Tameez Latib

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1.

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 9 \\ 3 & 7 & 11 & 14 \\ 4 & 9 & 14 & 19 \end{bmatrix}$$

We want to find the LU factorization, so first we pivot on C_{11}:

$$C_1 = L_1 C$$

Where

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}$$

So that

$$C_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

And now we pivot on $(C_1)_{22}$:

$$C_2 = L_2 C_1$$

Where

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

So that

$$C_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

And now we pivot on $(C_2)_{33}$:

$$C_3 = L_3 C_2$$

Where

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

So that

$$C_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we have that $U = C_3$ and $L = (L_3 L_2 L_1)^{-1} = L_1^{-1} L_2^{-1} L_3^{-1}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 \end{bmatrix}$$

$$U \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $A \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant, then if $A = D - R$, where D is a diagonal matrix, then

a. Show $||D^{-1}R||_\infty < 1$

Since A is strictly diagonally dominant,

$$|D_{ii}| = |A_{ii}| > \sum_{j=1, j \neq i}^{n} |A_{ij}| = \sum_{j=1, j \neq i}^{n} |R_{ij}|$$

for all i

Therefore,

$$\sum_{j=1, j \neq i}^{n} |(D^{-1}R)_{ij}| = \sum_{j=1, j \neq i}^{n} |(D^{-1})_{ii} R_{ij}| = \frac{\sum_{j=1, j \neq i}^{n} |R_{ij}|}{D_{ii}} < 1$$

for all i

Also note that $(D^{-1}R)_{ii} = 0$, so we then have

$$\sum_{j=1}^{n} |(D^{-1}R)_{ij}| < 1$$

for all i, so $||D^{-1}R||_\infty < 1$

b. Show the iteration

$$x_{k+1} = D^{-1}Rx_k + D^{-1}b$$

converges and converges to the solution of $Ax = b$

Since $||D^{-1}R||_\infty < 1$, we can apply fixed point theorem element-wise on x_k,