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Problem 13

I, Tameez Latib, declare that this work is my own. I did this work honestly and can fully stand behind everything that I have written.

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Suppose that osteoblasts secrete uncalcified bone matrix from x = 0 at a constant rate, so that all parts of the matrix move at speed s ¿ 0 (along the +x direction) Let?s suppose the crystals are uniformly deposited and spaced, have a spherical shape, and grow at a rate proportional to their surface area.

From this, let's model the Volume (of a spherical crystal) as a function of x and t, V(x, t). With V' (the growth) being proportional to surface area

$$(V(x,t))' \propto 4\pi r^2$$

Noting that $V(x,t) = 4/3\pi r^3$, Then $V^{2/3} \propto r^2$

Hence

$$(V(x,t))' \propto V^{2/3}$$

$$(V(x,t))' = c * V^{2/3}$$

Where c is a constant. We can also model the position x as a function of t; $x(t) = st + x(0)$ That is, the crystals move at a fixed speed s from their initial position x(0).

Now, by the chain rule, the left hand side is:

$$V' = V_t t' + V_x x' = V_t + sV_x$$

So now we have that

$$V_t + sV_x = cV^{2/3}$$

to non-dimensionalise,

$V = v^*\bar{v}$ $t = t^*\bar{t}$ $x = x^*\bar{x}$

Then:

$$\frac{\bar{v}}{\bar{t}}v^*_{t^*} + \frac{\bar{v}s}{\bar{x}}v^*_{x^*} = c\bar{v}^{2/3}v^{*2/3}$$

Letting $\bar{x} = s\bar{t}$, $\bar{v} = (c\bar{t})^3$,

$$v^*_{t^*} + \frac{\bar{t}s}{\bar{x}}v^*_{x^*} = c\frac{\bar{t}}{\bar{v}^{1/3}}v^{*2/3}$$

We obtain:

$$v^*_{t^*} + v^*_{x^*} = v^{*2/3}$$

For convenience, I will omit the '*'s, since there will be no further non-dimensionalising or converting back to dimensional form

Now to solve

$$v_t + v_x = v^{2/3}$$

To the constraints: $v(x,0) = 0, v(0,t) = 1$
First, let $x = x(\tau, \sigma)$, $t = t(\tau, \sigma)$, with σ parametrizing the boundary. Then

$$\frac{dv}{d\tau} = v_x \frac{\partial x}{\partial \tau} + v_t \frac{\partial t}{\partial \tau}$$

If we let

$$\frac{\partial x}{\partial \tau} = \frac{\partial t}{\partial \tau} = 1$$

Then

$$\frac{dv}{d\tau} = v^{2/3}$$

This differential equation can be solved, to get

$$v = \frac{(\tau + c)^3}{27}$$

Now to convert back, σ is the boundary, so

$$x(0,\sigma) = \begin{cases} 0 & \text{if } \sigma < 0 \\ \sigma & \text{if } \sigma > 0 \end{cases}$$

$$t(0,\sigma) = \begin{cases} -\sigma & \text{if } \sigma < 0 \\ 0 & \text{if } \sigma > 0 \end{cases}$$

Then we have that $x - t = \sigma$. Furthermore, since $\frac{\partial x}{\partial \tau} = 1$, $x = \tau + x(0,\sigma)$

$$\tau = x - x(0,\sigma) = \begin{cases} x - 0 & \text{if } \sigma < 0 \\ x - \sigma & \text{if } \sigma > 0 \end{cases}$$

$$\tau = \begin{cases} x & \text{if } x < t \\ t & \text{if } x > t \end{cases}$$

Then we have

$$v(x,t) = \begin{cases} (x+c)^3/27 & \text{if } x < t \\ (t+c)^3/27 & \text{if } x > t \end{cases}$$

For the initial conditions, v(0, t) = 1 and v(x, 0) = 0, This means (as $x = 0 < t$ and $t = 0 < x$)

$$v(x,t) = \begin{cases} (x+3)^3/27 & \text{if } x < t \\ (t)^3/27 & \text{if } x > t \end{cases}$$

However, x ¿ t means that we are trying to find the volume of a crystal where there is no matrix. So the only solution should be

$$v(x,t) = \frac{(x+3)^3}{27}$$