1. 2-D Parallel-plates Waveguide

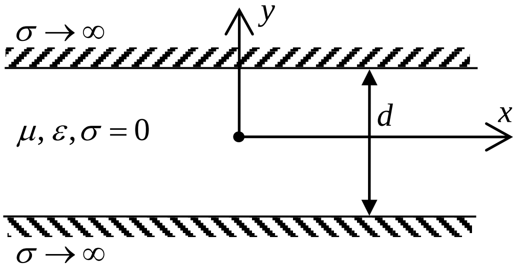

Figure 1. 2D parallel-plate loss-free waveguide

$$\Delta\vec{E} + \omega^2 \mu\varepsilon\vec{E} = 0 \tag{1}$$

E-polarisation: $\vec{E} = \vec{e}_z \cdot E_z(x, y)$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \omega^2 \mu\varepsilon E_z = 0 \tag{2}$$

X-propagation direction: $E_z(x, y) = E_{z0}(y) \cdot e^{-j\gamma x}$

$$\frac{\partial E_z}{\partial x} = -j\gamma \cdot E_{z0}(y) \cdot e^{-j\gamma x}, \frac{\partial^2 E_z}{\partial x^2} = -\gamma^2 \cdot E_{z0}(y) \cdot e^{-j\gamma x} \tag{3}$$

$$(2) \overset{(3)}{\Rightarrow} \frac{d^2 E_{z0}}{dy^2} + \left(\omega^2 \mu\varepsilon - \gamma^2\right) E_{z0} = 0 \tag{4}$$

$$k^2 = \omega^2 \mu\varepsilon - \gamma^2 \tag{5}$$

$$\frac{d^2 E_{z0}}{dy^2} + k^2 \cdot E_{z0} = 0 \tag{6}$$

$$E_{z0}(y) = C_1 \cdot \cos\left(k \cdot y\right) + C_2 \cdot \sin\left(k \cdot x\right) \tag{7}$$

$$E_z(x, y) = C_1 \cdot \cos\left(k \cdot y\right) \cdot e^{-j\gamma x} + C_2 \cdot \sin\left(k \cdot x\right) \cdot e^{-j\gamma x} = EvenModes + OddModes \tag{8}$$

Boundary condition:

$$y = \pm\frac{d}{2} \Rightarrow E_{z0} = 0 \tag{9}$$

Even Modes:	Odd Modes:
$\cos\left(k\cdot\frac{\pm d}{2}\right)=0\Rightarrow k\cdot\frac{d}{2}=\frac{2n+1}{2}\cdot\pi$ $\Rightarrow k=\frac{2n+1}{d}\cdot\pi,\ n=0,1,2,3,\ldots$	$\sin\left(k\cdot\frac{\pm d}{2}\right)=0\Rightarrow k\cdot\frac{d}{2}=n\cdot\pi$ $\Rightarrow k=\frac{2n}{d}\cdot\pi,\ n=1,2,3,\ldots$

$d=0.02m$
$\varepsilon=8.85\cdot10^{-12}F/m,\ \mu=12.56\cdot10^{-7}Tm/A$

$\gamma\in R\Rightarrow$ Propagating modes
$\gamma\in C\Rightarrow$ Evanescent modes

$$\gamma=\sqrt{\omega^2\mu\varepsilon-k^2}\Rightarrow \text{ Cut off condition: } \omega\geq\frac{k}{\sqrt{\mu\varepsilon}}=k\cdot v \qquad (5)$$

Even modes for $n=0$ (fundamental mode):

$$k=\frac{\pi}{d}=157\Rightarrow f\geq\frac{k\cdot v}{2\pi}=7.5\cdot10^{9}Hz$$

General boundary that is neither perpendicular to the x-axis nor to the y-axis:

$$E_z(\vec{r})=E_{zE}(\vec{r_t})\cdot e^{-j\vec{k_n}\cdot\vec{r}}+E_{zR}(\vec{r_t})\cdot e^{+j\vec{k_n}\cdot\vec{r}}$$
$$\omega^2\mu\varepsilon=k_t^2+k_n^2$$

$$\frac{\partial}{\partial n}\left(e^{-j\vec{k_n}\cdot\vec{r}}\right)=\nabla\left(e^{-j\vec{k_n}\cdot\vec{r}}\right)\cdot\vec{n}=-j\cdot e^{-j\vec{k_n}\cdot\vec{r}}\cdot\nabla\left(\vec{k_n}\cdot\vec{r}\right)\cdot\vec{n}=-j\cdot e^{-j\vec{k_n}\cdot\vec{r}}\cdot\left[\left(\vec{k_n}\cdot\nabla\right)\vec{r}\right]\cdot\vec{n}$$
$$-jk_n\cdot e^{-j\vec{k_n}\cdot\vec{r}}\cdot\left[\left(\vec{n}\cdot\nabla\right)\vec{r}\right]\cdot\vec{n}=-jk_n\cdot e^{-j\vec{k_n}\cdot\vec{r}}\cdot\left[\left(n_x\frac{\partial}{\partial x}+n_y\frac{\partial}{\partial y}+n_z\frac{\partial}{\partial z}\right)\vec{r}\right]\cdot\vec{n}=-jk_n\cdot e^{-j\vec{k_n}\cdot\vec{r}}\cdot\left[\vec{n}\right]\cdot\vec{n}$$

$$\frac{\partial}{\partial n}\left(e^{-j\vec{k_n}\cdot\vec{r}}\right)=-jk_n\cdot e^{-j\vec{k_n}\cdot\vec{r}}$$
$$\frac{\partial}{\partial n}\left(e^{+j\vec{k_n}\cdot\vec{r}}\right)=+jk_n\cdot e^{-j\vec{k_n}\cdot\vec{r}}$$

$$\frac{\partial E_z}{\partial n}(x,y)=-jk_n\cdot E_{zE}(\vec{r_t})\cdot e^{-j\vec{k_n}\cdot\vec{r}}+jk_n\cdot E_{zR}(\vec{r_t})\cdot e^{+j\vec{k_n}\cdot\vec{r}}$$
$$\frac{\partial E_z}{\partial n}(x,y)=-2jk_n\cdot E_{zE}(\vec{r_t})\cdot e^{-j\vec{k_n}\cdot\vec{r}}+jk_n\cdot E_{zE}(\vec{r_t})\cdot e^{-j\vec{k_n}\cdot\vec{r}}+jk_n\cdot E_{zR}(\vec{r_t})\cdot e^{+j\vec{k_n}\cdot\vec{r}}$$
$$\frac{\partial E_z}{\partial n}(x,y)=-2jk_n\cdot E_{zE}(\vec{r_t})\cdot e^{-j\vec{k_n}\cdot\vec{r}}+jk_n\cdot E_z(x,y)$$
$$\frac{\partial E_z}{\partial n}(x,y)-jk_n\cdot E_z(x,y)=-2jk_n\cdot E_{zE}(\vec{r_t})\cdot e^{-j\vec{k_n}\cdot\vec{r}}$$

The last equation is valid if the vector $\vec{n}$ has the same direction as the vector $\vec{k_n}$.

H-polarisation: $\vec{H} = \vec{e}_z \cdot H_z(x,y)$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \omega^2 \mu\varepsilon H_z = 0 \tag{10}$$

X-propagation direction: $H_z(x,y) = H_{z0}(y) \cdot e^{-jk_x x}$

$$\frac{\partial H_z}{\partial x} = -jk_x \cdot H_{z0}(y) \cdot e^{-jk_x x}, \frac{\partial^2 H_z}{\partial x^2} = -k_x{}^2 \cdot H_{z0}(y) \cdot e^{-jk_x x} \tag{11}$$

$$(2) \overset{(3)}{\Rightarrow} \frac{d^2 H_{z0}}{dy^2} + \left(\omega^2 \mu\varepsilon - k_x{}^2\right) H_{z0} = 0 \tag{12}$$

$$k_y{}^2 = \omega^2 \mu\varepsilon - k_x{}^2 \tag{13}$$

$$\frac{d^2 H_{z0}}{dy^2} + k_y{}^2 \cdot H_{z0} = 0 \tag{14}$$

$$H_{z0}(y) = C_1 \cdot \cos\left(k_y \cdot y\right) + C_2 \cdot \sin\left(k_y \cdot y\right) \tag{15}$$

$$H_z(x,y) = C_1 \cdot \cos\left(k_y \cdot y\right) \cdot e^{-jk_x x} + C_2 \cdot \sin\left(k_y \cdot y\right) \cdot e^{-jk_x x} = EvenModes + OddModes \tag{16}$$

Boundary condition:

$$y = \pm\frac{d}{2} \Rightarrow \frac{dH_{z0}}{dy} = 0 \;\Rightarrow \tag{17}$$

Odd Modes:	Even Modes:
$\cos\left(k_y \cdot \frac{\pm d}{2}\right) = 0 \Rightarrow k_y \cdot \frac{d}{2} = \frac{2n+1}{2} \cdot \pi$ $\Rightarrow k_y = \frac{2n+1}{d} \cdot \pi,\ n = 0,1,2,3,\ldots$	$\sin\left(k_y \cdot \frac{\pm d}{2}\right) = 0 \Rightarrow k_y \cdot \frac{d}{2} = n \cdot \pi$ $\Rightarrow k_y = \frac{2n}{d} \cdot \pi,\ n = 1,2,3,\ldots$

$d = 0.1m$

$\varepsilon = 8.85 \cdot 10^{-12} F/m,\ \mu = 12.56 \cdot 10^{-7} Tm/A$

$k_x \in R \Rightarrow$ Propagating modes

$k_y \in C \Rightarrow$ Evanescent modes

$$k_x = \sqrt{\omega^2 \mu\varepsilon - k_y{}^2} \;\Rightarrow\; \text{Cut off condition: } \omega \geq \frac{k_y}{\sqrt{\mu\varepsilon}} = k_y \cdot v \tag{18}$$

Odd modes for $n = 0$ (fundamental mode):

$$k_y = \frac{\pi}{d} = 31.4 \quad \Rightarrow \quad f \geq \frac{k_y \cdot v}{2\pi} = 1.5 \cdot 10^9 \, Hz$$

$$H_z^{ex}(x, y) = H_0 \cos\left(\frac{\pi}{d} \cdot y\right) \cdot e^{-jk_x x}$$

$$E_x = \frac{1}{j\omega\varepsilon} \frac{\partial H_z}{\partial y}$$

$$E_y = -\frac{1}{j\omega\varepsilon} \frac{\partial H_z}{\partial x} \quad \Rightarrow \quad E_y^{ex}(x, y) = -\frac{1}{j\omega\varepsilon}(-jk_x) H_z^{ex}(x, y) = \frac{k_x}{\omega\varepsilon} H_z^{ex}(x, y)$$

$$S_x^{ex} = \frac{1}{2} H_z^{ex} \cdot E_y^{ex} = \frac{1}{2} \frac{k_x}{\omega\varepsilon} H_z^{ex2}$$

$$P_{input}^{ex} = \int_{-d/2}^{d/2} S_x^{ex}(0, y) \, dy = \frac{1}{2} H_0^2 \frac{k_x}{\omega\varepsilon} \int_{-d/2}^{d/2} \cos^2\left(\frac{\pi}{d} \cdot y\right) dy = \frac{1}{2} H_0^2 \frac{k_x}{\omega\varepsilon} \frac{d}{\pi} \left(\frac{1}{2} t + \frac{1}{4} \sin(2t)\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{2} H_0^2 \frac{k_x}{\omega\varepsilon} \frac{d}{\pi} \frac{\pi}{2} = \frac{1}{2} H_0^2 \frac{k_x}{\omega\varepsilon} \frac{d}{2}$$

$$P_{input}^{ex} = 1 \quad \Rightarrow \quad H_0 = \sqrt{2} \cdot \sqrt{\frac{2\omega\varepsilon}{dk_x}}$$ (Normalization)