BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY, DHAKA

L-3/T-2 B. Sc. Engineering Examinations, 2007-2008

Sub : **CSE 301** (Mathematical Analysis for Computer Science)

Full Marks : 210 Time : 3 Hours

The figures in the margin indicate full marks.

USE SEPARATE SCRIPTS FOR EACH SECTION

SECTION – A

There are **FOUR** questions in this section. Answer any **THREE**.

1. (a) Consider the coin changing problem. There are three types of coins of value 1, 5 and 10 cents, each of them as many as you need. Let Pn, Nn, Dn represent the number of ways you can pay n cents by using the above three types of coins. Answer the followings: **(20)**

   (i) Derive the recursive equations for Pn, Nn, Dn. Show all your steps.

   (ii) Use these functions to find the number of ways you can pay 53 cents.

   Is the number of ways for paying 53 cents the same as the number of ways for paying 54 cents? Justify.

   (b) Consider stirling numbers of the first form $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, which represents the number of ways to partition a set of n distinct elements into k non-empty subsets. Explain how you can derive the recursive equation $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$. Then consider the second form of stirling numbers $\left[\begin{matrix} n \\ k \end{matrix} \right]$, which represents the number of ways to partition n distinct elements into k cycles (instead of subsets). Explain how you can derive the recursive equation $\left[\begin{matrix} n \\ k \end{matrix} \right] = (n-1)\left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]$. **(15)**

2. For each of Binomial, Geometric, Poisson and Exponential random variables, do the followings: **(35)**

   (i) Explain the meaning of the random variable.

   (ii) Write the probability mass function.

   (iii) Show that over all possible outcomes, the probability sum is 1.

   (iv) Derive the expectation.

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3. (a) Suppose that the probabilities of whether or not it rains today are as follows: **(20)**

- If it rained last two days, then probability of rain today = 0.9
- If it rained yesterday but not the day before, then probability of rain today = 0.7
- If it did not rain yesterday, but it had rained the day before, then the probability of rain today = 0.5
- If it did not rain for the last two days, then the probability of rain today = 0.1

Write the transition probability matrix. Find the probability that it will rain Thursday, given that it rained Monday and Tuesday. Find the limiting probabilities (i.e. the long term probabilities) of the followings:

(i) it rains for consecutive two days.

(ii) it does not rain for consecutive two days.

(iii) it rains and does not rain in alternate days.

(b) Consider a unisex 2-step beauty parlor with two chairs: chair 1 and chair 2. A customer enters the parlor if chair 1 is empty. When her/his getting service in chair 1 is completed, s/he will either go to chair 2 if it is empty or wait until it becomes empty. Suppose that customer enters at a Poisson rate of λ and the service rate in chair 1 and chair 2 are at Poisson rates of μ_1 and μ_2, respectively. Define the states of the beauty parlor and draw the state transition diagram. **(15)**

4. (a) Consider an M/M/1 queue which has Poisson arrival rate λ and service rate μ. The state of the system is determined by the number of people in the system. **(15)**

(i) Explain why for a particular state, in the long run are the leaving rate and the entering rate the same?

(ii) Using the above observation, find the expression for Pn, the probability of the system is in state n, in terms of λ and μ.

(iii) Find the average number of customer in the system, L, and the average waiting time in the system, W.

(b) We want to compare the following two queuing systems. Each system has two infinite queues and two servers. The first system is a combination of two M/M/1 queues and the second system is an M/M/2 queue. Derive the average waiting time for the second system. Then compare it with that of the first system (you can use the expression for the first system from the answer of part (a) of the Question. **(20)**

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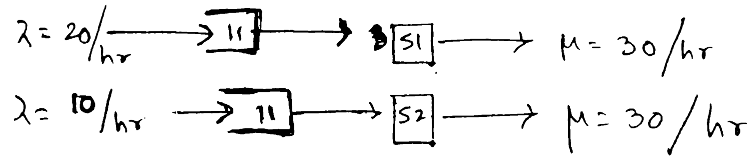
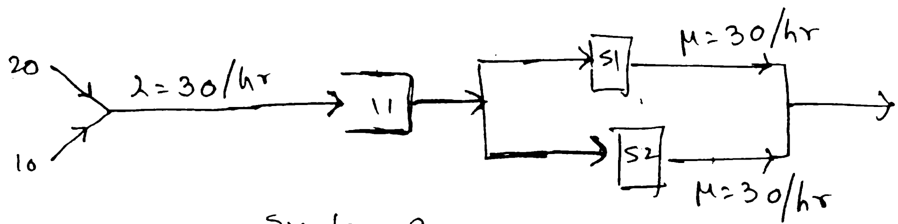


System 1



System 2

SECTION–B

There are **FOUR** questions in this section. Answer any **THREE**.

5. (a) Given a set of n-1 lines in the plane, explain how adding a new line can create n new regions. Also explain the case when it would create fewer than n regions. From here, write the necessary and sufficient condition for the maximum number of regions created by n lines in the plane and write a recursive equation. Then solve this equation to a closed form by any method you like. **(15)**

   (b) Prove by rearranging terms that **(12)**

$$\sum_{1\le j<k\le n} (a_j b_k - a_k b_j)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

   (c) Sum the series using perturbation method **(8)**

$$\sum_{1\le k\le 2n} (-1)^k k$$

6. (a) Give a recursive definition of Euclid's numbers. Give an example which shows that the Euclid's numbers are not always prime. Prove that any two Euclid numbers are relatively prime. Use this fact to prove further that there are infinitely many primes. **(12)**

   (b) Show that the expression **(15)**

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

   is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

   (c) Let $f_n = 2^{2^n} + 1$. Prove that $f_m \perp f_n$ if $m < n$. Here $\perp$ denotes relatively prime. **(8)**

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7. (a) Explain why this is true: $a \equiv b$ and $c \equiv d \Rightarrow ac \equiv bd$. Give a counter example which shows that the other direction is not true, that is $ac \equiv bd \Rightarrow a \equiv b$ and $c \equiv d$ is not true. What extra condition do you need to say that both are true? Justify. **(15)**

(b) Explain that the largest power of p that divides n! is $\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$.

From this equation, find the largest power of 3 that divides 100!. **(12)**

(c) Prove or disprove: **(8)**

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \le \lfloor 2x \rfloor + \lfloor 2y \rfloor.$$

8. (a) Prove that a harmonic number can be arbitrarily large. To prove this, use the following steps: **(15)**

(i) First, argue that it suffices to prove the equation:

$$H_{2^m} \ge 1 + \frac{m}{2}$$

(ii) Then prove $H_{2^{m+1}} \ge H_{2^m} + \frac{1}{2}$

(iii) From here, prove $H_{2^m} \ge 1 + \frac{m}{2}$.

(b) Prove using induction that **(8)**

$$\sum_{0 \le k < n} H_k = n H_n - n$$

(c) Show some examples such that $F_{2n} = F_n F_{n+1} + F_{n-1} F_n$, might be true. Assuming that this is true, prove that F_{kn} is multiple of F_n, for any $k \ge 2$. **(12)**