Formulating Models

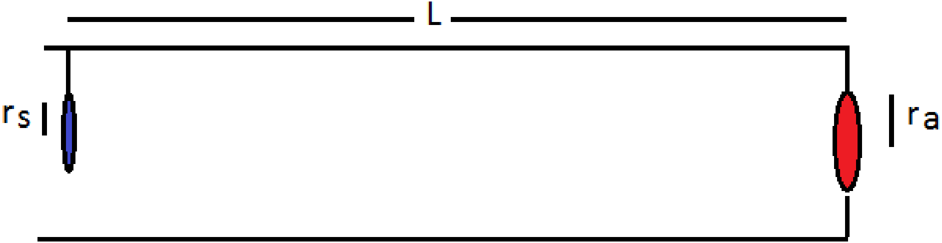
MATH 142, UCLA

Math Models

Mathematical modelling is the process of taking real world questions and observations from different systems such as in engineering, medicine, chemistry, physics, economics, and so on; determining quantitative or qualitative laws that govern the system; formulating these laws, perhaps with simplifications, in a more abstract form with mathematics; employing mathematical techniques to study the resulting systems from various perspectives; and from this analysis, inferring key behaviours that drive the systems, making predictions about new phenomena, and in some cases demonstrating a proof of concept. **Unlike purely theoretical mathematics, this must always be done with the real world in mind: answers should "make sense physically" at least in cases when an intuitive answer can be obtained for comparison and they should be capable of making predictions, etc.** Another part of modelling is to understand the limitations of the models used:
All models are wrong, but some are useful. $\sim$ George Box

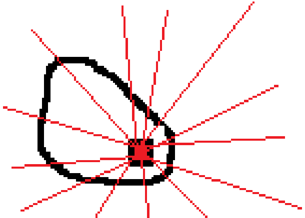
Beamline

In TRIUMF, a particle accelerator and physics research centre, a group of condensed matter physicists needed to build a passageway for a particle beam used to probe a superconductor. Within the passageway, a small sample of an exotic superconducting material of radius $r_s \approx 2$ cm would be held in place. The sample would be surrounded by a cylindrical steel vacuum chamber held at temperature 4 K and far down the chamber, at distance $L \approx 20$ m, a tiny aperture of radius $r_a \approx 5$ cm would allow the particles to enter. The superconductor would need to be kept colder than its surroundings and purchasing appropriate equipment to cool the sample required an estimate of the energy flowing into the sample at any instant. **How much is the sample heated by its surroundings?**



Beamline

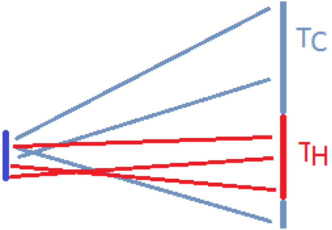
- Stefan-Boltzmann Law: the energy emitted by an object at temperature T K (Kelvins) per unit area of surface per unit time is
$$j = \epsilon\sigma T^4$$
where $\sigma \approx 5.67 \times 10^{-8}$ W m^{-2} K^{-4} is the Stefan-Boltzmann constant and $0 \leq \epsilon \leq 1$ is the emissivity.
- Energy flux in W m^{-2}, the amount of energy moving through a unit area per unit time in a given direction, due to a small area segment on an objects surface is the energy density output divided by $2\pi d^2$ where d is distance.
- The emissivity of the steel in the beamline is approximately 0.9.



Beamline

We need to make some assumptions and simplifications

- Worst case: sample absorbs all radiation it is exposed to.
- $r_a, r_s \ll L$: aspect ratio of sample size (or beam radius size) much smaller than length
- Sample can only see a narrow aperture of radius r_a at distance L away allowing input radiation at temperature $T_H \approx 300$ K (room temperature) with emissivity 1, and outside this aperture, in all directions, it sees the cold chamber of temperature $T_C \approx 4$ K.
- Idea: power absorbed by sample $\approx$ (power absorbed if it saw only T_c) - (power absorbed due to aperture at T_c) + (power absorbed due to aperture at T_H)

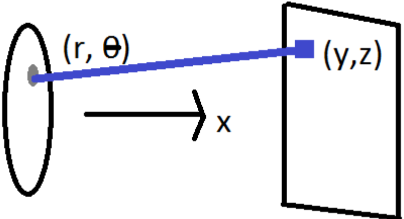


Beamline

- Only cold power:

$$P_C = \sigma \epsilon T_C^4 \int_0^{2\pi} \int_0^{r_s} r\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j(y, z; r, \theta) \mathrm{d}y \mathrm{d}z\right) \mathrm{d}r \mathrm{d}\theta$$

  where $j(y, z; r, \theta)$ is the energy moving into the sample at (r, θ) per unit time.
- $j(y, z; r, \theta) = -\hat{x} \cdot \frac{-\langle L, y - r\cos\theta, z - r\sin\theta \rangle / \sqrt{L^2 + (y - r\cos\theta)^2 + (z - r\sin\theta)^2}}{2\pi(L^2 + (y - r\cos\theta)^2 + (z - r\sin\theta)^2)}$
- By symmetry, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j(y, z; r, \theta) \mathrm{d}y \mathrm{d}z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L}{2\pi(L^2 + y^2 + z^2)^{3/2}} \mathrm{d}y \mathrm{d}z$ and in polar coordinates this is $\int_0^{2\pi} \int_0^{\infty} \frac{Ls}{2\pi(L^2 + s^2)^{3/2}} \mathrm{d}s \mathrm{d}\psi = 1$ so $P_C = \pi r_s^2 \sigma \epsilon T_C^4$.



Beamline

Through geometry, can show that power upon sample due to temperature T and emissivity ϵ with aperture radius r_a at distance L is

$$P = 2\pi \int_0^{r_s} r[\int_0^{r_a} \int_0^{2\pi} \frac{\sigma\epsilon T^4 L}{2\pi(L^2 + s^2\cos^2\theta + (r - s\sin\theta)^2)^{3/2}} s d\theta ds] dr.$$

Gruesome, even if it can be evaluated... But if $r_a \ll L$ then

$$\begin{aligned} P &= \frac{\sigma\epsilon T^4}{L^2} \int_0^{r_s} r[\int_0^{r_a} \int_0^{2\pi} (1 + O(s^2/L^2)) s d\theta ds] dr \\ &\sim \frac{\pi}{2}\sigma\epsilon T^4 r_s^2 (\frac{r_a}{L})^2 + O(r_a^4/L^4) \end{aligned}$$

Beamline

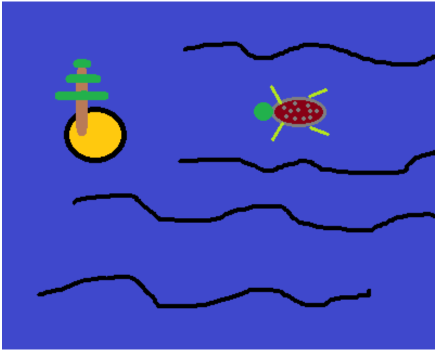
- In the worst case scenario, a power of

$$P \approx \pi r_s^2 \sigma \epsilon T_C^4 - \sigma \epsilon T_C^4 \frac{\pi}{2} r_s^2 (r_a/L)^2 + \sigma T_H^4 \frac{\pi}{2} r_s^2 (r_a/L)^2$$

  must be extracted from the sample to prevent heating.
- Room temperature head load is inversely proportional to L^2 and proportional to r_a^2: roughly speaking, if the tube length goes up by $\sim 10\%$ the room temperature heat load goes down by $\sim 20\%$.
- Makes sense: for small r_a/L, aperture angle is roughly 90 degrees and all room temperature radiation strikes sample in normal direction. Total power output by room temp sector is $\sigma T_H^4 \pi r_a^2$; and power flux should be approximately this divided by $2\pi L^2$.

Sea Turtle Navigation

[Based off modelling of Thomas Hillen at University of Alberta]: Ascension Island is a tiny island in the Atlantic Ocean where many sea turtles, originating in Brazil, travel some 2000 km to lay their eggs. Many biologists are puzzled by how the turtles make their way to the island in the first place and how strong their navigational skills must be. **What can be said about the turtles arriving to the island in relation to factors such as ocean current speeds, sea turtle swimming speed, and their navigational skills?**



Sea Turtle Navigation

- The turtles do not travel together; they all seem to go their own way.
- The turtles do not always travel straight towards the island; frequently they go off in the wrong direction and readjust their orientation.

Sea Turtle Navigation

To obtain a simple model, we make some assumptions.

- We consider a one-dimensional model with the island at $x = 0$ and the turtles initially at $x = x_0$ at time $t = 0$.
- We assume that there is an ocean current velocity v and that relative to the surrounding water, a turtle can swim with speed s.
- We assume that over each characteristic time interval Δt that a turtle is either swimming with the correct orientation at speed s relative to its surroundings with probability $0 \leq p \leq 1$ or with incorrect orientation with speed s relative to its surroundings with probability $1 - p$.
- Equivalently: assume a turtle spends a fraction p of its time going in the correct direction and the fraction $1 - p$ of its time going in the incorrect direction, both at its natural swimming speed with respect to the surrounding water.

Sea Turtle Navigation

Over a time Δt, a turtle is subject to a displacement

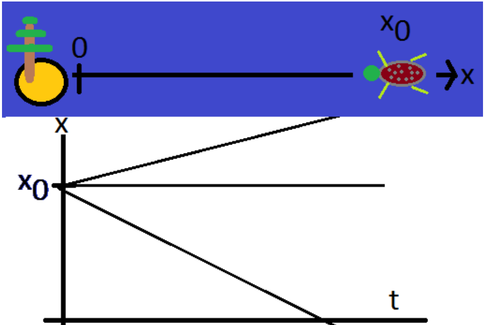
$$\Delta x = \Delta t(\underbrace{p(v-s)}_{\text{correct orientation with probability p}} + \underbrace{(1-p)(v+s)}_{\text{incorrect orientation with probability 1-p}})$$

so that roughly

$$\frac{dx}{dt} \approx v + (1-2p)s, \quad x(0) = x_0.$$

If the turtle does ever reach the island then the arrival time is $\frac{x_0}{(2p-1)s-v}$.

The turtle will arrive to the island iff $(2p-1)s - v > 0 \iff \frac{v}{s} < 2p - 1$.



Sea Turtle Navigation

- The turtle only makes it to the island if its average velocity is negative, i.e., that on average it moves towards the island.
- If $|v| > s$ then even with perfect navigation a turtle cannot reach the island when $v > 0$, and even with completely imperfect navigation a turtle will be dragged to the island when $v < 0$.
- Limitations: one dimensional is not realistic; ocean current velocities may vary in space and time; turtles may have various swimming speeds; the navigation strength may change with spatial location or even time, ...

Summary

Modelling techniques:

- breaking a value into more manageable components
- considering worst case scenario
- making a one-dimensional model

Concepts:

- black body radiation law and flux intensity vs distance
- relative velocity in a moving medium

Mathematical details:

- surface flux integrals
- polar coordinates
- Taylor series
- lines and intercepts