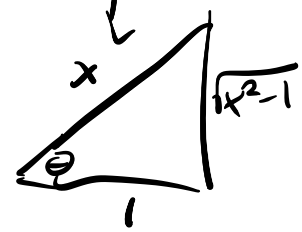
Solutions

$\sin^2 + \cos^2 = 1$
$\tan^2 + 1 = \sec^2$

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$$

Notice u-sub doesn't work and we have 2 squares. thus, Trig sub!

$$x^2-1 = \sec^2\theta - 1 = \tan^2\theta$$
$$\Rightarrow x = \sec\theta \Rightarrow 1/x = \cos\theta$$
$$dx = \sec\theta\tan\theta\,d\theta$$



$$= \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta$$
$$= \int \tan^2\theta\,d\theta$$
$$= \int \sec^2\theta - \theta\,d\theta = \tan\theta - \theta$$
$$= \sqrt{x^2-1} - \sec^{-1}(x)\Big|_1^2$$
$$= \sqrt{3} - \sec^{-1}(2) - \left(0 - \sec^{-1}(1)\right)$$
$$= \sqrt{3} - \pi/3 - 0 = \boxed{\sqrt{3} - \pi/3}$$

$y = \sec^{-1}(2)$
$\sec(y) = 2$
$\cos(y) = 1/2$
$y = \pi/3$

$y = \sec^{-1}(1)$
$\sec(y) = 1$
$\cos(y) = 1$
$y = 0$

u-sub doesn't work so polynomial in denom tells us partial fraction!

$$\int \frac{1}{y^2-4y-12}\, dy$$

$$\frac{1}{y^2-4y-12} = \frac{1}{(y-6)(y+2)} = \frac{A}{y-6} + \frac{B}{y+2}$$

$$\Rightarrow A(y+2) + B(y-6) = 1$$

$$\Rightarrow (1)\ A+B=0 \qquad 2*(1)-(2):$$

$$(2)\ 2A-6B=1 \qquad 8B=-1$$

$$B=-1/8$$

$$\Rightarrow A = 1/8$$

$$= \int \frac{1/8}{y-6} + \frac{-1/8}{y+2}\, dy$$

$$= 1/8 \ln|y-6| - 1/8 \ln|y+2| + C$$

$$\int \frac{\sin(\ln(x))}{x}\,dx \qquad \begin{aligned} u &= \ln(x) \\ du &= 1/x\,dx \end{aligned}$$

$$= \int \sin(u)\,du = -\cos(u) + C$$

$$= -\cos(\ln(x)) + C$$

$$\int_0^1 \frac{\sqrt{\tan^{-1}(x)}}{1+x^2}\,dx \qquad \begin{aligned} u &= \tan^{-1}(x) \\ du &= \frac{1}{1+x^2}\,dx \end{aligned}$$

$$= \int u^{1/2}\,du = \frac{2u^{3/2}}{3} = \frac{2\tan^{-1}(x)}{3}\Big|_0^1 = \frac{2}{3}\tan^{-1}(1) - \frac{2}{3}\tan^{-1}(0)$$

$$\begin{aligned} y &= \tan^{-1}(0) \\ \tan(y) &= 0 \\ \Rightarrow \sin(y) &= 0 \\ y &= 0 \end{aligned} \qquad \begin{aligned} y &= \tan^{-1}(1) \\ \tan(y) &= 1 \\ y &= \pi/4 \end{aligned}$$

$$= \frac{2}{3}\left(\frac{\pi}{4}\right) - 0 = \boxed{\frac{\pi}{6}}$$

$$\int \tan^5(x)\sec^3(x)\,dx = \int \tan^4(x)\sec^2(x)\,(\tan(x)\sec(x))\,dx$$

$$= \int (\sec^2(x)-1)^2 \sec^2(x)\,(\tan(x)\sec(x)\,dx)$$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$$= \int (u^2-1)^2 u^2\,du$$

$$= \int (u^4-2u^2+1)u^2\,du = \int u^6-2u^4+u^2\,du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C = \boxed{\frac{\sec^7(x)}{7} - \frac{2\sec^5(x)}{5} + \frac{\sec^3(x)}{3} + C}$$

$$\int x\sec(x)\tan(x)\,dx$$

notice $\sec(x)\tan(x)$ is derivative of $\sec(x)$! Hint for by parts.

$u = x \qquad dv = \sec(x)\tan(x)\,dx$

$du = dx \qquad v = \sec(x)$

$$= x\sec(x) - \int \sec(x)\,dx$$

$$= \boxed{x\sec(x) - \ln|\sec(x)+\tan(x)| + C}$$

improper!

$$\int_2^\infty \frac{dx}{x\ln(x)}$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \int \frac{1}{u}\,du = \ln|u| = \ln(\ln(x))\Big|_2^\infty$$

$$= \lim_{c\to\infty} \ln(\ln(c)) - \ln(\ln(2))$$

$$= \infty - \ln(\ln(2)) = \infty$$

$$\int_2^6 \frac{y}{\sqrt{y-2}}\,dy$$

This one is tricky!
1st ignore bounds and do a u-sub

$u = y-2 \Rightarrow y = u+2$

$du = dy$

$$\int \frac{u+2}{\sqrt{u}}\,du = \int u^{-1/2}(u+2)\,du$$

$$= \int u^{1/2} + 2u^{-1/2}\,du$$

$$= \frac{2u^{3/2}}{3} + 4u^{1/2} = \frac{2(y-2)^{3/2}}{3} + 4(y-2)^{1/2}\Big|_2^6$$

Notice no need for limit at 2 since both pieces are defined at $y=2$!

$$= \frac{2(4)^{3/2}}{3} + 4(4)^{1/2} - (0+0)$$

$$= \frac{2(8)}{3} + 4(2) = \frac{16}{3} + 8 = \boxed{\frac{40}{3}}$$

Really Hard!

$$\int \frac{x+1}{9x^2+6x+5}\,dx$$

$$= \int \frac{\frac{2}{3}\tan\theta + \frac{2}{3}}{4\sec^2\theta}\,\frac{2}{3}\sec^2\theta\,d\theta$$

$$= \frac{1}{9}\int \tan\theta + 1 \quad d\theta$$

$$= \frac{1}{9}\int \tan\theta\,d\theta + \frac{1}{9}\int d\theta$$

$$= \frac{1}{9}\int \frac{\sin\theta}{\cos\theta}\,d\theta + \frac{1}{9}\int d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= -\frac{1}{9}\int \frac{1}{u}\,du + \frac{1}{9}\theta$$

$$= -\frac{1}{9}\ln|\cos\theta| + \frac{1}{9}\theta$$

$$= \boxed{-\frac{1}{9}\ln\left|\frac{3(x+\frac{1}{3})}{\sqrt{4+9(x+\frac{1}{3})^2}}\right| + \frac{1}{9}\tan^{-1}\left(\frac{3}{2}(x+\tfrac{1}{3})\right) + C}$$

$$9(x^2 + \tfrac{2}{3}x) + 5$$

$$9(x+\tfrac{1}{3})^2 - 1 + 5$$

$$= 9(x+\tfrac{1}{3})^2 + 4$$

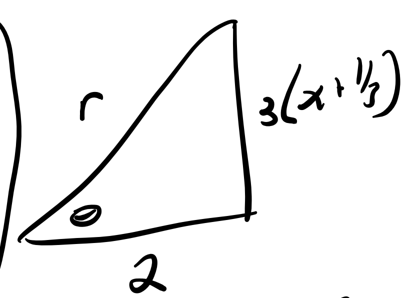
$$= 4\left(\frac{9}{4}(x+\tfrac{1}{3})^2 + 1\right)$$

$$= 4\left(\left(\frac{3}{2}(x+\tfrac{1}{3})\right)^2 + 1\right)$$

Let $\frac{3}{2}(x+\frac{1}{3}) = \tan\theta$ $\Rightarrow$ $x + \frac{1}{3} = \frac{2}{3}\tan\theta$

$$x = \frac{2}{3}\tan\theta - \frac{1}{3}$$

$$dx = \frac{2}{3}\sec^2\theta\,d\theta$$



$$r^2 = 4 + 9(x+\tfrac{1}{3})^2$$

$$r = \sqrt{4+9(x+\tfrac{1}{3})^2}$$