

Proof Repair across Quotient Type Equivalences

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Proofs in proof assistants like Coq can be brittle, breaking easily in response to changes. To address this, recent work introduced an algorithm and tool in Coq to automatically repair broken proofs in response to changes that correspond to type equivalences. However, many changes remained out of the scope of this algorithm and tool—especially changes in underlying *behavior*. We extend this proof repair algorithm so that it can express certain changes in behavior that were previously out of scope. We focus in particular on equivalences between *quotient types*—types equipped with a relation that describes what it means for any two elements of that type to be equal. Quotient type equivalences can be used to express interesting changes in representations of mathematical structures, as well as changes in the underlying implementations of data structures.

We extend this algorithm and tool to support quotient type equivalences in Coq. Notably, since Coq lacks quotient types entirely, our extensions use Coq’s setoid machinery to represent quotients externally. Specifically, (1) our extension to the algorithm supports new changes corresponding to setoids, and (2) our extension to the tool supports this new class of changes and further automates away some of the new proof obligations. We demonstrate our extensions on proof repair case studies for previously unsupported changes. We also perform manual proof repair in Cubical Agda, a language with a univalent metatheory, which allows us to construct the first ever internal proofs of correctness for proof repair.

Additional Key Words and Phrases: Proof Repair, Cubical Agda, Coq, Quotient Types

1 Introduction

Writing formal proofs in proof assistants like Coq, Agda, Lean, and Isabelle/HOL is a time-intensive task. Even once written, proofs may break in the face of minor changes in the datatypes, programs, and specifications they are about. User study data suggests that this process of writing and rewriting proofs is ubiquitous during proof development [22], and that it can be challenging to deal with even for experts.

Proof repair [19] aims to simplify this process by introducing algorithms and tools that fix formal proofs in response to breaking changes. Given an existing type A and some set of functions and theorems on that type, as well a new type B , proof repair seeks to generate the equivalent functions and theorems defined on the type B . Proof repair further requires that the new functions and proofs on B make no reference to the original functions and proofs on A , which distinguishes it from the more general *proof transfer*.

Prior work introduced PUMPKIN Pi, a Coq plugin for proof repair across changes in datatypes that can be described by type equivalences [21]. In this work, we extend proof repair to support equivalences between *quotient types* (Section 2). Recent work in Cubical Agda showed that certain relations describing changes in behavior can be adjusted to equivalences between quotient types [2].

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One specific example presented in that work uses two of the most basic queue representations. The first, one list queues, enqueue elements to the front of the list and dequeue elements from the back of the list. The second, two list queues, use a pair of lists. Elements are enqueued onto the front of the first list. When dequeuing, if the second list is empty, the first list is reversed onto the second, and then in any case the front element of the second list is removed. These types are not equivalent in any natural way, because multiple two list queues correspond to a single one list queue: for instance, the one list queue 1 corresponds to both $(1, [])$ and $([], \text{rev } 1)$. Angiuli et al. quotient two list queues so that, in the quotient type, $(1, [])$ and $([], \text{rev } 1)$ are in the same equivalence class, which makes the types equivalent. They are then able to perform proof transfer across that equivalence.

We wish to use this same approach to implement *proof repair* instead of *proof transfer*. Unlike Cubical Agda, Coq lacks quotient types entirely, so one cannot use the original PUMPKIN Pi transformation as-is to support this class of changes. To handle this, we represent quotient types externally using Coq’s setoid machinery, and we represent quotient type equivalences as setoid equivalences (Section 2). We then extend the proof transformation to support the newly generated equality proof obligations (Section 3).

By extending PUMPKIN Pi to quotient equivalences, users gain both expressive power and efficiency. Prior to this work, PUMPKIN Pi could capture a similar approach using sigma types instead of setoids. To do this, one would choose a canonical element for each equivalence class in the quotient, and then use the subtype of these elements instead of a quotient or setoid. However, that approach has severe drawbacks. In the two list queue example above, when using both quotient and setoids representations, the multiple representatives of each class allow for an amortized constant time dequeue operation. The subtype representation instead only allows for a linear time dequeue operation.

We implement the extended algorithm by way of an extension to the implementation of PUMPKIN Pi (Section 4). Our implementation includes new automation, both to support repair across this class of changes and to automate away some of the newly generated proof obligations corresponding to this class of changes. We demonstrate our extended implementation on three case studies that cannot be handled by prior proof repair work—two that are mathematical in nature and one that deals with changes in behavior (Section 5). Finally, we define correctness for proof repair and construct internal proofs of correct repair in Cubical Agda (Section 6). Our contributions are:

- (1) an **extension** to the algorithm for proof repair across type equivalences that supports quotient type equivalences represented via setoids,
- (2) an **implementation** of this extension in PUMPKIN Pi,
- (3) new **automation** in this implementation to minimize the proof burden of setoid-specific proof obligations,
- (4) a **demonstration** of the above supporting new use cases by way of three case studies, and
- (5) the first **construction** of internal proofs of correct repair.

Our code is available in supplementary material.

2 Quotients and Setoids

Our goal is to extend PUMPKIN Pi to perform repair between types equivalent up to quotients. To begin this task, we define quotient types:

Definition 2.1. A quotient type A / eqA is defined by a type A and an equivalence relation $\text{eqA} : A \rightarrow A \rightarrow \text{Prop}$. A / eqA has one constructor, $[\cdot] : A \rightarrow A / \text{eqA}$, and its elements are called the *equivalence classes* of the elements of A . A / eqA has the additional property that, for $a, b : A$, $[a] = [b]$ if $\text{eqA } a \ b$ is inhabited. Eliminating an element $q : A / \text{eqA}$ yields the

underlying $a : A$ from which it was constructed. However, the user must prove that the result of the computation is equal for any $a, b : A$ with $[a] = [b]$.

To give a concrete example, consider the quotient of \mathbb{N} by the equivalence relation eqN , where $\text{eqN } n \ m$ is inhabited if n and m have the same parity. The resulting type, denoted by mathematicians as $\mathbb{N}/2$, has two elements, $[0]$ and $[1]$; any other application of $[\cdot]$ produces a term equal to one of these. For $\mathbb{N}/2$, eliminating $[0]$ and eliminating $[2]$ must provably result in equal terms.

Quotient types can be challenging to implement within a language because of the equality property mentioned above. While some languages have native quotient types, many type systems, including Coq, do not support this property without the use of axioms. To circumvent this issue without introducing any axioms, we use setoids.

Definition 2.2. A setoid is the pair (A, eqA) . When the equivalence relation eqA is obvious from context, we may also call A a setoid as shorthand.

Notice that setoids have no special constructors, eliminators, or equality properties. An element of A is said to be an element of the setoid, and users of the setoid should compare elements using eqA instead of native equality. Because of this, rewriting is more difficult in setoids than in quotient types. Rewriting by an equality in a quotient type can be done using the equality eliminator, but rewriting cannot generally be done for arbitrary equivalence relations. To compensate for this, we define the notion of a proper function.

Definition 2.3. For setoids (A, eqA) and (B, eqB) , a function $f : A \rightarrow B$ is *proper* if, for all $a_1, a_2 : A$, $\text{eqA } a_1 \ a_2$ implies that $\text{eqB } (f \ a_1) \ (f \ a_2)$.

Using our example, $\mathbb{N}/2$, $f : \mathbb{N} \rightarrow \mathbb{N}$ is proper if inputs to f of the same parity produce outputs of the same parity. The successor function would be one example of such a proper function. When a term is composed of proper functions, it becomes possible to construct proofs for rewriting by equivalence relations in much the same way we can rewrite by equality. Coq provides automation in its standard library to support constructing these proofs. By using setoids in place of quotients, equivalence relations in place of equality, and proper functions in place of arbitrary functions, we mimic the functionality of quotients using setoids.

Our paper deals with repair across quotient type equivalences. To begin, we define type isomorphisms:

Definition 2.4. Types A and B are isomorphic if there exist functions $f : A \rightarrow B$ and $g : B \rightarrow A$ such that

- $\forall (a : A), g \ (f \ a) = a$
- $\forall (b : B), f \ (g \ b) = b$

Type equivalences are type isomorphisms with an additional adjoint property, which can be derived from any isomorphism. A quotient type equivalence is simply an equivalence between two quotient types. When using setoids to represent quotients, we must instead define a notion of a setoid equivalence:

Definition 2.5. Two setoids (A, eqA) and (B, eqB) are equivalent if there exist functions $f : A \rightarrow B$ and $g : B \rightarrow A$ satisfying the following properties:

- f and g are proper.
- $\forall (a : A), \text{eqA } (g \ (f \ a)) \ a$
- $\forall (b : B), \text{eqB } (f \ (g \ b)) \ b$

The function g is said to be inverse to f .

```

Inductive nat :=
  | 0 : nat
  | S : nat -> nat.

Inductive positive :=
  | x0 : positive -> positive
  | xI : positive -> positive
  | xH : positive.
Inductive N :=
  | N0 : N
  | Npos : positive -> N.

```

Fig. 1. The naturals represented in Coq, taken from the PUMPKIN Pi paper [21], in unary (left) and binary (right). In `positive`, `xH` is one, `x0` is appending a 0 to the right side of the binary representation, and `xI` is appending a 1 instead. Then, `N` is either 0 or a positive binary number.

Two setoids are equivalent if the quotient types they represent would be isomorphic. One example of a nontrivial setoid equivalence between the setoids $(\mathbb{N}, \text{eq}_{\mathbb{N}})$ and $(\text{Bool}, =)$ is given by the function $\text{isEven} : \mathbb{N} \rightarrow \text{Bool}$, sending even numbers to `true` and odd numbers to `false`. This function has inverse $g : \text{Bool} \rightarrow \mathbb{N}$ mapping `true` to 0 and `false` to 1. f is proper, since isEven respects parity, and g is automatically proper because Bool uses equality as its equivalence relation. The other conditions, $\text{isEven} (g \ b) = b$ and $\text{eq}_{\mathbb{N}} (g (\text{isEven } n)) \ n$, are both satisfied, and thus this indeed forms a setoid equivalence.

3 Approach: Proof Term Transformation

To reiterate, our goal is to perform proof repair across quotient type equivalences. Because we are working in Coq, which does not have quotient types, we represent quotients using setoids. Thus, given two types A and B which may either be types using standard equality or setoids using some equivalence relation, we wish to turn terms defined over A into terms defined over B which no longer have any references to A .

To make the proof repair problem more concrete, we consider a specific example from the PUMPKIN Pi paper. In Figure 1, we see two representations of the natural numbers. The first is a unary representation, while the second is a binary representation. Suppose that a user has begun proof development using the unary representation, implementing functions, such as addition, and theorems, such as that 0 is an identity for addition. Then, suppose that the user needs to switch to the binary representation later in development. Proof repair takes the functions and theorems defined over the unary natural numbers and turns them into functions defined over the binary natural numbers with the same behavior. These functions and theorems will make no reference to the unary natural numbers, so that the type can be deleted from the codebase entirely.

We will review how PUMPKIN Pi repairs proofs across type equivalences by directly transforming proof terms across those equivalences (Section 3.1). Then, we extend PUMPKIN Pi's transformation to support setoid equivalences (Section 3.2).

3.1 PUMPKIN Pi's Transformation

The PUMPKIN Pi transformation that we extend operates over terms in the type theory of Coq, the Calculus of Inductive Constructions (CIC_{ω}) [13]. CIC_{ω} extends the Calculus of Constructions [12] with inductive types. The grammar for CIC_{ω} is in Figure 2; the type rules are standard and omitted.

PUMPKIN Pi implements proof repair over terms in CIC_{ω} by transforming proof terms implemented over an old type A to instead be implemented over a new version of that type B . The key insight behind this transformation is that, by Lambek's theorem, any equivalence between types A

$\langle i \rangle \in \mathbb{N}, \langle v \rangle \in \text{Vars}, \langle s \rangle \in \{ \text{Prop}, \text{Set}, \text{Type} \langle i \rangle \}$
 $\langle t \rangle ::= \langle v \rangle \mid \langle s \rangle \mid \Pi (\langle v \rangle : \langle t \rangle) . \langle t \rangle \mid \lambda (\langle v \rangle : \langle t \rangle) . \langle t \rangle \mid \langle t \rangle \langle t \rangle \mid \text{Ind} (\langle v \rangle : \langle t \rangle) \{ \langle t \rangle, \dots, \langle t \rangle \}$
 $\mid \text{Constr} (\langle i \rangle, \langle t \rangle) \mid \text{Elim} (\langle t \rangle, \langle t \rangle) \{ \langle t \rangle, \dots, \langle t \rangle \}$

Fig. 2. The grammar of CIC_ω from PUMPKIN Pi [21], adapted from Timany and Jacobs [28]. The terms here are, in order: variables, sorts, dependent product types, functions, applications, inductive types, constructors, and eliminators.

and B can be decomposed into separate components that talk only about A and only about B [19]. Functions and proofs can be unified with applications of these components, reducing repair to a simple proof term transformation replacing components that talk about A with their counterparts over B [21].

PUMPKIN Pi calls each such decomposed equivalence a *configuration*, comprising pairs of the form:

$((\text{DepConstr}, \text{DepElim}), (\iota, \eta))$

for types on both sides of the equivalence.¹ DepConstr and DepElim are, respectively, constructors and eliminators for each type, termed *dependent constructors* and *dependent eliminators* by PUMPKIN Pi. The constructors must generate the elements of the type, and the eliminator must specify how to consume an element produced by the constructors. These constructors and eliminators must take the *same shape*, even if A and B themselves have different shapes.

For the type of unary naturals in Figure 1, we can choose depConstr for nat to be 0 and S, the constructors for nat . However, to repair to the binary naturals, we need to provide dependent constructors for \mathbb{N} . These constructors *must correspond across the equivalence*, so we arrive at constructors with these types:

Definition $\text{depConstrNZero} : \mathbb{N}$.

Definition $\text{depConstrNSuc} : \mathbb{N} \rightarrow \mathbb{N}$.

These dependent constructors *do not* share the type signatures of \mathbb{N} 's constructors, but rather are user-defined functions corresponding to nat 's constructors. Likewise, we can choose the eliminator for nat , nat_rect , as the depElim for nat , and the dependent eliminator for \mathbb{N} then takes the shape of nat 's eliminator:

Definition $\text{depElimN} : \forall (P : \mathbb{N} \rightarrow \text{Type}),$
 $(P \text{ depConstrNZero}) \rightarrow$
 $(\forall n : \mathbb{N}, P n \rightarrow P (\text{depConstrNSuc } n)) \rightarrow$
 $\forall n : \mathbb{N}, P n.$

We give a concrete example of repairing addition in Figure 3.

The fact that both dependent eliminators must have the same shape even when the underlying types do not is exactly why we need the remaining element of the configuration: ι . This gives the ι -reduction rules, which specify how to reduce an application of a dependent eliminator to a dependent constructor. When the shape of the underlying type is the same as the shape of the configuration components, as is true for nat , this will be definitional—the proof assistant will

¹While PUMPKIN Pi has an η term, meant to deal with η -expansions of constructors applied to eliminators, it appears to us that this term is not necessary. The example of nontrivial η given in the PUMPKIN Pi paper can be rewritten to instead have nontrivial ι and trivial η , and we are not aware of any cases where this cannot be done. Further, while the type of ι is always clearly defined, what η should be is unclear for many examples. Thus, for the purposes of this paper, η will always be trivial and we will ignore it.

Definition `addNat (a b : nat) :=`
`depElimNat`
`(fun _ => nat -> nat)`
`(fun b => b)`
`(fun a IH b => depConstrNatSuc (IH b))`
`a`
`b.`

Definition `addN (a b : N) :=`
`depElimN`
`(fun _ => N -> N)`
`(fun b => b)`
`(fun a IH b => depConstrNSuc (IH b))`
`a`
`b.`

Fig. 3. Repairing the addition function from unary (left) to binary (right) natural numbers.

$\Gamma \vdash t \uparrow t'$

$$\begin{array}{c}
 \text{DEP-ELIM} \\
 \frac{\Gamma \vdash a \uparrow b \quad \Gamma \vdash p_a \uparrow p_b \quad \Gamma \vdash \vec{f}_a \uparrow \vec{f}_b}{\Gamma \vdash \text{DepElim}(a, p_a) \vec{f}_a \uparrow \text{DepElim}(b, p_b) \vec{f}_b} \\
 \\
 \text{DEP-CONSTR} \\
 \frac{\Gamma \vdash \vec{t}_a \uparrow \vec{t}_b}{\Gamma \vdash \text{DepConstr}(j, A) \vec{t}_a \uparrow \text{DepConstr}(j, B) \vec{t}_b} \\
 \\
 \text{ETA} \qquad \qquad \qquad \text{IOTA} \qquad \qquad \qquad \text{EQUIVALENCE} \\
 \frac{}{\Gamma \vdash \text{Eta}(A) \uparrow \text{Eta}(B)} \quad \frac{\Gamma \vdash q_A \uparrow q_B \quad \Gamma \vdash \vec{t}_A \uparrow \vec{t}_B}{\Gamma \vdash \text{Iota}(j, A, q_A) \vec{t}_A \uparrow \text{Iota}(j, B, q_B) \vec{t}_B} \quad \frac{}{\Gamma \vdash A \uparrow B} \\
 \\
 \text{CONSTR} \qquad \qquad \qquad \text{IND} \qquad \qquad \qquad \text{APP} \\
 \frac{\Gamma \vdash T \uparrow T' \quad \Gamma \vdash \vec{t} \uparrow \vec{t}'}{\Gamma \vdash \text{Constr}(j, T) \vec{t} \uparrow \text{Constr}(j, T') \vec{t}'} \quad \frac{\Gamma \vdash T \uparrow T' \quad \Gamma \vdash \vec{C} \uparrow \vec{C}'}{\Gamma \vdash \text{Ind}(Ty : T) \vec{C} \uparrow \text{Ind}(Ty : T') \vec{C}'} \quad \frac{\Gamma \vdash f \uparrow f' \quad \Gamma \vdash t \uparrow t'}{\Gamma \vdash f t \uparrow f' t'} \\
 \\
 \text{ELIM} \qquad \qquad \qquad \text{LAM} \\
 \frac{\Gamma \vdash c \uparrow c' \quad \Gamma \vdash Q \uparrow Q' \quad \Gamma \vdash \vec{f} \uparrow \vec{f}'}{\Gamma \vdash \text{Elim}(c, Q) \vec{f} \uparrow \text{Elim}(c', Q') \vec{f}'} \quad \frac{\Gamma \vdash t \uparrow t' \quad \Gamma \vdash T \uparrow T' \quad \Gamma, t : T \vdash b \uparrow b'}{\Gamma \vdash \lambda(t : T). b \uparrow \lambda(t' : T'). b'} \\
 \\
 \text{PROD} \qquad \qquad \qquad \text{VAR} \\
 \frac{\Gamma \vdash t \uparrow t' \quad \Gamma \vdash T \uparrow T' \quad \Gamma, t : T \vdash b \uparrow b'}{\Gamma \vdash \Pi(t : T). b \uparrow \Pi(t' : T'). b'} \quad \frac{v \in \text{Vars}}{\Gamma \vdash v \uparrow v}
 \end{array}$$

Fig. 4. Transformation for repair across $A \approx B$ with configuration $((\text{DepConstr}, \text{DepElim}), (\text{Eta}, \text{Iota}))$, from previous work [19]. Our work adapts and extends this transformation.

handle it automatically. However, if the inductive structure has changed, as it has for \mathbb{N} , this will be a propositional equality.

Once we have defined the components of the configuration, we are ready to do repair. First, the functions we wish to repair are converted to explicitly refer to the configuration terms. Sometimes this happens via manual user annotation, and sometimes this happens via custom unification machinery inside of PUMPKIN Pi that does this automatically for some classes of changes (see Section 4.2). Then, we follow the syntactic transformation outlined in Figure 4. We will discuss what it means for the repaired term to be *correct* in Section 6.


```

344                                     Inductive unit_two :=
345 Inductive unit :=
346   | tt.
347                                     | one
348                                     | two.
348 Theorem rewrite_example :
349   forall (x : unit), eq x tt -> eq x tt.
350 Proof.
351   intros x H.
352   rewrite H.
353   reflexivity.
354 Qed.
355
356 fun (x : unit) (H : @eq unit x tt) =>
357   @eq_ind_r unit tt
358   (fun x0 : unit => @eq unit x0 tt)
359   (reflexivity(@eq unit) tt) x H
360
361
362                                     Definition eq_unit_two (u1 u2 : unit_two) :
363                                     Prop := True.
364
365                                     fun (x : unit_two)
366                                     (H : eq_unit_two x one) =>
367                                     [[LiftRewriter
368                                     (unit_two, one,
369                                     (fun x0 : unit_two =>
370                                     eq_unit_two x0 one),
371                                     (reflexivity(eq_unit_two) one), x, H)]]
372                                     : forall x : unit_two,
373                                     eq_unit_two x one ->
374                                     eq_unit_two x one

```

Fig. 6. A demonstration of repairing a rewrite in a proof. The left side is over the source type, which is simply the type with one element. We provide a sample Coq tactic script which performs a rewrite, and below it is the term produced. The right side shows the repaired term over the target setoid, which is the type with two elements which are equal under the equivalence relation.

proof. The rules `LIFTEEMPTY` and `LIFTCONS` describe how the environment is transformed. Then, applications of `@eq_rect` are replaced with the proof produced by the oracle (by `EQREWRITE`). We show an example of repairing such a rewrite in Figure 6.

If \equiv_C is some other equivalence relation, the user can still perform rewrites, except those rewrites behave differently. We denote such a rewrite using `StartRewrite(C, x, y, e, g, t)`. Here, $x, y : C$, $e : \equiv_C x y$, g is the type the user is rewriting, and t is the term the rewrite is applied to. This rewrite replaces every instance of x in the type of t with y . The `SETOIDREWRITE` rule assumes that our oracle `[[−]]` can perform a rewrite using the repaired data on the repaired term, which we denote by `LiftSetoidRewriter(D, x, y, e, g t)`. Then, instances of `StartRewrite(C, x, y, e, g, t)` repair to `[[LiftSetoidRewriter(D, x, y, e, g t)]]` (by `SETOIDREWRITE`).

Importantly, this extension to PUMPKIN Pi can only affect the trusted computing base of the proof system if the rewrite oracle has trust assumptions. Thus, by enforcing that the implementation of the oracle introduces no such assumptions, using this extension poses no risk of a soundness failure beyond that of the base type theory without any extensions.

4 Implementation

We implement this extension to the transformation by extending the PUMPKIN Pi Coq plugin (Section 4.1). Consistently with the original PUMPKIN Pi, our implementation relies on some user annotations (Section 4.2), and places some other restrictions on the format terms can take (Section 4.3). Our implementation includes custom automation to dispatch proper proofs specific to setoids (Section 4.4).

4.1 Extending PUMPKIN Pi

We extend the PUMPKIN Pi Coq plugin directly. Plugins are a method of adding functionality to Coq. They are written in OCaml and can interact directly with the Coq internal codebase. Plugins

can directly transform and produce terms; all terms that plugins produce are checked by Coq's type checker, and so cannot be ill typed.

PUMPKIN Pi has various classes of proof repair transformations across type equivalences for which it has specialized automation. We add an additional class, termed setoid lifting, to support our extended transformation. This class mostly reuses the existing transformation, but implements the new rules from Figure 5. Once the transformation rules were determined, the implementation was made straightforward thanks to PUMPKIN Pi's extensibility; in all, our extension adds 1659 lines of code, 510 of which are dedicated to proper proof generation implemented in `propergen.ml`. The core extensions to the transformation can be found in `lift.ml`, `liftconfig.ml`, and `liftrules.ml` in the supplementary material.

In our extended configuration, the proof that \equiv_C is an equivalence relation takes the form of instances of type classes `Equivalence \equiv_C` . This makes it possible to use Coq's setoid automation to implement the oracle that produces proofs of rewrites described in Section 3.2 with no axioms or other extensions to the trusted computing base. Coq has a tactic, called `setoid_rewrite`, which attempts to perform rewriting by an equivalence relation. However, because equivalence relations are not generally Leibniz, we must *prove* for each function we define that the function is proper, as defined in Section 2, if we wish to rewrite under applications of that function. In Coq, we prove this by creating an instance of the `Proper` type class. The `setoid_rewrite` tactic uses the `Proper` and `Equivalence` type class instances to search for proofs of rewrites, and thus we can use it as our oracle.

4.2 User Annotations

To perform repair, PUMPKIN Pi requires that users annotate their proofs explicitly with components of the configuration. These annotations are required to identify parts of the configuration, thereby decoupling the undecidable part of proof repair (configuration inference) from the decidable part (the proof term transformation itself). We inherit this requirement for our extension, and thus both the original configuration and to our extension to equivalences must be annotated. However, we have implemented automation into PUMPKIN Pi that generates many of the needed annotations for equivalence relations automatically.

Specifically, if no relation is provided for a given type, our extension defaults to strict equality. Then, we can automatically infer annotations corresponding to applications of equality `@eq C`, as well as applications of reflexivity `@eq_refl C`. The same holds for rewrites that fully apply any of Coq's equality eliminators. However, this annotation inference cannot deal with unapplied instances of `@eq` and `@eq_refl` that may later be specialized to `C`, or equality eliminators that are not fully applied. Such terms are considered improperly annotated.

When an equivalence relation is provided for a type, we provide a custom tactic `rewrite_annotate` which the user can use in place of `rewrite` in proofs, which automatically performs annotation while rewriting. This inserts a custom annotation term `START_REWRITE` before applying the rewrite, which our extension looks for to identify and repair rewrites. This constant takes two non-implicit arguments: the proof of equivalence by which the user is rewriting, and the type they are rewriting.

4.3 Restrictions

Like the original PUMPKIN Pi, we repair proof terms directly, and we do not directly repair terms involving pattern matching and recursion. PUMPKIN Pi includes some automation to transform pattern matching and recursion to induction, which is likewise bundled in our extension. We inherit PUMPKIN Pi's proof term decompiler, which makes it possible to get simple tactic proofs from repaired proof terms.

<pre> 442 Theorem depRec (C : Type) 443 (posP : ∀ (n : nat), C) 444 (negSucP : ∀ (n : nat), C) 445 (z : GZ) : C. 446 447 448 </pre>	<pre> Theorem depElimProp (P : GZ -> Prop) `(p : Proper (GZ -> Prop) (eq_GZ ==> iff) P) (posP : ∀ (n : nat), P (depConstrPos n)) (negSucP : ∀ (n : nat), P (depConstrNegSuc n)) (z : GZ) : P z. </pre>
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Fig. 7. The types of the two eliminators we use in one of our case studies. The left has non-dependently typed output, but can eliminate into Type, while the right has dependently typed output but only eliminates into Prop. The right eliminator also requires a proof that the motive is proper as a function from the setoid (GZ, eq_GZ) to the setoid (Prop, iff).

While the original proof repair work had a single `depElim` term on each side of the configuration, we have multiple eliminators. One eliminates into the sort `Type`, but is purely nondependent. The other is dependent, but only eliminates into the sort `Prop`, and requires that the motive of the eliminator be proven to be proper, considering `Prop` as a setoid with `iff`. The types of two of these eliminators for one of our case studies are in Figure 7. We do this because Coq’s setoid automation does not work for a dependently typed notion of an equivalence relation. Thus, we cannot use the setoid automation to perform rewrites on applications of functions we define with a dependently typed eliminator. For our `Prop`-sorted eliminator, this loss means that users cannot automatically perform rewrites on the *proofs* of propositions. In Coq, `Prop` is frequently treated as effectively proof irrelevant, so this loss is more acceptable.

In addition, our use of Coq’s setoid automation is facilitated through the use of Coq’s rewrite tactics. However, these tactics do not allow a rewrite which does not change the goal type, and so neither does our automation for repairing setoid rewrites. Furthermore, the setoid rewrite automation does not allow specifying a motive P . To perform setoid rewrites with a motive, a user trying to prove $P \ x \rightarrow P \ y$ can perform a substitution $P[z/y]$, where z is free in P , and then define $Q := \text{fun } z \Rightarrow P[z/y]$. Then, for another fresh variable w , the user can perform a `setoid_rewrite` to prove $H : \forall (w : B), Q \ w \ x \rightarrow Q \ w \ y$, and recover the desired rewrite proof as $H \ y$. Our extension uses this methodology when repairing rewrites with a motive to setoids.

4.4 Automating Proper Proofs

To perform rewrites on a term using Coq’s setoid automation, it is necessary to prove that the functions in that term are proper. Thus, when repairing terms, it is potentially necessary to have proper proofs for every previously repaired function. We implement automation that helps prove many functions to be proper automatically. Our automation must fail in some cases, though, since proving a general function is proper is undecidable. To prove this, for any proposition $P : \text{Prop}$, define $f : \text{bool} \rightarrow \text{Prop}$, $f \ b = \text{if } b \text{ then True else } P$. Generating a proof that f is proper, using `bool` as a setoid relating true and false, and `Prop` as a setoid with the relation `iff`, is equivalent to proving `iff P True`, which is undecidable.

Presently, our automation constructs proper proofs automatically in two practical cases. First, when f is a composition of proper functions, we introduce hypotheses stating that all arguments are equivalent, and then rewrite by these hypotheses. This is the approach taken by Coq’s `solve_proper` tactic, though we modify it slightly. Coq’s `solve_proper` fails if any of its inputs do not appear in the body of the function because the rewrite for that argument will fail. To avoid this, we use the `try` tactical when rewriting. We also try both Coq’s `rewrite` and `setoid_rewrite` tactics, while `solve_proper` only runs `setoid_rewrite`. While `setoid_rewrite` supports rewriting under binders in some instances, `rewrite` succeeds in some instances where `setoid_rewrite` fails.

```

491 Inductive Z : Set :=
492   | pos : nat -> Z
493   | negsuc : nat -> Z.
494
495 Definition GZ := nat * nat.
496 Definition eq_GZ (z1 z2 : GZ) := match z1, z2 with
497   | (a1, a2), (b1, b2) => a1 + b2 = a2 + b1
498   end.

```

Fig. 8. The types of our integer representations. We provide an instance of Equivalence `eq_GZ` for the case study in the supplementary material.

Second, suppose that f is an application of the eliminator of some inductive type with a constant motive. Then, if all of the inductive case arguments provided are proper functions, we can prove that the eliminator is a proper function from its base cases to its output. To make this concrete, we will consider an example. Let (C, eqC) be a setoid, fix some $P = \text{fun } _ \Rightarrow C$, and consider:

```

nat_rect P : ∀ (po : C) (ps : forall (n : nat) (pn : C), C) (n : nat), C

```

Then, we can automatically prove that:

```

Proper (eq ==> eqC ==> eqC) ps ->
  ∀ (n : nat), Proper (eqC ==> eqC) (fun po => nat_rect P po ps n)

```

by induction on n . The base case holds by assumption via the definition of `Proper`, and the inductive case holds by ps being proper.

This can be done in general for inductive types. Thus, our automation checks if f is an eliminator at the top level, and if so generates and tries to prove such a proper goal. If it succeeds, the automation attempts to show that the inductive cases (in the above example, ps) are proper using the rewriting strategy described above. In our case studies, this is necessary for automatically solving some of the generated proper goals.

5 Case Studies

We use our extended version of PUMPKIN Pi to automatically repair proofs on three case studies that use quotient type equivalences. First, we conduct repair between two representations of the integers (Section 5.1). Second, we study two common implementations of the queue data structure and how we can repair from one to the other (Section 5.2). Third, we repair between dense and sparse representations of polynomials with natural number coefficients (Section 5.3). In each of these case studies, the type to which we repair functions and theorems has a structure which enables efficient implementations of key operations. Our extension makes it possible to take advantage of this structure while using the repaired theorems, something impossible with prior versions of PUMPKIN Pi. All of the case study examples can be found in more detail in the supplementary material.

5.1 Adding, Fast and Slow

Our first case study is a usage of quotients that is foundational in mathematics. We consider a change in the type representing integers from the inductive type found in many standard libraries to the standard quotient based representation. We repair addition and proofs about addition from one representation to the other. Finally, we recover the repaired proofs for a more efficient version of addition over the repaired type. Our Coq implementation of this case study can be found in `grothendieck_int_equivalence_repair_tool.v`.

Types & Configuration. Our first representation, Z , is based on the default implementation of the integers in Cubical Agda: two copies of \mathbb{N} glued back-to-back. We will repair functions and proofs about Z to use a representation that may be more familiar to set theorists: viewing the

```

540 Definition addZ (z1 z2 : Z) : Z :=
541   depRecZ Z
542   (fun (p : nat) => add_posZ z1 p)
543   (fun (p : nat) => add_negsucZ z1 p)
544   z2.

```

```

Definition addGZ (z1 z2 : GZ) : GZ :=
  depRecGZ GZ
  (fun p : nat => add_posGZ z1 p)
  (fun p : nat => add_negsucGZ z1 p)
  z2.

```

Fig. 9. The annotated definition of addition on Z (left, adapted from the Cubical Agda standard library), and the repaired function defined over GZ (right). We omit the definitions of `add_posZ` and `add_negsucZ`, which are also automatically repaired.

```

550 Theorem add0LZ : ∀ z : Z,
551   z = addZ (depConstrZPos 0) z.

```

```

Theorem add0LGZ : ∀ z : GZ,
  eq_GZ z (addGZ (depConstrGZPos 0) z).

```

Fig. 10. An addition identity whose proof we repaired automatically.

integers as elements of $\mathbb{N} \times \mathbb{N} / \sim$, where $(x_1, x_2) \sim (y_1, y_2) \iff x_1 + y_2 = x_2 + y_1$. We call the resulting type GZ^2 , and the equivalence relation `eq_GZ`. The definitions are in Figure 8. Z and GZ are setoid equivalent using the definition from Section 2, by mapping `pos n` to (n, \emptyset) and `negsuc n` to $(\emptyset, S n)$.

We next decompose our isomorphism into a repair configuration consisting of dependent constructors, dependent eliminators, and ι -reduction rules for both types. The configuration differs from those found in the original PUMPKIN Pi examples in that there are two eliminators (Figure 7 from Section 4.3): `depRec` for eliminating into nondependent types, and `depElimProp` for eliminating into dependent types that reside in `Prop`. Furthermore, `depElimProp` on GZ has an extra proof obligation: the motive $P : GZ \rightarrow \text{Prop}$ must be a proper function, where the sort `Prop` is viewed as the setoid $(\text{Prop}, \text{iff})$. While in theory each of these eliminators need their own set of ι -reduction theorems, we provide them solely for `depRec`, since needing ι for `depElimProp` is not common and does not show up in our case study. The full configuration can be found in the supplementary material.

Function Repair. Next, we repair functions automatically using our extension of PUMPKIN Pi. For example, we repair addition from Z (Figure 9, left) to GZ (Figure 9, right) by running the following command:

```
Lift Z GZ in addZ as addGZ.
```

Note that the call to `depRecZ` is directly replaced with one to `depRecGZ`, and the functions `add_posZ` and `add_negsucZ` are replaced with their repaired analogues. We also repair the successor and predecessor functions.

Our extension of PUMPKIN Pi also automatically generates proofs that the repaired functions are proper. First, the user must prove that `depRec` is proper. Our extension uses this to generate proper proofs for functions applying `depRec`. For example, it generates the proof that addition is proper:

```
addGZ_proper : Proper (eq_GZ ==> eq_GZ ==> eq_GZ) addGZ.
```

Proof Repair. We automatically repair the proof `add0LZ`, which shows that 0 is a left identity for addition. Figure 10 shows the old and new theorem types. Note that, in the repaired theorem type, equality has been automatically replaced with an equivalence relation on the type, reflecting that

²In reference to Grothendieck, as this is the Grothendieck group of the natural numbers

```

589 Definition fastAddGZ (a b : GZ) := match b with
590   | (b1, b2) => match a with
591     | (a1, a2) => (a1 + b1, a2 + b2)
592   end
593 end.

```

Fig. 11. Our fast addition function on the repaired integers. The direct use of pattern matching is acceptable because this function is neither to be repaired nor a product of repair.

the repaired theorem is about setoid equality instead of eq. In addition, we repair a proof that 0 is a right identity for addition. The original and repaired proofs can be found in the supplementary material.

For now, there is a bit of extra work related to proper proof generation when our proofs apply `depElimPropGZ` (as `add0LZ` does). In particular, while our automation generates proper proofs for some of the motives passed to `depElimPropGZ`, there is not yet a way to automatically supply those proofs to PUMPKIN Pi so that it uses them when repairing applications of `depElimPropZ`. For now, we define a constant corresponding to the motive, which we then separately repair:

```
Lift Z GZ in add0LMotiveZ as add0LMotiveGZ.
```

The proof that this motive is proper is automatically generated. We then reconfigure PUMPKIN Pi to use these applications of `depElimPropZ` and `depElimPropGZ` for its eliminators:

```

612 Definition appliedDepElimPropZ :=
613   depElimPropZ add0LMotiveZ.
614 Definition appliedDepElimPropGZ :=
615   depElimPropGZ add0LMotiveGZ add0LMotive_proper.

```

We use `appliedDepElimPropZ` in our proof of `add0LZ`, and can repair the term. Presently, an implementation bug only surfacing in this case study forces us to repair the arguments to `appliedDepElimPropZ` before reconfiguring. Future versions can avoid the need for this workaround by automatically supplying the necessary proper proofs to `depElimProp` terms.

Further Steps. As is, our repaired addition function is inefficient. It uses the repaired eliminator for GZ, which inherits the inductive structure of the eliminator for Z. This repaired eliminator is slow, as it internally computes a canonical representative of the equivalence class of the given element.

We adapt our repaired proofs to use the more efficient addition function defined in Figure 11. Consistently with prior work in PUMPKIN Pi, to move between slow and fast implementations we take an ad hoc approach. First, we prove that both implementations of addition are pointwise equivalent in our setoid, producing a term `addEqualFastAdd`. Then, we rewrite across this equality to get from proofs of repaired theorems defined over slow addition to proofs of corresponding theorems defined over fast addition. For example, we use this methodology to translate the proof of `add0LGZ` into the proof of theorem:

```

633 Theorem fastAdd0LGZ : ∀ (z : GZ),
634   eq_GZ z (fastAddGZ (depConstrGZPos 0) z).

```

using only one rewrite by `addEqualFastAdd`.

```

638                                     Definition TLQ := list A * list A.
639                                     Definition insOrder (q : TLQ) := match q with
640                                     | (l1, l2) => l1 ++ rev l2
641                                     end.
642                                     Definition eq_queue (q1 q2 : TLQ) :=
643                                     insOrder q1 = insOrder q2.
644
645 Definition OLQ := list A.

```

Fig. 12. One list queues and two list queues. We provide an instance of Equivalence `eq_queue` in the supplementary material.

```

649 Definition enqueueOLQ (a : A) (q : OLQ) : OLQ :=
650   depConstrOLQInsert a q.
651
652 Definition dequeueHelpOLQ (outer : A)
653   (q : OLQ) (m : option (OLQ * A)) :
654   option (OLQ * A) :=
655   @option_rect
656     (OLQ * A)
657     (fun _ => option (OLQ * A))
658     (fun (p : (OLQ * A)) => Some
659       (depConstrOLQInsert outer (fst p) ,
660        (snd p)))
661     (Some (depConstrOLQEmpty, outer))
662     m.
663
664 Definition dequeueOLQ :
665   OLQ -> option (OLQ * A) :=
666   depRecOLQ (option (OLQ * A)) None
667   dequeueHelpOLQ.
668
669 Definition enqueueTLQ (a : A) (q : TLQ) :
670   TLQ :=
671   depConstrTLQInsert a q.
672
673 Definition dequeueHelpTLQ (outer : A)
674   (q : TLQ) (m : option (TLQ * A)) :
675   option (TLQ * A) :=
676   @option_rect
677     (TLQ * A)
678     (fun _ => option (TLQ * A))
679     (fun (p : (TLQ * A)) => Some
680       (depConstrTLQInsert outer (fst p) ,
681        (snd p)))
682     (Some (depConstrTLQEmpty, outer))
683     m.
684
685 Definition dequeueTLQ :
686   TLQ -> option (TLQ * A) :=
687   depRecTLQ (option (TLQ * A)) None
688   dequeueHelpTLQ.

```

Fig. 13. Definitions for enqueue and dequeue over OLQ on the left, and their repaired versions over TLQ on the right.

5.2 Variations on a Theme of Queues

Next, we repair functions and proofs across a change in implementation of a queue data structure. This is motivated by an example from Angiuli et al. [2], which showed that quotient types can be used to adjust certain relations more general than equivalences into equivalences for use with transport in Cubical Agda. That class of changes was cited in the PUMPKIN Pi paper as an example that could not be expressed naturally in Coq with the original framework. With our extensions to PUMPKIN Pi, we can express this using setoids. Our Coq implementation of this case study can be found in `two_list_queue_equivalence_repair_tool.v`.

Types & Configuration. Our first implementation OLQ represents queues using a single list. Elements enqueue at the front of the list and dequeue from the back of the list. This is simple, but the dequeue operation runs in linear time. Our second implementation TLQ uses a two list representation of queues. Elements enqueue at the front of the first list, and dequeue from the front of the second

```

687
688 Definition returnOrEnqOLQ (a : A)
689   (m : option (OLQ * A)) : (OLQ * A) :=
690   @option_rect
691     (OLQ * A)
692     (fun _ => prod OLQ A)
693     (fun (p : (OLQ * A)) =>
694       (enqueueOLQ a (fst p), snd p))
695     (depConstrOLQEmpty, a)
696     m.
697
698 Theorem dequeueEnqueue (a : A) (q : OLQ) :
699   dequeueOLQ (enqueueOLQ a q)
700   = Some (returnOrEnqOLQ a (dequeueOLQ q)).
701
702
703 Definition returnOrEnqTLQ : (a : A)
704   (m : option (TLQ * A)) : (TLQ * A) :=
705   @option_rect
706     (TLQ * A)
707     (fun _ => prod TLQ A)
708     (fun (p : (TLQ * A)) =>
709       (enqueueTLQ a (fst p), snd p))
710     (depConstrTLQEmpty, a)
711     m.
712
713 Theorem dequeueEnqueue (a : A) (q : TLQ) :
714   eq_deq_ret
715     (dequeueTLQ (enqueueTLQ a q))
716     (Some
717       (returnOrEnqTLQ a (dequeueTLQ q)))
718   = Some
719     (returnOrEnqTLQ a (dequeueTLQ q)).

```

Fig. 14. Main theorem relating dequeue and enqueue, stated for OLQ on the left and TLQ on the right. We repair this theorem's proof from OLQ to TLQ. Here, eq_deq_ret is the equivalence relation eq_queue lifted to the return type of dequeueTLQ.

list, reversing the first list onto the second when the second is empty. This defines an amortized constant time dequeue operation.

Each two list queue (l_1, l_2) corresponds to the queue $l_1 ++ (\text{rev } l_2)$, but multiple two list queues correspond to a single one list queue. Thus, we use the equivalence relation $(l_1, l_2) \sim (l_3, l_4) \iff l_1 ++ (\text{rev } l_2) = l_3 ++ (\text{rev } l_4)$ and consider TLQ as a setoid. These two types are setoid equivalent along the expected correspondence, which lets us define the repair configuration. See Figure 12 for the types, and the supplementary material for the repair configuration.

Function Repair. We now use our extension to repair functions across this change. We provide the standard queue API by repairing enqueueOLQ and dequeueOLQ, as well as the helper functions dequeueHelpOLQ and returnOrEnqOLQ. The first three can be found in Figure 13, and the last in Figure 14.

In the previous case study, our automation succeeded in generating every function's proper proof. This time, however, while the proofs that enqueueTLQ and dequeueTLQ are proper are generated automatically, the proper proofs for dequeueHelpTLQ and returnOrEnqTLQ failed to generate and needed to be supplied manually. Also, we need to define multiple equivalence relations, since the return type of dequeueTLQ is $\text{option } (\text{TLQ} * A)$ and our automation does not yet automatically lift the equivalence over TLQ to types including TLQ. The user must provide these equivalences to PUMPKIN Pi.

Proof Repair. We prove a theorem dequeueEnqueueOLQ, found in Figure 14, stating that enqueue and dequeue commute in the expected way, using returnOrEnq as a helper function. We repair this automatically, with none of the workarounds from the previous case study, since the proof of dequeueEnqueueOLQ is defined using the option_rect rather than depElimProp. We also repair the proof of dequeueEmptyOLQ, providing an algebraic specification for the repaired datatype.

Further Steps. As in the previous case study, we have finished repairing proofs, but our repaired implementation of dequeue is inefficient. Thus, we implement the fast version of dequeue described


```

736 Definition fastDequeueTLQ (q : TLQ) :=
737   let (l1, l2) := q in match l1, l2 with
738   | [], [] => None
739   | h1 :: t1, [] =>
740     Some (([], t1 (rev l1)), hd h1 (rev l1))
741   | _, h2 :: t2 => Some ((l1, t2), h2)
742 end.

```

Fig. 15. Fast dequeue function for two list queues.

```

746 Definition CLPoly := list nat.
747 Definition eq_CLPoly (l1 l2 : CLPoly) :=
748   removeLeadingZeros l1 =
749     removeLeadingZeros l2.
750
751 Definition CEPPoly := list (nat * nat).
752 Definition eq_CEPPoly (p1 p2 : CEPPoly) :=
753   forall (exp : nat),
754     coeff p1 exp = coeff p2 exp.

```

Fig. 16. Definitions and equivalence relations for CLPoly and CEPPoly. `removeLeadingZeros l` removes leading zeros from `l`, while `coeff p exp` is the `nth` degree coefficient of `p`. We provide instances of Equivalence `eq_CLPoly` and Equivalence `eq_CEPPoly` in the supplementary material.

```

756 Definition depConstrCLPoly
757   (l : opaque_list)
758   (p : noLeadingZeros l) : CLPoly.
759 Definition depRecCLPoly (C : Type)
760   (X : forall (l : opaque_list)
761     (p : noLeadingZeros l), C)
762   (p : CLPoly) : C.
763
764 Definition depConstrCEPPoly
765   (l : list nat)
766   (p : noLeadingZeros l) : CEPPoly.
767 Definition depRecCEPPoly (C : Type)
768   (X : forall (l : list nat)
769     (p : noLeadingZeros l), C)
770   (p : CEPPoly) : C.

```

Fig. 17. The types of `depConstr` and `depRec` for CLPoly on the left and CEPPoly on the right, where `opaque_list` is an alias for `list nat`.

earlier, found in Figure 15, and can follow the same methodology as in the previous case study to port repaired proofs to use this fast version. We first show pointwise equality of the functions, and then rewrite inside of our proofs, giving us repaired proofs about the fast dequeue function.

5.3 Polynomial Polynomials

For our third case study, we provide sparse and dense representations of univariate polynomials with natural number coefficients, both of which are represented using setoids. The former representation uses the simplest possible data type to represent a polynomial, while the latter representation will mimic the way polynomials are commonly written, which will utilize far less storage for polynomials of high degree with many terms having a coefficient of 0. We implement and repair addition of polynomials and evaluation of polynomials on a natural number, as well as proofs that addition is commutative and that evaluation respects addition. Our proofs can be found in `polynomial.v`.

Types & Configuration. Our first representation of polynomials, CLPoly (short for coefficient list polynomial), is as lists of natural numbers. The members of the list are the coefficients of the polynomial in order of decreasing degree. For example, $x^2 + 3$ is represented as `[1; 0; 3]`. Two

```

885 Definition addCLPoly (p1 p2 : CLPoly) :=
886   depRecCLPoly CLPoly
887     (fun (l : opaque_list)
888       (p : noLeadingZeros l) =>
889         depRecCLPoly CLPoly
890           (fun (l0 : opaque_list)
891             (p0 : noLeadingZeros l0) =>
892               depConstrCLPoly (addLists l l0)
893                 (addListsNoLeadingZeros l
894                   l0 p p0))
895             p2)
896     p1.

897 Definition evalCLPoly (p : CLPoly)
898   (n : nat) :=
899   depRecCLPoly nat
900     (fun (l : opaque_list)
901       (proof : noLeadingZeros l) =>
902         evalList l n)
903   p.

904
905 Definition addCEPPoly (p1 p2 : CEPPoly) :=
906   depRecCEPPoly CEPPoly
907     (fun (l : opaque_list)
908       (p : noLeadingZeros l) =>
909         depRecCEPPoly CEPPoly
910           (fun (l0 : opaque_list)
911             (p0 : noLeadingZeros l0) =>
912               depConstrCEPPoly (addLists l l0)
913                 (addListsNoLeadingZeros l
914                   l0 p p0))
915             p2)
916     p1.

917 Definition evalCEPPoly (p : CEPPoly)
918   (n : nat) :=
919   depRecCEPPoly nat
920     (fun (l : opaque_list)
921       (proof : noLeadingZeros l) =>
922         evalList l n)
923   p.

```

Fig. 18. The definitions of addition and evaluation for polynomials. The original, over CLPoly, is on the left, and the repaired version over CEPPoly is on the right.

```

924 Theorem addComm :
925   forall (p1 p2 : CLPoly),
926     eq_CLPoly (add p1 p2) (add p2 p1).

927 Theorem evalRespectsAdd :
928   forall (p1 p2 : CLPoly) (n : nat),
929     eval (add p1 p2) n =
930       (eval p1 n) + (eval p2 n).

```

Fig. 19. The types of addComm and evalRespectsAdd for CLPoly. These theorems were repaired to CEPPoly.

CLPolys are equivalent if they are equal after removing leading zeros. Our second representation, CEPPoly (short for coefficient-exponent pair polynomial), is lists of pairs of natural numbers. Each pair (c, exp) represents a monomial in a sum, with c the coefficient and exp the exponent of the monomial, with duplicate exponents allowed. Using the same example, $x^2 + 3$ would be represented as $[(1, 2); (3, 0)]$. Two members of CEPPoly are equivalent if the polynomial they represent has the same coefficients. Both representations are found in Figure 16.

Notice that every equivalence class in either of the setoids uniquely defines exactly one polynomial. Thus, the isomorphism between these setoids maps each member of the class representing polynomial p in one setoid to a member of the class representing polynomial p in the other setoid.

Next, we define a configuration for repair. PUMPKIN Pi's configurations are based on decomposing an equivalence between inductive types, but here, both types are setoids. We choose an inductive type equivalent to both of our setoids, and the configuration components for our setoids have the structure of that type. For this, we choose the type of canonical representatives of CLPoly, the sigma type:

```
{l : list nat | noLeadingZeros l}
```

The types of `depConstr` for both `CLPoly` and `CEPPoly` take the shape of the constructor of this type, as do the types of the eliminators and ι -reduction rules. We show this for part of our configuration for `CLPoly` and `CEPPoly` in Figure 17.

Function Repair. We use `CLPoly` as our source setoid and `CEPPoly` as our target setoid. We define `add` and `eval`, representing addition and evaluation, explicitly annotated with the components of the configuration. We automatically repair these functions using our extension to `PUMPKIN Pi`. The original definitions and the repaired versions can be found in Figure 18. Our extension automatically proves that `eval` is proper, but fails to show `add` is proper, so we do so manually.

Proof Repair. We repair proofs of `addComm`, which states that `add` is commutative, and `evalRespectsAdd`, which states that evaluation distributes over addition. The statements of these theorems can be found in Figure 19. Both use `depElimProp`, so we reconfigure `PUMPKIN Pi` to use specialized versions of `depElimProp`, as we did in the first case study. From there, both proofs repair automatically. Neither of these proofs use rewriting. Thus, to demonstrate lifting setoid rewrites to setoid rewrites, we also repair three simple proofs about polynomials which specifically use setoid rewriting.

Further Steps. The constructor for `CLPoly`, seen in Figure 17, accepts an argument of type `opaque_list`. Because `CLPoly` is internally `list nat`, `PUMPKIN Pi` will attempt to repair *all* instances of `list nat`. However, we define functions over `list nat` which we do not want to repair. Thus, we define an alias `opaque_list` for `list nat` and tell `PUMPKIN Pi` not to repair `opaque_list`, as well as any other functions and theorems that should not be repaired. This behavior and our strategy for dealing with it are inherited from `PUMPKIN Pi`.

6 Correctness

Up to now, we have been imprecise with what it means for repair to be conducted correctly. The original `PUMPKIN Pi` paper [21] outlines a definition for correctness of repair using univalent type theory. We review that definition here, giving the needed background in univalent type theory. Then, we implement this definition in `Cubical Agda`, a proof assistant which has the necessary univalent type theory. This allows us to, for the first time, construct proofs that functions and theorems were correctly repaired.

To implement correctness of repair, we need a notion of heterogeneous equality. The `PUMPKIN Pi` paper uses *dependent path equality* in a univalent type system for this purpose. Univalence states that equivalence is equivalent to equality [3, 8]: any equivalence of types corresponds to a unique proof of equality between those two types. We term these equalities *path equalities*. Given a path equality $p : A \equiv B$ between two types and given elements $a : A, b : B$ of those types, we can construct the type of dependent path equalities between those elements, notated `PathP p a b`. This type is inhabited if and only if the equivalence of types corresponding to p by univalence maps a to b .

The repair transformation is parametrized by an equivalence $f : A \rightarrow B$ between the old and new types. By univalence, this equivalence gives an equality between those types. This equality of types in turn extends to equalities of types built using those types: for instance, if $A \equiv B$, then $A \rightarrow C \equiv B \rightarrow C$. For any repaired term, then, there is an equality between the type of the original term and the type of the repaired term. We obtain this equality inductively in much the same way we do repair inductively. Whenever an inductive type appears in the old term, if that type is A , we use the equality proof corresponding to f . Whenever another inductive type appears, we use `refl` as the equality proof. When the other rules would apply, we have theorems constructing an equality proof from the proofs for the constituent terms. Thus, we can construct the type of

```

883 lamOK: {T} {F}
884   (f: (t: T i0) → F i0 t) (f': (t: T i1) → F i1 t)
885   (b≡b' : ∀ {t : T i0} {t' : T i1}
886     (t≡t' : PathP (λ i → T i) t t') →
887     PathP (λ i → F i (t≡t' i)) (f t) (f' t')) →
888   PathP (λ i → ∀ (t : T i) → F i t) f f'
889 lamOK {T} {F} f f' b≡b' = funExtDep b≡b'

```

Fig. 20. A theorem showing that the LAM rule is correct. Here, i , $i0$, and $i1$ are terms of the interval type, which is a primitive construct in cubical used to define path equalities. The rest is analogous to Figure 4: f is the left function in the rule, f' is the right function, $F\ i0$ is the type of f , $F\ i1$ is the type of f' , and all other subterms have the same names.

dependent path equalities between these terms. If that type is inhabited, we say that the term was repaired correctly.

Thus, to use this approach to construct proofs of correct repair, we need to work in a type system that supports this notion of dependent path equality. Coq does not support univalence without adding axioms to its type system. Instead, we implement this approach using Cubical Agda, which has univalence as a theorem. As a technical detail, we restrict ourselves to working with types which are *h-sets*: that is, types where uniqueness of identity proofs holds. As a bonus, Cubical Agda allows a direct implementation of quotient types via the following higher inductive type:

```

905 data _/_ (A : Type) (R : A → A → Type) : Type
906   [_] : (a : A) → A / R
907   eq/: (a1 a2 : A) → (r: R a1 a2) → [ a1 ] ≡ [ a2 ]
908   squash/ : (x y : A / R) → (p q : x ≡ y) → p ≡ q

```

The first constructor is the constructor for an element of a quotient type as described in Section 2, and the second encodes the equality property for quotient types. The third constructor enforces that quotient types are *h-sets*. We will use this type instead of setoids when specifying correctness of repair in Cubical Agda. As a result, the additional rules to repair equivalence relations from Figure 5 in Section 3 are not needed when conducting repair in Cubical Agda. The transformation from Figure 4 in Section 3 works, assuming the same annotations as in Coq, noting that we cannot reuse any of PUMPKIN Pi's automation in Cubical Agda. There is only one additional restriction: for a quotient type Q , every motive $P : Q \rightarrow \text{Set}$ comes with the requirement that $((x : Q) \rightarrow \text{isSet } (P\ x))$, to ensure that we actually do stay in the *h-set* fragment of Cubical Agda. To demonstrate that our repair methodology still works under this paradigm, we manually followed the transformation to repair the functions and proofs from the first two case studies in Section 5, which can be found in the supplementary materials in files `grothendieck_int_equiv.agda` and `equivalence_queue.agda` respectively.

Then, we go about proving theorems which state that each repair rule repairs terms correctly, given that the inputs to the rule are themselves correctly repaired. Some of these theorems are generic across all types. For example, we internally prove correctness of the LAM rule of the transformation from Figure 4 in Figure 20, which is generic across any repair instance.

Other rules are stated specifying the types being repaired. For example, we proved the repaired eliminator we defined for a simple quotient equivalence was correct. Our source type was \mathbb{N} , and our target was Int / rInt , where $\text{Int} = \mathbb{N} \uplus \mathbb{N}$ and rInt relates $\text{inl } n$ and $\text{inr } n$. The dependent constructors and eliminators correspond to the usual ones for \mathbb{N} . The correctness condition for a repaired eliminator for a given configuration was stated externally in the PUMPKIN Pi paper, but

```

932 elimOK :
933   ∀ (a : ℕ) (b : Int / rInt) (a≡b : PathP (λ i → ℕ≡Int/rInt i) a b) →
934   ∀ (PA : ℕ → Type) (PB : Int / rInt → Type) (PBSet : ∀ b → isSet (PB b)) →
935   ∀ (PA≡PB :
936     ∀ a b (a≡b : PathP (λ i → ℕ≡Int/rInt i) a b) →
937     PathP (λ i → Type) (PA a) (PB b)) →
938   ∀ (PA0 : PA zero) (PB0 : PB depConstrInt/rInt0) →
939   ∀ (PA0≡PB0 : PathP (λ i → PA≡PB zero depConstrInt/rInt0 depConstr0OK i) PA0 PB0) →
940   ∀ (PAS : ∀ a → PA a → PA (suc a)) (PBS : ∀ b → PB b → PB (depConstrInt/rIntS b)) →
941   ∀ (PAS≡PBS :
942     ∀ a b (IHa : PA a) (IHa : PB b) a≡b (IHa≡IHa : PathP (λ i → PA≡PB a b a≡b i) IHa IHa) →
943     PathP (λ i → PA≡PB (suc a) (depConstrInt/rIntS b) (depConstrSOK a b a≡b i)
944       (PAS a IHa)
945       (PBS b IHa)) →
946     PathP (λ i → PA≡PB a b a≡b i)
947     (Nat.elim {A = PA} PA0 PAS a)
948     (depElimSetInt/rInt PB PBSet PB0 PBS b))

```

Fig. 21. The theorem stating the correctness condition for the repaired dependent eliminator for a simple example type, which has been proven internally in Cubical Agda. This theorem shows that, if all the inputs to the eliminator correspond to each other across the isomorphism, then the output of the eliminator applications also corresponds across that isomorphism. Here, `depConstr0OK` and `depConstrSOK` are the correctness proofs of the repaired constructors, also proven internally.

it was not proven for any type. We adapted this theorem to Cubical Agda for our example and, for the first time, proved that it held. The type of the rule showing that the dependent eliminator repairs correctly can be found in Figure 21.

Using *these* rules, we were able to compose the correctness proofs to show the correctness of repaired *functions*, like addition:

```

970 addCorrect : ∀ (a b : ℕ) (a' b' : Int / rInt) →
971   ∀ (pa : PathP (λ i → Nat≡Int/rInt i) a a') (pb : PathP (λ i → Nat≡Int/rInt i) b b') →
972   PathP (λ i → Nat≡Int/rInt i) (add' a b) (addInt/rInt' a' b')

```

We also were able to prove theorems about these functions. For instance, below is the type of a term proving that our proof that addition is commutative on \mathbb{N} , `addCommNat`, was correctly repaired to a proof that addition is commutative on our new type:

```

981 addCommCorrect :
982   (a : ℕ) (a' : Int / rInt) (pa : PathP (λ i → Nat≡Int/rInt i) a a') →
983   (b : ℕ) (b' : Int / rInt) (pb : PathP (λ i → Nat≡Int/rInt i) b b') →
984   PathP (λ i → addCorrect a b a' b' pa pb i ≡ addCorrect b a b' a' pb pa i)
985   (addCommNat a b) (addCommInt/rInt a' b')

```

When constructing proofs of correct repair for theorems, because of difficulties when composing different `PathP`s, our rules for proving correct repair do not always compose directly in their current formulation. As a result, if Cubical Agda had the facilities for building automation, we would not currently have a complete procedure for constructing proofs of correct repair fully automatically. However, we are able to prove these theorems directly in such cases, demonstrating that our repair procedure produces the desired output. All of the rules we prove, as well as the proofs of correct repair in our example, can be found in `equivalence_int_abs.agda` in the supplementary material.

7 Related Work

Proof Repair. This work extends the PUMPKIN Pi [21] proof repair transformation and Coq plugin to support quotient type equivalences, a class of changes previously not supported. PUMPKIN Pi has some more mature automation for other classes of changes, like automatic search for configurations, that we do not yet extend to work for quotient type equivalences. Proof repair was first introduced in parallel by Ringer et al. [23] and Robert [24], with strong influence from program repair [18]. SISYPHUS [15] is a recent proof repair tool that, like our work, can handle changes in behavior (using a mix of dynamic and static techniques). However, SISYPHUS repairs proofs of imperative OCaml programs verified in Coq using an embedded separation logic, whereas our work repairs proofs that are written in Coq directly.

Univalent Foundations. Parts of this project are grounded in Cubical Agda, and parts assume a univalent metatheory. Cubical Agda is an implementation of cubical type theory [30]. Cubical type theory [3, 8, 11] was developed to give a constructive account of the univalence axiom. When working in Cubical Agda, we are able to state and prove internal correctness of parts of our repair transformation and have a computational interpretation of functional extensionality. Cubical type theory itself is a derivative of Voevodsky's homotopy type theory [29], which presents the univalence axiom non-constructively. Homotopy type theory has additionally been implemented in Coq as the HoTT library [5].

Proof Reuse and Transfer. Proof repair is an instance of proof reuse, which seeks to use existing proofs in new goals. Other work in proof reuse includes CoqEAL [10] which uses refinement relations to verify properties of efficient functions using proofs on functions that are easy to reason about. CoqEAL can handle relations more general than equivalences, but does not include support for porting proofs across those changes. In Isabelle/HOL, the Transfer package [16] uses automation to transfer proofs between types. Both approaches require the source and target type to remain in the codebase, unlike proof repair. A complementary approach is to design proofs to be more reusable or more robust to changes from the start [7, 14, 32]. More work on proof reuse can be found in the QED at Large [20] survey of proof engineering.

Work has been done to implement transfer tools in Coq that approximate or externally implement automation corresponding to univalent transport. Tabareau et al. [26] defines univalent parametricity, which allows transport of a restricted class of functions and theorems. Univalent parametricity implements an ad hoc form of transport that only sometimes requires functional extensionality, and in many cases is axiom-free. It also includes a form of type-directed search to transport terms by way of type classes, something that proof repair tools like PUMPKIN Pi and our extension still lack. Subsequent work introduces a white-box transformation [27] similar to the

repair transformation from PUMPKIN Pi, which Ringer [19] describes as developed in parallel with mutual influence. None of these support quotient type equivalences like our work does, though it is possible that by leaning further on functional extensionality, one could use these tools with quotient type equivalences.

More recent work called, Trocq [9] implements external transfer for Coq that directly supports relations more general than equivalences, like CoqEAL, but also supports proofs. Like PUMPKIN Pi, Trocq goes out of its way to avoid depending on axioms like univalence and functional extensionality. Trocq’s motivation of supporting transfer of proofs across relations more general than equivalences is similar to the motivation of our extensions to PUMPKIN Pi, with two differences: (1) our work supports a more limited class of relations that can be described as equivalences between quotient types, and (2) for that class of changes, by extending PUMPKIN Pi’s proof term transformation, our work makes it possible to remove the old version of a type after applying repair. The major benefit of our tool relative to Trocq comes from (2)—while all proof repair tools implement a kind of transfer, not all transfer methods implement repair, and Trocq does not implement repair.

Quotients and Equivalences. Our work uses quotient types to expand the scope of proof repair. Quotient types exist in other proofs assistants besides Cubical Agda, like Isabelle/HOL [17, 31], as well as Lean [4] by way of axioms. Bortin and L  th [6] use quotient types to construct theories in Isabelle, like multisets and finite sets as quotients of lists. Coq does not have quotient types, but it does have setoids [25], which do not explicitly form equivalence classes like quotients do. Setoid type theory uses a setoid model to justify the axioms needed to represent quotient types [1]. We draw on internal quotient types for our grounding in Cubical Agda, and we draw on setoids for our work in Coq.

Our idea for extending proof repair using quotient type equivalences to begin with comes from Angiuli et al. [2], which shows that certain relations more general than equivalences can be represented this way. The first example present in that paper is the queue example which we have also studied in our work. Because that work uses transport, it requires the user to keep both versions of the type in their codebase. We avoid that problem, but also have to reason more closely about the inductive structure of our types. In doing so, we extend proof repair to support a new class of changes described as missing from the original PUMPKIN Pi work [21].

8 Conclusions & Future Work

We extended PUMPKIN Pi to support changes represented by quotient type equivalences, enabling repair in situations previously untenable. The key challenge we overcame was supporting quotient types in a proof repair algorithm built for a type theory that does not have quotient types to begin with. We addressed this by representing quotient types using setoids, extending the PUMPKIN Pi algorithm and implementation to repair proofs about equivalence relations, and adding new automation to dispatch newly generated proof obligations. Our extension demonstrated success on three case studies not supported by the original PUMPKIN Pi. We also constructed the first internal correctness proofs for repair. We wish to continue to improve our extension’s automation and usability, and we hope to look at other kinds of types and relations that can be expressed even when the type theory lacks them, as quotient types can be by way of setoids. We hope this will open the door to proof repair for more sophisticated classes of changes.

9 Data-Availabilty Statement

This paper comes with an artifact containing the extension to PUMPKIN Pi we wrote, as well as all the case studies described within the paper. The artifact is functional, and all the case studies compile with no unsolved goals or added axioms. The artifact can be found in the supplementary

material uploaded with the submission, and we intend to submit the artifact for Artifact Evaluation. Instructions for using the artifact to run the case studies can be found in `README.quotients.md`.

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