# Linear Models and Proportionality

MATH 142, UCLA

# Linear Models and Proportionality

In trying to model complex phenomena with limited information, or to formulate a tractable model suitable for analysis and simple qualitative insights, linear models are often developed. This includes the assumption that some quantity is proportional to another, i.e., that the two quantities differ by a multiplicative factor In many cases, despite the nonlinear nature of the world in how consumer demand responds to price, of how a spring's restoring force responds to its change in length, and the likes, linear models are surprisingly insightful!

# Revenue, Cost, and Profit

We begin by considering a company with a monopoly: they sell a single product to consumers in a market with no competition. Think of SoCal Gas, or Time Warner Cables to some degree.

Revenue $R$ is the total amount of money (often per unit time) that a company earns through sales and services. For a single product line, the revenue generated by that product type is

$$R = pq$$

where $p$ is the price of a single item and $q$ is the number of items that sell per unit time (often called the demand).

The cost $C$ is the total expense in producing (and selling) a collection of items

$$C = C(q).$$

The profit $P$ is the net money earned by a company,

$$P = R - C.$$

# Revenue, Cost, and Profit

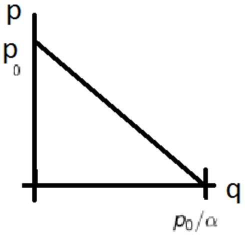
Demand is a function of price $q = q(p)$. Also, to achieve a given demand, there is an appropriate price so $p = p(q)$.
Assume that

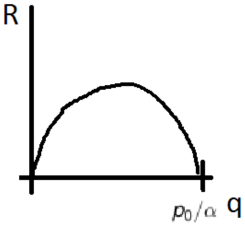
$$p(q) = p_0 - \alpha q$$

is linear with $p_0, \alpha > 0$: if demand is to rise, price must decline.
Treating $R$ as a continuous function of $q$,

$$R(q) = p_0 q - \alpha q^2$$

is maximized halfway between $q = 0$ where price is $p_0$ and $q = p_0/\alpha$ where price is 0.



# Revenue, Cost, and Profit

The function $R(q)$ in this case is a downward parabola: revenue begins at 0, steadily increases to a maximum point at some demand-price pair, and then decreases to 0.

The optimal price point is a delicate balance between raising the price to earn more revenue per item sold and deterring customers with the increasing prices.

# Revenue, Cost, and Profit

In maximizing profit, we set $P'(q) = R'(q) - C'(q) = 0$. Assuming of course that at such a point, $P'' < 0$, we have a maximum.

The quantities $R'(q)$ and $C'(q)$ are called the marginal revenue and marginal cost, respectively. They respectively represent the change in revenue in producing and selling one more item (per unit time) and change in cost in producing one more item (per unit time):
$R(q+1) - R(q) \approx R'(q)$, $C(q+1) - C(q) \approx C'(q)$.

At maximum profit, marginal revenue equals marginal cost. As long as the marginal revenue exceeds the marginal cost, then producing and selling one more item per unit time should bring in more revenue than it costs to produce one more item per unit time, and profits will increase. If the marginal revenue is less than the marginal cost, the profit will decline in producing and selling one more item per unit time. The sweet spot occurs when the the profit has risen all that it can, just before it declines.

# Mass-Spring System

The momentum of an object is its mass times its velocity. Velocity is the rate of change of position with respect to time.
Newton's second law: force is the defined as the rate of change of momentum. With constant mass, force is mass times acceleration (rate of change of its velocity with respect to time)

$$\underline{F} = \frac{\mathrm{d}\underline{p}}{\mathrm{d}t} = m\frac{\mathrm{d}^2\underline{x}}{\mathrm{d}t^2}$$

where $\underline{F}$ is the force (a vector), $m$ is the mass of an object, $\underline{x}$ is the position of an object (a vector), $\underline{p} = m\frac{\mathrm{d}\underline{x}}{\mathrm{d}t}$ is momentum, and $t$ is the time.

# Mass-Spring System

In terms of proportionalities, etc:

- The force $\underline{F}$ required to accelerate an object with acceleration $\underline{a}$ is proportional to the acceleration $\underline{a}$ with constant of proportionality $m$, the mass. $\underline{F} \propto \underline{a}$. By increasing the acceleration, the force must go up for a fixed mass and vice-versa: by increasing the force with mass fixed, the acceleration should go up.
- With $\underline{a} = \frac{1}{m}\underline{F}$, the acceleration is inversely proportional to the mass: the "constant of proportionality" is the force. For a given force, the acceleration of an object decreases as its mass increases and to increase the acceleration of an object with force fixed, its mass must decrease.

# Mass-Spring System

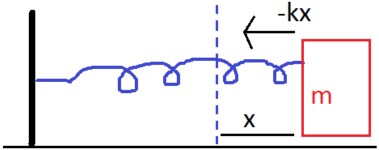
The units of mass $[m]$ are typically kilograms (kg). The units of position $[\underline{x}]$ are typically metres (m). The units of time $[t]$ are typically seconds (s). The units of force $[\underline{F}]$ are Newtons (N) where 1 N $=$ 1 kg m/s$^2$.

Springs are coils of material that have a preferred length: when they are stretched beyond this preferred length, they pull back, and when they are compressed shorter than this length, they push out. If $x$ denotes the position of one end of a spring with respect to its equilibrium (in the direction of the spring), the most naive assumption is that the spring force

$$F_{\text{spring}} = -kx$$

where $k$ is called the spring constant.
The units of the spring constant $[k]$ are N/m.

# Mass-Spring System




Consider a mass attached to a spring on a horizontal table (without friction), with $\underline{x} = x$ being the displacement from equilibrium in the direction of the spring. Then

$$mx''(t) = -kx \implies mx''(t) + kx(t) = 0$$

describes the motion of the mass.
This is a second-order linear ODE. Given initial conditions $x(0) = x_0$ the initial position and $x'(0) = v_0$ the initial velocity then

$$x(t) = x_0 \cos(\sqrt{k/m}t) + v_0\sqrt{m/k}\sin(\sqrt{k/m}t)$$

is the solution (by solving the second-order constant coefficient linear ODE).

# Mass-Spring System

Due to the spring's restoring force, pulling in when extended and pushing out when compressed, and the lack of friction, the mass will oscillate back and forth forever.

# Mass-Spring System

We can also learn more about the mass-spring system with "mathemagic":

$$\begin{aligned} mx''(t) + kx(t) = 0 &\implies mx'(t)x''(t) + kx(t)x'(t) = 0 \\ &\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2\right) = 0 \end{aligned}$$

We call $\frac{1}{2}mx'(t)^2$ the kinetic energy: energy of movement.
And we call $\frac{1}{2}kx(t)^2$ the potential energy of the spring: potential to do work (cause motion) within the spring.
The sum of the kinetic and potential energies is constant.

## Summary

Modelling techniques:

- picking a linear equation with the correct qualitative behaviour
- interpreting proportionality statements
- looking for symmetries

Concepts:

- profit, revenue, and cost
- marginals
- forces and accelerations
- springs
- conservation of energy

Mathematical details:

- optimization
- second-order constant coefficient linear ODEs
- multiplying by a derive and integrating trick