Precalc: I have attached a packet of problems that i used as a review from precaclulus. It is A LOT of problems! Please pick and choose the problems for yourself depending on what is difficult for you! This way we can work out what I should go over! The answers are also on Blackboard!

Limits: A limit is either a value, undefined, or $\pm\infty$. A limit is undefined if the right limit and the left limit are not equal. The first trouble students encounter is the difference between a limit and and left/right limit. It is best to explain with an example.

$$\lim_{x\to 0} 2x.$$

This is a simple example, but it should explain what is going on with a limit. When we say left limit, we mean approaching $x = 0$ with values from the left of 0. This means we go -2, -1, $-.5$, $-.25$, ... Clearly, the value of $2x$ when we approach from the left is 0! Similarly we get 0 from the right with this idea. Thus, the limit exists and is 0. Now let's take an example where this is not as obvious.

$$\lim_{x\to 2} f(x) \text{ where } f(x) \text{ is the following:}$$

$$f(x) = \begin{cases} x+1 & x \geq 2 \\ x+2 & x < 2 \end{cases}.$$

In this case we have an issue. If we approach 2 from the left, we use the bottom piece of this function ans get 4. However, if we approach from the right, we use the top piece of this function and get 3. Thus, the limit is undefined!

What about ∞? Let's see!

$$\lim_{x\to 0} \frac{1}{x^2}.$$

This limit is more interesting. First we notice this function is undefined at 0. However, we do have something happening here. If we consider the left limit, this consists of negative numbers. Thus, $1/x^2$ is positive. Now as we get closer to 0, those numbers are going to be smaller and smaller. For example, $\frac{1}{(-1/2)^2} = 4$, $\frac{1}{(-1/3)^2} = 9$, $\frac{1}{(-1/4)^2} = 16$, ... Notice that these are increasing! So, we say, the left limit is growing infinitely positive! The right limit by similar argument is doing the same! Thus, our limit is ∞. However, we must stop and consider the following: "Is infinity a number?" The answer for the context of this class is no. Thus, the limit does not exist, but we do know if grows inifintely large!

Our last case is if the variable is approaching $\pm\infty$. This may not be covered in your classes, but it is good to see.

$$\lim_{x\to \infty} \frac{1}{x}.$$

This time we cannot evaluate from the right since the right of infinity does not exist! So really what we are finding is the left limit of this function going toward infinity! What this looks like is: $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... You can see this is getting smaller and smaller as we increase x, but always positive. Thus, this limit is 0.

There are also many little tips and tricks that can help you in the case that your function is undefined at the limit point. These include multiplying top and bottom by the conjugate, simplifying via factoring, common denominator, dividing every term in top and bottom by the highest power of the varaible, and L'Hospital's Rule which will be a future thing.

1. $\lim\limits_{x \to -1} \dfrac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

   Solution. Let us ignore the lim for now. Notice when you plug in -1 this function is undefined. This means either the limit DNE or we need to manipulate. Since the denominator is a polynomial and is 0 at $x = -1$, that means -1 is a root and $x + 1$ is a factor of the polynomial. This should be our hint to try factoring.

   $$= \frac{(2x+1)(x+1)}{(x-3)(x+1)} = \frac{2x+1}{x-3}.$$

   Now we have gotten rid of the $x + 1$ which was what made the denominator 0. Thus, we can plug in our limit to get the answer.

   $$\frac{2(-1)+1}{(-1)-3} = \frac{1}{4}.$$

2. $\lim\limits_{x \to 0} \dfrac{1}{x} - \dfrac{1}{x^2 + x}$

   Solution. Once again notice that both fractions are undefined at $x = 0$. This means we need to get rid of the x in the denominator of both of these! Let's make our lives easier and make this one fraction.

   $$= \frac{x+1}{x(x+1)} - \frac{1}{x(x+1)} = \frac{x+1-1}{x(x+1)} = \frac{x}{x(x+1)} = \frac{1}{x+1}.$$

   Notice our x fell out just from simplifying! Now we can plug in 0.

   $$\frac{1}{0+1} = 1.$$

3. $\lim_{h\to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$

Solution. We will use the same tactics as the previous (common denominator) but only in the numerator for now.

$$\text{numerator} = \frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2} = \frac{x^2-(x+h)^2}{x^2(x+h)^2}.$$

Keep in mind our issue is with the denominator of h. Notice both sides of this difference in the numerator above will generate an x^2. This means it will cancel. Let us expand and cancel.

$$= \frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2} = \frac{-2xh-h^2}{x^2(x+h)^2}.$$

Another thing to note is that there is an h in both terms of the numerator above. Since we still have that trailing h in the denominator that we did not forget about, let's bring it back in and see if some cancelling can happen.

$$\text{original fraction} = \frac{\dfrac{h(-2x-h)}{x^2(x+h)^2}}{h} = \frac{h(-2x-h)}{x^2h(x+h)^2} = \frac{-2x-h}{x^2(x+h)^2}.$$

Now the troubling h is gone! Let's plug in 0 for h.

$$\frac{-2x-0}{x^2(x+0)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}.$$

Note: This is actually how you find the derivative of $1/x^2$ by definition!

4. $\lim_{x\to 0} \dfrac{\sqrt{9+x}-3}{x}$

Solution. The issue lies with the x in the denominator. Notice we cannot factor nor get a common denominator. The square root here is the key hint. This means we should use the conjugate! This is what that looks like.

$$\frac{(\sqrt{9+x}-3)(\sqrt{9+x}+3)}{x(\sqrt{9+x}+3)} = \frac{9+x-9}{x(\sqrt{9+x}+3)} = \frac{x}{x(\sqrt{9+x}+3)} = \frac{1}{\sqrt{9+x}+3}.$$

Notice our x problem is gone! Now we can plug in.

$$\frac{1}{\sqrt{9+0}+3} = \frac{1}{6}.$$

5. $\lim_{x \to -2} \dfrac{x^4 - 2}{2x^2 - 3x + 2}$

Solution. There is no problem here when I plug in $x = -2$! Thus, plug it in!

$$\frac{(-2)^4 - 2}{2(-2)^2 - 3(-2) + 2} = \frac{7}{8}.$$