Optimal Control Theory

Calculus of Variations

Fundamental Theorem

- Functional
  - Function of functions assigning to set of real numbers

(ex) $J(x) = \int_{t_0}^{t_f} x(t)\,dt$ $\qquad$ $J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\,dt$

- Increment of a functional

$$\Delta J \triangleq J(\mathbf{x} + \delta\mathbf{x}) - J(\mathbf{x}) \qquad \mathbf{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

- Variation of a functional

$$\Delta J(\mathbf{x}, \delta\mathbf{x}) = \boxed{\delta J(\mathbf{x}, \delta\mathbf{x})} + g(\mathbf{x}, \delta\mathbf{x}) \cdot \|\delta\mathbf{x}\|$$

δJ is linear in $\delta\mathbf{x}$

if $\lim_{\|\delta\mathbf{x}\| \to 0} \{g(\mathbf{x}, \delta\mathbf{x})\} = 0$ $\longrightarrow$ δJ is variation of J

(ex) $J(x) = \int_0^1 [x^2(t) + 2x(t)]\,dt$

$$\Delta J(x, \delta x) = J(x + \delta x) - J(x)$$
$$= \int_0^1 \{[x(t) + \delta x(t)]^2 + 2[x(t) + \delta x(t)]\}\,dt - \int_0^1 [x^2(t) + 2x(t)]\,dt$$
$$= \underbrace{\boxed{\int_0^1 \{[2x(t) + 2]\,\delta x(t)}}_{\delta J} + [\delta x(t)]^2\}\,dt$$

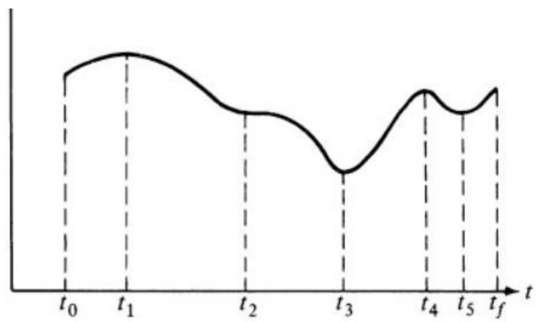
$$\Delta J(x, \delta x) = \boxed{\frac{\partial J}{\partial x}\delta x} + \frac{\partial^2 J}{\partial x^2}(\delta x)^2 + \cdots$$

Fundamental Theorem

- Fundamental theorem of the calculus of variations

If $\mathbf{x}^*$ is an extremal, then $\boxed{\delta J(\mathbf{x}^*, \delta\mathbf{x}) = 0}$ for all admissible $\delta\mathbf{x}$

necessary condition



t_1: global (absolute) maximum

t_3: global (absolute) minimum

t_4: local (relative) maximum

t_5: local (relative) minimum

Extrema for Functionals

- Find extrema for the functional

$$J(\mathbf{x}) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\, dt \qquad \mathbf{x}(t_o) = \mathbf{x}_o$$

$$\begin{aligned}\Delta J &= J(\mathbf{x}+\delta\mathbf{x}) - J(\mathbf{x}) \\ &= \int_{t_o}^{t_f+\delta t_f} g(\mathbf{x}(t)+\delta\mathbf{x}(t), \dot{\mathbf{x}}(t)+\delta\dot{\mathbf{x}}(t), t)\ dt - \int_{t_o}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\ dt \\ &= \int_{t_o}^{t_f}\{g(\mathbf{x}(t)+\delta\mathbf{x}(t), \dot{\mathbf{x}}(t)+\delta\dot{\mathbf{x}}(t), t) - g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\}\ dt \\ &\quad + \int_{t_f}^{t_f+\delta t_f} g(\mathbf{x}(t)+\delta\mathbf{x}(t), \dot{\mathbf{x}}(t)+\delta\dot{\mathbf{x}}(t), t)\ dt\end{aligned}$$

$$\delta J(\mathbf{x}, \delta\mathbf{x}) = \int_{t_o}^{t_f}\left\{\left[\frac{\partial g(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial \mathbf{x}}\right]^T \delta\mathbf{x}(t) + \left[\frac{\partial g(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial \dot{\mathbf{x}}}\right]^T \delta\dot{\mathbf{x}}(t)\right\}\ dt + g(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f)\ \delta t_f$$

Integration by parts $\int_a^b u(x)v'(x)\,dx = \left[u(x)v(x)\right]_a^b - \int_a^b u'(x)v(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx$

$$\int_{t_o}^{t_f}\left[\frac{\partial g(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial \dot{\mathbf{x}}}\right]^T \delta\dot{\mathbf{x}}(t)\,dt = \left[\frac{\partial g(\mathbf{x}(t_f),\dot{\mathbf{x}}(t_f),t_f)}{\partial \dot{\mathbf{x}}}\right]^T \delta\mathbf{x}(t_f) - \left[\frac{\partial g(\mathbf{x}(t_o),\dot{\mathbf{x}}(t_o),t_o)}{\partial \dot{\mathbf{x}}}\right]^T \underbrace{\cancel{\delta\mathbf{x}(t_o)}}_{0} - \int_{t_o}^{t_f}\frac{d}{dt}\left[\frac{\partial g(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial \dot{\mathbf{x}}}\right]^T \delta\mathbf{x}(t)\,dt$$

let $\delta\mathbf{x}(t_f) = \delta\mathbf{x}_f - \dot{\mathbf{x}}(t_f)\delta t_f$

$$= \left[\frac{\partial g(\mathbf{x}(t_f),\dot{\mathbf{x}}(t_f),t_f)}{\partial \dot{\mathbf{x}}}\right]^T \delta\mathbf{x}_f - \left[\frac{\partial g(\mathbf{x}(t_f),\dot{\mathbf{x}}(t_f),t_f)}{\partial \dot{\mathbf{x}}}\right]^T \dot{\mathbf{x}}(t_f)\delta t_f - \int_{t_o}^{t_f}\frac{d}{dt}\left[\frac{\partial g(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial \dot{\mathbf{x}}}\right]^T \delta\mathbf{x}(t)\,dt$$

Extrema for Functionals

- Fundamental theorem

$$\delta J(\mathbf{x}^*, \delta\mathbf{x}) = 0 = \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \delta\mathbf{x}_f$$
$$+ \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\right]\delta t_f$$
$$+ \int_{t_0}^{t_f} \left\{\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right]\right\}^T \delta\mathbf{x}(t)\, dt$$

- Euler equation

$$\boxed{\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right] = \mathbf{0}}$$

- Boundary condition

$$\boxed{\begin{aligned}&\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \delta\mathbf{x}_f \\ &+ \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\right]\delta t_f = 0\end{aligned}}$$

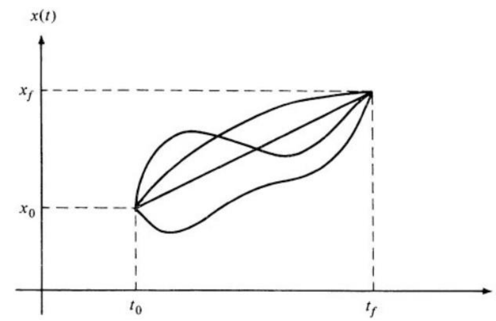
Extrema for Functionals

- Euler equation $$\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right] = \mathbf{0}$$

- Boundary condition

Problem description	*Substitution*	*Boundary conditions*	*Remarks*
1. $\mathbf{x}(t_f)$, t_f both specified (*Problem 1*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
2. $\mathbf{x}(t_f)$ free; t_f specified (*Problem 2*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
3. t_f free; $\mathbf{x}(t_f)$ specified (*Problem 3*)	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $- \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
4. t_f, $\mathbf{x}(t_f)$ free and independent (*Problem 4*)	—	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
5. t_f, $\mathbf{x}(t_f)$ free but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ (*Problem 4*)	$\delta\mathbf{x}_f = \frac{d\boldsymbol{\theta}}{dt}(t_f)\,\delta t_f$†	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $+ \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f) - \dot{\mathbf{x}}^*(t_f)\right] = 0$†	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f

Extrema for Functionals – Problem 1



(Problem 1) t_f specified, $x(t_f)$ specified

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \quad x(t_o) = x_o, x(t_f) = x_f,$$

Euler equation

$$\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right] = \mathbf{0} \quad \Rightarrow \quad \boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0}$$

Boundary condition $\boxed{x(t_f) = x_f,}$

$$\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \cancel{\delta \mathbf{x}_f}^{\,0}$$
$$+ \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\right] \cancel{\delta t_f}^{\,0} = 0$$

Extrema for Functionals – Problem 1

(Ex) Find an extremal for $J(x) = \int_0^{\pi/2} [\dot{x}^2(t) - x^2(t)]\, dt$ $\quad x(0) = 0$ and $x(\pi/2) = 1$

(sol)
$$0 = \frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right]$$
$$= -2x^*(t) - \frac{d}{dt}[2\dot{x}^*(t)],$$

$\Rightarrow$ $\ddot{x}^*(t) + x^*(t) = 0.$ $\quad s^2 + 1 = 0 \quad s = \pm j1$

$\Rightarrow$ $x^*(t) = c_3 \cos(t) + c_4 \sin(t),$

$\Rightarrow$ $x(0) = 0$ and $x(\pi/2) = 1$

$$0 = c_3 \cos(0) + c_4 \sin(0) \Longrightarrow c_3 = 0$$
$$1 = \overset{0}{c_3} \cos\left(\frac{\pi}{2}\right) + c_4 \sin\left(\frac{\pi}{2}\right) \Longrightarrow c_4 = 1$$

$\Rightarrow$ $x^*(t) = \sin(t)$

Extrema for Functionals – Problem 1

(Ex) Find an extremal curve for the functional

$$J(\mathbf{x}) = \int_0^{\pi/4} [x_1^2(t) + 4x_2^2(t) + \dot{x}_1(t)\dot{x}_2(t)]\,dt \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{x}\left(\frac{\pi}{4}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Euler equations $\quad 2x_1^*(t) - \ddot{x}_2^*(t) = 0 \qquad 8x_2^*(t) - \ddot{x}_1^*(t) = 0,$

$$\Rightarrow \quad x_1^*(t) = c_1\epsilon^{2t} + c_2\epsilon^{-2t} + c_3 \cos 2t + c_4 \sin 2t,$$

$$x_2^*(t) = \tfrac{1}{2}c_1\epsilon^{2t} + \tfrac{1}{2}c_2\epsilon^{-2t} - \tfrac{1}{2}c_3 \cos 2t - \tfrac{1}{2}c_4 \sin 2t.$$

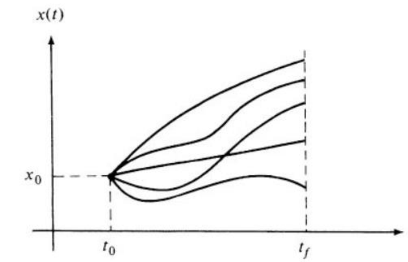
Boundary conditions $\quad x_1^*(0) = 0; \quad x_2^*(0) = 1; \quad x_1^*\left(\frac{\pi}{4}\right) = 1; \quad x_2^*\left(\frac{\pi}{4}\right) = 0$

$$\Rightarrow \quad c_1 = \frac{-\frac{1}{2} + \epsilon^{-\pi/2}}{\epsilon^{-\pi/2} - \epsilon^{\pi/2}}; \qquad c_2 = \frac{\frac{1}{2} - \epsilon^{\pi/2}}{\epsilon^{-\pi/2} - \epsilon^{\pi/2}}; \qquad c_3 = -1; \qquad c_4 = \frac{1}{2}$$

Extrema for Functionals – Problem 2

(Problem 2) t_f specified, $x(t_f)$ unspecified

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \qquad x(t_o) = x_o,\ t_f \text{ specified}$$



Euler equation

$$\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right] = \mathbf{0} \quad \Rightarrow \quad \boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0}$$

Boundary condition

$$\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \delta\mathbf{x}_f + \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\right] \cancelto{0}{\delta t_f} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0}$$

Extrema for Functionals – Problem 2

(Ex) Find an extremal for $J(x) = \int_0^2 [\dot{x}^2(t) + 2x(t)\dot{x}(t) + 4x^2(t)]\, dt$ $x(0) = 1$, and $x(2)$ is free

(sol) Euler equation $-\ddot{x}^*(t) + 4x^*(t) = 0$ $\Longrightarrow$ $x^*(t) = c_1\epsilon^{-2t} + c_2\epsilon^{2t}$.

Boundary condition $x(0) = 1$ $\Longrightarrow$ $c_1 + c_2 = 1$

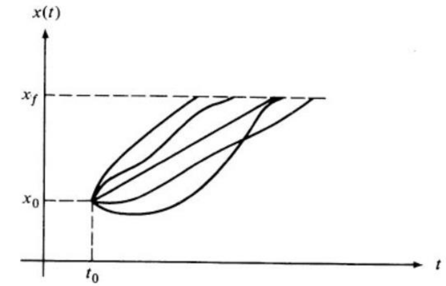
$$\frac{\partial g}{\partial \dot{x}}(x^*(2), \dot{x}^*(2)) = 0 \Longrightarrow \dot{x}^*(2) + x^*(2) = 0 \Longrightarrow -c_1\epsilon^{-4} + 3c_2\epsilon^4 = 0.$$

$$c_1 = \frac{3\epsilon^4}{\epsilon^{-4} + 3\epsilon^4}, \quad \text{and} \quad c_2 = \frac{\epsilon^{-4}}{\epsilon^{-4} + 3\epsilon^4}$$

Extrema for Functionals – Problem 3

(Problem 3) t_f unspecified, $x(t_f)$ specified

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \qquad x(t_o) = x_o, x(t_f) = x_f, t_f \text{ free}$$



Euler equation

$$\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right] = \mathbf{0} \quad \Rightarrow \quad \boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0}$$

Boundary condition

$$\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \cancelto{0}{\delta \mathbf{x}_f} + \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\right]\delta t_f = 0$$

$$\Rightarrow \quad \boxed{g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\dot{x}^*(t_f) = 0}$$

Extrema for Functionals – Problem 3

(ex) Find an extremal for $J(x) = \int_1^{t_f} [2x(t) + \frac{1}{2}\dot{x}^2(t)]\, dt$ $\quad x(1) = 4, x(t_f) = 4$, and $t_f > 1$

(sol) Euler equation $\quad \ddot{x}^*(t) = 2 \quad \Rightarrow \quad x^*(t) = t^2 + c_1 t + c_2.$

Boundary condition $\quad x(1) = 4 \quad \Rightarrow \quad x^*(1) = 4 = 1 + c_1 + c_2$, or $c_1 + c_2 = 3$

$x(t_f) = 4 \quad \Rightarrow \quad x^*(t_f) = 4 = t_f^2 + c_1 t_f + c_2$

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\dot{x}^*(t_f) = 0$$

$$\Rightarrow \quad 2x^*(t_f) - \tfrac{1}{2}\dot{x}^{*2}(t_f) = 2c_2 - \frac{c_1^2}{2} = 0$$

$x^*(t) = t^2 - 6t + 9, \quad$ and $\quad t_f = 5.$

Extrema for Functionals – Problem 4

(Problem 4-1) t_f unspecified, $x(t_f)$ unspecified

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \qquad x(t_o) = x_o, t_f \text{ and } x(t_f) \text{ free}$$

Euler equation

$$\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right] = \mathbf{0} \Rightarrow \boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0}$$

Boundary condition

$$\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \delta \mathbf{x}_f + \left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\right]\delta t_f = 0$$

$$\Rightarrow \boxed{\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0}$$

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right] \dot{x}^*(t_f) = 0. \Rightarrow \boxed{g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0}$$

Extrema for Functionals – Problem 4

(Problem 4-2) t_f unspecified, $x(t_f)$ unspecified

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \quad x(t_o) = x_o, t_f \text{ free,} \quad x(t_f) = \theta(t_f)$$

Euler equation

$$\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), t)\right] = \mathbf{0} \quad \Rightarrow \quad \boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0}$$

Boundary condition

$$\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \delta\mathbf{x}_f \longleftarrow \delta x_f \doteq \frac{d\theta}{dt}(t_f)\,\delta t_f$$

$$+\left[g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\right]\delta t_f = 0$$

$$\Rightarrow \quad \boxed{\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\left[\frac{d\theta}{dt}(t_f) - \dot{x}^*(t_f)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0}$$

Extrema for Functionals – Problem 4

(Ex) $J(x) = \int_{t_0}^{t_f} [1 + \dot{x}^2(t)]^{1/2}\, dt$ $x(0) = 0$ $x(t_f)$ is required to lie on the line $\theta(t) = -5t + 15$

(sol) Euler equation $\frac{d}{dt}\left[\frac{\dot{x}^*(t)}{[1 + \dot{x}^{*2}(t)]^{1/2}}\right] = 0 \Longrightarrow \ddot{x}^*(t) = 0 \Longrightarrow x^*(t) = c_1 t + c_2$

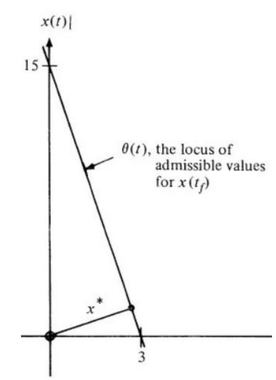
Boundary condition

$x^*(0) = 0 \Longrightarrow c_2 = 0.$

$\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\left[\frac{d\theta}{dt}(t_f) - \dot{x}^*(t_f)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \Longrightarrow -5\dot{x}^*(t_f) + 1 = 0$

$\Longrightarrow c_1 = \frac{1}{5}.$

$x^*(t_f) = \theta(t_f) \Longrightarrow \frac{1}{5}t_f = -5t_f + 15 \Longrightarrow t_f = \frac{75}{26} = 2.88$



Extrema for Functionals

(Q) $$J(\mathbf{x}) = \int_0^{\pi/4} [x_1^2(t) + \dot{x}_1(t)\dot{x}_2(t) + \dot{x}_2^2(t)]\, dt$$

$x_1(0) = 1;\quad x_1\left(\frac{\pi}{4}\right) = 2;$

$x_2(0) = \frac{3}{2};\quad x_2\left(\frac{\pi}{4}\right)$ free.

Constrained Extrema

- Constrained minimization of functional

$$J(\mathbf{w}) = \int_{t_0}^{t_f} g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)\, dt$$

w is an $(n+m) \times 1$ vector of functions

$$\boxed{f_i(\mathbf{w}(t), t) = 0, \qquad i = 1, 2, \ldots, n}$$

Point constraints

- Augmented functional

$$J_a(\mathbf{w}, \mathbf{p}) = \int_{t_0}^{t_f} \{g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + p_1(t)[f_1(\mathbf{w}(t), t)] + p_2(t)[f_2(\mathbf{w}(t), t)] + \cdots + p_n(t)[f_n(\mathbf{w}(t), t)]\}\, dt$$

$$= \int_{t_0}^{t_f} \{g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)[\mathbf{f}(\mathbf{w}(t), t)]\}\, dt$$

$p_1(t), \cdots, p_1(t)$: Lagrange multipliers

- Variation of functional

$$\delta J_a(\mathbf{w}, \delta\mathbf{w}, \mathbf{p}, \delta\mathbf{p}) = \int_{t_0}^{t_f} \left\{\left[\frac{\partial g^T}{\partial \mathbf{w}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)\left[\frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}(t), t)\right]\right] \delta\mathbf{w}(t) + \left[\frac{\partial g^T}{\partial \dot{\mathbf{w}}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)\right] \delta\dot{\mathbf{w}}(t) + [\mathbf{f}^T(\mathbf{w}(t), t)]\, \delta\mathbf{p}(t)\right\} dt$$

Integration by parts

$$= \int_{t_0}^{t_f} \left\{\left[\frac{\partial g^T}{\partial \mathbf{w}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)\left[\frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}(t), t)\right] - \frac{d}{dt}\left[\frac{\partial g^T}{\partial \dot{\mathbf{w}}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)\right]\right] \delta\mathbf{w}(t) + \cancelto{0}{[\mathbf{f}^T(\mathbf{w}(t), t)]}\, \delta\mathbf{p}(t)\right\} dt$$

- Euler equation

$$\frac{\partial g}{\partial \mathbf{w}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}^*(t), t)\right]^T \mathbf{p}^*(t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\mathbf{w}}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t)\right] = \mathbf{0}$$

Define $\boxed{g_a(\mathbf{w}(t), \dot{\mathbf{w}}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)[\mathbf{f}(\mathbf{w}(t), t)],}$ augmented function

$$\boxed{\frac{\partial g_a}{\partial \mathbf{w}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\mathbf{w}}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t)\right] = \mathbf{0}}$$

Constrained Extrema

(Ex) minimize $f(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2$

s.t $y_3 = y_1 y_2 + 5$

$y_1 + y_2 + y_3 = 1$

(sol) augmented function

$$f_a(y_1, y_2, y_3, p_1, p_2) = y_1^2 + y_2^2 + y_3^2 + p_1[y_1 y_2 + 5 - y_3] + p_2[y_1 + y_2 + y_3 - 1].$$

Euler equation

$$\begin{aligned} y_1^* + y_2^* + y_3^* - 1 &= 0 \\ y_1^* y_2^* + 5 - y_3^* &= 0 \\ 2y_1^* + p_1^* y_2^* + p_2^* &= 0 \\ 2y_2^* + p_1^* y_1^* + p_2^* &= 0 \\ 2y_3^* - p_1^* + p_2^* &= 0 \end{aligned} \quad \Longrightarrow \quad (y_1^*, y_2^*, y_3^*) = \begin{cases} (2, -2, 1) \\ \text{or} \\ (-2, 2, 1) \end{cases}$$

$$\Longrightarrow \quad f_{min} = 9$$

Constrained Extrema

- Constrained minimization of functional

$$J(\mathbf{w}) = \int_{t_0}^{t_f} g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)\, dt$$

$$\boxed{f_i(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) = 0, \qquad i = 1, 2, \ldots, n.}$$

w is an $(n + m) \times 1$ vector of functions

differential eq. constraints

- Augmented functional

$$\begin{aligned} J_a(\mathbf{w}, \mathbf{p}) &= \int_{t_0}^{t_f} \{g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + p_1(t)[f_1(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)] + p_2(t)[f_2(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)] + \cdots + p_n(t)[f_n(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)]\}\, dt \\ &= \int_{t_0}^{t_f} \{g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)[\mathbf{f}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)]\}\, dt. \\ &= \int_{t_0}^{t_f} g_a(\mathbf{w}(t), \dot{\mathbf{w}}(t), \mathbf{p}(t), t)\, dt \end{aligned}$$

$$\boxed{g_a(\mathbf{w}(t), \dot{\mathbf{w}}(t), \mathbf{p}(t), t) = g(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)[\mathbf{f}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)]}$$

- Variation of functional

$$\begin{aligned} \delta J_a(\mathbf{w}, \delta\mathbf{w}, \mathbf{p}, \delta\mathbf{p}) = \int_{t_0}^{t_f} \Bigg\{ &\left[\frac{\partial g^T}{\partial \mathbf{w}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)\left[\frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)\right] - \frac{d}{dt}\left[\frac{\partial g^T}{\partial \dot{\mathbf{w}}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t) + \mathbf{p}^T(t)\left[\frac{\partial \mathbf{f}}{\partial \dot{\mathbf{w}}}(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)\right]\right]\right] \delta\mathbf{w}(t) \\ &+ [\underbrace{\mathbf{f}^T(\mathbf{w}(t), \dot{\mathbf{w}}(t), t)}_{0}]\, \delta\mathbf{p}(t) \Bigg\}\, dt \end{aligned}$$

- Euler equation

$$\frac{\partial g}{\partial \mathbf{w}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t)\right]^T \mathbf{p}^*(t) - \frac{d}{dt}\left\{\frac{\partial g}{\partial \dot{\mathbf{w}}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t) + \left[\frac{\partial \mathbf{f}}{\partial \dot{\mathbf{w}}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), t)\right]^T \mathbf{p}^*(t)\right\} = \mathbf{0}$$

$$\boxed{\frac{\partial g_a}{\partial \mathbf{w}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\mathbf{w}}}(\mathbf{w}^*(t), \dot{\mathbf{w}}^*(t), \mathbf{p}^*(t), t)\right] = \mathbf{0}}$$

Constrained Extrema

(Ex) system

$$\dot{x}_1(t) = x_2(t) - x_1(t)$$
$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + u(t)$$

performance

$$J(\mathbf{x}, u) = \int_{t_0}^{t_f} \tfrac{1}{2}[x_1^2(t) + x_2^2(t) + u^2(t)]\, dt$$

(sol) $x_1 \triangleq w_1$, $x_2 \triangleq w_2$, and $u \triangleq w_3$,

$$J(\mathbf{w}) = \int_{t_0}^{t_f} \tfrac{1}{2}[w_1^2(t) + w_2^2(t) + w_3^2(t)]\, dt$$

$$\dot{w}_1(t) = w_2(t) - w_1(t)$$
$$\dot{w}_2(t) = -2w_1(t) - 3w_2(t) + w_3(t)$$

$\Rightarrow$

$$f_1(\mathbf{w}(t), \dot{\mathbf{w}}(t)) = w_2(t) - w_1(t) - \dot{w}_1(t) = 0$$
$$f_2(\mathbf{w}(t), \dot{\mathbf{w}}(t)) = -2w_1(t) - 3w_2(t) + w_3(t) - \dot{w}_2(t) = 0,$$

$$\begin{aligned} g_a(\mathbf{w}(t), \dot{\mathbf{w}}(t), \mathbf{p}(t)) = {} & \tfrac{1}{2}w_1^2(t) + \tfrac{1}{2}w_2^2(t) + \tfrac{1}{2}w_3^2(t) \\ & + p_1(t)[w_2(t) - w_1(t) - \dot{w}_1(t)] \\ & + p_2(t)[-2w_1(t) - 3w_2(t) + w_3(t) - \dot{w}_2(t)] \end{aligned}$$

Euler equation

$$\dot{p}_1^*(t) = -w_1^*(t) + p_1^*(t) + 2p_2^*(t)$$
$$\dot{p}_2^*(t) = -w_2^*(t) - p_1^*(t) + 3p_2^*(t)$$
$$w_3^*(t) + p_2^*(t) = 0.$$

Constrained Extrema

(Q) Determine the necessary condition for

$$J(\mathbf{w}) = \int_{t_0}^{t_f} [w_1^2(t) + w_1(t)w_2(t) + w_2^2(t) + w_3^2(t)]\,dt$$

$$\dot{\mathbf{w}}_1(t) = w_2(t)$$

$$\dot{w}_2(t) = -w_1(t) + [1 - w_1^2(t)]w_2(t) + w_3(t)$$