Optimal Control Theory

Variational Approach to Optimal Control Problems

Necessary Condition for Optimal Control

- Optimal control problem
  - System

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$\mathbf{x}$: $n \times 1$ state vector $\quad \mathbf{x}(t_0) = \mathbf{x}_0$
$\mathbf{u}$: $m \times 1$ control input vector

  - Performance measure

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

- Functional

$$h(\mathbf{x}(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}[h(\mathbf{x}(t), t)]\, dt + h(\mathbf{x}(t_0), t_0)$$

minimization does not affect

$$\Rightarrow \quad J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \underline{\frac{d}{dt}[h(\mathbf{x}(t), t)]} \right\} dt + \cancel{h(\mathbf{x}(t_0), t_0)}.$$

$$\Rightarrow \quad J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \underline{\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t)} \right\} dt.$$

chain rule

- Augmented Functional

$$J_a(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)] \right\} dt$$

$$= \int_{t_0}^{t_f} \{ g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \}\, dt$$

Necessary Condition for Optimal Control

- Variation of augmented functional

$$\delta J_a(\mathbf{u}^*) = 0 \longrightarrow$$

$$\begin{aligned}
\delta J_a(\mathbf{u}^*) = 0 = &\left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right]^T \delta \mathbf{x}_f \\
&+ \Big[g_a(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) \\
&- \left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\Big] \delta t_f \\
&+ \int_{t_0}^{t_f} \Big\{\Big[\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \\
&- \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T\Big] \delta \mathbf{x}(t) \\
&+ \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta \mathbf{u}(t) \\
&+ \left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta \mathbf{p}(t)\Big\}\, dt.
\end{aligned}$$

- Necessary condition for optimal control

$$\boxed{\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)]}$$

Hamiltonian

$$\boxed{\left.\begin{aligned}
\dot{\mathbf{x}}^*(t) &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
\dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
0 &= \frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)
\end{aligned}\right\} \begin{array}{l}\text{for all} \\ t \in [t_0, t_f]\end{array}}$$

$$\boxed{\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0}$$

boundary condition

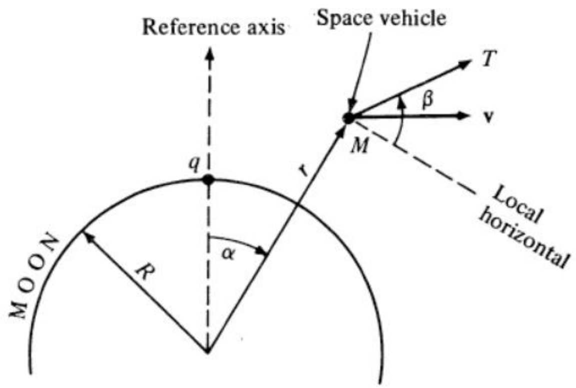
Necessary Condition for Optimal Control

- Boundary conditions

Problem	*Description*	*Substitution in Eq. (5.1-18)*	*Boundary-condition equations*	*Remarks*
t_f fixed	1. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$	$(2n + k)$ equations to determine the $2n$ constants of integration and the variables $d_1, \ldots, d_k$
t_f free	4. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	6. $\mathbf{x}(t_f)$ on the moving point $\boldsymbol{\theta}(t)$	$\delta\mathbf{x}_f = \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]\delta t_f$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $+ \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right] = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	7. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f
	8. $\mathbf{x}(t_f)$ on the moving surface $\mathbf{m}(\mathbf{x}(t), t) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right]$ $\mathbf{m}(\mathbf{x}^*(t_f), t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $= -\sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f.

Necessary Condition for Optimal Control

(Ex) space vehicle control



$x_1 \triangleq r, x_2 \triangleq \alpha, x_3 \triangleq \dot{r}$, and $x_4 \triangleq r\dot{\alpha}$

$u \triangleq \beta$

T: thrust magnitude
β: thrust direction

(state equation)

$$\dot{x}_1(t) = x_3(t)$$
$$\dot{x}_2(t) = \frac{x_4(t)}{x_1(t)}$$
$$\dot{x}_3(t) = \frac{x_4^2(t)}{x_1(t)} - \frac{g_0 R^2}{x_1^2(t)} + \left[\frac{T}{M}\right] \sin u(t)$$
$$\dot{x}_4(t) = -\frac{x_3(t)x_4(t)}{x_1(t)} + \left[\frac{T}{M}\right] \cos u(t).$$

(performance)

$$J(u) = \int_0^{t_f} dt$$

(Hamiltonian)

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)]$$
$$= 1 + p_1(t)x_3(t) + \frac{p_2(t)x_4(t)}{x_1(t)} + p_3(t)\left[\frac{x_4^2(t)}{x_1(t)} - \frac{g_0R^2}{x_1^2(t)} + \left[\frac{T}{M}\right]\sin u(t)\right] + p_4(t)\left[\frac{-x_3(t)x_4(t)}{x_1(t)} + \left[\frac{T}{M}\right]\cos u(t)\right].$$

(costate equation)

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \quad \Leftrightarrow$$

$$\dot{p}_1^*(t) = -\frac{\partial \mathscr{H}}{\partial x_1} = \frac{p_2^*(t)x_4^*(t)}{x_1^{*2}(t)} + p_3^*(t)\left[\frac{x_4^{*2}(t)}{x_1^{*2}(t)} - \frac{2g_0R^2}{x_1^{*3}(t)}\right] - \frac{p_4^*(t)x_3^*(t)x_4^*(t)}{x_1^{*2}(t)}$$
$$\dot{p}_2^*(t) = -\frac{\partial \mathscr{H}}{\partial x_2} = 0$$
$$\dot{p}_3^*(t) = -\frac{\partial \mathscr{H}}{\partial x_3} = -p_1^*(t) + \frac{p_4^*(t)x_4^*(t)}{x_1^*(t)}$$
$$\dot{p}_4^*(t) = -\frac{\partial \mathscr{H}}{\partial x_4} = -\frac{p_2^*(t)}{x_1^*(t)} - \frac{2p_3^*(t)x_4^*(t)}{x_1^*(t)} + \frac{p_4^*(t)x_3^*(t)}{x_1^*(t)}.$$

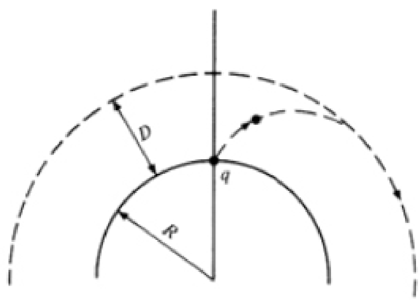
$$0 = \frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \quad \Leftrightarrow \quad 0 = \frac{\partial \mathscr{H}}{\partial u} = \left[\frac{T}{M}\right][p_3^*(t)\cos u^*(t) - p_4^*(t)\sin u^*(t)]$$

Necessary Condition for Optimal Control

(Ex) space vehicle control (continued)

$\Rightarrow$ $\boxed{u^*(t) = \tan^{-1}\frac{p_3^*(t)}{p_4^*(t)}}$ or $\sin u^*(t) = \frac{p_3^*(t)}{\sqrt{p_3^{*2}(t) + p_4^{*2}(t)}}$ $\cos u^*(t) = \frac{p_4^*(t)}{\sqrt{p_3^{*2}(t) + p_4^{*2}(t)}}$.

(boundary condition)



Orbit injection

t_f: free, $x_2(t_f)$: free

$\mathbf{x}(0) = [R\ 0\ 0\ 0]^T$

$x_1^*(t_f) = R + D$

$x_3^*(t_f) = 0$

$x_4^*(t_f) = \sqrt{\frac{g_0 R^2}{[R + D]}}$ $\leftarrow$ $M[r^*(t)\dot{\alpha}^*(t)]^2/r^*(t) = Mg_0R^2/r^{*2}(t)$ for $t \geq t_f$.

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]\delta t_f = 0$$

$\Rightarrow$

$$p_2^*(t_f) = 0$$

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f)) = 0$$

Necessary Condition for Optimal Control

(Q)

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_1(t) + [1 - x_1^2(t)]x_2(t) + u(t)$$

$$J = \int_0^1 \tfrac{1}{2}[2x_1^2(t) + x_2^2(t) + u^2(t)]\, dt.$$

The initial and final state values are specified.

Determine the costate equations for the system.

Linear Regulator Problems

- System model

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t);$$

- Performance measure

$$J = \tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{H}\mathbf{x}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)]\, dt;$$

$\mathbf{H}, \mathbf{Q}$: real symmetric positive semi-definite matrix
$\mathbf{R}$: real symmetric positive definite matrix

- Boundary condition
  - t_f: fixed, $\mathbf{x}(t_f)$: free

$$\mathbf{p}^*(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\boldsymbol{x}^*(t_f)) \Rightarrow \boxed{\mathbf{p}^*(t_f) = \mathbf{H}\,\mathbf{x}^*(t_f)}$$

- Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)] \Rightarrow$$

$$\boxed{\begin{aligned}\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = {} & \tfrac{1}{2}\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \tfrac{1}{2}\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t) \\ & + \mathbf{p}^T(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^T(t)\mathbf{B}(t)\mathbf{u}(t),\end{aligned}}$$

- Necessary condition

$$\left.\begin{aligned}\dot{\mathbf{x}}^*(t) &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \mathbf{0} &= \frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\end{aligned}\right\} \begin{array}{l}\text{for all} \\ t \in [t_0, t_f]\end{array} \Rightarrow$$

$$\boxed{\begin{aligned}\dot{\mathbf{x}}^*(t) &= \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\mathbf{p}^*(t) \\ \mathbf{0} &= \frac{\partial \mathscr{H}}{\partial \mathbf{u}} = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^T(t)\mathbf{p}^*(t).\end{aligned}}$$

Linear Regulator Problems

- Necessary condition (continued)

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial \mathscr{H}}{\partial \mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\mathbf{p}^*(t)$$

$$\mathbf{0} = \frac{\partial \mathscr{H}}{\partial \mathbf{u}} = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^T(t)\mathbf{p}^*(t).$$

$\Rightarrow$

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t)$$

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t)$$

Let $\boxed{\mathbf{p}^*(t) \triangleq \mathbf{K}(t)\mathbf{x}^*(t)}$ (by R.E.Kalman)

$$\dot{\mathbf{p}}(t) = \dot{\mathbf{K}}(t)\mathbf{x}(t) + \mathbf{K}(t)\dot{\mathbf{x}}(t) = -\mathbf{Q}(t)\mathbf{x}(t) - \mathbf{A}^T(t)\mathbf{p}(t)$$

$$\Leftrightarrow \quad \dot{\mathbf{K}}(t)\mathbf{x}(t) + \mathbf{K}(t)\{\mathbf{A}(t)\mathbf{x}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}(t)\} = -\mathbf{Q}(t)\mathbf{x}(t) - \mathbf{A}^T(t)\mathbf{p}(t)$$

$$\Leftrightarrow \quad \dot{\mathbf{K}}(t)\mathbf{x}(t) + \mathbf{K}(t)\{\mathbf{A}(t)\mathbf{x}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t)\} = -\mathbf{Q}(t)\mathbf{x}(t) - \mathbf{A}^T(t)\mathbf{K}(t)\mathbf{x}(t)$$

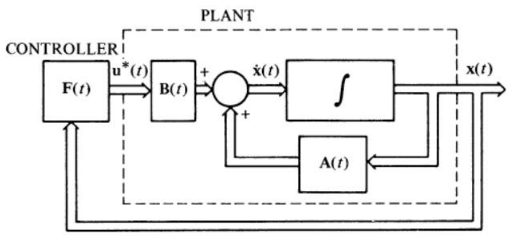
$$\Leftrightarrow \quad \boxed{\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t) - \mathbf{Q}(t) - \mathbf{A}^{\mathbf{T}}(t)\mathbf{K}(t)}$$

Ricatti equation

$$\mathbf{K}(t_f) = \mathbf{H}$$

Backward solution

$$\Rightarrow \quad \boxed{\begin{aligned}\mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) \\ &\triangleq \mathbf{F}(t)\mathbf{x}(t),\dagger\end{aligned}}$$



Linear Regulator Problems

(Ex) $\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = 2x_1(t) - x_2(t) + u(t)$

$$J(u) = \int_0^T \left[x_1^2(t) + \tfrac{1}{2}x_2^2(t) + \tfrac{1}{4}u^2(t)\right] dt.$$

$\mathbf{x}(0) = [-4\ 4]$

(sol) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, and $R = \frac{1}{2}$ $\mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\boxed{\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t) - \mathbf{Q}(t) - \mathbf{A}^{\mathbf{T}}(t)\mathbf{K}(t)} \quad \boxed{\mathbf{K}(t_f) = \mathbf{H}}$$

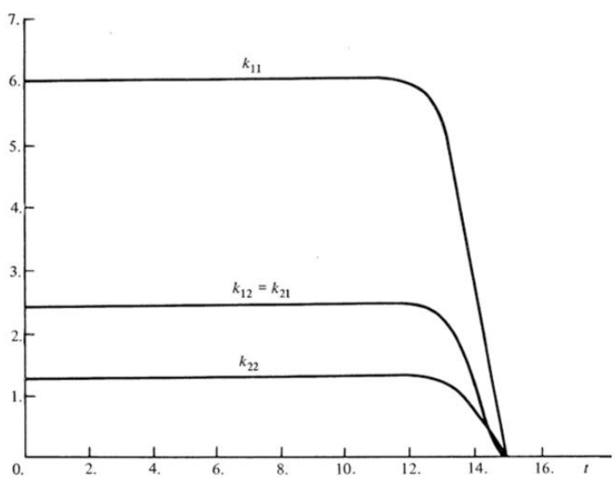
$\dot{k}_{11}(t) = 2[k_{12}^2(t) - 2k_{12}(t) - 1]$

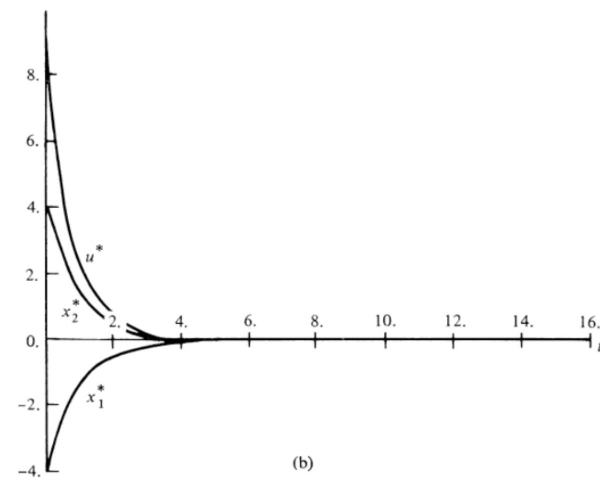
$\dot{k}_{12}(t) = 2k_{12}(t)k_{22}(t) - k_{11}(t) + k_{12}(t) - 2k_{22}(t)$

$\dot{k}_{22}(t) = 2k_{22}^2(t) - 2k_{12}(t) + 2k_{22}(t) - 1.$

$k_{11}(T) = k_{12}(T) = k_{22}(T) = 0$

$u^*(t) = -2[k_{12}(t) \quad k_{22}(t)]\mathbf{x}(t).$





Linear Tracking Problems

- System model

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t).$$

- Performance measure

$$J = \tfrac{1}{2}[\mathbf{x}(t_f) - \mathbf{r}(t_f)]^T\mathbf{H}[\mathbf{x}(t_f) - \mathbf{r}(t_f)] + \tfrac{1}{2}\int_{t_0}^{t_f} \{[\mathbf{x}(t) - \mathbf{r}(t)]^T\mathbf{Q}(t)[\mathbf{x}(t) - \mathbf{r}(t)] + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\}\, dt$$
$$\triangleq \tfrac{1}{2}\|\mathbf{x}(t_f) - \mathbf{r}(t_f)\|_{\mathbf{H}}^2 + \tfrac{1}{2}\int_{t_0}^{t_f} \{\|\mathbf{x}(t) - \mathbf{r}(t)\|_{\mathbf{Q}(t)}^2 + \|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2\}\, dt$$

- Boundary condition
  - t_f: fixed, $\mathbf{x}(t_f)$: free

$$\mathbf{p}^*(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\boldsymbol{x}^*(t_f)) \Rightarrow \boxed{\mathbf{p}^*(t_f) = \mathbf{H}\mathbf{x}^*(t_f) - \mathbf{H}\mathbf{r}(t_f)}$$

- Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)] \Rightarrow \boxed{\begin{aligned}\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) &= \tfrac{1}{2}\|\mathbf{x}(t) - \mathbf{r}(t)\|_{\mathbf{Q}(t)}^2 + \tfrac{1}{2}\|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2 \\ &\quad + \mathbf{p}^T(t)\mathbf{A}(t)\mathbf{x}(t) + \mathbf{p}^T(t)\mathbf{B}(t)\mathbf{u}(t).\end{aligned}}$$

- Necessary condition

$$\left.\begin{aligned}\dot{\mathbf{x}}^*(t) &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \mathbf{0} &= \frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\end{aligned}\right\} \text{for all } t \in [t_0, t_f] \Rightarrow \boxed{\begin{aligned}\dot{\mathbf{x}}^*(t) &= \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\mathbf{p}^*(t) + \mathbf{Q}(t)\mathbf{r}(t) \\ \mathbf{0} &= \frac{\partial \mathscr{H}}{\partial \mathbf{u}} = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^T(t)\mathbf{p}^*(t)\end{aligned}}$$

Linear Tracking Problems

- Necessary condition (continued)

$$\dot{\mathbf{x}}^*(t) = \mathbf{A}(t)\mathbf{x}^*(t) + \mathbf{B}(t)\mathbf{u}^*(t)$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial \mathscr{H}}{\partial \mathbf{x}} = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\mathbf{p}^*(t) + \mathbf{Q}(t)\mathbf{r}(t)$$

$$\mathbf{0} = \frac{\partial \mathscr{H}}{\partial \mathbf{u}} = \mathbf{R}(t)\mathbf{u}^*(t) + \mathbf{B}^T(t)\mathbf{p}^*(t)$$

$$\Rightarrow \quad \boxed{\begin{aligned} \mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t) \\ \dot{\mathbf{x}}^*(t) &= \mathbf{A}(t)\mathbf{x}^*(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{p}^*(t) \end{aligned}}$$

Let $\boxed{\mathbf{p}^*(t) = \mathbf{K}(t)\mathbf{x}^*(t) + \mathbf{s}(t)}$

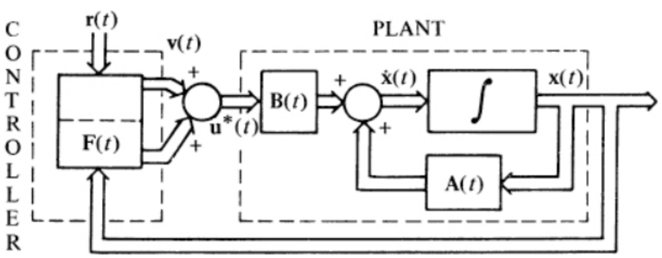
$$\Rightarrow \quad \boxed{\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{K}(t) - \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)}$$

Ricatti equation

$$\boxed{\dot{\mathbf{s}}(t) = -\left[\mathbf{A}^T(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\right]\mathbf{s}(t) + \mathbf{Q}(t)\mathbf{r}(t)}$$

$$\mathbf{K}(t_f) = \mathbf{H} \qquad \mathbf{s}(t_f) = -\mathbf{H}\mathbf{r}(t_f)$$

$$\Rightarrow \quad \boxed{\begin{aligned} \mathbf{u}^*(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) - \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t) \\ &\triangleq \mathbf{F}(t)\mathbf{x}(t) + \mathbf{v}(t), \end{aligned}}$$



Linear Tracking Problem

(Ex)

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = 2x_1(t) - x_2(t) + u(t)$$

$$J(u) = \int_0^T \{[x_1(t) - 0.2t]^2 + 0.025u^2(t)\}\, dt.$$

$$\mathbf{x}(0) = [-4\ 0]^T$$

(sol)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 0.05, \quad \text{and} \quad \mathbf{r}(t) = \begin{bmatrix} 0.2t \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{K}(t) - \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t) \qquad \mathbf{K}(T) = \mathbf{0}$$

$$\Rightarrow \quad \begin{aligned} \dot{k}_{11}(t) &= 20k_{12}^2(t) - 4k_{12}(t) - 2 \\ \dot{k}_{12}(t) &= 20k_{12}(t)k_{22}(t) - k_{11}(t) + k_{12}(t) - 2k_{22}(t) \\ \dot{k}_{22}(t) &= 20k_{22}^2(t) - 2k_{12}(t) + 2k_{22}(t) \end{aligned}$$

$$\dot{\mathbf{s}}(t) = -[\mathbf{A}^T(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)]\,\mathbf{s}(t) + \mathbf{Q}(t)\mathbf{r}(t) \qquad \mathbf{s}(T) = \mathbf{0}$$

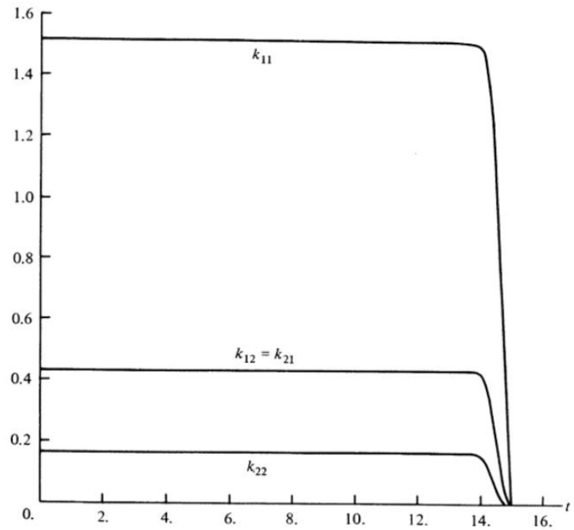
$$\Rightarrow \quad \begin{aligned} \dot{s}_1(t) &= 2[10k_{12}(t) - 1]s_2(t) + 0.4t \\ \dot{s}_2(t) &= -s_1(t) + [20k_{22}(t) + 1]s_2(t). \end{aligned}$$

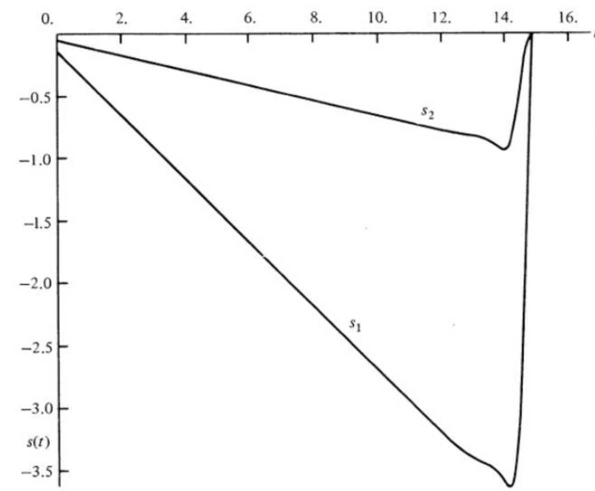
$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) - \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t)$$

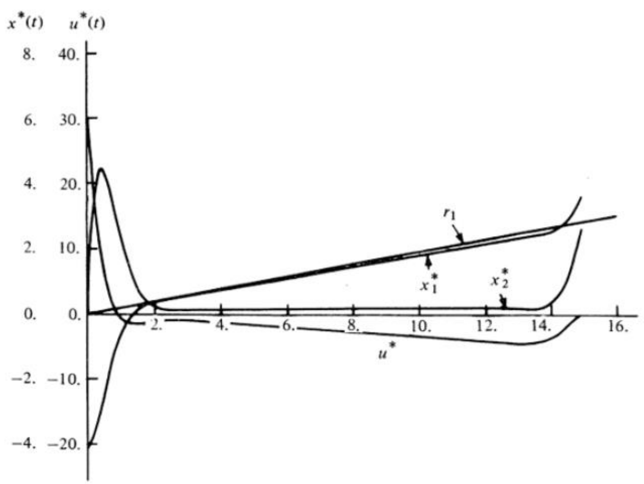
$$\Rightarrow \quad u^*(t) = -20[k_{12}(t)x_1(t) + k_{22}(t)x_2(t) + s_2(t)]$$

Linear Tracking Problem

(Ex) continued

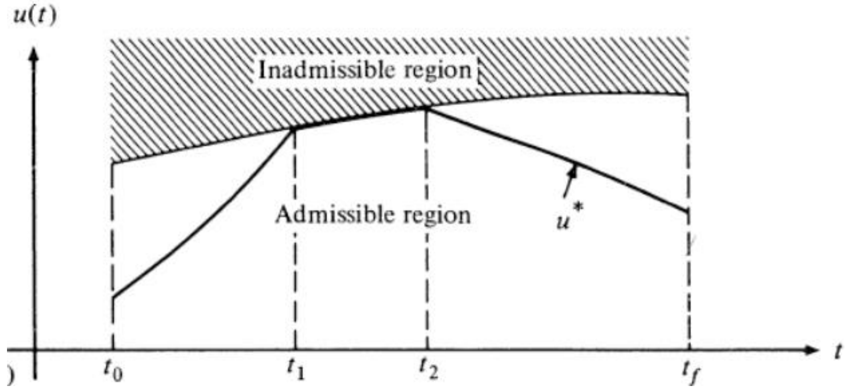






Pontryagin's Minimum Principle

- Bounded control



- Necessary condition for optimal control

$$\boxed{\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \leq \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t)}$$

for all $t \in [t_0, t_f]$ and for all *admissible control*

- Pontryagin's minimum principle
  - An optimal control must minimize the Hamiltonian

Pontryagin's Minimum Principle

- Necessary conditions for optimal control

Unconstrained control

$$\left.\begin{aligned}
\dot{\mathbf{x}}^*(t) &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
\dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
0 &= \frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)
\end{aligned}\right\} \begin{array}{l}\text{for all} \\ t \in [t_0, t_f]\end{array}$$

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0$$

Constrained control

$$\left.\begin{aligned}
\dot{\mathbf{x}}^*(t) &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
\dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\
\mathscr{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) &\le \mathscr{H}(\mathbf{x}^*(t), \mathbf{u}(t), \mathbf{p}^*(t), t) \\
&\qquad \text{for all admissible } \mathbf{u}(t)
\end{aligned}\right\} \begin{array}{l}\text{for all} \\ t \in [t_0, t_f]\end{array}$$

and

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta \mathbf{x}_f + \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0.$$

Pontryagin's Minimum Principle

(Ex) Consider the system having the state equations

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_2(t) + u(t),$$

with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$. The performance measure to be minimized is

$$J(u) = \int_{t_0}^{t_f} \tfrac{1}{2}[x_1^2(t) + u^2(t)]\,dt;$$

t_f is specified, and the final state $\mathbf{x}(t_f)$ is free.

(a) Unconstrained control

- Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \tfrac{1}{2}x_1^2(t) + \tfrac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t)$$

- Costate equation

$$\dot{p}_1^*(t) = -\frac{\partial \mathscr{H}}{\partial x_1} = -x_1^*(t)$$
$$\dot{p}_2^*(t) = -\frac{\partial \mathscr{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t)$$

- Optimal control

$$\frac{\partial \mathscr{H}}{\partial u} = u^*(t) + p_2^*(t) = 0 \qquad \frac{\partial^2 \mathscr{H}}{\partial u^2} = 1$$
$$\boxed{u^*(t) = -p_2^*(t)}$$

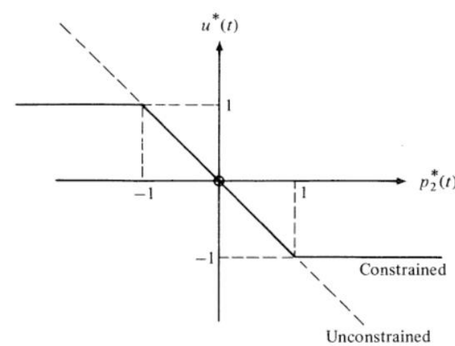
- Boundary condition

$$\mathbf{p}^*(t_f) = \mathbf{0}$$

(b) Constrained control

$$-1 \le u(t) \le +1 \qquad \text{for all } t \in [t_0, t_f].$$

$$\boxed{u^*(t) = \begin{cases} -1, & \text{for } 1 < p_2^*(t) \\ -p_2^*(t), & \text{for } -1 \le p_2^*(t) \le 1 \\ +1, & \text{for } p_2^*(t) < -1. \end{cases}}$$



Pontryagin's Minimum Principle

- Inequality constraints

$$\mathbf{f}(\mathbf{x}(t), t) \geq \mathbf{0} \qquad \mathbf{f}(\mathbf{x}(t), t) = [f_1(\mathbf{x}(t), t) \ \cdots f_l(\mathbf{x}(t), t)]^T$$

  – Define a new state $\dot{x}_{n+1}(t)$ by

$$\dot{x}_{n+1}(t) \triangleq [f_1(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_1) + [f_2(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_2) + \cdots + [f_l(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_l)$$

where

$$\mathbb{1}(-f_i) = \begin{cases} 0, & \text{for } f_i(\mathbf{x}(t), t) \geq 0 \\ 1, & \text{for } f_i(\mathbf{x}(t), t) < 0 \end{cases}$$

Heaviside step function

$\Rightarrow$ $\dot{x}_{n+1}(t)$=0 when all of the constraints are satisfied

  – Modified Hamiltionian

$$\begin{aligned} \mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) &= g(\mathbf{x}(t), \mathbf{u}(t), t) + p_1(t)a_1(\mathbf{x}(t), \mathbf{u}(t), t) + \cdots + p_n(t)a_n(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\boxed{+ p_{n+1}(t)\{[f_1(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_1) + \cdots + [f_l(\mathbf{x}(t), t)]^2 \mathbb{1}(-f_l)\}} \\ &\triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \end{aligned}$$

Pontryagin's Minimum Principle

(Ex)

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_2(t) + u(t)$$

$$J(u) = \int_{t_0}^{t_f} \tfrac{1}{2}[x_1^2(t) + u^2(t)]\,dt$$

t_f: given, $\mathbf{x}(t_f)$: free

$$-1 \le u(t) \le 1 \qquad \text{for } t \in [t_0, t_f]$$
$$-2 \le x_2(t) \le 2 \qquad \text{for } t \in [t_0, t_f].$$

(sol) Inequality constraints

$$-2 \le x_2(t) \le 2 \iff [x_2(t) + 2] \ge 0 \text{ and } [2 - x_2(t)] \ge 0 \iff \begin{aligned} f_1(\mathbf{x}(t)) &= [x_2(t) + 2] \ge 0 \\ f_2(\mathbf{x}(t)) &= [2 - x_2(t)] \ge 0 \end{aligned}$$

Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \tfrac{1}{2}x_1^2(t) + \tfrac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t)$$
$$+ p_3(t)\{[x_2(t) + 2]^2 \mathbb{1}(-x_2(t) - 2) + [2 - x_2(t)]^2 \mathbb{1}(x_2(t) - 2)\}$$

Minimum principle

Necessary condition

$$\dot{x}_1^*(t) = x_2^*(t), \qquad x_1^*(t_0) = x_{1_0}$$
$$\dot{x}_2^*(t) = -x_2^*(t) + u^*(t), \qquad x_2^*(t_0) = x_{2_0}$$
$$\dot{x}_3^*(t) = [x_2^*(t) + 2]^2 \mathbb{1}(-x_2^*(t) - 2) + [2 - x_2^*(t)]^2 \mathbb{1}(x_2^*(t) - 2), \qquad x_3^*(t_0) = 0$$
$$\dot{p}_1^*(t) = -\frac{\partial \mathscr{H}}{\partial x_1} = -x_1^*(t)$$
$$\dot{p}_2^*(t) = -\frac{\partial \mathscr{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t) - 2p_3^*(t)[x_2^*(t) + 2]\mathbb{1}(-x_2^*(t) - 2)$$
$$+ 2p_3^*(t)[2 - x_2^*(t)]\mathbb{1}(x_2^*(t) - 2)$$
$$\dot{p}_3^*(t) = -\frac{\partial \mathscr{H}}{\partial x_3} = 0 \Rightarrow p_3^*(t) = \text{a constant}$$
$$p_1^*(t_f) = p_2^*(t_f) = 0$$

Optimal control

$$u^*(t) = \begin{cases} -1, & \text{for } 1 < p_2^*(t) \\ -p_2^*(t), & \text{for } -1 \le p_2^*(t) \le 1 \\ +1, & \text{for } p_2^*(t) < -1. \end{cases}$$

$$\frac{\partial \mathcal{H}}{\partial u} = u^*(t) + p_2^*(t) = 0$$

Minimum-Time Problems

- Minimum-time problem

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t) \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$J(\mathbf{u}) = \int_{t_0}^{t_f} dt = t_f - t_0$$

$$M_{i-} \le u_i(t) \le M_{i+}, \qquad i = 1, 2, \ldots, m, \qquad t \in [t_0, t^*]$$

  - Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = 1 + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), t) + \mathbf{B}(\mathbf{x}(t), t)\mathbf{u}(t)]$$

  - Minimum principle

$$1 + \mathbf{p}^{*T}(t)[\mathbf{a}(\mathbf{x}^*(t), t) + \mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}^*(t)] \le 1 + \mathbf{p}^{*T}(t)[\mathbf{a}(\mathbf{x}^*(t), t) + \mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t)]$$

$$\Rightarrow \quad \mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}^*(t) \le \mathbf{p}^{*T}(t)\mathbf{B}(\mathbf{x}^*(t), t)\mathbf{u}(t)$$

  - Optimal control (time-optimal control)

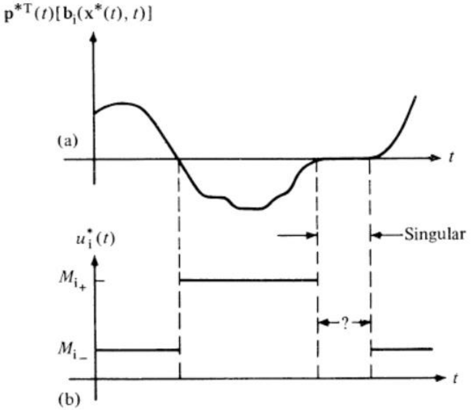
Let $\mathbf{B}(\mathbf{x}^*(t), t) = [\mathbf{b}_1(\mathbf{x}^*(t), t) \,\vdots\, \mathbf{b}_2(\mathbf{x}^*(t), t) \,\vdots\, \cdots \,\vdots\, \mathbf{b}_m(\mathbf{x}^*(t), t)]$

then

$$u_i^*(t) = \begin{cases} M_{i+}, & \text{for } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) < 0 \\ M_{i-}, & \text{for } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) > 0 \\ \text{Undetermined}, & \text{for } \mathbf{p}^{*T}(t)\mathbf{b}_i(\mathbf{x}^*(t), t) = 0 \end{cases}$$

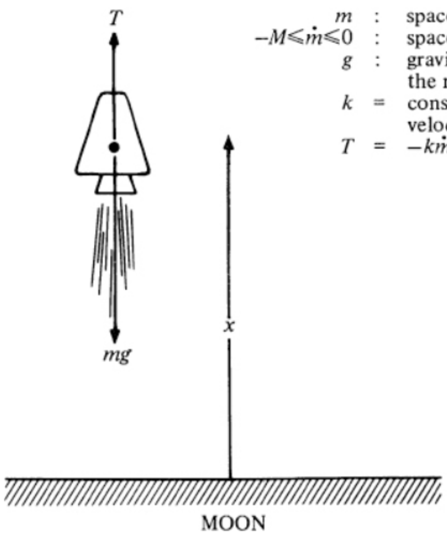
$$i = 1, 2, \ldots, m$$

Bang-bang control



Minimum-Time Problems

(Ex) lunar soft landing



m : spacecraft mass
$-M \leqslant \dot{m} \leqslant 0$: spacecraft mass rate
g : gravitational constant in the near field of the moon
k = constant: relative exhaust velocity of gases
T = $-k\dot{m}$: thrust

Equation of motion
$$m(t)\ddot{x}(t) = -gm(t) + T(t) = -gm(t) - k\dot{m}(t)$$

Mass rate constraint
$$-M \le \dot{m} \le 0$$

State equation
$$x_1 \triangleq x,\ x_2 \triangleq \dot{x},\ x_3 \triangleq m \qquad u \triangleq \dot{m}$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -g - \frac{k}{x_3(t)}u(t)$$
$$\dot{x}_3(t) = u(t).$$

Hamiltonian
$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = 1 + p_1(t)x_2(t) - gp_2(t) - \frac{kp_2(t)u(t)}{x_3(t)} + p_3(t)u(t)$$

Minimum principle
$$\mathbf{p}^{*T}(t)\,\mathbf{B}(\mathbf{x}^*(t), t) = -\frac{kp_2(t)}{x_3(t)} + p_3(t)$$

Optimal control
$$u^*(t) = \begin{cases} 0, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} < 0 \\ -M, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} > 0 \\ \text{Undetermined}, & \text{for } p_3^*(t) - \dfrac{kp_2^*(t)}{x_3^*(t)} = 0. \end{cases}$$

Minimum-Time Problems

(Ex) For a linear time-invariant system

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t) \qquad |u(t)| \le 1$$

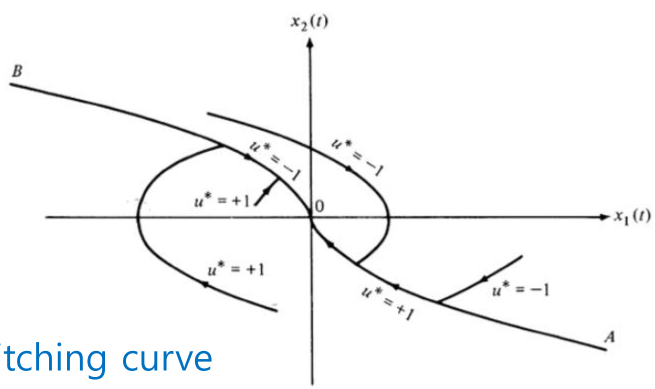
find the optimal control transfer the initial state x_0 to the origin (0,0) in minimum time.

(sol)

Hamiltonian $\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = 1 + p_1(t)x_2(t) + p_2(t)u(t)$

Minimum principle $\mathbf{p}^{*T}(t)\,\mathbf{B}(\mathbf{x}^*(t), t) = p_2(t)$

Optimal control $u^*(t) = \begin{cases} -1, & \text{for } p_2^*(t) > 0 \\ +1, & \text{for } p_2^*(t) < 0 \end{cases} \Rightarrow u^*(t) = \begin{cases} +1, & \text{for all } t \in [t_0, t^*], \text{ or} \\ -1, & \text{for all } t \in [t_0, t^*], \text{ or} \\ +1, & \text{for } t \in [t_0, t_1),^\dagger \text{ and } -1, \text{ for } t \in [t_1, t^*], \text{ or} \\ -1, & \text{for } t \in [t_0, t_1), \text{ and } +1, \text{ for } t \in [t_1, t^*]. \end{cases}$



Time-optimal switching curve

Minimum-Time Problems

(Q) Find the optimal control law for transferring the system

$$\dot{x}_1(t) = -x_1(t) - u(t)$$
$$\dot{x}_2(t) = -2x_2(t) - 2u(t)$$

from an arbitrary initial state to the origin in minimum time. The admissible controls are constrained by $|u(t)| \leq 1.0$.