Thoughts:

You just learned an antiderivative, but what does this look like geometrically or in terms of the original function? Well, that is where Riemann comes in. Riemann was trying to find a way to get an exact answer for the area below a curve bounded by the x-axis. All he kept doing was drawing rectangles and adding up their areas. Then smaller rectangles and adding their area, but it was never exact unless the function was a straight line. The smaller the rectangles, the more rectangles he used and the more accurate the number was. So, Riemann hypothesized that if you take these rectangles to infinitely skinny, but there be infinitely many of them, you will get the exact area under the curve. I don't know about you, but that smells like a limit to me!! This little history lesson is to let you know that your teachers are taking you for a walk in Riemann's shoes with this first section. They are making you find these estimates that Riemann did, so you can see how the limit works later. Let's look at an example.

Let's look at the curve $f(x)$ from 0 to 4, but use only 2 rectangles to estimate the area under this curve. Well, there are 2 ways to look at this; either you make the height of each rectangle the right endpoint or the left endpoint of these rectangles. For example, here our first rectangle is from 0 to 2 since I want equal length rectangles across my x-axis. Do I use 0 to get the height of the rectangle or 2 to get the height of the rectangle? This is where we can determine an upper bound and a lower bound for the area under the curve. For now let us use the left endpoints. Thus, our first rectangle has area $2f(0)$. Why? you are probably asking. This is because one side of the rectangle is 2 as we picked and the other is the height of the function at the left endpoint which we picked. That is $f(0)$! Now we have one more rectangle and, by similar process we should get $2f(2)$. Thus, our area using left endpoints is $2f(0) + 2f(2)$.

Ok, but what if I use 4 equally spaced rectangles? Then how tall are these rectangles? On our interval they will be 1 tall. This is because our interval is length 4. Using the same technique now, we get the area to be $1f(0) + 1f(1) + 1f(2) + 1f(3)$. Notice 4 pieces to the sum for each rectangle.

Let's try to generalize this. If my curve goes from a to b and I want n rectangles, what is the left endpoint area? Well the length of our interval is $b - a$. So, if we break this length up into n pieces, we get the length of each rectangle to be $\Delta = \dfrac{b-a}{n}$! Now we know the first height will be $f(a)$ if we use left endpoints. What will the second be? We determined Δ to be length of each rectangle on the x-axis. Thus, the next endpoint would be $a + \Delta$ and the next would be $a + 2\Delta$ and so on! Therefore, from the leftendpoints we get

$$A_L = \Delta(f(x_0) + f(x_1) + \ldots + f(x_{n-1})) \tag{1}$$

where $\Delta = \dfrac{b-a}{n}$ and $x_i = a + i\Delta$. Any guesses about right endpoint? Well here it is!

$$A_R = \Delta(f(x_1) + f(x_2) + \ldots + f(x_n)) \tag{2}$$

where $\Delta = \dfrac{b-a}{n}$ and $x_i = a + i\Delta$. This is not surprising if you ask me as we just shifted every x over 1.

Now you will here the magic words of "Overestimate" and "Underestimate". You can determine if A_R or A_L is one or the other by finding both and whichever is bigger is over and vice versa or by increasing decreasing intervals. This is only helpful if you have a solely increasing or decreasing function. On an increasing interval, a left endpoint will overestimate and vice versa! Switch it around with a decreasing interval and right endpoints overestimate. I would not rely on this fact, but lean into it if needed. The point here is the understand where Riemann gets the all mighty integral.

Definite Integral- Yes! It is finally here. I know. I know. You guys were super excited to leave derivatives so here you are! We said something about limits so let's bring that up! We define the definite integral as the area under the curve or by the previous section

Definition 0.1 (Definite Integral)**.**

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \Delta(f(x_1) + \ldots + f(x_n))$$

with $\Delta = \dfrac{b-a}{n}$ and $x_i = a + i\Delta$.

Now I do not want to go into too much detail about the definite integral, but I think it's worthwhile to give some properties such as:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

if c is between a and b. Another is

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

This will all lead to the biggest theorem in this class; the Fundamental Theorem of Calculus where the antiderivative makes a guest appearance.

Problems: Estimate the area under the curve for each interval and number of rectangles given. Then determine which is an overestimate and which is an underestimate.

1. $f(x) = 1 + x^2$ from $x = -1$ to 2 with 3 rectangles. Then do it with 6 rectangles.
2. $f(x) = x - 2\ln(x)$ from $x = 1$ to 5 with 4 rectangles. Then pick a number larger than 4 and do it with that many rectangles. Be strategic!
3. $f(x) = 2 + \sin(x)$ from $x = 0$ to π with 2, 4, and 8 rectangles.
4. For the next questions, give a formula for the definite integral as a limit with a sum. See if you can determine the limit! (This part is tricky since you will not have seen limits like this before, but I encourage you to try it since it is what comes next)
5. $\int_0^8 \sin(\sqrt{x})\,\mathrm{d}x$
6. $\int_0^2 \dfrac{x}{x+1}\,\mathrm{d}x$
7. $\int_2^5 (4 - 2x)\,\mathrm{d}x$
8. $\int_1^4 (x^2 - 4x + 2)\,\mathrm{d}x$
9. Try and show that $\int_a^b x\,\mathrm{d}x = \dfrac{b^2 - a^2}{2}$