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1.

$f(x) = 1/x, x_i = i+1, 0 \leq i \leq 2$, find Lagrange polynomial interpolating $(x_i, f(x_i))$ by

a. Lagrange formula:

We define

$$L_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)}$$

$$L_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)}$$

$$L_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)}$$

And our solution is

$$P(x) = L_0(x) * f(x_0) + L_1(x) * f(x_1) + L_2(x) * f(x_2)$$

$$P(x) = \frac{11}{6} - x + \frac{1}{6}x^2$$

And to check, $P(1) = 1, P(2) = 1/2, P(3) = 1/3$ So therefore this is the unique $P(x)$ corresponding to the interpolation

b. Now using Nevilles method:

We define the following polynomials, through an iterative process, where

$$P_{n_1,n_2,...n_m}(x_j) = f(x_j), j \in \{n_1, n_2, ...n_m\}$$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2}$$

$$P_2(x) = \frac{1}{3}$$

$$P_{0,1}(x) = \frac{P_1(x)(x-1) - P_0(x)(x-2)}{2-1} = \frac{3-x}{2}$$

$$P_{1,2}(x) = \frac{P_2(x)(x-2) - P_1(x)(x-3)}{3-2} = \frac{5-x}{6}$$

$$P_{1,2,3}(x) = \frac{P_{1,2}(x)(x-1) - P_{0,1}(x)(x-3)}{3-1} = \frac{11}{6} - x + \frac{1}{6}x^2$$

And this is the same P as before

c. Now using divided differences:

$$f[x_0] = 1$$
$$f[x_1] = \frac{1}{2}$$
$$f[x_2] = \frac{1}{3}$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{2-1} = \frac{-1}{2}$$
$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{3-2} = \frac{-1}{6}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - [x_0, x_1]}{3-1} = \frac{1}{6}$$

And our P is now

$$P(x) = f[x_0] + f[x_0, x_1](x-1) + f[x_0, x_1, x_2](x-1)(x-2) = \frac{11}{6} - x + \frac{1}{6}x^2$$

Which is again the same polynomial.
2. Find the natural cubic spline passing through $(-1, 1), (0, 1), (1, 2)$
Let

$$S_1(x) = a_1 + b_1(x+1) + c_1(x+1)^2 + d_1(x+1)^3$$
$$S_2(x) = a_2 + b_2(x) + c_2(x)^2 + d_2(x)^3$$

By the endpoint conditions,
$S_1(-1) = 1 \implies a_1 = 1$,
$S_2(0) = 1 \implies a_2 = 1$,
$S_1(0) = 1 \implies b_1 + c_1 + d_1 = 0$,
$S_2(1) = 2 \implies b_2 + c_2 + d_2 = 1$
Because it is natural, we have
$S_1''(-1) = 0 \implies c_1 = 0$,
$S_2''(1) = 0 \implies 2c_2 + 6d_2 = 0$
And the continuity of first/second derivatives give us the conditions
$S_1'(0) = S_2'(0) \implies b_1 + 2c_1 + 3d_1 = b_2$,
$S_1''(0) = S_2''(0) \implies 2c_1 + 6d_1 = 2c_2$
Doing some algebra we get
$d_1 = c_2/3, d_2 = -c_2/3, b_1 = -c_2/3, b_2 = 2c_2/3$
So then
$c_2 = 3/4$
Our final solution is

$$S_1(x) = 1 - \frac{1}{4}(x+1) + \frac{1}{4}(x+1)^3$$
$$S_2(x) = 1 + \frac{1}{2}(x) + \frac{3}{4}(x)^2 - \frac{1}{4}(x)^3$$

$$S(x) = \begin{cases} S_1(x) & -1 \leq x \leq 0 \\ S_2(x) & 0 \leq x \leq 1 \end{cases}$$

With $S(x)$ being our cubic spline

3. Prove the following theorem: Let $f \in C^1([a,b])$ and $x_0, x_1, ...x_n$ be n distinct nodes in $[a,b]$, and let

$$H(x) = \sum_{i=0}^{n}(f(x_i)H_{n,i}(x) + f'(x_i)\hat{H}_{n,i}(x))$$

where

$$H_{n,j}(x) = (1 - 2(x - x_j)L'_{n,j}(x_j))L^2_{n,j}(x)$$

$$\hat{H}_{n,j}(x) = (x - x_j)L^2_{n,j}(x)$$

Then $H(x_i) = f(x_i), H'(x_i) = f'(x_i)$ for $0 \leq i \leq n$

Note that if we plug in $x_i \in \{x_0, x_1, ...x_n\}$,

$$H_{n,j}(x_i) = (1 - 2(x_i - x_j)L'_{n,j}(x_j))L^2_{n,j}(x_i) = \delta_{i,j}$$

$$\hat{H}_{n,j}(x_i) = (x_i - x_j)L^2_{n,j}(x_i) = 0$$

Similarly, if we plug in x_i to these functions derivatives,

$$H'_{n,j}(x_i) = (-2L'_{n,j}(x_j))L^2_{n,j}(x_i) + 2L'_{n,j}(x_i) * (1 - 2(x_i - x_j)L'_{n,j}(x_j)) = 0$$

$$\hat{H}'_{n,j}(x_i) = L^2_{n,j}(x_i) + 2(x_i - x_j)L'_{n,j}(x_i) = \delta_{i,j}$$

Therefore

$$H(x_j) = \sum_{i=0}^{n}(f(x_i)H_{n,i}(x_j) + f'(x_i)\hat{H}_{n,i}(x_j))$$

$$= \sum_{i=0}^{n} f(x_i)\delta_{i,j} = f(x_j)$$

$$H'(x_j) = \sum_{i=0}^{n}(f(x_i)H'_{n,i}(x_j) + f'(x_i)\hat{H}'_{n,i}(x_j))$$

$$= \sum_{i=0}^{n} f'(x_i)\delta_{i,j} = f'(x_j)$$

4. Let $f \in C^3([x_0 - h, x_0 + h])$

4.a: $P(x)$, the lagrange polynomial interpolating nodes $x_0 - h, x_0, x_0 + h$

Define

$$L_-(x) = \frac{(x-x_0)(x-x_0-h)}{(x_0-h-x_0)(x_0-h-x_0-h)} = \frac{(x-x_0)(x-x_0-h)}{2h^2}$$
$$L_0(x) = \frac{(x-x_0+h)(x-x_0-h)}{(x_0-x_0+h)(x_0-x_0-h)} = \frac{(x-x_0+h)(x-x_0-h)}{-h^2}$$
$$L_+(x) = \frac{(x-x_0)(x-x_0+h)}{(x_0+h-x_0)(x_0+h-x_0+h)} = \frac{(x-x_0)(x-x_0+h)}{2h^2}$$

So then

$$P(x) = L_-(x)f(x_0-h) + L_0(x)f(x_0) + L_+(x)f(x_0+h)$$

4.b

$$E(x) = \frac{f^{(3)}(\xi(x))}{6}(x-x_0+h)(x-x_0)(x-x_0-h)$$

By the lagrange interpolation theorem, where $\xi(x) \in [x_0-h, x_0+h]$

4.c

$$f'(x_0) = P'(x_0) + E'(x_0)$$

Note that

$$L'_-(x_0) = \frac{-h}{2h^2} = \frac{-1}{2h}$$
$$L'_0(x_0) = 0$$
$$L'_+(x_0) = \frac{h}{2h^2} = \frac{1}{2h}$$

$$f'(x_0) = \frac{-f(x_0-h)+f(x_0+h)}{2h} - \frac{h^2 f^{(3)}(\xi(x))}{6}$$

where $\xi(x) \in [x_0-h, x_0+h]$

4.d

if f is a polynomial of degree less than or equal to 2, then $f^{(3)}(x) = 0 \implies E(x) = 0$. In other words,

$$f(x) = P(x)$$

and so their derivatives must also be equal;

$$f'(x) = P'(x)$$

4.e

$$f'(x_0) - P'(x_0) = E'(x_0) = -\frac{h^2 f^{(3)}(\xi(x))}{6}$$

where $\xi(x) \in [x_0-h, x_0+h]$

5. Refer to code