

Thoughts:

This section is always a mystery to me. People tend to struggle with this section when the concept is very simple. Teachers say it is because of students not liking word problems, but I disagree! I think it is best to lay out the common struggles with this section before I get into what this section is about.

1. Word problems
2. Finding the right equation (what I think the major issue is)
3. Differentiating between this and a related rates problem

These to me are the biggest issues for solving these types of questions. Because of this, once I finish talking about this topic, I will give ways to address these issues before you build up frustration about them and give a negative connotation to the subject.

First, what is optimization? optimization is just a fancy word for finding the maximum and the minimum. Thus, this section has everything to do with finding the ABSOLUTE extrema! The best intro is an example. This is a question I wrote for my business calculus final. I hope you find it as entertaining as I do.

Example 0.1. Mary wants to build a rectangular fenced area next to the intersection of two rivers so her little lambs do not need her to bring water. The area she found looks like the below with dashed lines representing the riversides.



If Mary has 1000 feet of fencing, what is the maximum area of the land surrounded by water and fencing?

Mary realized the above area was a safety hazard for her lambs since they could fall in the river. Now she wants to construct a fenced rectangle of area 312500 square feet in the rainforest of Costa Rica, where the air is so humid it can act as their water, and away from any hazardous rivers (She does not realize the abundant hazards of the rainforest; silly Mary). She does realize however that it will cost more for her to get fencing and is now concerned about this cost. For the first 3 sides, it costs \$8 per foot and there is a discount on the last side of \$2 per foot. What is minimum cost of the fencing?

You can see here we have an example of a question about maximums and minimums. Let's start with the first question. The first thing we do is identify the equations we can use. Here we are

talking about area so area of a rectangle is $A = lw$. The other thing we can talk about is perimeter. In this case we care about perimeter in which fencing is used! That means perimeter with fencing is $P = l + w$. However, we know P ! So we have $1000 = l + w$. If we solve for one of the variables, we can reduce our area equation down to one variable.

$$1000 = l + w \Rightarrow 1000 - l = w \Rightarrow A = l(1000 - l)$$

Now we can maximize by taking the derivative with respect to l and setting equal to 0.

$$\frac{dA}{dl} = 1000 - 2l = 0 \Rightarrow l = 500$$

It is standard to check that this is a maximum and not a minimum or just a critical point, but most of the time this is assumed. Now we find w via our previous equation and get $w = 500$ as well. Are we done? NO!!! We want the maximum area! Thus, we need to plug this into area to get that number! We get $\max A = 250000 \text{ ft}^2$.

The second question takes a little more thought. Once again we have the area referenced. Thus, an important equation is $A = lw$. We also have the area so, $312500 = lw$. Now we have information about cost. the first 3 sides will either be $2l + w$ or $2w + l$. Since we are maximizing, it is irrelevant which we choose. This gives us the equation $(2l + w)8 + 2w = C$ where C is cost. We want to minimize cost, so let's use the area formula to replace a variable in C and take the derivative with respect to what is left.

$$\begin{aligned} w = \frac{312500}{l} \Rightarrow C &= 16l + 10\frac{312500}{l} = 16l + \frac{3125000}{l} = 16l + 3125000l^{-1} \Rightarrow 16 - 3125000l^{-2} = 0 \\ \Rightarrow 16l^2 &= 3125000 \Rightarrow l = \frac{\sqrt{3125000}}{4} \end{aligned}$$

Now we repeat the process as the previous question with the cost equation to find w and find the minimum cost.

You can see this is not the craziest thing to do. The problem is students struggle. So, now I will address those issues I talked about at the beginning. The first issue I cannot do anything about. It just takes time to become familiar with the formate of word problems, but some suggestions are to first write out everything you know and what you need. This helps organize the information. This also helps with the second issue. Finding the equation(s) is just a matter of making sense of what you have and what you need! Try this out if you are stuck. The last issue is about the difference between related rate and optimization. Here is the key: MAX AND MIN is optimization. Done and done! You can also notice the difference when your related rates problem talks about rates, but optimization does not. HOWEVER, this is a dangerous game since rate is a loose word. I would go with the max and min difference before the rate idea. Looking for rates may be a good way to confirm optimization or not.

Problems:

1. Find two numbers whose product is 100 and whose sum is a minimum.

Solution. Let x and y be the numbers. We need $xy = 100$ and $A = x + y$ is minimal. Thus, we solve for x or y and plug into A .

$$x = \frac{100}{y} \Rightarrow A = \frac{100}{y} + y \Rightarrow \frac{dA}{dy} = \frac{-100}{y^2} + 1 = 0 \Rightarrow y^2 = 100 \Rightarrow y = 10, -10$$

Now we find x for each and determine the minimum. $x = 10$ or $x = -10$. Clearly the sum of 2 negative numbers is smaller so the minimal sum is -20 .

2. What is the minimum vertical difference between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

Solution. Vertical distance is just $y_1 - y_2$. Since we have the y 's we plug in.

$$d = x^2 + 1 - (x - x^2) = 2x^2 - x + 1$$

Now we find the derivative.

$$d' = 4x - 1 = 0 \Rightarrow x = \frac{1}{4}$$

This gives $y_1 = \frac{17}{16}$ and $y_2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$. Thus vertical distance is $V = \frac{14}{16} = \frac{7}{8}$.

3. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of length 3 cm and 4 cm if two sides of the rectangle lie along the legs.

Solution. This one is more tricky. First off, the last side is 5 by Pythagorean Theorem. Thus, if the sides of the inside rectangle are x and y , the area of the rectangle is $A = xy$. The Pythagorean Theorem also gives us $(4 - x)^2 + (3 - y)^2 = 25$. This should warrant a solution.

4. A piece of wire 10 cm long is cut into 2 pieces. One piece is bent into a square and the other bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) maximum (b) minimum? (This means try maximizing then try minimizing)
5. Find the point of the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Solution. $d = \sqrt{(x - 3)^2 + (y - 0)^2} = \sqrt{(x - 3)^2 + x} = \sqrt{x^2 - 5x + 9}$ by substituting in for y . Now derivative.

$$d' = (1/2)((x^2 - 5x + 9)^{(-1/2)}) \cdot (2x - 5) = \frac{2x - 5}{\sqrt{x^2 - 5x + 9}} = 0$$

This is 0 when $x = 5/2$. Thus, $y = \sqrt{5/2}$.

6. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

Solution. The tangent line slope is the derivative. $y' = 120x^2 - 15x^4$. We want to maximize this, so we take the derivative and set it equal to 0.

$$y'' = 240x - 60x^3 = 60x(4 - x^2) = 60x(2 - x)(2 + x) = 0 \Rightarrow x = 0, 2, -2$$

By first derivative test (number line), $x_1 = -2$ and $x_2 = 2$ are local maximums. Thus, we determine which makes the derivative largest by plugging in. However, they are equal upon plugging in (powers are even). Thus, the largest slope is $y = 120(4) - 15(16) = 120(4) - 60(4) = 4(60) = 240$.