# PROOF REPAIR

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ABSTRACT

# PROOF REPAIR



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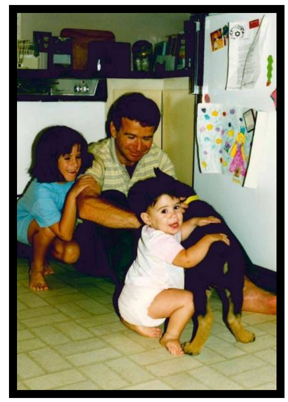
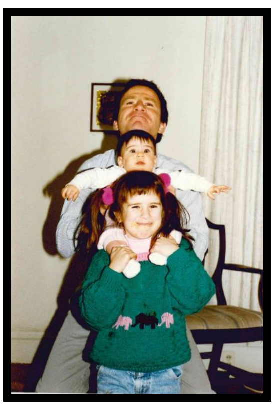
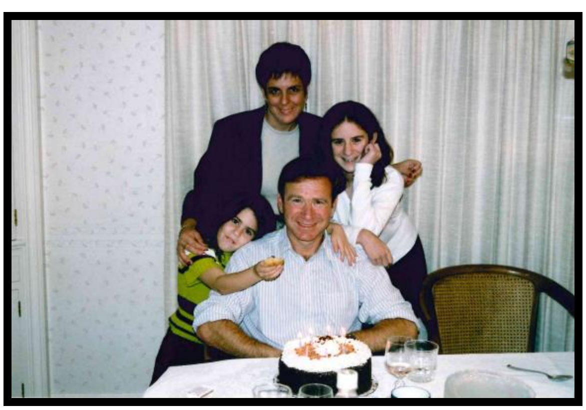
Abstract will go here.

To my family.




I love all of you.

# CONTENTS

# ACKNOWLEDGMENTS

I've always believed the acknowledgments section to be one of the most important parts of a paper. But there's never enough room to thank everyone I want to thank. Now that I have the chance—where do I begin?

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# 1

## INTRODUCTION

What would it take to empower programmers of all skill levels across all domains to formally prove the absence of costly or dangerous bugs in software systems—that is, to formally *verify* them?

Verification has already come a long way toward this since its inception, especially when it comes to the scale of systems that can be verified. The seL4 [66] verified operating system (OS) microkernel, for example, is the effort of a team of proof engineers spanning more than a million lines of proof, costing over 20 person-years. Given a famous 1977 critique of verification [42] (emphasis mine):

> *A sufficiently fanatical researcher* might be willing to devote *two or three years* to verifying a significant piece of software if he could be assured that the software would remain stable.

I could argue that, over 40 years, either verification has become easier, or researchers have become more fanatical. Unfortunately, not all has changed (emphasis still mine):

> But real-life programs need to be maintained and modified. There is *no reason to believe* that verifying a modified program is any easier than verifying the original the first time around.

As we will soon see, this remains so difficult that sometimes, even experts give up in the face of change.

This thesis aims to change that by taking advantage of a missed opportunity: tools for developing verified systems have no understanding of how these systems evolve over time, so they miss out on crucial information. This thesis introduces a new class of verification tools called *proof repair* tools. Proof repair tools understand how software systems evolve, and use the crucial information that evolution carries to automatically evolve proofs about those systems. This gives us reason to believe.

## 1.1 THESIS

Proof repair falls under the umbrella of *proof engineering*: the technologies that make it easier to develop and maintain verified systems. Much like software engineering scales programming to large systems, so proof engineering scales verification to large systems. In recent years, proof engineers have verified OS microkernels [66, 64], machine learning systems [**?**], distributed systems [**?**], constraint solvers [**?**], web browser kernels [**?**], compilers [72, 73], file systems [**?**], and even a quantum optimizer [**?**]. As we will soon see, practitioners have found these verified systems to be more robust and secure in deployment.

Proof engineering focuses in particular on verified systems that have been developed using special tools called *proof assistants* or interactive theorem provers (ITPs). Examples of proof assistants include Coq [30], Isabelle/HOL [59], HOL Light [55], and Lean [2]. The proof assistant that I focus on in this thesis will be the Coq proof assistant. A discussion of how this work carries over to other proof assistants is in Section 5.

To develop a verified system using a proof assistant like Coq, the proof engineer does three things:

1. implements a program using a functional programming language,

2. specifies what it means for the program to be correct, and

3. proves that the program satisfies the specification.

This proof assistant then automatically checks this proof with a small trusted part of its system [10, 11]. If the proof is correct, then the program satisfies its specification—it is *verified*.

Proof repair automatically fixes broken proofs in response to changes in programs and specifications. For example, a proof engineer who optimizes an algorithm may change the program, but not the specification; a proof engineer who adapts an OS to new hardware may change both. Even a small change to a program or specification can break many proofs, especially in large systems. Changing a verified library, for example, can break proofs about programs that depend on that library—and those breaking changes can be outside of the proof engineers' control.

Proof repair views these broken proofs as bugs that a tool can patch. In doing so, it shows that there *is* reason to believe that verifying a modified system should often, in practical use cases, be easier than verifying the original the first time around, even when proof engineers do not follow good development processes, or when change occurs outside of proof engineers' control. More formally:

> **Thesis**: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and

Figure 1: TODO

> apply to fix other proofs broken by the same change. A tool that automates this can save work for proof engineers relative to reference manual repairs in practical use cases.

## 1.2 APPROACH

The way that proof engineers typically write proofs can obfuscate the information that changes in programs, specifications, and proofs carry. The typical proof engineering workflow in Coq is interactive: The proof engineer passes Coq high-level search procedures called *tactics* (like `induction`), and Coq responds to each tactic by refining the current goal to some subgoal (like the proof obligation for the base case). This loop of tactics and goals continues until no goals remain, at which point the proof engineer has constructed a high-level sequence of tactics called a *proof script*. To check the proof, the proof assistant compiles it down to a low-level representation called a *proof term*, then checks that the proof term has the expected type. Figure 1 illustrates this workflow.

The high-level language of tactics can abstract away important details that a proof repair tool needs, but the low-level language of proof terms can be brittle and challenging to work with. Crucially, though, the low-level language comes with lots of structure and strong guarantees. My approach to proof repair works in the low-level language to take advantage of that. It then builds back up to the high-level language in the end.

By working at the low-level language, it is able to systematically and with strong guarantees extract and generalize the information that breaking changes carry, then apply those changes to fix other proofs broken by the same change. But by later building up to the high-level language, it can in the end produce proofs that integrate more naturally with proof engineering workflows.

This works using a combination of semantic differencing and program transformations in this low-level language. In particular, it uses a semantic differencing algorithm to extract information from a breaking change in a program, specification, or proof. It then combines that with program transformations to generalize and, in some cases, apply that information to fix other proofs broken by the same change. In the end, it uses a prototype decompiler to get from the low-level language back up to the high-level language, so that proof engineers can continue to work in that language going forward.

## 1.3 RESULTS

The technical results of this thesis are threefold:

1. the **design** of differencing algorithms & program transformations for proof repair,
2. an **implementation** of a proof repair tool suite, and
3. **case studies** to evaluate the tool suite on real proof repair scenarios.

Viewing the thesis statement as a theorem, the proof is as follows:

> **Thesis Proof**: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and apply to fix other proofs broken by the same change (by **design** and **implementation**). A tool that automates this can save work for proof engineers relative to reference manual repairs in practical use cases (by **case studies**).

DESIGN The design describes semantic differencing algorithms to extract information from breaking changes in verified systems, along with proof term transformations to generalize and, in some cases, apply the information to fix proofs broken by the change. The semantic differencing algorithms compare the old and new versions of a changed term or type, and from that find a diff that describes that information corresponding to that change; the transformations then use that diff to transform some term to a more general fix. The details vary by the class of change supported. These design is guided heavily by foundational developments in depenedent type theory; the theory is sprinkle throughout as appropriate. More details including limitations are in the corresponding chapters.

IMPLEMENTATION The implementation shows that in fact *a tool* can extract and generalize the information that changes carry, and then apply that information to fix other proofs broken by the same change. This implementation comes in the form of a proof repair tool suite for Coq called PUMPKIN PATCH (Proof Updater Mechanically Passing Knowledge Into New Proofs, Assisting the Coq Hacker). PUMPKIN PATCH implements two kinds of proof repair: proof repair by example (Chapter 3) and proof repair across type equivalences (Chapter 4). Notably, since all repairs that PUMPKIN PATCH produces are checked Coq in the end, PUMPKIN PATCH does not extend *Trusted Computing Base* (TCB): the set of unverified components that the correctnes of the proof development depends on [**?**]. In total, PUMPKIN PATCH is about 15000 lines of code implemented in OCaml. These 15000 lines of code consist of three plugins and a library, which together bridge the gap between the theory supported by design and the practical proof repair needed for the case studies. Toward that end, five notable features include:

1. a preprocessing tool to support features in the implementation language missing from the theory,
2. a prototype decompiler from proof terms to proof scripts for better workflow integration,
3. optimizations for efficiency,
4. meaningful error messages for usability, and
5. additional automation for applying patches.

More details and other features are in the corresponding chapters.

CASE STUDIES The case studies show that PUMPKIN PATCH can save work for proof engineers relative to reference manual repairs in practical use cases. TODO for all of the tools, summary, forward references.

## 1.4 READING GUIDE

How to read this thesis

Mapping of papers to chapters (conclusion paves path to the next era of verification)

Authorship statements for included paper materials, to credit coauthors. Discussion of "we" versus "I" or something.

Expected reader background & where to find more info

# 2

## MOTIVATING PROOF REPAIR

This thesis describes techniques and tools for automatically repairing broken proofs in a proof assistant. It focuses in particular on proofs about formally verified programs, though many of the techniques and tools carry over to mathematical proofs as well. Formal verification of a program can improve actual and perceived reliability. It can help the programmer think about the desired and actual behavior of the program, perhaps finding and fixing bugs in the process [91]. It can make explicit which parts of the system are trusted, and further decrease the burden of trust as more of the system is verified.

One noteworthy program verification success story is the CompCert [72, 73] verified optimizing C compiler. Both the back-end and front-end compilation passes of CompCert have been verified, ensuring the correctness of their composition [63]. CompCert has stood up to the trials of human trust: it has been used, for example, to compile code for safety-critical flight control software [48]. It has also stood up to rigorous testing: while the test generation tool Csmith [115] found 79 bugs in GCC and 202 bugs in LLVM, it was unable to find any bugs in the verified parts of CompCert.

CompCert, however, was not a simple endeavor: the original development comprised of approximately 35000 lines of Coq code; functionality accounted for only 13% of this, while specifications and proofs accounted for the other 87%. This is not unusual for large proof developments. The initial correctness proofs for an OS microkernel, for example, consisted of 480000 lines of specifications and proofs [64]. Proof engineering technologies make it possible to develop verified systems at this scale. See Ringer and friends 2019 [99] for a comprehensive overview of proof engineering.

Proof repair—the focus of this thesis—is a new proof engineering technology that focuses in particular on minimizing the burden of change as verified systems evolve over time. But for the sake of this chapter, I motivate proof repair not on a large verified system like a C compiler or an OS microkernel. Instead, I motivate it on a simple proof development: a list zip function accompanied by a formal proof that it preserves the lengths of its inputs. This is a small example, but it is worth noting that large proof developments like compilers and microkernels are often made up of many of these small examples built on top of each other.

The proof assistant that I motivate this on is Coq, since this is the proof assistant for which Pumpkin Patch is implemented. I motivate this using the small list zip example in three parts:

1. the workflow of and theory beneath *proof development* (Section 2.1),

2. some challenges of and approaches to *proof maintenance* (Section 2.2), and

3. the motivation for and approach to *proof repair* that follow (Section 2.3).

I will refer to the example and theory introduced in this chapter in later chapters, so it is good to at least skim this chapter regardless of Coq experience.

## 2.1 PROOF DEVELOPMENT

Before I motivate proof maintenance and repair, it helps to understand proof development in Coq to begin with. In the introduction, I briefly explained the workflow for using Coq to develop a verified system:

> The proof engineer does three things:
>
> 1. implements a program using a functional programming language,
> 2. specifies what it means for the program to be correct, and
> 3. proves that the program satisfies the specification.

That functional programming language is a rich functional programming language called *Gallina*. It is possile to use Gallina to write the program, the specification, and the proof—but writing the proof in Gallina can be challenging. Instead, proof engineers typically use Gallina to write only the program and specification, and write the proof interactively. I alluded to this in the introduction as well, when I explained the typical proof development workflow in Coq:

> The proof engineer passes Coq high-level search procedures called *tactics* (like `induction`), and Coq responds to each tactic by refining the current goal to some subgoal (like the proof obligation for the base case). This loop of tactics and goals continues until no goals remain, at which point the proof engineer has constructed a high-level sequence of tactics called a *proof script*. To check the proof,

```
Inductive nat :=
| O : nat
| S : nat → nat.
```

```
nat_rect :
  ∀ (P : nat → Type),
    P O → (* base case *)
    (∀ (n : nat),
      (* inductive case *)
       P n → P (S n)) →
    ∀ (n : nat), P n.
```

Figure 2: The type of natural numbers `nat` in Coq defined inductively by its two constructors (left), and the type of the corresponding eliminator or induction principle `nat_rect` that Coq generates (right).

> the proof assistant compiles it down to a low-level representation called a *proof term*, then checks that the proof term has the expected type.

The low-level language of proof terms in Coq is Gallina—the same rich functional programming language proof engineers use to write programs and specifications. The high-level language of proof scripts in Coq is a language called Ltac that we will soon see.

In this thesis, I will not teach you all of Coq. Good sources for learning more about Coq include the books Certified Programming with Dependent Types [24] and Software Foundations [96], and the survey paper by Ringer and friends 2019 [99]. What I will do is motivate this workflow on an example (Section 2.1.1) and explain the theory beneath (Section 2.1.2).

### 2.1.1 *The Workflow*

Before we can write our small verified program, we need some datatypes and functions that we can find in the Coq standard library (Section 2.1.1.1). We can then use these datatypes and functions to write the `zip` function and show that it preserves its length (Section 2.1.1.2).

#### 2.1.1.1 *Preliminaries*

To prove that the list zip function preserves its length, we need the `list` datatype and the `length` function. To write the `length` function, we need the `nat` datatype of unary natural numbers. All of these can be found in the Coq standard library.

Each of `nat` and `list` in Gallina is an *inductive type*: it is defined by its *constructors* (ways of constructing a term with that type). A `nat` (Figure 2, left), for example, is either `O` or the successor `S` of another `nat`; these are the two constructors of `nat`.

Every inductive type in Gallina comes equipped with an *eliminator* or induction principle that the proof engineer can use to write func-

```
Inductive list {T : Type} :=
| nil : list T
| cons :
    T → list T → list T.
```

```
list_rect :
  ∀ {T : Type} (P : list T → Type
    ),
    P nil → (* base case *)
    (∀ (t : T) (tl : list T),
      (* inductive case *)
      P tl → P (cons t tl)) →
    ∀ (l : list T), P l.
```

Figure 3: The type of polymorphic lists `list` in Coq defined inductively by its two constructors (left), and the type of the corresponding eliminator or induction principle `list_rect` that Coq generates (right). The curly brace notation means that the type parameter `T` is implicit in applications.

```
Fixpoint length {T} l :=
  match l with
  | nil => O
  | cons t tl =>
      S (length tl)
  end.
```

```
Definition length {T} l :=
  list_rect T (fun _ => nat)
    O
    (fun t tl (length_tl : nat) =>
      S length_tl)
    l.
```

Figure 4: The list `length` function, defined both by pattern matching and recursion (left) and using the eliminator `list_rect` (right).

tions and proofs about the datatype. For example, the eliminator for `nat` (Figure 2, right) is the standard induction principle for natural numbers, which Coq calls `nat_rect`. This eliminator states that a statement `P` (called the inductive *motive*) about the natural numbers holds for every number if it holds for `O` in the base case and, in the inductive case, assuming it holds for some `n`, it also holds for the successor `S n`.

A `list` (Figure 3, left) is similar to a `nat`, but with two differences: `list` is polymorphic over some type `T` (so we can have a list of natural numbers, for example, written `list nat`), and the second constructor adds a new element of the type `T` to the front of the list. Otherwise, `list` also has two constructors, `nil` and `cons`, where `nil` represents the empty list, and `cons` sticks a new element in front of any existing list. Similarly, the eliminator for `list` (Figure 3, right) looks like the eliminator for `nat` but with an argument corresponding to the parameter `T` over which `list` is polymorphic, and with an additional argument corresponding to the new element in the inductive case.

One interesting thing about the types of these eliminators `list_rect` and `nat_rect` include universal quantification over all inputs, written ∀. Gallina's type system is expressive enough to include universal quantification over inputs—I will explain how in Section 2.1.2.

We can use these eliminators to write functions and proofs, like the `length` function we will need for our proof development (Figure 4,

```
zip {T₁} {T₂} (l₁ : list T₁) (l₂ : list T₂) : list (T₁ * T₂) :=
  list_rect (fun _ : list T₁ => list T₂ → list (T₁ * T₂))
    (fun _ => nil)
    (fun t₁ tl₁ (zip_tl₁ : list T₂ → list (T₁ * T₂)) l₂ =>
      list_rect (fun _ : list T₂ => list (T₁ * T₂))
        nil
        (fun t₂ tl₂ (_ : list (T₁ * T₂)) =>
          cons (t₁, t₂) (zip_tl₁ tl₂))
        l₂)
    l₁
    l₂.
```

Figure 5: The list `zip` function, taken from an existing tool [**?**] and translated to use eliminators.

right). More standard is to write functions using pattern matching and guarded recursion, like the `length` function from the Coq standard library (Figure 4, left). Both of these two functions behave the same way, but the function on the left is perhaps a bit easier to understand from a traditional programming background: the `length` of the empty list `nil` is `0`, and the length of any other list is just the successor (`S`) of the result of recursively calling `length` on everything but the first element of the list. Indeed, `list_rect`—like all eliminators in Coq—is just a constant that refers to a function itself defined using pattern matching and guarded recursion. In fact, eliminators are equally expressive to pattern matching and guarded recursion [**?**].

For the sake of this thesis, however, I will assume *primitive eliminators*: eliminators that are a part of the core syntax and theory itself, and that do not reduce to terms that use pattern matching and guarded recursion. Likewise, when I show Gallina code, I will always use functions that apply eliminators rather than pattern matching, like the `length` function from Figure 4 on the right. I remove the `Definition` and `Fixpoint` keywords, since everything from here on out is a `Definition`. To handle practical code that uses pattern matching and guarded recursion, I preprocesss the code using a tool by my coauthor Nate Yazdani (more about this later). In the rest of the paper, I skip this preprocessing step in examples, but I describe it more in the implementation section later.

#### 2.1.1.2 *List Zip Preserves Length*

With `nat`, `list`, and `length` defined, we can now write our small verified program. We start by writing the `zip` program, then specify what it means to preserve its length, and then finally write an interactive proof that shows that specification actually holds. Coq checks this proof and lets us now that our proof is correct, so our `zip` function is verified.

```
Theorem zip_preserves_length :
  ∀ {T₁} {T₂} (l₁ : list T₁) (l₂
      : list T₂),
    length l₁ = length l₂ →
    length (zip l₁ l₂) = length
        l₁.
```

```
Theorem zip_preserves_length :
  ∀ {T₁} {T₂} (l₁ : list T₁) (l₂
      : list T₂),
    length (zip l₁ l₂) = min (
        length l₁) (length l₂).
```

Figure 6: Two possible specifications of a proof that `zip` preserves the length of the input lists.

PROGRAM The list `zip` function is in Figure 5. It takes as arguments two lists $l_1$ and $l_2$ of possibly different types $T_1$ and $T_2$, and zips them together into a list of pairs $(T_1 * T_2)$. For example, if the input lists are:

```
(* [1; 2; 3; 4] *)
l₁ := cons 1 (cons 2 (cons 3 (cons 4 nil))).

(* ["x"; "y"; "z"] *)
l₂ := cons "x" (cons "y" (cons "z" nil)).
```

then `zip` applied to those two lists returns:

```
(* [(1, "x"); (2, "y"); (3, "z")] *)
cons (1, "x") (cons (2, "y") (cons (3, "z") nil)).
```

It is worth noting that the implementation of `zip` has to make some decision with what to do with the extra `4` at the end—that is, how `zip` behaves when the input lists are different lengths. The decision that this implementation makes is to just ignore those extra elements.

Otherwise, the implementation is fairly standard. If $l_1$ is `nil` (base case of the outer induction), `zip` returns `nil`. Otherwise (inductive case of the outer induction), if $l_2$ is `nil` (base case of the inner induction), `zip` returns `nil`. If $l_2$ is anything else (inductive case of the inner induction), `zip` combines the first two elements of each list into a pair $((t_1, t_2))$, then sticks that in front of (using `cons`) the result of recursively calling `zip` on the tails of each list ($zip\_tl_1\ tl_2$).

SPECIFICATION Once we have written our `zip` function, we can then specify what we want to prove about it: that the `zip` function preserves the lengths of the inputs $l_1$ and $l_2$. We do this by defining a type `zip_preserves_length` (Figure 6, left), which in Coq we state as a `Theorem`. This theorem takes advantage of Gallina's rich type system to quantify over all possible input lists $l_1$ and $l_2$. It says that if the lengths of the inputs are the same, then the length of the output is the same as the lengths of the inputs. Our proof will soon show that this type is inhabited, and so this statement is true.

It is worth noting that this step of choosing a specification is a bit of an art—we have some freedom when we choose our specification. We could just as well have chosen a different version of `zip_preserves_length` (Figure 6, right) that states that the length of the output is the *minimum* of the lengths of the inputs (using `min` from the Coq standard library). This is also true for our `zip` implementation, and in fact it is stronger—it implies the original theorem as well. But regardless of which version we choose, we then get to the fun part of actually writing our proof.

PROOF As I mentioned earlier, it is possible to write proofs directly in Gallina—but this can be difficult. Instead, it is more common to write proofs interactively using the tactic language Ltac. Each tactic is effectively a search procedure for a proof term, given the context and goals at each step of the proof. The way that this works is, after we state the theorem that we want to prove:

```
Theorem zip_preserves_length :
  ∀ {T₁} {T₂} (l₁ : list T₁) (l₂ : list T₂),
    length l₁ = length l₂ →
    length (zip l₁ l₂) = length l₁.
```

we then add one more word:

```
Proof.
```

then step down past that word inside of an IDE. The IDE then drops into an interactive proof mode. In that proof mode, it tracks the context of the proof so far, along with the goal we want to prove. After each tactic we type and step past, Coq responds by refining the goal into some subgoal and updating the context. We continue this until no goals remain. The survey paper [99] has a good overview of the tactic language in Coq and in other proof assistants, plus different interfaces and IDEs for writing proofs interactively and screenshots of them in action.

Figure 7 shows a proof script for this theorem (left), along with the corresponding proof term (right). As we can see, the proof term is quite complicated. Thankfully, the details do not matter to us, since we can write the high-level proof script on the left instead. Even though this proof script is still a bit manual for the sake of demonstration, it is much simpler than the low-level proof term.

To write the proof script on the left of Figure 7, we start by stepping past `Proof` in our IDE. After this, our initial context (above the line) is empty, and our initial goal (below the line) is the original theorem:

```
______________________________________(1/1)
∀ {T₁} {T₂} (l₁ : list T₁) (l₂ : list T₂),
  length l₁ = length l₂ →
  length (zip l₁ l₂) = length l₁.
```

We start this proof with the introduction tactic `intros`:

```
  intros T₁ T₂ l₁.
```

This is essentially the equivalent of the natural language proof strategy "assume arbitrary $T_1$, $T_2$, and $l_1$." That is, it moves the universally quantified arguments from our goal into our context:

```
Theorem zip_preserves_length :
  ∀ {T₁} {T₂} (l₁ : list T₁) (l₂ : list T₂),
    length l₁ = length l₂ →
    length (zip l₁ l₂) = length l₁.
Proof.
  intros T1 T2 l1.
  induction l₁ as [|t₁ tl₁ IHtl₁].
  - auto.
  - intros l2. induction l2 as [|t₂ tl₂ IHtl₂].
    + intros H. auto.
    + intros H. simpl. rewrite IHtl₁; auto.
Defined.
```

```
zip_preserves_length :
  ∀ {T₁} {T₂} (l₁ : list T₁) (l₂ : list T₂),
    length l₁ = length l₂ →
    length (zip l₁ l₂) = length l₁
:=
fun (T₁ T₂ : Type) (l₁ : list T₁) (l₂ : list T₂) =>
  list_rect
    (fun (l₁ : list T₁) =>
      ∀ (l₂ : list T₂),
        length l₁ = length l₂ →
        length (zip l₁ l₂) = length l₁)
    (fun (l₂ : list T₂) _ => eq_refl)
    (fun (t₁ : T₁) (tl₁ : list T₁)
         (IHtl₁ :
           ∀ (l₂ : list T2),
             length tl₁ = length l₂ →
             length (zip tl₁ l₂) = length tl₁)
         (l₂ : list T₂) =>
      list_rect
        (fun (l₂ : list T₂) =>
          length (cons t₁ tl₁) = length l₂ →
          length (zip (cons t₁ tl₁) l₂) = length (cons t₁ tl₁))
        (fun (H : length T₁ (cons t₁ tl₁) = length nil) => eq_sym
             H)
        (fun (t₂ : T₂) (tl₂ : list T₂) _ (H : length (cons t₁ tl₁)
             = length (cons t₂ tl₂)) =>
          eq_ind_r
            (fun (n : nat) => S n = S (length tl₁))
            eq_refl
            (IHtl₁ tl₂ (eq_add_S (length tl₁) (length tl₂) H)))
        l2)
  l1
  l2.
```

Figure 7: A proof script (left) and corresponding proof term (right) in Coq that shows that the list zip function preserves its length.

```
T₁ : Type
T₂ : Type
l₁ : list T₁
______________________________________(1/1)
∀ (l₂ : list T₂),
  length l₁ = length l₂ →
  length (zip l₁ l₂) = length l₁.
```

From this state, we can induct over the input list (choosing names for variables Coq introduces in the inductive case):

```
  induction l₁ as [|t₁ tl₁ IHtl₁].
```

This breaks into two subgoals and subcontexts: one for the base case and one for the inductive case. The base case:

```
T₁ : Type
T₂ : Type
______________________________________(1/2)
∀ l₂ : list T₂,
  length nil = length l₃ →
  length (zip nil l₂) = length nil.
```

holds by reflexivity, which the `auto` tactic takes care of.

In the inductive case:

```
T₁ : Type
T₂ : Type
t₁ : T₁
tl₁ : list T₁
IHtl₁ : ∀ l₂ : list T₂,
        length tl₁ = length l₂ →
        length (zip tl₁ l₂) = length tl₁
______________________________________(2/2)
∀ l₂ : list T2,
  length (cons t₁ tl₁) = length l₂ →
  length (zip (cons t₁ tl₁) l₂) = length (cons t₁ tl₁).
```

we again use `intros` and `induction`, this time to induct over $\texttt{l}_2$. This again produces two subgoals: one for the base case and one for the inductive case. The base case has an absurd hypothesis, which we introduce as `H` and then use `auto` to show our conclusion holds. The inductive case holds by simplification and rewriting by the inductive hypothesis $\texttt{IHtl}_1$.

After this, no goals remain, so our proof is done; we can write `Defined`. What happens when we write `Defined` is that Coq produces the proof term on the right of Figure 7. It then checks the type of that term and ensures that it is exactly the theorem we have stated. Since it is, Coq lets us know that our proof is correct, so our `zip` function is verified. Thankfully, though, we never have to write the low-level proof term ourselves; we see proofs as these high-level proof csripts.

Some correspondence between the proof script and proof term may already be clear. For example, every call to `induction` in the proof script shows up as an application of the eliminator `list_rect` in the proof term. In Section **??**, I will explain this connection in more detail by introducing a prototype decompiler from proof terms back up to

$\langle i \rangle \in \mathbb{N},\ \langle v \rangle \in \text{Vars},\ \langle s \rangle \in \{\ \text{Prop, Set, Type}\langle i \rangle\ \}$

$\langle t \rangle ::= \langle v \rangle \ \mid\ \langle s \rangle \ \mid\ \Pi\,(\langle v \rangle : \langle t \rangle)\,.\,\langle t \rangle \ \mid\ \lambda\,(\langle v \rangle : \langle t \rangle)\,.\,\langle t \rangle \ \mid\ \langle t \rangle\ \langle t \rangle$

Figure 8: Syntax for $\text{CoC}_\omega$ with (from left to right) variables, sorts, dependent types, functions, and application.

$\langle t \rangle ::= \ldots \ \mid\ \text{Ind}\ (\langle v \rangle : \langle t \rangle)\{\langle t \rangle, \ldots, \langle t \rangle\} \ \mid\ \text{Constr}\ (\langle i \rangle, \langle t \rangle) \ \mid\ \text{Elim}(\langle t \rangle, \langle t \rangle)\{\langle t \rangle, \ldots, \langle t \rangle\}$

Figure 9: $\text{CIC}_\omega$ is $\text{CoC}_\omega$ with inductive types, inductive constructors, and primitive eliminators.

proof scripts. This decompiler makes it possible for Pumpkin to work over highly structured Gallina terms, but produce proof scripts that the proof engineer can use going forward.

These tactics are a form of *proof automation*, and they do indeed make proof development easier than writing raw proof terms. Unfortunately, this flavor of proof automation a bit naive when it comes to *maintaining* proofs in the face of change. Proof repair is a new form of proof automation that understands how proofs evolve over time. But it is exactly the rich structure of the type theory beneath Gallina that makes a principled approach to proof repair possible.

### 2.1.2 *The Theory Beneath*

The type theory that Gallina implements is $\text{CIC}_\omega$, or the Calculus of Inductive Constructions. $\text{CIC}_\omega$ is based on the Calculus of Constructions (CoC), a variant of the lambda calculus with polymorphism (types that dependent on types) and dependent types (types that depend on terms) [31]. CoC with an infinite universe hierarchy is called $\text{CoC}_\omega$. The syntax for $\text{CoC}_\omega$ is in Figure 8. Note that whereas in Gallina we represent universal quantification over terms or types with $\forall$, here we represent it with $\Pi$, as is standard.

The syntax for $\text{CIC}_\omega$ is in Figure 9), building on syntax from an existing paper [107]; the type theory is standard and omitted. $\text{CIC}_\omega$ extends $\text{CoC}_\omega$ with inductive types [32]. As in Gallina, inductive types are defined by their constructors and eliminators. Consider the inductive type `nat` of unary natural numbers that we saw in Figure 2, this time in $\text{CIC}_\omega$:

```
Ind (nat : Set) {
  nat,
  nat → nat
}
```

where the `O` constructor type is the zeroth element in the list, and the `S` constructor type is the first element. Accordingly, the terms:

```
Constr (0, nat)
```

and:

```
Constr (1, nat)
```

refer to the constructors `O` and `S`, respectively.

As in Gallina, `nat` comes associated with an eliminator. Unlike in Gallina, here we truly assume primitive eliminators—that these eliminators do not reduce at all. Instead, we represent them explicitly with the `Elim` construct. Thus, to eliminate over a natural number `n` with motive `P`, we write:

```
Elim (n, P) {
  f_O,
  f_S
}
```

where functions $f_O$ and $f_S$ prove the base and inductive cases, respectively. When `n`, `P`, $f_O$, and $f_S$ are arbitrary, this statement has the same type as `nat_rect` in Gallina.

Gallina implements $\text{CIC}_\omega$, but with a few important differences. More information is on the website, but two differences are relevant to repair: The first is that Gallina lacks primitive eliminators, as we mentioned earlier. The second notable difference is that Gallina has constants that define terms—later on, this will help with building optimizations for repair tools.

Otherwise, a proof repair tool for Gallina can harness the power of $\text{CIC}_\omega$. This type theory is fairly simple, but $\Pi$ makes it possible to quantify over both terms and types, so that we can state powerful theorems and prove that they old. Inductive types make it possible to write proofs by induction. Both of these constructs mean that terms in Gallina are extremely structured, and as we will soon see, that structure makes a proof repair tool's job much easier. For now, though, let us return to our proof development and demonsrate why we need a repair tool to begin with.

## 2.2 PROOF MAINTENANCE

Problem is when something changes—change something in running example

There are a lot of development processes people use to make proofs less likely to break to begin with (survey paper)

But still, even with these, the reality: This happens all the time (REPLICA)

And in fact not just after developing a proof, but during development too (REPLICA)

And breaks proofs even for experts (REPLICA)

And it's an extra big problem when you have a large development and the changes are outside of your control

Hence Social Processes

Why automation breaks, even with good development processes

Hence proof repair—smarter automation

## 2.3 PROOF REPAIR

Name inspired by program repair, but quite different as we'll soon see.

Recall thesis: Changes in programs, specifications, and proofs carry information that a tool can extract, generalize, and apply to fix other proofs broken by the same change. A tool that automates this can save work for proof engineers relative to reference manual repairs in practical use cases.

Proof repair accomplishes this using a combination of differencing and program transformations.

Differencing extracts the information from the change in program, specification, or proof.

The transformations then generalize that information to a more general fix for other proofs broken by the same change.

The details of applying the fix vary by the kind of fix, as we'll soon see.

Crucially, all of this happens over the proof terms in this rich language we saw in the Development section. This is kind of the key insight that makes it all work.

This is great because this language gives us so much information and certainty. This helps us with two of the biggest challenges from program repair. (generals related work)

But it's also challenging because this language is so unforgiving. Plus, in the end, we need these tactic proofs, not just proof terms. So we can't just reuse program repair tools. (generals related work)

So next two chapters will show two tools in our tool suite that work this way, how they handle these challenges, and how they save work.

# 3

# PROOF REPAIR BY EXAMPLE

The first tool (PUMPKIN PATCH) focuses on changes in programs and specifications, though these changes are limited in scope as we'll see later.

What this tool does is, when programs and specifications change and this breaks a lot of proofs, it lets the proof engineer fix just one of those proofs. It then generalizes the example patch into something that can fix other proofs broken by the same change.

So in other words, the information from those changes is carried in the difference between the old and new version of the example patched proof. PUMPKIN PATCH generalizes that information.

Application can be automated in some cases at the end, or it can be manual.

The work saved is shown retroactively on case studies replaying changes from large proof devleopments in Git. Results for this tool are preliminary compared to what we'll see later, since this was the first prototype.

## 3.1 MOTIVATING EXAMPLE

Traditional proof automation considers only the current state of theorems, proofs, and definitions. This is a missed opportunity: verification projects are rarely static. Like other software, these projects evolve over time.

With traditional proof automation, the burden of change largely falls on proof engineers. This does not have to be true. Proof automation can view theorems, proofs, and definitions as fluid entities: when a proof or specification changes, a tool can search the difference between the old and new versions for a *reusable patch* that can fix broken proofs.

WITHOUT PROOF REPAIR Experienced Coq programmers use design principles and custom tactics to make proofs resilient to change. These techniques are useful for large proof developments, but they place the burden of change on the programmer. This can be problematic when change occurs outside of the programmer's control.

```
Definition IZR (z:Z) : R :=
  match z with
  | Z0 => 0
  | Zpos n => INR (Pos.to_nat n)
  | Zneg n => - INR (Pos.to_nat n)
  end.
```

```
Definition IZR (z:Z) : R :=
  match z with
  | Z0 => 0
  | Zpos n => IPR n
  | Zneg n => - IPR n
  end.
```

Figure 10: Old (left) and new (right) definitions of IZR in Coq. The old definition applies injection from naturals to reals and conversion of positives to naturals; the new definition applies injection from positives to reals.

Consider a commit from the Coq 8.7 release [83]. This commit redefined injection from integers to reals (Figure 10). This change broke 18 proofs in the standard library.

The Coq developer who committed the change fixed the broken proofs, then made an additional 12 commits to address the change in `coq-contribs`, a regression suite of projects that the Coq developers maintain as versions change. Many of these changes were simple. For example, the developer wrote a lemma that describes the change:

```
Lemma INR_IPR : ∀p, INR (Pos.to_nat p) = IPR p.
```

The developer then used this lemma to fix broken proofs within the standard library. For example, one proof broke on this line:

```
rewrite Pos2Nat.inj_sub by trivial.✗
```

It succeeded with the lemma:

```
rewrite <- 3!INR_IPR, Pos2Nat.inj_sub by trivial.✓
```

These changes are outside-facing: Coq users have to make similar changes to their own proofs when they update from Coq 8.6 to Coq 8.7. The Coq developer can update some tactics to account for this, but it is impossible to account for every tactic that users could use. Furthermore, while the developer responsible for the changes knows about the lemma that describes the change, the Coq user does not. The Coq user must determine how the definition has changed and how to address the change, perhaps by reading documentation or by talking to the developers.

WITH PROOF REPAIR When a user updates the Coq standard library, a proof repair tool can determine that the definition has changed, then analyze changes in the standard library and in `coq-contribs` that resulted from the change in definition (in this case, rewriting by the lemma). It can extract a reusable patch from those changes, which it can automatically apply within broken user proofs. The user never has to consider how the definition has changed.

## 3.2 APPROACH

In the example from Section 3.1, we can see how the example change in one proof carries enough information to fix other proofs broken by the same change (namely the rewrite by `INR\_IPR`). So a tool can extract that, generalize it, and use it to fix other proofs broken by the same change.

The key insight behind PUMPKIN's approach is that this is true more generally. To use PUMPKIN, the programmer modifies a single proof script to provide an *example* of how to adapt a proof to a change. PUMPKIN extracts that information into a *patch candidate*—which is localized to the context of the example, but not enough to fix other proofs broken by the change. It then generalizes that candidate into a *reusable patch*: a function that can be used to fix other broken proofs broken by the same change, which PUMPKIN defines as a Coq term.

In other words, looking back to the thesis statement, the information shows up in the difference between versions of the example patched proof. PUMPKIN can extract and generalize that information. Application works with hint databases or is manual. Here is the system diagram for PUMPKIN. The PUMPKIN repository contains a detailed user guide.

As mentioned earlier, PUMPKIN does this using a combination of semantic differencing and program transformations. Differencing looks at the difference between versions of the example patched proof for this information, and finds the candidate. Then, program transformations modify that candidate to produce the reusable proof patch.

And of course all of this happens over proof terms, since tactics might hide necessary in information. Of course this is hard to see on the example from Section 3.1, since we were lucky enough to see the difference in tactics here. Let's look at a toy example for which that isn't true.

To motivate this workflow, consider using PUMPKIN to search the proofs in Figure 11 for a patch between conclusions. Except we will show a place where the lemma is actually applied. Note that the tactics don't change even though the terms do—and even though the change could break other proofs.

So what do we do? We invoke the plugin using `old` and `new` as the example change:

```
Patch Proof old new as patch.
```

PUMPKIN first determines the type that a patch from `new` to `old` should have. To determine this, it semantically *diffs* the types and finds this goal type (line 2):

```
∀ n m p, n <= m -> m <= p -> n <= p -> n <= p + 1
```

It then breaks each inductive proof into cases and determines an intermediate goal type for the candidate. In the base case, for example,

```
Theorem old: ∀ (n m p : nat),
  n <= m -> m <= p ->
  n <= p + 1.                    (* P p *)
Proof.
  intros. induction H0.
  - auto with arith.
  - constructor. auto.
Qed.

fun (n m p : nat) (H : n <= m) (H0 : m <= p) =>
  le_ind
    m                                    (* m *)
    (fun p0 => n <= p0 + 1)     (* P *)
    (le_plus_trans n m 1 H)     (* : P m *)
    (fun (m0 : nat) (_ : m <= m0) (IHle : n <= m0 + 1) =>
      le_S n (m0 + 1) IHle)
    p                                    (* p *)
    H0
```

```
Theorem new: ∀ (n m p : nat), n <= m -> m <= p ->
  n <= p.
  (* P' p *)
Proof.
  intros. induction H0.
  - auto with arith.
  - constructor. auto.
Qed.

fun (n m p : nat) (H : n <= m) (H0 : m <= p) =>
  le_ind
    m
  (* m *)
    (fun p0 => n <= p0)
  (* P' *)
    H
  (* : P' m *)
    (fun (m0 : nat) (_ : m <= m0) (IHle : n <= m0) =>
      le_S n m0 IHle)
    p
  (* p *)
    H0
```

Figure 11: Two proofs with different conclusions (top) and the corresponding proof terms (bottom) with relevant type information. We highlight the change in theorem conclusion and the difference in terms that corresponds to a patch.

it *diffs* the types and determines that a candidate between the base cases of `new` and `old` should have this type (lines 11 and 12):

```
(fun p0 => n <= p0) m -> (fun p0 => n <= p0 + 1) m
```

It then *diffs* the terms (line 13) for such a candidate:

```
fun n m p H0 H1 =>
  (fun (H : n <= m) => le_plus_trans n m 1 H)
: ∀ n m p, n <= m -> m <= p -> n <= m -> n <= m + 1
```

This candidate is close, but it is not yet a patch. This candidate maps base case to base case (it is applied to `m`); the patch should map conclusion to conclusion (it should be applied to `p`).

This is where the transformations come in. There are four:

1. *Patch specialization* to arguments
2. *Patch abstraction* of arguments or functions
3. *Patch inversion* to reverse a patch
4. *Lemma factoring* to break a term into parts

Here, Pumpkin *abstracts* this candidate by `m` (line 11), which lifts it out of the base case:

```
fun n0 n m p H0 H1 =>
  (fun (H : n <= n0) => le_plus_trans n n0 1 H)
: ∀ n0 n m p, n <= m -> m <= p -> n <= n0 -> n <= n0 + 1
```

Pumpkin then *specializes* this candidate to `p` (line 16), the argument to the conclusion of `le_ind`. This produces a patch:

```
patch n m p H0 H1 :=
  (fun (H : n <= p) => le_plus_trans n p 1 H)
: ∀ n m p, n <= m -> m <= p -> n <= p -> n <= p + 1
```

The user can then use `patch` to fix other broken proofs. For example, given a proof that applies `old`, the user can use `patch` to prove the same conclusion by applying `new`:

```
apply old.✓
apply patch. apply new.✓
```

This can happen automatically through hint databases.

This simple example uses only two transformations. The other transformations help turn candidates into patches in similar ways. We discuss all of this in detail later.

**CONFIGURATION** The components come together to form a proof patch finding procedure:

**Pseudocode:** find_patch(term, term′, direction)

1: *diff* types of term and term′ for goals
2: *diff* term and term′ for candidates
3: **if** there are candidates **then**
4:     *factor*, *abstract*, *specialize*, and/or *invert* candidates
5:     **if** there are patches **then return** patches
6: **return** failure

PUMPKIN infers a *configuration* from the example change. This configuration customizes the highlighted lines for an entire class of changes: It determines what to diff on lines 1 and 2, and how to use the components on line 4.

For example, to find a patch for Figure 11, PUMPKIN used the configuration for changes in conclusions of two proofs that induct over the same hypothesis. Given two such proofs:

```
∀ x, H x -> P x
∀ x, H x -> P' x
```

PUMPKIN searches for a patch with this type:

```
∀ x, H x -> P' x -> P x
```

using this configuration:

```
1: diff conclusion types for goals
2: diff conclusion terms for candidates
3: if there are candidates then
4:     abstract and then specialize candidates
```

Later we will see real-world examples that demonstrate more configurations.

## 3.3 DIFFERENCING

The tool should be able to identify the semantic difference between terms. The semantic difference is the difference between two terms that corresponds to the difference between their types. Consider the base case terms in Figure 11 (line 13):

```
le_plus_trans n m 1 H : n <= m + 1
                    H : n <= m
```

The semantic differencing component first identifies the difference in their types, or the *goal type*:

```
n <= m -> n <= m + 1
```

It then finds a difference in terms that has that type:

```
fun (H : n <= m) => le_plus_trans n m 1 H
```

This is the *candidate* for a reusable patch that the other components modify to find a patch.

Differencing operates over terms and types. Differencing tactics is insufficient, since tactics and hints may mask patches (line 5).[1] Furthermore, differencing is aware of the semantics of terms and types. Simply exploring the syntactic difference makes it hard to identify which changes are meaningful. For example, in the inductive case (line 14), the inductive hypothesis changes:

```
... (IHle : n <= m0 + 1) ...
... (IHle : n <= m0) ...
```

1 Since this is a simple example, replaying an existing tactic happens to work. There are additional examples in the repository (`Cex.v`).

However, the type of `IHle` changes for *any* two inductive proofs over `le` with different conclusions. A syntactic differencing component may identify this change as a candidate. Our semantic differencing component knows that it can ignore this change.

Plus parts of Inside the Core, Testing Boundaries, Future Work

How differencing works in detail

Limitations and whether they're addressed in later tools yet or not

## 3.4 TRANSFORMATION

**PATCH SPECIALIZATION** The tool should be able to specialize a patch candidate to specific arguments as determined by the differences in terms. To find a patch for Figure 11, for example, PUMPKIN must specialize the patch candidate to `p` to produce the final patch.

**PATCH ABSTRACTION** A tool should be able to abstract patch candidates of this form by the common argument:

```
candidate : P' t -> P t
candidate_abs : ∀ t0, P' t0 -> P t0
```

and it should be able to abstract patch candidates of this form by the common function:

```
candidate : P t' -> P t
candidate_abs : ∀ P0, P0 t' -> P0 t
```

This is necessary because the tool may find candidates in an applied form. For example, when searching for a patch between the proofs in Figure 11, PUMPKIN finds a candidate in the difference of base cases. To produce a patch, PUMPKIN must abstract the candidate by the argument `m`. Abstracting candidates is not always possible; abstraction will necessarily be a collection of heuristics.

**PATCH INVERSION** The tool should be able to invert a patch candidate. This is necessary to search for isomorphisms. It is also necessary to search for implications between propositionally equal types, since candidates may appear in the wrong direction. For example, consider two list lemmas (we write `length` as `len`):

```
old : ∀ l' l, len (l' ++ l) = len l' + len l
new : ∀ l' l, len (l' ++ l) = len l' + len (rev l)
```

If PUMPKIN searches the difference in proofs of these lemmas for a patch from the conclusion of `new` to the conclusion of `old`, it may find a candidate *backwards*:

```
candidate l' l (H : old l' l) :=
  eq_ind_r ... (rev_length l)
: ∀ l' l, old l' l -> new l' l
```

The component can invert this to get the patch:

```
patch l' l (H : new l' l) :=
  eq_ind_r ... (eq_sym (rev_length l))
: ∀ l' l, new l' l -> old l' l
```

We can then use this patch to port proofs. For example, if we add this patch to a hint database [1], we can port this proof:

```
Theorem app_rev_len : ∀ l l',
  len (rev (l' ++ l)) = len (rev l) + len (rev l').
Proof.
  intros. rewrite rev_app_distr. apply old.✓
Qed.
```

to this proof:

```
Theorem app_rev_len : ∀ l l',
  len (rev (l' ++ l)) = len (rev l) + len (rev l').
Proof.
  intros. rewrite rev_app_distr. apply new.✓
Qed.
```

Rewrites like `candidate` are *invertible*: We can invert any rewrite in one direction by rewriting in the opposite direction. In contrast, it is not possible to invert the patch PUMPKIN found for Figure 11. Inversion will necessarily sometimes fail, since not all terms are invertible.

LEMMA FACTORING The tool should be able to factor a term into a sequence of lemmas. This can help break other problems, like abstraction, into smaller subproblems. It is also necessary to invert certain terms. Consider inverting an arbitrary sequence of two rewrites:

```
t := eq_ind_r G ... (eq_ind_r F ...)
```

We can view `t` as a term that composes two functions:

```
g := eq_ind_r G ...
f := eq_ind_r F ...
t := g ∘ f
```

The inverse of `t` is the following:

$$t^{-1} := f^{-1} \circ g^{-1}$$

To invert `t`, PUMPKIN identifies the factors `[f; g]`, inverts each factor to $[f^{-1}; g^{-1}]$, then folds and applies the inverse factors in the opposite direction.

plus parts of PUMPKIN PATCH Inside the Core, Testing Boundaries, Future Work

How the four transformations work in detail

Limitations and whether they're addressed in later tools yet or not

## 3.5 IMPLEMENTATION

parts of PUMPKIN PATCH Inside the Core, plus more

### 3.5.1 *Tool Details*

While our system is a very early prototype under active development, we have made the source code available on Github.[2] The interested

2 http://github.com/uwplse/PUMPKIN-PATCH/tree/cpp18

reader can follow along in the repository. Our prototype has no impact on the trusted computing base (Section 3.5.2.1).

#### 3.5.1.1 *Semantic Differencing*

We implement semantic differencing over *trees*: PUMPKIN compiles each proof term into a tree (`evaluation.ml`). In these trees, every node is a type context, and every edge is an extension to that type context with a new term.[3] Correspondingly, type differencing (to identify goal types) compares nodes, and term differencing (to find candidates) compares edges.

The component (`differencing.ml`) uses these nodes and edges to prioritize semantically relevant differences. At the lowest level, it calls a primitive differencing function which checks if it can substitute one term within another term to find a function between their types.

The key benefit to this model is that it gives us a natural way to express inductive proofs, so that differencing can efficiently identify good candidates. Consider, for example, searching for a patch between conclusions of two inductive proofs of theorems about the natural numbers:

```
nat_ind P ... (fun (IH : P n) => ...) : ∀ n, P n
nat_ind P' ... (fun (IH : P' n) => ...) : ∀ n, P' n
```

In each case, the component diffs the terms in the dotted edges of the tree for `nat_ind` (Figure 12) to try to find a term that maps between conclusions of that case:

```
P' 0 -> P 0         (* base case candidate *)
P' (S n) -> P (S n) (* inductive case candidate *)
```

The component also knows that the change in the type of `IH` is inconsequential (it occurs for any change in conclusion). Furthermore, it knows that `IH` cannot show up as a hypothesis in the patch, so it attempts to remove any occurrences of `IH` in any candidate.

When the component finds a candidate, it knows `P'` and `P` as well as the arguments `0` or `(S n)`. This makes it simple to query abstraction for the final patch:

```
∀ n, P' n -> P n
```

The differencing component is *lazy*: it compiles terms into trees one step at a time. It then *expands* each tree as needed to find candidates (`expansion.ml`). For example, consider searching two functions for a patch between conclusions:

```
fun (t : T) => b
fun (t' : T) => b'
```

Differencing introduces a single term of type `T` to a common environment, then expands and recursively diffs the bodies `b` and `b'` in that environment.

3 These trees are inspired by categorical models of dependent type theory [54].

```
∀ P,
 P 0 ->
 (∀ n, P n -> P (S n)) ->
 ∀ n, P n
```

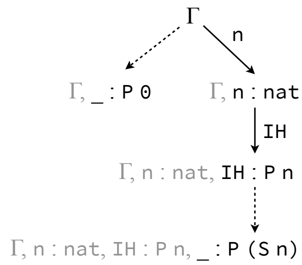


Figure 12: The type of (left) and tree for (right) the induction principle `nat_ind`. The solid edges represent hypotheses, and the dotted edges represent the proof obligations for each case in an inductive proof.

The tool always maintains pointers to easily switch between the tree and AST representations of the terms. This representation enables extensibility.

#### 3.5.1.2 *Transformations*

PATCH SPECIALIZATION Specialization (`specialize.ml`) takes a patch candidate and some arguments, all of which are Coq terms. It applies the candidate to the arguments, then it $\beta\iota$-reduces [26] the result using Coq's `Reduction.nf_betaiota` function. It is the job of the patch finding procedure to provide both the candidate and the arguments.

PATCH ABSTRACTION Abstraction (`abstraction.ml`) takes a patch candidate, the goal type, and the function arguments or function to abstract. It first generalizes the candidate, wrapping it inside of a lambda from the type of the term to abstract. Then, it substitutes terms inside the body with the abstract term. It continues to do this until there is nothing left to abstract, then filters results by the goal type. Consider, for example, abstracting this candidate by `m`:

```
fun (H : n <= m) => le_plus_trans n m 1 H
: n <= m -> n <= m + 1
```

The generalization step wraps this in a lambda from some `nat`, the type of `m`:

```
fun (n0 : nat) =>
  (fun (H : n <= m) => le_plus_trans n m 1 H)
: ∀ n0, n <= m -> n <= m + 1
```

The substitution step replaces `m` with `n0`:

```
fun (n0 : nat) =>
  (fun (H : n <= n0) => le_plus_trans n n0 1 H)
: ∀ n0, n <= n0 -> n <= n0 + 1
```

Abstraction uses a list of *abstraction strategies* to determine what subterms to substitute. In this case, the simplest strategy works: The tool replaces all terms that are convertible to the concrete argument m with the abstract argument n0, which produces a single candidate. Type-checking this candidate confirms that it is a patch.

In some cases, the simplest strategy is not sufficient, even when it is possible to abstract the term. It may be possible to produce a patch only by abstracting *some* of the subterms convertible to the argument or function (we show an example of this in Section **??**), or the term may not contain any subterms convertible to the argument or function at all. We implement several strategies to account for this. The combinations strategy, for example, tries all combinations of substituting only some of the convertible subterms with the abstract argument. The pattern-based strategy substitutes subterms that match a certain pattern with a term that corresponds to that pattern.

It is the job of the patch finding procedure to provide the candidate and the terms to abstract. In addition, each configuration includes a list of strategies. The configuration for changes in conclusions, for example, starts with the simplest strategy, and moves on to more complex strategies only if that strategy fails. This design makes abstraction simple to extend with new strategies and simple to call with different strategies for different classes of changes.

PATCH INVERSION Patch inversion (inverting.ml) exploits symmetry to try to reverse the conclusions of a candidate patch. It first factors the candidate using the factoring component, then calls the primitive inversion function on each factor, then finally folds the resulting list in reverse. The primitive inversion function exploits symmetry. For example, equality is symmetric, so the component can invert any application of eq_ind or eq_ind_r (any rewrite). Indeed, eq_ind and eq_ind_r are inverses, and are related by symmetry:

```
eq_ind_r A x P (H : P x) y (H0 : y = x) :=
  eq_ind x (fun y0 : A => P y0) H y (eq_sym H0)
```

If inversion does not recognize that the type is symmetric, it swaps subterms and type-checks the result to see if it is an inverse.

LEMMA FACTORING The lemma factoring component (factoring .ml) searches within a term for its factors. For example, if the term composes two functions, it returns both factors:

```
 t : X -> Z              (* term *)
[f : X -> Y; g : Y -> Z] (* factors *)
```

In this case, the component takes the composite term and X as arguments. It first searches as deep as possible for a term of type X -> Y for some Y. If it finds such a term, then it recursively searches for a term with type Y -> Z. It maintains all possible paths of factors along the way, and it discards any paths that cannot reach Z.

The current implementation can handle paths with more than two factors, but it fails when Y depends on X. Other components may benefit from dependent factoring; we leave this to future work.

#### 3.5.1.3 *Inside the Procedure*

The implementation (`patcher.ml4`) of the procedure from Section **??** starts with a preprocessing step which compiles the proof terms to trees (like the tree in Figure 12). It then searches for candidates one step at a time, expanding the trees when necessary.

The Pumpkin prototype exposes the patch finding procedure to users through the Coq command `Patch Proof`. Pumpkin automatically infers which configuration to use for the procedure from the example change. For example, to find a patch for the case study in Section 3.6.1, we used this command:

```
Patch Proof Old.unsigned_range unsigned_range as patch.
```

Pumpkin analyzed both versions of `unsigned_range` and determined that a constructor of the `int` type changed (Figure 13), so it initialized the configuration for changes in constructors.

Internally, Pumpkin represents configurations as sets of options, which it passes to the procedure. The procedure uses these options to determine how to compose components (for example, whether to abstract candidates) and how to customize components (for example, whether semantic differencing should look for an intermediate lemma). To implement new configurations for different classes of changes, we simply tweak the options.

### 3.5.2 *Workflow Integration*

Needed: hints and so on, any work done since, the Git interface, whatever.

#### 3.5.2.1 *Trusted Computing Base*

A common concern for Coq plugins is an increase in the trusted computing base. The Coq developers provide a safe plugin API in Coq 8.7 to address this [43]. Our prototype takes this into consideration: While Pumpkin does not yet support Coq 8.7, it only calls the internal Coq functions that the developers plan to expose in the safe API [69]. Furthermore, Coq type-checks terms that plugins produce. Since Pumpkin does not modify the type checker, it cannot produce an ill-typed term.

## 3.6 RESULTS

Needed: key technical results

We used the Pumpkin prototype to emulate three motivating scenarios from real-world code:

1. **Updating definitions** within a project (CompCert, Section 3.6.1)

```
Record int : Type :=
  mkint { intval: Z; intrange
      : 0 <= intval < modulus
       }.
```

```
Record int : Type :=
  mkint { intval: Z; intrange
      : -1 < intval < modulus
       }.
```

Figure 13: Old (left) and new (right) definitions of `int` in CompCert.

2. **Porting definitions** between libraries
   (Software Foundations, Section 3.6.2)

3. **Updating proof assistant versions**
   (Coq Standard Library, Section 3.6.3)

The code we chose for these scenarios demonstrated different classes of changes. For each case, we describe how PUMPKIN configures the procedure to use the core components for that class of changes. Our experiences with these scenarios suggest that patches are useful and that the components are effective and flexible.

IDENTIFYING CHANGES We identified Git commits from popular Coq projects that demonstrated each scenario. These commits updated proofs in response to breaking changes. We emulated each scenario as follows:

1. *Replay* an example proof update for PUMPKIN

2. *Search* the example for a patch using PUMPKIN

3. *Apply* the patch to fix a different broken proof

Our goal was to simulate incremental use of a patch finding tool, at the level of a small change or a commit that follows best practices. We favored commits with changes that we could isolate. When isolating examples for PUMPKIN, we replayed changes from the bottom up, as if we were making the changes ourselves. This means that we did not always make the same change as the user. For example, the real change from Section 3.6.1 updated multiple definitions; we updated only one.

PUMPKIN is a proof-of-concept and does not yet handle some kinds of proofs. In each scenario, we made minor modifications to proofs so that we could use PUMPKIN (for example, using induction instead of destruction). PUMPKIN does not yet handle structural changes like adding constructors or parameters, so we focused on changes that preserve structure, like modifying constructors. Chapter 4 describes an extension to PUMPKIN that supports changes in structure.

### 3.6.1 *Updating Definitions*

Coq programmers sometimes make changes to definitions that break proofs within the same project. To emulate this use case, we identified

```
Fixpoint bin_to_nat (b : bin) :
    nat :=
  match b with
  | BO => O
  | B2 b' => 2 * (bin_to_nat b
     ')
  | B21 b' => 1 + 2 * (
     bin_to_nat b')
  end.
```

```
Fixpoint bin_to_nat (b : bin) :
    nat :=
  match b with
  | BO => O
  | B2 b' => (bin_to_nat b') +
      (bin_to_nat b')
  | B21 b' => S ((bin_to_nat b
     ') + (bin_to_nat b'))
  end.
```

Figure 14: Definitions of `bin_to_nat` for Users A (left) and B (right).

a CompCert commit [74] with a breaking change to `int` (Figure 13). We used PUMPKIN to find a patch that corresponds to the change in `int`. The patch PUMPKIN found fixed broken inductive proofs.

REPLAY We used the proof of `unsigned_range` as the example for PUMPKIN. The proof failed with the new `int`:

```
Theorem unsigned_range:
  ∀(i : int), 0 <= unsigned i < modulus.
Proof.
  intros i. induction i using int_ind; auto.✗
```

We replayed the change to `unsigned_range`:

```
  intros i. induction i using int_ind. simpl.  omega.✓
```

SEARCH We used PUMPKIN to search the example for a patch that corresponds to the change in `int`. It found a patch with this type:

```
∀ z : Z, -1 < z < modulus -> 0 <= z < modulus
```

APPLY After changing the definition of `int`, the proof of the theorem `repr_unsigned` failed on the last tactic:

```
Theorem repr_unsigned:
  ∀(i : int), repr (unsigned i) = i.
Proof.
  ... apply Zmod_small; auto.✗
```

Manually trying `omega`—the tactic which helped us in the proof of `unsigned_range`—did not succeed. We added the patch that PUMPKIN found to a hint database. The proof of the theorem `repr_unsigned` then went through:

```
... apply Zmod_small; auto.✓
```

#### 3.6.1.1 *Configuration*

This scenario used the configuration for changes in constructors of an inductive type. Given such a change:

```
Inductive T := ... | C : ... -> H -> T
Inductive T' := ... | C : ... -> H' -> T'
```

PUMPKIN searches two inductive proofs of theorems:

```
∀ (t : T), P t
∀ (t : T'), P t
```

for an isomorphism[4] between the constructors:

```
... -> H -> H'
... -> H' -> H
```

The user can apply these patches within the inductive case that corresponds to the constructor `C` to fix other broken proofs that induct over the changed type. PUMPKIN uses this configuration for changes in constructors:

1: *diff* inductive constructors for goals
2: use *all components* to recursively search for changes in conclusions of the corresponding case of the proof
3: **if** there are candidates **then**
4: try to *invert* the patch to find an isomorphism

### 3.6.2 *Porting Definitions*

Coq programmers sometimes port theorems and proofs to use definitions from different libraries. To simulate this, we used PUMPKIN to port two solutions [3, 9] to an exercise in Software Foundations to each use the other solution's definition of the fixpoint `bin_to_nat` (Figure 14). We demonstrate one direction; the opposite was similar.

REPLAY We used the proof of `bin_to_nat_pres_incr` from User A as the example for PUMPKIN. User A cut an inline lemma in an inductive case and proved it using a rewrite:

```
assert (∀a, S (a + S (a + 0)) = S (S (a + (a + 0)))).
- ... rewrite <- plus_n_O. rewrite -> plus_comm.
```

When we ported User A's solution to use User B's definition of `bin_to_nat`, the application of this inline lemma failed. We changed the conclusion of the inline lemma and removed the corresponding rewrite:

```
assert (∀a, S (a + S a) = S (S (a + a))).
- ... rewrite -> plus_comm.
```

SEARCH We used PUMPKIN to search the example for a patch that corresponds to the change in `bin_to_nat`. It found an isomorphism:

```
∀P b, P (bin_to_nat b) -> P (bin_to_nat b + 0)
∀P b, P (bin_to_nat b + 0) -> P (bin_to_nat b)
```

APPLY After porting to User B's definition, a rewrite in the proof of the theorem `normalize_correctness` failed:

```
Theorem normalize_correctness:
  ∀b, nat_to_bin (bin_to_nat b) = normalize b.
Proof.
  ... rewrite -> plus_0_r.X
```

4 If PUMPKIN finds just one implication, it returns that.

Attempting the obvious patch from the difference in tactics—rewriting by `plus_n_O`—failed. Applying the patch that PUMPKIN found fixed the broken proof:

```
... apply patch_inv. rewrite -> plus_0_r.✓
```

In this case, since we ported User A's definition to a simpler definition,[5] PUMPKIN found a patch that was not the most natural patch. The natural patch would be to remove the rewrite, just as we removed a different rewrite from the example proof. This did not occur when we ported User B's definition, which suggests that in the future, a patch finding tool may help inform novice users which definition is simpler: It can factor the proof, then inform the user if two factors are inverses. Tactic-level changes do not provide enough information to determine this; the tool must have a semantic understanding of the terms.

#### 3.6.2.1 *Configuration*

This scenario used the configuration for changes in cases of a fixpoint. Given such a change:

```
Fixpoint f ... := ... | g x
Fixpoint f' ... := ... | g x'
```

PUMPKIN searches two proofs of theorems:

```
∀ ..., P (f ...)
∀ ..., P (f' ...)
```

for an isomorphism that corresponds to the change:

```
∀P, P x -> P x'
∀P, P x' -> P x
```

The user can apply these patches to fix other broken proofs about the fixpoint.

The key feature that differentiates these from the patches we have encountered so far is that these patches hold for *all* `P`; for changes in fixpoint cases, the procedure abstracts candidates by `P`, not by its arguments. PUMPKIN uses this configuration for changes in fixpoint cases:

---

1: *diff* fixpoint cases for goals
2: use *all components* to recursively search an intermediate lemma for a change in conclusions
3: **if** there are candidates **then**
4: *specialize* and *factor* the candidate
*abstract* the factors by functions
try to *invert* the patch to find an isomorphism

---

For the prototype, we require the user to cut the intermediate lemma explicitly and to pass its type and arguments. In the future, an improved semantic differencing component can infer both the

---

5 User A uses `*`; User B uses `+`. For arbitrary `n`, the term `2 * n` reduces to `n + (n + 0)`, which does not reduce any further.

```
Definition divide p q := ∃          Definition divide p q := ∃
    r, p * r = q.                       r, q = r * p.
```

Figure 15: Old (left) and new (right) definitions of `divide` in Coq.

intermediate lemma and the arguments: It can search within the proof for some proof of a function that is applied to the fixpoint.

### 3.6.3 *Updating Proof Assistant Versions*

Coq sometimes makes changes to its standard library that break backwards-compatibility. To test the plausibility of using a patch finding tool for proof assistant version updates, we identified a breaking change in the Coq standard library [75]. The commit changed the definition of `divide` prior to the Coq 8.4 release (Figure 15). The change broke 46 proofs in the standard library. We used PUMPKIN to find an isomorphism that corresponds to the change in `divide`. The isomorphism PUMPKIN found fixed broken proofs.

REPLAY We used the proof of `mod_divide` as the example for PUMPKIN. The proof broke with the new `divide`:

```
Theorem mod_divide:
  ∀ a b, b~=0 -> (a mod b == 0 <-> (divide b a)).
Proof.
  ... rewrite (div_mod a b Hb) at 2.X
```

We replayed changes to `mod_divide`:

```
  ... rewrite mul_comm.  symmetry.
  rewrite (div_mod a b Hb) at 2.✓
```

SEARCH We used PUMPKIN to search the example for a patch that corresponds to the change in `divide`. It found an isomorphism:

```
∀r p q, p * r = q -> q = r * p
∀r p q, q = r * p -> p * r = q
```

APPLY The proof of the theorem `Zmod_divides` broke after rewriting by the changed theorem `mod_divide`:

```
Theorem Zmod_divides:
  ∀a b, b<>0 -> (a mod b = 0 <-> ∃c, a = b * c).
Proof.
  ... split; intros (c,Hc); exists c; auto.X
```

Adding the patches PUMPKIN found to a hint database made the proof go through:

```
  ... split; intros (c,Hc); exists c; auto.✓
```

#### 3.6.3.1 *Configuration*

This scenario used the configuration for changes in dependent arguments to constructors. PUMPKIN searches two proofs that apply the same constructor to different dependent arguments:

```
... (C (P x)) ...
... (C (P' x)) ...
```

for an isomorphism between the arguments:

```
∀ x, P x -> P' x
∀ x, P' x -> P x
```

The user can apply these patches to patch proofs that apply the constructor (in this case study, to fix broken proofs that instantiate `divide` with some specific `r`).

So far, we have encountered changes of this form as arguments to an induction principle; in this case, the change is an argument to a constructor. A patch between arguments to an induction principle maps directly between conclusions of the new and old theorem without induction; a patch between constructors does not. For example, for `divide`, we can find a patch with this form:

```
∀x, P x -> P' x
```

However, without using the induction principle for `exists`, we can't use that patch to prove this:

```
(∃x, P x) -> (∃ x, P' x)
```

This changes the goal type that semantic differencing determines. PUMPKIN uses this configuration for changes in constructor arguments:

1: *diff* constructor arguments for goals
2: use *all components* to recursively search those arguments for changes in conclusions

3: **if** there are candidates **then**
4: *abstract* the candidate
*factor* and try to *invert* the patch to find an isomorphism

For the prototype, the model of constructors for the semantic differencing component is limited, so we ask the user to provide the type of the change in argument (to guide line 2). We can extend semantic differencing to remove this restriction.

## 3.7 CONCLUSION

Rehashing thesis and how we do it

What we haven't accomplished yet at this point (parts of PUMPKIN PATCH future work).

The PUMPKIN PATCH prototype did not apply the patches that it finds, handle changes in structure, or include support for tactics beyond the use of hints. The next chapter addresses these limitations.

# 4

# PROOF REPAIR ACROSS TYPE EQUIVALENCES

This extension to the suite adds support for a broad class of changes in datatypes, handling a large class of practical repair scenarios. What this tool (PUMPKIN Pi) does is, when datatypes change and this breaks a lot of proofs, it generalizes the change in datatype itself (possibly with some user input) so that it can automatically fix proofs broken by the change in datatype.

So in other words, the information from those changes is carried in the difference between the old and new version of the changed datatype, possibly with some user input.

PUMPKIN Pi generalizes that information and applies it automatically.

The work saved is shown on a lot of case studies (see Table from PUMPKIN Pi).

## 4.1 MOTIVATING EXAMPLE

Consider a simple example of using PUMPKIN Pi: repairing proofs after swapping the two constructors of the `list` datatype (Figure 16). This is inspired by a similar change from a user study of proof engineers (Section 4.6). Even such a simple change can cause trouble, as in this proof from the Coq standard library (comments ours for clarity):[1]

```
Lemma rev_app_distr {A} :
  ∀ (x y : list A), rev (x ++ y) = rev y ++ rev x.
```

1 We use induction instead of pattern matching.

```
Inductive list (T : Type) :
    Type :=
| nil : list T
| cons : T → list T → list
    T.
```

```
Inductive list (T : Type) :
    Type :=
| cons : T → list T → list
    T
| nil : list T.
```

Figure 16: A change from the old version of `list` (left) to the new version of `list` (right). The old version of `list` is an inductive datatype that is either empty (the `nil` constructor), or the result of placing an element in front of another `list` (the `cons` constructor). The change swaps these two constructors (orange).

```
swap T (l : Old.list T) : New.
    list T :=
  Old.list_rect T (fun (l :
      Old.list T) => New.list T
      )
    New.nil
    (fun t _ (IHl : New.list T)
        => New.cons T t IHl)
    l.
```

```
swap⁻¹ T (l : New.list T) :
    Old.list T :=
  New.list_rect T (fun (l :
      New.list T) => Old.list T
      )
    (fun t _ (IHl : Old.list T)
        => Old.cons T t IHl)
    Old.nil
    l.
```

```
Lemma section: ∀ T (l : Old.
    list T),
  swap⁻¹ T (swap T l) = l.
Proof.
  intros T l. symmetry.
      induction l as [ |a l0 H
      ].
  - auto.
  - simpl. rewrite ← H. auto.
Qed.
```

```
Lemma retraction: ∀ T (l : New.
    list T),
  swap T (swap⁻¹ T l) = l.
Proof.
  intros T l. symmetry.
      induction l as [t l0 H|
      ].
  - simpl. rewrite ← H. auto.
  - auto.
Qed.
```

Figure 17: Two functions between `Old.list` and `New.list` (top) that form an equivalence (bottom).

```
Proof. (* by induction over x and y *)
  induction x as [| a l IHl].
  (* x nil:  *) induction y as [| a l IHl].
  (* y nil:  *) simpl. auto.
  (* y cons *) simpl. rewrite app_nil_r; auto.
  (* both cons:  *) intro y. simpl.
  rewrite (IHl y). rewrite app_assoc; trivial.
Qed.
```

This lemma says that appending (++) two lists and reversing (`rev`) the result behaves the same as appending the reverse of the second list onto the reverse of the first list. The proof script works by induction over the input lists `x` and `y`: In the base case for both `x` and `y`, the result holds by reflexivity. In the base case for `x` and the inductive case for `y`, the result follows from the existing lemma `app_nil_r`. Finally, in the inductive case for both `x` and `y`, the result follows by the inductive hypothesis and the existing lemma `app_assoc`.

When we change the `list` type, this proof no longer works. To repair this proof with Pumpkin Pi, we run this command:

```
Repair Old.list New.list in rev_app_distr.
```

assuming the old and new list types from Figure 16 are in modules `Old` and `New`. This suggests a proof script that succeeds (in light blue to denote Pumpkin Pi produces it automatically):

```
Proof. (* by induction over x and y *)
  intros x. induction x as [a l IHl| ]; intro y0.
  - (* both cons:  *) simpl. rewrite IHl. simpl.
    rewrite app_assoc. auto.
  - (* x nil:  *) induction y0 as [a l H| ].
    + (* y cons:  *) simpl. rewrite app_nil_r. auto.
    + (* y nil:  *) auto.
Qed.
```

where the dependencies (`rev`, `++`, `app_assoc`, and `app_nil_r`) have also been updated automatically ①. If we would like, we can manually modify this to something that more closely matches the style of the original proof script:

```
Proof. (* by induction over x and y *)
  induction x as [a l IHl|].
  (* both cons:  *) intro y. simpl.
  rewrite (IHl y). rewrite app_assoc; trivial.
  (* x nil:  *) induction y as [a l IHl|].
  (* y cons:  *) simpl. rewrite app_nil_r; auto.
  (* y nil:  *) simpl. auto.
Qed.
```

We can even repair the entire list module from the Coq standard library all at once by running the `Repair module` command ①. When we are done, we can get rid of `Old.list`.

The key to success is taking advantage of Coq's structured proof term language: Coq compiles every proof script to a proof term in a rich functional programming language called Gallina—Pumpkin Pi repairs that term. Pumpkin Pi then decompiles the repaired proof term (with optional hints from the original proof script) back to a suggested proof script that the proof engineer can maintain.

In contrast, updating the poorly structured proof script directly would not be straightforward. Even for the simple proof script above, grouping tactics by line, there are $6! = 720$ permutations of this proof script. It is not clear which lines to swap since these tactics do not have a semantics beyond the searches their evaluation performs. Furthermore, just swapping lines is not enough: even for such a simple change, we must also swap arguments, so `induction x as [| a l IHl ]` becomes `induction x as [a l IHl|]`. A recent thesis [100] describes the challenges of repairing tactics in detail. Pumpkin Pi's approach circumvents this challenge.

## 4.2 APPROACH

Pumpkin Pi can do much more than permute constructors. Given an equivalence between types $A$ and $B$, Pumpkin Pi repairs functions and proofs defined over $A$ to instead refer to $B$. It does this in a way that allows for removing references to $A$, which is essential for proof repair, since $A$ may be an old version of an updated type.

Like I mentioned earlier, this also works using differencing and program transformations over proof terms. Here, differencing thus looks at the difference between versions of the changed datatype, and finds something called a type equivalence (Section 4.2.1). Sometimes differencing is automatic, and sometimes it's manual. Then, program transformation ports proofs across the equivalence directly (Section 4.2.2). So they take care of application.

Figure 18 shows how this comes together when the proof engineer invokes Pumpkin Pi:

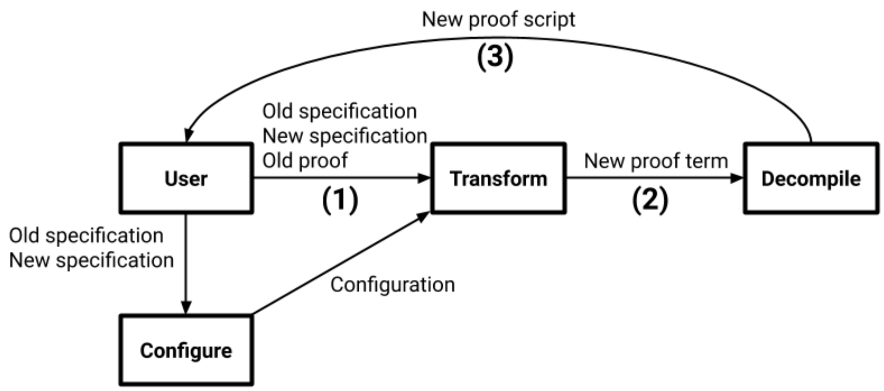


Figure 18: The workflow for Pumpkin Pi.

1. The proof engineer **Configure**s Pumpkin Pi, either manually or automatically.
2. The configured **Transform** transforms the old proof term into the new proof term.
3. **Decompile** suggests a new proof script.

There are currently four search procedures for automatic configuration implemented in Pumpkin Pi (see Table 1 on page 69). Manual configuration makes it possible for the proof engineer to configure the transformation to any equivalence, even without a search procedure. Section 4.6 shows examples of both workflows applied to real scenarios

### 4.2.1 *Scope: Type Equivalences*

Pumpkin Pi repairs proofs in response to changes in types that correspond to *type equivalences* [108], or pairs of functions that map between two types and are mutual inverses.[2] When a type equivalence between types $A$ and $B$ exists, those types are *equivalent* (denoted $A \simeq B$). Figure 17 shows a type equivalence between the two versions of `list` from Figure 16 that Pumpkin Pi discovered and proved automatically (1).

To give some intuition for what kinds of changes can be described by equivalences, we preview two changes below. See Table 1 on page 69 for more examples.

**Factoring out Constructors.** Consider changing the type `I` to the type `J` in Figure 19. `J` can be viewed as `I` with its two constructors `A` and `B` pulled out to a new argument of type `bool` for a single constructor. With Pumpkin Pi, the proof engineer can repair functions and proofs about `I` to instead use `J`, as long as she configures Pumpkin Pi to describe which constructor of `I` maps to `true` and which maps to `false`. This information about constructor mappings induces an equivalence

2 The adjoint follows, and Pumpkin Pi includes machinery to prove it (10) (23).

`I` $\simeq$ `J` across which Pumpkin Pi repairs functions and proofs. File ② shows an example of this, mapping `A` to `true` and `B` to false, and repairing proofs of De Morgan's laws.

**Adding a Dependent Index.** At first glance, the word *equivalence* may seem to imply that Pumpkin Pi can support only changes in which the proof engineer does not add or remove information. But equivalences are more powerful than they may seem. Consider, for example, changing a list to a length-indexed vector (Figure 20). Pumpkin Pi can repair functions and proofs about lists to functions and proofs about vectors of particular lengths ③, since $\Sigma$`(l:list T).length l = n` $\simeq$ `vector T n`. From the proof engineer's perspective, after updating specifications from `list` to `vector`, to fix her functions and proofs, she must additionally prove invariants about the lengths of her lists. Pumpkin Pi makes it easy to separate out that proof obligation, then automates the rest.

More generally, in homotopy type theory, with the help of quotient types, it is possible to form an equivalence from a relation, even when the relation is not an equivalence [5]. While Coq lacks quotient types, it is possible to achieve a similar outcome and use Pumpkin Pi for changes that add or remove information when those changes can be expressed as equivalences between $\Sigma$ types or sum types.

### 4.2.2 *Goal: Transport with a Twist*

The goal of Pumpkin Pi is to implement a kind of proof reuse known as *transport* [108], but in a way that is suitable for repair. Informally, transport takes a term $t$ and produces a term $t'$ that is the same as $t$ modulo an equivalence $A \simeq B$. If $t$ is a function, then $t'$ behaves the same way modulo the equivalence; if $t$ is a proof, then $t'$ proves the same theorem the same way modulo the equivalence.

When transport across $A \simeq B$ takes $t$ to $t'$, we say that $t$ and $t'$ are *equal up to transport* across that equivalence (denoted $t \equiv_{A \simeq B} t'$).[3] In Section 4.1, the original append function ++ over `Old.list` and the repaired append function ++ over `New.list` that Pumpkin Pi produces are equal up to transport across the equivalence from Figure 17, since (by `app_ok` ①):

```
∀ T (l1 l2 : Old.list T),
  swap T (l1 ++ l2) = (swap T l1) ++ (swap T l2).
```

The original `rev_app_distr` is equal to the repaired proof up to transport, since both prove the same thing the same way up to the equivalence, and up to the changes in ++ and `rev`.

3 This notation should be interpreted in a metatheory with *univalence*—a property that Coq lacks—or it should be approximated in Coq. The details of transport with univalence are in the Homotopy Type Theory book [108], and an approximation in Coq is in the univalent parametricity framework paper [105]. For equivalent $A$ and $B$, there can be many equivalences $A \simeq B$. Equality up to transport is across a *particular* equivalence, but we erase this in the notation.

```
Inductive I :=
| A : I
| B : I.
```

```
Inductive J :=
| makeJ : bool → J.
```

Figure 19: The old type `I` (left) is either `A` or `B`. The new type `J` (right) is `I` with `A` and `B` factored out to `bool` (orange).

```
Inductive list (T : Type) :
    Type :=
| nil : list T
| cons : T → list T → list T.
```

```
Inductive vector (T : Type) : nat →
    Type :=
| nil : vector T 0
| cons : T → ∀ (n : nat), vector T
    n → vector T (S n).
```

Figure 20: A vector (bottom) is a list (top) indexed by its length (orange). Vectors effectively make it possible to enforce length invariants about lists at compile time.

Transport typically works by applying the functions that make up the equivalence to convert inputs and outputs between types. This approach would not be suitable for repair, since it does not make it possible to remove the old type $A$. Pumpkin Pi implements transport in a way that allows for removing references to $A$—by proof term transformation.

## 4.3 DIFFERENCING

At the heart of Pumpkin Pi is a configurable proof term transformation for transporting proofs across equivalences (4). It is a generalization of the transformation from an earlier version of Pumpkin Pi called Devoid [?], which solved this problem a particular class of equivalences.

The transformation takes as input a deconstructed equivalence that we call a *configuration*. This section introduces the configuration (Section 4.3.1), defines the transformation that builds on that (Section **??**), then specifies correctness criteria for the configuration (Section 4.3.2). Section **??** describes the additional work needed to implement this transformation.

**Conventions.** All terms that we introduce in this section are in the Calculus of Inductive Constructions ($\text{CIC}_\omega$), the type theory that Coq's proof term language Gallina implements. $\text{CIC}_\omega$ is based on the

```
DepConstr(0, list T) : list T
    := Constr(0, list T).
DepConstr(1, list T) t l :
    list T :=
  Constr (1, list T) t l.

DepElim(l, P) { Pnil, Pcons } :
     P l :=
  Elim(l, P) { Pnil, Pcons }.
```

```
DepConstr(0, list T) : list T
    := Constr(1, list T).
DepConstr(1, list T) t l :
    list T :=
  Constr(0, list T) t l.

DepElim(l, P) { Pnil, Pcons } :
     P l :=
  Elim(l, P) { Pcons, Pnil }.
```

Figure 21: The dependent constructors and eliminators for old (left) and new (right) `list`, with the difference in orange.

Calculus of Constructions (CoC), a variant of the lambda calculus with polymorphism (types that depend on types) and dependent types (types that depend on terms) [31]. $\text{CIC}_\omega$ extends CoC with inductive types [32]. Inductive types are defined solely by their constructors (like `nil` and `cons` for `list`) and eliminators (like the induction principle for `list`); this section assumes that these eliminators are primitive.

The syntax for $\text{CIC}_\omega$ with primitive eliminators is in Figure 24. The typing rules are standard. We assume inductive types $\Sigma$ with constructor $\exists$ and projections $\pi_l$ and $\pi_r$, and an equality type = with constructor `eq_refl`. We use $\vec{t}$ and $\{t_1, \ldots, t_n\}$ to denote lists of terms.

### 4.3.1 *The Configuration*

The configuration is the key to building a proof term transformation that implements transport in a way that is suitable for repair. Each configuration corresponds to an equivalence $A \simeq B$. It deconstructs the equivalence into things that talk about $A$, and things that talk about $B$. It does so in a way that hides details specific to the equivalence, like the order or number of arguments to an induction principle or type.

At a high level, the configuration helps the transformation achieve two goals: preserve equality up to transport across the equivalence between $A$ and $B$, and produce well-typed terms. This configuration is a pair of pairs:

```
((DepConstr, DepElim), (Eta, Iota))
```

each of which corresponds to one of the two goals: `DepConstr` and `DepElim` define how to transform constructors and eliminators, thereby preserving the equivalence, and `Eta` and `Iota` define how to transform $\eta$-expansion and $\iota$-reduction of constructors and eliminators, thereby producing well-typed terms. Each of these is defined in $\text{CIC}_\omega$ for each equivalence.

Carlo theory will go here: basically the names aren't coincidences, it's because this corresponds to an initial algebra, so it's more natural when you have inductive types but more general than that. Draw diagram, explain what each part corresponds to.

**Preserving the Equivalence.** To preserve the equivalence, the configuration ports terms over $A$ to terms over $B$ by viewing each term of type $B$ as if it were an $A$. This way, the rest of the transformation can replace values of $A$ with values of $B$, and inductive proofs about $A$ with inductive proofs about $B$, all without changing the order or number of arguments.

The two configuration parts responsible for this are `DepConstr` and `DepElim` (*dependent constructors* and *eliminators*). These describe how to construct and eliminate $A$ and $B$, wrapping the types with a common inductive structure. The transformation requires the same number of dependent constructors and cases in dependent eliminators for $A$ and

$B$, even if $A$ and $B$ are types with different numbers of constructors ($A$ and $B$ need not even be inductive; see Sections 4.3.2 and 4.6).

For the `list` change from Section 4.1, the configuration that Pumpkin Pi discovers uses the dependent constructors and eliminators in Figure 21. The dependent constructors for `Old.list` are the normal constructors with the order unchanged, while the dependent constructors for `New.list` swap constructors back to the original order. Similarly, the dependent eliminator for `Old.list` is the normal eliminator for `Old.list`, while the dependent eliminator for `New.list` swaps cases.

As the name hints, these constructors and eliminators can be dependent. Consider the type of vectors of some length:

```
packed_vect T := Σ(n : nat).vector T n.
```

Pumpkin Pi can port proofs across the equivalence between this type and `list T` (3). The dependent constructors Pumpkin Pi discovers pack the index into an existential, like:

```
DepConstr(0, packed_vect) : packed_vect T :=
  ∃ (Constr(0, nat)) (Constr(0, vector T)).
```

and the eliminator it discovers eliminates the projections:

```
DepElim(s, P) { f₀ f₁ } : P (∃ (πₗ s) (πᵣ s)) :=
  Elim(πᵣ s, λ(n : nat)(v : vector T n).P (∃ n v)) {
    f₀,
    (λ(t : T)(n : nat)(v : vector T n).f₁ t (∃ n v))
  }.
```

In both these examples, the interesting work moves into the configuration: the configuration for the first swaps constructors and cases, and the configuration for the second maps constructors and cases over `list` to constructors and cases over `packed_vect`. That way, the transformation need not add, drop, or reorder arguments. Furthermore, both examples use automatic configuration, so Pumpkin Pi's **Configure** component discovers `DepConstr` and `DepElim` from just the types $A$ and $B$, taking care of even the difficult work.

**Producing Well-Typed Terms.** The other configuration parts `Eta` and `Iota` deal with producing well-typed terms, in particular by transporting equalities. $\mathrm{CIC}_{\omega}$ distinguishes between two important kinds of equality: those that hold by reduction (*definitional* equality), and those that hold by proof (*propositional* equality). That is, two terms `t` and `t'` of type `T` are definitionally equal if they reduce to the same normal form, and propositionally equal if there is a proof that `t = t'` using the inductive equality type `=` at type `T`. Definitionally equal terms are necessarily propositionally equal, but the converse is not in general true.

When a datatype changes, sometimes, definitional equalities defined over the old version of that type must become propositional. A naive proof term transformation may fail to generate well-typed terms if it does not account for this. Otherwise, if the transformation transforms

```
Inductive nat :=
| O : nat
| S : nat → nat.
```

```
Inductive positive :=
| xI : positive → positive
| xO : positive → positive
| xH : positive.

Inductive N :=
| N0 : N
| Npos : positive → N.
```

Figure 22: A unary natural number `nat` (left) is either zero (`O`) or the successor of some other natural number (`S`). A binary natural number `N` (right) is either zero (`N0`) or a positive binary number (`Npos`), where a positive binary number is either 1 (`xH`), or the result of shifting left and adding 1 (`xI`) or 0 (`xO`). Unary and binary natural numbers are equivalent, but have different inductive structures. Consequentially, definitional equalities over `nat` may become propositional over `N`.

a term `t : T` to some `t' : T'`, it does not necessarily transform `T` to `T'` [106].

`Eta` and `Iota` describe how to transport equalities. More formally, they define $\eta$-expansion and $\iota$-reduction of $A$ and $B$, which may be propositional rather than definitional, and so must be explicit in the transformation. $\eta$-expansion describes how to expand a term to apply a constructor to an eliminator in a way that preserves propositional equality, and is important for defining dependent eliminators [93]. $\iota$-reduction ($\beta$-reduction for inductive types) describes how to reduce an elimination of a constructor [92].

The configuration for the change from `list` to `packed_vect` has propositional `Eta`. It uses $\eta$-expansion for $\Sigma$:

`Eta(packed_vect) :=` $\lambda$`(s:packed_vect).`$\exists$ `(`$\pi_l$ `s) (`$\pi_r$ `s).`

which is propositional and not definitional in Coq. Thanks to this, we can forego the assumption that our language has primitive projections (definitional $\eta$ for $\Sigma$).

Each `Iota`—one per constructor—describes and proves the $\iota$-reduction behavior of `DepElim` on the corresponding case. This is needed, for example, to port proofs about unary numbers `nat` to proofs about binary numbers `N` (Figure 22). While we can define `DepConstr` and `DepElim` to induce an equivalence between them ⑤, we run into trouble reasoning about applications of `DepElim`, since proofs about `nat` that hold by reflexivity do not necessarily hold by reflexivity over `N`. For example, in Coq, while `S (n + m) = S n + m` holds by reflexivity over `nat`, when we define `+` with `DepElim` over `N`, the corresponding theorem over `N` does not hold by reflexivity.

To transform proofs about `nat` to proofs about `N`, we must transform *definitional* $\iota$-reduction over `nat` to *propositional* $\iota$-reduction over `N`. For our choice of `DepConstr` and `DepElim`, $\iota$-reduction is definitional over `nat`, since a proof of:

```
∀ P p0 ps n,
  DepElim(DepConstr(1, nat) n, P) { p0, ps } =
  ps n (DepElim(n, P) { p0, ps }).
```

holds by reflexivity. `Iota` for `nat` in the `S` case is a rewrite by that proof by reflexivity (5), with type:

```
∀ P p0 ps n (Q: P (DepConstr(1, nat) n) → s),
  Iota(1, nat, Q) :
    Q (ps n (DepElim(n, P) { p0, ps })) →
    Q (DepElim(DepConstr(1, nat) n, P) { p0, ps }).
```

In contrast, $\iota$ for `N` is propositional, since the theorem:

```
∀ P p0 ps n,
  DepElim(DepConstr(1, N) n, P) { p0, ps } =
  ps n (DepElim(n, P) { p0, ps }).
```

no longer holds by reflexivity. `Iota` for `N` is a rewrite by the propositional equality that proves this theorem (5), with type:

```
∀ P p0 ps n (Q: P (DepConstr(1, N) n) → s),
  Iota(1, N, Q) :
    Q (ps n (DepElim(n, P) { p0, ps })) →
    Q (DepElim(DepConstr(1, N) n, P) { p0, ps }).
```

By replacing `Iota` over `nat` with `Iota` over `N`, the transformation replaces rewrites by reflexivity over `nat` to rewrites by propositional equalities over `N`. That way, `DepElim` behaves the same over `nat` and `N`.

Taken together over both $A$ and $B$, `Iota` describes how the inductive structures of $A$ and $B$ differ. The transformation requires that `DepElim` over $A$ and over $B$ have the same structure as each other, so if $A$ and $B$ *themselves* have the same inductive structure (if they are *ornaments* [80]), then if $\iota$ is definitional for $A$, it will be possible to choose `DepElim` with definitional $\iota$ for $B$. Otherwise, if $A$ and $B$ (like `nat` and `N`) have different inductive structures, then definitional $\iota$ over one would become propositional $\iota$ over the other.

### 4.3.2 *Specifying Correct Configurations*

Choosing a configuration necessarily depends in some way on the proof engineer's intentions: there can be infinitely many equivalences that correspond to a change, only some of which are useful (for example (7), any $A$ is equivalent to `unit` refined by $A$). And there can be many configurations that correspond to an equivalence, some of which will produce terms that are more useful or efficient than others (consider `DepElim` converting through several intermediate types).

While we cannot control for intentions, we *can* specify what it means for a chosen configuration to be correct: Fix a configuration. Let `f` be the function that uses `DepElim` to eliminate $A$ and `DepConstr` to construct $B$, and let `g` be similar. Figure 23 specifies the correctness criteria for the configuration. These criteria relate `DepConstr`, `DepElim`, `Eta`, and `Iota` in a way that preserves equivalence coherently with equality.

```
section: ∀ (a : A), g (f a)
     = a.
retraction: ∀ (b : B), f (g
     b) = b.

constr_ok: ∀ j x⃗_A x⃗_B, x⃗_A
     ≡_{A≃B} x⃗_B →
  DepConstr(j, A) x⃗_A ≡_{A≃B}
      DepConstr(j, B) x⃗_B.

elim_ok: ∀ a b P_A P_B f⃗_A f⃗_B,
  a ≡_{A≃B} b →
  P_A ≡_{(A→s)≃(B→s)} P_B →
  ∀ j, f⃗_A[j] ≡_{ξ(A,P_A,j)≃ξ(B,P_B,j)}
       f⃗_B[j]→
  DepElim(a, P_A) f⃗_A
      ≡_{(Pa)≃(Pb)} DepElim(b, P
      _B) f⃗_A.
```

```
elim_eta(A): ∀ a P f⃗, DepElim(a, P)
     f⃗ : P (Eta(A) a).
eta_ok(A): ∀ (a : A), Eta(A) a = a.

iota_ok(A): ∀ j P f⃗ x⃗ (Q: P(Eta(A)
     (DepConstr(j, A) x⃗)) → s),
  Iota(A, j, Q) :
    Q (DepElim(DepConstr(j, A) x⃗, P)
         f⃗) →
    Q (rew ← eta_ok(A) (DepConstr(j
        , A) x⃗) in
      (f⃗[j]...(DepElim(IH_0, P) f⃗)...(
          DepElim(IH_n, P) f⃗)...)).
```

Figure 23: Correctness criteria for a configuration to ensure that the transformation preserves equivalence (left) coherently with equality (right, shown for $A$; $B$ is similar). `f` and `g` are defined in text. $s$, $\vec{f}$, $\vec{x}$, and $\vec{\mathtt{IH}}$ represent sorts, eliminator cases, constructor arguments, and inductive hypotheses. $\xi$ $(A, P, j)$ is the type of `DepElim(A, P)` at `DepConstr(j, A)` (similarly for $B$).

**Equivalence.** To preserve the equivalence (Figure 23, left), `DepConstr` and `DepElim` must form an equivalence (`section` and `retraction` must hold for `f` and `g`). `DepConstr` over $A$ and $B$ must be equal up to transport across that equivalence (`constr_ok`), and similarly for `DepElim` (`elim_ok`). Intuitively, `constr_ok` and `elim_ok` guarantee that the transformation correctly transports dependent constructors and dependent eliminators, as doing so will preserve equality up to transport for those subterms. This makes it possible for the transformation to avoid applying `f` and `g`, instead porting terms from $A$ directly to $B$.

**Equality.** To ensure coherence with equality (Figure 23, right), `Eta` and `Iota` must prove $\eta$ and $\iota$. That is, `Eta` must have the same definitional behavior as the dependent eliminator (`elim_eta`), and must behave like identity (`eta_ok`). Each `Iota` must prove and rewrite along the simplification (*refolding* [17]) behavior that corresponds to a case of the dependent eliminator (`iota_ok`). This makes it possible for the transformation to avoid applying `section` and `retraction`.

**Correctness.** With these correctness criteria for a configuration, we get the completeness result (proven in Coq ⑧) that every equivalence induces a configuration. We also obtain an algorithm for the soundness result that every configuration induces an equivalence.

The algorithm to prove `section` is as follows (`retraction` is similar): replace `a` with `Eta(A) a` by `eta_ok(A)`. Then, induct using `DepElim` over $A$. For each case $i$, the proof obligation is to show that `g (f a)` is equal to `a`, where `a` is `DepConstr(A, i)` applied to the non-inductive arguments (by `elim_eta(A)`). Expand the right-hand side using `Iota(A, i)`, then expand it again using `Iota(B, i)` (destructing over each `eta_ok` to apply the corresponding `Iota`). The result follows by definition of `g` and `f`, and by reflexivity.

#### 4.3.3 *Search Procedures*

PUMPKIN Pi implements four search procedures for automatic configuration (6). Three of the four procedures are based on the search procedure from DEVOID [**?**], while the remaining procedure instantiates the types $A$ and $B$ of a generic configuration that can be defined inside of Coq directly.

The algorithm above is essentially what **Configure** uses to generate functions `f` and `g` for the automatic configurations (9), and also generate proofs `section` and `retraction` that these functions form an equivalence (10). To minimize dependencies, PUMPKIN Pi does not produce proofs of `constr_ok` and `elim_ok` directly, as stating these theorems cleanly would require either a special framework [105] or a univalent type theory [108]. If the proof engineer wishes, it is possible to prove these in individual cases (8), but this is not necessary in order to use PUMPKIN Pi.

##### 4.3.3.1 *Algebraic Ornaments*

Differencing in DEVOID discovers equivalences that correspond to *algebraic ornaments*. An algebraic ornament relates an inductive type $A$ to an indexed version of that type $B$ with a new index of type $I_B$, where the new index is fully determined by a unique fold over $A$. For example, `vector` is exactly `list` with a new index of type `nat`, where the new index is fully determined by the `length` function. Consequentially, there are two functions:

```
ltv : list T                  → Σ(n : nat).vector T n.
vtl : Σ(n : nat).vector T n → list T.
```

that are mutual inverses:

```
∀ (l : list T),                vtl (ltv l) = l.
∀ (v : Σ(n : nat).vector T n), ltv (vtl v) = v.
```

and therefore form the type equivalence from Section **??**. Moreover, since the new index is fully determined by `length`, we can relate `length` to `ltv`:

```
∀ (l : list T), length l = πl (ltv l).
```

In general, we can view an algebraic ornament as a type equivalence:

$$A\ \vec{i}\ \simeq\ \Sigma(n : I_B\ \vec{i}).B\ (\texttt{index}\ n\ \vec{i})$$

where $\vec{i}$ are the indices of $A$, $I_B$ is a function over those indices, and the `index` operation inserts the new index $n$ at the right offset. Such a type equivalence consists of two functions [108]:

```
promote : A i⃗                           →  Σ(n : I_B i⃗).B (index n i⃗).
forget  : Σ(n : I_B i⃗).B (index n i⃗)  →  A i⃗.
```

that are mutual inverses:[4]

```
section    : ∀ (a : A i⃗),                          forget (promote
     a) = a.
retraction : ∀ (b_Σ : Σ(n : I_B i⃗).B (index n i⃗)), promote (forget
     b_Σ) = b_Σ.
```

An algebraic ornament is additionally equipped with an indexer, which is a unique fold:

```
indexer : A i⃗  →  I_B i⃗.
```

which projects the promoted index:

```
coherence : ∀(a : A i⃗), indexer a = π_l (promote a).
```

Following existing work [68], we call this equivalence the *ornamental promotion isomorphism*; when it holds and the indexer exists, we say that $B$ is an algebraic ornament of $A$.

`Find ornament` searches for algebraic ornaments between types and is, to the best of our knowledge, the first search algorithm for ornaments.

In their original form, ornaments are a programming mechanism: Given a type $A$, an ornament determines some new type $B$. We invert this process for algebraic ornaments: Given types $A$ and $B$, Devoid searches for an ornament between them. This is possible for algebraic ornaments precisely because the indexer is extensionally unique. For example, all possible indexers for `list` and `vector` must compute the length of a list; if we were to try doubling the length instead, we would not be able to satisfy the equivalence.

`Find ornament` takes two inductive types and searches for the components of the ornamental promotion isomorphism between them:

- **Inputs**: Inductive types $A$ and $B$, assuming:
  - $B$ is an algebraic ornament of $A$,
  - $B$ has the same number of constructors in the same order as $A$,
  - $A$ and $B$ do not contain recursive references to themselves under products, and
  - for every recursive reference to $A$ in $A$, there is exactly one new hypothesis in $B$, which is exactly the new index of the corresponding recursive reference in $B$.
- **Outputs**: Functions `promote`, `forget`, and `indexer`, guaranteeing:

4 The adjunction condition follows from section and retraction.

- the outputs form the ornamental promotion isomorphism between the inputs.

`Find ornament` includes an option to generate a proof that the outputs form the ornamental promotion isomorphism; by default, this option is false, since `Lift` does not need this proof.

**Presentation.** We present both algorithms relationally, using a set of judgments; to turn these relations into algorithms, prioritize the rules by running the derivations in order, falling back to the original term when no rules match. The default rule for a list of terms is to run the derivation on each element of the list individually.

**Notes on Syntax.** The language the algorithms operate over is $\text{CIC}_\omega$ with primitive eliminators; this is a simplified version of the type theory underlying Coq. Figure 24 contains the syntax (which includes variables, sorts, product types, functions, inductive types, constructors, and eliminators), as well as the syntax for some judgments and operations, the rules for which are standard and thus omitted. For simplicity of presentation, we assume variables are names; we assume that all names are fresh. As in Coq, we assume the existence of an inductive type $\Sigma$ for sigma types with projections $\pi_l$ and $\pi_r$; for simplicity, we assume projections are primitive. Throughout, we use $\vec{i}$ and $\{t_1, \ldots, t_n\}$ to denote lists of terms, and we use $\vec{i}[j]$ to denote accessing the element of the list $\vec{i}$ at offset $j$.

**Common Definitions.** The algorithms assume list insertion and removal functions `insert` and `remove`, plus two functions DEVOID implements: `off` computes the offset of the new index of type $I_B$ in $B$'s indices, and `new` determines whether a hypothesis in a case of the eliminator type of $B$ is new. Figure 25 contains other common definitions, the names for which are reserved: The `index` and `deindex` functions insert an index into and remove an index from a list at the index computed by `off`. Input type $A$ expands to an inductive type with indices of types $\vec{X_A}$, sort $\mathrm{s}_A$, and constructors $\{\mathrm{C}_{A_1}, \ldots, \mathrm{C}_{A_n}\}$. $\mathrm{P}_A$ denotes the type of the motive of the eliminator of $A$, and each $\mathrm{E}_{A_i}$ denotes the type of the eliminator for the $i$th constructor of $A$. Analogous names are also reserved for input type $B$.

The `Find ornament` algorithm implements the specification. It builds on three intermediate steps: one to generate each of `indexer`, `promote`, and `forget`. Figure 26 shows the algorithm for generating `indexer`. The algorithms for generating `promote` and `forget` are similar; Figure 27 shows only the derivations for generating `promote` that are different from those for generating `indexer`, and the derivations for generating `forget` are omitted.

| | |
|---|---|
| $\langle i \rangle \in \mathbb{N}$, $\langle v \rangle \in$ Vars, $\langle s \rangle \in \{$ Prop, Set, Type$\langle i \rangle$ $\}$ | $\Gamma \vdash t : T$ `// type checking` |
| | $\Gamma \vdash t_1 \equiv_{\beta\delta\iota} t_2$ `// definitional equality` |
| $\langle t \rangle ::= \langle v \rangle \mid \langle s \rangle \mid \Pi\ (\langle v \rangle : \langle t \rangle)\ .\ \langle t \rangle \mid$ | $t_\beta$ `// beta-reduction` |
| $\lambda\ (\langle v \rangle : \langle t \rangle)\ .\ \langle t \rangle \mid \langle t \rangle\ \langle t \rangle \mid$ | $t_{\beta\delta\iota}$ `// normalization` |
| Ind $(\langle v \rangle : \langle t \rangle)\{\langle t \rangle, \ldots, \langle t \rangle\} \mid$ Constr $(\langle i \rangle, \langle t \rangle) \mid$ | $t\ [y\ /\ x]$ `// substitution` |
| Elim$(\langle t \rangle, \langle t \rangle)\{\langle t \rangle, \ldots, \langle t \rangle\}$ | $\xi\ (I, Q, c, C)$ `// type of eliminator` |

Figure 24: $\mathrm{CIC}_\omega$ syntax (left, from existing work [107]) and judgments and operations (right).

$A$ := $\mathrm{Ind}(Ty_A : \Pi(\vec{i_A} : \vec{X_A}).\mathrm{s}_A)\{\mathrm{C}_{A_1}, \ldots, \mathrm{C}_{A_n}\}$
$B$ := $\mathrm{Ind}(Ty_B : \Pi(\vec{i_B} : \vec{X_B}).\mathrm{s}_B)\{\mathrm{C}_{B_1}, \ldots, \mathrm{C}_{B_n}\}$
$\forall 1 \leq i \leq n$,
  $\mathrm{E}_{A_i}\ (p_A : \mathrm{P}_A)$ := $\xi(A,\ p_A,\ \mathrm{Constr}(i,\ A),\ C_{A_i})$
  $\mathrm{E}_{B_i}\ (p_B : \mathrm{P}_B)$ := $\xi(B,\ p_B,\ \mathrm{Constr}(i,\ B),\ C_{B_i})$

$\mathrm{P}_A$ := $\Pi(\vec{i_A} : \vec{X_A})(a : A\ \ \vec{i_A}).\mathrm{s}_A$
$\mathrm{P}_B$ := $\Pi(\vec{i_B} : \vec{X_B})(b : B\ \ \vec{i_B}).\mathrm{s}_B$

index := insert (off $A$ $B$)
deindex := remove (off $A$ $B$)

Figure 25: Common definitions for both algorithms.

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_{i_m} t}$$

$$\text{Index-Motive}\quad \frac{}{\Gamma \vdash (A, B) \Downarrow_{i_m} \lambda(\vec{i_A} : \vec{X_A})(a : A\ \vec{i_A}).(I_B\ \vec{i_A})_\beta}$$

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_{i_c} t}$$

$$\text{Index-Conclusion}\quad \frac{}{\Gamma \vdash (p_A\ \vec{i_A}\ a,\ p_B\ \vec{i_B}\ b) \Downarrow_{i_c} \vec{i_B}[\text{off}\ A\ B]}$$

$$\text{Index-Hypothesis}\quad \frac{\text{new}\ n_B\ b_B \qquad \Gamma,\ n_B : t_B \vdash (\Pi(n_A : t_A).b_A,\ b_B) \Downarrow_{i_c} t}{\Gamma \vdash (\Pi(n_A : t_A).b_A,\ \Pi(n_B : t_B).b_B) \Downarrow_{i_c} t}$$

$$\text{Index-IH}\quad \frac{\Gamma \vdash (A, B) \Downarrow_{i_m} p \qquad \Gamma,\ n_A : p\ \vec{i_A}\ a \vdash (b_A,\ b_B[n_A/\vec{i_B}[\text{off}\ A\ B]]) \Downarrow_{i_c} t}{\Gamma \vdash (\Pi(n_A : p_A\ \vec{i_A}\ a).b_A,\ \Pi(n_B : p_B\ \vec{i_B}\ b).b_B) \Downarrow_{i_c} \lambda(n_A : p\ \vec{i_A}\ a).t}$$

$$\text{Index-Prod}\quad \frac{\Gamma,\ n_A : t_A \vdash (b_A,\ b_B[n_A/n_B]) \Downarrow_{i_c} t}{\Gamma \vdash (\Pi(n_A : t_A).b_A,\ \Pi(n_B : t_B).b_B) \Downarrow_{i_c} \lambda(n_A : t_A).t}$$

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_{i} t}$$

$$\text{Index-Ind}\quad \frac{\Gamma \vdash (A,\ B) \Downarrow_{i_m} p \qquad \Gamma,\ p_A : \text{P}_A,\ p_B : \text{P}_B \vdash \{(\text{E}_{A_1}\ p_A,\ \text{E}_{B_1}\ p_B), \ldots, (\text{E}_{A_n}\ p_A,\ \text{E}_{B_n}\ p_B)\} \Downarrow_{i_c} \vec{f}}{\Gamma \vdash (A,\ B) \Downarrow_i \lambda(\vec{i_a} : \vec{X_A})(a : A\ \vec{i_a}).\text{Elim}(a, p)\vec{f}}$$

Figure 26: Identifying the indexer function.

SEARCHING FOR THE INDEXER Search generates the `indexer` by traversing the types of the eliminators for $A$ and $B$ in parallel using the algorithm from Figure 26, which consists of three judgments: one to generate the motive, one to generate each case, and one to compose the motive and cases.

**Generating the Motive.** The $(T_A,\ T_B) \Downarrow_{i_m} t$ judgment consists of only the derivation Index-Motive, which computes the indexer motive from the types $A$ and $B$ (expanded in Figure 25). It does this by constructing a function with $A$ and its indices as premises, and the type $I_B$ in the conclusion with the appropriate indices. Consider `list` and `vector`:

```
list T := Ind (Ty_A : Type) {...}     vector T := Ind (Ty_B : Π
    (n : nat).Type) {...}
```

For these types, INDEX-MOTIVE computes the motive:

```
λ (l:list T) . nat
```

**Generating Each Case.** The $\Gamma \vdash (T_A, T_B) \Downarrow_{i_c} t$ judgment generates each case of the indexer by traversing in parallel the corresponding cases of the eliminator types for $A$ and $B$. It consists of four derivations: INDEX-CONCLUSION handles base cases and conclusions of inductive cases, while INDEX-HYPOTHESIS, INDEX-IH, and INDEX-PROD recurse into products.

INDEX-HYPOTHESIS handles each new hypothesis that corresponds to a new index in an inductive hypothesis of an inductive case of the eliminator type for $B$. It adds the new index to the environment, then recurses into the body of only the type for which the index already exists. For example, in the inductive case of `list` and `vector`, `new` determines that `n` is the new hypothesis. INDEX-HYPOTHESIS then recurses into the body of only the `vector` case:

```
Π (t_l:T) (l:list T) (IH_l:p_A l), ...     Π (t_v:T) (v:vector T n)
    (IH_v:p_B n v), ...
```

INDEX-PROD is next. It recurses into product types when the hypothesis is neither a new index nor an inductive hypothesis. Here, it runs twice, recursing into the body and substituting names until it hits the inductive hypothesis for both types:

```
Π (IH_l:p_A l), p_A (cons t_l l)           Π (IH_v:p_B n l), p_B
    (S n) (consV n t_l l)
```

INDEX-IH then takes over. It substitutes the new motive in the inductive hypothesis, then recurses into both bodies, substituting the new inductive hypothesis for the index in the eliminator type for $B$. Here, it substitutes the new motive for `p`$_A$ in the type of `IH`$_l$, extends the environment with `IH`$_l$, then substitutes `IH`$_l$ for `n`, so that it recurses on these types:

```
p_A (cons t_l l)                           p_B (S IH_l) (consV IH_l t_l l)
```

Finally, INDEX-CONCLUSION computes the conclusion by taking the index of motive $p_B$ at off $A$ $B$, here `S IH`$_l$. In total, this produces a function that computes the length of `cons t l`:

```
λ (t_l:T) (l:list T) (IH_l:(λ (l:list T).nat) l).S IH_l
```

**Composing the Result.** The $\Gamma \vdash (T_A, T_B) \Downarrow_i t$ judgment consists of only INDEX-IND, which identifies the motive and each case using the other two judgments, then composes the result. In the case of `list` and `vector`, this produces a function that computes the length of a list:

```
λ (l:list T).Elim(l, λ (l:list T).nat)
  {0, λ (t_l:T) (l:list T) (IH_l:(λ (l:list T).nat) l).S IH_l}
```

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_{p_m} t}$$

Promote-Motive

$$\frac{\Gamma \vdash (A,\ B) \Downarrow_i \pi}{\Gamma \vdash (A,\ B) \Downarrow_{p_m} \lambda(\vec{i_a} : \vec{X_A})(a : A\ \vec{i_a}).B\ (\text{index}\ (\pi\ \vec{i_a}\ a)\ \vec{i_a})}$$

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_{p_c} t}$$

Promote-Conclusion

$$\frac{}{\Gamma \vdash (p_A\ \vec{i_A}\ a,\ p_B\ \vec{i_B}\ b) \Downarrow_{p_c} b}$$

Promote-IH

$$\frac{\Gamma \vdash (A,\ B) \Downarrow_i \pi \qquad \Gamma \vdash (A,\ B) \Downarrow_{p_m} p \qquad \Gamma,\ n_A : p\ \vec{i_A}\ a \vdash (b_A,\ b_B[n_A/b][\pi\ \vec{i_A}\ a/\vec{i_B}[\text{off}\ A\ B]]) \Downarrow_{p_c} t}{\Gamma \vdash (\Pi(n_A : p_A\ \vec{i_A}\ a).b_A,\ \Pi(n_B : p_B\ \vec{i_B}\ b).b_B) \Downarrow_{p_c} \lambda(n_A : p\ \vec{i_A}\ a).t}$$

$$\boxed{\Gamma \vdash (T_A,\ T_B) \Downarrow_p t}$$

Promote-Ind

$$\frac{\Gamma \vdash (A,\ B) \Downarrow_i \pi \qquad \Gamma \vdash (A,\ B) \Downarrow_{p_m} p \qquad \Gamma,\ p_A : \mathrm{P}_A,\ p_B : \mathrm{P}_B \vdash \{(\mathrm{E}_{A_1}\ p_A,\ \mathrm{E}_{B_1}\ p_B), \ldots, (\mathrm{E}_{A_n}\ p_A,\ \mathrm{E}_{B_n}\ p_B)\} \Downarrow_{p_c} \vec{f}}{\Gamma \vdash (A,\ B) \Downarrow_p \lambda(\vec{i_A} : \vec{X_A})(a : A\ \vec{i_A}).\exists\ (\pi\ \vec{i_A}\ a)\ (\text{Elim}(a, p)\vec{f})}$$

Figure 27: Identifying the promotion function.

#### 4.3.3.2 *Searching for Promote and Forget*

Figure 27 shows the interesting derivations for the judgment $(T_A,\ T_B) \Downarrow_p t$ that searches for `promote`: Promote-Motive identifies the motive as $B$ with a new index (which it computes using `indexer`, denoted by metavariable $\pi$). When Promote-IH recurses, it substitutes the inductive hypothesis for the term rather than for its index, and it substitutes the new index (which it also computes using `indexer`) inside of that term. Promote-Conclusion returns the entire term, rather than its index. Finally, Promote-Ind not only recurses into each case, but also packs the result.

The omitted derivations to search for `forget` are similar, except that the domain and range are switched. Consequentially, `indexer` is never needed; Forget-Motive removes the index rather than inserting it, and Forget-IH no longer substitutes the index. Additionally, Forget-Hypothesis adds the hypothesis for the new index rather than skipping it, and Forget-Ind eliminates over the projection rather than packing the result.

CORE SEARCH ALGORITHM The core search algorithm produces `indexer`, `promote`, and `forget`, then composes them into a tuple. This tuple is how Devoid represents ornaments internally. Devoid has options (used in `Example.v`) that tell search to generate proofs that its outputs are correct, thereby increasing confidence in and usefulness of those outputs. The proof of `coherence` is reflexivity. The intuition behind the automation to prove `section` and `retraction` (`equivalence.ml`) is that `promote` and `forget` map along corresponding constructors, so inductive cases preserve equalities. Thus, each inductive case of these proofs is generated by a fold that rewrites each recursive reference, with reflexivity as identity.

#### 4.3.3.3 *Other Search Procedures*

Brief explanation of these and how differencing works for them in detail based on algebraic ornaments

DESIGNING NEW SEARCH PROCEDURES How hard, how useful

### 4.3.4 *Limitations*

Limitations and whether they're addressed in other tools yet or not

## 4.4 TRANSFORMATION

Figure 33 shows the proof term transformation $\Gamma \vdash t \Uparrow t'$ that forms the core of PUMPKIN Pi. The transformation is parameterized over equivalent types $A$ and $B$ (EQUIVALENCE) as well as the configuration. It assumes $\eta$-expanded functions. It implicitly constructs an updated context $\Gamma'$ in which to interpret $t'$, but this is not needed for computation.

The proof term transformation is (perhaps deceptively) simple by design: it moves the bulk of the work into the configuration, and represents the configuration explicitly. Of course, typical proof terms in Coq do not apply these configuration terms explicitly. PUMPKIN Pi does some additional work using *unification heuristics* to get real proof terms into this format before running the transformation. It then runs the proof term transformation, which transports proofs across the equivalence that corresponds to the configuration.

**Unification Heuristics.** The transformation does not fully describe the search procedure for transforming terms that PUMPKIN Pi implements. Before running the transformation, PUMPKIN Pi *unifies* subterms with particular $A$ (fixing parameters and indices), and with applications of configuration terms over $A$. The transformation then transforms configuration terms over $A$ to configuration terms over $B$. Reducing the result produces the output term defined over $B$.

Figure 29 shows this with the list append function ++ from Section 4.1. To update ++ (top left), PUMPKIN Pi unifies `Old.list T` with $A$, and `Constr` and `Elim` with `DepConstr` and `DepElim` (bottom left). After

$$\boxed{\Gamma \vdash t \Uparrow t'}$$

$$\textsc{Dep-Elim}\quad \frac{\Gamma \vdash a \Uparrow b \qquad \Gamma \vdash p_a \Uparrow p_b \qquad \Gamma \vdash \vec{f_a} \Uparrow \vec{f_b}}{\Gamma \vdash \text{DepElim}(a,\ p_a)\vec{f_a} \Uparrow \text{DepElim}(b,\ p_b)\vec{f_b}}$$

$$\textsc{Dep-Constr}\quad \frac{\Gamma \vdash \vec{t_a} \Uparrow \vec{t_b}}{\Gamma \vdash \text{DepConstr}(j,\ A)\ \vec{t_a} \Uparrow \text{DepConstr}(j,\ B)\ \vec{t_b}}$$

$$\textsc{Eta}\quad \frac{}{\Gamma \vdash \text{Eta}(A) \Uparrow \text{Eta}(B)}$$

$$\textsc{Iota}\quad \frac{\Gamma \vdash q_A \Uparrow q_B \qquad \Gamma \vdash \vec{t_A} \Uparrow \vec{t_B}}{\Gamma \vdash \text{Iota}(j,\ A,\ q_A)\ \vec{t_A} \Uparrow \text{Iota}(j,\ B,\ q_B)\ \vec{t_B}}$$

$$\textsc{Equivalence}\quad \frac{}{\Gamma \vdash A \Uparrow B}$$

$$\textsc{Constr}\quad \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma \vdash \vec{t} \Uparrow \vec{t'}}{\Gamma \vdash \text{Constr}(j,\ T)\ \vec{t} \Uparrow \text{Constr}(j,\ T')\ \vec{t'}}$$

$$\textsc{Ind}\quad \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma \vdash \vec{C} \Uparrow \vec{C'}}{\Gamma \vdash \text{Ind}(Ty : T)\vec{C} \Uparrow \text{Ind}(Ty : T')\vec{C'}}$$

$$\textsc{App}\quad \frac{\Gamma \vdash f \Uparrow f' \qquad \Gamma \vdash t \Uparrow t'}{\Gamma \vdash f t \Uparrow f' t'}$$

$$\textsc{Elim}\quad \frac{\Gamma \vdash c \Uparrow c' \qquad \Gamma \vdash Q \Uparrow Q' \qquad \Gamma \vdash \vec{f} \Uparrow \vec{f'}}{\Gamma \vdash \text{Elim}(c, Q)\vec{f} \Uparrow \text{Elim}(c', Q')\vec{f'}}$$

$$\textsc{Lam}\quad \frac{\Gamma \vdash t \Uparrow t' \qquad \Gamma \vdash T \Uparrow T' \qquad \Gamma,\ t : T \vdash b \Uparrow b'}{\Gamma \vdash \lambda(t : T).b \Uparrow \lambda(t' : T').b'}$$

$$\textsc{Prod}\quad \frac{\Gamma \vdash t \Uparrow t' \qquad \Gamma \vdash T \Uparrow T' \qquad \Gamma,\ t : T \vdash b \Uparrow b'}{\Gamma \vdash \Pi(t : T).b \Uparrow \Pi(t' : T').b'}$$

$$\textsc{Var}\quad \frac{v \in \text{Vars}}{\Gamma \vdash v \Uparrow v}$$

Figure 28: Transformation for transporting terms across $A \simeq B$ with configuration `((DepConstr, DepElim), (Eta, Iota))`.

```
(* 1: original term *)
λ (T : Type) (l m : Old.list T)
  .
 Elim(l, λ(l: Old.list T).Old.
     list T → Old.list T)) {
   (λ m . m),
   (λ t _ IHl m . Constr(1, Old.
       list T) t (IHl m))
 } m.
```

```
(* 2: after unifying with
    configuration *)
λ (T : Type) (l m : A) .
 DepElim(l, λ(l: A).A → A)) {
   (λ m . m)
   (λ t _ IHl m . DepConstr(1,
       A) t (IHl m))
 } m.
```

```
(* 3: after transforming *)
λ (T : Type) (l m : B) .
 DepElim(l, λ(l: B).B → B)) {
   (λ m . m)
   (λ t _ IHl m . DepConstr(1,
       B) t (IHl m))
 } m.
```

```
(* 4: reduced to final term *)
λ (T : Type) (l m : New.list T)
  .
 Elim(l, λ(l: New.list T).New.
     list T → New.list T)) {
   (λ t _ IHl m . Constr(0, New.
       list T) t (IHl m)),
   (λ m . m)
 } m.
```

Figure 29: Swapping cases of the append function, counterclockwise, the input term: 1) unmodified, 2) unified with the configuration, 3) ported to the updated type, and 4) reduced to the output.

unification, the transformation recursively substitutes $B$ for $A$, which moves `DepConstr` and `DepElim` to construct and eliminate over the updated type (bottom right). This reduces to a term with swapped constructors and cases over `New.list T` (top right).

In this case, unification is straightforward. This can be more challenging when configuration terms are dependent. This is especially pronounced with definitional `Eta` and `Iota`, which typically are implicit (reduced) in real code. To handle this, Pumpkin Pi implements custom *unification heuristics* for each search procedure that unify subterms with applications of configuration terms, and that instantiate parameters and dependent indices in those subterms 6. The transformation in turn assumes that all existing parameters and indices are determined and instantiated by the time it runs.

Pumpkin Pi falls back to Coq's unification for manual configuration and when these custom heuristics fail. When even Coq's unification is not enough, Pumpkin Pi relies on proof engineers to provide hints in the form of annotations 5.

**Algebraic Ornaments.** Consider instantiating the transformation to algebraic ornaments. We show only one direction of the algorithm, promoting from $A$ to packed $B$; the forgetful direction is similar. The core algorithm (Figure 33) builds on a set of common definitions (Figure 30) and two intermediate judgments: one to lift eliminators (Figure 31) and one to lift constructors (Figure 32).

**Common Definitions.** The common definitions (Figure 30) define some useful syntax: $\uparrow$ applies `promote`, $\downarrow$ applies `forget`, and $\pi_{I_B}$ applies `indexer`. $\exists_{I_B}$ packs a term of type $B$ into an existential with the

$$
\begin{array}{ll}
\uparrow \quad \{\vec{i_a} : \vec{X_A}\} := \texttt{promote}\ \vec{i_a}. & \downarrow \\
\quad\ \{\vec{i_b} : \vec{X_B}\} := \texttt{forget}\ \vec{i_b}. & \\
\pi_{I_B}\ \{\vec{i_a} : \vec{X_A}\} := \texttt{indexer}\ \vec{i_a}. & \exists_{I_B}\ \{\vec{i_b} : \vec{X_B}\}\ (b : B\ \vec{i_b}) \\
\quad := \exists\ \vec{i_b}\texttt{[off] b}. & \\
\uparrow_B\ := \pi_r \circ \uparrow. & \downarrow_A\ := \downarrow \circ \exists_{I_B}. \\
\uparrow_{I_B} := \pi_l \circ \uparrow. & \downarrow_{I_B} := \pi_{I_B} \circ \downarrow_A.
\end{array}
$$

Figure 30: Common definitions for the core lifting algorithm.

index at the appropriate offset. $\uparrow_B$ and $\uparrow_{I_B}$ promote and then project; $\downarrow_A$ packs and forgets, and $\downarrow_{I_B}$ packs, forgets, and then applies `indexer` to project the index.

LIFTING ELIMINATORS The $\Gamma \vdash t \Uparrow_E t'$ judgment (Figure 31) defines rules for lifting the motive and case of an eliminator, changing the *domain of induction* from $A$ to $B$. The intuition is that any term of type $A$ is the result of forgetting some term of type packed $B$. Then, since $A$ and $B$ have the same inductive structure, we can lift the eliminator of $A$ to the eliminator of $B$, and move that forgetfulness *inside of each case*. For example, the following terms are propositionally equal:

```
Elim((↓A b), pA){
  fnil,
  (λ(tl:T)(l:list T)(IHl:
      pA l).
    fcons tl l IHl)
}
```

```
Elim(b, λ(n:nat)(v:vector T n).pA
    (↓A v)){
  fnil,
  (λ(n:nat)(tv:T)(v:vector T n)(IHv:pA
      (↓A v)).
    fcons tv (↓A v) IHv)
}
```

The induction rules implement this transformation. Case lifts a case of the eliminator by first recursively lifting the motive, then using the lifted motive to compute the type of the new case, and then using that type to compute the body of the new case. In the example above, when lifting the inductive case, it first recursively lifts the motive $p_A$ using Motive, which drops the index, packs and forgets the argument of type $B$, and then $\beta$-reduces the result, eliminating references to $B$. This produces the new motive:

```
λ(n:nat)(v:vector T n).pA (↓A v)
```

which Case then uses to compute the type of the inductive case of the eliminator for $B$:

```
Π(tv:T)(n:nat)(v:vector T n)(IHv:pA (↓A v)).pA (↓A (consV tv (
    S n) v))
```

The $\Gamma \vdash (t,\ T) \Uparrow_{E_x} t'$ judgment then uses that type to compute the lifted function body. It computes this in a similar way to Motive, except that there are as many indices to drop and arguments to pack and forget as there are inductive hypotheses, and these do not occur in predictable places, so more rules are involved. This computes the new function:

```
λ(n:nat)(tv:T)(v:vector T n)(IHv:pA (↓A v)).fcons tv (↓A v) IHv
```

$$\boxed{\Gamma \vdash (t,\ T) \Uparrow_{E_x} t'}$$

$$\textsc{Drop-Index}\quad \frac{\text{new } n\ b \qquad \Gamma,\ n : t \vdash (f,\ b) \Uparrow_{E_x} b'}{\Gamma \vdash (f,\ \Pi(n : t).b) \Uparrow_{E_x} \lambda(n : t).b'}$$

$$\textsc{Forget-Arg}\quad \frac{\Gamma \vdash \vec{i} : \vec{X_B} \qquad \Gamma,\ n : B\ \vec{i} \vdash ((f\ (\downarrow_A n))_\beta,\ b) \Uparrow_{E_x} b'}{\Gamma \vdash (f,\ \Pi(n : B\ \vec{i}).b) \Uparrow_{E_x} \lambda(n : B\ \vec{i}).b'}$$

$$\textsc{Arg}\quad \frac{\Gamma,\ n : t \vdash ((f\ n)_\beta,\ b) \Uparrow_{E_x} b'}{\Gamma \vdash (f,\ \Pi(n : t).b) \Uparrow_{E_x} \lambda(n : t).b'} \qquad \textsc{Concl}\quad \frac{}{\Gamma \vdash (t,\ p_B\ \vec{y}) \Uparrow_{E_x} t}$$

$$\boxed{\Gamma \vdash t \Uparrow_E t'}$$

$$\textsc{Motive}\quad \frac{\Gamma \vdash p_A : \mathrm{P}_A}{\Gamma \vdash p_A \Uparrow_E \lambda(\vec{i} : \vec{X_B})(b : B\ \vec{i}).(p_A\ (\text{deindex}\ \vec{i})\ (\downarrow_A b))_\beta}$$

$$\textsc{Case}\quad \frac{\Gamma \vdash p_A : \mathrm{P}_A \qquad \Gamma \vdash f_i : \mathrm{E}_{A_i}\ p_A \qquad \Gamma \vdash p_A \Uparrow_E p_B \qquad \Gamma \vdash (f_i,\ \mathrm{E}_{B_i}\ p_B) \Uparrow_{\mathrm{E_x}} f_i'}{\Gamma \vdash f_i \Uparrow_E f_i'}$$

Figure 31: Lifting eliminators.

$$\boxed{\Gamma \vdash t \Uparrow_C t'}$$

$$\textsc{Normalize}\quad \frac{}{\Gamma \vdash \text{Constr}(j,\ A)\ \vec{x} \Uparrow_C (\uparrow (\text{Constr}(j,\ A)\ \vec{x}))_{\beta\delta\iota}}$$

Figure 32: Lifting constructors.

LIFTING CONSTRUCTORS The $\Gamma \vdash t \Uparrow_C t'$ judgment (Figure 32) lifts applications of constructors of $A$ to applications of constructors of $B$. This judgment computes one step of the promotion, leaving the recursive lifting of the arguments to the final algorithm. Using the same types, in the base case:

$\uparrow$ `nil` $\equiv_{\beta\delta\iota}$ $\exists$ `O nilV`

and in the inductive case:

$\uparrow$ `(cons t l)` $\equiv_{\beta\delta\iota}$ $\exists$ `(S (`$\uparrow_{I_B}$ `l)) (consV (`$\uparrow_{I_B}$ `l) t (`$\uparrow_B$ `l))`

This derivation consists of only one rule: NORMALIZE, which normalizes the promotion of the constructor. This is guaranteed to succeed because the application of the constructor is fully $\eta$-expanded. The core algorithm later internalizes the promotion functions in the result.

CORE LIFTING ALGORITHM The core algorithm (Figure 33) builds on these intermediate judgments. The interesting derivations for correctness are the first six: LIFT-ELIM and LIFT-CONSTR use the judgments for lifting eliminators and constructors of $A$. INTERNALIZE internalizes the explicit `promote` functions from the lifted constructors to recursive applications of the algorithm. RETRACTION and COHERENCE use the respective properties of the ornamental promotion isomorphism metatheoretically: the first to drop the explicit `forget` functions from the lifted eliminators, and the second to lift the `indexer` to a projection (in the forgetful direction, SECTION replaces RETRACTION). Finally, EQUIVALENCE lifts $A$ along the equivalence to packed $B$. The remaining derivations recurse predictably.

**Specifying a Correct Transformation.** The implementation of this transformation in PUMPKIN Pi produces a term that Coq type checks, and so does not add to the trusted computing base. As PUMPKIN Pi is an engineering tool, there is no need to formally prove the transformation correct, though doing so would be satisfying. The goal of such a proof would be to show that if $\Gamma \vdash t \Uparrow t'$, then $t$ and $t'$ are equal up to transport, and $t'$ refers to $B$ in place of $A$. The key steps in this transformation that make this possible are porting terms along the configuration (DEP-CONSTR, DEP-ELIM, ETA, and IOTA). For metatheoretical reasons, without additional axioms, a proof of this theorem in Coq can only be approximated [105]. It would be possible to generate per-transformation proofs of correctness, but this does not serve an engineering need.

### 4.4.1 *Limitations*

Limitations and whether they're addressed in other tools yet

$$\boxed{\Gamma \vdash t \Uparrow t'}$$

$$\textsc{Lift-Elim}\ \frac{\begin{array}{l}\Gamma \vdash \vec{i} : \vec{X_A} \qquad \Gamma \vdash a : A\ \vec{i} \\ \Gamma \vdash p_a \Uparrow_E p' \qquad \Gamma \vdash \vec{f_a}\ \Uparrow_E \vec{f'} \\ \Gamma \vdash p' \Uparrow p_b \qquad \Gamma \vdash \vec{f'} \Uparrow \vec{f_b} \qquad \Gamma \vdash a \Uparrow b_\Sigma\end{array}}{\Gamma \vdash \text{Elim}(a,\ p_a)\vec{f_a} \Uparrow \text{Elim}(\pi_r\ b_\Sigma,\ p_b)\vec{f_b}}$$

$$\textsc{Lift-Constr}\ \frac{\begin{array}{l}\Gamma \vdash \vec{i} : \vec{X_A} \qquad \Gamma \vdash \text{Constr}(j,\ A)\ \vec{t_a} : A\ \vec{i} \\ \Gamma \vdash \text{Constr}(j,\ A)\ \vec{t_a} \Uparrow_C t' \\ \Gamma \vdash t' \Uparrow t''\end{array}}{\Gamma \vdash \text{Constr}(j,\ A)\ \vec{t_a} \Uparrow t''}$$

$$\textsc{Internalize}\ \frac{\Gamma \vdash a \Uparrow b_\Sigma}{\Gamma \vdash \uparrow\ a \Uparrow b_\Sigma}$$

$$\textsc{Retraction}\ \frac{\Gamma \vdash\ b_\Sigma \Uparrow b'_\Sigma}{\Gamma \vdash \downarrow\ b_\Sigma \Uparrow b'_\Sigma}$$

$$\textsc{Coherence}\ \frac{\Gamma \vdash \vec{i} : \vec{X_A} \qquad \Gamma \vdash a : A\ \vec{i} \qquad \Gamma \vdash a \Uparrow b_\Sigma}{\Gamma \vdash \pi_{I_B}\ a\ \Uparrow (\pi_l\ b_\Sigma)_\beta}$$

$$\textsc{Equivalence}\ \frac{\Gamma \vdash \vec{i} : \vec{X_A}}{\Gamma \vdash A\ \vec{i} \Uparrow \Sigma(n : (I_B\ \vec{i})_\beta).B\ (\text{index}\ n\ \vec{i})}$$

$$\textsc{Constr}\ \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma \vdash \vec{t} \Uparrow \vec{t'}}{\Gamma \vdash \text{Constr}(j,\ T)\ \vec{t} \Uparrow \text{Constr}(j,\ T')\ \vec{t'}}$$

$$\textsc{Ind}\ \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma \vdash \vec{C} \Uparrow \vec{C'}}{\Gamma \vdash \text{Ind}(Ty : T)\vec{C} \Uparrow \text{Ind}(Ty : T')\vec{C'}}$$

$$\textsc{Elim}\ \frac{\Gamma \vdash c \Uparrow c' \qquad \Gamma \vdash Q \Uparrow Q' \qquad \Gamma \vdash \vec{f} \Uparrow \vec{f'}}{\Gamma \vdash \text{Elim}(c, Q)\vec{f} \Uparrow \text{Elim}(c', Q')\vec{f'}}$$

$$\textsc{App}\ \frac{\Gamma \vdash f \Uparrow f' \qquad \Gamma \vdash t \Uparrow t'}{\Gamma \vdash f t \Uparrow f' t'}$$

$$\textsc{Lam}\ \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma,\ t : T \vdash b \Uparrow b'}{\Gamma \vdash \lambda(t : T).b \Uparrow \lambda(t : T').b'}$$

$$\textsc{Prod}\ \frac{\Gamma \vdash T \Uparrow T' \qquad \Gamma,\ t : T \vdash b \Uparrow b'}{\Gamma \vdash \Pi(t : T).b \Uparrow \Pi(t : T').b'}$$

Figure 33: Core lifting algorithm.

## 4.5 IMPLEMENTATION

Parts of PUMPKIN Pi and DEVOID implementation, plus more (still need to arrange, fill in, and so on)

### 4.5.1 *Tool Details*

Implemented in blah blah blah, and so on.

### 4.5.2 *Workflow Integration*

#### 4.5.2.1 *Configure*

#### 4.5.2.2 *Transform*

**Termination.** When a subterm unifies with a configuration term, this suggests that PUMPKIN Pi *can* transform the subterm, but it does not necessarily mean that it *should*. In some cases, doing so would result in nontermination. For example, if $B$ is a refinement of $A$, then we can always run EQUIVALENCE over and over again, forever. We thus include some simple termination checks in our code (12).

**Intent.** Even when termination is guaranteed, whether to transform a subterm depends on intent. That is, PUMPKIN Pi automates the case of porting *every* $A$ to $B$, but proof engineers sometimes wish to port only *some* $A$s to $B$s. PUMPKIN Pi has some support for this using an interactive workflow (13), with plans for automatic support in the future.

**From CIC$_\omega$ to Coq.** The implementation (4) of the transformation handles language differences to scale from CIC$_\omega$ to Coq. We use the existing `Preprocess` [?] command to turn pattern matching and fixpoints into eliminators. We handle refolding of constants in constructors using `DepConstr`.

**Reaching Real Proof Engineers.** Many of our design decisions in implementing PUMPKIN Pi were informed by our partnership with an industrial proof engineer (Section 4.6). For example, the proof engineer rarely had the patience to wait more than ten seconds for PUMPKIN Pi to port a term, so we implemented optional aggressive caching, even caching intermediate subterms encountered while running the transformation (14). We also added a cache to tell PUMPKIN Pi not to $\delta$-reduce certain terms (14). With these caches, the proof engineer found PUMPKIN Pi efficient enough to use on a code base with tens of thousands of lines of code and proof.

The experiences of proof engineers also inspired new features. For example, we implemented a search procedure to generate custom eliminators to help reason about types like $\Sigma$`(l : list T).length l`

$\langle v \rangle \in$ Vars, $\langle t \rangle \in \text{CIC}_\omega$

$\langle p \rangle$ ::= intro $\langle v \rangle$ | rewrite $\langle t \rangle$ $\langle t \rangle$ | symmetry | apply $\langle t \rangle$ | induction $\langle t \rangle$ $\langle t \rangle$ { $\langle p \rangle, \ldots, \langle p \rangle$ } | split { $\langle p \rangle$, $\langle p \rangle$ } | left | right | $\langle p \rangle$ . $\langle p \rangle$

Figure 34: Qtac syntax.

$$\boxed{\Gamma \vdash t \Rightarrow p}$$

$$\textsc{Intro}\ \frac{\Gamma,\ n : T \vdash b \Rightarrow p}{\Gamma \vdash \lambda(n : T).b \Rightarrow \text{intro } n.\ p} \qquad \textsc{Symmetry}\ \frac{\Gamma \vdash H \Rightarrow p}{\Gamma \vdash \texttt{eq\_sym}\ H \Rightarrow \text{symmetry}.\ p}$$

$$\textsc{Split}\ \frac{\Gamma \vdash l \Rightarrow p \qquad \Gamma \vdash r \Rightarrow q}{\Gamma \vdash \text{Constr}(0,\ \wedge)\ l\ r \Rightarrow \text{split}\{p, q\}.}$$

$$\textsc{Left}\ \frac{\Gamma \vdash H \Rightarrow p}{\Gamma \vdash \text{Constr}(0,\ \vee)\ H \Rightarrow \text{left}.\ p} \qquad \textsc{Right}\ \frac{\Gamma \vdash H \Rightarrow p}{\Gamma \vdash \text{Constr}(1,\ \vee)\ H \Rightarrow \text{right}.\ p}$$

$$\textsc{Rewrite}\ \frac{\Gamma \vdash H_1 : x = y \qquad \Gamma \vdash H_2 \Rightarrow p}{\Gamma \vdash \text{Elim}(H_1,\ P)\{x,\ H_2,\ y\} \Rightarrow \text{symmetry. rewrite } P\ H_1.\ p}$$

$$\textsc{Induction}\ \frac{\Gamma \vdash \vec{f} \Rightarrow \vec{p}}{\Gamma \vdash \text{Elim}(t,\ P)\ \vec{f} \Rightarrow \text{induction } P\ t\ \vec{p}} \qquad \textsc{Apply}\ \frac{\Gamma \vdash t \Rightarrow p}{\Gamma \vdash f t \Rightarrow \text{apply } f.\ p}$$

$$\textsc{Base}\ \frac{}{\Gamma \vdash t \Rightarrow \text{apply } t}$$

Figure 35: Qtac decompiler semantics.

`= n` by reasoning separately about the projections (15). We added informative error messages (22) to help the proof engineer distinguish between user errors and bugs. These features helped with workflow integration.

#### 4.5.2.3 *Decompile*

**Transform** produces a proof term, while the proof engineer typically writes and maintains proof scripts made up of tactics. We improve usability thanks to the realization that, since Coq's proof term language Gallina is very structured, we can decompile these Gallina terms to suggested Ltac proof scripts for the proof engineer to maintain.

**Decompile** implements a prototype of this translation (11): it translates a proof term to a suggested proof script that attempts to prove the same theorem the same way. Note that this problem is not well defined: while there is always a proof script that works (applying the proof term with the `apply` tactic), the result is often qualitatively unreadable. This is the baseline behavior to which the decompiler defaults. The goal of the decompiler is to improve on that baseline as much as possible, or else suggest a proof script that is close enough to correct that the proof engineer can manually massage it into something that works and is maintainable.

**Decompile** achieves this in two passes: The first pass decompiles proof terms to proof scripts that use a predefined set of tactics. The second pass improves on suggested tactics by simplifying arguments, substituting tacticals, and using hints like custom tactics and decision procedures.

**First Pass: Basic Proof Scripts.** The first pass takes Coq terms and produces tactics in Ltac, the proof script language for Coq. Ltac can be confusing to reason about, since Ltac tactics can refer to Gallina terms, and the semantics of Ltac depends both on the semantics of Gallina and on the implementation of proof search procedures written in OCaml. To give a sense of how the first pass works without the clutter of these details, we start by defining a mini decompiler that implements a simplified version of the first pass. Section **??** explains how we scale this to the implementation.

The mini decompiler takes $\text{CIC}_\omega$ terms and produces tactics in a mini version of Ltac which we call Qtac. The syntax for Qtac is in Figure 34. Qtac includes hypothesis introduction (`intro`), rewriting (`rewrite`), symmetry of equality (`symmetry`), application of a term to prove the goal (`apply`), induction (`induction`), case splitting of conjunctions (`split`), constructors of disjunctions (`left` and `right`), and composition (`.`). Unlike in Ltac, `induction` and `rewrite` take a motive explicitly (rather than relying on unification), and `apply` creates a new subgoal for each function argument.

The semantics for the mini decompiler $\Gamma \vdash t \Rightarrow p$ are in Figure 35 (assuming $=$, `eq_sym`, $\wedge$, and $\vee$ are defined as in Coq). As with the real decompiler, the mini decompiler defaults to the proof script that applies the entire proof term with `apply` (Base). Otherwise, it improves on that behavior by recursing over the proof term and constructing a proof script using a predefined set of tactics.

For the mini decompiler, this is straightforward: Lambda terms become introduction (Intro). Applications of `eq_sym` become symmetry of equality (Symmetry). Constructors of conjunction and disjunction map to the respective tactics (Split, Left, and Right). Applications of equality eliminators compose symmetry (to orient the rewrite direction) with rewrites (Rewrite), and all other applications of eliminators become induction (Induction). The remaining applications become apply tactics (Apply). In all cases, the decompiler recurses, breaking into cases, until only the Base case holds.

While the mini decompiler is very simple, only a few small changes are needed to move this to Coq. The generated proof term of `rev_app_distr`

```
fun (y0 :  list A) =>
  list_rect _ _ (fun a l H =>
    eq_ind_r _ eq_refl (app_nil_r (rev l) (a::[])))
    eq_refl
    y0
```

```
- intro y0. induction y0 as [a l H|].
  + simpl.  rewrite app_nil_r. auto.
  + auto.
```

Figure 36: Proof term (top) and decompiled proof script (bottom) for the base case of `rev_app_distr` (Section 4.1), with corresponding terms and tactics grouped by color & number.

from Section 4.1, for example, consists only of induction, rewriting, simplification, and reflexivity (solved by `auto`). Figure 36 shows the proof term for the base case of `rev_app_distr` alongside the proof script that Pumpkin Pi suggests. This script is fairly low-level and close to the proof term, but it is already something that the proof engineer can step through to understand, modify, and maintain. There are few differences from the mini decompiler needed to produce this, for example handling of rewrites in both directions (`eq_ind_r` as opposed to `eq_ind`), simplifying rewrites, and turning applications of `eq_refl` into `reflexivity` or `auto`.

**Second Pass: Better Proof Scripts.** The implementation of **Decompile** first runs something similar to the mini decompiler, then modifies the suggested tactics to produce a more natural proof script (11). For example, it cancels out sequences of `intros` and `revert`, inserts semicolons, and removes extra arguments to `apply` and `rewrite`. It can also take tactics from the proof engineer (like part of the old proof script) as hints, then iteratively replace tactics with those hints, checking for correctness. This makes it possible for suggested scripts to include custom tactics and decision procedures.

**From Qtac to Ltac.** The mini decompiler assumes more predictable versions of `rewrite` and `induction` than those in Coq. **Decompile** includes additional logic to reason about these tactics (11). For example, Qtac assumes that there is only one `rewrite` direction. Ltac has two rewrite directions, and so the decompiler infers the direction from the motive.

Qtac also assumes that both tactics take the inductive motive explicitly, while in Coq, both tactics infer the motive automatically. Consequentially, Coq sometimes fails to infer the correct motive. To handle induction, the decompiler strategically uses `revert` to manipulate the goal so that Coq can better infer the motive. To handle rewrites, it uses `simpl` to refold the goal before rewriting. Neither of these approaches is guaranteed to work, so the proof engineer may sometimes need to tweak the suggested proof script appropriately. Even if we pass Coq's induction principle an explicit motive, Coq still sometimes fails due to unrepresented assumptions. Long term, using another tactic like `change` or `refine` before applying these tactics may help with cases for which Coq cannot infer the correct motive.

**From CIC$_\omega$ to Coq.** Scaling the decompiler to Coq introduces let bindings, which are generated by tactics like `rewrite in`, `apply in`, and `pose`. **Decompile** implements (11) support for `rewrite in` and `apply in` similarly to how it supports `rewrite` and `apply`, except that it ensures that the unmanipulated hypothesis does not occur in the body of the let expression, it swaps the direction of the rewrite, and it recurses into any generated subgoals. In all other cases, it uses `pose`, a catch-all for let bindings.

**Forfeiting Soundness.** While there is a way to always produce a correct proof script, **Decompile** deliberately forfeits soundness to suggest more useful tactics. For example, it may suggest the `induction` tactic, but leave the step of motive inference to the proof engineer. We have found these suggested tactics easier to work with (Section 4.6). Note that in the case the suggested proof script is not quite correct, it is still possible to use the generated proof term directly.

**Pretty Printing.** After decompiling proof terms, **Decompile** pretty prints the result (11). Like the mini decompiler, **Decompile** represents its output using a predefined grammar of Ltac tactics, albeit one that is larger than Qtac, and that also includes tacticals. It maintains the recursive proof structure for formatting. Pumpkin Pi keeps all output terms from **Transform** in the Coq environment in case the decompiler does not succeed. Once the proof engineer has the new proof, she can remove the old one.

## 4.6 RESULTS

This section summarizes eight case studies using Pumpkin Pi, corresponding to the eight rows in Table 1. These case studies highlight Pumpkin Pi's flexibility in handling diverse scenarios, the success of automatic configuration for better workflow integration, the preliminary success of the prototype decompiler, and clear paths to better serving proof engineers. Detailed walkthroughs are in the code.

**Algebraic Ornaments: Lists to Packed Vectors.** The transformation in Pumpkin Pi is a generalization of the transformation from Devoid. Devoid supported proof reuse across *algebraic ornaments*, which describe relations between two inductive types, where one type is the other indexed by a fold [80]. A standard example is the relation between a list and a length-indexed vector (Figure 20).

Pumpkin Pi implements a search procedure for automatic configuration of algebraic ornaments. The result is all functionality from Devoid, plus tactic suggestions. In file (3), we used this to port functions and a proof from lists to vectors of *some* length, since `list T` $\simeq$

| Class | Config. | Examples | Sav. | Repair Tools |
|---|---|---|---|---|
| Algebraic Ornaments | Auto | List to Packed Vector, hs-to-coq (3) | ☺ | Pumpkin Pi, Devoid, UP |
| | | List to Packed Vector, Std. Library (16) | ☺ | Pumpkin Pi, Devoid, UP |
| Unpack Sigma Types | Auto | Vector of Particular Length, hs-to-coq (3) | ☺ | Pumpkin Pi, UP |
| Tuples & Records | Auto | Simple Records (13) | ☺ | Pumpkin Pi, UP |
| | | Parameterized Records (17) | ☺ | Pumpkin Pi, UP |
| | | Industrial Use (18) | ☺ | Pumpkin Pi, UP |
| Permute Constructors | Auto | List, Standard Library (1) | ☺ | Pumpkin Pi, UP |
| | | Modifying a PL, REPLica Benchmark (1) | 😐 | Pumpkin Pi, UP |
| | | Large Ambiguous Enum (1) | 😐 | Pumpkin Pi, UP |
| Add new Constructors | Mixed | PL Extension, REPLica Benchmark (19) | ☹ | Pumpkin Pi |
| Factor out Constructors | Manual | External Example (2) | ☺ | Pumpkin Pi, UP |
| Permute Hypotheses | Manual | External Example (20) | ☹ | Pumpkin Pi, UP |
| Change Ind. Structure | Manual | Unary to Binary, Classic Benchmark (5) | 😐 | Pumpkin Pi, Magaud |
| | | Vector to Finite Set, External Example (21) | ☺ | Pumpkin Pi |

Table 1: Some changes using Pumpkin Pi (left to right): class of changes, kind of configuration, examples, whether using Pumpkin Pi saved development time relative to reference manual repairs (☺ if yes, 😐 if comparable, ☹ if no), and Coq tools we know of that support repair along (Repair) or automatic proof of (Search) the equivalence corresponding to each example. Tools considered are Devoid [?], the Univalent Parametricity (UP) white-box transformation [106], and the tool from Magaud & Bertot 2000 [78]. Pumpkin Pi is the only one that suggests tactics. More nuanced comparisons to these and more are in Section 5.

```
Inductive Term : Set :=
| Var : Identifier → Term
| Int : Z → Term
| Eq : Term → Term → Term
| Plus : Term → Term → Term
| Times : Term → Term → Term
| Minus : Term → Term → Term
| Choose : Identifier → Term
    → Term.
```

```
Inductive Term : Set :=
| Var : Identifier → Term
| Bool : Identifier → Term
| Eq : Term → Term → Term
| Int : Z → Term
| Plus : Term → Term → Term
| Times : Term → Term → Term
| Minus : Term → Term → Term
| Choose : Identifier → Term
    → Term.
```

Figure 37: A simple language (left) and the same language with two swapped constructors and an added constructor (right).

`packed_vect T`. The decompiler helped us write proofs in the order of hours that we had found too hard to write by hand, though the suggested tactics did need massaging.

**Unpack Sigma Types: Vectors of Particular Lengths.** In the same file ③, we then ported functions and proofs to vectors of a *particular* length, like `vector T n`. DEVOID had left this step to the proof engineer. We supported this in PUMPKIN Pi by chaining the previous change with an automatic configuration for unpacking sigma types. By composition, this transported proofs across the equivalence from Section **??**.

Two tricks helped with workflow integration for this change: 1) have the search procedure view `vector T n` as `Σ(v : vector T m).n = m` for some `m`, then let PUMPKIN Pi instantiate those equalities via unification heuristics, and 2) generate a custom eliminator for combining list terms with length invariants. The resulting workflow works not just for lists and vectors, but for any algebraic ornament, automating manual effort from DEVOID. The suggested tactics were helpful for writing proofs in the order of hours that we had struggled with manually over the course of days, but only after massaging. More effort is needed to improve tactic suggestions for dependent types.

**Tuples & Records: Industrial Use.** An industrial proof engineer at the company Galois has been using PUMPKIN Pi in proving correct an implementation of the TLS handshake protocol. Galois had been using a custom solver-aided verification language to prove correct C programs, but had found that at times, the constraint solvers got stuck. They had built a compiler that translates their language into Coq's specification language Gallina, that way proof engineers could finish stuck proofs interactively using Coq. However, due to language differences, they had found the generated Gallina programs and specifications difficult to work with.

The proof engineer used PUMPKIN Pi to port the automatically generated functions and specifications to more human-readable functions and specifications, wrote Coq proofs about those functions and specifications, then used PUMPKIN Pi to port those proofs back to proofs about the original functions and specifications. So far, they have used at least three automatic configurations, but they most often used an automatic configuration for porting compiler-produced anonymous tuples to named records, as in file (18). The workflow was a bit nonstandard, so there was little need for tactic suggestions. The proof engineer reported an initial time investment learning how to use PUMPKIN Pi, followed by later returns.

**Permute Constructors: Modifying a Language.** The swapping example from Section 4.1 was inspired by benchmarks from the REPLICA user study of proof engineers [**?**]. A change from one of the benchmarks is in Figure 37. The proof engineer had a simple language represented by an inductive type `Term`, as well as some definitions and proofs about the language. The proof engineer swapped two constructors in the language, and added a new constructor `Bool`.

This case study and the next case study break this change into two parts. In the first part, we used PUMPKIN Pi with automatic configuration to repair functions and proofs about the language after swapping the constructors (1). With a bit of human guidance to choose the permutation from a list of suggestions, PUMPKIN Pi repaired everything, though the original tactics would have also worked, so there was not a difference in development time.

**Add new Constructors: Extending a Language.** We then used PUMPKIN Pi to repair functions after adding the new constructor in Figure 37, separating out the proof obligations for the new constructor from the old terms (19). This change combined manual and automatic configuration. We defined an inductive type `Diff` and (using partial automation) a configuration to port the terms across the equivalence `Old.Term + Diff` $\simeq$ `New.Term`. This resulted in case explosion, but was formulaic, and pointed to a clear path for automation of this class of changes. The repaired functions guaranteed preservation of the behavior of the original functions.

Adding constructors was less simple than swapping. For example, PUMPKIN Pi did not yet save us time over the proof engineer from the user study; fully automating the configuration would have helped significantly. In addition, the repaired terms were (unlike in the swap case) inefficient compared to human-written terms. For now, they make good regression tests for the human-written terms—in the future, we hope to automate the discovery of the more efficient terms, or use the refinement framework CoqEAL [28] to get between proofs of the inefficient and efficient terms.

**Factor out Constructors: External Example.** The change from Figure 19 came at the request of a non-author. We supported this using a manual configuration that described which constructor to map to `true` and which constructor to map to `false` (2). The configuration was very simple for us to write, and the repaired tactics were immediately useful. The development time savings were on the order of minutes for a small proof development. Since most of the modest development time went into writing the configuration, we expect time savings would increase for a larger development.

**Permute Hypotheses: External Example.** The change in (20) came at the request of a different non-author (a cubical type theory expert), and shows how to use Pumpkin Pi to swap two hypotheses of a type, since $\texttt{T1} \rightarrow \texttt{T2} \rightarrow \texttt{T3} \simeq \texttt{T2} \rightarrow \texttt{T1} \rightarrow \texttt{T3}$. This configuration was manual. Since neither type was inductive, this change used the generic construction for any equivalence. This worked well, but necessitated some manual annotation due to the lack of custom unification heuristics for manual configuration, and so did not yet save development time, and likely still would not have had the proof development been larger. Supporting custom unification heuristics would improve this workflow.

**Change Inductive Structure: Unary to Binary.** In (5), we used Pumpkin Pi to support a classic example of changing inductive structure: updating unary to binary numbers, as in Figure 22. Binary numbers allow for a fast addition function, found in the Coq standard library. In the style of Magaud & Bertot 2000 [78], we used Pumpkin Pi to derive a slow binary addition function that does not refer to `nat`, and to port proofs from unary to slow binary addition. We then showed that the ported theorems hold over fast binary addition.

The configuration for `N` used definitions from the Coq standard library for `DepConstr` and `DepElim` that had the desired behavior with no changes. `Iota` over the successor case was a rewrite by a lemma from the standard library that reduced the successor case of the eliminator that we used for `DepElim`:

```
N.peano_rect_succ : ∀ P pO pS n,
  N.peano_rect P pO pS (N.succ n) =
  pS n (N.peano_rect P pO pS n).
```

The need for nontrivial `Iota` comes from the fact that `N` and `nat` have different inductive structures. By writing a manual configuration with this `Iota`, it was possible for us to implement this transformation that had been its own tool.

While porting addition from `nat` to `N` was automatic after configuring Pumpkin Pi, porting proofs about addition took more work. Due to the lack of unification heuristics for manual configuration, we had to annotate the proof term to tell Pumpkin Pi that implicit casts in the inductive cases of proofs were applications of `Iota` over `nat`. These annotations were formulaic, but tricky to write. Unification heuristics would go a long way toward improving the workflow.

After annotating, we obtained automatically repaired proofs about slow binary addition, which we found simple to port to fast binary addition. We hope to automate this last step in the future using CoqEAL. Repaired tactics were partially useful, but failed to understand custom eliminators like `N.peano_rect`, and to generate useful tactics for applications of `Iota`; both of these are clear paths to more useful tactics. The development time for this proof with Pumpkin Pi was comparable to reference manual repairs by external proof engineers. Custom unification heuristics would help bring returns on investment for experts in this use case.

## 4.7 CONCLUSION

Rehashing thesis and how we do it

What we got here beyond what we had in PUMPKIN PATCH, segue into next chapter

# 5

# RELATED WORK

TODO somewhere here or elsewhere (if elsewhere, fix references): talk about what lessons carry over to automated theorem provers, and which lessons carry over to other ITPs, and what work is needed to reach those tools.

## 5.1 PROGRAMS

### *Program Refactoring*

Refactoring [84].

### *Program Repair*

Proof repair can be viewed as a form of *program repair* [89, 50] for proof assistants. Proof assistants like Coq are a good fit for program repair: A recent paper [98] recommends that program repair tools draw on extra information such as specifications or example patches. In Coq, specifications and examples are rich and widely available: specifications thanks to dependent types, and examples thanks to constructivism.

Program repair tools for languages with non-dependent type systems [95, 76, 71, 82, 88] may have applications in the context of a dependently typed language. Similarly, our work may have applications within program repair in these languages: Future applications of our approach may repurpose it to repair programs for functional languages.

### *Ornaments*

Devoid automates discovery of and lifting across algebraic ornaments in a higher-order dependently typed language. In the decade since the discovery of ornaments [80], there have been a number of formalizations and embedded implementations of ornaments [36, 67, 37, 68, 35]. Devoid is the first tool for ornamentation to operate over a non-embedded dependently typed language. It essentially moves the automation-heavy approach of Ornamentation in ML [113], which operates on non-embedded ML code, into the type theory that forms the basis of theorem provers like Coq. In doing so, it takes advantage of the properties of algebraic ornaments [80]. It also introduces the first search algorithm to identify ornaments, which in the past was identified as a "gap" in the literature [68].

### *Programming by Example*

Our approach generalizes an example that the programmer provides. This is similar to programming by example, a subfield of program synthesis [53]. This field addresses different challenges in different logics, but may drive solutions to similar problems in a dependently typed language.

### *Differencing & Incremental Computation*

Existing work in differencing and incremental computation may help improve our semantic differencing component. Type-directed diffing [87] finds differences in algebraic data types. Semantics-based change impact analysis [7] models semantic differences between documents. Differential assertion checking [70] analyzes different versions of a program for relative correctness with respect to a specification. Incremental $\lambda$-calculus [20] introduces a general model for program changes. All of these may be useful for improving semantic differencing.

## 5.2 PROOFS

### *Proof Reuse*

Our approach reimagines the problem of proof reuse in the context of proof automation. While we focus on changes that occur over time, traditional proof reuse techniques can help improve our approach.

Proof reuse for extended inductive types [15] adapts proof obligations to structural changes in inductive types. Later work [90] proposes a method to generate proofs for new constructors. These approaches may be useful when extending the differencing component to handle structural changes. Existing work in theorem reuse and proof generalization [46, 97, 60] abstracts existing proofs for reusability, and may be useful for improving the abstraction component. Our work focuses on the components critical to searching for patches; these complementary approaches can drive improvements to the components.

A few proof reuse tools work by proof term transformation and so can be used for repair. Existing work [60] describes a transformation that generalizes theorems in Isabelle/HOL. PUMPKIN Pi generalizes the transformation from DEVOID [?], which transformed proofs along algebraic ornaments [80]. Magaud & Bertot 2000 [78] implement a proof term transformation between unary and binary numbers. Both of these fit into PUMPKIN Pi configurations, and none suggests tactics in Coq like PUMPKIN Pi does. The expansion algorithm from Magaud & Bertot 2000 [78] may help guide the design of unification heuristics in PUMPKIN Pi.

Existing work in proof reuse focuses on transferring proofs between isomorphisms, either through extending the type system [12] or through an automatic method [79]. This is later generalized and implemented in Isabelle [56] and Coq [?, ?]; later methods can also handle implications.

The widely used Transfer [56] package supports proof reuse in Isabelle/HOL. Transfer works by combining a set of extensible transfer rules with a type inference algorithm. Transfer is not yet suitable for repair, as it necessitates maintaining references to both datatypes. One possible path toward implementing proof repair in Isabelle/HOL may be to reify proof terms using something like Isabelle/HOL-Proofs, apply a transformation based on Transfer, and then (as in REPLICA) decompile those terms to automation that does not apply Transfer or refer to the old datatype in any way.

CoqEAL [28] transforms functions across relations in Coq, and these relations can be more general than PUMPKIN Pi's equivalences. However, while PUMPKIN Pi supports both functions and proofs, CoqEAL supports only simple functions due to the problem that `Iota` addresses. CoqEAL may be most useful to chain with PUMPKIN Pi to get faster functions. Both CoqEAL and recent ornaments work [112] may help with better workflow support for changes that do not correspond to equivalences.

The PUMPKIN Pi transformation implements transport. Transport is realizable as a function given univalence [108]. UP [105] approximates it in Coq, only sometimes relying on functional extensionality. While powerful, neither approach removes references to the old type.

Recent work [106] extends UP with a white-box transformation that may work for repair. This imposes proof obligations on the proof engineer beyond those imposed by REPLICA, and it includes neither search procedures for equivalences nor tactic script generation. It also does not support changes in inductive structure, instead relying on its original black-box functionality; `Iota` solves this in REPLICA. The most fruitful progress may come from combining these tools.

DEVOID identifies and lifts proofs along a specific equivalence similar to that from existing ornaments work [68]. The need to automatically lift functions and proofs across equivalences and other relations is a long-standing challenge for proof engineers [78, 13, 77, 57, 116, 29]. The univalence axiom from Homotopy Type Theory [108] enables transparent transport of proofs; cubical type theory [27] gives univalence a constructive interpretation.

The problem that we solve is fundamentally about proof reuse, which applies software reuse principles to ITPs. There is a wealth of work in proof reuse, from tactic languages [47] and logical frameworks [21], to tools for proof abstraction and generalization [**?**, 61], to domain-specific methodologies [38] and frameworks [39].

Devoid focuses on the specific problem of reuse when adding fully-determined indices to types. Other approaches to this problem include combinators which definitionally reduce to desirable terms [44] in the language Cedille, and automatic generation of conversion functions in Ghostbuster [81] for GADTs in Haskell. Our work focuses on a type theory different from both of these, in which the properties that allow for such combinators in Cedille are not present, and in which dependent types introduce challenges not present in Haskell.

Devoid is not the first tool to combine search with reuse. Optician [86] synthesizes bidirectional string transformations; a similar approach may help extend tooling to handle transformations for low-level data. Pumpkin Patch [**?**] searches the difference in proofs for patches that can be used to repair proofs broken by changes; Devoid uses a similar approach to identify functions that form an equivalence. The resulting tools are complementary: Devoid supports the addition of indices and hypotheses, which Pumpkin Patch does not support; Pumpkin Patch supports changes in values, which Devoid does not support.

*Proof Evolution*

There is a small body of work on change and dependency management for verification, both to evaluate impact of potential changes and maximize reuse [58, 6] and to optimize build performance [22]. These approaches may help isolate changes, which is necessary to identify future benchmarks, integrate with CI systems, and fully support version updates.

*Proof Refactoring*

Proof repair is related to proof refactoring [110]. The proof refactoring tool Levity [16] for Isabelle/HOL has seen large-scale industrial use. Levity focuses on a different task: moving lemmas. Chick [100] and RefactorAgda [111] are proof refactoring tools in a Gallina-like language and in Agda, respectively. These tools support primarily syntactic changes and do not have tactic support.

A few proof refactoring tools operate directly over tactics: POLAR [45] refactors proof scripts in languages based on Isabelle/Isar [109], CoqPIE [101] is an IDE with support for simple refactorings of Ltac scripts, and Tactician [4] is a refactoring tool for switching between tactics and tacticals. This approach is not tractable for more complex changes [100].

### *Proof Design*

Much work focuses on designing proofs to be robust to change, rather than fixing broken proofs. This can take the form of design principles, like using information hiding techniques [114, 65] or any of the structures [**?**, 104, 102] for encoding interfaces in Coq. CertiKOS [52] introduces the idea of a deep specification to ease verification of large systems. Design principles for specific domains (like formal metatheory [8, 40, 41]) can also make verification more tractable. Design and repair are complementary: design requires foresight, while repair can occur retroactively. Repair can help with changes that occur outside of the proof engineer's control, or with changes that are difficult to protect against even with informed design.

Another approach to this is to use heavy proof automation, for example through program-specific proof automation [25] or general-purpose hammers [14, 94, 62, 33]. The degree to which proof engineers rely on automation varies, as seen in the data from a user study [**?**]. Automation-heavy proof engineering styles localize the burden of change to the automation, but can result in terms that are large and slow to type check, and tactics that can be difficult to debug. While these approaches are complementary, more work is needed for REPLICA to better support developments in this style.

### *Proof Automation*

We address a missed opportunity in proof automation for ITP: searching for patches that can fix broken proofs. This is complementary to existing automation techniques. Nonetheless, there is a wealth of work in proof automation that makes proofs more resilient to change. Powerful tactics like `crush` [24] can make proofs more resilient to changes. Hammers like Isabelle's sledgehammer [94] can make proofs agnostic to some low-level changes. Recent work [34] paves the way for a hammer in Coq. Even the most powerful tactics cannot address all changes; our hope is to open more possibilities for automation.

Powerful project-specific tactics [24, 23] can help prevent low-level maintenance tasks. Writing these tactics requires good engineering [51] and domain-specific knowledge, and these tactics still sometimes break in the face of change. A future patching tool may be able to repair tactics; the debugging process for adapting a tactic is not too dissimilar to providing an example to a tool.

Rippling [19] is a technique for automating inductive proofs that uses restricted rewrite rules to guide the inductive hypothesis toward the conclusion; this may guide improvements to the differencing, abstraction, and specialization components. The abstraction and factoring components address specific classes of unification problems; recent developments to higher-order unification [85] may help improve these components. Lean [103] introduces the first congruence closure algorithm for dependent type theory that relies only on the Uniqueness of Identity Proofs (UIP) axiom. While UIP is not fundamental to Coq, it is frequently assumed as an axiom; when it is, it may be tractable to use a similar algorithm to improve the tool.

GALILEO [18] repairs faulty physics theories in the context of a classical higher-order logic (HOL); there is preliminary work extending this style of repair to mathematical proofs. Knowledge-sharing methods [49] can adapt some proofs across different representations of HOL. These complementary approaches may guide extensions to support decidable domains and classical logics.

# 6

# CONCLUSIONS & FUTURE WORK

Reflect on thesis statement and explain how we got it exactly now that you know everything

But I want to spend the resst of this thesis talking about the next era of verification so I can write out a bunch of ideas for students who might want to work with me

## THE NEXT ERA: PROOF ENGINEERING FOR ALL

Future Work from many papers, plus research statement, DARPA thoughts, plus more, but trimmed down a lot

What I want in the long run, how this all fits in, is a world of proof engineering for all. From research statement, three rings (four including experts in the center).

And what we have so far with my thesis is a world where it's easier for experts and a bit easier for practitioners, but there's still a lot left to go building on it.

So here are 12 short future project summaries that reach each of these tiers, building that world. Super please contact me if any of these seem fun to you.

### *Proof Engineering for Experts*

Unifying theme: lateral reach. Some examples:

MORE PROOF ASSISTANTS Thoughts from PUMPKIN Pi on Isabelle/HOL, future work from PUMPKIN PATCH.

MORE CHANGES Version updates, isolating large changes (PUMPKIN PATCH), relations more general than equivalences (PUMPKIN Pi).

MORE STYLES ML for decompiler (PUMPKIN Pi, REPLICA) : more for diverse proof styles (PUMPKIN PATCH). Note that this is a WIP, but sketch out project, challenges, future ideas, expectations, evaluation a bit.

*Proof Engineering for Practitioners*

Unifying theme: usability. Some examples:

AUTOMATION More search procedures for automatic configuration, e-graphs from PUMPKIN Pi, custom unification heuristics.

INTEGRATION IDE & CI integration, HCI for repair.

EVALUATION repair challenge, user studies ideas (PUMPKIN PATCH, REPLICA, panel w/ Benjamin Pierce, QED at large). (maybe look for more ideas, this can be merged with integration if need be).

*Proof Engineering for Software Engineers*

Unifying theme: mixed methods verification, or the 2030 vision from Twitter thread. Some examples:

GRADUAL VERIFICATION A continuum from testing to verification, tools to help with that.

TOOL-ASSISTED PROOF DEVELOPMENT Tool-assisted development to follow good design principles for verificattion (James Wilcox conversation, final REPLICA takeaway).

SPECIFICATION INFERENCE Analysis to infer specs (TA1).

*Proof Engineering for New Domains*

Unifying theme: collaboration, new abstractions for new domains). Some examples:

MACHINE LEARNING Fairification & other ML correctness properties. Some stuff here but more.

CRYPTOGRAPHY Lots of stuff here but not thinking broadly enough. What about cryptographic proof systems? ZK and beyond. Recall email thread.

SOMETHING ELSE Look for more in survey paper, email, DARPA TAs, Twitter. Healthcare perhaps?

# INDEX

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