Tameez Latib
Problem 15
I, Tameez Latib, declare that this work is my own. I did this work honestly and can fully stand behind everything that I have written.
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Given that cars obey the principle:

$$u(x,t) = u_{max}(1 - \frac{\rho(x,t)}{\rho_{max}})$$

To non-dimensionalise this model,
$u = \bar{u}v$
$\rho = \bar{\rho}p$
Then:

$$v(x,t) = \frac{u_{max}}{\bar{u}}(1 - \frac{p(x,t)\bar{\rho}}{\rho_{max}})$$

By the inviscid burgers equation,

$$v_t + (\frac{1}{2}v^2)_x = 0$$

$$(\frac{u_{max}}{\bar{u}}(1 - \frac{p(x,t)\bar{\rho}}{\rho_{max}}))_t + (\frac{1}{2}(\frac{u_{max}}{\bar{u}}(1 - \frac{p(x,t)\bar{\rho}}{\rho_{max}}))^2)_x = 0$$

Since constants are differentiated to 0, (and letting $u_{max}\bar{\rho} = \rho_{max}\bar{u}$)

$$(-p)_t + (\frac{1}{2}(\frac{u_{max}}{\bar{u}}(1 - \frac{p\bar{\rho}}{\rho_{max}}))^2)_x = 0$$

$$(-p)_t + (\frac{1}{2}(-2p + p^2\frac{\bar{\rho}}{\rho_{max}}))_x = 0$$

Letting $\bar{\rho} = 2\rho_{max}$

$$(-p)_t + (-p + p^2)_x = 0$$

$$(p)_t + (p - p^2)_x = 0$$

Solving this with the constraint that

$$p(x,0) = \begin{cases} 0 & \text{if } x < -1 \\ 1 & \text{if } -1 < x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Which describes traffic building up at a red light (at x = 0) which becomes green at t = 0.
Since

$$(p)_t + (p - p^2)_x = 0$$

is nonlinear, let's use characteristics find a curve X(t) and then have solutions in the form p(X(t), t)

Note that

$$(p)_t + (1 - 2p)p_x = 0$$

Then if we let X'(t) = (1-2p), $(X(t) = (1 - 2p)t)$ then we have

$$\frac{dp}{dt}(X(t), t) = p_t + (1 - 2p)p_x = 0$$

And so

$$p(x, t) = p(x - (1 - 2p)t, 0)$$

But there is a shock now since the value does not match when x = 0 (from right we have p = 0, from left we have p = 1)

So by rankine-hugoniot condition,

$$s'(t) = \frac{[p - p^2]}{[p]}$$

since $p^- = 1$ and $p^+ = 0$,

$$s'(t) = \frac{0 - (1 - 1^2)}{0 - 1} = 0$$

$s(t) = 0$ as $s(0) = 0$ (where it originates)

[This doesn't make much sense, the shock is 0 meaning that p = 1 for x ¡ 0 and p = 0 for x ¿ 0 but this is only true at t = 0. After this point, cars pass the x = 0 point and so the density should be positive.]

There is also a rarefaction with no values near x = -1 as to the left the curves are straight vertical lines and to the right they extend towards the right. In this case, the rarefaction begins at t = 0 as well, so we have

$n = x/t, p(x, t) = r(n)$

Since

$$p_t = -\frac{x}{t^2}r'(n)$$

$$p_x = \frac{1}{t}r'(n)$$

Then the equation

$$(p)_t + (1 - 2p)p_x = 0$$

Can be written as

$$-\frac{x}{t^2}r'(n) + (1 - 2p)\frac{1}{t}r'(n) = 0$$

Since r'(n) is not 0,

$$1 - 2p = n = \frac{x}{t}$$

$$p = \frac{t - x}{2t}$$

Where this solution is valid when $-1 < x < -t$ (as x = -t, p = 1, corresponding to the curves already found)

The rarefaction fans out solutions from x = -1 (left) with p = 0 at t = 0 and x = -1 (right) with p = 1 at t = 0

From this, we have that

$$p(x,t) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{t-x}{2t} & \text{if } -1 < x < -t \\ 1 & \text{if } -t < x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Since it is a triangle wave, it's very symmetric and so we only need to take the RMS from 0 to T/4

$f(x) = 4ax/T$

$f^2 = 16a^2x^2/T^2$ and so

$$RMS = \sqrt{4/T * \int_0^{T/4} 16a^2x^2/T^2 dt} = \sqrt{\frac{64a^2(T/4)^3}{3T^3}}$$

$$RMS = a\sqrt{3}$$

And $V_{pp} = 2a$, so

$$\frac{RMS}{V_{pp}} = \frac{\sqrt{3}}{2}$$