Thoughts: Why are they called improper integrals? Well, the easy answer is that they are not actually an integral. So, in this light, these integrals do not have a solution. What we do instead, is get an approximate solution by use of limits! There are two reasons why an integral would be improper. The first is the obvious case in that infinity is not a number that can be reached. Thus, the integral is not something that can be evaulated. The other case is if at some point on the integral bounds, the function is undefined. This is once again obvious once you identify this issue!

The trick: Whenever (1) you see infinity or (2) a function is undefined at a bound of the integral, you will be taking a limit. It is that simple! Let's get a little more detailed about how to handle each case.

1. $\int_a^b f(x)\,\mathrm{d}x$. Say $f(x)$ is defined on $[a,b]$ and the antiderivative of $f(x)$ is $F(x)$. Now Choose either $a=-\infty$ or $b=\infty$. We will choose $b=\infty$ but the process is the same. What we get is

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{c\to\infty}\int_a^c f(x)\,\mathrm{d}x$$

   The last question is when to take the limit and that is the best part of all of this! You can ignore that limit symbol until the very end. Apply the limit once you have literally everything including placing the bounds into $F(c)-F(a)$. Once you have done this, then take the limit.

   NOTE: If it happens that both ∞ and $-\infty$ are your bounds, choose a random number, I'd choose 0, and split the integral up around that as a sum. Then proceed as above for each integral!

2. $\int_a^b f(x)\,\mathrm{d}x$. Say $f(x)$ is defined on $[a,b)$ but not at b and the antiderivative of $f(x)$ is $F(x)$. Then

$$\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx.$$

   Say $f(x)$ is defined on $(a,b]$ but not at a and the antiderivative of $f(x)$ is $F(x)$. Then

$$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx.$$

   These may not be as obvious as to why at first glance, but consider what is happening. In the first piece, f is undefined at b and my integral is of everything LEFT of b. That is why we take the limit from the LEFT. Similarly, for the second piece, we see the integral is for everything to the RIGHT of a so we use the RIGHT limit of a. Not so bad! If there is a point in the middle that is undefined, break the integral up around that point as a sum and apply the rules above to both sides!

Problems:

1. $\displaystyle\int_1^{\infty} \frac{\ln(x)}{x}\,\mathrm{d}x$

Solution.

$$\begin{aligned}
\int_1^{\infty} \frac{\ln(x)}{x}\,\mathrm{d}x &= \lim_{c\to\infty} \int_1^{c} \frac{\ln(x)}{x}\,\mathrm{d}x \\
&= \lim_{c\to\infty} \left.\frac{\ln^2(x)}{2}\right|_1^c \\
&= \lim_{c\to\infty} \frac{\ln^2(c)}{2} - \frac{0}{2} = \infty.
\end{aligned}$$

2. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,\mathrm{d}x$

Solution. In this case, we break up the integral across a number to apply our limit twice.

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,\mathrm{d}x = \lim_{c\to-\infty} \int_c^0 \frac{x^2}{9+x^6}\,\mathrm{d}x + \lim_{d\to\infty} \int_0^d \frac{x^2}{9+x^6}\,\mathrm{d}x.$$

The integral itself is not obvious, but notice we have 2 squares on the bottom summed together. This is your hint to either do a trig integral or create a arctan integral. My suggestion is an arctan integral since it will only require a small u-sub to do. Let $u = x^3$. Then $du = 3x^2dx$ and $du/3 = x^2dx$. Thus, forgetting the limits for now,

$$\begin{aligned}
&= \int_c^0 \frac{x^2}{9+x^6}\,\mathrm{d}x + \int_0^d \frac{x^2}{9+x^6}\,\mathrm{d}x \\
&= (1/3)\int \frac{1}{9+u^2} du + (1/3)\int \frac{1}{9+u^2} du \\
&= (1/9)\arctan(u/3) + (1/9)\arctan(u/3) \\
&= (1/9)\left[\left.\arctan(x^3/3)\right|_c^0 + \left.\arctan(x^3/3)\right|_0^d\right] \\
&= (1/9)\left[\lim_{c\to-\infty} \arctan(c^3/3) + \lim_{d\to\infty} \arctan(d^3/3)\right] \\
&= (1/9)(\pi/2 + \pi/2) = \pi/9.
\end{aligned}$$

3. $\displaystyle\int_0^{9} \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x$

Solution. Notice this is undefined at 1 and 1 is between our integral bounds. Thus,

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x.$$

Now we can use limits going toward 1 from the left and right to finish this.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x &= \lim_{c\to 1^-}\int_0^c \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x + \lim_{d\to 1^+}\int_d^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x \\
&= \lim_{c\to 1^-}(3/2)(x-1)^{(2/3)}\Big|_0^c + \lim_{d\to 1^+}(3/2)(x-1)^{(2/3)}\Big|_d^9 \\
&= 0-(3/2)(-1)^{(2/3)} + (3/2)8^{(2/3)} - 0 = -(3/2)+6.
\end{aligned}$$

4. $\displaystyle\int_0^1 \frac{3}{x^5}\,\mathrm{d}x$

   Solution. Undefined at 0! Thus,

$$\begin{aligned}
\int_0^1 \frac{3}{x^5}\,\mathrm{d}x &= \lim_{c\to 0^+}\int_c^1 \frac{3}{x^5}\,\mathrm{d}x \\
&= \lim_{c\to 0^+}(-3/4)(1/x^4)\Big|_c^1 \\
&= \lim_{c\to 0^+}(-3/4)-(-3/4)(1/c^4) = \infty.
\end{aligned}$$

5. $\displaystyle\int_0^{2\pi} \sec^2(x)\,\mathrm{d}x$

   Solution. Since $\sec(x) = 1/\cos(x)$, this is undefined when $\cos(x) = 0$ on the interval. Thus,

$$\begin{aligned}
&= \lim_{c\to(\pi/2)^-}\int_0^c \sec^2(x)\,\mathrm{d}x + \lim_{d\to(\pi/2)^+}\int_d^\pi \sec^2(x)\,\mathrm{d}x + \lim_{w\to(3\pi/2)^-}\int_\pi^w \sec^2(x)\,\mathrm{d}x + \lim_{v\to(3\pi/2)^+}\int_v^{2\pi} \sec^2(x)\,\mathrm{d}x \\
&= \tan(c)-\tan(0)+\tan(\pi)-\tan(d)+\tan(w)-\tan(\pi)+\tan(2\pi)-\tan(v) \\
&= \infty - 0 + 0 - (-\infty) + \infty - 0 + 0 - (-\infty) = \infty.
\end{aligned}$$