HW #1 - Math Stats 2 - Tommy Stell

1.) $f(x) = \begin{cases} 2xe^{-x^2} & x>0 \\ 0 & \text{else} \end{cases}$ & $Y = X^2$

a.) find dist. fxn. of Y $\qquad F_Y(y) = F_X(g^{-1}(y)) \quad y = g(x)$

$y = g(x) = x^2 \rightarrow g^{-1}(y) = \sqrt{y}$

$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq y^{1/2}) = \int_{-\infty}^{y^{1/2}} f_X(x)\,dx$

$F_Y(y) = \int_0^{y^{1/2}} 2xe^{-x^2}\,dx \qquad u = -x^2 \quad du = -2x\,dx$

$F_Y(y) = -\int_0^{-y} e^u\,du = -e^u \Big|_0^{-y} = -e^{-y} - -e^0$

$F_Y(y) = \begin{cases} 1-e^{-y} & y>0 \\ 0 & \text{else} \end{cases}$

b.) $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(1-e^{-y})$

$f_Y(y) = \begin{cases} e^{-y} & y>0 \\ 0 & \text{else} \end{cases}$

2.) $f_{X_i}(x_i) = \begin{cases} \frac{1}{\theta} e^{-x_i/\theta} & x_i > 0 \\ 0 & \text{else} \end{cases}$, $i \in \{1, 2\}$,, iid

show if $\theta_i = 1$ then $Z = \frac{X_1}{X_1 + X_2}$ has $u(z; 0, 1)$

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2) = \begin{cases} e^{-x_1} e^{-x_2} & x_1, x_2 > 0 \\ 0 & \text{else} \end{cases}$$

$$f_{X_1,X_2}(x_1, x_2) = e^{-(x_1 + x_2)}$$

$$f_Z(z) = \int_0^\infty f_{X_1,X_2}(x_1, x_2) \left| \frac{\partial x_2}{\partial z} \right| dx_1 \qquad x_2 = x_1\left(\frac{1-z}{z}\right) = \frac{x_1}{z} - x_1$$

$$\frac{\partial x_2}{\partial z} = -\frac{x_1}{z^2}$$

$$f_Z(z) = \int_0^\infty e^{-x_1/z} \left| -\frac{x_1}{z^2} \right| dx_1$$

$$f_Z(z) = \int_0^\infty \frac{1}{z} \frac{x_1}{z} e^{-x_1/z}\, dx_1 \qquad u = \frac{x_1}{z} \qquad du = \frac{1}{z} dx_1$$

$$f_Z(z) = \int_0^\infty u e^{-u}\, du$$ which by def. $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$

$$g_Z(z) = \Gamma(2) = 1$$

so $g_Z(z) = u(z; 0, 1)$ for $0 < z < 1$

3.) if $f_X(x) = b(x; 3, \frac{1}{3}) = \binom{3}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{3-x}$ $x = 0, 1, 2, 3$

x	0	1	2	3
$f_X(x)$	$8/27$	$12/27$	$6/27$	$1/27$

a.) $Y = \frac{X}{1+X}$

y	0	$1/2$	$2/3$	$3/4$
$f_Y(y)$	$8/27$	$12/27$	$6/27$	$1/27$

b.) $u = (1-x)^4$

u	1	0	1	2
$f_u(u)$	$8/27$	$12/27$	$6/27$	$1/27$

$\Downarrow$

u	0	1	2
$f_u(u)$	$12/27$	$14/27$	$1/27$

4.) $f_X(x) = u(x; 0, 1)$ show that $y = -2\ln(x)$ has Γ-dist., find parameters

$$f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

for $y = g(x) = -2\ln(x) = \ln(x^{-2})$

$x = e^{-y/2}$

$x' = -\frac{1}{2}e^{-y/2}$

$$f_Y(y) = f_X(e^{-y/2})\left(\tfrac{1}{2}e^{-y/2}\right)$$

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y>0 \\ 0 & \text{else} \end{cases}$$

$$F_Y(y) = \int_0^\infty \tfrac{1}{2}e^{-y/2}\,dy$$

$$G(x;\alpha,\beta) = \begin{cases} \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\,dx \\ 0 & \text{else} \end{cases}$$

set $\alpha = 1$, $\beta = 2$

$$G(x;1,2) = \begin{cases} \int_0^\infty \frac{1}{2}x^0 e^{-x/2}\,dx & x>0 \\ 0 & \text{else} \end{cases}$$

So $F_Y(y) = G(y; 1, 2)$

5.) $f_{X_1,X_2}(x_1,x_2) = \begin{cases} 4x_1x_2 & 0<x_1,x_2<1 \\ 0 & \text{else} \end{cases}$

find jpdf of $Y_1 = X_1^2 \quad Y_2 = X_1X_2$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(w_1(y_1,y_2), w_2(y_1,y_2)) \cdot \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$x_1 = \sqrt{y_1} \quad y_2 = \sqrt{y_1}\, x_2 \rightarrow x_2 = \frac{y_2}{\sqrt{y_1}}$

$$J = \begin{vmatrix} \frac{1}{2}\frac{1}{\sqrt{y_1}} & 0 \\ -\frac{1}{2}\frac{y_2}{y_1^{3/2}} & \frac{1}{\sqrt{y_1}} \end{vmatrix} = \frac{1}{2y_1}$$

$$f_{Y_1,Y_2}(y_1,y_2) = 4(\sqrt{y_1})\left(\frac{y_2}{\sqrt{y_1}}\right)\left(\frac{1}{2y_1}\right)$$

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{2y_2}{y_1} & 0<y_2^2<y_1<1 \\ 0 & \text{else} \end{cases}$$

6.) $X, Y \Rightarrow$ identical Γ-dist. which are iid

a.) find jpdf of $u = \frac{x}{x+y}$ & $v = x+y$

$$x = uv \qquad g(x;\alpha,\beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad x>0 \quad 0 \text{ else}$$
$$y = v(1-u)$$

$$f_{x,y}(x,y) = \underbrace{\left(\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\right)^2}_{K} x^{\alpha-1} y^{\alpha-1} e^{-x/\beta} e^{-y/\beta} = K(xy)^{\alpha-1} e^{-\left(\frac{x+y}{\beta}\right)}$$

$$f_{u,v}(u,v) = f_{x,y}(w_x(u,v), w_y(u,v))\,|J|$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = |v - vu + vu| = |v|$$

$$f_{u,v}(u,v) = K(uv^2(1-u))^{\alpha-1} e^{-v/\beta} \quad \begin{matrix} 0<u<1 \\ 0<v \end{matrix} \quad 0 \text{ else}$$

b.) $h(u) = \int_{-\infty}^{\infty} f_{u,v}(u,v)\,dv = K(u(1-u))^{\alpha-1} \int_0^{\infty} v^{2\alpha-1} e^{-v/\beta}\,dv$

$$= K(u(1-u))^{\alpha-1} \beta^{2\alpha} \int_0^{\infty} \frac{1}{\beta^{2\alpha}} v^{2\alpha-1} e^{-v/\beta}\,dv = K(u(1-u))^{\alpha-1} \beta^{2\alpha}\Gamma(2\alpha)$$

$$= \frac{1}{\beta^{2\alpha}\Gamma(\alpha)} \beta^{2\alpha}\Gamma(2\alpha)\, u^{\alpha-1}(1-u)^{\alpha-1}$$

$$= \frac{\Gamma(2\alpha)\, u^{\alpha-1}(1-u)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha)}$$

$h(u) = \beta(u;\alpha,\alpha) \rightarrow$ Beta-dist w/ $\beta = \alpha$