 26). PUMPKIN Pi can repair functions and proofs about lists to functions and proofs about vectors of particular lengths ③, since `Σ(l:list T) .length l = n ≃ vector T n`. From the proof engineer's perspective, after updating specifications from `list` to `vector`, to fix her functions and proofs, she must additionally prove invariants about the lengths of her lists. PUMPKIN Pi makes it easy to separate out that proof obligation, then automates the rest.

More generally, in homotopy type theory, with the help of quotient types, it is possible to form an equivalence from a relation, even when the relation is not an equivalence [4]. While Coq lacks quotient types, it is possible to achieve a similar outcome and use PUMPKIN Pi for changes that add or remove information when those changes can be expressed as equivalences between Σ types or sum types.

4.2.3 *Differencing: Equivalences from Changes*

Differencing in PUMPKIN Pi is what configures the proof term transformation with a particular type equivalence. By default, when the proof engineer invokes the `Repair` command, differencing runs automatically. For example, when we invoked:

```
Repair Old.list New.list in rev_app_distr.
```

in Section **??**, differencing discovered and proved the equivalence in Figure 24. In total, PUMPKIN Pi currently implements four search procedures for automatic configuration; I will explain these more in Sections 4.3 and 4.5. Each search procedure automates differencing

for an entire class of changes that correspond to type equivalences. In the end, it returns a configuration that corresponds to the equivalence (see Section 4.3 for details), along with the equivalence itself.

Manual configuration makes it possible for the proof engineer to skip differencing, so that PUMPKIN Pi is not limited by the search procedures currently implemented. Manual configuration supports any change that can be described by a type equivalence. To configure PUMPKIN Pi manually, the proof engineer can pass the configuration correspond to the equivalence to PUMPKIN Pi's `Configure Repair` command before invoking `Repair`. This is what I did for the change in Figure 25 ②.

SUMMARY In summary, differencing has the following specification:

- **Inputs**: types A and B, assuming:
  - there is a type equivalence describing the change from A to B (possibly with some new information), and
  - the corresponding change is in a class currently supported by a search procedure for automatic configuration.
- **Outputs**:
  - functions `f` and `g`,
  - proofs `section` and `retraction`, and
  - a configuration c,

  guaranteeing:
  - `f` and `g` form an equivalence that describes the change from A to B (possibly with some new information),
  - `section` and `retraction` prove that `f` and `g` form an equivalence, and
  - c is a decomposition of the equivalence.

The new information for the change in Figure 26, for example, is the length of the list. Differencing discovers the equivalence corresponding to this change, as well as a configuration c that is a decomposition of this equivalence. Section 4.3 describes what it means for c to be a decomposition of the equivalence, and proves that this is possible for any equivalence. Manual configuration, on the other hand, takes in the configuration directly. In either case, the transformation uses this configuration to automatically repair broken proofs.

4.2.4 *Transformation: Transport with a Twist*

PUMPKIN Pi repairs proofs in response to these changes by implementing a kind of proof reuse known as *transport* [49], but in a way that is

suitable for repair. Informally, transport takes a term t and produces a term t' that is the same as t modulo an equivalence $A \simeq B$. If t is a function, then t' behaves the same way modulo the equivalence; if t is a proof, then t' proves the same theorem the same way modulo the equivalence.

When transport across $A \simeq B$ takes t to t', I say that t and t' are *equal up to transport* across that equivalence (denoted $t \equiv_{A \simeq B} t'$).[4] In Section 4.1, the original append function `++` over `Old.list` and the repaired append function `++` over `New.list` that PUMPKIN Pi produces are equal up to transport across the equivalence from Figure 24, since (by `app_ok` ①):

```
∀ T (l1 l2 : Old.list T),
  swap T (l1 ++ l2) = (swap T l1) ++ (swap T l2).
```

The original `rev_app_distr` is equal to the repaired proof up to transport, since both prove the same thing the same way up to the equivalence, and up to the changes in `++` and `rev`.

Transport typically works by applying the functions that make up the equivalence to convert inputs and outputs between types. This approach would not be suitable for repair, since it does not make it possible to remove the old type A. PUMPKIN Pi implements transport in a way that allows for removing references to A—by proof term transformation.

SUMMARY In summary, the transformation has the following specification:

- **Inputs**:
  - types A and B,
  - the configuration c, and
  - a proof term t,

  assuming c corresponds to a type equivalence describing the change from A to B (possibly with some new information).
- **Outputs**: a proof term t', guaranteeing:
  - t' refers to B in place of A, and
  - t and t' are equal up to transport along the equivalence formed by c.

This specification glazes over a few important issues, most notably that B may refer to A (so PUMPKIN Pi has specialized termination

4 This notation should be interpreted in a metatheory with *univalence*—a property that Coq lacks—or it should be approximated in Coq. The details of transport with univalence are in the Homotopy Type Theory book [49], and an approximation in Coq is in the univalent parametricity framework paper [46]. For equivalent A and B, there can be many equivalences $A \simeq B$. Equality up to transport is across a *particular* equivalence, but I erase this in the notation.

```
DepConstr(0, list T) : list T
    := Constr(0, list T).
DepConstr(1, list T) t l :
    list T :=
  Constr (1, list T) t l.

DepElim(l, P) { pnil, pcons } :
    P l :=
  Elim(l, P) { pnil, pcons }.
```

```
DepConstr(0, list T) : list T
    := Constr(1, list T).
DepConstr(1, list T) t l :
    list T :=
  Constr(0, list T) t l.

DepElim(l, P) { pnil, pcons } :
    P l :=
  Elim(l, P) { pcons, pnil }.
```

Figure 27: The dependent constructors and eliminators for old (left) and new (right) `list`, with the difference in orange.

logic), and that it may be desirable to port only *some* instances of A to B (but PUMPKIN Pi by default ports *all* instances). I discuss these more in Sections 4.4 and 4.5.

4.3 DIFFERENCING

Differencing—whether done by the tool (automatic configuration) or by the proof engineer (manual configuration)—identifies and proves a type equivalence. But differening further decomposes that equivalence into a form called a *configuration*. The configuration is the key to building a proof term transformation that implements transport in a way that is suitable for repair.

Each configuration corresponds to an equivalence $A \simeq B$. It deconstructs the equivalence into things that talk about A, and things that talk about B. It does so in a way that hides details specific to the equivalence, like the order or number of arguments to an induction principle or type.

At a high level, the configuration helps the transformation achieve two goals:

1. preserve equality up to transport across the equivalence between A and B (Section 4.3.1), and
2. produce well-typed terms (Section 4.3.2).

To differencing, this configuration is a pair of pairs of terms:

```
((DepConstr, DepElim), (Eta, Iota))
```

each of which corresponds to one of the two goals: `DepConstr` and `DepElim` define how to transform constructors and eliminators, thereby preserving the equivalence, and `Eta` and `Iota` define how to transform η-expansion and ι-reduction of constructors and eliminators, thereby producing well-typed terms.

Formally, every correct configuration corresponds to an initial algebra of an endofunctor, and correctness—that every configurations and equivalences are isomorphic—follows by Lambek's theorem (Sec-

tion 4.3.3).[5] Each search procedure for automatic configuration produces both the configuration and the equivalence that it induces (Section 4.3.4). Manual configuration takes as input the configuration directly.

All terms that I introduce in this section are in CIC_ω with primitive eliminators. Section 4.3.5 revisits the limitations of differencing from PUMPKIN, and Section 4.5 describes how I scale this from CIC_ω to Coq.

4.3.1 *Preserving the Equivalence*

To preserve the equivalence, the configuration maps terms over A to terms over B by viewing each term of type B as if it were an A. This way, the transformation in Section 4.4 can replace values of A with values of B, and inductive proofs about A with inductive proofs about B, all without changing the order or number of arguments.

The two configuration parts responsible for this are `DepConstr` and `DepElim` (*dependent constructors* and *eliminators*). These describe how to construct and eliminate A and B, wrapping the types with a common inductive structure. The transformation requires the same number of dependent constructors and cases in dependent eliminators for A and B, even if A and B are types with different numbers of constructors (A and B need not even be inductive; see Sections 4.3.3 and 4.6).

For the `list` change from Section 4.1, the configuration that PUMPKIN Pi discovers uses the dependent constructors and eliminators in Figure 27. The dependent constructors for `Old.list` are the normal constructors with the order unchanged, while the dependent constructors for `New.list` swap constructors back to the original order. Similarly, the dependent eliminator for `Old.list` is the normal eliminator for `Old.list`, while the dependent eliminator for `New.list` swaps cases.

As the name hints, these constructors and eliminators can be dependent. Consider the type of vectors of some length:

```
Σ(n : nat).vector T n
```

PUMPKIN Pi can port proofs across the equivalence between this type and `list T` ③. The dependent constructors PUMPKIN Pi discovers pack the index into an existential, like:

```
DepConstr(0, Σ(n : nat).vector T n) : Σ(n : nat).vector T n
    :=
  ∃ (Constr(0, nat)) (Constr(0, vector T)).
```

and the eliminator it discovers eliminates the projections:

```
DepElim(s, P) { f₀ f₁ } : P (∃ (π_l s) (π_r s)) :=
```

5 When I met with Michael Shulman over coffee in San Diego a few years ago, he at various points said that a preliminary version of this work "feels like univalence," "feels like coherence," and "feels like an endofunctor." All three were correct! But I didn't understand the connection to endofunctors by way of Lambek's until a few months before writing this thesis. Anders Mörtberg and Carlo Angiuli identified this connection, and Carlo explained it to me in great detail. It is quite beautiful!

```
Elim(π_r s, λ(n : nat)(v : vector T n).P (∃ n v)) {
  f_0,
  (λ(t : T)(n : nat)(v : vector T n).f_1 t (∃ n v))
}.
```

In both these examples, the interesting work moves into the configuration: the configuration for the first swaps constructors and cases, and the configuration for the second maps constructors and cases over `list` to constructors and cases over `packed_vect`. That way, the transformation need not add, drop, or reorder arguments—it can truly be generic over type equivalences. Furthermore, both examples use automatic configuration, so differencing in PUMPKIN Pi's **Configure** component discovers `DepConstr` and `DepElim` from just the types A and B, taking care of even the difficult work.

4.3.2 *Producing Well-Typed Terms*

The other configuration parts `Eta` and `Iota` deal with producing well-typed terms, in particular by transporting equalities. CIC$_\omega$ distinguishes between two important kinds of equality: those that hold by reduction (*definitional* equality), and those that hold by proof (*propositional* equality). That is, two terms `t` and `t'` of type `T` are definitionally equal if they reduce to the same normal form, and propositionally equal if there is a proof that `t = t'` using the inductive equality type `=` at type `T`. Definitionally equal terms are necessarily propositionally equal, but the converse is not in general true.

When a datatype changes, sometimes, definitional equalities defined over the old version of that type must become propositional. A naive proof term transformation may fail to generate well-typed terms if it does not account for this. Otherwise, if the transformation transforms a term `t : T` to some `t' : T'`, it does not necessarily transform `T` to `T'` [47].

`Eta` and `Iota` describe how to transport equalities. More formally, they define η-expansion and ι-reduction of A and B, which may be propositional rather than definitional, and so must be explicit in the transformation. η-expansion describes how to expand a term to apply a constructor to an eliminator in a way that preserves propositional equality, and is important for defining dependent eliminators [39]. ι-reduction (β-reduction for inductive types) describes how to reduce an elimination of a constructor [38].

The configuration for the change from lists to vectors of some length has propositional `Eta`. It uses η-expansion for Σ:

```
Eta(Σ (n : nat) . vector T n) := λ(s : Σ (n : nat) . vector T
    n).∃ (π_l s) (π_r s).
```

which is propositional and not definitional in Coq. Thanks to this, we can forego the assumption that our language has primitive projections (definitional η for Σ).

```
                                  Inductive positive :=
                                  | xI : positive → positive
                                  | xO : positive → positive
                                  | xH : positive.

Inductive nat :=                  Inductive N :=
| O : nat                         | N0 : N
| S : nat → nat.                  | Npos : positive → N.
```

Figure 28: A unary natural number `nat` (left) is either zero (`O`) or the successor of some other natural number (`S`). A binary natural number `N` (right) is either zero (`N0`) or a positive binary number (`Npos`), where a positive binary number is either 1 (`xH`), or the result of shifting left and adding 1 (`xI`) or 0 (`xO`). Unary and binary natural numbers are equivalent, but have different inductive structures. Consequentially, definitional equalities over `nat` may become propositional over `N`.

Each `Iota`—one per constructor—describes and proves the ι-reduction behavior of `DepElim` on the corresponding case. This is needed, for example, to port proofs about unary numbers `nat` to proofs about binary numbers `N` (Figure 28). While we can define `DepConstr` and `DepElim` to induce an equivalence between them ⑤, we run into trouble reasoning about applications of `DepElim`, since proofs about `nat` that hold by reflexivity do not necessarily hold by reflexivity over `N`. For example, in Coq, while `S (n + m) = S n + m` holds by reflexivity over `nat`, when we define `+` with `DepElim` over `N`, the corresponding theorem over `N` does not hold by reflexivity.

To transform proofs about `nat` to proofs about `N`, we must transform *definitional* ι-reduction over `nat` to *propositional* ι-reduction over `N`. For our choice of `DepConstr` and `DepElim`, ι-reduction is definitional over `nat`, since a proof of:

```
∀ P pO pS n,
  DepElim(DepConstr(1, nat) n, P) { pO, pS } =
  pS n (DepElim(n, P) { pO, pS }).
```

holds by reflexivity. `Iota` for `nat` in the `S` case is a rewrite by that proof by reflexivity ⑤, with type:

```
∀ P pO pS n (Q: P (DepConstr(1, nat) n) → s),
  Iota(1, nat, Q) :
    Q (pS n (DepElim(n, P) { pO, pS })) →
    Q (DepElim(DepConstr(1, nat) n, P) { pO, pS }).
```

In contrast, ι for `N` is propositional, since the theorem:

```
∀ P pO pS n,
  DepElim(DepConstr(1, N) n, P) { pO, pS } =
  pS n (DepElim(n, P) { pO, pS }).
```

no longer holds by reflexivity. `Iota` for `N` is a rewrite by the propositional equality that proves this theorem ⑤, with type:

```
∀ P pO pS n (Q: P (DepConstr(1, N) n) → s),
  Iota(1, N, Q) :
    Q (pS n (DepElim(n, P) { pO, pS })) →
```

```
section: ∀ (a : A), g (f a)
     = a.
retraction: ∀ (b : B), f (g
     b) = b.

constr_ok: ∀ j x⃗_A x⃗_B, x⃗_A
    ≡_{A≃B} x⃗_B →
  DepConstr(j, A) x⃗_A ≡_{A≃B}
      DepConstr(j, B) x⃗_B.

elim_ok: ∀ a b P_A P_B f⃗_A f⃗_B,
  a ≡_{A≃B} b →
  P_A ≡_{(A→s)≃(B→s)} P_B →
  ∀ j, f⃗_A[j] ≡_{ξ(A,P_A,j)≃ξ(B,P_B,j)}
       f⃗_B[j] →
  DepElim(a, P_A) f⃗_A
      ≡_{(Pa)≃(Pb)} DepElim(b, P
      _B) f⃗_A.
```

```
elim_eta(A): ∀ a P f⃗, DepElim(a, P)
     f⃗ : P (Eta(A) a).
eta_ok(A): ∀ (a : A), Eta(A) a = a.

iota_ok(A): ∀ j P f⃗ x⃗ (Q: P(Eta(A)
    (DepConstr(j, A) x⃗)) → s),
  Iota(A, j, Q) :
    Q (DepElim(DepConstr(j, A) x⃗, P)
         f⃗) →
    Q (rew ← eta_ok(A) (DepConstr(j
       , A) x⃗) in
      (f⃗[j]...(DepElim(IH_0, P) f⃗)...(
         DepElim(IH_n, P) f⃗)...)).
```

Figure 29: Correctness criteria for a configuration to ensure that the transformation preserves equivalence (left) coherently with equality (right, shown for A; B is similar). `f` and `g` are defined in text. s, $\vec{f}$, $\vec{x}$, and $\vec{\text{IH}}$ represent sorts, eliminator cases, constructor arguments, and inductive hypotheses. ξ (A, P, j) is the type of `DepElim(A, P)` at `DepConstr(j, A)` (similarly for B).

```
Q (DepElim(DepConstr(1, N) n, P) { p_0, p_S }).
```

By replacing `Iota` over `nat` with `Iota` over `N`, the transformation replaces rewrites by reflexivity over `nat` to rewrites by propositional equalities over `N`. That way, `DepElim` behaves the same over `nat` and `N`.

Taken together over both A and B, `Iota` describes how the inductive structures of A and B differ. The transformation requires that `DepElim` over A and over B have the same structure as each other, so if A and B *themselves* have the same inductive structure (if they are *ornaments* [34]), then if ι is definitional for A, it will be possible to choose `DepElim` with definitional ι for B. Otherwise, if A and B (like `nat` and `N`) have different inductive structures, then definitional ι over one would become propositional ι over the other.

4.3.3 *Configurations as Initial Algebras*

(Carlo theory will be integrated into here when I have time: basically the names aren't coincidences, it's because this corresponds to an initial algebra, so it's more natural when you have inductive types but more general than that. Draw diagram, explain what each part corresponds to. Explain that correctness is just Lambek's!)

Choosing a configuration necessarily depends in some way on the proof engineer's intentions: there can be infinitely many equivalences

that correspond to a change, only some of which are useful (for example ⑦, any A is equivalent to `unit` refined by A). And there can be many configurations that correspond to an equivalence, some of which will produce terms that are more useful or efficient than others (consider `DepElim` converting through several intermediate types).

While we cannot control for intentions, we *can* specify what it means for a chosen configuration to be correct: Fix a configuration. Let `f` be the function that uses `DepElim` to eliminate A and `DepConstr` to construct B, and let `g` be similar. Figure 29 specifies the correctness criteria for the configuration. These criteria relate `DepConstr`, `DepElim`, `Eta`, and `Iota` in a way that preserves equivalence coherently with equality.

EQUIVALENCE To preserve the equivalence (Figure 29, left), `DepConstr` and `DepElim` must form an equivalence (`section` and `retraction` must hold for `f` and `g`). `DepConstr` over A and B must be equal up to transport across that equivalence (`constr_ok`), and similarly for `DepElim` (`elim_ok`). Intuitively, `constr_ok` and `elim_ok` guarantee that the transformation correctly transports dependent constructors and dependent eliminators, as doing so will preserve equality up to transport for those subterms. This makes it possible for the transformation to avoid applying `f` and `g`, instead porting terms from A directly to B.

EQUALITY To ensure coherence with equality (Figure 29, right), `Eta` and `Iota` must prove η and ι. That is, `Eta` must have the same definitional behavior as the dependent eliminator (`elim_eta`), and must behave like identity (`eta_ok`). Each `Iota` must prove and rewrite along the simplification (*refolding* [9]) behavior that corresponds to a case of the dependent eliminator (`iota_ok`). This makes it possible for the transformation to avoid applying `section` and `retraction`.

CORRECTNESS With these correctness criteria for a configuration, we get the completeness result (proven in Coq ⑧) that every equivalence induces a configuration. We also obtain an algorithm for the soundness result that every configuration induces an equivalence.

The algorithm to prove `section` is as follows (`retraction` is similar): replace `a` with `Eta(A) a` by `eta_ok(A)`. Then, induct using `DepElim` over A. For each case i, the proof obligation is to show that `g ( f a)` is equal to `a`, where `a` is `DepConstr(A, i)` applied to the non-inductive arguments (by `elim_eta(A)`). Expand the right-hand side using `Iota(A, i)`, then expand it again using `Iota(B, i)` (destructing over each `eta_ok` to apply the corresponding `Iota`). The result follows by definition of `g` and `f`, and by reflexivity.

EQUIVALENCES FROM CONFIGURATIONS The algorithm above is essentially what differencing uses for each search procedure to gen-

erate functions `f` and `g` for the automatic configurations ⑨, and also generate proofs `section` and `retraction` that these functions form an equivalence ⑩. To minimize dependencies, PUMPKIN Pi does not produce proofs of `constr_ok` and `elim_ok` directly, as stating these theorems cleanly would require either a special framework [46] or a univalent type theory [49]. If the proof engineer wishes, it is possible to prove these in individual cases ⑧, but this is not necessary in order to use PUMPKIN Pi.

4.3.4 *Search Procedures*

PUMPKIN Pi implements four search procedures for automatic configuration ⑥:

1. algebraic ornaments,
2. unpacking Σ types,
3. swapping constructors, and
4. moving between nested pairs and records.

As a courtesy to the reader, in this section, I detail just the first search procedure as an example. In Section 4.5, I will briefly describe the other search procedures, as well as what is needed to extend PUMPKIN Pi with new search procedures. I will also explain how the search procedures are implemented.

The first search procedure discovers equivalences that correspond to *algebraic ornaments*. An algebraic ornament relates an inductive type to an indexed version of that type, where the new index is fully determined by a unique fold over A. For example, `vector` is exactly `list` with a new index of type `nat`, where the new index is fully determined by the `length` function (recall Figure 26 on page 67). The equivalence that we already saw in Section 4.2.2 follows from this:

```
Σ(l : list T).length l = n ≡ vector T n
```

Alternatively, we can say that a list is equivalent to a vector of *some* length:

```
list T ≡ Σ(n : nat).vector T n
```

As usual, this equivalence is made up of two functions `f` and `g`, along with proofs `section` and `retraction`. In addition, for algebraic ornaments, there is a proof of this theorem:

```
∀ (l : list T), length l = π_l (f l)
```

which states that the `length` function is coherent with this equivalence.

In Section 4.6, I will show you a case study of porting functions and proofs from lists to length-indexed vectors. Nominally this works by porting the functions and proofs along the equivalence from Section 4.2.2, but in practice this works by chaining two different automatic configurations with some human input. The first configuration

uses a search procedure that discovers the equivalence between lists and vectors of some length above, as well as the proof of coherence. The second configuration uses a search procedure that discovers how to unpack vectors of some length to vectors of a *particular* length.

The first configuration nicely demonstrates how differencing works, so let us look at this in detail. Assume the existence of inductive types A and A_I, related by an algebraic ornament with the index of type I. In the scope of this thesis, further assume that A and I are not indexed types.[6] Then there is a type equivalence:

$A \simeq \Sigma(i : I).A_I\ i$

In addition, there is an indexer, which is a unique fold:

`indexer` $: A \to I$.

which projects the promoted index:

`coherence` $: \forall(a : A),$ `indexer` $a = \pi_l$ (`f` a).

Following existing work, I call this equivalence the *ornamental promotion isomorphism* [28]; when it holds and the indexer exists, I say that A_I is an algebraic ornament of A.

In their original form, ornaments are a programming mechanism: given a type A, an ornament determines some new type A_I. Differencing inverts this process for algebraic ornaments: given types A and A_I, it searches for the configuration that induces the ornamental promotion isomorphism between them. This is possible for algebraic ornaments precisely because the indexer is extensionally unique. For example, all possible indexers for `list` and `vector` must compute the length of a list; if we were to try doubling the length instead, we would not be able to satisfy the equivalence.

COMMON DEFINITIONS The algorithm assumes a function `new` that determines whether a hypothesis in a case of the eliminator type of A_I is new. Figure 30 contains other common definitions, the names for which are reserved: Input type A expands to an inductive type with constructors $\{C_{A_1}, \ldots, C_{A_n}\}$. P_A denotes the type of the motive of the eliminator of A, and each E_{A_j} denotes the type of the eliminator for the jth constructor of A. Analogous names are also reserved for input type A_I. The type B is A_I at some index of type I.

For historical reasons, differencing generates the equivalence first, then derives the configuration, rather than the other way around. It builds on three intermediate steps: one to generate each of `indexer`, `f`,

6 The original paper that this is from lets all of A, A_I, and I be *indexed* inductive types, with the new index of type I appearing anywhere within the list of indices of A_I; the implementation makes the same decision. I felt that this was very important to show in detail when I wrote that paper, since indices are often omitted, even though handling them is one of the trickiest parts of implementing an algorithm like this. In this thesis, however, I decided to simplify the presentation and assume the types are not indexed, and so the new index is the only index of A_I. I do recommend checking out the original paper if you would really like to implement something like this over indexed types—it is formalized those who are *sufficiently fanatical*.

$$
\begin{aligned}
&A \;:=\; \mathrm{Ind}(Ty_A : \mathrm{s}_A)\{\mathrm{C}_{A_1}, \ldots, \mathrm{C}_{A_n}\} \\
&A_I \;:=\; \mathrm{Ind}(Ty_{A_I} : (\Pi(i : I).\mathrm{s}_{A_I}))\{\mathrm{C}_{A_{I_1}}, \ldots, \mathrm{C}_{A_{I_n}}\} \\
&B \;:=\; \Sigma(i : I).A_I\, i
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{P}_A \;:=\; \Pi(a : A).\mathrm{s}_A \\
&\mathrm{P}_{A_I} \;:=\; \Pi(i : I)(a_i : A_I\;\; i).\mathrm{s}_{A_I}
\end{aligned}
$$

$$
\begin{aligned}
&\forall 1 \leq j \leq n, \\
&\quad \mathrm{E}_{A_j}\;\; (p_A : \mathrm{P}_A) \;:=\; \xi(A,\;\; p_A,\;\; \mathrm{Constr}(j,\;\; A),\;\; C_{A_j}) \\
&\quad \mathrm{E}_{B_j}\;\; (p_{A_I} : \mathrm{P}_{A_I}) \;:=\; \xi(A_I,\;\; p_{A_I},\;\; \mathrm{Constr}(j,\;\; A_I),\;\; C_{A_{I_j}})
\end{aligned}
$$

Figure 30: Common definitions.

and `g`. It then uses that to build the configuration. Figure 31 shows the algorithm for generating `indexer`. The algorithms for generating `f` and `g` are similar; Figure 32 shows only the derivations for generating `f` that are different from those for generating `indexer`, and the derivations for generating `g` are omitted.

4.3.4.1 *Differencing for the Indexer*

Then differencing algorithm generates the `indexer` by traversing the types of the eliminators for A and A_I in parallel using the algorithm from Figure 31, which consists of three judgments: one to generate the motive, one to generate each case, and one to compose the motive and cases.

GENERATING THE MOTIVE The $(T_A,\; T_{A_I}) \Downarrow_{i_m} t$ judgment consists of only the derivation INDEX-MOTIVE, which computes the indexer motive from the types A and A_I (expanded in Figure 30). It does this by constructing a function from A to I. Consider `list` and `vector`:

```
list T := Ind (Ty_A : Type) {...}     vector T := Ind (Ty_B : Π
    (n :  nat).Type) {...}
```

For these types, INDEX-MOTIVE computes the motive:

```
λ (l :  list T) . nat
```

GENERATING EACH CASE The $\Gamma \vdash (T_A,\; T_{A_I}) \Downarrow_{i_c} t$ judgment generates each case of the indexer by traversing in parallel the corresponding cases of the eliminator types for A and A_I. It consists of four derivations: INDEX-CONCLUSION handles base cases and conclusions of inductive cases, while INDEX-HYPOTHESIS, INDEX-IH, and INDEX-PROD recurse into products.

INDEX-HYPOTHESIS handles each new hypothesis that corresponds to a new index in an inductive hypothesis of an inductive case of the eliminator type for A_I. It adds the new index to the environment, then recurses into the body of only the type for which the index already exists. For example, in the inductive case of `list` and `vector`,

$$\boxed{\Gamma \vdash (T_A,\ T_{A_I}) \Downarrow_{i_m} t}$$

INDEX-MOTIVE

$$\frac{}{\Gamma \vdash (A, A_I) \Downarrow_{i_m} \lambda(a : A).I}$$

$$\boxed{\Gamma \vdash (T_A,\ T_{A_I}) \Downarrow_{i_c} t}$$

INDEX-CONCLUSION

$$\frac{}{\Gamma \vdash (p_A\ a,\ p_{A_I}\ i\ a_i) \Downarrow_{i_c} i}$$

INDEX-HYPOTHESIS

new $n_{A_I}\ b_{A_I}$

$$\frac{\Gamma,\ n_{A_I} : t_{A_I} \vdash (\Pi(n_A : t_A).b_A,\ b_{A_I}) \Downarrow_{i_c} t}{\Gamma \vdash (\Pi(n_A : t_A).b_A,\ \Pi(n_{A_I} : t_{A_I}).b_{A_I}) \Downarrow_{i_c} t}$$

INDEX-IH

$$\frac{\Gamma \vdash (A, A_I) \Downarrow_{i_m} p \qquad \Gamma,\ n_A : p\ a \vdash (b_A,\ b_{A_I}[n_A/i]) \Downarrow_{i_c} t}{\Gamma \vdash (\Pi(n_A : p_A\ a).b_A,\ \Pi(n_{A_I} : p_{A_I}\ i\ b).b_{A_I}) \Downarrow_{i_c} \lambda(n_A : p\ a).t}$$

INDEX-PROD

$$\frac{\Gamma,\ n_A : t_A \vdash (b_A,\ b_{A_I}[n_A/n_{A_I}]) \Downarrow_{i_c} t}{\Gamma \vdash (\Pi(n_A : t_A).b_A,\ \Pi(n_{A_I} : t_{A_I}).b_{A_I}) \Downarrow_{i_c} \lambda(n_A : t_A).t}$$

$$\boxed{\Gamma \vdash (T_A,\ T_{A_I}) \Downarrow_i t}$$

INDEX-IND

$$\frac{\Gamma \vdash (A,\ A_I) \Downarrow_{i_m} p \qquad \Gamma,\ p_A : \mathrm{P}_A,\ p_{A_I} : \mathrm{P}_{A_I} \vdash \{(\mathrm{E}_{A_1}\ p_A,\ \mathrm{E}_{A_{I_1}}\ p_{A_I}), \ldots, (\mathrm{E}_{A_n}\ p_A,\ \mathrm{E}_{A_{I_n}}\ p_{A_I})\} \Downarrow_{i_c} \vec{f}}{\Gamma \vdash (A,\ A_I) \Downarrow_i \lambda(a : A).\mathrm{Elim}(a, p)\vec{f}}$$

Figure 31: Differencing for the indexer function.

`new` determines that `n` is the new hypothesis. INDEX-HYPOTHESIS then recurses into the body of only the `vector` case:

```
Π (t_l:T) (l : list T) (IH_l : p_A l), ...    Π (t_v : T) (v :
    vector T n) (IH_v : p_A_I n v), ...
```

INDEX-PROD is next. It recurses into product types when the hypothesis is neither a new index nor an inductive hypothesis. Here, it runs twice, recursing into the body and substituting names until it hits the inductive hypothesis for both types:

```
Π (IH_l : p_A l), p_A (cons t_l l)          Π (IH_v : p_A_I n l), p
    A_I (S n) (consV n t_l l)
```

INDEX-IH then takes over. It substitutes the new motive in the inductive hypothesis, then recurses into both bodies, substituting the new inductive hypothesis for the index in the eliminator type for A_I. Here, it substitutes the new motive for `p_A` in the type of `IH_l`, extends the environment with `IH_l`, then substitutes `IH_l` for `n`, so that it recurses on these types:

```
p_A (cons t_l l)                          p_A_I (S IH_l) (consV IH_l t_l
    l)
```

Finally, INDEX-CONCLUSION computes the conclusion by taking the new index of the application of the motive p_{A_I}, here `S IH_l`. In total, this produces a function that computes the length of `cons t l`:

```
λ (t_l : T) (l : list T) (IH_l : (λ (l : list T) . nat) l) . S IH_l
```

COMPOSING THE RESULT The $\Gamma \vdash (T_A,\ T_{A_I}) \Downarrow_i t$ judgment consists of only INDEX-IND, which identifies the motive and each case using the other two judgments, then composes the result. In the case of `list` and `vector`, this produces a function that computes the length of a list:

```
λ (l : list T).Elim(l, λ (l : list T) . nat)
  {0, λ (t_l : T) (l : list T) (IH_l : (λ (l : list T) . nat) l) . S IH_l}
```

4.3.4.2 *Differencing for the Configuration*

As mentioned earlier, for historical reasons, differencing in PUMPKIN Pi discovers the equivalence parts `f` and `g` first, then uses those functions to discover the configuration. It also proves that these functions form an equivalence, and that the indexer is coherent with the equivalence.

DISCOVERING THE EQUIVALENCE PARTS Figure 32 shows the interesting derivations for the judgment $(T_A,\ T_B) \Downarrow_f t$ that searches for `f`: F-MOTIVE identifies the motive as B with a new index (which it computes using `indexer`, denoted by metavariable π). When F-IH recurses, it substitutes the inductive hypothesis for the term rather than for its index, and it substitutes the new index (which it also computes using `indexer`) inside of that term. F-CONCLUSION returns the entire term, rather than its index. Finally, F-IND not only recurses into each case, but also packs the result.

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_{f_m} t}$$

F-MOTIVE
$$\frac{\Gamma \vdash (A,\ A_I) \Downarrow_i \pi}{\Gamma \vdash (A,\ A_I) \Downarrow_{f_m} \lambda(a : A).A_I\ (\pi\ \vec{i_a}\ a)}$$

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_{f_c} t}$$

F-CONCLUSION
$$\frac{}{\Gamma \vdash (p_A\ a,\ p_{A_I}\ i\ a_i) \Downarrow_{f_c} a_i}$$

F-IH
$$\frac{\begin{array}{l}\Gamma \vdash (A,\ A_I) \Downarrow_i \pi \qquad \Gamma \vdash (A,\ A_I) \Downarrow_{f_m} p \\ \Gamma,\ n_A : p\ a \vdash (b_A,\ b_{A_I}[n_A/a_i][\pi\ a/i]) \Downarrow_{p_c} t\end{array}}{\begin{array}{l}\Gamma \vdash (\Pi(n_A : p_A\ a).b_A,\ \Pi(n_{A_I} : p_{A_I}\ i\ a_i).b_{A_I}) \\ \quad \Downarrow_{p_c} \lambda(n_A : p\ a).t\end{array}}$$

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_f t}$$

F-IND
$$\frac{\begin{array}{l}\Gamma \vdash (A,\ A_I) \Downarrow_i \pi \qquad \Gamma \vdash (A,\ A_I) \Downarrow_{f_m} p \\ \Gamma,\ p_A : \mathrm{P}_A,\ p_{A_I} : \mathrm{P}_{A_I} \vdash \{(\mathrm{E}_{A_1}\ p_A,\ \mathrm{E}_{A_{I_1}}\ p_{A_I}), \ldots, (\mathrm{E}_{A_n}\ p_A,\ \mathrm{E}_{A_{I_n}}\ p_{A_I})\} \Downarrow_{p_c} \vec{f}\end{array}}{\Gamma \vdash (A,\ A_I) \Downarrow_p \lambda(a : A).\exists\ (\pi\ a)\ (\mathrm{Elim}(a, p)\vec{f})}$$

Figure 32: Differencing for `f`.

The omitted derivations to difference for `g` are similar, except that the domain and range are switched. Consequentially, `indexer` is never needed; G-MOTIVE removes the index rather than inserting it, and G-IH no longer substitutes the index. Additionally, G-HYPOTHESIS adds the hypothesis for the new index rather than skipping it, and G-IND eliminates over the projection rather than packing the result.

DERIVING THE CONFIGURATION DEPCONSTR and DEPELIM over A are just the standard constructors and eliminators for A. To derive DEPCONSTR over B, differencing takes each constructor of A, applies `f` to the conclusion of that constructor, and normalizes the result. It then ports the hypotheses of the resulting constructor to use B in place of A, and drops the remaining applications of `f` and the indexer in the body, replacing them instead with the projections π_r and π_l, respectively.

For example, earlier, letting A be lists, A_I be vectors, and B be vectors of some length, I noted that the empty constructor of B packs the constructor of A_I into an existential:

```
DepConstr(0, Σ(n : nat).vector T n) : Σ(n : nat).vector T n
    :=
  ∃ (Constr(0, nat)) (Constr(0, vector T)).
```

This is the same as applying the function `f` that PUMPKIN Pi derives to the empty list constructor `Constr(0, list T)`, and then normalizing the result. On the other hand, the `cons` constructor over B not just packs the result into an existential, but also takes B itself as an argument rather than A:

```
DepConstr(1, Σ(n : nat).vector T n) : T → Σ(n : nat).vector
    T n → Σ(n : nat).vector T n :=
```

```
λ (t : T) . λ (l : Σ(n : nat).vector T n) .
  ∃ ((Constr (1, nat)) (π_l l)) ((Constr (1, vector T)) t (π_l
      l) (π_r l))).
```

To derive this, differencing applies `f` to the conclusion of the constructor of A and normalizes the result:

```
λ (t : T) . λ (l : list T) .
  ∃ ((Constr (1, nat)) (π l)) ((Constr (1, vector T)) t (π l)
     (f l))).
```

It then lifts the hypothesis of type `list T`, and removes remaining references to the indexer π and to `f`, replacing then instead with the projections π_l and π_r.

Deriving `DepElim` over B works similarly, except that it lifts not just the hypotheses of types A, but also the motive and inductive hypotheses. This produces the dependent eliminator I presented earlier:

```
DepElim(s, P) { f_0 f_1 } : P (∃ (π_l s) (π_r s)) :=
  Elim(π_r s, λ(n : nat)(v : vector T n).P (∃ n v)) {
    f_0,
    (λ(t : T)(n : nat)(v : vector T n).f_1 t (∃ n v))
  }.
```

Differencing discovers `Eta` and `Iota` directly. For any algebraic ornament, `Eta` is the standard `Eta` expansion for Σ types:

```
Eta(B) := λ(b : B).∃ (π_l b) (π_r b).
```

Each `Iota` follows by rewriting by reflexivity, since A and A_I have the same inductive structure.

PROVING CORRECTNESS In the end, PUMPKIN Pi generates proofs of `section` and `retraction`, as well as the coherence theorem `coherence` for algebraic ornaments. This proves the correctness property that the configuration induces an equivalence, thereby increasing confidence in the output of the search procedure. The proof of coherence follows by reflexivity, thanks to the construction of `f` applying the indexer as the left projection. The proofs of `section` and `retraction` follow from the algorithm presented earlier.

4.3.5 *Revisiting Limitations*

Differencing in PUMPKIN had a two fundamental limitations: its heuristic-based nature and the difficulty of isolating changes. With PUMPKIN Pi in the PUMPKIN PATCH repair suite, these limitations are partially addressed, but remain as challenges. As in the previous chapter, limitations that are due to the choice of implementation strategy are in Section 4.5.1.2.

HEURISTICS The PUMPKIN differencing algorithms were heuristic-based. The differencing algorithms here are still heuristic-based, as is fundamental. But some progress has been made, as the domain of each search procedure is defined as a large class of type equivalences,

drawing on inspiration from type theory and from the needs of real proof engineers. For example, algebraic ornaments are a flexible class of changes drawing on about a decade of type theory literature. In addition, a number of the algorithms presented in this section, like the algorithm the prove an equivalence from a configuration, are generic over any possible type equivalence. Most notably, that the configuration can describe any possible type equivalence means that the heuristics in PUMPKIN Pi are now abstracted away from the transformation, whereas for PUMPKIN the heuristics leaked into the transformations as well.

ISOLATING CHANGES While PUMPKIN Pi adds support for a large class of changes, the challenge of isolating changes from PUMPKIN's differencing algorithms remain. Most search procedures allow for only a single change at once, and further impose some syntactic restrictions on input types and how they change. However, the presence of manual configuration in PUMPKIN Pi now makes it possible for proof engineers to describe the changes themselves, circumventing this difficulty. Some search procedures for automatic configuration, like the one for swapping constructors, further prompt the human for input to choose among multiple choices and resolve ambiguities. This fits with the typical interactive workflows of proof engineers, but makes it possible to dodge many of the challenges of isolating changes seen earlier.

4.4 TRANSFORMATION

At the heart of PUMPKIN Pi is a configurable proof term transformation for transporting proofs across equivalences ④. Figure 33 shows this proof term transformation. The transformation $\Gamma \vdash t \Uparrow t'$ takes some term t defined over the old version of a type to a new term t' defined over the new version of the type. It is parameterized over equivalent types A and B (EQUIVALENCE) as well as the configuration for that equivalence. It assumes η-expanded functions. It implicitly constructs an updated context Γ' in which to interpret t', but this is not needed for computation.

The proof term transformation is (perhaps deceptively) simple by design: it moves the bulk of the work into the configuration, and represents the configuration explicitly. This configuration either comes from automatic configuration (like the search procedure in the previous section), or directly from the proof engineer. Of course, in both of these cases, typical proof terms in Coq do not apply these configuration terms explicitly. PUMPKIN Pi does some additional work using *unification heuristics* to get real proof terms into this format before running the transformation (Section 4.4.1). It then runs the proof term transformation, which transports proofs across the equivalence that

$\boxed{\Gamma \vdash t \Uparrow t'}$

$$\textsc{Dep-Elim}\ \frac{\Gamma \vdash a \Uparrow b \qquad \Gamma \vdash p_a \Uparrow p_b \qquad \Gamma \vdash \vec{f}_a \ \Uparrow \vec{f}_b}{\Gamma \vdash \mathrm{DepElim}(a,\ p_a)\vec{f}_a \Uparrow \mathrm{DepElim}(b,\ p_b)\vec{f}_b}$$

$$\textsc{Dep-Constr}\ \frac{\Gamma \vdash \vec{t}_a \Uparrow \vec{t}_b}{\Gamma \vdash \mathrm{DepConstr}(j,\ A)\ \vec{t}_a \Uparrow \mathrm{DepConstr}(j,\ B)\ \vec{t}_b} \qquad \textsc{Eta}\ \frac{}{\Gamma \vdash \mathrm{Eta}(A) \Uparrow \mathrm{Eta}(B)}$$

$$\textsc{Iota}\ \frac{\Gamma \vdash q_A \Uparrow q_B \qquad \Gamma \vdash \vec{t_A} \Uparrow \vec{t_B}}{\Gamma \vdash \mathrm{Iota}(j,\ A,\ q_A)\ \vec{t_A} \Uparrow \mathrm{Iota}(j,\ B,\ q_B)\ \vec{t_B}} \qquad \textsc{Equivalence}\ \frac{}{\Gamma \vdash A \ \Uparrow B}$$

$$\textsc{Constr}\ \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma \vdash \vec{t} \Uparrow \vec{t'}}{\Gamma \vdash \mathrm{Constr}(j,\ T)\ \vec{t} \Uparrow \mathrm{Constr}(j,\ T')\ \vec{t'}} \qquad \textsc{Ind}\ \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma \vdash \vec{C} \Uparrow \vec{C'}}{\Gamma \vdash \mathrm{Ind}(Ty : T)\vec{C} \Uparrow \mathrm{Ind}(Ty : T')\vec{C'}}$$

$$\textsc{App}\ \frac{\Gamma \vdash f \Uparrow f' \qquad \Gamma \vdash t \Uparrow t'}{\Gamma \vdash f t \Uparrow f' t'} \qquad \textsc{Elim}\ \frac{\Gamma \vdash c \Uparrow c' \qquad \Gamma \vdash Q \Uparrow Q' \qquad \Gamma \vdash \vec{f} \Uparrow \vec{f'}}{\Gamma \vdash \mathrm{Elim}(c, Q)\vec{f} \Uparrow \mathrm{Elim}(c', Q')\vec{f'}}$$

$$\textsc{Lam}\ \frac{\Gamma \vdash t \Uparrow t' \qquad \Gamma \vdash T \Uparrow T' \qquad \Gamma,\ t : T \vdash b \Uparrow b'}{\Gamma \vdash \lambda(t : T).b \Uparrow \lambda(t' : T').b'}$$

$$\textsc{Prod}\ \frac{\Gamma \vdash t \Uparrow t' \qquad \Gamma \vdash T \Uparrow T' \qquad \Gamma,\ t : T \vdash b \Uparrow b'}{\Gamma \vdash \Pi(t : T).b \Uparrow \Pi(t' : T').b'} \qquad \textsc{Var}\ \frac{v \in \mathrm{Vars}}{\Gamma \vdash v \Uparrow v}$$

Figure 33: Transformation for transporting terms across $A \simeq B$ with configuration `((DepConstr, DepElim), (Eta, Iota))`.

```
(* 1: original term *)
λ (T : Type) (l m : Old.list T)
    .
 Elim(l, λ(l: Old.list T).Old.
     list T → Old.list T)) {
   (λ m . m),
   (λ t _ IHl m . Constr(1, Old.
      list T) t (IHl m))
 } m.

(* 2: after unifying with
    configuration *)
λ (T : Type) (l m : A) .
 DepElim(l, λ(l: A).A → A)) {
   (λ m . m)
   (λ t _ IHl m . DepConstr(1,
      A) t (IHl m))
 } m.
```

```
(* 4: reduced to final term *)
λ (T : Type) (l m : New.list T)
    .
 Elim(l, λ(l: New.list T).New.
     list T → New.list T)) {
   (λ t _ IHl m . Constr(0, New.
      list T) t (IHl m)),
   (λ m . m)
 } m.

(* 3: after transforming *)
λ (T : Type) (l m : B) .
 DepElim(l, λ(l: B).B → B)) {
   (λ m . m)
   (λ t _ IHl m . DepConstr(1,
      B) t (IHl m))
 } m.
```

Figure 34: Swapping cases of the append function, counterclockwise, the input term: 1) unmodified, 2) unified with the configuration, 3) ported to the updated type, and 4) reduced to the output.

corresponds to the configuration, replacing A with B (Section 4.4.2). In doing so, it addresses several of the limitations from the PUMPKIN transformations (Section 4.4.3).

4.4.1 *Unification Heuristics*

The transformation does not fully describe the search procedure for transforming terms that PUMPKIN Pi implements. Before running the transformation, PUMPKIN Pi *unifies* subterms with particular A (fixing parameters and indices), and with applications of configuration terms over A. The transformation then transforms configuration terms over A to configuration terms over B. Reducing the result produces the output term defined over B.

Figure 34 shows this with the list append function ++ from Section 4.1. To update ++ (top left), PUMPKIN Pi unifies `Old.list T` with A, and `Constr` and `Elim` with `DepConstr` and `DepElim` (bottom left). After unification, the transformation recursively substitutes B for A, which moves `DepConstr` and `DepElim` to construct and eliminate over the updated type (bottom right). This reduces to a term with swapped constructors and cases over `New.list T` (top right).

In this case, unification is straightforward. This can be more challenging when configuration terms are dependent. This is especially pronounced with definitional `Eta` and `Iota`, which typically are implicit (reduced) in real code. To handle this, PUMPKIN Pi implements custom *unification heuristics* for each search procedure that unify subterms with applications of configuration terms, and that instantiate parameters and dependent indices in those subterms ⑥. The transfor-

mation in turn assumes that all existing parameters and indices are determined and instantiated by the time it runs.

PUMPKIN Pi falls back to Coq's unification for manual configuration and when these custom heuristics fail. When even Coq's unification is not enough, PUMPKIN Pi relies on proof engineers to provide hints in the form of annotations (5).

4.4.2 *Specifying a Correct Transformation*

The implementation of this transformation in PUMPKIN Pi produces a term that Coq type checks, and so does not add to the TCB. As PUMPKIN Pi is an engineering tool, there is no need to formally prove the transformation correct, though doing so would be satisfying. The goal of such a proof would be to show that if $\Gamma \vdash t \Uparrow t'$, then t and t' are equal up to transport, and t' refers to B in place of A. The key steps in this transformation that make this possible are porting terms along the configuration (DEP-CONSTR, DEP-ELIM, ETA, and IOTA). The rest is straightforward. For metatheoretical reasons, without additional axioms, a proof of this theorem in Coq can only be approximated [46]. It would be possible to generate per-transformation proofs of correctness, but this does not serve an engineering need.

4.4.3 *Revisiting Limitations*

The transformations in PUMPKIN struggled with undecidability and modeling diverse proof styles. The PUMPKIN Pi transformation partially addresses this. As in the previous chapter, limitations that are due to the choice of implementation strategy are in Section 4.5.1.3.

UNDECIDABILITY PUMPKIN Pi cannot totally get rid of undecidability, but what it can do is move the burden of undecidability outside of the details of the algorithm. This is what the unification heuristics accomplish. Most notably, since unification heuristics are abstracted from the transformation itself, this makes it relatively simple to hook in a human-assisted workflow using annotations. PUMPKIN Pi implements this. Still, the PUMPKIN Pi transformation does struggle sometimes with termination. I decribe this more in Section 4.5.

MODELING DIVERSE PROOF STYLES Unlike the procedures from PUMPKIN, the PUMPKIN Pi transformation is generic over the proof term structure. This benefit again comes from decoupling unification from the transformation, and from the fact that the transformation is defined over any possible Gallina term. The only difficulties show up at the level of implementation details, which I will describe in Section 4.5.

4.5 IMPLEMENTATION

The source code of the PUMPKIN Pi plugin is on Github. It also ships with the PUMPKIN PATCH plugin suite by default. As with PUMPKIN, the latest version supports Coq 8.8, with Coq 8.9.1 support in a branch. This section decribes the implementation of the core functionality of the PUMPKIN Pi plugin (Section 4.5.1), along with important features for workflow integration (Section 4.5.2). The interested reader can follow along in the repository.

4.5.1 *Tool Details*

The implementation (Section 4.5.1.1) of the `Repair` command looks up the configuration for the types, invoking a search procedure for automatic configuration if relevant. The configuration, whether obtained manually or automatically, is cached for future calls to `Repair`. The implementations of the differencing search procedures for automatic configuration (Section 4.5.1.2) have freedom over the details of how they are implemented, as long as they return configurations in the end. The transformation (Section 4.5.1.3) operates directly over proof terms in Gallina, with the cached configuration also defined as proof terms in Gallina. As with PUMPKIN, the PUMPKIN Pi extension to PUMPKIN PATCH does not extend the TCB in any way (Section 4.5.1.4).

4.5.1.1 *The Command*

The implementation of PUMPKIN Pi exposes the repair workflow to proof engineers through the `Repair` command. This invokes the workflow from Figure 23 on page 65. When the proof engineer invokes the command:

```
Repair A B in old_proof as new_proof.
```

The first step—CONFIGURE—looks up A and B in a cache of configurations. If it finds one, it uses that; otherwise, it runs the differencing search procedures for automatic configuration. If that fails, it prompts the proof engineer to supply a configuration manually. Proof engineers can supply manual configurations by defining them as Gallina terms and providing them to the `Configure Repair` command.

Once PUMPKIN Pi has found a configuration, the next step—TRANSFORM—runs the transformation. It transforms `old_proof` into some new proof term, then defines the new proof term as `new_proof`. It is also possile to ask PUMPKIN Pi to automatically generate a name. The final step—DECOMPILE—runs in the end to produce a suggested proof script.

There are a few other useful commands in PUMPKIN Pi that are detailed in the repository. Notably, the `Repair Module` command operates over entire modules at once and defines new modules. The

`Preprocess` command converts functions to use eliminators, so that the transformation can assume primitive eliminators. The `Lift` and `Lift Module` commands skip the decompilation step, when only the term is desired in the end. Several options make it possible to control whether PUMPKIN Pi generates correctness proofs, as well as how aggressively it runs optimizations, and what it considers in its optimizations. The repository includes more information on all of these.

4.5.1.2 *Differencing*

As mentioned in Section 4.3.4, differencing inside of PUMPKIN Pi implements four search procedures for automatic configuration ⑥:

1. algebraic ornaments,
2. unpacking Σ types,
3. swapping constructors, and
4. moving between nested pairs and records.

I already detailed the first of these in Section 4.3.4. The second is often used in combination with the first, to get from equivalences of this form:

$A \equiv \Sigma(i : I).A_I i$

to equivalences of this form:

$\Sigma(a : A).\pi a = i \equiv A_I i$

where π is the indexer. The first two configurations are used in combination, for example, to repair proofs in response to the change from lists to vectors of a particular length in Figure 26, as long as the proof engineer proves the missing length invariant. This differencing algorithm is implemented by simply instantiating the types A and B of a generic configuration that can be defined inside of Coq directly.

The third configuration—invoked in the example in Section 4.1—constructs a swap map that maps the constructors of A to the swapped constructors of B, then otherwise runs an algorithm much like the algorithm for algebraic ornaments. When there are multiple possible swap maps, it returns an ordered list of possible mappings to choose from, and prompts the proof engineer to select one. It also lets the proof engineer supply the swap map directly, and from that can derive the configuration, the equivalence, and the proof. The fourth search procedure is similar to the search procedure for algebraic ornaments, but with some additional logic to handle nested tuples.

All search procedures have the historical detail of defining the equivalence first, then deriving the configuration from it, rather than the other way around. For simplicity, the search procedures currently implemented inside of PUMPKIN Pi make some syntactic assumptions about the input and output types. The details of these restrictions can be found inside of the repository.

Defining new search procedures comes with a lot of freedom, as long as the search procedures produce a configuration in the end. I found it simple to add configuration four in a matter of a few days in response to a request by an industrial proof engineer, reusing existing functions defined in other search procedures. I would not expect the average proof engineer to be able to do the same, and I have not yet asked anyone else to write a search procedure to gauge the effort involved for others. It is notable that manual configuration is always an option when search procedures do not exist, but the results in Section 4.6 suggest that automatic configuration is very helpful for saving developer effort, and so worth implementing for useful classes of equivalences.

4.5.1.3 *Transformation*

The implementation ④ of the transformation from Section 4.4 operates over terms in Gallina. It takes as input two types A and B, along with a configuration that induces an equivalence between them. A and B may not necessarily be the inputs to differencing—in the case of algebraic ornaments, for example, differencing takes A and A_I as inputs, but the transformation takes A and B as inputs, where B is $\Sigma(i : I).A_I i$.

The unification heuristics run before the transformation ⑥, and in the end return which transformation derivation to run, and with which arguments ⑫. For the most part, the derivations are implemented in a way that corresponds to the derivations in Figure 33 on page 84, but with a few differences highlighted below.

FROM CIC_ω TO GALLINA The implementation ④ of the transformation handles language differences to scale from CIC_ω to Gallina. For example, recall that the transformation assumes primitive eliminators, while Gallina implements eliminators in terms of pattern matching and fixpoints. To handle terms that use these features, PUMPKIN Pi includes a `Preprocess` command that translates these terms into corresponding eliminator applications.[7] This command can preprocess a definition (like `length` from Figure 7 on page 17) or an entire module (like `List`, as shown in `ListToVect.v`) for lifting. It currently supports fixpoints that are structurally recursive on only immediate substructures.[8] To translate such a fixpoint, it first extracts a motive, then generates each case by partially reducing the function's

7 My collaborator Nate Yazdani wrote this command, and it is available as a standalone plugin on Github. I helped him design and test the command, but did not write any of the code.

8 This is enough to preprocess many practical terms, including the entire `List` module. But it is not as general as it could be. A more general translation may help PUMPKIN Pi support more terms, and discussions with Coq developers a couple of years ago suggest that the implementation of such a translation building on work from the equations [45] plugin was in progress. I do not know the current status of that project.

body under a hypothetical context for the constructor arguments. The implementation documents this and other language differences.

OPTIMIZATIONS In addition to cases for the derivations, the implementation of the transformation includes a number of cases that correspond to optimizations. The code denotes these explicitly (4) and explains them in detail in comments (12). For example, the transformation assumes that all terms are fully η-expanded. Sometimes, however, η-expansion is not necessary. For efficiency, rather than fully η-expand ahead of time, PUMPKIN Pi η-expands lazily, only when it is necessary for correctness. The `LazyEta` optimization implements this. Section 4.5.2.2 describes some other optimizations included for the sake of integration with proof engineering workflows.

TERMINATION When a subterm unifies with a configuration term, this suggests that PUMPKIN Pi *can* transform the subterm, but it does not necessarily mean that it *should*. In some cases, doing so would result in nontermination. For example, if B is a refinement of A, then the transformation can always run EQUIVALENCE over and over again, forever. PUMPKIN Pi thus includes some simple termination checks in the code (12).

INTENT Even when termination is guaranteed, whether to transform a subterm depends on intent. That is, PUMPKIN Pi automates the case of porting *every* A to B, but proof engineers sometimes wish to port only *some* As to Bs. PUMPKIN Pi has some support for this using an interactive workflow (13), with plans for automatic support in the future.

4.5.1.4 *Trusted Computing Base*

As with PUMPKIN, PUMPKIN Pi is implemented as a Coq plugin, and produces terms that Coq type checks in the end. PUMPKIN Pi does not modify the type checker, and furthermore does not add any axioms, so it does not increase the TCB. This is perhaps even more notable for PUMPKIN Pi, since all other implementations of transport across equivalences that I am aware of at least sometimes add axioms. Since PUMPKIN Pi implements transport as a proof term transformation, it is able to circumvent this and not increase the TCB in any way.

4.5.2 *Workflow Integration*

So far, I have described the core functionality of PUMPKIN Pi. But PUMPKIN Pi has many additional features for the sake of integration with typical proof engineering workflows. Most notable of these is the decompiler from Gallina to Ltac (Section 4.5.2.1), which helps PUMPKIN Pi produce a suggested proof script that the proof engineer

$\langle v \rangle \in$ Vars, $\langle t \rangle \in \text{CIC}_\omega$

$\langle p \rangle ::=$ intro $\langle v \rangle$ | rewrite $\langle t \rangle$ $\langle t \rangle$ | symmetry | apply $\langle t \rangle$ | induction $\langle t \rangle$ $\langle t \rangle$ { $\langle p \rangle, \ldots, \langle p \rangle$ } | split { $\langle p \rangle, \langle p \rangle$ } | left | right | $\langle p \rangle$. $\langle p \rangle$

Figure 35: Qtac syntax.

$\boxed{\Gamma \vdash t \Rightarrow p}$

INTRO
$$\frac{\Gamma,\ n : T \vdash b \Rightarrow p}{\Gamma \vdash \lambda(n : T).b \Rightarrow \text{intro } n.\ p}$$

SYMMETRY
$$\frac{\Gamma \vdash H \Rightarrow p}{\Gamma \vdash \texttt{eq\_sym}\ H \Rightarrow \text{symmetry. } p}$$

SPLIT
$$\frac{\Gamma \vdash l \Rightarrow p \qquad \Gamma \vdash r \Rightarrow q}{\Gamma \vdash \text{Constr}(0,\ \wedge)\ l\ r \Rightarrow \text{split}\{p, q\}.}$$

LEFT
$$\frac{\Gamma \vdash H \Rightarrow p}{\Gamma \vdash \text{Constr}(0,\ \vee)\ H \Rightarrow \text{left. } p}$$

RIGHT
$$\frac{\Gamma \vdash H \Rightarrow p}{\Gamma \vdash \text{Constr}(1,\ \vee)\ H \Rightarrow \text{right. } p}$$

REWRITE
$$\frac{\Gamma \vdash H_1 : x = y \qquad \Gamma \vdash H_2 \Rightarrow p}{\Gamma \vdash \text{Elim}(H_1,\ P)\{x,\ H_2,\ y\} \Rightarrow \text{symmetry. rewrite } P\ H_1.\ p}$$

INDUCTION
$$\frac{\Gamma \vdash \vec{f} \Rightarrow \vec{p}}{\Gamma \vdash \text{Elim}(t,\ P)\ \vec{f} \Rightarrow \text{induction } P\ t\ \vec{p}}$$

APPLY
$$\frac{\Gamma \vdash t \Rightarrow p}{\Gamma \vdash ft \Rightarrow \text{apply } f.\ p}$$

BASE
$$\frac{}{\Gamma \vdash t \Rightarrow \text{apply } t}$$

Figure 36: Qtac decompiler semantics.

can maintain in the end. This and other features help PUMPKIN Pi reach real proof engineers (Section 4.5.2.2).

4.5.2.1 *Decompiling to Tactics*

Transform produces a proof term, while the proof engineer typically writes and maintains proof scripts made up of tactics. PUMPKIN Pi improves usability thanks to the realization that, since Coq's proof term language Gallina is very structured, it is possible to decompile these Gallina terms to suggested Ltac proof scripts for the proof engineer to maintain.

Decompile implements a prototype of this translation (11): it translates a proof term to a suggested proof script that attempts to prove the same theorem the same way. Note that this problem is not well defined: while there is always a proof script that works (applying the proof term with the `apply` tactic), the result is often qualitatively

unreadable. This is the baseline behavior to which the decompiler defaults. The goal of the decompiler is to improve on that baseline as much as possible, or else suggest a proof script that is close enough to correct that the proof engineer can manually massage it into something that works and is maintainable.

Decompile achieves this in two passes: The first pass decompiles proof terms to proof scripts that use a predefined set of tactics. The second pass improves on suggested tactics by simplifying arguments, substituting tacticals, and using hints like custom tactics and decision procedures.

FIRST PASS: BASIC PROOF SCRIPTS The first pass takes Coq terms and produces tactics in Ltac, the proof script language for Coq. Ltac can be confusing to reason about, since Ltac tactics can refer to Gallina terms, and the semantics of Ltac depends both on the semantics of Gallina and on the implementation of proof search procedures written in OCaml. To give a sense of how the first pass works without the clutter of these details, I start by defining a mini decompiler that implements a simplified version of the first pass. I then explain how RanDair scaled this to the implementation.

The mini decompiler takes CIC_ω terms and produces tactics in a mini version of Ltac which I call Qtac. The syntax for Qtac is in Figure 35. Qtac includes hypothesis introduction (`intro`), rewriting (`rewrite`), symmetry of equality (`symmetry`), application of a term to prove the goal (`apply`), induction (`induction`), case splitting of conjunctions (`split`), constructors of disjunctions (`left` and `right`), and composition (`.`). Unlike in Ltac, `induction` and `rewrite` take a motive explicitly (rather than relying on unification), and `apply` creates a new subgoal for each function argument.

The semantics for the mini decompiler $\Gamma \vdash t \Rightarrow p$ are in Figure 36 (assuming $=$, `eq_sym`, $\wedge$, and $\vee$ are defined as in Coq). As with the real decompiler, the mini decompiler defaults to the proof script that applies the entire proof term with `apply` (BASE). Otherwise, it improves on that behavior by recursing over the proof term and constructing a proof script using a predefined set of tactics.

For the mini decompiler, this is straightforward: Lambda terms become introduction (INTRO). Applications of `eq_sym` become symmetry of equality (SYMMETRY). Constructors of conjunction and disjunction map to the respective tactics (SPLIT, LEFT, and RIGHT). Applications of equality eliminators compose symmetry (to orient the rewrite direction) with rewrites (REWRITE), and all other applications of eliminators become induction (INDUCTION). The remaining applications become apply tactics (APPLY). In all cases, the decompiler recurses, breaking into cases, until only the BASE case holds.

While the mini decompiler is very simple, only a few small changes are needed for RanDair to move this to Coq. The generated proof

```
fun (y0 :  list A)[1] =>
  list_rect[2] _ _  (fun a l H[2] =>
    eq_ind_r[3] _ eq_refl[4] (app_nil_r (rev l) (a::[]))[3])
    eq_refl[5]
    y0[2]

- intro y0.[1] induction y0 as [a l H|].[2]
  + simpl.  rewrite app_nil_r.[3] auto.[4]
  + auto.[5]
```

Figure 37: Proof term (top) and decompiled proof script (bottom) for the base case of `rev_app_distr` (Section 4.1), with corresponding terms and tactics grouped by color & number.

term of `rev_app_distr` from Section 4.1, for example, consists only of induction, rewriting, simplification, and reflexivity (solved by `auto`). Figure 37 shows the proof term for the base case of `rev_app_distr` alongside the proof script that PUMPKIN Pi suggests. This script is fairly low-level and close to the proof term, but it is already something that the proof engineer can step through to understand, modify, and maintain. There are few differences from the mini decompiler needed to produce this, for example handling of rewrites in both directions (`eq_ind_r` as opposed to `eq_ind`), simplifying rewrites, and turning applications of `eq_refl` into `reflexivity` or `auto`.

SECOND PASS: BETTER PROOF SCRIPTS The implementation of **Decompile** first runs something similar to the mini decompiler, then modifies the suggested tactics to produce a more natural proof script (11). For example, it cancels out sequences of `intros` and `revert`, inserts semicolons, and removes extra arguments to `apply` and `rewrite`. It can also take tactics from the proof engineer (like part of the old proof script) as hints, then iteratively replace tactics with those hints, checking for correctness. This makes it possible for suggested scripts to include custom tactics and decision procedures.

FROM QTAC TO LTAC The mini decompiler assumes more predictable versions of `rewrite` and `induction` than those in Coq. **Decompile** includes additional logic to reason about these tactics (11). For example, Qtac assumes that there is only one `rewrite` direction. Ltac has two rewrite directions, and so the decompiler infers the direction from the motive.

Qtac also assumes that both tactics take the inductive motive explicitly, while in Coq, both tactics infer the motive automatically. Consequentially, Coq sometimes fails to infer the correct motive. To handle induction, the decompiler strategically uses `revert` to manipulate the goal so that Coq can better infer the motive. To handle rewrites, it uses `simpl` to simplify the goal before rewriting. Neither of these approaches is guaranteed to work, so the proof engineer may sometimes need to tweak the suggested proof script appropriately.

Even if RanDair passes Coq's induction principle an explicit motive, Coq still sometimes fails due to unrepresented assumptions. Long term, using another tactic like `change` or `refine` before applying these tactics may help with cases for which Coq cannot infer the correct motive.

FROM CIC$_\omega$ TO COQ Scaling the decompiler to Coq introduces let bindings, which are generated by tactics like `rewrite in`, `apply in`, and `pose`. **Decompile** implements (11) support for `rewrite in` and `apply in` similarly to how it supports `rewrite` and `apply`, except that it ensures that the unmanipulated hypothesis does not occur in the body of the let expression, it swaps the direction of the rewrite, and it recurses into any generated subgoals. In all other cases, it uses `pose`, a catch-all for let bindings.

FORFEITING SOUNDNESS While there is a way to always produce a correct proof script, by default, **Decompile** deliberately forfeits soundness to suggest more useful tactics. For example, it may suggest the `induction` tactic, but leave the step of motive inference to the proof engineer. I have found these suggested tactics easier to work with (Section 4.6). Note that in the case the suggested proof script is not quite correct, it is still possible to use the generated proof term directly. Still, in the event that soundness is desirable, RanDair has since implemented a simplified sound version of the decompiler in a branch.

PRETTY PRINTING After decompiling proof terms, **Decompile** pretty prints the result (11). Like the mini decompiler, **Decompile** represents its output using a predefined grammar of Ltac tactics, albeit one that is larger than Qtac, and that also includes tacticals. It maintains the recursive proof structure for formatting. PUMPKIN Pi keeps all output terms from **Transform** in the Coq environment in case the decompiler does not succeed. Once the proof engineer has the new proof, she can remove the old one.

4.5.2.2 *Reaching Real Proof Engineers*

In the end, the goal of workflow integration is to reach real proof engineers. Many of my design decisions in implementing PUMPKIN Pi were informed by my partnership with an industrial proof engineer (Section 4.6). For example, the proof engineer rarely had the patience to wait more than ten seconds for PUMPKIN Pi to port a term, so I implemented optional aggressive caching, even caching intermediate subterms encountered while running the transformation (14). I also added a cache to tell PUMPKIN Pi not to δ-reduce certain terms (14). With these caches, the proof engineer found PUMPKIN Pi efficient

enough to use on a code base with tens of thousands of lines of code and proof.

The experiences of proof engineers also inspired new features. For example, I implemented a search procedure to generate custom eliminators to help reason about types like `Σ(l : list T).length l = n` by reasoning separately about the projections (15). I added informative error messages (22) to help the proof engineer distinguish between user errors and bugs. Nate implemented machinery for whole module processing to handle entire libraries at once. These features helped with workflow integration.

4.6 RESULTS

PUMPKIN Pi is flexible and useful. It can help and in fact has helped proof engineers save work on a variety of real proof repair scenarios (Section 4.6.1). In addition, the approach taken has measurable benefits in terms of both work savings and performance relative to a comparable tool for the class of changes on which Nate and I have done an extended evaluation (Section 4.6.2).

4.6.1 PUMPKIN P*i Eight Ways*

This section summarizes eight case studies using PUMPKIN Pi, corresponding to the eight rows in Table 1. These case studies highlight PUMPKIN Pi's flexibility in handling diverse scenarios, the success of automatic configuration for better workflow integration, the preliminary success of the prototype decompiler, and clear paths to better serving proof engineers. Detailed walkthroughs are in the code.

ALGEBRAIC ORNAMENTS: LISTS TO PACKED VECTORS PUMPKIN Pi implements a search procedure for automatic configuration of algebraic ornaments, detailed in Section 4.3.4. In file (3), I used this to port functions and a proof from lists to vectors of *some* length, since `list T` ≃ `Σ(n : nat).vector T n`. The decompiler helped me write proofs in the order of hours that I had found too hard to write by hand, though the suggested tactics did need massaging.

UNPACK SIGMA TYPES: VECTORS OF PARTICULAR LENGTHS In the same file (3), I then ported functions and proofs to vectors of a *particular* length, like `vector T n`. I supported this in PUMPKIN Pi by chaining the previous change with an automatic configuration for unpacking sigma types. By composition, this transported proofs across the equivalence from Section **??**.

Two tricks helped with workflow integration for this change: 1) have the search procedure view `vector T n` as `Σ(v : vector T m).n = m` for some `m`, then let PUMPKIN Pi instantiate those equalities via unification

Class	Config.	Examples	Sav.	Repair T
Algebraic Ornaments	Auto	List to Packed Vector, hs-to-coq (3)	☺	Pumpkin
		List to Packed Vector, Std. Library (16)	☺	Pumpkin
Unpack Sigma Types	Auto	Vector of Particular Length, hs-to-coq (3)	☺	Pumpkin
Tuples & Records	Auto	Simple Records (13)	☺	Pumpkin
		Parameterized Records (17)	☺	Pumpkin
		Industrial Use (18)	☺	Pumpkin
Permute Constructors	Auto	List, Standard Library (1)	☺	Pumpkin
		Modifying a PL, REPLica Benchmark (1)	😐	Pumpkin
		Large Ambiguous Enum (1)	😐	Pumpkin
Add new Constructors	Mixed	PL Extension, REPLica Benchmark (19)	☹	Pumpkin
Factor out Constructors	Manual	External Example (2)	☺	Pumpkin
Permute Hypotheses	Manual	External Example (20)	☹	Pumpkin
Change Ind. Structure	Manual	Unary to Binary, Classic Benchmark (5)	😐	Pumpkin
		Vector to Finite Set, External Example (21)	☺	Pumpkin

Table 1: Some changes using Pumpkin Pi (left to right): class of changes, kind of configuration, examples, whether using Pumpkin Pi saved development time relative to reference manual repairs (☺ if yes, 😐 if comparable, ☹ if no), and Coq tools we know of that support repair along (Repair) or automatic proof of (Search) the equivalence corresponding to each example. Tools considered are Devoid [?], the Univalent Parametricity (UP) white-box transformation [47], and the tool from Magaud & Bertot 2000 [33]. Pumpkin Pi is the only one that suggests tactics. More nuanced comparisons to these and more are in Chapter 5.

```
Inductive Term : Set :=
| Var : Identifier → Term
| Int : Z → Term
| Eq : Term → Term → Term
| Plus : Term → Term → Term
| Times : Term → Term → Term
| Minus : Term → Term → Term
| Choose : Identifier → Term
    → Term.
```

```
Inductive Term : Set :=
| Var : Identifier → Term
| Bool : Identifier → Term
| Eq : Term → Term → Term
| Int : Z → Term
| Plus : Term → Term → Term
| Times : Term → Term → Term
| Minus : Term → Term → Term
| Choose : Identifier → Term
    → Term.
```

Figure 38: A simple language (left) and the same language with two swapped constructors and an added constructor (right).

heuristics, and 2) generate a custom eliminator for combining list terms with length invariants. The resulting workflow works not just for lists and vectors, but for any algebraic ornament, automating otherwise manual effort. The suggested tactics were helpful for writing proofs in the order of hours that I had struggled with manually over the course of days, but only after massaging. More effort is needed to improve tactic suggestions for dependent types.

TUPLES & RECORDS: INDUSTRIAL USE An industrial proof engineer at the company Galois has been using PUMPKIN Pi in proving correct an implementation of the TLS handshake protocol. Galois had been using a custom solver-aided verification language to prove correct C programs, but had found that at times, the constraint solvers got stuck. They had built a compiler that translates their language into Coq's specification language Gallina, that way proof engineers could finish stuck proofs interactively using Coq. However, due to language differences, they had found the generated Gallina programs and specifications difficult to work with.

The proof engineer used PUMPKIN Pi to port the automatically generated functions and specifications to more human-readable functions and specifications, wrote Coq proofs about those functions and specifications, then used PUMPKIN Pi to port those proofs back to proofs about the original functions and specifications. So far, they have used at least three automatic configurations, but they most often used an automatic configuration for porting compiler-produced anonymous tuples to named records, as in file (18). The workflow was a bit nonstandard, so there was little need for tactic suggestions. The proof engineer reported an initial time investment learning how to use PUMPKIN Pi, followed by later returns.

PERMUTE CONSTRUCTORS: MODIFYING A LANGUAGE The swapping example from Section 4.1 was inspired by benchmarks from the REPLICA user study of proof engineers seen earlier. A change from one of the benchmarks is in Figure 38. The proof engineer had a simple language represented by an inductive type `Term`, as well as some definitions and proofs about the language. The proof engineer swapped two constructors in the language, and added a new constructor `Bool`.

This case study and the next case study break this change into two parts. In the first part, I used PUMPKIN Pi with automatic configuration to repair functions and proofs about the language after swapping the constructors ①. With a bit of human guidance to choose the permutation from a list of suggestions, PUMPKIN Pi repaired everything, though the original tactics would have also worked, so there was not a difference in development time.

ADD NEW CONSTRUCTORS: EXTENDING A LANGUAGE I then used PUMPKIN Pi to repair functions after adding the new constructor in Figure 38, separating out the proof obligations for the new constructor from the old terms (19). This change combined manual and automatic configuration. I defined an inductive type `Diff` and (using partial automation) a configuration to port the terms across the equivalence `Old.Term + Diff` $\simeq$ `New.Term`. This resulted in case explosion, but was formulaic, and pointed to a clear path for automation of this class of changes. The repaired functions guaranteed preservation of the behavior of the original functions.

Adding constructors was less simple than swapping. For example, PUMPKIN Pi did not yet save us time over the proof engineer from the user study; fully automating the configuration would have helped significantly. In addition, the repaired terms were (unlike in the swap case) inefficient compared to human-written terms. For now, they make good regression tests for the human-written terms—in the future, I hope to automate the discovery of the more efficient terms, or use the refinement framework CoqEAL [13] to get between proofs of the inefficient and efficient terms.

FACTOR OUT CONSTRUCTORS: EXTERNAL EXAMPLE The change from Figure 25 came at the request of an anonymous reviewer. I supported this using a manual configuration that described which constructor to map to `true` and which constructor to map to `false` (2). The configuration was very simple for me to write, and the repaired tactics were immediately useful. The development time savings were on the order of minutes for a small proof development. Since most of the modest development time went into writing the configuration, I expect time savings would increase for a larger development.

PERMUTE HYPOTHESES: EXTERNAL EXAMPLE The change in (20) came at the request of Anders Mörtberg, a cubical type theory expert. It shows how to use PUMPKIN Pi to swap two hypotheses of a type, since `T1` $\to$ `T2` $\to$ `T3` $\simeq$ `T2` $\to$ `T1` $\to$ `T3`. This configuration was manual. Since neither type was inductive, this change used the generic construction for any equivalence. This worked well, but necessitated some manual annotation due to the lack of custom unification heuristics for manual configuration, and so did not yet save development time, and likely still would not have had the proof development been larger. Supporting custom unification heuristics would improve this workflow.

CHANGE INDUCTIVE STRUCTURE: UNARY TO BINARY In (5), I used PUMPKIN Pi to support a classic example of changing inductive structure: updating unary to binary numbers, as in Figure 28. Binary numbers allow for a fast addition function, found in the Coq

standard library. In the style of Magaud & Bertot 2000 [33], I used PUMPKIN Pi to derive a slow binary addition function that does not refer to `nat`, and to port proofs from unary to slow binary addition. I then showed that the ported theorems hold over fast binary addition.

The configuration for `N` used definitions from the Coq standard library for `DepConstr` and `DepElim` that had the desired behavior with no changes. `Iota` over the successor case was a rewrite by a lemma from the standard library that reduced the successor case of the eliminator that I used for `DepElim`:

```
N.peano_rect_succ : ∀ P pO pS n,
  N.peano_rect P pO pS (N.succ n) =
  pS n (N.peano_rect P pO pS n).
```

The need for nontrivial `Iota` comes from the fact that `N` and `nat` have different inductive structures. By writing a manual configuration with this `Iota`, it was possible to instantiate the PUMPKIN Pi transformation to the transformation that had been its own tool.

While porting addition from `nat` to `N` was automatic after configuring PUMPKIN Pi, porting proofs about addition took more work. Due to the lack of unification heuristics for manual configuration, I had to annotate the proof term to tell PUMPKIN Pi that implicit casts in the inductive cases of proofs were applications of `Iota` over `nat`. These annotations were formulaic, but tricky to write. Unification heuristics would go a long way toward improving the workflow.

After annotating, I obtained automatically repaired proofs about slow binary addition, which I found simple to port to fast binary addition. I hope to automate this last step in the future using CoqEAL. Repaired tactics were partially useful, but failed to understand custom eliminators like `N.peano_rect`, and to generate useful tactics for applications of `Iota`; both of these are clear paths to more useful tactics. The development time for this proof with PUMPKIN Pi was comparable to reference manual repairs by external proof engineers. Custom unification heuristics would help bring returns on investment for experts in this use case.

4.6.2 *Evaluation:* PUMPKIN P*i for Transport*

PUMPKIN Pi implements transport across equivalences in a way that is suitable for repair. Nate and I compared PUMPKIN Pi to another tool for transport across equivalences in Coq, using algebraic ornaments as an example. We used PUMPKIN Pi to automatically discover and transport functions and proofs along the equivalences corresponding to these ornaments for two scenarios:

1. Single Iteration: from binary trees to sized binary trees
2. Multiple Iterations: from binary trees to binary search trees to AVL trees

At the time, the decompiler was not yet implemented, so we focused solely on proof terms. For comparison, we also used the ornaments that Pumpkin Pi discovered to transport functions and proofs using the UP black-box transformation [46]. Pumpkin Pi produced faster functions and smaller terms, especially when composing multiple iterations of repair. In addition, Pumpkin Pi imposed little burden on the user, and the equivalences Pumpkin Pi discovered proved useful to UP.

We chose UP for comparison because it also implements transport across equivalences in Coq, and because doing so demonstrates the benefits of specialized automation for classes of equivalences like ornaments. At the time of evaluation, the UP white-box transformation did not exist, so we used the black-box transformation. Using the white-box transformation would likely improve the performance of functions and proofs generated by UP when applicable. Our experiences suggest that it is possible to use both tools in concert. Chapter 5 discusses UP in more detail.

SETUP The code is in the `eval` folder of the repository. For each scenario, Nate ran Pumpkin Pi to search for an ornament, and then transported functions and proofs along that ornament using both Pumpkin Pi and UP. He noted the amount of user interaction (Section 4.6.2.1), and together we measured the performance of transported terms (Section 4.6.2.2). To test the performance of transported terms, we tested runtime by taking the median of ten runs using `Time Eval vm_compute` with test values in Coq 8.8.0, and we tested size by normalizing and running `coqwc` on the result.[9]

In the first scenario, Nate transported traversal functions along with proofs that their outputs are permutations of each other from binary trees (`tree`) to sized binary trees (`Sized.tree`). In the second scenario, he transported the traversal functions to AVL trees (`avl`) through four intermediate types (one for each new index), and he lifted a search function from BSTs (`bst`) to AVL trees through one intermediate type. Both scenarios considered only full binary trees.

To fit `bst` and `avl` into algebraic ornaments for Pumpkin Pi, we decided to use boolean indices to track invariants. While the resulting types are not the most natural definitions, this scenario demonstrates that it is possible to express interesting changes to structured types as algebraic ornaments, and that lifting across these types in Pumpkin Pi produces efficient functions.

4.6.2.1 *User Experience*

For each intermediate type in each scenario, Nate used Pumpkin Pi to discover the equivalence. This was enough for Pumpkin Pi

9 i5-5300U, at 2.30GHz, 16 GB RAM

		10	100	1000	10000	100000
`preorder`	**Unlifted**	0.0	0.0	0.0	3.0 (1.00x)	37.0 (1.00x)
	Pumpkin Pi	0.0	0.0	0.0	3.0 (1.00x)	35.0 (0.95x)
	UP	0.0	1.0	27.0	486.5 (162.17x)	8078.5 (218.33x)

Figure 39: Median runtime (ms) of the original (`tree`) and transported (`Sized.tree`) `preorder` over ten runs with test inputs ranging from about 10 to about 10000 nodes.

		10	100	1000	10000	100000
`preorder`	**Unlifted**	0.0	0.0	0.0	3.0 (1.00x)	37.0 (1.00x)
	Pumpkin Pi	71.5	71.0	69.0	75.0 (25.00x)	109.0 (2.95x)
	UP	1.0	11.0	152.0	2976.5 (992.17x)	56636.5 (1530.72x)
`search`	**Unlifted**	0.0	0.0	2.0 (1.00x)	3.0 (1.00x)	29.0 (1.00x)
	Pumpkin Pi	12.0	14.0	12.0 (6.00x)	15.0 (5.00x)	50.0 (1.72x)
	UP	1.0	5.0	67.0 (33.50x)	1062.0 (354.00x)	15370.5 (530.02x)

Figure 40: Median runtime (ms) of the original (`tree`) and transported (`avl`) `preorder`, plus the original (`bst`) and transported (`avl`) `search`, over ten runs with inputs ranging from about 10 to about 100000 nodes.

to lift functions and proofs with no additional proof burden and no additional axioms. To use UP without Pumpkin Pi, Nate would have had to prove the equivalence by hand, but instead he was able to use the equivalence generated by Pumpkin Pi. In addition, to use UP, he had to prove univalent parametricity of each inductive type; these proofs were small, but required specialized knowledge. To lift the proof of the theorem `pre_permutes` using UP, he had to prove the univalent parametric relation between the unlifted and lifted versions of the functions that the theorem referenced; this pulled in the functional extensionality axiom, which was not necessary using Pumpkin Pi.

In the second scenario, to simulate the incremental workflow Pumpkin Pi requires, Nate transported along each intermediate type, then unpacked the result. For example, the ornament from `bst` to `avl` passed through an intermediate type; Nate lifted `search` to this type first, unpacked the result, and then repeated this process. In this scenario, using UP differently or using Pumpkin Pi with a manual configuration could have saved some work relative to the workflow chosen, since with those workflows, it is possible to skip the intermediate type;[10] Pumpkin Pi with automatic configuration is best fit where an incremental workflow is desirable.

4.6.2.2 *Performance*

Relative to UP, Pumpkin Pi produced faster functions. Figure 39 summarizes runtime in the first scenario for `preorder`, and Figure 40 summarizes runtime in the second scenario for `preorder` and `search`. The `inorder` and `postorder` functions performed similarly to `preorder`. The functions Pumpkin Pi produced imposed modest overhead for smaller inputs, but were tens to hundreds of times faster than the functions that UP produced for larger inputs. This performance gap was more pronounced over multiple iterations of lifting.

Pumpkin Pi also produced smaller terms: in the first scenario, 13 vs. 25 LOC for `preorder`, 12 vs. 24 LOC for `inorder`, and 17 vs. 29 LOC for `postorder`; and in the second scenario, 21 vs. 120 LOC for `preorder`, 20 vs. 119 LOC for `inorder`, 24 vs. 125 LOC for `postorder`, and 31 vs. 52 LOC for `search`. In the first scenario, the transported proof of `pre_permutes` using Pumpkin Pi was 85 LOC; the transported proof of `pre_permutes` using UP was 1463184 LOC.

I suspect Pumpkin Pi provided these performance benefits because it directly transformed induction principles, whereas the UP black-box transformation implemented transport in a standard way, defining transported functions in terms of the original functions. The multiple iteration case in particular highlights this, since UP's black box approach makes lifted terms much slower and larger as the number of iterations increases, while Pumpkin Pi's approach does not.[11]

4.7 CONCLUSION

The Pumpkin Pi plugin addresses the main limitations of the Pumpkin prototype: Pumpkin had limited support for patch application, supported a narrow class of changes, did not support typical proof engineering workflows like tactics, and only retroactively *could have* helped proof engineers in a few practical use cases. Pumpkin Pi, in contrast, supports patch application in a principled manner, broadens the scope of Pumpkin Patch to include a large and flexible class of changes, is built with workflow integration including tactics in mind,

10 The performances of the terms that UP produces are sensitive to the equivalence used; for a 100 node tree, this alternate workflow in UP produced a search function which is hundreds of times slower and traversal functions which are thousands of times slower than the functions that Pumpkin Pi produced. In addition, the lifted proof of `pre_permutes` using UP failed to normalize with a timeout of one hour.

11 After this evaluation and several discussions with the authors, UP later introduced the white-box transformation, which is similar to the Pumpkin Pi transformation [47], though it does not yet support arbitrary equivalences (the black-box transformation of course does). At the time of the evaluation, Pumpkin Pi also did not yet support arbitrary equivalences, though the UP black-box transformation did. The development of both the UP white-box transformation and the generalized Pumpkin Pi transformation happened with frequent conversations between the authors of both papers, and doubtlessly involved mutual influence on one another. It was wonderful, and involved multiple trips to France. I will discuss UP more in Chapter 5.

and can save and in fact already has saved work for proof engineers in practical use cases.

At this point, it is fair to say that my thesis holds:

> **Thesis**: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and apply to fix other proofs broken by the same change (Sections 3.2, 3.3, 3.4, 4.2, 4.3,and 4.4). A tool that automates this (Sections 3.5 and 4.5) can save work for proof engineers relative to reference manual repairs in practical use cases (Sections 3.6 and 4.6).

And so there really is *reason to believe*.

It is, importantly, not just me who believes this now. Consider a recent article by a proof engineer saying just this (emphasis mine, again):

> We have *reason to think* such proof repair is tractable. Rather than trying to synthesize a complete proof from nothing—a problem known to be immensely difficult—we start from a correct proof of fairly similar software. We will be attempting proof reconstruction *within a known neighborhood*.

The proof engineer credited my work on Twitter, but noted that there ought to be much, much more work in this space. I agree, and I am excited to share some ideas for this in Chapter 6.

But first, I would like to back up a bit and talk about other work in this space, in part because doing so is standard, and in part because it has been and continues to be a wonderful source of inspiration. I would not skip this chapter—remember that I wrote a whole survey paper of proof engineering, so I take related work quite seriously. If you are still awake, please enjoy Chapter 5.

5

RELATED WORK

Proof repair can be viewed as program repair (Section 5.2) for the domain of proof engineering (Section 5.1). Work in type theory, program synthesis, differencing, program transformations, automated theorem proving, and user studies has inspired and continues to inspire improvements to proof repair tools and techniques (Section 5.3).

5.1 PROOF ENGINEERING

Proof engineering refers to the technologies that make it easier to develop and maintain systems verified using proof assistants. Proof repair falls into the domain of proof engineering. This section briefly describes work in proof engineering relevant to my work on proof repair: proof assistants other than Coq (Section 5.1.1), other technologies for proof evolution (Section 5.1.2), proof reuse (Section 5.1.3), proof design (Section 5.1.4), and other kinds of proof automation (Section 5.1.5. See my survey paper on proof engineering for a detailed overview of the field [42].

5.1.1 *Proof Assistants*

Introduction, definition, de bruijn.

Summary of proof assistants.

I will discuss how these differ and the implications for proof repair.

FOUNDATIONS Most notably, proof terms. But also differences in logics (constructivism, then refer back to it later).

AESTHETICS Like different tactic languages.

5.1.2 *Proof Evolution*

PROOF REFACTORING

PROOF REPAIR

DEPENDENCY MANAGEMENT

5.1.3 *Proof Reuse*

PROOF TRANSFORMATION

5.1.4 *Proof Design*

5.1.5 *Proof Automation*

5.2 PROGRAM REPAIR

Proof repair can be viewed as a form of *program repair* [36, 19] for proof assistants. Proof assistants like Coq are an especially good fit for program repair (Section 5.2.1). While it is not straightforward to apply existing program repair techniques to proof assistants, looking to them for inspiration may help improve proof repair tools more in the future (Section 5.2.2).

5.2.1 *A Good Fit*

A recent survey of program repair distinguishes between repair tools based on the *oracle* they use to judge whether a patch is correct. For example, proof repair is a kind of *specification-based* repair, since it uses a specification (a goal type derived from differencing) as an oracle. Program repair tools sometimes use other oracles—commonly, test suites (*test-based* repair).

A recent review [41] of a popular test-based program repair tool [**?**] and its variants shows that most of the reported patches generated by the tool are not correct. These observations are later reaffirmed in a different setting [**?**]. In response, the paper recommends that program repair tools draw on extra information, like specifications or example patches. In Coq, specifications and examples are rich and widely available: specifications thanks to dependent types, and examples thanks to constructivism. This shows why proof repair in Coq is an especially good fit for program repair.

SPECIFICATIONS One limitation of test-based program repair tools is that tests in evaluation suites are often underspecified, so it can be hard to know when a patch to a program is correct. For example, the review notes that some tests in the evaluation suite for the tool check whether a program terminates with an exit code of 0, but do not check the program output. In addition, patches are often overfit to the tests in the test suite; additional tests expose problems with those patches. In fact, some patches are outright harmful, as they introduce new problems which the test suite does not check for.

In contrast, in the world of proof repair, there is always a specification to work with—the theorem being proven—so a proof repair tool does not need to rely on tests. Furthermore, the scope of properties that can be specified in proof assistants like Coq is especially large thanks to its experessive type system CIC_ω, with polymorphism and dependent types. The richness of the type theory further makes it possible for PUMPKIN PATCH to check itself along the way and make sure it is on the right track.

Still, there is always a chance that the specification itself must change in order for proof repair to work, as I showed with PUMPKIN Pi. In those cases, it is helpful to have some assurance that the specifications the tool produces are meaningful—in the case of PUMPKIN Pi, that the old and new specifications are equal up to transport along the change in the datatype. It is also useful to have a human in the loop to check specifications in the end, as PUMPKIN Pi does. Section 5.2.2 discusses other specification-based repair tools, as well as other repair tools that bring a human into the loop.

EXAMPLES Another challenge for test-based program repair tools is defining the correct search space and searching efficiently within it. For example, the review found that running the same tool on strengthened versions of the test suites produced no patches at all in the time alloted. One possible reason for this is that the tools could not search for the correct patches efficiently enough. Example-based techniques can help navigate a large search space quickly.

PUMPKIN PATCH uses examples—in the former of changes to datatypes or proofs—to derive patches. The constructive nature of the logic CIC_ω beneath Coq makes this especially appealing and powerful, since it means (by definition) that proofs must be constructed from basic principles. So existence proofs, for example, must be accompanied by a witness.[1] Each of these witnesses is in effect an example that PUMPKIN PATCH can extract and generalize, narrowing down the search space of possible repairs.

Thanks to the richness of the type theory, PUMPKIN PATCH can in practical use cases repair proofs by generalizing a very small number of examples, like a single example patched proof, or a single example change to a datatype. Section 5.2.2 describes other example-based repair tools.

5.2.2 *Techniques for Inspiration*

This section discusses techniques from existing program repair tools that are relevant to proof repair. It focuses in particular on what a recent survey [36] of program repair calls behavioral repair, or patching the code, rather than state repair, or patching the dynamic behav-

1 This is known as the *existence property*.

ior. Among behavioral repair tools, it focuses on regression repair, specification-based repair, repair by example, and other techniques that bring a human into the loop.

REGRESSION REPAIR Regression repair tools target regression bugs, or bugs introduced by changes in code. In some sense, proof repair is a kind of regression repair, as it repairs proofs that used to succeed, but after some change, no longer do.

SPECIFICATION-BASED REPAIR

REPAIR BY EXAMPLE

HUMAN IN THE LOOP Some tools avoid using test suites to judge correctness of candidate patches, and instead bring the programmer into the loop. For example, both ReAssert [**?**] and QACrashFix [**?**] suggest repairs directly to the programmer. Opad [**?**] provides an alternative to expecting the test-based repair tool to make use of extra information. Instead, Opad automatically detects overfitted patches and improves existing test suites. The evaluation shows that using Opad to improve test suites improves the performance of several test-based program repair tools.

5.3 MORE INSPIRATION

Other inspiration comes from type theory (Section 5.3.1), program synthesis (Section 5.3.2), differencing (Section 5.3.3), program transformations (Section 5.3.4), automated theorem proving (Section 5.3.5), and user studies (Section 5.3.6).

5.3.1 *Type Theory*

ORNAMENTS

TRANSPORT

PARAMETRICITY

5.3.2 *Program Synthesis*

PROGRAMMING BY EXAMPLE

NEUROSYMBOLIC PROGRAMMING

5.3.3 *Differencing*

5.3.4 *Program Transformations*

5.3.5 *Automated Theorem Proving*

E-GRAPHS

5.3.6 *User Studies*

6

CONCLUSIONS & FUTURE WORK

Reflect on thesis statement and explain how we got it exactly now that you know everything

But I want to spend the resst of this thesis talking about the next era of verification so I can write out a bunch of ideas for students who might want to work with me

THE NEXT ERA: PROOF ENGINEERING FOR ALL

Future Work from many papers, plus research statement, DARPA thoughts, plus more, but trimmed down a lot

What I want in the long run, how this all fits in, is a world of proof engineering for all. From research statement, three rings (four including experts in the center).

And what we have so far with my thesis is a world where it's easier for experts and a bit easier for practitioners, but there's still a lot left to go building on it.

So here are 12 short future project summaries that reach each of these tiers, building that world. Super please contact me if any of these seem fun to you.

Proof Engineering for Experts

Unifying theme: lateral reach. Some examples:

MORE PROOF ASSISTANTS Thoughts from PUMPKIN Pi on Isabelle/HOL, future work from PUMPKIN PATCH.

MORE CHANGES Version updates, isolating large changes (PUMPKIN PATCH), relations more general than equivalences (PUMPKIN Pi).

MORE STYLES ML for decompiler (PUMPKIN Pi, REPLICA) : more for diverse proof styles (PUMPKIN PATCH). Note that this is a WIP, but sketch out project, challenges, future ideas, expectations, evaluation a bit.

Proof Engineering for Practitioners

Unifying theme: usability. Some examples:

AUTOMATION More search procedures for automatic configuration, e-graphs from PUMPKIN Pi, custom unification heuristics.

INTEGRATION IDE & CI integration, HCI for repair.

EVALUATION repair challenge, user studies ideas (PUMPKIN PATCH, REPLICA, panel w/ Benjamin Pierce, QED at large). (maybe look for more ideas, this can be merged with integration if need be).

Proof Engineering for Software Engineers

Unifying theme: mixed methods verification, or the 2030 vision from Twitter thread. Some examples:

GRADUAL VERIFICATION A continuum from testing to verification, tools to help with that.

TOOL-ASSISTED PROOF DEVELOPMENT Tool-assisted development to follow good design principles for verificattion (James Wilcox conversation, final REPLICA takeaway).

SPECIFICATION INFERENCE Analysis to infer specs (TA1).

Proof Engineering for New Domains

Unifying theme: collaboration, new abstractions for new domains). Some examples:

MACHINE LEARNING Fairification & other ML correctness properties. Some stuff here but more.

CRYPTOGRAPHY Lots of stuff here but not thinking broadly enough. What about cryptographic proof systems? ZK and beyond. Recall email thread.

SOMETHING ELSE Look for more in survey paper, email, DARPA TAs, Twitter. Healthcare perhaps?

INDEX

Collaborators
    Nathaniel Yazdani, 7, 95, 99, 100
    RanDair Porter, 7, 92, 94
Concepts
    Algebraic Ornaments, 76, 95
Conventions
    Primitive Eliminators, 17, 19, 20, 71, 88, 89

Languages
    Gallina, 10–12, 14, 18, 64
    Ltac, 11, 12

Type Theories
    Calculus of Inductive Constructions, 14, 18, 71

BIBLIOGRAPHY

[1] Coq reference manual, section 8.9: Controlling automation, 2017.

[2] Lean theorem prover, 2017.

[3] User A. Software foundations solution, 2017.

[4] Carlo Angiuli, Evan Cavallo, Anders Mörtberg, and Max Zeuner. Internalizing representation independence with univalence, 2020.

[5] Serge Autexier, Dieter Hutter, and Till Mossakowski. Verification, induction termination analysis. chapter Change Management for Heterogeneous Development Graphs, pages 54–80. Springer-Verlag, Berlin, Heidelberg, 2010.

[6] User B. Software foundations solution, 2017.

[7] Henk Barendregt and Erik Barendsen. Autarkic computations in formal proofs. *Journal of Automated Reasoning*, 28(3):321–336, 2002.

[8] Henk Barendregt and Freek Wiedijk. The challenge of computer mathematics. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 363(1835):2351–2375, 2005.

[9] Pierre Boutillier. *New tool to compute with inductive in Coq*. Theses, Université Paris-Diderot - Paris VII, February 2014.

[10] Ahmet Celik, Karl Palmskog, and Milos Gligoric. icoq: Regression proof selection for large-scale verification projects. In *Proceedings of the 32Nd IEEE/ACM International Conference on Automated Software Engineering*, ASE 2017, pages 171–182, Piscataway, NJ, USA, 2017. IEEE Press.

[11] Adam Chlipala. *Certified Programming with Dependent Types - A Pragmatic Introduction to the Coq Proof Assistant*. MIT Press, 2013.

[12] Adam Chlipala. Library equality, 2017.

[13] Cyril Cohen, Maxime Dénès, and Anders Mörtberg. Refinements for Free! In *Certified Programs and Proofs*, pages 147 – 162, Melbourne, Australia, December 2013.

[14] Coq Development Team. The Coq proof assistant, 1989-2021.

[15] T. Coquand and Gérard Huet. The calculus of constructions. Technical Report RR-0530, INRIA, May 1986.

[16] Thierry Coquand and Christine Paulin. Inductively defined types. In Per Martin-Löf and Grigori Mints, editors, *COLOG-88*, pages 50–66, Berlin, Heidelberg, 1990. Springer Berlin Heidelberg.

[17] Richard A. DeMillo, Richard J. Lipton, and Alan J. Perlis. Social processes and proofs of theorems and programs. In *Proceedings of the 4th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages*, POPL '77, pages 206–214, New York, NY, USA, 1977. ACM.

[18] Ricardo Bedin França, Denis Favre-Felix, Xavier Leroy, Marc Pantel, and Jean Souyris. Towards Formally Verified Optimizing Compilation in Flight Control Software. In *Bringing Theory to Practice: Predictability and Performance in Embedded Systems*, volume 18 of *OpenAccess Series in Informatics (OASIcs)*, pages 59–68, Dagstuhl, Germany, 2011. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[19] Luca Gazzola, Daniela Micucci, and Leonardo Mariani. Automatic software repair: A survey. In *Proceedings of the 40th International Conference on Software Engineering*, ICSE '18, pages 1219–1219, New York, NY, USA, 2018. ACM.

[20] Sumit Gulwani, Oleksandr Polozov, and Rishabh Singh. Program synthesis. *Foundations and Trends in Programming Languages*, 4(1-2):1–119, 2017.

[21] Martin Hofmann. Syntax and semantics of dependent types. In *Semantics and Logics of Computation*, pages 79–130. Cambridge University Press, 1997.

[22] HOL Light Development Team. HOL Light, 1996-2021.

[23] D. Hutter. Management of change in structured verification. In *ASE 2000*, pages 23–31, Sept 2000.

[24] Isabelle Development Team. Isabelle, 1994-2021.

[25] Daniel Kästner, Xavier Leroy, Sandrine Blazy, Bernhard Schommer, Michael Schmidt, and Christian Ferdinand. Closing the gap – the formally verified optimizing compiler CompCert. In *SSS'17: Safety-critical Systems Symposium 2017*, Developments in System Safety Engineering: Proceedings of the Twenty-fifth Safety-critical Systems Symposium, pages 163–180, Bristol, United Kingdom, February 2017. CreateSpace.

[26] Gerwin Klein, June Andronick, Kevin Elphinstone, Toby Murray, Thomas Sewell, Rafal Kolanski, and Gernot Heiser. Comprehensive formal verification of an OS microkernel. *ACM Trans. Comput. Syst.*, 32(1):2:1–2:70, February 2014.

[27] Gerwin Klein, Kevin Elphinstone, Gernot Heiser, June Andronick, David Cock, Philip Derrin, Dhammika Elkaduwe, Kai Engelhardt, Rafal Kolanski, Michael Norrish, Thomas Sewell, Harvey Tuch, and Simon Winwood. seL4: Formal verification of an OS kernel. In *Proceedings of the ACM SIGOPS 22Nd Symposium on Operating Systems Principles*, SOSP '09, pages 207–220, New York, NY, USA, 2009. ACM.

[28] Hsiang-Shang Ko and Jeremy Gibbons. Programming with ornaments. *Journal of Functional Programming*, 27, 2016.

[29] Xavier Leroy. Formal certification of a compiler back-end or: Programming a compiler with a proof assistant. In *Conference Record of the 33rd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '06, pages 42–54, New York, NY, USA, 2006. ACM.

[30] Xavier Leroy. Formal verification of a realistic compiler. *Commun. ACM*, 52(7):107–115, July 2009.

[31] Xavier Leroy. Commit to compcert: lib/integers.v, 2013.

[32] letouzey. Commit to coq: change definition of divide (compat with znumtheory), 2011.

[33] Nicolas Magaud and Yves Bertot. Changing data structures in type theory: A study of natural numbers. In *International Workshop on Types for Proofs and Programs*, pages 181–196. Springer, 2000.

[34] Conor McBride. Ornamental algebras, algebraic ornaments, 2011.

[35] Guillaume Melquiond. Commit to coq: Make izr use a compact representation of integers, 2017.

[36] Martin Monperrus. Automatic software repair: A bibliography. *ACM Comput. Surv.*, 51(1):17:1–17:24, January 2018.

[37] Toby Murray and P. C. van Oorschot. BP: Formal proofs, the fine print and side effects. In *IEEE Cybersecurity Development (SecDev)*, pages 1–10, Sep. 2018.

[38] nLab authors. beta-reduction. `http://ncatlab.org/nlab/show/beta-reduction`, July 2020. Revision 6.

[39] nLab authors. eta-conversion. `http://ncatlab.org/nlab/show/eta-conversion`, July 2020. Revision 12.

[40] Benjamin C. Pierce, Arthur Azevedo de Amorim, Chris Casinghino, Marco Gaboardi, Michael Greenberg, Catălin Hriţcu, Vilhelm Sjöberg, and Brent Yorgey. *Software Foundations*. Electronic textbook, 2016. Version 4.0. `http://www.cis.upenn.edu/ bcpierce/sf`.

[41] Zichao Qi, Fan Long, Sara Achour, and Martin Rinard. An analysis of patch plausibility and correctness for generate-and-validate patch generation systems. In *Proceedings of the 2015 International Symposium on Software Testing and Analysis*, ISSTA 2015, pages 24–36, New York, NY, USA, 2015. ACM.

[42] Talia Ringer, Karl Palmskog, Ilya Sergey, Milos Gligoric, and Zachary Tatlock. Qed at large: A survey of engineering of formally verified software. *Foundations and Trends® in Programming Languages*, 5(2-3):102–281, 2019.

[43] Talia Ringer and Nathaniel Yazdani. Pumpkin-git, 2018.

[44] Valentin Robert. *Front-end tooling for building and maintaining dependently-typed functional programs*. PhD thesis, UC San Diego, 2018.

[45] Matthieu Sozeau. Equations: A dependent pattern-matching compiler. In Matt Kaufmann and Lawrence C. Paulson, editors, *Interactive Theorem Proving*, pages 419–434, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.

[46] Nicolas Tabareau, Éric Tanter, and Matthieu Sozeau. Equivalences for free: Univalent parametricity for effective transport. *Proc. ACM Program. Lang.*, 2(ICFP):92:1–92:29, July 2018.

[47] Nicolas Tabareau, Éric Tanter, and Matthieu Sozeau. The marriage of univalence and parametricity, 2019.

[48] Amin Timany and Bart Jacobs. First steps towards cumulative inductive types in CIC. In *ICTAC*, 2015.

[49] Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.

[50] Karin Wibergh. Automatic refactoring for agda. Master's thesis, Chalmers University of Technology and University of Gothenburg, 2019.

[51] Xuejun Yang, Yang Chen, Eric Eide, and John Regehr. Finding and understanding bugs in C compilers. In *Proceedings of the 32nd ACM SIGPLAN Conference on Programming Language Design and Implementation*, PLDI '11, pages 283–294, New York, NY, USA, 2011. ACM.