Thoughts:
This is actually a very important topic in mathematics and really gives value to series in general. The best way to start is by defining a power series.

Definition 0.1 (Power Series)**.** A series in the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \dots.$$

OR a power series centered at a:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \dots.$$

NOTE: If there is a constant in front of the x, you need to get rid of it!
Seems simple enough. The question is convergence of these series! The easiest example of a power series is geometric series! These have strict rules for convergence which leads us to more ideas of convergence for the more general set of power series. This leads us to a big theorem.

Theorem 0.1. *Let* $S = \sum_{n=0}^{\infty} c_n (x-a)^n$*. We have 3 cases:*

1. *The series converges when* $x = a$*.*
2. *The series converges for all* x*.*
3. *There is a positive number* R *where the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

The last case may seem a little strange. If we think about geometric series this makes sense! our x has to be less than 1! The last case is the most interesting case and thus has a few definitions associated. For example, the R is the **radius of convergence**. In other words, the series converges on the interval $(a-R, a+R)$. Now finding this radius of convergence is the key part. The nice thing is you will either always use a ratio test or a root test to do so! Here is an example:

$$\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{2^n}.$$

Notice $\frac{3^n}{2^n}$ is our c_n here. Let's try using root test!

$$\left| \frac{3^n (x+4)^n}{2^n} \right|^{1/n} = \left| \frac{3(x+4)}{2} \right| = (3/2)\,|x+4|$$

For this to converge by the root test, $(3/2)\,|x+4| < 1$. Thus,

$$(3/2)\,|x+4| < 1 \Rightarrow |x+4| < \frac{2}{3}.$$

Therefore, our radius of convergence is $2/3$. To find the radius of convergence there is still more work to be done. The radius of convergence with the root test has given us

$$-\frac{2}{3} < x + 4 < \frac{2}{3} \Rightarrow -\frac{14}{3} < x < -\frac{10}{3}.$$

WE ARE NOT DONE! The root test tells us nothing about the end points! Thus, we have to test them individually. Let's start with the left side of the inequality.

$$\sum_{n=1}^{\infty} \frac{3^n(-(14/3)+4)^n}{2^n} = \sum_{n=1}^{\infty} \frac{3^n(-2/3)^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n.$$

This sum is divergent! Now you can probably guess what is happening at the other end point! The other end point gives

$$\sum_{n=1}^{\infty} 1$$

which is divergent. Therefore, the interval we originally had is the interval of convergence!

Problems: Find the radius of convergence and the interval of convergence.

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2 x^n}{2^n}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{b^n}{\ln(n)}(x-a)^n, \ b>0$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n, \ k$ positive integer

8. Challenge: Find the domain of

$$A(x) = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots .$$

9. Suppose the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence for the power series $\sum c_n x^{2n}$?