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Problem 10

I, Tameez Latib, declare that this work is my own. I did this work honestly and can fully stand behind everything that I have written.

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Given a pendulum with the following information: Thin rod, length L = 1m; mass of rod is negligible, mass at pivot point m = 1kg, at t = 0, $\theta = 0$, and 400 J Kinetic Energy is imparted into the pendulum.

We know $E = 1/2 * mv^2$

Therefore $v = \sqrt{(2E/m)}$

Also, $-mgLsin(\theta) = torque_{net} = I\theta''$

As the net force is due to gravity, and taking the magnitude of the cross product of r and f (to get torque), we get the left hand side.

Here, $I = mL^2$,

So our differential equation is

Also, $-g/L * sin(\theta) = \theta''$

From $v = \sqrt{(2E/m)}$, this tells us $\theta'(0) = 1/L * \sqrt{(2E/m)} = \sqrt{\frac{2E}{mL^2}}$

Let's nondimensionalise the model, by letting

$$t = \tau * \bar{t} = \tau * \sqrt{\frac{mL^2}{2E}}$$

I.e. $\theta'(0) = 1/\bar{t}$

And note θ is already unit-less, so we have

$$-\frac{g}{L} * sin(\theta) = \theta''/\bar{t}^2$$

$$-\frac{g\bar{t}^2}{L} * sin(\theta) = \theta''$$

Let $\epsilon = \frac{g\bar{t}^2}{L}$, so

$$-\epsilon * sin(\theta) = \theta''$$

Since $\epsilon < 1$, we make the ansantz that $\theta = \theta_0 + \epsilon * \theta_1 ...$

Furthermore, note the taylor expansion:

$$sin(\theta) = x - x^3/6 + ...$$

So we have

$$0 = \theta_0''$$

Subject to $\theta_0(0) = 0$, $\theta_0'(0) = 1$ (Since $\theta_0'(t = 0) = 1/\bar{t}$). Then it is obvious that

$$\theta_0 = \tau$$

And our other differential equation is:

$$-\epsilon(\theta_0 + O(\epsilon) - (\theta_0 + O(\epsilon))^3/6 + ...) = \epsilon * \theta_1''$$

Grouping all the O(1) terms,

$$(\theta_0) - (\theta_0))^3/6 + ...) = \theta_1''$$

And the left side is simply the taylor series for sin, so

$$-sin(\theta_0) = \theta_1''$$

$$\theta_1'' = -sin(\tau)$$

$$\theta_1 = sin(\tau) + C\tau + D$$

Given that $\theta_1(0) = 0$, $\theta_1'(0) = 0$
Then D = 0 and C = -1
So the final solution is

$$\theta = \tau + \epsilon(sin(\tau) - \tau)$$