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1. a) We have that $p_n \to p^* = 0$ and that $p_{n+1} = ln(p_n+1)/2, p_0 = 1$
Now we want to show that p_n converges linearly.
If we set

$$g(x) = \frac{1}{2} ln(x+1)$$

We can apply the fixed point theorem if $|g'(x)| < 1$ for all $x \in I$ and g maps I to itself for some interval I

Let $I = [0,2]$. $g(I) = [0, 0.346] \in I$ and

$$g'(x) = \frac{1}{2(x+1)}$$

$g'(I) = [\frac{1}{6}, 0.5]$ So $|g'(x)| < 1$ for all $x \in I$.

Therefore the fixed point theorem gives us a unique fixed point, and that $p_{n+1} = ln(p_n+1)/2$ converges linearly.

b) We have that $p_n \to p^* = 1$ and that $p_n = 1 + 2^{1-n} + (n+2)^{-n}, p_0 = 4$
Now we want to show that p_n converges linearly.

$$\lim_{n\to\infty} \frac{|p_{n+1} - p^*|}{|p_n - p^*|} = \lim_{n\to\infty} \frac{|2^{1-(n+1)} + ((n+1)+2)^{-(n+1)}|}{|2^{1-n} + (n+2)^{-n|}}$$

$$= \lim_{n\to\infty} \frac{|2^{-n} + (n+3)^{-n-1}|}{|2^{1-n} + (n+2)^{-n}|} = 1$$

So therefore the sequences converges linearly.
2 We have that $p_n = 10^{-2^n}$
Now we want to show that p_n converges and that it converges quadratically.
Clearly

$$\lim_{n\to\infty} 10^{-2^n} = 0$$

So now we only need to prove quadratic convergence:

$$\lim_{n\to\infty} \frac{|10^{-2^(n+1)}|}{|10^{(-2^n)^\alpha}|} = \lim_{n\to\infty} |10^{-(2^n)*2+(2^n)*\alpha}| = 1$$

for $\alpha = 2$. Therefore we have quadratic convergence
3. We have $f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 3$
a) Lagrange interpolation to find polynomial f:
We define

$$L_1(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} = \frac{(x-2)(x-3)(x-4)}{-6}$$

$$L_2(x) = \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} = \frac{(x-1)(x-3)(x-4)}{2}$$

$$L_3(x) = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} = \frac{(x-1)(x-2)(x-4)}{-2}$$

$$L_4(x) = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} = \frac{(x-1)(x-2)(x-3)}{6}$$

Then we have

$$f(x) = L_1(x)f(1) + L_2(x)f(2) + L_3(x)f(3) + L_4(x)f(4)$$

$$f(x) = 2L_1(x) + L_2(x) + 4L_3(x) + 3L_4(x)$$

Which is a polynomial of degree 3 that satisfies the 4 equations:
$f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 3$
b) Using Nevilles method we construct

$$P_1(x) = 2$$
$$P_2(x) = 1$$
$$P_3(x) = 4$$
$$P_4(x) = 3$$

$$P_{12}(x) = \frac{(x-1)P_2(x) - (x-2)P_1(x)}{2-1} = -x+3$$
$$P_{23}(x) = \frac{(x-2)P_3(x) - (x-3)P_2(x)}{3-2} = 3x-5$$
$$P_{34}(x) = \frac{(x-3)P_4(x) - (x-4)P_3(x)}{4-3} = -x+7$$

$$P_{123}(x) = \frac{(x-1)P_{23}(x) - (x-3)P_{12}(x)}{3-1} = 2x^2 - 7x + 7$$
$$P_{234}(x) = \frac{(x-2)P_{34}(x) - (x-4)P_{23}(x)}{4-2} = -2x^2 + 13x - 17$$

$$f(x) = \frac{(x-1)P_{234}(x) - (x-4)P_{123}(x)}{4-1} = \frac{1}{3}(-4x^3 + 30x^2 - 65x + 45)$$

Which is a polynomial of degree 3 that satisfies the 4 equations:
$f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 3$

Furthermore, this is the same polynomial as constructed in a) since we have 4 equations and 4 unknowns (coefficients of $x^3, x^2, x, 1$) so they must be the same polynomial.

4. Refer to code and graphs