1 Discussion Nov 13th, 2025, Linz

Evolution problem: Let $\hat{\boldsymbol{y}} \in L^2(0,T;\mathbb{R}^N)$ a given target and $\gamma, \nu > 0$ two penalization parameters. We minimize the cost functional

$$\min_{\boldsymbol{y},\boldsymbol{u}} J(\boldsymbol{y},\boldsymbol{u}) := \frac{1}{2}\int_0^\top |\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)|^2\,dt + \frac{\gamma}{2}|\boldsymbol{y}(T) - \hat{\boldsymbol{y}}(T)|^2 + \frac{\nu}{2}\int_0^\top |\boldsymbol{u}(t)|^2\,dt, \tag{1}$$

such that

$$\dot{\boldsymbol{y}}(t) + A\boldsymbol{y}(t) = \boldsymbol{u}(t), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{2}$$

Here, $\boldsymbol{y}(t), \hat{\boldsymbol{y}}(t), \boldsymbol{u}(t) \in \mathbb{R}^N$ are column vectors, $|\cdot|$ denotes the Euclidean norm, $A \in \mathbb{R}^{N\times N}$ can be seen as the discretization matrix of some spatial operators, and $\boldsymbol{y}_0 \in \mathbb{R}^N$ is a given initial vector. This problem can be seen as the semi-discretization in space of some linear PDE-constrained optimization problems.

First-order optimality system: The standard approach to treat this problem is to introduce a Lagrange multiplier $\boldsymbol{p} \in L^2(0,T;\mathbb{R}^N)$, and write the Lagrangian as

$$\mathcal{L}(\boldsymbol{y},\boldsymbol{p},\boldsymbol{u}) = J(\boldsymbol{y},\boldsymbol{u}) - \int_0^\top \boldsymbol{p}^\top(t)\left(\dot{\boldsymbol{y}}(t) + A\boldsymbol{y}(t) - \boldsymbol{u}(t)\right)\,dt.$$

For any variation $\delta\boldsymbol{y} \in C^\infty(0,T;\mathbb{R}^N)$ with $\delta\boldsymbol{y}(0) = 0$, we have

$$\begin{aligned}\partial_{\boldsymbol{y}}\mathcal{L}[\delta\boldsymbol{y}] = &\int_0^\top \left(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)\right)^\top \delta\boldsymbol{y}(t)\,dt + \gamma\left(\boldsymbol{y}(T) - \hat{\boldsymbol{y}}(T)\right)^\top \delta\boldsymbol{y}(T)\\ &+ \int_0^\top \dot{\boldsymbol{p}}^\top(t)\delta\boldsymbol{y}(t)\,dt - \int_0^\top (A^\top\boldsymbol{p})^\top(t)\delta\boldsymbol{y}(t)\,dt - \boldsymbol{p}^\top(T)\delta\boldsymbol{y}(T) + \boldsymbol{p}^\top(0)\delta\boldsymbol{y}(0).\end{aligned}$$

Equating it to zero gives

$$-\dot{\boldsymbol{p}}(t) + A^\top\boldsymbol{p}(t) = \boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t), \quad \boldsymbol{p}(T) = \gamma\left(\boldsymbol{y}(T) - \hat{\boldsymbol{y}}(T)\right).$$

Similarly, for any variation $\delta\boldsymbol{u} \in C^\infty(0,T;\mathbb{R}^N)$, we have

$$\partial_{\boldsymbol{u}}\mathcal{L}[\delta\boldsymbol{u}] = \int_0^\top \nu\boldsymbol{u}(t)^\top\delta\boldsymbol{u}(t)\,dt + \int_0^\top \boldsymbol{p}^\top(t)\delta\boldsymbol{u}(t)\,dt.$$

Equating it to zero gives

$$\boldsymbol{p}(t) + \nu\boldsymbol{u}(t) = 0.$$

We then obtain the optimality system

$$\begin{aligned}\dot{\boldsymbol{y}}(t) + A\boldsymbol{y}(t) &= \boldsymbol{u}(t), & \boldsymbol{y}(0) &= \boldsymbol{y}_0,\\ -\dot{\boldsymbol{p}}(t) + A^\top\boldsymbol{p}(t) &= \boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t), & \boldsymbol{p}(T) &= \gamma\left(\boldsymbol{y}(T) - \hat{\boldsymbol{y}}(T)\right),\\ \nu\boldsymbol{u}(t) &= -\boldsymbol{p}(t),\end{aligned} \tag{3}$$

which can also be written in the integral form as

$$\begin{aligned}&\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_0^\top \left(-A\boldsymbol{y}(t) + \boldsymbol{u}(t)\right)\,dt,\\ &\boldsymbol{p}(t) = \boldsymbol{p}(T) + \int_t^\top \Big(-A^\top\boldsymbol{p}(t) + \boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)\Big)\,dt,\\ &\boldsymbol{u}(t) = -\boldsymbol{p}(t)/\nu, \quad t \in (0,T).\end{aligned}$$

Reduced cost functional: To enter in the same optimization framework proposed in [1], we introduce the solution operator $\mathcal{S} : L^2(0,T;\mathbb{R}^N) \to L^2(0,T;\mathbb{R}^N)$ such that $\mathcal{S}\boldsymbol{u} = \boldsymbol{y}$. Substituting $\mathcal{S}\boldsymbol{u}$ into the cost functional gives

$$J(\boldsymbol{u}) = J(\mathcal{S}\boldsymbol{u},\boldsymbol{u}) = \frac{1}{2}\int_0^\top |\mathcal{S}\boldsymbol{u}(t) - \hat{\boldsymbol{y}}(t)|^2\,dt + \frac{\gamma}{2}|\mathcal{S}\boldsymbol{u}(T) - \hat{\boldsymbol{y}}(T)|^2 + \frac{\nu}{2}\int_0^\top |\boldsymbol{u}(t)|^2\,dt,$$

where we still use J to denote the reduced cost functional. We can then write the optimization problem as $\min_{\boldsymbol{u}} J(\boldsymbol{u})$ and use the elimination approach proposed in [1].

Decomposition: There are two different ways to decompose the problem and write

$$J(\boldsymbol{u}_1, \boldsymbol{u}_2).$$

If we decompose the control variable $\boldsymbol{u}$ into two controls $\boldsymbol{u}_1(t) \in \mathbb{R}^{N_1}$ and $\boldsymbol{u}_2(t) \in \mathbb{R}^{N_2}$ with $N_1 + N_2 = N$. This is exactly in the spirit of [1], which corresponds to a space decomposition. In this case, the functional $J : L^2(0, T; \mathbb{R}^{N_1}) \times L^2(0, T; \mathbb{R}^{N_2}) \to \mathbb{R}$. Instead, one can also decompose the time interval $(0, T)$ into two subintervals $Q_1 := (T_0, T_1)$ and $Q_2 := (T_1, T_2)$ with $T_0 = 0$, $T_2 = T$ and $T_1 \in (0, T)$. This corresponds to a time decomposition, and the two associated controls $\boldsymbol{u}_1(t) \in \mathbb{R}^N$, $t \in Q_1$ and $\boldsymbol{u}_2(t) \in \mathbb{R}^N$, $t \in Q_2$. In this case, the functional $J : L^2(T_0, T_1; \mathbb{R}^N) \times L^2(T_1, T_2; \mathbb{R}^N) \to \mathbb{R}$.

As we are interested in the second case, let us derive the algorithm following the approach proposed in [1]. Assuming that for every $\boldsymbol{u}_1$ the equation $\nabla_{\boldsymbol{u}_2} J(\boldsymbol{u}_1, \boldsymbol{u}_2) = 0$ admits a unique solution $\boldsymbol{u}_2$ and that $\nabla_{\boldsymbol{u}_2\boldsymbol{u}_2} J(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is invertible for every $(\boldsymbol{u}_1, \boldsymbol{u}_2)$, then applying the implicit function theorem, there exists a continuously differentiable mapping $h : L^2(T_0, T_1; \mathbb{R}^N) \to L^2(T_1, T_2; \mathbb{R}^N)$ such that we can eliminate $\boldsymbol{u}_2$ and obtain $\nabla_{\boldsymbol{u}_2} J(\boldsymbol{u}_1, h(\boldsymbol{u}_1)) = 0$. We may apply the Newton iteration to solve the reduced optimality condition $F(\boldsymbol{u}_1) = \nabla_{\boldsymbol{u}_1} J(\boldsymbol{u}_1, h(\boldsymbol{u}_1)) = 0$. For iteration index $k = 0, 1, \ldots$, one solves

$$\boldsymbol{u}_1^{k+1} = \boldsymbol{u}_1^k - \left(JF(\boldsymbol{u}_1^k)\right)^{-1} \nabla_{\boldsymbol{u}_1} J(\boldsymbol{u}_1^k, h(\boldsymbol{u}_1^k)),$$

which is exactly what has been shown in [1, Eq. 4].

As also discussed in [1], our ultimate goal is to solve the optimization problem. Thus, one can also perform such variable elimination on the objective function J, that is, $\tilde{J}(\boldsymbol{u}_1) := J(\boldsymbol{u}_1, h(\boldsymbol{u}_1))$, and then apply a gradient descent method to solve the minimization problem as

$$\boldsymbol{u}_1^{k+1} = \boldsymbol{u}_1^k - \alpha \nabla \tilde{J}(\boldsymbol{u}_1^k). \tag{4}$$

with α the step size satisfies $\tilde{J}(\boldsymbol{u}_1^k - \alpha \nabla \tilde{J}(\boldsymbol{u}_1^k)) < \tilde{J}(\boldsymbol{u}_1^k)$.

Algorithm: The solving process can be resumed as:

1. For a given $\boldsymbol{u}_1^k$, one can find the state variable $\boldsymbol{y}_1^k$ with

$$\boldsymbol{y}_1^k(t) = \boldsymbol{y}_0 + \int_0^{\top} \left(-A\boldsymbol{y}_1^k(t) + \boldsymbol{u}_1^k(t)\right) dt.$$

   This consists in applying the solution operator $\mathcal{S}_1 \boldsymbol{u}_1^k$.

2. Using the fact that $\boldsymbol{y}_2^k(T_1) = \boldsymbol{y}_1^k(T_1)$, one solves the system

$$\begin{aligned}
\boldsymbol{y}_2^k(t) &= \boldsymbol{y}_2^k(T_1) + \int_{T_1}^{t} \left(-A\boldsymbol{y}_2^k(t) + \boldsymbol{u}_2^k(t)\right) dt, \\
\boldsymbol{p}_2^k(t) &= \boldsymbol{p}_2^k(T_2) + \int_{t}^{T_2} \left(-A^{\top}\boldsymbol{p}_2^k(t) + \boldsymbol{y}_2^k(t) - \hat{\boldsymbol{y}}(t)\right) dt, \\
\boldsymbol{u}_2^k(t) &= -\boldsymbol{p}_2^k(t)/\nu, \quad t \in (T_1, T_2).
\end{aligned}$$

   The above part consists in evaluating the mapping $\boldsymbol{u}_2^k = h(\boldsymbol{u}_1^k)$.

3. Update the control variable in Q_1 with

$$\boldsymbol{u}_1^{k+1} = \boldsymbol{u}_1^k - \alpha \left(\nabla_{\boldsymbol{u}_1} \tilde{J}\left(\boldsymbol{u}_1^k, h(\boldsymbol{u}_1^k)\right) + \nabla_{\boldsymbol{u}_2} \tilde{J}\left(\boldsymbol{u}_1^k, h(\boldsymbol{u}_1^k)\right) h'(\boldsymbol{u}_1^k)\right).$$

   As the implicit mapping h fulfils $\nabla_{\boldsymbol{u}_2} J(\boldsymbol{u}_1^k, h(\boldsymbol{u}_1^k)) = 0$. The update can then be written as

$$\boldsymbol{u}_1^{k+1} = \boldsymbol{u}_1^k - \alpha \nabla_{\boldsymbol{u}_1} \tilde{J}\left(\boldsymbol{u}_1^k, h(\boldsymbol{u}_1^k)\right) = \boldsymbol{u}_1^k - \alpha(\boldsymbol{p}_1^k + \nu \boldsymbol{u}_1^k),$$

where $\boldsymbol{p}_1^k$ is given by

$$\boldsymbol{p}_1^k(t) = \boldsymbol{p}_1^k(T_1) + \int_t^{T_1} \left(-A^\top \boldsymbol{p}_1^k(t) + \boldsymbol{y}_1^k(t) - \hat{\boldsymbol{y}}(t)\right) dt,$$

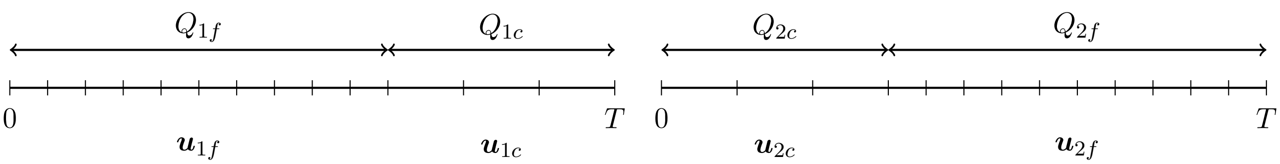
with $\boldsymbol{p}_1^k(T_1) = \boldsymbol{p}_2^k(T_1)$.

Here, we are in the case with an exact solve of $\nabla_{\boldsymbol{u}_2} J(\boldsymbol{u}_1, \boldsymbol{u}_2) = 0$, as $\nabla_{\boldsymbol{u}_2} J(\boldsymbol{u}_1, \boldsymbol{u}_2) = \nu \boldsymbol{u}_2^k + \boldsymbol{p}_2^k$.

2 Discussion Dec 16th, 2025, Zoom

We talked about the idea we had in Linz, and Gabriele mentioned his work with Parisa, which could be adapted to our case. Here is the idea (based on my understanding).

Coarse-fine decomposition: We consider two controls $u_1 := (\boldsymbol{u}_{1f}, \boldsymbol{u}_{1c})^\top$ and $\boldsymbol{u}_2 := (\boldsymbol{u}_{2c}, \boldsymbol{u}_{2f})^\top$, where $\boldsymbol{u}_1$ is defined in the domain $Q_1 := Q_{1f} \cup Q_{1c}$, and $\boldsymbol{u}_2$ is defined in the domain $Q_2 := Q_{2c} \cup Q_{2f}$. Note that $Q_1 = Q_2 = Q = [0, T]$. Below is an illustration of these two decompositions.



For each decomposition framework, we apply the gradient descent algorithm (4) to compute the optimal control $\boldsymbol{u}$, that is,

$$\boldsymbol{u}_{1f}^{k+1} = \boldsymbol{u}_{1f}^k - \alpha \nabla_{\boldsymbol{u}_{1f}} J\left(\boldsymbol{u}_{1f}^{k+1}, h\left(\boldsymbol{u}_{1f}^{k+1}\right)\right), \quad \boldsymbol{u}_{2f}^{k+1} = \boldsymbol{u}_{2f}^k - \alpha \nabla_{\boldsymbol{u}_{2f}} J\left(h\left(\boldsymbol{u}_{2f}^{k+1}\right), \boldsymbol{u}_{2f}^{k+1}\right). \tag{5}$$

Consider two restriction matrices R_1 and R_2 such that each part of $\boldsymbol{u}_j$ in the fine grid satisfies $\boldsymbol{u}_{1f}^k = R_1 \boldsymbol{u}^k$ and $\boldsymbol{u}_{2f}^k = R_2 \boldsymbol{u}^k$. Consider also two partition matrices $\tilde{R}_1$ and $\tilde{R}_2$ such that each new iteration is given by $\boldsymbol{u}^{k+1} = \tilde{R}_1^\top \boldsymbol{u}_{1f}^{k+1} + \tilde{R}_2^\top \boldsymbol{u}_{2f}^{k+1}$. Since there is an overlap between Q_{1f} and Q_{2f}, we need these partition matrices to average $\boldsymbol{u}_{jf}^k$ on the overlap. The restriction matrices and partition matrices satisfy $\tilde{R}_1^\top R_1 + \tilde{R}_2^\top R_2 = I_N$.

We define the matrix associated with the time discrete gradient ∇J by $H := \nu I_N + (A^\top)^{-1} A^{-1}$. In Q_1, the matrix $H^1 = H$ is written as

$$H^1 := \begin{bmatrix} H_{ff}^1 & H_{fc}^1 \\ H_{cf}^1 & H_{cc}^1 \end{bmatrix}.$$

Using the Schur complement, we can write the gradient descent of $\boldsymbol{u}_{1f}^k$ in (5) as

$$\boldsymbol{u}_{1f}^{k+1} = \boldsymbol{u}_{1f}^k - \alpha \left(H_{ff}^1 - H_{fc}^1 (H_{cc}^1)^{-1} H_{cf}^1\right) \boldsymbol{u}_{1f}^k.$$

Similarly, we have the matrix in Q_2 given by

$$H^2 := \begin{bmatrix} H_{cc}^2 & H_{cf}^2 \\ H_{fc}^2 & H_{ff}^2 \end{bmatrix},$$

and the gradient descent of $\boldsymbol{u}_{2f}^k$ in (5) can be written as

$$\boldsymbol{u}_{2f}^{k+1} = \boldsymbol{u}_{2f}^k - \alpha \left(H_{ff}^2 - H_{fc}^2 (H_{cc}^2)^{-1} H_{cf}^2\right) \boldsymbol{u}_{2f}^k.$$

Combining these two gradient descent parts gives the next update of $\boldsymbol{u}^k$ as

$$\boldsymbol{u}^{k+1} = \tilde{R}_1^\top \boldsymbol{u}_{1f}^{k+1} + \tilde{R}_2^\top \boldsymbol{u}_{2f}^{k+1} = \left[\tilde{R}_1^\top \left(I_1 - \alpha\left(H_{ff}^1 - H_{fc}^1 (H_{cc}^1)^{-1} H_{cf}^1\right)\right) R_1 + \tilde{R}_2^\top \left(I_2 - \alpha\left(H_{ff}^2 - H_{fc}^2 (H_{cc}^2)^{-1} H_{cf}^2\right)\right) R_2\right] \boldsymbol{u}^k.$$

We said to test this iterative matrix numerically with only fine grids everywhere to see its spectral radius. If the results are good, we can then talk about the next step.

As I accidentally test first the elliptic case, which is in the MATLAB code `EllipIterMatRho`, I also describe the elliptic problem briefly.

Elliptic problem: Let $\hat{y} \in L^2(\Omega)$ be a given target and $\nu > 0$ a penalization parameter. We minimize the cost functional

$$J(y,u) = \frac{1}{2}\|y - \hat{y}\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}. \tag{6}$$

such that

$$-\Delta y = u \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega. \tag{7}$$

The Lagrangian is given by

$$\mathcal{L}(y,u,p) = \frac{1}{2}\int_\Omega (y(x) - \hat{y}(x))^2 \, dx + \frac{\nu}{2}\int_\Omega (u(x))^2 \, dx - \int_\Omega \nabla p(x) \cdot \nabla y(x) \, dx + \int_\Omega p(x) u(x) \, dx.$$

The optimality system is given by

$$\begin{aligned} -\Delta y &= u && \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ -\Delta p &= y - \hat{y} && \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \\ \nu u + p &= 0 && \text{in } \Omega. \end{aligned} \tag{8}$$

3 Discussion Jan 20th, 2026, Zoom

We mainly discussed the elliptic case and how to assemble the matrix without inverting globally the state matrix A and the adjoint matrix $A^\top$ in H. Since the role of the matrices H^1_{fc} and H^1_{cf} in the MATLAB code `EllipIterMatRho` or `ParaIterMatRho` is unclear, we talked about analyzing the problem at the continuous level and seeing if we can invert "a partial fine partial coarse $-\Delta$" instead of "a full $-\Delta$". The idea is the following (based on my understanding).

Continuous analysis: We assume that there exist two operators $B_1 : \Omega_1 \to \Omega$ and $B_2 : \Omega_2 \to \Omega$ such that the elliptic optimal control problem (6)-(7) can be rewritten as: minimizing the cost functional

$$J(y_1, y_2, u) = \frac{1}{2}\|y_1 + y_2 - \hat{y}\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_1\|^2_{L^2(\Omega_1)} + \frac{\nu}{2}\|u_2\|^2_{L^2(\Omega_2)},$$

such that

$$-\Delta y_1 = B_1 u_1 \text{ in } \Omega, \quad y_1 = 0 \text{ on } \partial\Omega, \qquad -\Delta y_2 = B_2 u_2 \text{ in } \Omega, \quad y_2 = 0 \text{ on } \partial\Omega.$$

Here, we have $y_1 + y_2 = y$, but $u_1 + u_2 \neq u$, instead $B_1 u_1 + B_2 u_2 = u$.

For a given u_1, we derive the optimality system associated with u_2,

$$\begin{aligned} -\Delta y_2 &= B_2 u_2 && \text{in } \Omega, \quad y_2 = 0 \quad \text{on } \partial\Omega, \\ -\Delta p_2 &= y_1 + y_2 - \hat{y} && \text{in } \Omega, \quad p_2 = 0 \quad \text{on } \partial\Omega, \\ \nu u_2 + B_2^\top p_2 &= 0 && \text{in } \Omega, \end{aligned}$$

where $B_2^\top : \Omega \to \Omega_2$. Applying a gradient descent to solve it, one gets

$$u_2^{k+1} = u_2^k - \alpha\left(\nu u_2^k + B_2^\top\left((-\Delta)^{-2} B_2 u_2^k + (-\Delta)^{-1} y_1^k - (-\Delta)^{-1}\hat{y}\right)\right).$$

Meanwhile, we derive the optimality system associated with u_1 for a given u_2,

$$\begin{aligned} -\Delta y_1 &= B_1 u_1 && \text{in } \Omega, \quad y_1 = 0 \quad \text{on } \partial\Omega, \\ -\Delta p_1 &= y_1 + y_2 - \hat{y} && \text{in } \Omega, \quad p_1 = 0 \quad \text{on } \partial\Omega, \\ \nu u_1 + B_1^\top p_1 &= 0 && \text{in } \Omega, \end{aligned}$$

where $B_1^\top : \Omega \to \Omega_1$. Applying a gradient descent to solve it, one gets

$$u_1^{k+1} = u_1^k - \alpha\left(\nu u_1^k + B_1^\top\left((-\Delta)^{-2} B_1 u_1^k + (-\Delta)^{-1} y_2^k - (-\Delta)^{-1}\hat{y}\right)\right).$$

The next iteration u^{k+1} is then defined by

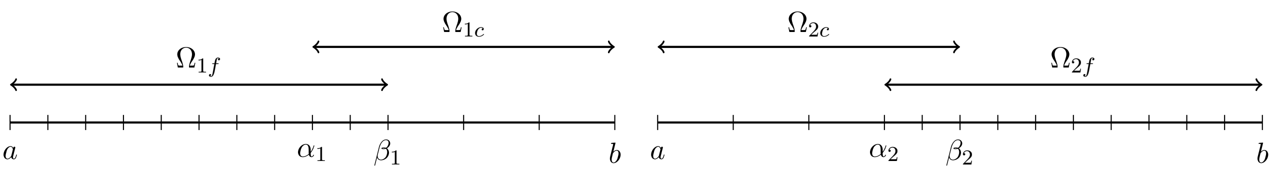
$$\begin{aligned}
u^{k+1} =& B_1 u_1^{k+1} + B_2 u_2^{k+1} \\
=& u^k - \alpha\nu u^k - \alpha(-\Delta)^{-2} u^k - \alpha(-\Delta)^{-1} y^k + 2\alpha(-\Delta)^{-1}\hat{y}, \\
=& \left(1 - \alpha(\nu + (-\Delta)^{-2})\right) u^k + \alpha(-\Delta)^{-1}(2\hat{y} - y^k)
\end{aligned}$$

where we need $B_1 B_1^\top = Id$ and $B_2 B_2^\top = Id$.

Question: 1. What are B_1 and B_2 ? In particular, the cost functional is kind of saying that $\|u\|^2_{L^2(\Omega)} = \|B_1 u_1 + B_2 u_2\|^2_{L^2(\Omega)} = \|u_1\|^2_{L^2(\Omega_1)} + \|u_2\|^2_{L^2(\Omega_2)}$. 2. I do not fully understand what "a partial fine partial coarse $-\Delta$" means at the continuous level.

From my side, the role of the two operators B_1, B_2 is unclear to me, and we do not have any mesh (fine or coarse) at the continuous level. Therefore, I try to think if there are some alternatives. Our goal is to avoid solving a global $-\Delta$, and probably later use this coarse-fine decomposition idea in the discrete setting. Based on this, I have the following idea using classic domain decomposition.

Idea from DD: We take the linear elliptic optimal control problem (6)-(7), or more precesily the associated optimality system (8). Consider two decompositions of the domain (a, b): $\Omega_1 = (a, \beta_1) \cup (\alpha_1, b)$ and $\Omega_2 = (a, \beta_2) \cup (\alpha_2, b)$, e.g see the illustration below.



They are very similar to the case that we talked about on Dec 16th, 2025. The only differences are that it is only in the space now, and there is an overlap in order to use the alternating Schwarz framework. Note that it is not necessary to have the overlap in the general decomposition setting.

For Ω_1, we use the alternating Schwarz framework to decompose the optimality system (8), which reads,

$$\begin{aligned}
-\partial_{xx} y_{1f} &= u_{1f} && \text{in } (a, \beta_1), & -\partial_{xx} y_{1c} &= u_{1c} && \text{in } (\alpha_1, b), \\
y_{1f}(a) &= 0,\ y_{1f}(\beta_1) = y_{1c}(\beta_1), && & y_{1c}(b) &= 0,\ y_{1c}(\alpha_1) = y_{1f}(\alpha_1), && \\
-\partial_{xx} p_{1f} &= y_{1f} - \hat{y} && \text{in } (a, \beta_1), & -\partial_{xx} p_{1c} &= y_{1c} - \hat{y} && \text{in } (\alpha_1, b), \\
p_{1f}(a) &= 0,\ p_{1f}(\beta_1) = p_{1c}(\beta_1), && & p_{1c}(b) &= 0,\ p_{1c}(\alpha_1) = p_{1f}(\alpha_1), && \\
\nu u_{1f} + p_{1f} &= 0 && \text{in } (a, \beta_1), & \nu u_{1c} + p_{1c} &= 0 && \text{in } (\alpha_1, b),
\end{aligned} \tag{9}$$

Similarly, we also apply such an alternating Schwarz framework in Ω_2, which reads,

$$\begin{aligned}
-\partial_{xx} y_{2c} &= u_{2c} && \text{in } (a, \beta_2), & -\partial_{xx} y_{2f} &= u_{2f} && \text{in } (\alpha_2, b), \\
y_{2c}(a) &= 0,\ y_{2c}(\beta_2) = y_{2f}(\beta_2), && & y_{2f}(b) &= 0,\ y_{2f}(\alpha_2) = y_{2c}(\alpha_2), && \\
-\partial_{xx} p_{2c} &= y_{2c} - \hat{y} && \text{in } (a, \beta_2), & -\partial_{xx} p_{2f} &= y_{2f} - \hat{y} && \text{in } (\alpha_1, b), \\
p_{2c}(a) &= 0,\ p_{2c}(\beta_2) = p_{2f}(\beta_2), && & p_{2f}(b) &= 0,\ p_{2f}(\alpha_2) = p_{2c}(\alpha_2), && \\
\nu u_{2c} + p_{2c} &= 0 && \text{in } (a, \beta_2), & \nu u_{2f} + p_{2f} &= 0 && \text{in } (\alpha_2, b),
\end{aligned} \tag{10}$$

Here, each $-\Delta$ (or each $-\partial_{xx}$) is only defined in each subdomain, and thus they are local. We now define our control variables by $u_1 := (u_{1f}, u_{1c})$ and $u_2 := (u_{2c}, u_{2f})$, we can then follow the same reasoning discussed in Section 2 to analyze the behavior of the gradient descent iteration at the continuous level for the variable $u := \tilde{\mathcal{R}}_1 u_{1f} + \tilde{\mathcal{R}}_2 u_{2f}$, where $\tilde{\mathcal{R}}_1, \tilde{\mathcal{R}}_2$ are the partition operators correspond to the partion matrices $\tilde{R}_1$ and $\tilde{R}_2$.

I do not know if this makes sense. Please let me know your opinion.

4 Discretization and numerical tests

As I use two different approaches in the numerical tests, I briefly describe both approaches.

Discretize-then-optimize: We describe here the discretize-then-optimize approach to solve the problem (1)-(2). Let $0 = t_0 < t_1 < \ldots < t_M = T$ with uniform time step $\Delta t = T/M$. Denote $\boldsymbol{y}_m \approx \boldsymbol{y}(t_m)$, $\hat{\boldsymbol{y}}_m \approx \hat{\boldsymbol{y}}(t_m)$, $\boldsymbol{u}_m \approx \boldsymbol{u}(t_m)$ and $\boldsymbol{p}_m \approx \boldsymbol{p}(t_m)$. Applying the Crank-Nicolson time integration method for (2) gives

$$\frac{\boldsymbol{y}_{m+1} - \boldsymbol{y}_m}{\Delta t} + A\frac{\boldsymbol{y}_{m+1} + \boldsymbol{y}_m}{2} = \frac{\boldsymbol{u}_{m+1} + \boldsymbol{u}_m}{2} \;\Leftrightarrow\; \left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_{m+1} - \left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_m = \frac{\Delta t}{2}(\boldsymbol{u}_{m+1} + \boldsymbol{u}_m), \tag{11}$$

for $m = 0, \ldots, M-1$ and a given $\boldsymbol{y}_0$. To keep consistence with the Crank-Nicolson method, we use the trapezodial rule for numerical integration of the cost function (1) and find

$$J_M(\boldsymbol{y}, \boldsymbol{u}) := \frac{\Delta t}{4}\sum_{m=0}^{M-1}\left(|\boldsymbol{y}_{m+1} - \hat{\boldsymbol{y}}_{m+1}|^2 + |\boldsymbol{y}_m - \hat{\boldsymbol{y}}_m|^2\right) + \frac{\gamma}{2}|\boldsymbol{y}_M - \hat{\boldsymbol{y}}_M|^2 + \frac{\nu\Delta t}{4}\sum_{m=0}^{M-1}\left(|\boldsymbol{u}_{m+1}|^2 + |\boldsymbol{u}_m|^2\right). \tag{12}$$

The discrete Lagrangian then reads

$$\mathcal{L} = J_M - \sum_{m=0}^{M-1}\boldsymbol{p}_{m+1}^\top\left(\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_{m+1} - \left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_m - \frac{\Delta t}{2}(\boldsymbol{u}_{m+1} + \boldsymbol{u}_m)\right). \tag{13}$$

To obtain the discrete adjoint equation, one needs to do the "discrete integration by parts" in (13), that is

$$\begin{aligned}
&\sum_{m=0}^{M-1}\boldsymbol{p}_{m+1}^\top\left(\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_{m+1} - \left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_m\right)\\
&= \sum_{m=1}^{M}\boldsymbol{p}_m^\top\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_m - \sum_{m=0}^{M-1}\boldsymbol{p}_{m+1}^\top\left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_m\\
&= \sum_{m=1}^{M-1}\left(\left(I_N + \frac{\Delta t}{2}A\right)^\top\boldsymbol{p}_m - \left(I_N - \frac{\Delta t}{2}A\right)^\top\boldsymbol{p}_{m+1}\right)^\top\boldsymbol{y}_m + \boldsymbol{p}_M^\top\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_M - \boldsymbol{p}_1^\top\left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_0.
\end{aligned}$$

Meanwhile, we re-write the sum over m of $\boldsymbol{y}_m$ in (12),

$$\frac{\Delta t}{4}\sum_{m=0}^{M-1}|\boldsymbol{y}_{m+1} - \hat{\boldsymbol{y}}_{m+1}|^2 + \frac{\Delta t}{4}\sum_{m=0}^{M-1}|\boldsymbol{y}_m - \hat{\boldsymbol{y}}_m|^2 = \frac{\Delta t}{2}\sum_{m=1}^{M-1}|\boldsymbol{y}_m - \hat{\boldsymbol{y}}_m|^2 + \frac{\Delta t}{4}|\boldsymbol{y}_M - \hat{\boldsymbol{y}}_M|^2 + \frac{\Delta t}{4}|\boldsymbol{y}_0 - \hat{\boldsymbol{y}}_0|^2.$$

We derive now the discrete adjoint equation

$$\partial_{\boldsymbol{y}_m}\mathcal{L} = -\left(\left(I_N + \frac{\Delta t}{2}A\right)^\top\boldsymbol{p}_m - \left(I_N - \frac{\Delta t}{2}A\right)^\top\boldsymbol{p}_{m+1}\right) + \Delta t(\boldsymbol{y}_m - \hat{\boldsymbol{y}}_m), \quad m = 1, \ldots, M-1,$$

with the final condition

$$\partial_{\boldsymbol{y}_M}\mathcal{L} = -\left(I_N + \frac{\Delta t}{2}A\right)^\top\boldsymbol{p}_M + \frac{\Delta t}{2}(\boldsymbol{y}_M - \hat{\boldsymbol{y}}_M) + \gamma(\boldsymbol{y}_M - \hat{\boldsymbol{y}}_M).$$

We treat in a similar way of the sum related to $\boldsymbol{u}_m$ in (12)-(13),

$$\begin{aligned}
\frac{\nu\Delta t}{4}\sum_{m=0}^{M-1}|\boldsymbol{u}_{m+1}|^2 + \frac{\nu\Delta t}{4}\sum_{m=0}^{M-1}|\boldsymbol{u}_m|^2 &= \frac{\nu\Delta t}{2}\sum_{m=1}^{M-1}|\boldsymbol{u}_m|^2 + \frac{\nu\Delta t}{4}|\boldsymbol{u}_M|^2 + \frac{\nu\Delta t}{4}|\boldsymbol{u}_0|^2,\\
\frac{\Delta t}{2}\sum_{m=0}^{M-1}\boldsymbol{p}_{m+1}^\top\boldsymbol{u}_{m+1} + \frac{\Delta t}{2}\sum_{m=0}^{M-1}\boldsymbol{p}_{m+1}^\top\boldsymbol{u}_m &= \frac{\Delta t}{2}\sum_{m=1}^{M-1}(\boldsymbol{p}_m^\top + \boldsymbol{p}_{m+1}^\top)\boldsymbol{u}_m + \frac{\Delta t}{2}\boldsymbol{p}_M^\top\boldsymbol{u}_M + \frac{\Delta t}{2}\boldsymbol{p}_1^\top\boldsymbol{u}_0.
\end{aligned}$$

This then gives the discrete optimality condition

$$\partial_{\boldsymbol{u}_m}\mathcal{L} = \frac{\Delta t}{2}(\boldsymbol{p}_m + \boldsymbol{p}_{m+1}) + \nu\Delta t\boldsymbol{u}_m, \quad m = 1, \ldots, M-1, \quad \partial_{\boldsymbol{u}_M}\mathcal{L} = \frac{\nu\Delta t}{2}\boldsymbol{u}_M + \frac{\Delta t}{2}\boldsymbol{p}_M, \quad \partial_{\boldsymbol{u}_0}\mathcal{L} = \frac{\nu\Delta t}{2}\boldsymbol{u}_0 + \frac{\Delta t}{2}\boldsymbol{p}_1.$$

Equating these partial derivatives to zero gives the discrete optimality system using the Crank-Nicolson method,

$$
\begin{aligned}
\left(I_N+\frac{\Delta t}{2}A\right)\boldsymbol{y}_{m+1}-\left(I_N-\frac{\Delta t}{2}A\right)\boldsymbol{y}_m &= \frac{\Delta t}{2}(\boldsymbol{u}_{m+1}+\boldsymbol{u}_m), && m=0,\dots,M-1,\\
\left(I_N+\frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_m-\left(I_N-\frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_{m+1} &= \Delta t(\boldsymbol{y}_m-\hat{\boldsymbol{y}}_m), && m=1,\dots,M-1,\\
\left(I_N+\frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_M &= \left(\frac{\Delta t}{2}+\gamma\right)(\boldsymbol{y}_M-\hat{\boldsymbol{y}}_M), \\
\nu\boldsymbol{u}_m &= -\frac{\boldsymbol{p}_m+\boldsymbol{p}_{m+1}}{2}, && m=1,\dots,M-1,\\
\nu\boldsymbol{u}_M &= -\boldsymbol{p}_M, \quad \nu\boldsymbol{u}_0=-\boldsymbol{p}_1.
\end{aligned}
\tag{14}
$$

Denote by $\boldsymbol{X}_1 := (\boldsymbol{y}_1,\dots,\boldsymbol{y}_M)^\top$, $\boldsymbol{X}_2 := (\boldsymbol{p}_1,\dots,\boldsymbol{p}_M)^\top$ and $\boldsymbol{X}_3 := (\boldsymbol{u}_0,\dots,\boldsymbol{u}_M)^\top$. Note that there is no $\boldsymbol{p}_0$ in the discretize-then-optimize approach, since the discrete adjoint state starts from $\boldsymbol{p}_1$ associated with the first CN discrete state equation. Note also that the size of $\boldsymbol{X}_3$ is different from the others. The all-at-once block matrix form is given by

$$
\tilde{A}\boldsymbol{X}=\boldsymbol{F},\quad \tilde{A}=\begin{bmatrix}\tilde{A}_{11} & \mathbf{0} & \tilde{A}_{13}\\ \tilde{A}_{21} & \tilde{A}_{22} & \mathbf{0}\\ \mathbf{0} & \tilde{A}_{32} & \tilde{A}_{33}\end{bmatrix},\quad \boldsymbol{X}=\begin{bmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_2\\ \boldsymbol{X}_3\end{bmatrix},\quad \boldsymbol{F}=\begin{bmatrix}\boldsymbol{F}_1\\ \boldsymbol{F}_2\\ \mathbf{0}\end{bmatrix}.
$$

Each block matrix $\tilde{A}_{ij}$ is given by

$$
\tilde{A}_{11}=\begin{bmatrix}
I_N+\frac{\Delta t}{2}A & & & & \\
-\left(I_N-\frac{\Delta t}{2}A\right) & I_N+\frac{\Delta t}{2}A & & & \\
 & \ddots & \ddots & & \\
 & & -\left(I_N-\frac{\Delta t}{2}A\right) & I_N+\frac{\Delta t}{2}A & \\
 & & & -\left(I_N-\frac{\Delta t}{2}A\right) & I_N+\frac{\Delta t}{2}A
\end{bmatrix},
$$

$$
\tilde{A}_{13}=\begin{bmatrix}
-\frac{\Delta t}{2}I_N & -\frac{\Delta t}{2}I_N & & \\
 & \ddots & \ddots & \\
 & & -\frac{\Delta t}{2}I_N & -\frac{\Delta t}{2}I_N
\end{bmatrix},\quad
\tilde{A}_{21}=\begin{bmatrix}
-\Delta t I_N & & & & \\
 & -\Delta t I_N & & & \\
 & & \ddots & & \\
 & & & -\Delta t I_N & \\
 & & & & -\left(\frac{\Delta t}{2}+\gamma\right)I_N
\end{bmatrix},
$$

$$
\tilde{A}_{22}=\begin{bmatrix}
I_N+\frac{\Delta t}{2}A^\top & -\left(I_N-\frac{\Delta t}{2}A^\top\right) & & & \\
 & I_N+\frac{\Delta t}{2}A^\top & -\left(I_N-\frac{\Delta t}{2}A^\top\right) & & \\
 & & \ddots & \ddots & \\
 & & & I_N+\frac{\Delta t}{2}A^\top & -\left(I_N-\frac{\Delta t}{2}A^\top\right)\\
 & & & & I_N+\frac{\Delta t}{2}A^\top
\end{bmatrix},
$$

$$
\tilde{A}_{32}=\begin{bmatrix}
I_N & & & \\
\frac{1}{2}I_N & \frac{1}{2}I_N & & \\
 & \ddots & \ddots & \\
 & & \frac{1}{2}I_N & \frac{1}{2}I_N\\
 & & & I_N
\end{bmatrix},\quad
\tilde{A}_{33}=\begin{bmatrix}
\nu I_N & & \\
 & \ddots & \\
 & & \nu I_N
\end{bmatrix}.
$$

Note that $\tilde{A}_{13}\in\mathbb{R}^{NM\times N(M+1)}$ and $\tilde{A}_{32}\in\mathbb{R}^{N(M+1)\times NM}$ are two rectangular matrices. Note also that $\tilde{A}_{22} = \tilde{A}_{11}^\top\in\mathbb{R}^{NM\times NM}$. The right-hand side vector is given by

$$
\boldsymbol{F}_1=\begin{bmatrix}\left(I_N-\frac{\Delta t}{2}A\right)\boldsymbol{y}_0\\ 0\\ \vdots\\ 0\end{bmatrix},\quad
\boldsymbol{F}_2=\begin{bmatrix}-\Delta t\hat{\boldsymbol{y}}_1\\ \vdots\\ -\Delta t\hat{\boldsymbol{y}}_{M-1}\\ -\left(\frac{\Delta t}{2}+\gamma\right)\hat{\boldsymbol{y}}_M\end{bmatrix}.
$$

Bring everything to the control variable, in this case $\boldsymbol{X}_3$, gives

$$(\tilde{A}_{33} + \tilde{A}_{32}\tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{13})\boldsymbol{X}_3 + \tilde{A}_{32}\tilde{A}_{22}^{-1}\boldsymbol{F}_2 - \tilde{A}_{32}\tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\boldsymbol{F}_1 = 0.$$

Therefore, the matrix H appeared in the Discussion Dec 16th, 2025 (Section 2) should be $H := \tilde{A}_{33} + \tilde{A}_{32}\tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{13}$ using the discretize-then-optimize approach.

Reduced system: We can substitute $\boldsymbol{u}_m$ by $\boldsymbol{p}_m$ to obtain the discrete reduced optimality system

$$\begin{aligned}
\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_{m+1} - \left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_m + \frac{\Delta t}{4\nu}(\boldsymbol{p}_m + 2\boldsymbol{p}_{m+1} + \boldsymbol{p}_{m+2}) &= 0, && m = 1,\dots,M-2,\\
\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_1 - \left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_0 + \frac{\Delta t}{4\nu}(3\boldsymbol{p}_1 + \boldsymbol{p}_2) &= 0,\\
\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_M - \left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_{M-1} + \frac{\Delta t}{4\nu}(3\boldsymbol{p}_M + \boldsymbol{p}_{M-1}) &= 0,\\
\left(I_N + \frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_m - \left(I_N - \frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_{m+1} - \Delta t\boldsymbol{y}_m &= -\Delta t\hat{\boldsymbol{y}}_m, && m = 1,\dots,M-1,\\
\left(I_N + \frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_M - \left(\frac{\Delta t}{2} + \gamma\right)\boldsymbol{y}_M &= -\left(\frac{\Delta t}{2} + \gamma\right)\hat{\boldsymbol{y}}_M.
\end{aligned} \tag{15}$$

The block matrix form of the reduced system is given by

$$\tilde{A}\boldsymbol{X} = \boldsymbol{F}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{bmatrix},$$

with

$$\tilde{A}_{12} = \begin{bmatrix} \frac{3\Delta t}{4\nu}I_N & \frac{\Delta t}{4\nu}I_N & & & \\ \frac{\Delta t}{4\nu}I_N & \frac{\Delta t}{2\nu}I_N & \frac{\Delta t}{4\nu}I_N & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\Delta t}{4\nu}I_N & \frac{\Delta t}{2\nu}I_N & \frac{\Delta t}{4\nu}I_N \\ & & & \frac{\Delta t}{4\nu}I_N & \frac{3\Delta t}{4\nu}I_N \end{bmatrix}.$$

The rest matrices are the same as in the full all-at-once matrix form. This reduced form is used in the MATLAB code `DtovsOtd`, where these block matrices are constructed by the function `BuildCNMatrixDto`, and the right-hand side vector is constructed by the function `BuildCNRhsDto`. This serves as the implemented solver for the discretize-then-optimize approach.

Optimize-then-discretize: Note that the discrete optimality system obtained in (14) is different from applying directly the Crank-Nicolson method to discretize the optimality system (3). For comparison, we also show the optimize-then-discretize approach to solve the problem (1)-(2), which consists of discretizing directly the optimality system (3) using the Crank-Nicolson method.

$$\begin{aligned}
\frac{\boldsymbol{y}_{m+1} - \boldsymbol{y}_m}{\Delta t} + \frac{A}{2}(\boldsymbol{y}_{m+1} + \boldsymbol{y}_m) &= \frac{1}{2}(\boldsymbol{u}_{m+1} + \boldsymbol{u}_m), && m = 0,\dots,M-1,\\
-\frac{\boldsymbol{p}_{m+1} - \boldsymbol{p}_m}{\Delta t} + \frac{A^\top}{2}(\boldsymbol{p}_{m+1} + \boldsymbol{p}_m) &= \frac{1}{2}(\boldsymbol{y}_{m+1} - \hat{\boldsymbol{y}}_{m+1} + \boldsymbol{y}_m - \hat{\boldsymbol{y}}_m), && m = 0,\dots,M-1,\\
\boldsymbol{p}_M &= \gamma(\boldsymbol{y}_M - \hat{\boldsymbol{y}}_M),\\
\nu\boldsymbol{u}_M &= -\boldsymbol{p}_M, && m = 0,\dots,M.
\end{aligned} \tag{16}$$

As in the discretize-then-optimize approach, we denote by $\boldsymbol{X}_1 := (\boldsymbol{y}_0,\dots,\boldsymbol{y}_M)^\top$, $\boldsymbol{X}_2 := (\boldsymbol{p}_0,\dots,\boldsymbol{p}_M)^\top$ and $\boldsymbol{X}_3 := (\boldsymbol{u}_0,\dots,\boldsymbol{u}_M)^\top$. Note that all $\boldsymbol{X}_j$ here have the same size, as we discretize the adjoint equation directly, and we have the $\boldsymbol{p}_0$ term. Using the same notations as in the previous approach, the all-at-once block matrix form is given by

$$\tilde{A}\boldsymbol{X} = \boldsymbol{F}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \boldsymbol{0} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \boldsymbol{X}_3 \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{0} \end{bmatrix}.$$

Each block matrix $\tilde{A}_{ij} \in \mathbb{R}^{N(M+1)\times N(M+1)}$ is given by

$$
\tilde{A}_{11} = \begin{bmatrix} I_N & & & & \\ -\left(I_N - \frac{\Delta t}{2}A\right) & I_N + \frac{\Delta t}{2}A & & & \\ & \ddots & \ddots & & \\ & & -\left(I_N - \frac{\Delta t}{2}A\right) & I_N + \frac{\Delta t}{2}A & \\ & & & -\left(I_N - \frac{\Delta t}{2}A\right) & I_N + \frac{\Delta t}{2}A \end{bmatrix},
$$

$$
\tilde{A}_{13} = \begin{bmatrix} 0 & & & \\ -\frac{\Delta t}{2}I_N & -\frac{\Delta t}{2}I_N & & \\ & \ddots & \ddots & \\ & & -\frac{\Delta t}{2}I_N & -\frac{\Delta t}{2}I_N \end{bmatrix}, \quad \tilde{A}_{21} = \begin{bmatrix} -\frac{\Delta t}{2}I_N & -\frac{\Delta t}{2}I_N & & & \\ & -\frac{\Delta t}{2}I_N & -\frac{\Delta t}{2}I_N & & \\ & & \ddots & \ddots & \\ & & & -\frac{\Delta t}{2}I_N & -\frac{\Delta t}{2}I_N \\ & & & & -\gamma I_N \end{bmatrix},
$$

$$
\tilde{A}_{22} = \begin{bmatrix} I_N + \frac{\Delta t}{2}A^\top & -\left(I_N - \frac{\Delta t}{2}A^\top\right) & & & \\ & I_N + \frac{\Delta t}{2}A^\top & -\left(I_N - \frac{\Delta t}{2}A^\top\right) & & \\ & & \ddots & \ddots & \\ & & & I_N + \frac{\Delta t}{2}A^\top & -\left(I_N - \frac{\Delta t}{2}A^\top\right) \\ & & & & I_N \end{bmatrix},
$$

$$
\tilde{A}_{32} = \begin{bmatrix} I_N & & \\ & \ddots & \\ & & I_N \end{bmatrix}, \quad \tilde{A}_{33} = \begin{bmatrix} \nu I_N & & \\ & \ddots & \\ & & \nu I_N \end{bmatrix}.
$$

The right-hand side vector is given by

$$
\boldsymbol{F}_1 = \begin{bmatrix} \boldsymbol{y}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{F}_2 = \begin{bmatrix} -\frac{\Delta t}{2}(\hat{\boldsymbol{y}}_0 + \hat{\boldsymbol{y}}_1) \\ \vdots \\ -\frac{\Delta t}{2}(\hat{\boldsymbol{y}}_{M-1} + \hat{\boldsymbol{y}}_M) \\ -\gamma \hat{\boldsymbol{y}}_M \end{bmatrix}.
$$

Bring everything to the control variable $\boldsymbol{X}_3$ gives

$$
(\tilde{A}_{33} + \tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{13})\boldsymbol{X}_3 + \tilde{A}_{22}^{-1}\boldsymbol{F}_2 - \tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\boldsymbol{F}_1 = 0.
$$

Therefore, the matrix H appeared in the discussion on Dec 16th, 2025 (Section 2) should be $H := \tilde{A}_{33} + \tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{13}$ using the optimize-then-discretize approach.

Reduced system: We can substitute $\boldsymbol{u}_m$ by $\boldsymbol{p}_m$ to obtain

$$
\begin{aligned}
\left(I_N + \frac{\Delta t}{2}A\right)\boldsymbol{y}_{m+1} - \left(I_N - \frac{\Delta t}{2}A\right)\boldsymbol{y}_m + \frac{\Delta t}{2\nu}(\boldsymbol{p}_m + \boldsymbol{p}_{m+1}) &= 0, && m = 0,\dots,M-1, \\
\left(I_N + \frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_m - \left(I_N - \frac{\Delta t}{2}A^\top\right)\boldsymbol{p}_{m+1} - \frac{\Delta t}{2}(\boldsymbol{y}_{m+1} + \boldsymbol{y}_m) &= -\frac{\Delta t}{2}(\hat{\boldsymbol{y}}_{m+1} + \hat{\boldsymbol{y}}_m), && m = 0,\dots,M-1, \\
\boldsymbol{p}_M - \gamma \boldsymbol{y}_M &= -\gamma \hat{\boldsymbol{y}}_M.
\end{aligned} \tag{17}
$$

The block matrix form of the reduced system is given by

$$
\tilde{A}\boldsymbol{X} = \boldsymbol{F}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{bmatrix},
$$

with

$$
\tilde{A}_{12} = \begin{bmatrix} 0 & & & & \\ \frac{\Delta t}{2\nu}I_N & \frac{\Delta t}{2\nu}I_N & & & \\ & \ddots & \ddots & & \\ & & \frac{\Delta t}{2\nu}I_N & \frac{\Delta t}{2\nu}I_N & \\ & & & \frac{\Delta t}{2\nu}I_N & \frac{\Delta t}{2\nu}I_N \end{bmatrix}.
$$

The rest matrices are the same as in the full all-at-once matrix form. This reduced form is used in the MATLAB code `DtovsOtd`, where these block matrices are constructed by the function `BuildCNMatrixOtd`, and the right-hand side vector is constructed by the function `BuildCNRhsOtd`. This serves as the implemented solver for the optimize-then-discretize approach.

Description of MATLAB code `DtovsOtd`: To test these two approaches, we use the following manufactured solutions,

$$y(x,t) = \sin(\pi x)(2t^2+t), \quad \hat{y}(x,t) = \nu \sin(\pi x)((\pi^4+1/\nu)(2t^2+t)-4), \quad p(x,t) = -\nu \sin(\pi x)(\pi^2(2t^2+t)+4t+1),$$

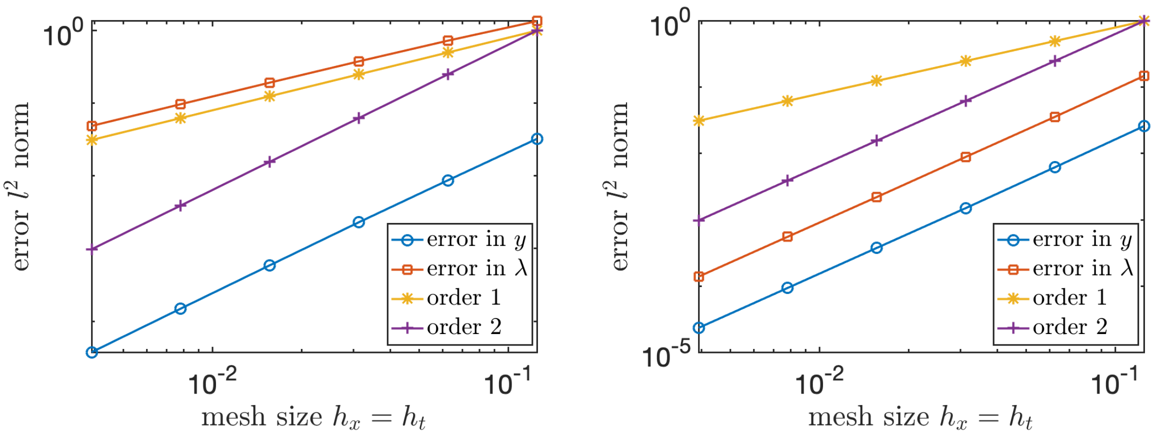
and a special penalization parameter γ to ensure the final condition,

$$\gamma = \frac{\pi^2(2T^2+T)+4T+1}{\pi^4(2T^2+T)-4},$$

for $x \in (0,1)$. These solutions satisfy the 1D reduced optimality system

$$\begin{aligned} \partial_t y - \partial_{xx} y &= -\nu^{-1} p, \quad y(0,t) = y(1,t) = 0, \quad y(x,0) = 0, \\ -\partial_t p - \partial_{xx} p &= y - \hat{y}, \quad\;\; p(0,t) = p(1,t) = 0, \quad p(x,T) = \gamma(y(x,T) - \hat{y}(x,T)). \end{aligned}$$

We set $T = 1$, $\nu = 1$ and $h_t = h_x = \{2^{-3}, \ldots, 2^{-8}\}$.



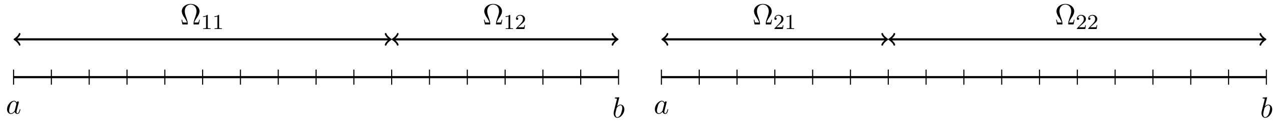
On the left is the error decay when refining the mesh for the discretize-then-optimize approach (15), and on the right is the case for the optimize-then-discretize approach (17). It is second-order for the optimize-then-discretize approach, as we use CN scheme to discretize the forward-backward system. However, it is only first-order for the adjoint variable in the discretize-then-optimize approach.

Description of MATLAB code `EllipIterMatRho`: Consider a one dimensional domain $\Omega = (a,b)$, we discretize $-\Delta$ by the centered finite difference that leads to a symmetric tri-diagonal matrix $A \in \mathbb{R}^{N\times N}$. The discrete all-at-one system then reads

$$\begin{bmatrix} A & 0 & -I_N \\ -I_N & A^\top & 0 \\ 0 & I_N & \nu I_N \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{p} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -\hat{\boldsymbol{y}} \\ 0 \end{bmatrix}, \tag{18}$$

or equivalently $H\boldsymbol{u} = A^\top \hat{\boldsymbol{y}}$ with $H := \nu I_N + (AA^\top)^{-1}$.

I follow the idea proposed in Section 2, except that I tested for the above linear elliptic optimal control problem. We consider two decompositions of the domain Ω: $\Omega_1 = \Omega_{11} \cup \Omega_{12}$ and $\Omega_2 = \Omega_{21} \cup \Omega_{22}$ with $\Omega_1 = \Omega_2 = \Omega$.



In each domain Ω_j, $j = 1, 2$, we solve the same optimality system (18) using a different decomposition. This corresponds to two control vectors $\boldsymbol{u}_1 := (\boldsymbol{u}_{11}, \boldsymbol{u}_{12})^\top$ and $\boldsymbol{u}_2 := (\boldsymbol{u}_{21}, \boldsymbol{u}_{22})^\top$, each satisfies a linear system $H^j \boldsymbol{u}_j = A^\top \hat{\boldsymbol{y}}$ associated with the problem defined in Ω_j, $j = 1, 2$. Based on the structure of Ω_j, the two matrices satisfy $H^1 = H^2 = H$ and are given by

$$H^1 = \begin{bmatrix} H^1_{11} & H^1_{12} \\ H^1_{21} & H^1_{22} \end{bmatrix}, \quad H^2 = \begin{bmatrix} H^2_{11} & H^2_{12} \\ H^2_{21} & H^2_{22} \end{bmatrix},$$

where each block is a truncation of the matrix H. The Schur complement S_1 defined by $S_1 := H^1_{11} - H^1_{12}(H^1_{22})^{-1}H^1_{21}$ is used to solve for $\boldsymbol{u}_{11}$, and the Schur complement S_1 defined by $S_2 := H^2_{22} - H^2_{21}(H^2_{11})^{-1}H^2_{12}$ is used to solve for $\boldsymbol{u}_{22}$. We apply the gradient descent algorithm to solve both $\boldsymbol{u}_{11}$ and $\boldsymbol{u}_{22}$ iteratively as explained in Section 2, and compute the spectral radius of the iterative matrix $\tilde{R}_1^\top(I - \alpha_1 S_1)R_1 + \tilde{R}_2^\top(I - \alpha_2 S_2)R_2$.

Description of MATLAB code `ParaIterMatRho`: This part tries to test the idea given in the discussion on Dec 16th, 2025 (Section 2) using only a fine mesh everywhere. The matrix H associated with the gradient is different depending on the two approaches. In the discretize-then-optimize approach, $H := \tilde{A}_{33} + \tilde{A}_{32}\tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{13}$; in the optimize-then-discretize approach, $H := \tilde{A}_{33} + \tilde{A}_{22}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{13}$. Details are given above for each approach. Apart from that, the rest follows the same steps as in the elliptic case, and we compute the spectral radius of the associated iterative matrix $\tilde{R}_1^\top(I - \alpha_1 S_1)R_1 + \tilde{R}_2^\top(I - \alpha_2 S_2)R_2$.

References

[1] G. Ciaramella and T. Vanzan, *Variable reduction as a nonlinear preconditioning approach for optimization problems*, in Domain Decomposition Methods in Science and Engineering XXVIII, Springer Cham, 2025.