Difference Equations

MATH 142, UCLA

Difference Equations

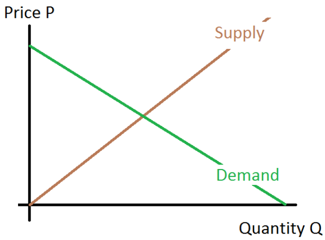
Some systems logically evolve in discrete steps such as, for example, paying off a mortgage. Other systems behave almost as though they happen in discrete steps, such as how a rumor spreads in a city. The resulting systems, often expressed at discrete times, give rise to difference equations. Like their continuous counterparts, differential equations, many difference equations cannot be solved and one resorts to numerical or graphical techniques to gain insights into the systems.

Supply and Demand

We consider a competitive market: a market where different companies sell goods of the same type and quality. All consumers have complete knowledge of the market and prices each company charges and all companies have access to the same internal procedures and technologies.
Let's consider the strawberry market (usually a market such as wheat is used). For simplicity, we assume that each year strawberry farmers have control over their yield, by their choice of man-hours and equipment.
Given any fixed price for strawberries, an individual farmer knows how many strawberries to produce to maximize their profit. The combination of all their strawberries is the market supply for a given price.
On the market as a whole, for a given price, a total quantity of strawberries will be purchased by consumers, their demand.

Supply and Demand

In this competitive market, when the market yield is q, individual farmer's should charge exactly $p(q)$ where $p(q)$ is the price consumers are willing to pay such that the entie market supply is purchased.



Supply and Demand

Consider the strawberry market. Each year, based on the selling price from the previous year, the farmers as whole decide on how many pints of strawberries to produce the following year by assuming the same selling price such that

$$s_n = \alpha p_{n-1}$$

where s denotes the supply and p the price with $\alpha > 0$.
At the market each year, the demand of the consumers is

$$d_n = \beta - \gamma p_n$$

with $\beta, \gamma > 0$, and where the price is found so as to equate the supply and demand.
Then

$$p_n = \frac{\beta}{\gamma} - \frac{\alpha}{\gamma} p_{n-1}$$

describes the relationship between the old and new prices per pint of strawberries.
Let's imagine that $p_0 = \$2$, $\alpha = 11000$ pints/\$, $\beta = 100000$ pints, $\gamma = 10000$ pints/\$.

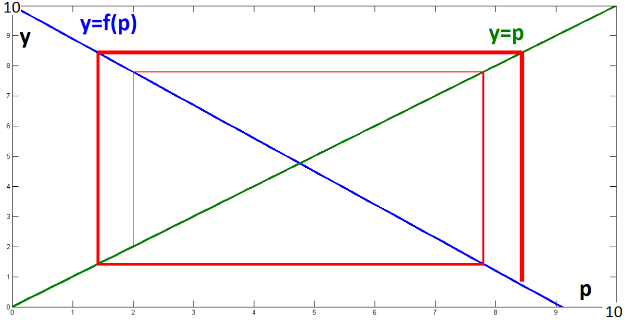
Supply and Demand

The supply and demand only make sense when they are positive, and the price should always be positive. Thus, $0 < p < \beta/\gamma = 10$. Outside of this range, the model is not valid.
We have the sequence $\{p_n = 10 - 1.1p_{n-1}\}_{n=1,2,3,\ldots}, \quad p_0 = 2$.
Recurrence equations such as $p_{n+1} = 10 - 11p_n/10$ are called difference equations: can also be written as $p_{n+1} - p_n = 10 - p_n/10$.
In this case, we find the sequence
$2, 7.8, 1.42, 0.7182, 9.20998, -0.130978$...

Supply and Demand

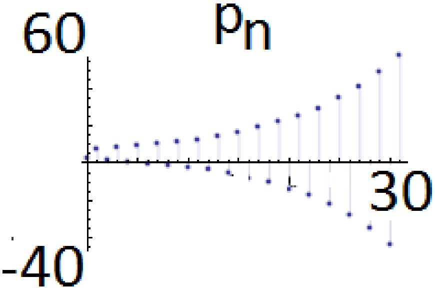
We can depict the situation graphically. Let $f(p) = 10 - 1.1p$. Start with p_0 and produce a cobweb diagram:

- Given a p-coordinate p_i, evaluate $p_{i+1} = f(p_i)$. Find the point (p_i, p_{i+1}) on curve $y = f(p)$.
- Trace a curve from (p_i, p_{i+1}) to intersect $y = p$ at (p_{i+1}, p_{i+1}) so the new p-coordinate is the new p-value. Repeat...
  (Trace from (p_{i+1}, p_{i+1}) to intersect $y = f(p)$ at $(p_{i+1}, f(p_{i+1}) = p_{i+2})$ to find p_{i+2}, etc.)



Supply and Demand

We can analyze this problem from an analytic perspective, too.
We wish to find p_n explicitly in terms of n. These are often difficult or impossible to solve analytically. In this case, we can solve the equation.
First, we find p^* such that $p^* = 10 - 11p^*/10$, a fixed point. We have $p^* = 100/21$.
Now, we consider a different sequence $a_n = p_n - p^*$ and find $a_{n+1} = -11a_n/10$.
If $a_0 = p_0 - p^*$ is given, $a_1 = -11a_0/10$, $a_2 = -11a_1/10 = (11/10)^2 a_0$, and in general $a_n = (-11/10)^n a_0$ so that $p_n = \frac{100}{21} - \frac{58}{21}(\frac{-11}{10})^n$.



Supply and Demand

The model, while valid, suggests that prices can fluctuate wildly about the equilibrium.

Validity is lost due to the price dropping below zero: a different formulation of the supply and demand equations could salvage this somewhat.

Effectively, what happens, starting with $p_0 = 2$ is that an abundance of strawberries were produced and farmers had to sell strawberries very cheaply one year. The next year, given how little they earned per pint, they are less inclined to produce as many strawberries and produce fewer the next year. With fewer strawberries available in total, the farmers can charge more and still sell everything they produced. With this higher selling price, they intend to produce more strawberries the next year at a high price, but they must lower their price to sell all they have produced, and the cycle goes on and on.

Spreading of a Rumor

Consider a community of people that often interact with each other: a school, small village, etc. If someone "knows something" another doesn't and they interact for some length of time, it's likely that information, correct or not, will get passed on. To model this process mathematically, we will define some variables.

Let the community consist of N individuals wherein each day they go about their business and have casual conversations with others. If there's a rumor going around and someone knows about it, with each individual they speak with, there is a probability p that they will share the information. Let r_0 denote the number of individuals who know the rumor at $t = 0$ and r_j denote the number who know the rumor at time $j\Delta t$ where Δt is the average social interaction time: the time where one individual interacts with another.

Spreading of a Rumor

Under some assumptions, at time $j\Delta t$, an individual who knows the rumor will, on average, pass the rumor on to $p\frac{N-r_j}{N-1}$ individuals over the next time interval.

Roughly then, we have:

$$r_{j+1} - r_j = pr_j\frac{N-r_j}{N-1}$$

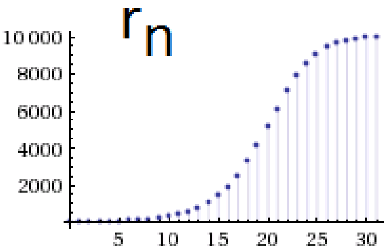
or that

$$r_{j+1} = ((1 + p\frac{N}{N-1}) - \frac{pr_j}{N-1})r_j$$

Technically $r_{j+1} = \min\{N, ((1 + p\frac{N}{N-1}) - \frac{pr_j}{N-1})r_j\}$

Spreading of a Rumor

With $N = 10000$, $r_0 = 10$, and $p = 0.4$, we plot the result.



Stability Theorem: if $x_{n+1} = f(x_n)$ is a recurrence relation with **fixed point** $x^* = f(x^*)$ then the point is asymptoticaly stable, respectively unstable, if $|f'(x^*)| < 1$, respectively, $|f'(x^*)| > 1$.

Asymptotically stable: there is some $\delta > 0$ such that for every x_0 obeying $|x_0 - x^*| < \delta$ then $|x_{n+1} - x^*| < |x_n - x^*|$ and $x_n \to x^*$ as $n \to \infty$. (start close enough to x^*, approach x^*)

Unstable: for every $\delta > 0$, there is some x obeying $|x - x^*| < \delta$ but where $|f(x) - x^*| > |x - x^*|$ (no matter how close to x^* we start, we can move away)

Spreading of a Rumor

Both $r^* = 0$ and $r^* = N$ are fixed points for
$f(r) = r(1 + p\frac{N}{N-1} - \frac{pr}{N-1}) = (1 + p\frac{N}{N-1})r - \frac{pr^2}{N-1}$.
$f'(0) = (1 + p\frac{N}{N-1}) > 0$ so 0 is unstable.
$f'(N) = 1 - p\frac{N}{N-1} \in (1 - \frac{N}{N-1}) \subset (-1, 1)$, for $0 < p \leq 1$, so N is stable.
Note that with $p = 0$, $f(r) = r$.
For any initial number of people who know the rumor, that number will tend to grow, moving away from the case where very few know the rumor $r = 0$ and gravitating to the case where everyone knows the rumor $r = N$.

Summary

Modelling techniques:

- postulating how a future state depends upon the past
- observing computer-aided and analytic techniques to draw conclusions

Concepts:

- supply and demand
- interactions among a well-mixed population

Mathematical details:

- plots of recurrence relations: cobwebs or value vs term number
- stability of fixed points
- solving a linear one-step recurrence with constant coefficients